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Generalised geometry for supersymmetric flux backgrounds

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Declaration

I hereby certify that, to the best of my knowledge, all of the material in this dissertation which is not my own work has been properly acknowledged. This research described has been done in collaboration with Daniel Waldram, Michela Petrini, Mariana Graña and Maxime Gabella. The presentation closely follows our papers [1–3].

Abstract

We present a geometric description of flux backgrounds in supergravity that preserve eight supercharges using the language of (exceptional) generalised geometry. These "exceptional Calabi–Yau" geometries generalise complex, symplectic and hyper-Kähler geometries, where integrability is equivalent to supersymmetry for the background. The integrability conditions take the form of vanishing moment maps for the "generalised diffeomorphism group", and the moduli spaces of structures appear as hyper-Kähler and symplectic quotients. Our formalism applies to generic $D = 4, 5, 6$ backgrounds preserving eight supercharges in both type II and eleven-dimensional supergravity. We include a number of examples of flux backgrounds that can be reformulated as exceptional Calabi–Yau geometries.

We extend this analysis and show that generic AdS flux backgrounds in $D = 4, 5$ are also described by exceptional generalised geometry, giving what one might call "exceptional Sasaki–Einstein" geometry. These backgrounds always admit a "generalised Reeb vector" that generates a Killing symmetry of the background, corresponding to the R-symmetry of the dual field theory. We also discuss the relation between generalised structures and quantities in the dual field theory.

We then consider deformations of these generalised structures. For AdS_5 backgrounds in type IIB, a first-order deformation amounts to turning on three-form fluxes that preserve supersymmetry. We find the general form of these fluxes for any Sasaki–Einstein space and show that higher-order deformations are obstructed by the moment map for the symmetry group of the undeformed background. In the dual field theory, this corresponds to finding those marginal deformations that are exactly marginal. We give a number of examples and match to known expressions in the literature. We also apply our formalism to AdS_4 backgrounds in M-theory, where the first-order deformation amounts to turning on a four-form flux that preserves supersymmetry.

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Chapter 1

Introduction

In this introductory chapter, we begin with an overview of string theory and the problems it may solve. We then discuss supergravity backgrounds with an emphasis on those that preserve supersymmetry and comment on the use of G-structures. Finally, we review the use of generalised geometry for backgrounds with flux.

1.1 Physics in the twenty-first century

Modern theoretical physics is built upon two great pillars: general relativity and quantum mechanics. General relativity describes classical gravity and its interaction with matter. The essential idea of this theory is that geometry controls the physics. Matter can warp and bend spacetime, while gravity itself is a manifestation of spacetime curvature. This geometric description of a physical phenomenon is appealing as it allows us to use a range of mathematical tools to understand physics.

Unfortunately there is a problem with this point of view. We believe the universe and the laws of physics that describe it are fundamentally quantum mechanical. Instead of the geometric picture suggested by relativity, we should think of physics in terms of quantum fields and interactions. There is no good reason to think that gravity is special in this respect, so we must treat it quantum mechanically too. We can do this by considering small fluctuations of the metric tensor g around a fixed background geometry η

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}, \tag{1.1}$$

where the coupling κ^2 is proportional to Newton's constant G and $h_{\mu\nu}$ is the graviton field which parametrises the fluctuations away from $\eta_{\mu\nu}$. As $g_{\mu\nu}$ describes a physical system up to diffeomorphisms, the fluctuation $h_{\mu\nu}$ must be a spin-two gauge field that couples to a rank-two symmetric tensor, $T_{\mu\nu}$, known as the stress-energy tensor. We can try to quantise the dynamics of $h_{\mu\nu}$ by taking a path integral over all possible field configurations. Unfortunately there is a problem with quantising gravity in this naive way as the coupling constant κ has negative mass dimension and so will grow at higher energies, rendering the theory non-renormalisable. Put another way, the gravitational coupling G has units of length squared, so the dimensionless coupling that controls the strength of interactions is G/ℓ^2, where ℓ is the characteristic length scale probed by a given physical process. As we probe smaller length scales the coupling grows without bound. Fatally, these problems are not restricted to high-energy scattering processes, but also appear in loop diagrams where the momenta of virtual particles are integrated over all possible values. Due to this, pure gravity diverges at two-loop order [4], and the problem is worse when matter or gauge fields are included, where the divergence appears at one loop [5–8]. Initially there was some hope that combining gravity and supersymmetry to give supergravity might help with renormalisation, but even this only delays the appearance of divergences to higher loop orders.[1]

Despite these problems, we still have an effective field theory description of quantum gravity that is perfectly good up to some energy scale E far below the Planck scale $M_\mathrm{P} = \sqrt{\hbar c/G}$. The effective theory is governed by an action that is an expansion in powers of curvatures

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big(M_\mathrm{P}^2 R + a_1 R^2 + a_2 R_{\mu\nu}R^{\mu\nu} + a_3 M_\mathrm{P}^{-2} R^3 + \ldots + \mathcal{L}_\text{matter}\Big), \tag{1.2}$$

where the a_i are an infinite set of couplings and $\mathcal{L}_\text{matter}$ contains particle physics and matter couplings up to the Planck scale. At low energies, there is an expansion in powers of E/M_P. At each order in the expansion, only a finite number of couplings are needed to calculate the amplitude for a given physical process, and we can perform a finite number of experiments to measure these couplings, after which our theory is predictive for that energy scale. This is just as good as any other effective field theory, such as the pion description of the strong force or the Fermi theory of weak interactions. The trouble appears for $E \approx M_\mathrm{P}$: there is no controlled expansion in powers of E/M_P and all the couplings are equally important. We can no longer carry out a finite number of experiments to measure the couplings and so the theory is no longer predictive. The effective field theory description breaks down at energies near the Planck scale, exactly the regime we need to probe to see quantum gravity effects.

There are a number of proposals that hope to fix these problems. As with the Fermi theory at the electroweak scale, it might be that new degrees of freedom appear at the Planck scale that smear out and soften graviton scattering. An immediate objection to this is that we do not know the degrees of freedom in $\mathcal{L}_\text{matter}$ even below the Planck scale, so our chances of stumbling across the correct ultraviolet (UV) degrees of freedom without a guiding principle are vanishingly small. One way around this is to hope that at high energies, gravity is described by an interacting UV fixed point [10]. At the fixed point, the infinite number of couplings are actually determined by a finite number of parameters, so

[1] The perturbative renormalisability of $\mathcal{N} = 8$ supergravity is an open question. Regardless, there are non-perturbative arguments that show the theory is inconsistent and requires a UV completion by M-theory on a seven-torus [9].

that the theory is again predictive. However, the dynamics at the fixed point will almost certainly be strongly coupled and so fall outside of the perturbative regime. There are technical problems with studying such fixed points non-perturbatively, such as the Gribov ambiguity [11] or the use of the exact renormalisation group equation, but more worryingly there is a fundamental issue with trying to understand quantum gravity in this way. It is common lore that there is no regime in which pure quantum gravity effects are important while matter interactions can be neglected – quantum gravity does not have a "decoupling regime". Thanks to this, even if we manage to find a UV fixed point in pure gravity, when we add matter to the theory the nature and even the existence of the fixed point can change. Again, we need to know the matter content of our theory right up to the energy scale associated with the fixed point and we are back to where we started, hoping to stumble on the correct theory. There seems to be no consistent way to view gravity as coming from a field theory of particles.

The apparent miracle of string theory is that it provides a seemingly UV-finite theory that includes gravity and specifies its matter content. The spectrum of a quantised string contains an infinite tower of massive higher-spin excitations that provide a UV completion of the effective field theory for quantum gravity and completely constrain the curvature and matter couplings. The massive higher-spin fields can be thought of as the gauge fields for a tower of spontaneously broken higher-spin gauge symmetries. On general grounds such a theory should be UV finite. At high energies, the string excitations are approximately massless and the higher-spin gauge symmetry is restored. The Coleman–Mandula theorem then forces the S-matrix to be trivial, so that scattering in the UV is soft and free of divergences. This is one of many reasons to be hopeful that string theory can provide a "theory of everything" or at least point us in the right direction.

Of course, until string theory is tested in experiments we must remember that mathematical beauty is not sufficient for its validity. Nature may really be described by some effective field theory up to the Planck scale, with a UV theory controlled by an infinite number of parameters that take some seemingly arbitrary values in our universe. The universe is not guaranteed be understandable by humans. But with the promise of string theory and much left to study, it seems somewhat premature to give up quite yet.

1.2 String theory

String theory is a quantum theory of gravity and matter, where the fundamental constituents of the universe are no longer particles, but extended objects known as strings. Strings can be open or closed, with their length scale is given by the string length ℓ_{s}. The dynamics of the theory is essentially fixed by minimising the area of the string as it propagates in spacetime. With an eye towards geometry, string theory is especially interesting as the spaces or "backgrounds" on which strings can propagate are highly restricted.

The first hints of string theory were seen while searching for an S-matrix description of the strong interaction. At the time, particle accelerators were producing an abundance of hadronic resonances that exhibited a nearly linear relationship between their spin and mass squared, $J \simeq M^2\alpha'$, with the constant of proportionality α' dubbed the Regge slope. Veneziano [12] proposed an expression for the scattering amplitude for the resonances that reproduced the Regge slope and obeyed the crossing relation between the s- and t-channels seen in experiments. This and later work came to be known as the dual resonance model. Interestingly, the amplitude was well behaved in the UV due to its pole structure, which could be viewed as coming from an infinite tower of massive higher-spin states. Soon after, it was shown that a generalisation of the Veneziano amplitude could be understood as coming from single-particle states of infinitely many harmonic oscillators [13–16], and so the amplitude might be interpreted as the tree-level contribution from a full quantum theory. It was soon pointed out that such an amplitude was consistent with that of a quantised relativistic string [17–19]. Unfortunately, unitarity of the theory required that it exist in a 26-dimensional spacetime, which posed problems for constructing realistic hadron models. This, together with QCD's newfound success, reduced the appeal of the dual resonance model as a theory of the strong interaction. Fortunately this model was destined for greater things. One of the original criticisms of the theory was the appearance of a massless spin-two particle in the spectrum that could not be removed. Upon identifying this excitation with the graviton, this apparent drawback became a reason to study the theory further.

String theory is usually formulated in terms of a conformal field theory (CFT) on the string's two-dimensional worldsheet [20]. The dynamics of the classical theory are governed by the Nambu–Goto action, which is simply the integrated worldsheet area. The Nambu–Goto action is not easily quantised as it contains a square root of the worldsheet fields. Instead, we use the Polyakov action which is classically equivalent to the Nambu–Goto action but is more easily quantised [21]. The Polyakov action is a Weyl invariant non-linear sigma model with a d-dimensional target space. Using the Weyl and diffeomorphism symmetries, one can gauge fix the action to obtain a flat worldsheet metric and d worldsheet scalar fields that can be thought of as maps from the worldsheet to the target space. The massless excitations of the string viewed from the target space are a scalar, a symmetric rank-2 tensor and an antisymmetric rank-2 tensor, which we identify as the dilaton, the metric and the Kalb–Ramond B field. From the worldsheet perspective, vacuum expectation values (VEVs) of these fields appear as couplings in the worldsheet action.

String theory has one free parameter $\ell_{\mathrm{s}} = \sqrt{\alpha'}$, the characteristic length scale of the string. Naively, the string coupling g_{s} is also a free parameter but its value is actually fixed by the dilaton. We understand string theory best as a perturbative double expansion in both g_{s} and $1/\alpha'$. The string coupling measures the strength of string interactions. In particular, it tells us how to weight different worldsheet topologies in the genus expansion for the string S-matrix. This is a perturbative expansion that makes sense only for small string coupling,[2] which is equivalent to requiring the string length to be much smaller than the Planck length. The expansion in $1/\alpha'$ is somewhat different. For a string in flat space, the worldsheet CFT is free and is easily solved. If instead the string is on a curved background, the CFT is an interacting theory where the target space metric appears in the action as an infinite set of couplings. The CFT is weakly coupled when these couplings are small, which is true when the target space is curved on a length scale much larger than the string length. If the target space is highly curved compared to the string length, the worldsheet theory is strongly coupled, but it is still well defined as a path integral and can be tackled using non-perturbative methods.

As with any quantum field theory, a symmetry of the classical action may fail to be a symmetry of the full quantum theory. Such a symmetry is called anomalous. An anomaly in a local symmetry indicates the quantum theory is inconsistent. Since the Weyl invariance of the Polyakov action is a local symmetry, the corresponding anomaly must vanish for the theory to be consistent. The Weyl anomaly appears in the trace of the worldsheet stress tensor, which is equivalent to the beta functions for the worldsheet couplings. The vanishing of the Weyl anomaly defines what we mean by a consistent string theory background: it is a target space for which the beta functions all vanish.[3] At one-loop, the beta functions fix $d = 26$ and reduce to the Einstein equations for the target space coupled to the dilaton and the B field [22].

Though an encouraging first step towards a theory of quantum gravity, the bosonic string cannot be the full story as it does not admit spacetime fermions and the ground state of the theory is tachyonic. We can solve both of these problems by introducing worldsheet fermions to obtain supersymmetry on the worldsheet. The resulting superstring theory still has a tachyon but it can be consistently removed using the so-called GSO projection, leaving a theory with spacetime supersymmetry (and fermions) which is free of anomalies in ten spacetime dimensions [23, 24].

There are in fact five consistent superstring theories, all with ten-dimensional target spaces: type I, type IIA, type IIB and heterotic with $\mathrm{SO}(32)$ or $\mathrm{E}_8 \times \mathrm{E}_8$ gauge group [23,25]. It was realised in the early 90s that all five theories are linked by a web of dualities [26] and further conjectured that they were actually different limits of a fundamental non-perturbative theory which has come to be known as M-theory [27].

There are two type II theories, so-called as they have $\mathcal{N} = 2$ supersymmetry in ten dimensions, obtained by different choices of GSO projection on the string worldsheet. The difference between them amounts to a choice of chirality for the spacetime fermions, particularly the two gravitini: in type IIA they have opposite chirality, giving $\mathcal{N} = (1, 1)$ supersymmetry; in type IIB they have the same chirality, giving $\mathcal{N} = (2, 0)$ supersymmetry.

[2] If the string coupling is large, the perturbative expansion we use to define the S-matrix is not well defined and we have to rely on path integral formulation of the string. String field theory provides such a formulation.

[3] The β-functions can also vanish in $d \neq 26$ if one has a dilaton with a large gradient. This solution does not describe our spacetime, which is approximately static and homogeneous, so we will only consider string theory in the critical dimension.

The two theories share their NS-NS sector, which contains a rank-two symmetric tensor, a rank-two antisymmetric tensor and a scalar field; these are more commonly known as the spacetime metric g, the B field and the dilaton ϕ. The R-R sector of each theory is filled out by $(p+1)$-form potentials C_{p+1} which give rise to $(p+2)$-form field strengths F_{p+2}, where p is even for type IIA and odd for type IIB.

One can integrate a field strength over a non-trivial cycle to find the corresponding conserved charge. A natural question to ask is what are the objects that carry these charges? In analogy with standard electromagnetism, a $(p+1)$-form potential will couple electrically to a $(p+1)$-dimensional hypersurface or magnetically to a $(6-p)$-dimensional hypersurface [28,29]. The hypersurfaces that source R-R charge are known as Dp-branes [30]. D-branes correspond to endpoints of open strings: an open string with $(9-p)$ Dirichlet boundary conditions has ends that live on Dp-branes. Furthermore, the massless modes of the open strings that end on the brane give rise to a gauge theory on the worldvolume of the brane [31]. Strings themselves are $(1+1)$-dimensional hypersurfaces and so source a two-form potential common to both type IIA and IIB, namely the B field. The B field also couples magnetically to a $(5+1)$-dimensional hypersurface known as the NS5-brane.

Type I string theory can be obtained as an orientifold of type IIB string theory in the presence of 32 half D9-branes to cancel anomalies [32,33]. The resulting theory contains unoriented open and closed strings, and the bosonic spacetime degrees of freedom are the metric, dilaton and the R-R two-form potential. From this, we see the theory has D1-, D5- and D9-branes. The theory admits $\mathcal{N}=1$ supersymmetry in ten dimensions and anomaly cancellation implies the presence of an SO(32) gauge group, coming from the 32 half D9-branes.

Heterotic string theory arises by taking the left-moving modes to be those of bosonic string theory and the right-movers to be those of superstring theory. The extra modes of the left-movers give one-form gauge potentials, where anomaly cancellation implies the gauge group must be SO(32) or $\mathrm{E}_8 \times \mathrm{E}_8$ [25,34]. There are no open strings and so no D-branes in this theory.

As we have mentioned, the five distinct string theories are thought to be connected by a web of duality transformations, known as S- and T-dualities. These dualities connect apparently distinct descriptions of the same physical system.

S-duality is a strong-weak duality that connects a strongly coupled description to a weakly coupled description, so is non-perturbative in g_s. For example, type IIB string theory is self-dual under S-duality, so it's weak and strong coupling limits are the same [35,36]. Type I string theory with string coupling g_s is S-dual to the SO(32) heterotic string with coupling $1/g_\mathrm{s}$. [37]. M-theory is S-dual to both the type IIA and the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic strings [26,38–40].

T-duality exchanges small and large radii in the target space, so is non-perturbative in α'. For example, the SO(32) and $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic strings are connected via T-duality [41], as are type IIA and IIB when compactified on a circle [28,42]. One can also combine T-duality and S-duality to give a so-called U-duality transformation [26], which is non-perturbative in both g_s and α'.

As with the bosonic string, the superstrings's target space must satisfy certain conditions to maintain worldsheet Weyl invariance, namely that the β-functions for the worldsheet couplings all vanish. To one-loop, the β-functions are simply the equations of motion for the massless excitations of the string in the target space. The remarkable fact is that the equations of motion are actually those of the known ten-dimensional supergravity theories [43]. This is not a surprise. As the string length is taken to zero, we expect to recover a point-particle limit, and indeed as $\alpha' \to 0$ all superstring theories admit a supergravity limit. Furthermore, despite us not having a definition of M-theory its low-energy limit is thought to be the unique eleven-dimensional $\mathcal{N} = 1$ supergravity [38,39,44]. In summary, we can study supergravity to better understand the low-energy behaviour of string theory and M-theory.

The supergravity limit gives us access to the massless, perturbative degrees of freedom of the corresponding string theory. Branes do not fall into this subsector. Taking type II string theory for example, D-brane masses scale as $1/g_s$, so they are "heavy" in the weakly coupled supergravity limit $g_s \to 0$ and no longer seen as perturbative degrees of freedom. Despite this, we cannot ignore their effects as the massless degrees of freedom we are interested in can interact with D-branes, which source R-R charge and are the endpoints of open strings. Instead of fields, D-branes appear in supergravity as non-perturbative solitons or solutions to the equations of motion [30,45]. M-theory also admits branes, the M2- and M5-branes, which appear in eleven-dimensional supergravity as solutions to the equations of motion. These brane solutions correspond to Bogomol'nyi–Prasad–Sommerfield (BPS) states [27,46]. In fact, they are half-BPS states and so are annihilated by half of the supersymmetry generators.

If string theory is to provide a description of our universe, an obvious impediment is the requirement that the theory lives in ten dimensions. At the energy scales we can probe, the universe looks very much four-dimensional. The standard way to recover four dimensions at low energies is to assume the ten-dimensional spacetime is a (possibly warped) product $\mathbb{R}^{3,1} \times M$, where $\mathbb{R}^{3,1}$ is four-dimensional Minkowski space and M is a six-dimensional compact space. If the volume of M is small compared with the energy scale of measurements, the resulting theory is effectively four-dimensional. This procedure is known as compactification and is the string theory realisation of the Kaluza–Klein mechanism. Importantly for us, the details of the four-dimensional effective theory depend on the choice of internal space M. In particular, the presence of supersymmetry at the compactification scale is fixed by the topological and differential structure of M. As we now discuss, there are a number of reasons why we might want to focus on compactifications that allow for supersymmetry.

Supersymmetry has been an important ingredient in particle phenomenology for some time, with a preference for $\mathcal{N} = 1$ supersymmetry that is spontaneously broken at low energies. Despite the lack of experimental evidence for its presence, we continue to use supersymmetry in our models as it solves a number of problems with the standard model. In the majority of grand unified theories (GUTs), the three gauge couplings of the standard model should unify once above the GUT scale. Unfortunately, if we follow the running of the gauge couplings within the standard model we find the couplings do not meet. Supersymmetry alters the running of the couplings so that they meet, giving gauge coupling unification and the possibility of deriving the standard model from a grand unified theory. In addition, the modified running of the couplings may provide a mechanism for electroweak symmetry breaking. Another problem is that, generically, we expect the Higgs mass to receive loop corrections that push it up to the Planck scale [47 –49]. If we want a 125 GeV Higgs, we are forced to tune the parameters of the standard model to an unnatural level. Supersymmetry solves this hierarchy problem as the loop corrections from bosonic and fermionic fields cancel each other [50, 51]. Supersymmetry also provides natural dark matter candidates if the lightest supersymmetric particle is stable [52, 53]. These are some of the reasons to focus on understanding those compactifications that lead to supersymmetric theories in four dimensions. More generally, as theorists we are also interested in supersymmetric compactifications to any number of dimensions.

As we have mentioned, string theory admits a number of surprising dualities, but testing such dualities is difficult. In most cases, we have only a perturbative description of the relevant theories at weak coupling. The dualities however can map between weak and strong coupling or perturbative and non-perturbative physics. For this reason, performing calculations to check the general validity of these dualities is difficult, if not impossible with our current knowledge. One way around this problem is to exploit supersymmetry. In the presence of sufficient supersymmetry there are quantities that can be calculated exactly on both sides of the duality and then compared for consistency. These quantities are normally interpreted as counting BPS states [54–57]. The dualities have survived all such checks to date and give a compelling reason to study supersymmetric backgrounds.

There has also been great interest in understanding supersymmetric backgrounds thanks to the AdS/CFT correspondence [58–60]. The AdS/CFT correspondence conjectures that type IIB string theory on a space that is the product of anti-de Sitter space (AdS) and a compact manifold is equivalent to a conformal field theory (CFT) living on the boundary of the AdS space. In a looser form, it relates string theory (or M-theory) on a geometry which is asymptotically AdS to a field theory living on the boundary, giving an explicit example of the holographic principle [61, 62]. The original setup is a stack of N D3-branes placed in flat $\mathbb{R}^{9,1}$, where the worldvolume of the D3-branes spans $\mathbb{R}^{3,1}$ and the transverse directions are $\mathbb{R}^6 = \mathbb{R}^+ \times \mathrm{S}^5$. At small string coupling, the gravitational backreaction is negligible and the branes are described in terms of a $\mathrm{U}(N)$ superconformal field theory on their worldvolume, coming from open strings stretching between the branes. For D3-branes, this worldvolume theory is in fact $\mathcal{N} = 4$ super Yang–Mills (SYM) [31]. At large string coupling, the backreaction of the branes introduces five-form flux and a relative warp factor between the worldvolume and transverse directions, giving a black brane solution in type IIB string theory. Near to the branes, in the near-horizon limit, the ten-dimensional geometry approaches $\mathrm{AdS}_5 \times \mathrm{S}^5$. The point is that these two descriptions of the branes are actually the same. Super Yang–Mills and string theory are very different theories, so it is important to understand the range of validity of each description. The 't Hooft coupling of the worldvolume theory is $\lambda = g_{\mathrm{YM}}^2 N$, where the Yang–Mills coupling g_{YM} of the worldvolume theory is related to the string coupling of the bulk theory by $g_{\mathrm{YM}}^2 = 4\pi g_{\mathrm{s}}$. In the near-horizon limit, the radius of both AdS_5 and S^5 is $R^4 = 4\pi g_{\mathrm{s}} N \alpha'^2$ so that

$$\lambda = \left(\frac{R}{\ell_{\mathrm{s}}}\right)^4, \tag{1.3}$$

where $\ell_{\mathrm{s}} = \sqrt{\alpha'}$ is the string length. Furthermore, the Planck length ℓ_{P} and the AdS_5 radius are related by

$$\left(\frac{\ell_{\mathrm{P}}}{R}\right)^4 = \frac{\pi^2}{\sqrt{2}N}. \tag{1.4}$$

In the supergravity approximation where $\ell_{\mathrm{s}} \ll R$, stringy corrections are absent leaving us with type IIB supergravity that is dual to a strongly coupled gauge theory with $\lambda \gg 1$. Conversely, strongly curved backgrounds for which the supergravity description breaks down should be described by a weakly coupled gauge theory. The gauge theory simplifies further in the limit $N \to \infty$ for fixed λ. This is known as the 't Hooft limit and corresponds to taking only planar diagrams in the gauge theory [63]. On the string side, from (1.4) we see this corresponds to $R \gg \ell_{\mathrm{P}}$ so that quantum gravity corrections are small, which corresponds to taking only genus-zero string diagrams. For $\lambda \gg 1$ and $N \gg 1$, we then have a strongly coupled gauge theory in the planar limit that is dual to classical supergravity. The point to remember is that the ten-dimensional background is actually a supersymmetric flux background as the D3-branes source N units of five-form flux that thread AdS_5 and the five-sphere. There are generalisations of this setup to geometries that come from replacing $\mathbb{R}^+ \times \mathrm{S}^5$ with conical Calabi–Yau spaces, where the compact five-dimensional space is Sasaki–Einstein [64]. A better understanding of the most general supersymmetric AdS flux backgrounds would give us a plethora of new examples and great insight into the AdS/CFT correspondence. This alone may be considered reason enough to study supersymmetric backgrounds.

String theory is a promising approach to understanding quantum gravity and has many other applications, including quantum field theory, mathematics, condensed matter physics and black-hole physics. In all of these areas, the presence of supersymmetry and the existence of supersymmetric string backgrounds are key to understanding the physics and making calculations tractable. In this thesis, we will focus on trying to understand these supersymmetric backgrounds in the supergravity limit.

1.3 Supergravity backgrounds

The first investigations into supergravity backgrounds focussed on those without flux that preserve supersymmetry. Assuming a four-dimensional Minkowski factor, the required spaces were found to be special holonomy manifolds [65, 66]. The archetypal example is a six-dimensional Calabi–Yau manifold [65] which admits a single covariantly constant spinor, a Killing spinor, and so has SU(3) holonomy. In this case, the geometry can be equivalently described by a holomorphic three-form Ω and a symplectic two-form ω, both constructed as bilinears of the Killing spinor. The covariant constancy of the spinor then translates to the integrability conditions $\mathrm{d}\Omega = \mathrm{d}\omega = 0$. In particular, integrability of Ω implies the manifold is complex, and the tools of complex and algebraic geometry can then be used to construct examples and calculate important physical properties, such as moduli spaces, particle spectra and couplings [67–69].

A large class of phenomenologically promising models come from combining these spaces with the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string, as compactifying this theory on a six-dimensional Calabi–Yau manifold leads to an $\mathcal{N} = 1$ effective theory in four dimensions with the possibility of chiral fermions [65]. Moreover, the standard model gauge group can be embedded in $\mathrm{E}_8 \times \mathrm{E}_8$, leading to realistic string models [70]. In the standard embedding, the SU(3) holonomy group of the Calabi–Yau threefold embeds in one of the E_8 factors, and the commutant of SU(3) inside E_8 is an E_6 gauge group that can accommodate the required $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. The particle content, such as the number of standard model generations, is then fixed by topological data of the Calabi–Yau. One can also carry out a similar program for M-theory on G_2 manifolds with singularities [66, 71–73].

Such compactifications generically lead to a number of massless scalar fields, known as moduli. For the example of Calabi–Yau spaces, the moduli correspond to the possible ways to deform either the Kähler structure or the complex structure of the threefold [69]. Generically, no potential is generated for such fields and so their VEVs are not fixed by any dynamical mechanism. This is a problem as the fields can appear as couplings in the low-energy effective action, leading to a large number of undetermined parameters in the effective four-dimensional theory. Not only is this undesirable from an aesthetic viewpoint, it is also at odds with observations: such massless fields can lead to long-range forces, which we do not observe. Furthermore, such fields would dramatically change early universe cosmology as the moduli would generically have VEVs of order the weak scale and would dominate reheating, causing problems for large-scale structure formation [74, 75].

To avoid these problems, one must introduce a potential for the moduli so that they become massive and decouple from the effective low-energy physics. Ideally, this potential should be dynamically generated so that the VEVs of the moduli are fixed by the background and not by some arbitrary choice of potential. One way to do this is to include fluxes on the internal space. These flux compactifications were originally an extension of Calabi–Yau compactifications for the heterotic string [76–78] and then extended to

M-theory in [79] (see [80] for a review). The use of fluxes to fix moduli was first studied in [81], and it indeed seems that all moduli may be fixed in this way in some supergravity limits [82–85].

Flux compactifications are also attractive as they allow us to obtain realistic models from type II theories, since fluxes generically reduce the four-dimensional gauge group and break supersymmetry [86, 87]. This has led to $\mathcal{N} = 1$ models from flux-deformed Calabi–Yau backgrounds and culminated in a whole new subfield known as F-theory [88]. In addition, fluxes generically lead to a warp factor for the four-dimensional spacetime, which has been suggested as a way to explain gauge hierarchies [89].

Flux backgrounds are also important for the AdS/CFT correspondence. Generically flux compactifications are dual to confining gauge theories, so they may provide a way to describe QCD (or QCD-like theories) via string theory [90–92]. Furthermore, the most studied examples of AdS/CFT are those for which the gravity solution is AdS times an odd-dimensional sphere with a top-form flux [64]. Turning on extra fluxes while preserving supersymmetry is then equivalent to deforming the gauge theory by marginal operators. The full supergravity solutions dual to these deformed theories are known in only a few cases [93], so a full understanding of supersymmetric flux backgrounds is essential for testing AdS/CFT.

For a time, flux compactifications were thought to be a fruitless endeavour thanks to a number of no-go theorems that ruled out compactifications to Minkowski or de Sitter spacetimes when fluxes are present [89, 94–98]. The no-go theorems apply to generic compactifications and do not assume supersymmetry. As an example, consider the Einstein equation restricted to the four-dimensional spacetime

$$\mathrm{e}^{-2\Delta}R + T_{\text{flux}} = 2\,\nabla^2\mathrm{e}^{2\Delta}, \tag{1.5}$$

where R is the Ricci scalar for the unwarped four-dimensional metric, T_{flux} is the contribution due to fluxes on the internal space and Δ is the warp factor for the four-dimensional metric. Upon integrating this equation over the internal compact space without boundary, the right-hand side vanishes. Since T_{flux} is positive definite (it depends on the squares of the fluxes) we must have $R < 0$ for consistency; this rules out both Minkowski and de Sitter solutions. One can escape these results by including stringy corrections to the Einstein equations or localised negative tension sources, such as orientifold planes, that provide a negative contribution to T_{flux}. The possibility of avoiding the no-go theorems reinvigorated the investigation of flux compactifications.

Having discussed the positives of flux compactifications, we must admit the program has some problems. Despite the large body of work on flux compactifications, there are still issues with constructing stable (or even metastable) de Sitter vacua. The most promising way forward is the KKLT proposal [99] which suggests a way to produce metastable de Sitter vacua with small positive cosmological constant. Unfortunately, there are now suspicions that the full backreacted solution is unstable [100, 101]. Furthermore, assuming de Sitter vacua can be constructed, there is still the problem of the string landscape [102]. Given the possible choices of fluxes, branes and singularities, there are likely to be an extremely large number of admissible vacua and, for the time being, we have no idea whether there is some dynamical mechanism for vacuum selection. There has been some work on statistical studies of the landscape [103], while a more ambitious approach is to classify possible backgrounds. One possible application of the work in this thesis is to the classification of generic flux backgrounds, particularly those that preserve eight supercharges.

1.4 Generalised geometry

String theory is a theory of extended objects and this extra complexity can lead to new and unexpected symmetries. In particular, its low-energy supergravity limit is not only a theory of gravity, it also admits fluxes which are derived from local potentials that are defined up to gauge transformations. Thanks to this, supergravity has not only diffeomorphism invariance, but also p-form gauge symmetries. This extra structure hints that we should come up with a new language that treats these symmetries on an equal footing, much as the language of differential geometry is suited to the diffeomorphism invariance of gravity. As we will see, this language is generalised geometry.

The conventional notion of a G-structure has already provided a useful way to analyse flux backgrounds [97, 104, 105]. While the manifold M no longer has special holonomy, the Killing spinor bilinears still define a set of tensors that are invariant under $G \subset \mathrm{SO}(d;\mathbb{R})$, where d is the dimension of M. The fluxes then appear as an obstruction to integrability of the G-structure. Formally this is encoded in the intrinsic torsion, and only when this vanishes does the background have special holonomy. For generic backgrounds, the structure is only locally defined since there can be points where the stabiliser group of the Killing spinors changes.

A natural question is whether there is an analogous geometric description of generic supersymmetric flux compactifications in terms of integrable structures. The basic point of this thesis is that there is actually a natural geometry in which supersymmetry for a generic flux background again corresponds to integrable, globally defined G-structures. In the context of type II reductions to four dimensions, this defines the natural string-theoretic generalisation of the notion of a Calabi–Yau manifold to backgrounds including both NS-NS and R-R flux.[4]

A description in terms of integrable structures is important since it provides approaches for tackling problems such as analysing the deformations and moduli spaces of arbitrary

[4] As we have mentioned, there are general "no-go" theorems [95, 97, 98, 106, 107] that, in the absence of sources, exclude reductions on a compact space to a Minkowski or de Sitter backgrounds when fluxes are present. Thus, the backgrounds in this thesis are generically either non-compact or have an anti-de Sitter spacetime.

flux backgrounds, as well as constructing new examples. We also note that it would not only give generalisations of the classical target-space theories of the topological string A and B models to include R-R modes, but also defines a corresponding pair of theories in M-theory.

Focussing for the moment on $\mathcal{N}=2$, $D=4$ backgrounds, in the case of NS-NS flux such a reformulation has already appeared under the name of generalised complex geometry [108–111]. Here one considers structures on a generalised tangent bundle $E \simeq TM \oplus T^*M$, admitting a natural $\mathrm{O}(d,d)$ metric. For a large class of supersymmetric backgrounds with non-trivial two-form B field and dilaton ϕ, the holomorphic and symplectic forms generalise to a pair of $\mathrm{O}(6,6)$ pure spinors $\Phi^{\pm} \in \Gamma(\wedge^{\pm} T^*M)$, each defining an $\mathrm{SU}(3,3) \subset \mathrm{O}(6,6)$ structure. The pure spinors satisfy compatibility conditions that imply that together they define an $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure. The $\mathcal{N}=2$ Killing spinor equations then imply $\mathrm{d}\Phi^{\pm}=0$, and one says the $\mathrm{SU}(3)\times\mathrm{SU}(3)$ structure is integrable [111]. Each such integrable $\Phi^{\pm}$ defines a generalised complex structure [108] and the integrable $\mathrm{SU}(3)\times\mathrm{SU}(3)$ structure is known as a generalised Calabi–Yau metric structure [109]. This language has been useful for a whole range of applications including addressing deformations [109, 112–114], topological strings and sigma models [115–119], T-duality and mirror symmetry [120–124], non-geometric backgrounds [125, 126], steps towards classifying flux backgrounds [127–131] and the AdS/CFT correspondence [132–134].

Generalised complex geometry is ideal for reformulating the NS-NS sector of type II supergravity, but the R-R fields do not enter on the same footing as the B field or dilaton. To include R-R fluxes and M-theory compactifications, we need to consider $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ or exceptional generalised geometry [135, 136]. The generalised tangent space is further extended to include the R-R gauge symmetries, such that it admits a natural action of $\mathrm{E}_{d(d)} \times \mathbb{R}^+$. This extension gives a unified geometrical description of type II and M-theory restricted to a $(d-1)$- or d-dimensional manifold respectively [137, 138], invariant under local transformations by the maximal compact subgroup H_d of $\mathrm{E}_{d(d)} \times \mathbb{R}^+$. The bosonic symmetries combine in a generalised Lie derivative and there is a generalised metric, invariant under H_d, that encodes all the bosonic degrees of freedom. One can find a generalised connection D that is the analogue of the Levi-Civita connection, such that the full bosonic action is equal to the corresponding generalised Ricci scalar, and the fermion equations of motion and supersymmetry variations can all be written in terms of D.

Exceptional generalised geometry is particularly suited to describing generic supersymmetric backgrounds with flux [126, 139–142]. In the case without flux, we know that the underlying structure is that of special holonomy. A similar idea holds for backgrounds with flux in generalised geometry. Minkowski backgrounds with any number of supersymmetries are now known to be in correspondence with generalised special holonomy spaces [143, 144], while the same result is known to hold true for minimally supersymmetric AdS backgrounds [145] and is suspected to hold for any number of supersymmetries. As with conventional special holonomy, we can think of these spaces as having torsion-free generalised G-structures defined by a set of invariant tensors that satisfy some integrability conditions. We will see that exceptional generalised geometry provides such a description.

The appearance of the exceptional groups can be understood by considering the maximal $\mathcal{N} = 8$, $d = 4$ supergravity. This theory has a global $E_{7(7)}$ and a local $SU(8)$ symmetry. These groups can be realised directly in eleven-dimensional supergravity by assuming a $4 + 7$ split of the underlying spacetime, leading to a breaking of $\mathrm{Spin}(10, 1)$ to $\mathrm{Spin}(3, 1) \times \mathrm{Spin}(7)$ [146]. One can then enhance the $\mathrm{Spin}(7)$ to a local $SU(8)$ symmetry, where $SU(8)$ is the maximal compact subgroup of $E_{7(7)}$. It has been shown that one can repeat this for the other $E_{d(d)}$ groups [147–150]. In a sense, exceptional generalised geometry gives an infinite-dimensional extension of these supergravities where the action of these hidden groups is geometrised.

We note that there is a long history of considering exceptional groups in supergravity, in many cases by positing the existence of extra coordinates [146, 149, 151–156]. More recently, there has been great interest in double field theory, where these extra coordinates play a central role [157–166]. One aim of double field theory is to construct an extension of supergravity in which T-duality is manifest. The inspiration for this is given by considering string compactifications on a d-dimensional torus, where T-duality acts by exchanging the momentum and winding modes of the string. One defines coordinates that are dual to the winding modes, giving both the usual compact coordinates on the torus x^m and another set of coordinates $\tilde{x}^m$ that parametrise a dual torus. Taken together, we have a "doubled", $2d$-dimensional torus. The fields of the theory can then depend on all coordinates of this extended space, and there is an action of the T-duality group $O(d, d)$ on the fields. However, these fields are constrained: to recover the known diffeomorphisms and gauge transformations one must impose a "section condition" that removes the dependence on half of the coordinates. Locally, after imposing the section condition, doubled field theory reduces to generalised geometry. However, double field theory can describe more complicated global configurations, such as T-folds where one can patch together the doubled space using T-dualities, in addition to the usual transition functions [167–169].

The study of double field theory has not been restricted to torus compactifications, however there are a number of problems with the theory on more general backgrounds [126, 166, 170, 171]. For toroidal backgrounds, the coordinates dual to the winding modes have a clear physical interpretation. For more general backgrounds there are generically no non-trivial cycles and so no windings modes. The question is then what the extra coordinates correspond to physically and whether the doubled space has any interpretation.[5] For this reason, we will not comment further on the relation between our constructions and double field theory.

[5] An alternative proposal which is consistent is that of Hull [166] in which the doubled space is physical but the geometric structures exist only on the local quotient space.

1.5 Plan of the thesis

We begin in chapter 2 with a brief review of supersymmetric backgrounds in supergravity, focussing first on the case without flux. We then explain how the geometric description of such spaces can be extended to generalised Calabi–Yau geometries in the presence of NS-NS flux, and propose that the correct language for understanding the most general flux backgrounds is exceptional generalised geometry.

In chapter 3 we define the analogue of Calabi–Yau geometry for generic $D = 4$, $\mathcal{N} = 2$ flux backgrounds in type II supergravity and M-theory. We begin by discussing backgrounds with a four-dimensional Minkowski spacetime and show that there are generalisations of the complex and symplectic structures for generic flux backgrounds. Such "exceptional Calabi–Yau" geometries are determined by two generalised objects that parametrise hyper- and vector-multiplet degrees of freedom, where supersymmetry of the background is equivalent to integrability of these generalised structures. We discuss how these ideas follow from gauged supergravity and the concept of generalised intrinsic torsion, and how they can be used to explore the moduli space of solutions. We then extend our construction to $D = 5$ and $D = 6$ flux backgrounds preserving eight supercharges, where analogous structures appear.

In chapter 4 we repeat our analysis for generic AdS flux backgrounds preserving eight supercharges in $D = 4$ and $D = 5$. Again, they are described by a pair of globally defined, generalised structures with integrability conditions that are equivalent to supersymmetry. We give a number of explicit examples of such "exceptional Sasaki–Einstein" backgrounds in type IIB supergravity and M-theory. In particular, we give the complete analysis of the generic AdS_5 M-theory backgrounds. We also briefly discuss the structure of the moduli space of solutions. In all cases, one structure defines a "generalised Reeb vector" that generates a Killing symmetry of the background corresponding to the R-symmetry of the dual field theory, and in addition encodes the generic contact structures that appear in the $D = 4$ M-theory and $D = 5$ type IIB cases. Finally, we investigate the relation between generalised structures and quantities in the dual field theory, showing that the central charge and R-charge of BPS wrapped-brane states are both encoded by the generalised Reeb vector, as well as discussing how volume minimisation (the dual of a- and $\mathcal{F}$-maximisation) is encoded.

In chapter 5 we apply our formalism to the study of exactly marginal deformations of $\mathcal{N} = 1$ SCFTs that are dual to generic AdS_5 flux backgrounds in type IIB or eleven-dimensional supergravity and show there is a geometric interpretation of the known gauge theory results. Focussing on Sasaki–Einstein backgrounds in type IIB supergravity we find an explicit, first-order expression for the three-form flux dual to the marginal deformations. We then show that our expression for the three-form flux matches those in the literature and the obstruction conditions match the one-loop beta functions of the dual SCFT.

In chapter 6 we extend this analysis to $d = 3$, $\mathcal{N} = 2$ superconformal field theories that arise on a stack of M2-branes at a conical singularity. The supergravity backgrounds are of the form $\mathrm{AdS}_4 \times M$, where M is a seven-dimensional Sasaki–Einstein manifold. Again, we find an explicit expression for the first-order four-form flux that is dual to the marginal deformations. We also show that our expression for the four-form flux matches those in the literature.

Finally, in chapter 7 we summarise the main points of this thesis and discuss some open problems and directions for future work.

Chapter 2

Supergravity backgrounds and generalised geometry

In this chapter, we review supersymmetric backgrounds and generalised geometry. We briefly summarise the standard $D = 4$, $\mathcal{N} = 2$ example of Calabi–Yau backgrounds in type II theories, discuss how the notion of Calabi–Yau extends when one includes NS-NS flux and mention the problems that arise when one includes all fluxes. We finish with a short review of generalised geometry in preparation for the next chapter.

2.1 Supersymmetric backgrounds

A supergravity background is a solution to the classical supergravity equations of motion. If we are to connect with phenomenology or AdS/CFT, we should look for solutions which are a (warped) product of an internal space M with a maximally symmetric external spacetime, such as Minkowski or AdS. In order to preserve the Poincaré or AdS symmetry of the external spacetime, we must set all fermionic fields to zero so the background is purely bosonic. As we outlined in the previous chapter, supersymmetric backgrounds are a key ingredient in both string phenomenology and the AdS/CFT correspondence. A background is supersymmetric if all the supergravity fields (and hence the solution) are invariant under the supersymmetry transformations. Recall that the supersymmetry transformations depend upon a choice of supersymmetry parameter ε and take the schematic form

$$\delta(\text{boson}) = \varepsilon(\text{fermion}), \qquad \delta(\text{fermion}) = \varepsilon(\text{boson}). \tag{2.1}$$

The variations of the bosonic fields always contain a fermionic field, and since we have set these to zero the variations automatically vanish. The non-trivial conditions come from the variations of the fermionic fields. Supersymmetry of the background is then equivalent to the existence of a non-vanishing spinor ε for which the supersymmetry variations vanish.

Generically, the variations take the form of differential and algebraic conditions

$$\mathcal{D}\varepsilon = 0, \qquad \mathcal{P}\varepsilon = 0,$$

where $\mathcal{D}$ is a supergravity connection and $\mathcal{P}$ is an endomorphism of the spinor bundle, both given in terms of the metric, fluxes and any other bosonic fields. These equations are known as the Killing spinor equations, and a background is supersymmetric if it admits one or more solutions, known as Killing spinors.

The product form of the metric implies that the ten-dimensional supersymmetry parameter ε schematically takes the form of a tensor product of a spinor η on Minkowski or AdS and an internal spinor χ on M:

$$\varepsilon = \eta \otimes \chi.$$

The decomposition of the ten-dimensional spinor splits the Killing spinor equations into conditions for η on the external spacetime and conditions for χ on M. We then say that the background preserves a number of real supercharges equal to the real degrees of freedom of η times the number of independent solutions of the Killing spinor equations on M. The basic question is then how the existence of Killing spinors on M restricts its geometry and fluxes.

2.1.1 Backgrounds without flux

The classic example of an $\mathcal{N} = 2$ background is type II string theory with vanishing fluxes, where M is a Calabi–Yau threefold. The background metric takes the form of a product

$$\mathrm{d}s^2_{10} = \mathrm{d}s^2(\mathbb{R}^{3,1}) + \mathrm{d}s^2(M), \tag{2.2}$$

and we have set all fluxes and the warp factor to zero. The only degrees of freedom on the internal manifold are its metric, so that the effective four-dimensional theory that would arise from compactification on this background depends upon the geometry of M alone.

With the aim of finding $\mathcal{N} = 2$ supersymmetry in the effective four-dimensional theory, the two ten-dimensional spinors decompose as

$$\begin{aligned} \varepsilon_1 &= \eta_1^+ \otimes \chi_1^+ + \eta_1^- \otimes \chi_1^-, \\ \varepsilon_2 &= \eta_2^+ \otimes \chi_2^\mp + \eta_2^- \otimes \chi_2^\pm, \end{aligned} \tag{2.3}$$

where the $\pm$ subscripts denote chirality, the upper/lower choice of chirality refers to type IIA/IIB, and η_i and χ_i are Weyl spinors in four and six dimensions, so that η_i^- and χ_i^- are the conjugates of η_i^+ and χ_i^+. Note that this is not the most general spinor ansatz [172], but it will suffice for this chapter.

The mere existence of non-vanishing spinor fields on M imposes a topological condition, namely a reduction of the structure group of M to a subgroup of $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$. Viewing χ_1^+ as a four-component Weyl spinor with norm $|\chi_1|^2 = \bar{\chi}_1^+\chi_1^+$, the frames in which it can be written as $\chi_1^+ = (|\chi_1|, 0, 0, 0)$ form an SU(3) structure. The second internal spinor χ_2^+ can then be parallel, nowhere parallel or a mix of the two depending on the position on the manifold M. If the second spinor is nowhere parallel to χ_1^+, the SU(3) frames in which $\chi_2^+ = (0, |\chi_2|, 0, 0)$ themselves form an SU(2) structure, and will lead to an $\mathcal{N} = 4$ effective four-dimensional theory. Instead, we will consider the case where the two internal spinors are parallel. In other words, there is a single non-vanishing spinor χ^+ on M that defines an SU(3) structure and two spinors $\eta_{1,2}$ on the external spacetime, resulting in eight supercharges and $\mathcal{N} = 2$ supersymmetry in four dimensions.

Equivalently, one can think of the SU(3) structure in terms of the invariant tensors defined by χ^+. Normalising the spinor so that $\bar{\chi}^+\chi^+ = 1$, the tensor $J^m{}_n = \mathrm{i}\bar{\chi}^+\gamma^m{}_n\chi^+$ is a section of the endomorphism bundle of TM and satisfies $J^2 = -\mathbb{1}$, hence it defines an almost complex structure. Furthermore, the metric is automatically Hermitian with respect to this almost complex structure, and lowering an index of J with the metric defines a two-form $\omega_{mn} = -J_{mn}$. One can also use the spinor to construct a nowhere-vanishing three-form $\Omega_{mnp} = \bar{\chi}^+\gamma_{mnp}\chi^-$. Using Fierz identities, one can check the action of J on ω and Ω implies that they are $(1,1)$- and $(3,0)$-forms, and so satisfy $\omega \wedge \Omega = 0$. Furthermore, the normalisation of χ^+ implies $\frac{1}{3!}\omega \wedge \omega \wedge \omega = \frac{1}{8}\mathrm{i}\Omega \wedge \bar{\Omega}$. The forms ω and Ω are invariant under $\mathrm{Sp}(6;\mathbb{R})$ and $\mathrm{SL}(3;\mathbb{C})$ subgroups of $\mathrm{GL}(6;\mathbb{R})$ respectively, and together they are invariant under $\mathrm{SU}(3) \subset \mathrm{SO}(6)$. The different structure groups embed as

$$\begin{array}{ccc} \mathrm{GL}(6;\mathbb{R}) & \supset & \mathrm{Sp}(6;\mathbb{R}) \text{ for } \omega \\ \cup & & \cup \\ \mathrm{SL}(3;\mathbb{C}) \text{ for } \Omega & \supset & \mathrm{SU}(3) \text{ for } \{\omega, \Omega\} \end{array} \tag{2.4}$$

Assuming the fluxes, warp factor and dilaton are set to zero, the Killing spinor equations in spacetime are trivial. We simply take η_i^+ to be constant spinors. The Killing spinors equations on the internal manifold reduce to

$$\nabla\chi^+ = 0.$$

In other words, the internal manifold M must admit covariantly constant spinors. This means that the Levi-Civita connection ∇ is compatible with the SU(3) structure defined by the spinors, and, since the Levi-Civita connection is torsion-free, the SU(3) structure has vanishing intrinsic torsion. The internal manifold M must thus have SU(3) special holonomy.

We can also understand this in terms of the invariant forms ω and Ω. The almost complex structure is covariantly constant, $\nabla J = 0$, since it is defined in terms of χ^+ which is itself covariantly constant, and so the almost complex structure is integrable. As ∇ is torsion-free, we can replace ∂ with ∇ in the exterior derivative, from which we see the covariant constancy of J implies $\mathrm{d}\omega = 0$. Thus, M is actually Kähler with Kähler form ω. As Ω is constructed from χ^+, we also have $\mathrm{d}\Omega = 0$ which implies Ω is a holomorphic three-form. This means the components of the three-form in complex coordinates are $\Omega_{ijk} = f(z)\epsilon_{ijk}$, where $f(z)$ is a holomorphic, nowhere-vanishing section of the canonical line bundle. The Ricci form for the Hermitian metric compatible with ω and the Levi-Civita connection is $\mathcal{R} = \mathrm{i}\partial\bar{\partial}\log\sqrt{g}$. From the expression for Ω, we have that the norm of the holomorphic three-form satisfies $\|\Omega\|^2 = |f(z)|^2/\sqrt{g}$, which together with $\bar{\partial}f = 0$ implies

$$\mathcal{R} = -\mathrm{i}\partial\bar{\partial}\log\|\Omega\|^2. \tag{2.5}$$

As $\|\Omega\|^2$ is a globally defined, nowhere-vanishing function, $\mathcal{R}$ is exact and so the manifold M has vanishing first Chern class.[1] Kähler manifolds with vanishing first Chern class are Calabi–Yau manifolds, which are known to admit a Ricci-flat metric in the same Kähler class [173–176]. The definition of the structure in terms of spinors makes this obvious as, since Ω is defined using χ^+ which has constant norm, Ω has constant norm and thus $\mathcal{R}$ actually vanishes.

In summary, one can build two differential forms as bilinears in χ^+, a symplectic form ω and a holomorphic three-form Ω, which together define a torsion-free SU(3) structure on M. On a more practical level, the crucial points are that the manifold is complex, allowing the use of algebraic geometry, and the existence theorem for the metric, which guarantees that as long as the Ricci form is exact, there exists a Ricci-flat metric. Upon including fluxes, we lose these mathematical tools. The SU(3) structure has torsion and is not always globally defined. The structure can interpolate between SU(2) and SU(3) depending on whether the internal spinors are parallel or not. To understand the general case, we now discuss a simple generalisation of Calabi–Yau that allows for NS-NS flux while retaining a geometric interpretation.

2.1.2 Generalising the notion of a Calabi–Yau structure

Generic flux solutions of the $\mathcal{N} = 2$ Killing spinor equations can be thought of as string-theory generalisations of the conventional notion of a Calabi–Yau manifold to backgrounds including both NS-NS and R-R fluxes. The simplest extension is to consider generic NS-NS backgrounds by including the dilaton and three-form flux $H = \mathrm{d}B$.

The solution is characterised by a pair of spinors (χ_1^+, χ_2^+), each stabilised by a different SU(3) subgroup of Spin(6) $\simeq$ SU(4). Generically the common subgroup leaving both χ_i^+ invariant is SU(2). However, since the norm between the spinors can vary over M, there can be points where the spinors are parallel and the stabiliser group enhances to SU(3). Backgrounds where this happens are called "type-changing" [108, 109]. The presence of

[1] One can also see this from the relation $c_1(M) = c_1(T^{1,0}M) = -c_1(K_M)$, where c_1 is the first Chern class and K_M is the canonical line bundle over M. Since Ω is a nowhere-vanishing section, K_M is trivial. A trivial bundle admits a flat connection, so $c_1(K_M) = 0$ and hence the first Chern class of the manifold vanishes.

two spinors χ_i^+ means that the differential forms constructed from the spinor bilinears are more intricate than in the Calabi–Yau case. The background can be characterised by two polyforms [111]

$$\Phi^+ = \mathrm{e}^{-\phi}\mathrm{e}^{-B}(\chi_1^+ \otimes \bar{\chi}_2^+) \in \Gamma(\wedge^+ T^*M), \tag{2.6}$$
$$\Phi^- = \mathrm{e}^{-\phi}\mathrm{e}^{-B}(\chi_1^+ \otimes \bar{\chi}_2^-) \in \Gamma(\wedge^- T^*M), \tag{2.7}$$

where $\wedge^+ T^*M$ and $\wedge^- T^*M$ are the bundles of even- and odd-degree forms respectively. The polyforms satisfy a pair of compatibility conditions (A.11) and the Killing spinor equations are equivalent to the integrability conditions

$$\mathrm{d}\Phi^+ = 0, \qquad \mathrm{d}\Phi^- = 0, \tag{2.8}$$

which define what is known as a generalised Calabi–Yau metric. A conventional Calabi–Yau background is of course a special case, given by taking

$$\Phi^+ = \mathrm{e}^{-\phi}\mathrm{e}^{-B}\mathrm{e}^{-\mathrm{i}\omega}, \qquad \Phi^- = \mathrm{i}\,\mathrm{e}^{-\phi}\mathrm{e}^{-B}\Omega, \tag{2.9}$$

with B closed and ϕ constant. We see that Φ^+ generalises the symplectic structure and Φ^- generalises the complex structure.

As we will now see, the geometric interpretation of these conditions is given by generalised geometry [108–111].

2.2 Generalised geometry

Generalised geometry is the study of structures on a vector bundle E over a manifold M, where E is formed from the tangent bundle, cotangent bundle and products thereof. The original formulation of generalised geometry was given by Hitchin [108] and codified by Gualtieri [109] into what we now call $\mathrm{O}(d,d) \times \mathbb{R}^+$ generalised geometry or generalised complex geometry. The original motivation was to define geometric structures that include both complex and symplectic geometry as limiting cases. In this case, the larger vector bundle is $E \simeq TM \oplus T^*M$ which admits a natural $\mathrm{O}(d,d)$ metric on it sections, coming from the obvious pairing of vectors with one-forms. This endows E with an $\mathrm{O}(d,d)$ structure. Much like conventional vectors, one can define a bracket on sections of E to give a generalisation of the Lie bracket, known as the Courant bracket. The automorphisms of the Courant bracket are not only diffeomorphisms but also closed shifts of the B field, or what we know as gauge transformations. In this way, the gauge symmetries of the NS-NS sector of type II supergravity are built into the geometric description. Using the generalisation of vectors and Lie brackets, one can proceed by analogy with conventional differential geometry, defining connections on E and a generalised metric, which defines an $\mathrm{O}(d) \times \mathrm{O}(d) \subset \mathrm{O}(d,d)$ structure. The key point is that the structures that arise are directly applicable to physics, in particular to string theory and supergravity.

Let us return to the generalised Calabi–Yau metric example of the previous section and briefly sketch how the pair of closed polyforms define a torsion-free structure in $\mathrm{O}(6,6)\times\mathbb{R}^+$ generalised geometry [108, 109, 111]. The generalised tangent bundle $E \simeq TM \oplus T^*M$ admits a natural $\mathrm{O}(6,6)$ metric η. The two polyforms $\Phi^\pm$ can then be viewed as sections of the positive and negative helicity $\mathrm{Spin}(6,6)$ spinor bundles[2] associated to E, each stabilised by a different $\mathrm{SU}(3,3)$ subgroup of $\mathrm{Spin}(6,6)$. Therefore, each $\Phi^\pm$ individually defines a generalised $\mathrm{SU}(3,3)$ structure. The compatibility conditions imply that their common stability group is $\mathrm{SU}(3)\times\mathrm{SU}(3)$, so that the various structure groups embed as

$$\begin{array}{ccc} \mathrm{O}(6,6)\times\mathbb{R}^+ & \supset & \mathrm{SU}(3,3)_+ \text{ for } \Phi^+ \\ \cup & & \cup \\ \mathrm{SU}(3,3)_- \text{ for } \Phi^- & \supset & \mathrm{SU}(3)\times\mathrm{SU}(3) \text{ for } \{\Phi^+,\Phi^-\} \end{array} \tag{2.10}$$

Note that the two $\mathrm{SU}(3)$ stabiliser groups are precisely the groups preserving χ_1^+ and χ_2^- in (2.3). The integrability conditions $\mathrm{d}\Phi^\pm = 0$ are equivalent to the existence of a torsion-free generalised connection compatible with the relevant $\mathrm{SU}(3,3)_\pm$ structure.

It is natural to ask how these structures and their integrability conditions are extended when one considers generic backgrounds, for example including R-R fluxes. These are the questions we address in chapter 3. In identifying the relevant objects in the generalised geometry, and how they connect to conventional notions of G-structures, it will be useful to have a range of examples of $\mathcal{N}=2$ backgrounds. To this end, a number of simple cases, with and without R-R fluxes and in both type II and M-theory, are summarised in appendix A, along with more details of the Calabi–Yau and generalised Calabi–Yau metric cases.

One can think of $\mathrm{O}(d,d)\times\mathbb{R}^+$ generalised geometry as geometrising the NS-NS sector of supergravity. If we want to describe generic flux backgrounds with R-R fluxes in type II theories or four-form flux in M-theory, the relevant extension is $\mathrm{E}_{d(d)}\times\mathbb{R}^+$ generalised geometry.

2.3 $\mathrm{E}_{d(d)}\times\mathbb{R}^+$ generalised geometry

$\mathrm{E}_{d(d)}\times\mathbb{R}^+$ or exceptional generalised geometry is the study of structures on a vector bundle known as a generalised tangent bundle E, where E admits a unique action of the $\mathrm{E}_{d(d)}$ group [135, 136]. We can define a generalised frame bundle $\tilde{F}$ for E as an $\mathrm{E}_{d(d)}\times\mathbb{R}^+$ principal bundle. There is a generalised Lie derivative [136, 137, 178] which encodes the infinitesimal symmetries, diffeomorphisms and gauge transformations, of the supergravity theory, and one can use it to define generalised torsion and the analogue of the Levi-Civita connection [137, 138]. Generalised tensors are defined as sections of vector bundles

[2] In making this identification there is an arbitrary scaling factor that can be viewed as promoting the $\mathrm{O}(6,6)$ action to an $\mathrm{O}(6,6)\times\mathbb{R}^+$ action, corresponding to the dilaton degree of freedom [124, 177].

transforming in some representation of $\mathrm{E}_{d(d)} \times \mathbb{R}^+$. A generalised G-structure is then defined by a set of generalised tensors that are invariant under the action of a subgroup $G \subset \mathrm{E}_{d(d)}$. Equivalently, it is a choice of G principal sub-bundle of the generalised frame bundle $\tilde{P}_G \subset \tilde{F}$. The notion of an integrable generalised structure as one with vanishing intrinsic torsion then follows in analogy to the conventional case [143]. We now summarise the key points we need, relegating some details to appendix B.

For M-theory on a manifold M of dimension $d \leq 7$, the generalised tangent bundle is

$$E \simeq TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M \oplus (T^*M \otimes \wedge^7 T^*M). \tag{2.11}$$

For a type II theory on a $(d-1)$-dimensional manifold M, the generalised tangent bundle is

$$E \simeq TM \oplus T^*M \oplus \wedge^{\pm} T^*M \oplus \wedge^5 T^*M \oplus (T^*M \otimes \wedge^6 T^*M), \tag{2.12}$$

where $\pm$ refers to even- or odd-degree forms for type IIA or IIB respectively. For type IIA, this is just a dimensional reduction of the M-theory case. For type IIB, this can be rewritten in a way that stresses the $\mathrm{SL}(2;\mathbb{R})$ symmetry as

$$E \simeq TM \oplus (S \otimes T^*M) \oplus \wedge^3 T^*M \oplus (S \otimes \wedge^5 T^*M) \oplus (T^*M \otimes \wedge^6 T^*M), \tag{2.13}$$

where S is an $\mathbb{R}^2$ bundle transforming as a doublet of $\mathrm{SL}(2;\mathbb{R})$. In all cases the generalised tangent bundle is an $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ vector bundle. For example, for $d = 7$ it transforms in the $\mathbf{56_1}$ representation, where the subscript denotes the $\mathbb{R}^+$ weight. By definition, a scalar field of weight p, transforming in the representation $\mathbf{1}_{\boldsymbol{p}}$, is a section of $(\det T^*M)^{p/(9-d)}$.[3]

The generalised frame bundle $\tilde{F}$ is an $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ principal bundle constructed from frames for E. One defines generalised tensors as sections of the vector bundles associated with different $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ representations. Of particular interest is the adjoint bundle $\operatorname{ad}\tilde{F}$, corresponding to the adjoint representation of $\mathrm{E}_{d(d)} \times \mathbb{R}^+$. In M-theory we have

$$\operatorname{ad}\tilde{F} \simeq \mathbb{R} \oplus (TM \otimes T^*M) \oplus \wedge^3 T^*M \oplus \wedge^6 T^*M \oplus \wedge^3 TM \oplus \wedge^6 TM, \tag{2.14}$$

while in type II

$$\begin{aligned} \operatorname{ad}\tilde{F} \simeq \mathbb{R} \oplus & \left[\mathbb{R} \oplus \wedge^6 TM \oplus \wedge^6 T^*M\right] \\ & \oplus \left[(TM \otimes T^*M) \oplus \wedge^2 T^*M \oplus \wedge^2 TM\right] \oplus \left[\wedge^{\mp} TM \oplus \wedge^{\mp} T^*M\right], \end{aligned} \tag{2.15}$$

where the upper and lower signs refer to type IIA and type IIB respectively. For IIB this

[3]Since supersymmetric backgrounds are orientable, we can assume $\det T^*M$ is trivial. A discussion of fractional density bundles can be found in [179].

can also be written as

$$\begin{aligned}\operatorname{ad}\tilde{F} \simeq \mathbb{R} \oplus (TM \otimes T^*M) \oplus (S \otimes S^*)_0 \oplus (S \otimes \wedge^2 TM) \oplus (S \otimes \wedge^2 T^*M) \\ \oplus \wedge^4 TM \oplus \wedge^4 T^*M \oplus (S \otimes \wedge^6 TM) \oplus (S \otimes \wedge^6 T^*M),\end{aligned} \tag{2.16}$$

where the subscript on $(S \otimes S^*)_0$ indicates that one takes the traceless part. For $d = 7$ these bundles transform in the $\mathbf{1_0} + \mathbf{133_0}$ representation, where the singlet is the part generating the $\mathbb{R}^+$ action.

The generalised tangent bundle is actually defined as an extension, so that there is a non-trivial patching between the tensor components. In M-theory, on the overlap of two local patches $U_i \cap U_j$ of M, a generalised vector $V \in \Gamma(E)$ is patched by

$$V_{(i)} = \mathrm{e}^{\mathrm{d}\Lambda_{(ij)} + \mathrm{d}\tilde{\Lambda}_{(ij)}} V_{(j)}, \tag{2.17}$$

where $\Lambda_{(ij)}$ and $\tilde{\Lambda}_{(ij)}$ are locally two- and five-forms respectively, which can be identified as sections of $\operatorname{ad}\tilde{F}$, so that $\mathrm{e}^{\mathrm{d}\Lambda_{(ij)} + \mathrm{d}\tilde{\Lambda}_{(ij)}}$ is the exponentiated adjoint action. The isomorphisms (2.11) and (2.14) depend on a pair of potentials $A \in \Gamma(\wedge^3 T^*M)$ and $\tilde{A} \in \Gamma(\wedge^6 T^*M)$ via the exponentiated adjoint action

$$V = \mathrm{e}^{A + \tilde{A}} \tilde{V}, \qquad R = \mathrm{e}^{A + \tilde{A}} \tilde{R}\, \mathrm{e}^{-A - \tilde{A}}, \tag{2.18}$$

where $V \in \Gamma(E)$ and $R \in \Gamma(\operatorname{ad}\tilde{F})$, the "untwisted" objects $\tilde{V}$ and $\tilde{R}$ are sections of $TM \oplus \wedge^2 T^*M \oplus \cdots$ and $\mathbb{R} \oplus (TM \otimes T^*M) \oplus \cdots$ respectively, and A and $\tilde{A}$ are patched by

$$A_{(i)} = A_{(j)} + \mathrm{d}\Lambda_{(ij)}, \qquad \tilde{A}_{(i)} = \tilde{A}_{(j)} + \mathrm{d}\tilde{\Lambda}_{(ij)} - \tfrac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge A_{(j)}. \tag{2.19}$$

The corresponding gauge-invariant field strengths

$$F = \mathrm{d}A, \qquad \tilde{F} = \mathrm{d}\tilde{A} - \tfrac{1}{2} A \wedge F, \tag{2.20}$$

are precisely the supergravity objects defined in (B.24). The type II theories are similarly patched. For type IIB we have

$$V_{(i)} = \mathrm{e}^{\mathrm{d}\Lambda^i_{(ij)} + \mathrm{d}\tilde{\Lambda}_{(ij)}} V_{(j)}, \tag{2.21}$$

where $\Lambda^i_{(ij)}$ and $\tilde{\Lambda}_{(ij)}$ are locally a pair of one-forms and a three-form respectively. The relations between the twisted and untwisted objects are written as

$$V = \mathrm{e}^{B^i + C} \tilde{V}, \qquad R = \mathrm{e}^{B^i + C} \tilde{R}\, \mathrm{e}^{-B^i - C}, \tag{2.22}$$

with the corresponding three- and five-form field strengths given by

$$F^i = \mathrm{d}B^i, \qquad F = \mathrm{d}C + \tfrac{1}{2}\epsilon_{ij} B^i \wedge F^j, \tag{2.23}$$

where $F^1 = H$, $F^2 = F_3$ and F are the usual supergravity field strengths, defined in (B.53). We discuss how to include a non-zero axion-dilaton in appendix B.3, following [139].

The differential structure of the generalised tangent bundle is captured by a generalisation of the Lie derivative that encodes the bosonic symmetries of supergravity, namely diffeomorphisms and form-field gauge transformations. Given a generalised vector field $V \in \Gamma(E)$, one can define the action of the generalised Lie derivative (or Dorfman derivative) L_V on any generalised tensor. For example, its action on generalised vectors is given in (B.16) and (B.45), and on sections of $\operatorname{ad}\tilde{F}$ in (B.17) and (B.46). The generalised Lie derivative endows E with the structure of a Leibniz algebroid [180] and will play an essential role in defining the integrability conditions on the generalised structures.

Chapter 3

Exceptional Calabi–Yau backgrounds

In this chapter we define the analogue of Calabi–Yau geometry for generic $D = 4$, $\mathcal{N} = 2$ flux backgrounds in type II supergravity and M-theory. We show that solutions of the Killing spinor equations are in one-to-one correspondence with integrable, globally defined structures in $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ generalised geometry. Such "exceptional Calabi–Yau" geometries are determined by two generalised objects that parametrise hyper- and vector-multiplet degrees of freedom and generalise conventional complex, symplectic and hyper-Kähler geometries. The integrability conditions for both hyper- and vector-multiplet structures are given by the vanishing of moment maps for the "generalised diffeomorphism group" of diffeomorphisms combined with gauge transformations. We give a number of explicit examples and discuss the structure of the moduli spaces of solutions. We then extend our construction to $D = 5$ and $D = 6$ flux backgrounds preserving eight supercharges, where similar structures appear.

3.1 Introduction

We are searching for a generalisation of the notion of a Calabi–Yau manifold to backgrounds including both NS-NS and R-R flux. We will show that exceptional generalised geometry [135–138] gives precisely such a reformulation: the supersymmetric background defines an *integrable* generalised structure, which we call an "exceptional Calabi–Yau" (ECY) geometry.[1] The tensors ω and Ω are replaced by a pair of generalised structures that interpolate between complex, symplectic and hyper-Kähler geometries. With respect to the $\mathcal{N} = 2$ supersymmetry, one structure is naturally associated to hypermultiplets and the other to vector multiplets, and the integrability conditions, defined using generalised intrinsic torsion [143], have an elegant interpretation in terms of moment maps.

[1] In this thesis, we take "integrable" to mean first-order integrability or, equivalently, vanishing intrinsic torsion. We have not made any investigations into whether first-order integrability implies full integrability for the structures we consider, or even how to define obstructions to higher-order integrability.

The two generalised structures defining generic $\mathcal{N}=2$, $D=4$ backgrounds are invariant under $\mathrm{Spin}^*(12)$ and $\mathrm{E}_{6(2)}$ subgroups of the $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ acting on the generalised tangent space. We refer to them as H and V structures respectively, standing for "hypermultiplet" and "vector-multiplet". If compatible, together they define an HV structure that is invariant under SU(6). It is then natural to define an ECY geometry as one that admits an *integrable* HV structure. Such structures were first introduced in the context of type II theories in [181]. Since the supersymmetry parameters transform under $\mathrm{H}_7 = \mathrm{SU}(8)$ in the exceptional generalised geometry, the SU(6) structure appears as SU(6) is the stabiliser group of a pair of Killing spinors. Some steps towards rephrasing supersymmetry in terms of integrable generalised structures in the $\mathcal{N}=1$ case, where the structure is SU(7), were taken in [136] in M-theory and in [181] in type II. The full set of $\mathcal{N}=1$ conditions, written using a particular generalised connection, were given in [182], and this was extended to $\mathcal{N}=2$ in [172]. The four-dimensional effective theories in both $N=1$ and $N=2$ have been considered in [136, 181, 183].

For each structure, we show that the integrability conditions correspond to the existence of a torsion-free G-compatible generalised connection. This follows the analysis of [143] where it was shown that there is a natural definition of intrinsic torsion for generalised G-structures, and one can define generalised special holonomy as structures with $G \subset \mathrm{H}_d$ and vanishing generalised intrinsic torsion. Supersymmetric backgrounds of type II and eleven-dimensional supergravity in various dimensions are constrained to have generalised special holonomy in both the Minkowski [143, 144] and AdS [145] case. Here, we use the same notion of generalised intrinsic torsion to prove that our integrability conditions are equivalent to the Killing spinor equations.

As first noted in [181], the infinite-dimensional spaces of hypermultiplet and vector-multiplet structures admit hyper-Kähler and special Kähler metrics respectively. Strikingly, we find that the integrability conditions for each can be formulated as the vanishing of the corresponding moment maps for the action of the generalised diffeomorphism group. The moduli spaces of structures are then given by a hyper-Kähler or symplectic quotient. For ECY geometries there is an additional integrability condition that involves both structures. That differential conditions appear as moment maps on infinite-dimensional spaces is a ubiquitous phenomenon [184, 185]. Examples include the Atiyah–Bott description of flat gauge connections on a Riemann surface [186], the Donaldson–Uhlenbeck–Yau equations [187–189], the Hitchin equations [190], and even the equations for Kähler–Einstein metrics [191, 192]. In our case we see that there are also moment maps for geometries defining generalisations of complex and symplectic structures that, in addition, use the full (generalised) diffeomorphism group.

Physically the appearance of moment maps is natural. It is possible to reformulate the full ten- or eleven-dimensional supergravity as a four-dimensional $\mathcal{N}=2$ theory [123, 181, 193]. The $\mathrm{Spin}^*(12)$ structures then naturally parametrise an infinite-dimensional space of hypermultiplets, while the $\mathrm{E}_{6(2)}$ structures encode an infinite-dimensional space of vector multiplets. This is the origin of our names for the two types of structures. The $\mathcal{N} = 2$ theory will be gauged, and supersymmetry implies that the gauging defines a triplet of moment maps on the hypermultiplets and a single moment map on the vector multiplets (see for example [194]). This structure was already noted in [181], where it was pointed out that the gauged symmetry was simply the R-R gauge transformations. However, for generic backgrounds, as we show here, not only the R-R gauge transformations but actually the whole set of generalised diffeomorphisms are gauged, including NS-NS gauge transformations and conventional diffeomorphisms. The integrability conditions can then be directly translated into the vanishing of the gaugino, hyperino and gravitino variations, following a similar analysis for $\mathcal{N} = 1$ backgrounds in [136, 181, 182, 195]. In making this translation we partly rephrase the standard conditions, as given in [196–198], showing that the gaugino variation generically implies a vanishing of the vector-multiplet moment map.

Our formalism also applies to both type II and M-theory backgrounds in $D = 5$ and $D = 6$ preserving eight supercharges. The hypermultiplet structure is always of the same form, but the second generalised structure that is compatible with it is dependent on the case in hand. As we discuss in the next chapter, AdS backgrounds can also be described in this formalism.

Starting in section 3.2 we define the relevant generalised structures for $\mathcal{N} = 2$, $D = 4$ backgrounds. We discuss the integrability conditions in section 3.3. We give a number of examples that we hope will clarify some of the more abstract constructions. More technical aspects, such as the equivalence of integrability with torsion-free G-structures, the origin of the integrability conditions from gauged supergravity and the moduli space of supersymmetric compactifications, are all in section 3.4. We discuss the extension to $D = 5, 6$ backgrounds in section 3.5.

3.1.1 Supersymmetric backgrounds in type II and M-theory

We consider type II and M-theory spacetimes of the form $\mathbb{R}^{D-1,1} \times M$, with a warped product metric

$$\mathrm{d}s^2 = \mathrm{e}^{2\Delta}\mathrm{d}s^2(\mathbb{R}^{D-1,1}) + \mathrm{d}s^2(M), \tag{3.1}$$

where Δ is a scalar function on M. Initially we will assume $D = 4$ and hence M is six-dimensional for type II and seven-dimensional for M-theory. For the type II theories we use the string frame metric so that the warp factors for type II and M-theory are related by $\Delta_{\mathrm{II}} = \Delta_{\mathrm{M}} + \frac{1}{3}\phi$, where ϕ is the dilaton. We allow generic fluxes compatible with the Lorentz symmetry of $\mathbb{R}^{3,1}$. Thus for M-theory, of the eleven-dimensional four-form flux $\mathcal{F}$ we keep the components

$$F_{m_1\dots m_4} = \mathcal{F}_{m_1\dots m_4}, \qquad \tilde{F}_{m_1\dots m_7} = (\star\mathcal{F})_{m_1\dots m_7}, \tag{3.2}$$

where $m = 1, \dots, 7$ are indices on M, while for type II we use the democratic formalism [199] and keep only the flux components that lie entirely on M.

In M-theory, the eleven-dimensional spinors ε can be decomposed into four- and seven-dimensional spinors η_i^+ and ϵ_i respectively according to

$$\varepsilon = \eta_1^+ \otimes \epsilon_1 + \eta_2^+ \otimes \epsilon_2 + \text{c.c.} \tag{3.3}$$

where $\pm$ denotes the chirality of $\eta_i^\pm$ and we add the charge conjugate. The internal spinor ϵ is complex, and can be thought of as a pair of real Spin(7) spinors $\epsilon = \operatorname{Re}\epsilon + \mathrm{i}\operatorname{Im}\epsilon$. The Killing spinor equations read [200–203]

$$\begin{aligned} \nabla_m\epsilon + \tfrac{1}{288}F_{n_1\dots n_4}(\gamma_m{}^{n_1\dots n_4} - 8\delta_m{}^{n_1}\gamma^{n_2n_3n_4})\epsilon - \tfrac{1}{12}\tfrac{1}{6!}\tilde{F}_{mn_1\dots n_6}\gamma^{n_1\dots n_6}\epsilon = 0, \\ \gamma^m\nabla_m\epsilon + (\partial_m\Delta)\gamma^m\epsilon - \tfrac{1}{96}F_{m_1\dots m_4}\gamma^{m_1\dots m_4}\epsilon - \tfrac{1}{4}\tfrac{1}{7!}\tilde{F}_{m_1\dots m_7}\gamma^{m_1\dots m_7}\epsilon = 0, \end{aligned} \tag{3.4}$$

where ∇ is the Levi-Civita connection for the metric on M and γ^m are the $\text{Cliff}(7;\mathbb{R})$ gamma matrices. These imply that $\tilde{F}$ vanishes for Minkowski backgrounds [200], since it can be supported only by a cosmological constant.

There are similar expressions for the Killing spinor equations in type II (see for example [111]). In this case, there are a pair of real ten-dimensional spinors $\{\varepsilon_1, \varepsilon_2\}$. The most general decomposition under $\text{Spin}(3,1) \times \text{Spin}(6)$ is [172]

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \eta_1^+ \otimes \begin{pmatrix} \chi_1^+ \\ \tilde{\chi}_1^\mp \end{pmatrix} + \eta_2^+ \otimes \begin{pmatrix} \tilde{\chi}_2^+ \\ \chi_2^\mp \end{pmatrix} + \text{c.c.} \tag{3.5}$$

where $\pm$ denotes the chirality, we add the charge conjugate and the upper and lower signs refer to type IIA and IIB respectively. This choice of sign corresponds to the two different embeddings of $\text{Spin}(6) \simeq \text{SU}(4) \subset \text{SU}(8)$: one for type IIA and one for type IIB, corresponding to the decompositions $\mathbf{8} = \mathbf{4} + \bar{\mathbf{4}}$ and $\mathbf{8} = \mathbf{4} + \mathbf{4}$ respectively. We see the internal spinors can be combined into two, complex, eight-component objects

$$\epsilon_1 = \begin{pmatrix} \chi_1^+ \\ \tilde{\chi}_1^\mp \end{pmatrix}, \qquad \epsilon_2 = \begin{pmatrix} \tilde{\chi}_2^+ \\ \chi_2^\mp \end{pmatrix}, \tag{3.6}$$

which for type IIA is simply the lift to the $d = 7$ complex spinors of the M-theory case. The standard spinor ansatz (2.3) corresponds to taking $\tilde{\chi}_i = 0$.

In both type II and M-theory, the gamma matrices generate an action of SU(8) on the eight-component spinors ϵ_i. For $\mathcal{N} = 2$ backgrounds we have two independent solutions, ϵ_1 and ϵ_2, to the Killing spinor equations. With respect to the SU(8) action, the solutions are thus invariant under an SU(6) subgroup. In $\text{E}_{7(7)} \times \mathbb{R}^+$ generalised geometry this SU(8) action is a local symmetry [137, 138]. From this perspective, as stressed in [143, 172, 181], we can view the $\mathcal{N} = 2$ background as defining a generalised SU(6) structure

$$\mathcal{N} = 2 \text{ background } \{\epsilon_1, \epsilon_2\} \quad \Longleftrightarrow \quad \text{generalised SU(6) structure.} \tag{3.7}$$

Understanding how this SU(6) structure is defined and its integrability conditions, along with the analogous structures in $D = 5$ and $D = 6$, will be the central goal of this chapter.

3.2 $\mathrm{E}_{7(7)}$ structures

We now show that a generic $\mathcal{N} = 2$, $D = 4$ background defines a pair of generalised structures in $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ generalised geometry. For type II backgrounds this pair was first identified in [181]. We will turn to the integrability conditions in the next section.

The idea of a generalised G-structure is as follows. In conventional geometry, the generic structure group of the tangent bundle TM of a d-dimensional manifold M is $\mathrm{GL}(d;\mathbb{R})$. The existence of a G-structure implies that the structure group reduces to $G \subset \mathrm{GL}(d;\mathbb{R})$. It can be defined by a set of tensors $\{\Xi\}$ that are stabilised by the action of G, or alternatively as a principal G-sub-bundle P_G of the $\mathrm{GL}(d;\mathbb{R})$ frame bundle F. In generalised geometry, one considers an extended tangent bundle E which admits the action of a group larger than $\mathrm{GL}(d;\mathbb{R})$. For us the relevant generalised geometry will have an action of $\mathrm{E}_{7(7)} \times \mathbb{R}^+$. One can define frames for E and a corresponding principal $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ bundle, called the generalised frame bundle $\tilde{F}$. A generalised G-structure is then defined by a set of generalised tensors that are invariant under the action of a subgroup $G \subset \mathrm{E}_{7(7)} \times \mathbb{R}^+$. Equivalently, it is a principal G-sub-bundle $\tilde{P}_G$, of the generalised frame bundle $\tilde{F}$.

The two generalised G-structures relevant to $\mathcal{N} = 2$, $D = 4$ backgrounds are[2,3]

$$\begin{aligned} &\textit{hypermultiplet structure}, \ J_\alpha && G = \mathrm{Spin}^*(12), \\ &\textit{vector-multiplet structure}, \ K && G = \mathrm{E}_{6(2)}. \end{aligned} \tag{3.8}$$

We will often refer to these as H and V structures respectively. As we will see, we can impose two compatibility conditions between the structures such that their common stabiliser group is $\mathrm{Spin}^*(12) \cap \mathrm{E}_{6(2)} = \mathrm{SU}(6)$, defining

$$\textit{HV structure}, \ \{J_\alpha, K\} \qquad G = \mathrm{SU}(6). \tag{3.9}$$

We see that the generalisation of the embeddings (2.4) and (2.10) for Calabi–Yau and generalised Calabi–Yau metrics respectively is given by

$$\begin{array}{ccc} \mathrm{E}_{7(7)} \times \mathbb{R}^+ & \supset & \mathrm{Spin}^*(12) \text{ for } J_\alpha \\ \cup & & \cup \\ \mathrm{E}_{6(2)} \text{ for } K & \supset & \mathrm{SU}(6) \text{ for } \{J_\alpha, K\} \end{array} \tag{3.10}$$

The SU(6) group is the same one that stabilises the pair of SU(8) Killing spinors $\{\epsilon_1, \epsilon_2\}$.

[2] In [181] these were denoted K_α and $\lambda = 2\,\mathrm{Re}\,L$ respectively.

[3] $\mathrm{Spin}^*(12)$ is the double cover of $\mathrm{SO}^*(12)$, the latter corresponding to a particular real form of the complex $\mathrm{SO}(12;\mathbb{C})$ Lie algebra [204].

		hyper		vector	
E	G_{frame}	G	Ξ	G	Ξ
TM	$\mathrm{GL}(6)$	$\mathrm{Sp}(6;\mathbb{R})$	ω	$\mathrm{SL}(3;\mathbb{C})$	Ω
$TM \oplus T^*M$	$\mathrm{O}(6,6)\times\mathbb{R}^+$	$\mathrm{SU}(3,3)_+$	Φ^+	$\mathrm{SU}(3,3)_-$	Φ^-
$TM \oplus T^*M \oplus \wedge^- T^*M \oplus \ldots$	$\mathrm{E}_{7(7)}\times\mathbb{R}^+$	$\mathrm{Spin}^*(12)$	J_α	$\mathrm{E}_{6(2)}$	K

Table 3.1: The (generalised) tangent bundles and G-structures in conventional, generalised complex and exceptional generalised geometry for type IIB supergravity. We include the group G_{frame} that acts on the (generalised) frame bundle, the reduced structure group G of the symplectic, complex, generalised complex, vector- or hypermultiplet structure, and the invariant object Ξ that defines the structure.

These structures are generalisations of the symplectic and complex structures on Calabi–Yau manifolds in type II compactifications. Focussing on type IIB, in table 3.2 we list the (generalised) tangent bundles and structures that appear in conventional and generalised $\mathrm{O}(d,d)\times\mathbb{R}^+$ and $\mathrm{E}_{7(7)}\times\mathbb{R}^+$ geometries. We see that the H structure generalises the symplectic structure ω (or the pure spinor Φ^+), while the V structure generalises the complex structure Ω (or the pure spinor Φ^-). For type IIA the situation is reversed, and the V and H structures generalise ω and Ω respectively.

Recall that the moduli spaces of (integrable) symplectic and complex structures of Calabi–Yau manifolds are associated with $\mathcal{N}=2$, $D=4$ hypermultiplets and vector multiplets in type II theories. The same thing happens here: the moduli space of integrable $\mathrm{Spin}^*(12)$ structures defines fields in hypermultiplets and that of integrable $\mathrm{E}_{6(2)}$ structures defines fields in vector multiplets, hence the names. In fact, one can also consider the infinite-dimensional space of all such structures, without imposing any integrability conditions, and these too can naturally be associated with hypermultiplets and vector multiplets. As described in [123, 181, 193], one can view this structure as arising from a rewriting of the full ten- or eleven-dimensional theory, analogous to the construction in [146], but with only eight supercharges manifest. The local $\mathrm{SO}(9,1)$ Lorentz symmetry is broken and the degrees of freedom can be repackaged into $\mathcal{N}=2$, $D=4$ multiplets. However, since all modes are kept – there is no Kaluza–Klein truncation – the hyper- and vector-multiplet spaces are infinite dimensional.

We now define H and V structures, discuss the infinite-dimensional spaces of structures, and, in each case, show how the various examples of $\mathcal{N}=2$, $D=4$ backgrounds given in appendix A define J_α and K.

3.2.1 Hypermultiplet structures

The idea of a hypermultiplet structure (or H structure) was first introduced in [181] in the context of type II theories. Formally we have:

Definition. An $E_{7(7)}$ *hypermultiplet structure* is a $\mathrm{Spin}^*(12) \subset E_{7(7)} \times \mathbb{R}^+$ generalised structure.

In other words, it is a $\mathrm{Spin}^*(12)$ principal sub-bundle $\tilde{P}_{\mathrm{Spin}^*(12)}$ of the generalised frame bundle $\tilde{F}$. More concretely, we can define the structure by choosing a set of invariant generalised tensors. The relevant objects are a triplet of sections of a weighted adjoint bundle

$$J_\alpha \in \Gamma\big(\mathrm{ad}\,\tilde{F} \otimes (\det T^*M)^{1/2}\big) \qquad \alpha = 1, 2, 3, \tag{3.11}$$

such that they transform in the $\mathbf{133_1}$ representation of $E_{7(7)} \times \mathbb{R}^+$. We require them to define a *highest weight* $\mathfrak{su}_2$ subalgebra of $\mathfrak{e}_{7(7)}$, which is the necessary and sufficient condition for them to be invariant under $\mathrm{Spin}^*(12)$. We can write the algebra as

$$[J_\alpha, J_\beta] = 2\kappa\epsilon_{\alpha\beta\gamma}J_\gamma, \tag{3.12}$$

where κ is a section of $(\det T^*M)^{1/2}$ and the commutator is simply the commutator in the adjoint representation of $E_{7(7)} \times \mathbb{R}^+$, defined in (B.11) and (B.40). The norms of the J_α, calculated using the $\mathfrak{e}_{7(7)}$ Killing form given in (B.31) and (B.60), are then fixed to be

$$\mathrm{tr}(J_\alpha J_\beta) = -\kappa^2\delta_{\alpha\beta}. \tag{3.13}$$

As described in [181], decomposing under the $\mathrm{SU}(8)$ subgroup[4] of $E_{7(7)}$, one can view the corresponding "untwisted" objects $\tilde{J}_\alpha$ as being constructed from bilinears of the Killing spinors ϵ_i of the form $\sigma_\alpha^{ij}\epsilon_i\bar{\epsilon}_j$, where σ_α^{ij} are the Pauli matrices.

A key point for us, first noted in [181], is that the infinite-dimensional space of H structures admits a natural hyper-Kähler metric. To define the space of structures, note that, at a particular point $x \in M$, the structure $J_\alpha|_x$ is invariant under $\mathrm{Spin}^*(12)$ so it can be viewed as fixing a point in the homogeneous space

$$J_\alpha|_x \in W = E_{7(7)} \times \mathbb{R}^+/\,\mathrm{Spin}^*(12). \tag{3.14}$$

One can then consider the fibre bundle of homogeneous spaces

$$\begin{array}{ccc} W & \longrightarrow & Z_{\mathrm{H}} \\ & & \downarrow \\ & & M \end{array} \tag{3.15}$$

constructed by taking a quotient $Z_{\mathrm{H}} = \tilde{F}/G$ of the generalised frame bundle $\tilde{F}$ by the structure group $G = \mathrm{Spin}^*(12)$. Choosing an H structure is equivalent to choosing a section of Z_{H}. Thus the infinite-dimensional space of all possible H structures is simply

[4] The actual subgroup is $\mathrm{SU}(8)/\mathbb{Z}_2$ but the discrete group factors are not important for the work in this thesis.

the space of smooth sections,

$$\text{space of hypermultiplet structures } \mathcal{A}_{\mathrm{H}} = \Gamma(Z_{\mathrm{H}}). \tag{3.16}$$

Crucially, the fibres W of Z_{H} are themselves pseudo-Riemannian hyper-Kähler spaces. In fact W is a hyper-Kähler cone over a pseudo-Riemannian symmetric quaternionic-Kähler space, also known as a Wolf space,

$$W/\mathbb{H}^* = \mathrm{E}_{7(7)}/(\mathrm{Spin}^*(12) \times \mathrm{SU}(2)), \tag{3.17}$$

where the action of the quaternions $\mathbb{H}^*$ mods out by $\mathrm{SU}(2) \times \mathbb{R}^+$. The Riemannian symmetric quaternionic-Kähler spaces were first considered by Wolf in [205] and classified by Alekseevsky in [206], while the pseudo-Riemannian case was analysed by Alekseevsky and Cortés [207], and (3.17) is indeed included in their list. Recall that one can always construct a hyper-Kähler cone, known as the Swann bundle, over any quaternionic-Kähler space [208]. In this case the cone directions are simply the SU(2) bundle together with the overall $\mathbb{R}^+$ scaling. The hyper-Kähler geometry on W, as first described in [209], is summarised in appendix E.2.

The hyper-Kähler geometry on $\mathcal{A}_{\mathrm{H}}$ is inherited directly from the hyper-Kähler geometry of the W fibres of Z_{H}. This is in much the same way that the infinite-dimensional space of smooth Riemannian metrics on a compact d-dimensional manifold (which can be viewed as the space of sections of a $\mathrm{GL}(d;\mathbb{R})/\mathrm{O}(d)$ homogeneous fibre bundle) is itself a Riemannian space [210–212]. The construction follows that on W. Concretely, consider a point $\sigma \in \mathcal{A}_{\mathrm{H}}$, corresponding to a choice of section $\sigma(x) \in \Gamma(Z_{\mathrm{H}})$. Equivalently, given a point $\sigma \in \mathcal{A}_{\mathrm{H}}$ we have a triplet of sections $J_\alpha(x)$. Formally, one can think of $J_\alpha(x)[\sigma]$ as a triplet of functions on $\mathcal{A}_{\mathrm{H}}$ taking values in the space of sections $\Gamma(\operatorname{ad}\tilde{F} \otimes (\det T^*M)^{1/2})$

$$J_\alpha \colon \mathcal{A}_{\mathrm{H}} \to \Gamma\big(\operatorname{ad}\tilde{F} \otimes (\det T^*M)^{1/2}\big). \tag{3.18}$$

The tangent space $T_\sigma\mathcal{A}_{\mathrm{H}}$ at σ is spanned by vectors $v \in T_\sigma\mathcal{A}_{\mathrm{H}}$ that can be viewed as a small deformation of the structure $J_\alpha(x)$. Formally, we can define the change $v_\alpha(x)$ in $J_\alpha(x)$, given by v acting on the section-valued functions J_α, that is $v_\alpha = v(J_\alpha) = \imath_v \delta J_\alpha$, where δ is the exterior derivative on $\mathcal{A}_{\mathrm{H}}$. By definition, $v_\alpha(x)$ is a section of $\operatorname{ad}\tilde{F} \otimes (\det T^*M)^{1/2}$. At each point σ it can always be written as

$$v_\alpha(x) = [R(x), J_\alpha(x)], \tag{3.19}$$

where $R(x)$ is a section of the $\mathfrak{e}_{7(7)} \oplus \mathbb{R}$ adjoint bundle $\operatorname{ad}\tilde{F}$. Note that only elements that are not in $\mathfrak{spin}^*_{12}$ actually generate non-zero v_α. Decomposing $\operatorname{ad}\tilde{F} \simeq \operatorname{ad}\tilde{P}_{\mathrm{Spin}^*(12)} \oplus \operatorname{ad}\tilde{P}^{\perp}_{\mathrm{Spin}^*(12)}$, where $\tilde{P}_{\mathrm{Spin}^*(12)}$ is the generalised G-structure defined by J_α, this means formally we can also identify

$$T_\sigma \mathcal{A}_{\mathrm{H}} \simeq \Gamma\big(\operatorname{ad} \tilde{P}^{\perp}_{\mathrm{Spin}^*(12)} \otimes (\det T^*M)^{1/2}\big). \tag{3.20}$$

Given two tangent vectors $v, w \in T_\sigma \mathcal{A}_{\mathrm{H}}$, we then define a triplet of symplectic forms at the point $\sigma \in \mathcal{A}_{\mathrm{H}}$, such that the symplectic products between v and w are given by

$$\Omega_\alpha(v, w) = \epsilon_{\alpha\beta\gamma} \int_M \operatorname{tr}(v_\beta w_\gamma). \tag{3.21}$$

Recall that $v_\alpha(x)$ and $w_\alpha(x)$ are sections of $\operatorname{ad} \tilde{F} \otimes (\det T^*M)^{1/2}$. Thus $\operatorname{tr}(v_\beta w_\gamma)$ is a section of $\det T^*M$ and can indeed be integrated over M. These forms define the hyper-Kähler structure.

The geometry on $\mathcal{A}_{\mathrm{H}}$ is actually itself a hyper-Kähler cone. There is a *global* $\mathrm{SU}(2) \times \mathbb{R}^+$ action that rotates and rescales the structures J_α. This means that one can define a hyper-Kähler potential [208], a real function χ which is a Kähler potential for each of the three symplectic structures. On $\mathcal{A}_{\mathrm{H}}$ it is given by the functional

$$\chi = \tfrac{1}{2} \int_M \kappa^2, \tag{3.22}$$

where κ^2 is the density that depends on the choice of structure $\sigma(x) \in \Gamma(Z_{\mathrm{H}})$ through (3.12). In terms of the Killing spinors ϵ_i, the global SU(2) symmetry corresponds to the fact that, under the decompositions (3.3) and (3.5), the ϵ_i are determined only up to global U(2) rotations of the pair of four-dimensional spinors η_i^+. Thus the global SU(2) action on J_α is simply part of the four-dimensional $\mathcal{N} = 2$ R-symmetry. The global $\mathbb{R}^+$ rescaling corresponds to shifting the warp factor Δ in (3.1) by a constant, and then absorbing this in a constant conformal rescaling of the flat metric $\mathrm{d}s^2(\mathbb{R}^{3,1})$. Modding out by these symmetries, we see that the physical space of structures is actually an infinite-dimensional quaternionic-Kähler space. As we have mentioned, this structure on $\mathcal{A}_{\mathrm{H}}$ can be viewed, following [123, 181, 193], as a rewriting of the full ten- or eleven-dimensional supergravity theory as a four-dimensional $\mathcal{N} = 2$ theory coupled to an infinite number of hypermultiplets, corresponding to the full tower of Kaluza–Klein modes parametrising $\mathcal{A}_{\mathrm{H}}$. Physically, the Swann bundle structure corresponds to coupling hypermultiplets to superconformal gravity [213–215].

3.2.2 Vector-multiplet structures

Vector-multiplet structures (or V structures) were also first introduced in [181] in the context of type II theories. Formally we have:

Definition. An $\mathrm{E}_{7(7)}$ *vector-multiplet structure* is an $\mathrm{E}_{6(2)} \subset \mathrm{E}_{7(7)} \times \mathbb{R}^+$ generalised structure.

In other words, it is an $\mathrm{E}_{6(2)}$ principal sub-bundle $\tilde{P}_{\mathrm{E}_{6(2)}}$ of the generalised frame bundle $\tilde{F}$. The corresponding invariant generalised tensor is a section of the generalised tangent bundle

$$K \in \Gamma(E), \tag{3.23}$$

which we recall transforms in the $\mathbf{56_1}$ representation of $\mathrm{E}_{7(7)} \times \mathbb{R}^+$. This tensor is almost generic, the only requirement is that it satisfies

$$q(K) > 0, \tag{3.24}$$

where q is the quartic invariant of $\mathrm{E}_{7(7)}$.[5] This ensures that the stabiliser group is $\mathrm{E}_{6(2)}$ [216]. As will see below when we discuss the geometry of the space of V structures following [181], one can use $q(K)$ to construct a second invariant generalised vector $\hat{K}$, and it is often convenient to consider the complex object

$$X = K + \mathrm{i}\hat{K}. \tag{3.25}$$

Decomposing under the SU(8) subgroup of $\mathrm{E}_{7(7)}$, one can view the corresponding "untwisted" objects $\tilde{X}$ as being constructed from bilinears of the Killing spinors ϵ_i of the form $\epsilon^{ij}\epsilon_i\epsilon_j^{\mathrm{T}} = \epsilon_1\epsilon_2^{\mathrm{T}} - \epsilon_2\epsilon_1^{\mathrm{T}}$.

In this case, the infinite-dimensional space of V structures admits a natural rigid (or affine) special Kähler metric [181]. The structure $K|_x$ at a particular point $x \in M$ fixes a point in the homogeneous space

$$K|_x \in P = \mathrm{E}_{7(7)} \times \mathbb{R}^+/\,\mathrm{E}_{6(2)}. \tag{3.26}$$

One can then consider the fibre bundle of homogeneous spaces

$$\begin{array}{ccc} P & \longrightarrow & Z_{\mathrm{V}} \\ & & \big\downarrow \\ & & M \end{array} \tag{3.27}$$

constructed by taking a quotient $Z_{\mathrm{V}} = \tilde{F}/G$ of the generalised frame bundle $\tilde{F}$ by the structure group $G = \mathrm{E}_{6(2)}$. Choosing a V structure is equivalent to choosing a section of Z_{V}. Thus the infinite-dimensional space of all possible V structures is simply the space of smooth sections,

$$\text{space of vector-multiplet structures } \mathcal{A}_{\mathrm{V}} = \Gamma(Z_{\mathrm{V}}). \tag{3.28}$$

The space of K is an open subset of $\Gamma(E)$, thus we can identify the space of V structures

[5] Recall that $\mathrm{E}_{7(7)}$ can be defined as the group preserving a symplectic invariant s and a symmetric quartic invariant q. Given the $\mathbb{R}^+$ weight of E, note that $q(K) \in \Gamma((\det T^*M)^2)$.

as

$$\mathcal{A}_{\mathrm{V}} = \{K \in \Gamma(E) : q(K) > 0\}. \tag{3.29}$$

Note that $\Gamma(E)$ is a vector space, and hence we have a natural set of local flat coordinates on $\mathcal{A}_{\mathrm{V}}$, fixed by choosing a frame for E. The decomposition into conventional tensors as in (B.6) and (B.35) is an example of such a choice.

The special Kähler metric on $\mathcal{A}_{\mathrm{V}}$ is again inherited from the special Kähler metric on P, the homogeneous space fibres of Z_{V}. (Special Kähler geometry is reviewed in [217, 218] and summarised in appendix E.1.) Recall that one can always define a complex cone over a local special Kähler manifold to give the corresponding rigid special Kähler manifold. The Riemannian symmetric spaces that admit local special Kähler metrics were analysed in [219, 220] and include the case $\mathrm{E}_{7(-25)}/(\mathrm{E}_6 \times \mathrm{U}(1))$. Here we need a pseudo-Riemannian form based on $\mathrm{E}_{7(7)}$, so the relevant space is

$$P/\mathbb{C}^* = \mathrm{E}_{7(7)}/(\mathrm{E}_{6(2)} \times \mathrm{U}(1)). \tag{3.30}$$

Here the $\mathbb{C}^*$ action is generated by the $\mathrm{U}(1)$ bundle together with the overall $\mathbb{R}^+$ scaling. The rigid special Kähler geometry on $\mathcal{A}_{\mathrm{V}}$ can be formulated in analogy to Hitchin's construction of the metric on the space of $\mathrm{SL}(3;\mathbb{C})$ structures [221] and $\mathrm{SU}(3,3)$ structures [108]. The space P is a "prehomogeneous vector space" [222], that is, it is an open orbit of $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ in the real $\mathbf{56_1}$ representation. The open subset is defined by the condition $q(K) > 0$. Consider a point $K \in \mathcal{A}_{\mathrm{V}}$. The vectors in the tangent space $T_K\mathcal{A}_{\mathrm{V}}$ at K can be viewed as a small deformation of K, which are just sections of E, hence $T_K\mathcal{A}_{\mathrm{V}} \simeq \Gamma(E)$. Given $v, w \in T_K\mathcal{A}_{\mathrm{V}}$, the fibre-wise $\mathrm{E}_{7(7)}$ symplectic invariant s then defines a symplectic form Ω on $\mathcal{A}_{\mathrm{V}}$ by

$$\Omega(v, w) = \int_M s(v, w), \tag{3.31}$$

where, since sections of E are weighted objects, $s(v, w)$ is a section of $\det T^*M$ and hence it can be integrated over M. As reviewed in appendix E.1, special Kähler geometry requires the existence of a flat connection preserving Ω. Here, the vector-space structure of $\Gamma(\ E)$ provides natural flat coordinates on $\mathcal{A}_{\mathrm{V}}$, and hence defines a flat connection with respect to which Ω is by definition constant. We can then use the quartic invariant to define a function H that determines the complex structure and hence the metric (E.4). We define the real Hitchin functional

$$H = \int_M \sqrt{q(K)}, \tag{3.32}$$

where again the weight of K means that $\sqrt{q(K)} \in \Gamma(\det T^*M)$. This defines a second invariant tensor $\hat{K} \in \Gamma(E) \simeq T_K\mathcal{A}_{\mathrm{V}}$ as the corresponding Hamiltonian vector field

$$\imath_{\hat{K}}\Omega = -\delta H, \tag{3.33}$$

where δ is the exterior derivative on $\mathcal{A}_{\mathrm{V}}$, and hence an invariant complex generalised vector

$X = K + \mathrm{i}\hat{K}$. The two real invariants correspond to the two singlets in the decomposition $\mathbf{56} = \mathbf{1} + \mathbf{1} + \mathbf{27} + \overline{\mathbf{27}}$ under $\mathrm{E}_{6(2)} \subset \mathrm{E}_{7(7)}$. The metric on $\mathcal{A}_{\mathrm{V}}$ is given by the Hessian

$$H_{MN} = -\frac{\delta H}{\delta K^M \delta K^N}, \tag{3.34}$$

where $M = 1, \dots, 56$ denote the components of K. The definition of the metric is equivalent to choosing a complex structure given by $\mathcal{I}^M{}_N = -\delta\hat{K}^M/\delta K^N$, and implies that $-H$ is the Kähler potential for the special Kähler metric on $\mathcal{A}_{\mathrm{V}}$.[6] In these expressions we are using the flat coordinates on $\mathcal{A}_{\mathrm{V}}$ defined by the vector space structure on $\Gamma(E)$. To see the more conventional description of special Kähler geometry in terms of a holomorphic prepotential $\mathcal{F}$, one needs to switch to a particular class of complex coordinates, as described in [218].

On any rigid special Kähler geometry there is a global $\mathbb{C}^*$ symmetry, such that the quotient space is, by definition, a local special Kähler geometry. On $\mathcal{A}_{\mathrm{V}}$, the action of $\mathbb{C}^*$ is constant rescaling and phase-rotation of the invariant tensor X. The U(1) part is simply the overall U(1) factor of the four-dimensional $\mathcal{N} = 2$ R-symmetry, while, as for the hypermultiplet structure, the $\mathbb{R}^+$ action is a reparametrisation of the warp factor Δ. Modding out by this symmetry, the physical space of structures $\mathcal{A}_{\mathrm{V}}/\mathbb{C}^*$ is an infinite-dimensional local special Kähler space. This is in line with the discussion of [123, 181, 193], where we view $\mathcal{A}_{\mathrm{V}}/\mathbb{C}^*$ as the space of vector-multiplet degrees of freedom, coming from rewriting the full ten- or eleven-dimensional supergravity theory as a four-dimensional $\mathcal{N} = 2$ theory. Physically, the cone structure on $\mathcal{A}_{\mathrm{V}}$ corresponds to coupling the vector multiplets to superconformal gravity [213–215].

3.2.3 Exceptional Calabi–Yau structures

In the previous sections, we defined two generalised structures that give the extension of complex and symplectic geometry of Calabi–Yau manifolds for generic flux solutions, but alone these are not enough to characterise a supersymmetric background. Recall that $\mathcal{N} = 2$ backgrounds define a generalised SU(6) structure [143, 181], this SU(6) being the same group that stabilises the $\mathcal{N} = 2$ Killing spinors. Formally we define:

Definition. An $\mathrm{E}_{7(7)}$ *HV structure* is an $\mathrm{SU}(6) \subset \mathrm{E}_{7(7)} \times \mathbb{R}^+$ generalised structure.

In other words, an SU(6) principal sub-bundle $\tilde{P}_{\mathrm{SU}(6)}$ of the generalised frame bundle $\tilde{F}$. If the SU(6) structure is integrable, we refer to it as "exceptional Calabi–Yau" or ECY. For type II backgrounds it is the flux generalisation of a Calabi–Yau three-fold, while for M-theory it is the generalisation of the product of a Calabi–Yau three-fold and S^1.
As in the simpler Calabi–Yau case, to ensure that the background is indeed $\mathcal{N} = 2$ we need to impose a compatibility condition between the H and V structures such that together they define a generalised SU(6) structure. The common stabiliser group $\mathrm{Spin}^*(12) \cap \mathrm{E}_{6(2)}$ of the pair $\{J_\alpha, K\}$ is SU(6) if and only if J_α and K satisfy two compatibility conditions.

[6] Note that our conventions for the $\mathrm{E}_{7(7)}$ symplectic form mean that the metric here is $\frac{1}{8}$ that in [181]. Also our normalisation of the quartic invariant is fixed relative to the symplectic form by the relation (E.7).

Definition. The two structures J_α and K are *compatible* if together they define an $\mathrm{SU}(6) \subset \mathrm{E}_{7(7)} \times \mathbb{R}^+$ generalised structure. The necessary and sufficient conditions are [181]

$$\begin{aligned} J_\alpha \cdot K &= 0, \\ \mathrm{tr}(J_\alpha J_\beta) &= -2\sqrt{q(K)}\,\delta_{\alpha\beta}, \end{aligned} \tag{3.35}$$

where $\cdot$ is the adjoint action $\mathbf{133} \times \mathbf{56} \to \mathbf{56}$, given in (B.10) and (B.39).[7]

These constraints can be thought of as the generalisations of the conditions (A.2) between symplectic and complex structures on a Calabi–Yau manifold. Note that they are equivalent to

$$J_+ \cdot X = J_- \cdot X = 0, \tag{3.36}$$

where $J_\pm = J_1 \pm \mathrm{i} J_2$, and the normalisation condition

$$\tfrac{1}{2}\mathrm{i}\, s(X, \bar{X}) = \kappa^2, \tag{3.37}$$

respectively, where κ is the factor appearing in (3.12) and $s(\cdot,\cdot)$ is the $\mathrm{E}_{7(7)}$ symplectic invariant, given in (B.30) and (B.59).

3.2.4 Examples of $\mathrm{E}_{7(7)}$ structures

We now show how the examples of $\mathcal{N} = 2$ supersymmetric backgrounds described in appendix A each define H and V structures. We hope this will give a sense of the variety of geometries that can be described. In the same way that generalised complex structures can be thought of as interpolating between complex and symplectic structures, we will see that H structures can interpolate between these and conventional hyper-Kähler structures. Similarly, V structures cover a wide range of possibilities, interpolating between complex, symplectic and simple product structures. We will also check that the structures are compatible, and so define an HV or generalised SU(6) structure. Although we do not give the details, the structures can be calculated explicitly as Killing spinor bilinears using the decomposition of $\mathrm{E}_{7(7)}$ under SU(8).

Throughout this section we will use the "musical isomorphism" to raise indices with the background metric g on M. For example, if ω is a two-form, $\omega^\sharp$ is the corresponding bivector $(\omega^\sharp)^{mn} = g^{mp} g^{nq} \omega_{pq}$. Note that when the flux is non-trivial, since the compatibility and normalisation conditions are $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ covariant, we can always check them using the untwisted structures. For example, the compatibility condition in M-theory is

$$J_\alpha \cdot K = (\mathrm{e}^{A+\tilde{A}} \tilde{J}_\alpha \mathrm{e}^{-A-\tilde{A}}) \cdot (\mathrm{e}^{A+\tilde{A}} \tilde{K}) = \mathrm{e}^{A+\tilde{A}} (\tilde{J}_\alpha \cdot \tilde{K}) = 0 \quad \Leftrightarrow \quad \tilde{J}_\alpha \cdot \tilde{K} = 0. \tag{3.38}$$

For the following examples, one can check the $\mathfrak{su}_2$ algebra (3.12) and normalisation (3.13) of the J_α using (B.11) and (B.31) for M-theory, and (B.40) and (B.60) for type IIB. The

[7] The second compatibility condition in (3.35) implies that $\mathbb{R}^+$ actions on J_α and X are correlated.

normalisation (3.37) of X (or K) can be checked using the symplectic invariant, given by (B.30) for M-theory and (B.59) for type IIB. Finally, one can check compatibility of the structures (3.36) using the adjoint action, given by (B.10) for M-theory and (B.39) for type IIB.

Calabi–Yau manifolds in type IIB

Consider first type IIB on a Calabi–Yau manifold M. The H structure is defined by the symplectic form ω on M. The decomposition of the adjoint bundle $\mathrm{ad}\,\tilde{F}$ in this case follows (2.16). The H structure is given by

$$\begin{aligned} J_+ &= \tfrac{1}{2}\kappa n^i \omega - \tfrac{1}{2}\mathrm{i}\kappa n^i \omega^\sharp + \tfrac{1}{12}\mathrm{i}\kappa n^i \omega\wedge\omega\wedge\omega + \tfrac{1}{12}\kappa n^i \omega^\sharp\wedge\omega^\sharp\wedge\omega^\sharp, \\ J_3 &= \tfrac{1}{2}\kappa\,\hat{\tau}^i{}_j - \tfrac{1}{4}\kappa\,\omega\wedge\omega + \tfrac{1}{4}\kappa\,\omega^\sharp\wedge\omega^\sharp, \end{aligned} \tag{3.39}$$

where the $\mathrm{SL}(2;\mathbb{R})$ doublet $n^i = (-\mathrm{i}, 1)^i$ is a section of S, $\hat{\tau} = -\mathrm{i}\sigma_2$ is a section of $(S\otimes S^*)_0$, where σ_2 is the second Pauli matrix, and the density is simply $\kappa^2 = \mathrm{vol}_6$, where $\mathrm{vol}_6 = \frac{1}{3!}\omega\wedge\omega\wedge\omega$ is the volume form on M. Note that J_3 can be thought of as a combination of two U(1) actions embedded in $\mathrm{E}_{7(7)}$, the first generated by $\hat{\tau}$ in $\mathfrak{sl}_2$ and the second generated by $\omega\wedge\omega - \omega^\sharp\wedge\omega^\sharp$. Since $\omega^\sharp = \omega^{-1}$, J_α is completely determined by ω alone.

Recall that in type IIB the generalised tangent bundle E has a decomposition into tensors, given in (2.13). For a Calabi–Yau background, the V structure is defined by the holomorphic three-form Ω simply as

$$X = \Omega. \tag{3.40}$$

We can also check the compatibility conditions given the form of J_α in (3.39). The adjoint action (B.39) gives

$$J_+\cdot X \propto -\mathrm{i}n^i\omega^\sharp\lrcorner\Omega + n^i\Omega\wedge\omega, \qquad J_-\cdot X \propto -\mathrm{i}\bar{n}^i\omega^\sharp\lrcorner\Omega + \bar{n}^i\Omega\wedge\omega. \tag{3.41}$$

These vanish if and only if $\omega\wedge\Omega = \omega\wedge\bar{\Omega} = 0$, from which we recover the standard compatibility condition for an SU(3) structure.

$\mathbf{CY_3\times S^1}$ in M-theory

For type IIA compactifications on Calabi–Yau three-folds, the complex structure should define the H structure. If we add the M-theory circle to this case, we expect the holomorphic three-form Ω and the complex structure I to appear in J_α – this is indeed the case. Using the decomposition (2.14), we find

$$\begin{aligned} J_+ &= \tfrac{1}{2}\kappa\,\Omega - \tfrac{1}{2}\kappa\,\Omega^\sharp, \\ J_3 &= \tfrac{1}{2}\kappa\, I - \tfrac{1}{16}\mathrm{i}\kappa\,\Omega\wedge\bar{\Omega} - \tfrac{1}{16}\mathrm{i}\kappa\,\Omega^\sharp\wedge\bar{\Omega}^\sharp, \end{aligned} \tag{3.42}$$

where the density is just the volume form $\kappa^2 = \mathrm{vol}_7 = \frac{1}{8}\mathrm{i}\Omega \wedge \bar{\Omega} \wedge \zeta$.

The symplectic structure on the Calabi–Yau manifold determines the V structure. Using the decomposition (2.11), we find

$$X = \zeta^\sharp + \mathrm{i}\omega - \tfrac{1}{2}\zeta \wedge \omega \wedge \omega - \mathrm{i}\zeta \otimes \mathrm{vol}_7 \,. \tag{3.43}$$

Using the adjoint action (B.10) and the algebraic conditions $\imath_{\zeta^\sharp}\Omega = 0$, $\imath_{\zeta^\sharp}\omega = 0$ and $\omega \wedge \Omega = 0$, it is straightforward to show that the compatibility conditions are satisfied.

Generalised Calabi–Yau metrics in type II

This is the case first considered in [181]. The H structure is determined by the $\mathrm{SU}(3,3)_\pm$ structure pure spinors Φ^- and Φ^+ in type IIA and type IIB respectively. To see the embedding it is natural to use the decomposition of $\mathrm{E}_{7(7)}$ under $\mathrm{SL}(2;\mathbb{R}) \times \mathrm{O}(6,6)$. The adjoint bundle was given in (2.15). The three sets of terms in brackets correspond to the decomposition $\mathbf{133} = (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{66}) + (\mathbf{2},\mathbf{32}^\mp)$, while the first term is just the singlet $(\mathbf{1},\mathbf{1})$ generating the $\mathbb{R}^+$ action.

The H structure is given by[8]

$$\begin{aligned} J_+ &= u^i \Phi^\mp, \\ J_3 &= \kappa(u^i \bar{u}_j + \bar{u}^i u_j) - \tfrac{1}{2}\kappa\, \mathcal{J}^\mp, \end{aligned} \tag{3.44}$$

where the upper/lower choice of sign in $\Phi^\mp$ gives the type IIA/IIB embedding, and we have defined

$$u^i = \frac{1}{2}\begin{pmatrix} -\mathrm{i}\kappa \\ \kappa^{-1} \end{pmatrix}^i \in \Gamma((\det T^*M)^{1/2} \otimes (\mathbb{R} \oplus \wedge^6 TM)), \tag{3.45}$$

with

$$\kappa^2 = \tfrac{1}{8}\mathrm{i}\langle \Phi^\pm, \bar{\Phi}^\pm \rangle, \tag{3.46}$$

where $u_i = \epsilon_{ij} u^j$, so that $u^i \bar{u}_i = -\frac{\mathrm{i}}{2}$, and we are using the isomorphism $\wedge^\pm TM \simeq \wedge^6 TM \otimes \wedge^\pm T^*M$. The object $\mathcal{J}^\pm$, transforming in the $\mathrm{O}(6,6)$ adjoint representation $(\mathbf{1},\mathbf{66})$, is the generalised complex structure defined in (A.13). It is important to note that the NS-NS B field is included in the definition of the pure spinors so that the objects J_α are honest sections of the twisted bundle $\mathrm{ad}\,\tilde{F}$.

Using the adjoint action and the $\mathfrak{e}_{7(7)}$ Killing form in section 3 of [181], one can check that the triplet satisfies the $\mathfrak{su}_2$ algebra (3.12) and is correctly normalised (3.13). The embedding reduces to the previous examples in that, for type IIA, the pure spinor Φ^- corresponding to the complex structure embeds in J_α and, for type IIB, we find J_α contains the symplectic structure. Note that upon taking a conventional symplectic structure, we expect this to reduce to the type IIB case of section 3.2.4. It is important to note that the $\mathrm{SL}(2;\mathbb{R})$ factor in each case is different: for type IIB it is S-duality, while for the

[8] Note that with our conventions, the $\mathbf{32}^\mp$ component $C^\mp$ here is equal to $\sqrt{2}$ times the $C^\mp$ used in [181].

generalised complex structure it is the commutant of the $O(6,6)$ action. Taking this into account, it is straightforward to show the two cases match after including a constant $SU(2)$ rotation of the J_α.

The V structure is determined by the generalised complex structure as [181]

$$X = \Phi^{\pm}, \tag{3.47}$$

where the upper/lower choice of sign in $\Phi^{\pm}$ gives the type IIA/IIB embedding. Using the symplectic invariant in section 3 of [181], rescaled by a factor of $1/4$, one can check this satisfies the normalisation condition (3.37). Notice that upon taking a conventional complex structure, this does indeed reduce to the case of section 3.2.4.

For J_+ in (3.44), the adjoint action in section 3 of [181] gives

$$J_+ \cdot X \propto u^i \langle \Phi^{\mp}, \Gamma^A \Phi^{\pm} \rangle, \qquad J_- \cdot X \propto \bar{u}^i \langle \bar{\Phi}^{\mp}, \Gamma^A \Phi^{\pm} \rangle. \tag{3.48}$$

These vanish if $\langle \Phi^{\pm}, \Gamma^A \Phi^{\mp} \rangle = \langle \bar{\Phi}^{\pm}, \Gamma^A \Phi^{\mp} \rangle = 0$. We recover the compatibility conditions (A.11) for $\{\Phi^+, \Phi^-\}$ to define an $SU(3) \times SU(3)$ structure.

D3-branes on HK $\times\, \mathbb{R}^2$ in type IIB

In this case, the hyper-Kähler geometry on M provides a natural candidate for realising the $\mathfrak{su}_2$ algebra. Using the structures defined in appendix A.4, we start by defining the *untwisted* structure

$$\tilde{J}_\alpha = -\tfrac{1}{2}\kappa\, I_\alpha - \tfrac{1}{2}\kappa\, \omega_\alpha \wedge \zeta_1 \wedge \zeta_2 + \tfrac{1}{2}\kappa\, \omega^\sharp_\alpha \wedge \zeta^\sharp_1 \wedge \zeta^\sharp_2, \tag{3.49}$$

where $\kappa^2 = e^{2\Delta} \operatorname{vol}_6$ includes the warp factor. The actual structure is a section of the twisted bundle $\operatorname{ad} \tilde{F}$, and includes the four-form potential C and two-form potentials B^i via the adjoint action as in (2.22)

$$J_\alpha = e^{B^i + C} \tilde{J}_\alpha e^{-B^i - C}. \tag{3.50}$$

We see explicitly that H structures can also encode hyper-Kähler geometries.

X essentially defines the structure of the $\mathbb{R}^2$ factor, since the hyper-Kähler structure was already encoded in J_α. We first define the untwisted object

$$\tilde{X} = \bar{n}^i e^{\Delta}(\zeta_1 - i\zeta_2) + i\bar{n}^i e^{\Delta}(\zeta_1 - i\zeta_2) \wedge \operatorname{vol}_4, \tag{3.51}$$

where $n^i = (-i, 1)^i$ and $\frac{1}{2}\omega_\alpha \wedge \omega_\beta = \delta_{\alpha\beta} \operatorname{vol}_4$. The presence of five- and three-form flux means the actual structure is a section of the twisted bundle E

$$X = e^{B^i + C} \tilde{X}. \tag{3.52}$$

We can check the compatibility condition with J_α in (3.49). This can be done using the twisted or untwisted forms, since the twisting is an $E_{7(7)} \times \mathbb{R}^+$ transformation. We find

$$\begin{aligned}\tilde{J}_\alpha \cdot \tilde{X} \propto -\bar{n}^i I_\alpha \cdot (\zeta_1 - \mathrm{i}\zeta_2) - \mathrm{i}\bar{n}^i(\omega_\alpha^\sharp \wedge \zeta_1^\sharp \wedge \zeta_2^\sharp) \lrcorner \big((\zeta_1 - \mathrm{i}\zeta_2) \wedge \mathrm{vol}_4\big) \\ - \mathrm{i}\bar{n}^i I_\alpha \cdot \big((\zeta_1 - \mathrm{i}\zeta_2) \wedge \mathrm{vol}_4\big).\end{aligned} \tag{3.53}$$

This vanishes as $I_\alpha \cdot \zeta_i = I_\alpha \cdot \mathrm{vol}_4 = 0$ and $\zeta_i^\sharp \lrcorner \omega_\alpha = 0$.

Wrapped M5-branes on HK $\times$ $\mathbb{R}^3$ in M-theory

The final example is that of wrapped M5-branes. As discussed in appendix A.5, the geometry admits two different sets of Killing spinors depending on whether the M5-branes wrap $\mathbb{R}^2$ or a Kähler two-cycle in the hyper-Kähler geometry. These lead to two different H structures.

Let us consider the Kähler two-cycle case first. Using the structure defined in appendix A.5, we can define the untwisted H structure as

$$\begin{aligned}\tilde{J}_\alpha = -\tfrac{1}{2}\kappa\, r_\alpha + \tfrac{1}{2}\kappa\, \omega_3 \wedge \zeta_\alpha - \tfrac{1}{2}\kappa\, \omega_3^\sharp \wedge \zeta_\alpha^\sharp \\ - \tfrac{1}{4}\kappa\, \epsilon_{\alpha\beta\gamma}\zeta_\beta \wedge \zeta_\gamma \wedge \mathrm{vol}_4 - \tfrac{1}{4}\kappa\, \epsilon_{\alpha\beta\gamma}\zeta_\beta^\sharp \wedge \zeta_\gamma^\sharp \wedge \mathrm{vol}_4^\sharp,\end{aligned} \tag{3.54}$$

where $\kappa = \mathrm{e}^{2\Delta}\,\mathrm{vol}_7$ and the tensors

$$r_\alpha = \epsilon_{\alpha\beta\gamma}\zeta_\beta^\sharp \otimes \zeta_\gamma \in \Gamma(TM \otimes T^*M), \tag{3.55}$$

generate the SO(3) rotations on $\mathbb{R}^3$. The V structure is defined by the untwisted object

$$\tilde{X} = \mathrm{e}^\Delta \Omega + \mathrm{i}\,\mathrm{e}^\Delta \Omega \wedge \mathrm{vol}_3, \tag{3.56}$$

where $\Omega = \omega_2 + \mathrm{i}\omega_1$.

For M5-branes wrapped on $\mathbb{R}^2$, the untwisted structures are

$$\tilde{J}_\alpha = -\tfrac{1}{2}\kappa\, I_\alpha - \tfrac{1}{2}\kappa\, \omega_\alpha \wedge \zeta_3 + \tfrac{1}{2}\kappa\, \omega_\alpha^\sharp \wedge \zeta_3^\sharp, \tag{3.57}$$

where again $\kappa^2 = \mathrm{e}^{2\Delta}\,\mathrm{vol}_7$, and

$$\begin{aligned}\tilde{X} = \mathrm{e}^\Delta(\zeta_1^\sharp + \mathrm{i}\zeta_2^\sharp) + \mathrm{e}^\Delta(\zeta_1 + \mathrm{i}\zeta_2) \wedge \zeta_3 - \mathrm{e}^\Delta(\zeta_1 + \mathrm{i}\zeta_2) \wedge \mathrm{vol}_4 \\ - \mathrm{i}\,\mathrm{e}^\Delta(\zeta_1 + \mathrm{i}\zeta_2) \otimes \mathrm{vol}_7\,.\end{aligned} \tag{3.58}$$

In both cases there is a non-trivial four-form flux, so that the actual twisted structures depend on the three-form potential A and, as in (2.18), are given by

$$J_\alpha = \mathrm{e}^A \tilde{J}_\alpha \mathrm{e}^{-A}, \qquad X = \mathrm{e}^A \tilde{X}. \tag{3.59}$$

It is easy to check that in both cases the algebra (3.12), normalisation and compatibility conditions are all satisfied.

3.3 Integrability

Having given the algebraic definitions of hyper- and vector-multiplet structures, we now need to find the differential conditions on them that imply the background is supersymmetric. Formulations in terms of specific generalised connections have already appeared in [172,181]. Here we would like to write conditions that use only the underlying differential geometry, in the same way that $\mathrm{d}\omega = \mathrm{d}\Omega = 0$ depends only on the exterior derivative. The key ingredient will be the action of the group of generalised diffeomorphisms GDiff. Infinitesimally, this action is generated by the generalised Lie derivative L_V, and we will see that all the conditions are encoded using this operator.

We will show that the hypermultiplet conditions arise as moment maps for the action of GDiff on the space of structures $\mathcal{A}_{\mathrm{H}}$. These maps were already partially identified in [181]. As we prove in section 3.4, in the language of G-structures, they are equivalent to requiring that the generalised $\mathrm{Spin}^*(12)$ structure is torsion-free. The vector-multiplet condition similarly implies that the generalised $\mathrm{E}_{6(2)}$ structure is torsion-free. Finally we consider integrability for an HV structure. Given integrable H and V structures, there is an additional requirement for the generalised SU(6) structure, defined by the pair $\{J_\alpha, K\}$, to be torsion-free. In other words, the existence of compatible torsion-free $\mathrm{Spin}^*(12)$ and $\mathrm{E}_{6(2)}$ structures is not sufficient to imply that the SU(6) structure is torsion-free. While not inconsistent with the general G-structure formalism, this is in contrast with the Calabi–Yau case, where the combination of integrable and compatible symplectic and complex structures is enough to imply the manifold is Calabi–Yau.

3.3.1 Integrability of the hypermultiplet structure

We now introduce moment maps for the action of generalised diffeomorphisms on the infinite-dimensional space of H structures. An H structure is then integrable if the corresponding moment maps vanish.

We denote the group of generalised diffeomorphisms – diffeomorphisms and form-field gauge transformations – by GDiff. Infinitesimally it is generated by the generalised Lie derivative L_V, where V is a generalised vector, that is, a section of E. Thus roughly we can identify the Lie algebra $\mathfrak{gdiff}$ with the space of sections $\Gamma(E)$. Actually this is not quite correct since there is a kernel in the map $\Gamma(E) \to \mathfrak{gdiff}$. For example, in M-theory, on a local patch U_i of M, we see from (B.16) that the component $\tau \in \Gamma(T^*U_i \otimes \wedge^7 T^*U_i)$ in V does not contribute to L_V. Similarly, if the components $\omega \in \Gamma(\wedge^2 T^*U_i)$ and $\sigma \in \Gamma(\wedge^5 T^*U_i)$ are closed they do not contribute. In what follows, it is nonetheless convenient to parametrise elements of $\mathfrak{gdiff}$ by $V \in \Gamma(E)$ remembering that this map is not an isomorphism.

Suppose that $\sigma(x) \in \mathcal{A}_{\mathrm{H}}$ is a particular choice of H structure parametrised by the triplet J_α. The change in structure generated by $\mathfrak{gdiff}$ is $\delta J_\alpha = L_V J_\alpha$, which can be viewed as an element of the tangent space $T_\sigma \mathcal{A}_{\mathrm{H}}$. Thus we have a map

$$\rho\colon \mathfrak{gdiff} \to T_\sigma \mathcal{A}_{\mathrm{H}}, \tag{3.60}$$

such that, acting on the triplet of section-valued functions J_α defined in (3.18), the vector ρ_V generates a change in J_α

$$\rho_V(J_\alpha) = L_V J_\alpha. \tag{3.61}$$

Given an arbitrary vector field $w \in T_\sigma \mathcal{A}_{\mathrm{H}}$, we have, from (3.21), that

$$\imath_{\rho_V} \Omega_\alpha(w) = \Omega_\alpha(\rho_V, w) = \epsilon_{\alpha\beta\gamma} \int_M \operatorname{tr}\big((L_V J_\beta) w_\gamma\big). \tag{3.62}$$

If $\pi \in \Gamma(\wedge^7 T^* M)$ is a top-form, so that it transforms in the $\mathbf{1_2}$ representation of $\mathrm{E}_{7(7)} \times \mathbb{R}^+$, then by definition

$$\int_M L_V \pi = \int_M \mathcal{L}_v \pi = 0, \tag{3.63}$$

where $\mathcal{L}_v$ is the conventional Lie derivative and $v \in \Gamma(TM)$ is the vector component of the generalised vector $V \in \Gamma(E)$. Using the Leibniz property of L_V, we then have

$$\begin{aligned} \imath_{\rho_V} \Omega_\alpha(w) &= \tfrac{1}{2} \epsilon_{\alpha\beta\gamma} \int_M \operatorname{tr}\big[(L_V J_\beta) w_\gamma - J_\beta (L_V w_\gamma)\big] \\ &= -\tfrac{1}{2} \epsilon_{\alpha\beta\gamma} \int_M \operatorname{tr}\big[w_\beta (L_V J_\gamma) + J_\beta (L_V w_\gamma)\big] \\ &= \imath_w \delta \mu_\alpha(V), \end{aligned} \tag{3.64}$$

where δ is the exterior derivative on $\mathcal{A}_{\mathrm{H}}$, that is, a functional derivative such that by definition $\imath_w \delta J_\alpha = w_\alpha$, and

$$\mu_\alpha(V) \coloneqq -\tfrac{1}{2} \epsilon_{\alpha\beta\gamma} \int_M \operatorname{tr}(J_\beta L_V J_\gamma), \tag{3.65}$$

is a triplet of moment maps. With this result we can define what we mean by an integrable structure:

Definition. An *integrable* or *torsion-free* hypermultiplet structure J_α is one satisfying

$$\mu_\alpha(V) = 0 \qquad \text{for all } V \in \Gamma(E), \tag{3.66}$$

where $\mu_\alpha(V)$ is given by (3.65).

As we will show in section 3.4, these conditions are equivalent to J_α admitting a torsion-free, compatible generalised connection. They are also the differential conditions on J_α implied by the requirement that the background admits Killing spinors preserving $\mathcal{N} = 2$

supersymmetry in four dimensions.

3.3.2 Integrability of the vector-multiplet structure

The integrability condition for the vector-multiplet structure K also depends on the generalised Lie derivative, but in a very direct way. Recall that $K \in \Gamma(E)$, thus we can consider the generalised Lie derivative along K, namely L_K.

Definition An *integrable* or *torsion-free* vector-multiplet structure K is one satisfying

$$L_K K = 0, \tag{3.67}$$

or, in other words, K is invariant under the generalised diffeomorphism generated by itself.

As we will show in section 3.4, these conditions are equivalent to there being a torsion-free generalised connection compatible with the generalised $\mathrm{E}_{6(2)}$ structure defined by K. Furthermore, it is easy to see that it implies $L_K \hat{K} = 0$. In addition, using the results of appendix C, we see that the generalised Lie derivative $L_X X$, where $X = K + \mathrm{i}\hat{K}$, is identically zero for any vector-multiplet structure K. Hence the integrability condition (3.67) is equivalent to

$$L_X \bar{X} = 0. \tag{3.68}$$

Again, (3.67) is implied by the existence of $\mathcal{N} = 2$ Killing spinors. In section 3.4.3, we will show that (3.67) is actually equivalent to the vanishing of a moment map for the action of GDiff on $\mathcal{A}_{\mathrm{V}}$.

3.3.3 Exceptional Calabi–Yau structures

Finally, we can consider the integrability conditions for the HV structure, defined by a compatible pair $\{J_\alpha, K\}$.

Definition. An ECY geometry admits an *integrable* or *torsion-free* HV structure $\{J_\alpha, K\}$, such that J_α and K are separately integrable and in addition

$$L_X J_\alpha = 0, \tag{3.69}$$

or, in other words, the J_α are also invariant under the generalised diffeomorphisms generated by K and $\hat{K}$.

As we will show in section 3.4, these conditions are equivalent to there being a torsion-free generalised connection compatible with the generalised SU(6) structure, defined by $\{J_\alpha, K\}$. Using the results of [143], this implies that these conditions are equivalent to the existence of $\mathcal{N} = 2$ Killing spinors.

It is important to note that the pair of compatible and integrable H and V structures is not enough to imply that the existence of an ECY geometry. This is because there can be a kernel in the torsion map, as can happen for conventional G-structures.[9]

3.3.4 Examples of integrable structures

We now return to our examples of supersymmetric $\mathcal{N}=2$ backgrounds and show in each case that the relevant integrability conditions (3.66), (3.68) and (3.69) are satisfied. For the examples of Calabi–Yau in type IIB and $\mathrm{CY}_3 \times \mathrm{S}^1$ in M-theory, we show that the conditions are necessary and sufficient using a decomposition into SU(3) torsion classes. The torsion classes are more complicated for the other examples, and so we show only that the supersymmetric backgrounds give examples of integrable structures. Instead, the equivalence of integrability and $\mathcal{N}=2$ supersymmetry is shown using generalised intrinsic torsion in section 3.4.

There are a number of convenient calculational tricks we will use. First note that in the (J_+, J_-, J_3) basis, the moment map conditions are naturally written as the combinations

$$\mu_3 := \tfrac{\mathrm{i}}{2}\int_M \operatorname{tr}(J_- L_V J_+) = 0, \qquad \mu_+ := -\mathrm{i}\int_M \operatorname{tr}(J_3 L_V J_+) = 0, \tag{3.70}$$

and $L_X J_\alpha$ is equivalent to $L_X J_+ = L_X J_- = 0$. We also note that, from the form of the generalised Lie derivative (B.15) and the adjoint projection (B.14) (and the corresponding expressions (B.44) and (B.43) in type IIB), acting on any generalised tensor α

$$L_V \alpha = \mathcal{L}_v \alpha - R\cdot\alpha, \tag{3.71}$$

where $R \in \Gamma(\operatorname{ad}\tilde{F})$, $R\cdot\alpha$ is the adjoint action, v is the vector component of V, $\mathcal{L}_v$ is the conventional Lie derivative and

$$R = \begin{cases} \mathrm{d}\omega + \mathrm{d}\sigma & \text{for M-theory,} \\ \mathrm{d}\lambda^i + \mathrm{d}\rho + \mathrm{d}\sigma^i & \text{for type IIB,} \end{cases} \tag{3.72}$$

where we are using the standard decompositions of V given in (B.6) and (B.35). Using the identity $\operatorname{tr}(A[B,C]) = \operatorname{tr}(B[C,A])$ and the algebra (3.12), this allows us to rewrite the moment maps (3.65) as

$$\begin{aligned} \mu_\alpha(V) &= -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\int_M \operatorname{tr}\big(J_\beta(\mathcal{L}_v J_\gamma - [R, J_\gamma])\big) \\ &= -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\int_M \operatorname{tr}(J_\beta \mathcal{L}_v J_\gamma) - 2\int_M \kappa\operatorname{tr}(R J_\alpha). \end{aligned} \tag{3.73}$$

The final tool is that, when the background has flux, it is often useful to write

[9] See appendix C of [98] for an explicit example of a non-integrable product structure defined by the product of two compatible, integrable complex structures.

the conditions using the untwisted structures $\tilde{J}_\alpha$ and $\tilde{X}$. For this we need the twisted generalised Lie derivative $\hat{L}_{\tilde{V}}$.[10] This is just the induced action of L_V on untwisted fields, and is given in (B.25) for M-theory and (B.54) for type IIB. It has the same form as L_V but includes correction terms involving the fluxes due to the p-form potentials. This can be written as a modified R in (3.73), given by

$$\tilde{R} = \begin{cases} \mathrm{d}\tilde{\omega} - \imath_{\tilde{v}} F + \mathrm{d}\tilde{\sigma} - \imath_{\tilde{v}}\tilde{F} + \tilde{\omega}\wedge F & \text{for M-theory} \\ \mathrm{d}\tilde{\lambda}^i - \imath_{\tilde{v}} F^i + \mathrm{d}\tilde{\rho} - \imath_{\tilde{v}} F - \epsilon_{ij}\tilde{\lambda}^i \wedge F^j + \mathrm{d}\tilde{\sigma}^i + \tilde{\lambda}^i \wedge F - \tilde{\rho}\wedge F^i & \text{for type IIB} \end{cases} . \tag{3.74}$$

The conditions for integrability on the untwisted structures are simply

$$\mu_a(\tilde{V}) = -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\int_M \mathrm{tr}\big(\tilde{J}_\beta \hat{L}_{\tilde{V}} \tilde{J}_\gamma\big) = 0 \quad \forall \tilde{V}, \qquad \hat{L}_{\tilde{X}}\bar{\tilde{X}} = 0, \qquad \hat{L}_{\tilde{X}}\tilde{J}_\alpha = 0. \tag{3.75}$$

Calabi–Yau in type IIB

Consider first the hypermultiplet structure (3.39). Parametrising $\tilde{V}$ as in (B.35), we get conditions for each component $\tilde{v}$, $\tilde{\lambda}^i$, $\tilde{\rho}$ and $\tilde{\sigma}^i$. From the second term in (3.73), taking each of the form-field components in turn, we find the non-zero moment maps are

$$\begin{aligned} \mu_+(\tilde{\lambda}^i) &\propto \int_M \epsilon_{ij} n^j \kappa^2 \omega^\sharp \lrcorner \mathrm{d}\tilde{\lambda}^i \propto \int_M \epsilon_{ij} n^j \omega\wedge\omega\wedge \mathrm{d}\tilde{\lambda}^i \propto \int_M \epsilon_{ij} n^j \mathrm{d}\omega\wedge\omega\wedge\tilde{\lambda}^i = 0, \\ \mu_+(\tilde{\sigma}^i) &\propto \int_M \epsilon_{ij} n^j \kappa^2 \,\mathrm{vol}_6^\sharp \lrcorner \mathrm{d}\tilde{\sigma}^i \propto \int_M \epsilon_{ij} n^j \mathrm{d}\tilde{\sigma}^i = 0, \end{aligned} \tag{3.76}$$

where we use $\kappa^2 = \mathrm{vol}_6$ so $\kappa^2\omega^\sharp \propto \omega\wedge\omega$, and for $\tilde{\rho}$

$$\mu_3(\tilde{\rho}) \propto \int_M \kappa^2 (\omega^\sharp\wedge\omega^\sharp)\lrcorner \mathrm{d}\tilde{\rho} \propto \int_M \omega\wedge \mathrm{d}\tilde{\rho} \propto \int_M \mathrm{d}\omega\wedge\tilde{\rho} = 0, \tag{3.77}$$

where we use $\kappa^2\omega^\sharp\wedge\omega^\sharp \propto \omega$. From this we recover $\mathrm{d}\omega = 0$. For the vector component $\tilde{v}$ the only non-zero contribution is

$$\begin{aligned} \mu_3(\tilde{v}) &\propto \int_M \kappa\,\omega^\sharp \lrcorner \mathcal{L}_{\tilde{v}}(\kappa\omega) - \mathcal{L}_{\tilde{v}}(\kappa\,\omega^\sharp)\lrcorner(\kappa\,\omega) + \kappa\,\mathrm{vol}_6^\sharp \lrcorner \mathcal{L}_{\tilde{v}}(\kappa\,\mathrm{vol}_6) - \mathcal{L}_{\tilde{v}}(\kappa\,\mathrm{vol}_6^\sharp)\lrcorner\kappa\,\mathrm{vol}_6 \\ &\propto \int_M \tfrac{1}{2}\omega\wedge\omega\wedge\mathcal{L}_{\tilde{v}}\omega + \mathcal{L}_{\tilde{v}}\,\mathrm{vol}_6 = 0, \end{aligned} \tag{3.78}$$

which can be seen to vanish using $\mathrm{d}\omega = 0$, $\mathcal{L}_{\tilde{v}}\omega = \imath_{\tilde{v}}\mathrm{d}\omega + \mathrm{d}\imath_{\tilde{v}}\omega$ and integrating by parts.

Turning to the conditions on X given by (3.40), from (B.45) only the τ component of the Dorfman derivative is non-trivial

$$L_X\bar{X} = \jmath\bar{\Omega}\wedge\mathrm{d}\Omega = 0. \tag{3.79}$$

[10]The nomenclature here is confusing: the twisted generalised Lie derivative acts on untwisted fields.

Notice that the integrability condition is considerable weaker than requiring an integrable $\mathrm{SL}(3;\mathbb{C})$ structure – it only requires that the $(3,1)$ part of $\mathrm{d}\Omega$ vanishes. In the intrinsic torsion language of [223], only the $\mathcal{W}_5$ component is set to zero, so that the underlying almost complex structure is unconstrained.

The pair $\{J_\alpha, K\}$ define an integrable generalised $\mathrm{SU}(6)$ structure if they are individually integrable and also satisfy (3.69). From (B.46), we have

$$\begin{aligned} L_X J_+ &\propto \mathrm{i} n^i \omega^\sharp \lrcorner \mathrm{d}\Omega - n^i \omega \wedge \mathrm{d}\Omega = 0, \\ L_X J_3 &\propto -\tfrac{1}{2}(\omega^\sharp \wedge \omega^\sharp) \lrcorner \mathrm{d}\Omega - j(\omega^\sharp \wedge \omega^\sharp) \lrcorner j \mathrm{d}\Omega + \tfrac{1}{2}\mathbb{1}(\omega^\sharp \wedge \omega^\sharp) \lrcorner \mathrm{d}\Omega = 0, \end{aligned} \tag{3.80}$$

which sets the remaining type-$(2,2)$ components of $\mathrm{d}\Omega$ to zero. Taken together, we have $\mathrm{d}\omega = \mathrm{d}\Omega = 0$, as expected.

$\mathbf{CY_3 \times S^1}$ in M-theory

Consider first the hypermultiplet structure (3.42). Parametrising $\tilde{V}$ as in (B.6), the form field components $\tilde{\omega}$ and $\tilde{\sigma}$ in the second term in (3.73) give the non-zero moment maps

$$\begin{aligned} \mu_+(\tilde{\omega}) &\propto \int_M \kappa^2 \Omega^\sharp \lrcorner \mathrm{d}\tilde{\omega} \propto \int_M \zeta \wedge \Omega \wedge \mathrm{d}\tilde{\omega} \propto \int_M \mathrm{d}(\zeta \wedge \Omega) \wedge \tilde{\omega} = 0, \\ \mu_3(\tilde{\sigma}) &\propto \int_M \kappa^2 (\Omega^\sharp \wedge \bar{\Omega}^\sharp) \lrcorner \mathrm{d}\tilde{\sigma} \propto \int_M \zeta \wedge \mathrm{d}\tilde{\sigma} \propto \int_M \mathrm{d}\zeta \wedge \tilde{\sigma} = 0, \end{aligned} \tag{3.81}$$

which give $\mathrm{d}\zeta = 0$ and $\zeta \wedge \mathrm{d}\Omega = 0$ as conditions, and where we have used $\kappa^2 \Omega^\sharp \propto \zeta \wedge \Omega$ and $\kappa^2 \Omega^\sharp \wedge \bar{\Omega}^\sharp \propto \zeta$. In the intrinsic torsion language of [224], this fixes the components $\{\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_5\}$ and $\{R, V_1, T_1, \mathcal{W}_0\}$ to zero. The vector contribution is

$$\begin{aligned} \mu_3(\tilde{v}) &\propto \int_M \kappa\, \bar{\Omega}^\sharp \lrcorner \mathcal{L}_{\tilde{v}}(\kappa\, \Omega) + \mathcal{L}_{\tilde{v}}(\kappa\, \Omega^\sharp) \lrcorner (\kappa\, \bar{\Omega}) \\ &\propto \int_M \zeta \wedge \bar{\Omega} \wedge \mathcal{L}_{\tilde{v}} \Omega + \zeta \wedge \Omega \wedge \mathcal{L}_{\tilde{v}} \bar{\Omega} \\ &\propto \int_M \imath_{\tilde{v}} \zeta\, \mathrm{d}\Omega \wedge \bar{\Omega} = 0, \end{aligned} \tag{3.82}$$

where we have used $\int \mathcal{L}_{\tilde{v}} \kappa^2 = 0$ and the previous conditions to reach the final line. This fixes the torsion class E to zero.

Turning to the conditions on X given by (3.43), upon using the algebraic relations we find (3.68) simplifies to $\mathrm{d}\omega \wedge \omega = 0$, which requires the torsion classes $\{\mathcal{W}_4, E + \bar{E}, V_2, T_2\}$ to vanish. Notice that this is weaker than requiring an integrable $\mathrm{Sp}(6;\mathbb{R})$ structure. One can also explicitly check that (3.67) and (3.68) constrain the same torsion classes, and that $L_X X = 0$ vanishes identically.

Finally, we have the additional condition that ensures the HV structure is integrable and so defines an ECY geometry (3.69). Upon imposing the previous conditions, this forces the remaining torsion classes to vanish. Taken together, we find ζ, ω and Ω are closed, and that $\zeta^\sharp$ is a Killing vector

$$\mathcal{L}_{\zeta^\sharp}\omega = 0, \qquad \mathcal{L}_{\zeta^\sharp}\Omega = 0. \tag{3.83}$$

Generalised Calabi–Yau metrics in type II

Throughout we will use the expressions given in appendix B of [181], generalised to describe both type IIA and IIB. The generalised vector decomposes as $V = v + \Lambda + \tilde{\Lambda} + \tau + \Lambda^\pm$ where v is a vector, Λ a one-form, $\tilde{\Lambda}$ a five-form, τ is a one-form density and $\Lambda^\pm$ are sums of even or odd forms. From the $\mathrm{e}^{B+\tilde{B}+C^\pm}$ action we conclude that in the splitting (3.73) we have

$$R = \mathrm{d}\Lambda + (\mathrm{d}\tilde{\Lambda})_{1\ldots6}v^i v_j + v^i \mathrm{d}\Lambda^\pm, \tag{3.84}$$

where $v^i = (1,0)$, $\mathrm{d}\Lambda$ acts as a "B-transform", and the upper sign refers to type IIA and the lower to type IIB. Thus in the moment maps for J_α given in (3.44), we have the non-zero contributions, using the trace formula given in section 3.1 of [181] and $u^i v_i = \kappa^{-1}$,

$$\mu_+(\Lambda^\pm) \propto \int_M \langle \mathrm{d}\Lambda^\pm, \Phi^\mp\rangle \propto \int_M \langle \Lambda^\pm, \mathrm{d}\Phi^\mp\rangle = 0, \tag{3.85}$$

and

$$\begin{aligned}\mu_3(\Lambda) &\propto \int_M \langle \Phi^\mp, \mathrm{d}\Lambda\wedge\Phi^\mp\rangle \propto \int_M \langle \mathrm{d}\Phi^\mp, \Lambda\wedge\Phi^\mp\rangle + \langle\Phi^\mp, \Lambda\wedge \mathrm{d}\Phi^\mp\rangle = 0,\\ \mu_3(\tilde{\Lambda}) &\propto \int_M \mathrm{d}\tilde{\Lambda} = 0,\end{aligned} \tag{3.86}$$

where in the first line we have used the expression (A.13) for $\mathcal{J}^{\pm A}{}_B$. From these we recover $\mathrm{d}\Phi^\mp = 0$. For the vector component we have

$$\mu_3(v) \propto \int_M \epsilon_{ij}\langle \bar{u}^i\bar{\Phi}^\mp, \mathcal{L}_v(u^i\Phi^\mp)\rangle \propto \int_M \langle\bar{\Phi}^\mp, \mathcal{L}_v\Phi^\mp\rangle = 0, \tag{3.87}$$

where we have used the identity $\epsilon_{ij}\bar{u}^i\mathcal{L}_v u^j = 0$. Using $\mathcal{L}_v\Phi^\mp = \imath_v\mathrm{d}\Phi^\mp + \mathrm{d}\imath_v\Phi^\mp$ and integration by parts, we see that this indeed vanishes.

For the conditions involving X given by (3.47), using (3.73) we have

$$L_X\alpha \propto (v^i\mathrm{d}\Phi^\pm)\cdot\alpha = 0, \tag{3.88}$$

where α is any generalised tensor, $\cdot$ is the relevant adjoint action and we have imposed $\mathrm{d}\Phi^\pm = 0$. Hence (3.68) and (3.69) are both satisfied.

D3-branes on HK $\times\,\mathbb{R}^2$ in type IIB

We have a non-trivial five-form flux F in this case, so it is convenient to use the untwisted structures and twisted generalised Lie derivative. Focussing on $\tilde{J}_\alpha$ given in (3.49), from (3.73) and (3.74), the only non-zero form-field contribution to the moment maps is

$$\mu_\alpha(\tilde{\rho}) \propto \int_M \kappa^2(\omega_\alpha^\sharp \wedge \zeta_1^\sharp \wedge \zeta_2^\sharp) \lrcorner \mathrm{d}\tilde{\rho} \propto \int_M \mathrm{e}^{2\Delta}\omega_\alpha \wedge \mathrm{d}\tilde{\rho} \propto \int_M \mathrm{d}\big(\mathrm{e}^{2\Delta}\omega_\alpha\big) \wedge \tilde{\rho} = 0. \tag{3.89}$$

We recover $\mathrm{d}(\mathrm{e}^{2\Delta}\omega_\alpha) = 0$. The $\tilde{v}$ condition is considerably more complicated and involves the five-form flux F through the term $\imath_{\tilde{v}}F$ in (3.74). After some manipulation, using in particular that $\epsilon_{\alpha\beta\gamma}\,\mathrm{tr}(I_\beta \mathcal{L}_v I_\gamma) = -\epsilon_{\alpha\beta\gamma}(\omega_\beta^\sharp \lrcorner \mathcal{L}_v \omega_\gamma)$, one finds

$$\mu_\alpha(\tilde{v}) \propto \int_M \mathrm{e}^{2\Delta}\omega_\alpha \wedge \imath_{\tilde{v}}F + 2\,\mathrm{e}^{2\Delta}\epsilon_{\alpha\beta\gamma}\mathrm{d}\Delta \wedge \omega_\beta \wedge \imath_{\tilde{v}}\omega_\gamma \wedge \zeta_1 \wedge \zeta_2. \tag{3.90}$$

This vanishes for $\mathrm{d}\Delta = -\frac{1}{4} \star F$, or more precisely it fixes the components of $\mathrm{d}\Delta$ that are transverse to $\zeta_{1,2}$.

For $\tilde{X}$ given in (3.51) and using (3.74), acting on any untwisted generalised tensor $\tilde{\alpha}$ we have

$$\hat{L}_{\tilde{X}}\tilde{\alpha} = -\tilde{R} \cdot \tilde{\alpha} = 0, \tag{3.91}$$

since we have

$$\tilde{R} = \bar{n}^i \mathrm{d}\big(\mathrm{e}^{\Delta}(\zeta_1 - \mathrm{i}\zeta_2)\big) + \mathrm{i}\bar{n}^i \mathrm{d}\big(\mathrm{e}^{\Delta}(\zeta_1 - \mathrm{i}\zeta_2) \wedge \mathrm{vol}_4\big) + \bar{n}^i \mathrm{e}^{\Delta}(\zeta_1 - \mathrm{i}\zeta_2) \wedge F = 0. \tag{3.92}$$

We have used $\mathrm{d}(\mathrm{e}^{\Delta}\zeta_i) = 0$ and $\mathrm{d}(\mathrm{e}^{4\Delta}\,\mathrm{vol}_4) = 0$ so that the last two terms simplify to

$$4\mathrm{i}\,\mathrm{d}\Delta \wedge (\zeta_1 - \mathrm{i}\zeta_2) \wedge \mathrm{vol}_4 = (\zeta_1 - \mathrm{i}\zeta_2) \wedge F, \tag{3.93}$$

which vanishes for $\mathrm{d}\Delta = -\frac{1}{4} \star F$, or more precisely it fixes the components of $\mathrm{d}\Delta$ that are in the direction of $\zeta_{1,2}$. Hence the conditions (3.68) and (3.69) are both satisfied.

We also note that it is simple to extend our description to include imaginary self-dual three-form flux, as first considered in [225–227] and analysed in detail in the case of hyper-Kähler manifolds times $\mathbb{R}^2$ in [228]. The metric, five-form flux and axion-dilaton are of the same form as for our example, but the warp factor is no longer harmonic and there is a non-zero three-form flux on M

$$F^1 + \mathrm{i}F^2 = \mathrm{d}\gamma_I(z) \wedge \tau_I, \tag{3.94}$$

where $\gamma_I(z)$ are analytic functions of $z = x + \mathrm{i}y$, and τ_I are harmonic anti-self-dual two-forms on the hyper-Kähler space. The moment maps are altered only in the $\tilde{\rho}$ component, thanks to the $-\epsilon_{ij}\tilde{\lambda}^i \wedge F^j$ contribution to $\tilde{R}$ in the presence of three-form flux, giving a term

$$\int_M (\tilde{\lambda}^1 \wedge \omega_\alpha \wedge F^2 - \tilde{\lambda}^2 \wedge \omega_\alpha \wedge F^1), \tag{3.95}$$

which vanishes as the wedge product of a self-dual two-form ω_α with an anti-self-dual two-form τ_I is zero. The $\hat{L}_{\tilde{X}}$ expression is also altered thanks to the same correction, giving an extra term

$$\epsilon_{ij}\bar{n}^i(\zeta_1 - \mathrm{i}\zeta_2) \wedge F^j = -(\zeta_1 - \mathrm{i}\zeta_2) \wedge (F^1 - \mathrm{i}F^2). \tag{3.96}$$

But this also vanishes as $F^1 + \mathrm{i}F^2 = \gamma_I'(z)\mathrm{d}z \wedge \tau_I$, and $\mathrm{d}z = \mathrm{e}^{-\Delta}(\zeta_1 + \mathrm{i}\zeta_2)$. Hence we still have $\hat{L}_{\tilde{X}}\tilde{\alpha} = 0$ for any tensor $\tilde{\alpha}$.

Wrapped M5-branes on HK $\times\, \mathbb{R}^3$ in M-theory

In both cases we have a non-trivial four-form flux F, and so it is convenient to use untwisted structures and the twisted generalised Lie derivative.

We first consider M5-branes wrapping a Kähler two-cycle in the hyper-Kähler. Using the form of $\tilde{J}_\alpha$ given in (3.54), together with (3.73) and (3.74), the contribution to the moment maps from $\tilde{\sigma}$ is

$$\mu_\alpha(\tilde{\sigma}) \propto \int_M \kappa^2 \epsilon_{\alpha\beta\gamma}(\mathrm{vol}_4^\sharp \wedge \zeta_\beta^\sharp \wedge \zeta_\gamma^\sharp) \lrcorner \mathrm{d}\tilde{\sigma} \propto \int_M \mathrm{e}^{2\Delta}\zeta_\alpha \wedge \mathrm{d}\tilde{\sigma} \propto \int_M \mathrm{d}(\mathrm{e}^{2\Delta}\zeta_\alpha) \wedge \tilde{\sigma}. \tag{3.97}$$

We recover $\mathrm{d}(\mathrm{e}^{2\Delta}\zeta_i) = 0$ for $i = 1, 2, 3$. The terms in the moment maps due to $\tilde{\omega}$ are

$$\begin{aligned} \mu_\alpha(\tilde{\omega}) &\propto \int_M \tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\kappa^2(\mathrm{vol}_4^\sharp \wedge \zeta_\beta^\sharp \wedge \zeta_\gamma^\sharp) \lrcorner (\tilde{\omega} \wedge F) - \kappa^2(\omega_3^\sharp \wedge \zeta_\alpha^\sharp) \lrcorner \mathrm{d}\tilde{\omega} \\ &\propto \int_M \mathrm{e}^{2\Delta}\zeta_\alpha \wedge F \wedge \tilde{\omega} + \tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\mathrm{d}(\mathrm{e}^{2\Delta}\omega_3 \wedge \zeta_\beta \wedge \zeta_\gamma) \wedge \tilde{\omega}. \end{aligned} \tag{3.98}$$

This vanishes upon using the expressions for the flux $F = \mathrm{e}^{-4\Delta} \star \mathrm{d}(\mathrm{e}^{4\Delta}\omega_3)$ and the exterior derivatives of the ζ_i. Again, the $\tilde{v}$ condition is more complicated and involves the four-form flux F through the term $\imath_{\tilde{v}}F$ in (3.74). After some manipulation, one finds

$$\begin{aligned} \mu_\alpha(\tilde{v}) \propto \int_M & 12\mathrm{d}\Delta \wedge \mathrm{vol}_4 \wedge \zeta_\alpha \wedge (\zeta_1 \wedge \imath_{\tilde{v}}\zeta_1 + \zeta_2 \wedge \imath_{\tilde{v}}\zeta_2 + \zeta_3 \wedge \imath_{\tilde{v}}\zeta_3) \\ & + \epsilon_{\alpha\beta\gamma}\omega_3 \wedge \zeta_\beta \wedge \zeta_\gamma \wedge \imath_{\tilde{v}}F \end{aligned} \tag{3.99}$$

Again, this vanishes after imposing the conditions from (A.28).

Now consider the conditions that depend on $\tilde{X}$. For $\tilde{X}$ given in (3.56), acting on any untwisted generalised tensor $\tilde{\alpha}$ we have

$$L_{\tilde{X}}\tilde{\alpha} = -\tilde{R} \cdot \tilde{\alpha}, \tag{3.100}$$

where $\tilde{R}$ is given by

$$\tilde{R} = \mathrm{d}(\mathrm{e}^\Delta\Omega) + \mathrm{d}(\mathrm{i}\,\mathrm{e}^\Delta\Omega \wedge \mathrm{vol}_3) + \mathrm{e}^\Delta\Omega \wedge F. \tag{3.101}$$

But $\tilde{R}$ itself vanishes as

$$\begin{aligned} \mathrm{d}(\mathrm{e}^{\Delta}\Omega) &= 0, \\ \mathrm{d}(\mathrm{i}\,\mathrm{e}^{\Delta}\Omega \wedge \mathrm{vol}_3) + \mathrm{e}^{\Delta}\Omega \wedge F &= 0, \end{aligned} \tag{3.102}$$

where we have used the expressions for the flux F and the torsion conditions on ω_α and ζ_i from (A.28). Hence, both (3.68) and (3.69) are satisfied.

Next we consider M5-branes wrapping an $\mathbb{R}^2$ plane in $\mathbb{R}^3$. Using the form of $\tilde{J}_\alpha$ given in (3.57), together with (3.73) and (3.74), the non-zero form-field contribution to the moment maps is

$$\mu_\alpha(\tilde{\omega}) \propto \int_M \kappa^2(\omega^\sharp_\alpha \wedge \zeta^\sharp_3) \lrcorner \mathrm{d}\tilde{\omega} \propto \int_M \mathrm{e}^{2\Delta}\omega_\alpha \wedge \zeta_1 \wedge \zeta_2 \wedge \mathrm{d}\tilde{\omega} \propto \int_M \mathrm{d}(\mathrm{e}^{2\Delta}\omega_\alpha \wedge \zeta_1 \wedge \zeta_2) \wedge \tilde{\omega}. \tag{3.103}$$

This vanishes after using the expressions in (A.31). Again, the $\tilde{v}$ condition is more complicated and involves the four-form flux F through the term $\imath_{\tilde{v}}F$ in (3.74). After some manipulation, one finds

$$\mu_\alpha(V) = \int_M 12\epsilon_{\alpha\beta\gamma}\mathrm{e}^{2\Delta}\mathrm{d}\Delta \wedge \omega_\beta \wedge \mathrm{vol}_3 \wedge \imath_{\tilde{v}}\omega_\gamma - 4\mathrm{e}^{2\Delta}\omega_\alpha \wedge \zeta_1 \wedge \zeta_2 \wedge \imath_{\tilde{v}}F. \tag{3.104}$$

This vanishes for $\star F = \mathrm{e}^{-4\Delta}\mathrm{d}(\mathrm{e}^{4\Delta}\zeta_1 \wedge \zeta_2)$, or more precisely it fixes the components of $\mathrm{d}\Delta$ that are transverse to $\zeta_{1,2,3}$.

For $\tilde{X}$ given in (3.58), acting on any untwisted generalised tensor $\tilde{\alpha}$ we have

$$L_{\tilde{X}}\tilde{\alpha} = \mathcal{L}_{\mathrm{e}^{\Delta}(\zeta^\sharp_1 + \mathrm{i}\zeta^\sharp_2)}\tilde{\alpha} - \tilde{R}\cdot\tilde{\alpha}, \tag{3.105}$$

where $\tilde{R}$ is given by

$$\begin{aligned} \tilde{R} = {} & \mathrm{d}\big[\mathrm{e}^{\Delta}(\zeta_1 + \mathrm{i}\zeta_2)\wedge\zeta_3\big] - \mathrm{e}^{\Delta}(\zeta^\sharp_1 + \mathrm{i}\zeta^\sharp_2)\lrcorner F - \mathrm{d}\big[\mathrm{e}^{\Delta}(\zeta_1 + \mathrm{i}\zeta_2)\wedge\mathrm{vol}_4\big] \\ & + \mathrm{e}^{\Delta}(\zeta_1 + \mathrm{i}\zeta_2)\wedge\zeta_3\wedge F. \end{aligned} \tag{3.106}$$

But $\tilde{R}$ vanishes as

$$\begin{aligned} \mathrm{d}(\mathrm{e}^{\Delta}\zeta_1 \wedge \zeta_3) &= 0, \\ \zeta^\sharp_1 \lrcorner F &= 0, \\ \mathrm{d}(\mathrm{e}^{\Delta}\zeta_1 \wedge \mathrm{vol}_4) - \mathrm{e}^{\Delta}\zeta_1 \wedge \zeta_3 \wedge F &= 0, \end{aligned} \tag{3.107}$$

with similar expressions for ζ_2. The generalised Lie derivative along $\tilde{X}$ then reduces to the Lie derivative along $\mathrm{e}^{\Delta}(\zeta^\sharp_1 + \mathrm{i}\zeta^\sharp_2)$, and we note that Δ does not depend on the coordinates x or y, so that $\zeta^\sharp_{1,2}\lrcorner\mathrm{d}\Delta = 0$. It is then simple to check that the Lie derivative along $\mathrm{e}^{\Delta}(\zeta^\sharp_1 + \mathrm{i}\zeta^\sharp_2)$ preserves both $\bar{\tilde{X}}$ and J_α, and so both (3.68) and (3.69) are satisfied.

3.4 Generalised intrinsic torsion, supersymmetry and moduli spaces

In this section, we analyse the integrability conditions for the hyper- and vector-multiplet structures using the notion of generalised intrinsic torsion, first introduced in generality in [143] and for a specific heterotic extension of $\mathrm{O}(d,d)\times\mathbb{R}^+$ generalised geometry in [140]. This will allow us to do two things: first to show that each integrability condition is equivalent to the existence of a torsion-free generalised connection compatible with the relevant structure, and second to prove, using the results of [143], that the full set of conditions defining an ECY geometry are equivalent to solving the $\mathcal{N}=2$ Killing spinor equations.

We then show that the integrability conditions have a simple interpretation in terms of rewriting the full ten- or eleven-dimensional supergravity theory in terms of an $\mathcal{N}=2$, $D=4$ gauged supergravity coupled to an infinite number of hyper- and vector-multiplets, as considered in [123, 181, 193]. Finally we discuss some general aspects of the moduli spaces of structures.

3.4.1 Generalised intrinsic torsion and integrability

We start by recalling the definition of generalised intrinsic torsion given in [143]. Let $\tilde{P}_G\subset\tilde{F}$ be a principal sub-bundle of the generalised frame bundle $\tilde{F}$ defining a generalised G-structure. It is always possible to find a generalised connection $\hat{D}$ that is compatible with $\tilde{P}_G$, however in general it will not be torsion-free. Recall that the generalised torsion T of $\hat{D}$ is defined, given any generalised tensor α and generalised vector $V\in\Gamma(E)$, by [137]

$$T(V)\cdot\alpha = L^{\hat{D}}_V\alpha - L_V\alpha, \tag{3.108}$$

where the torsion is viewed as a map $T\colon E\to\operatorname{ad}\tilde{F}$ and $T(V)$ acts in the adjoint representation on α. Here $L^{\hat{D}}_V$ is the generalised Lie derivative with the partial derivative replaced with the covariant derivative $\hat{D}$, that is, acting on any generalised tensor α,

$$L^{\hat{D}}_V\alpha = (V\cdot\hat{D})\alpha - (\hat{D}\times_{\mathrm{ad}} V)\cdot\alpha. \tag{3.109}$$

Let $W\subset E^*\otimes\operatorname{ad}\tilde{F}$ be the space of generalised torsions. For $\mathrm{E}_{7(7)}\times\mathbb{R}^+$ generalised geometry, we have [137]

$$W\simeq E^*\oplus K, \tag{3.110}$$

where the dual generalised tangent bundle E^* transforms as $\mathbf{56_{-1}}$ and K is the generalised tensor bundle corresponding to the $\mathbf{912_{-1}}$ representation. For other $\mathrm{E}_{d(d)}\times\mathbb{R}^+$ groups the representations appearing in W are listed in [137].

By definition, any other generalised connection $\hat{D}'$ compatible with $\tilde{P}_G$ can be written as $\hat{D}' = \hat{D} + \Sigma$, where

$$\Sigma = \hat{D} - \hat{D}' \in \Gamma(K_G), \qquad \text{with } K_G = E^* \otimes \operatorname{ad} \tilde{P}_G. \tag{3.111}$$

We then define a map $\tau\colon K_G \to W$ as the difference of the torsions of $\hat{D}$ and $\hat{D}'$,

$$\tau(\Sigma) = T - T' \in \Gamma(W). \tag{3.112}$$

In general, the map τ will not fill out the whole of W. Defining the image

$$W_G = \operatorname{im} \tau \subseteq W, \tag{3.113}$$

we can then define the space of the generalised intrinsic torsion, in exact analogy to ordinary geometry, as the part of W not spanned by W_G, that is

$$W_{\text{int}}^G = W/W_G. \tag{3.114}$$

Given any G-compatible connection $\hat{D}$, we say that the generalised intrinsic torsion T_{int}^G, of the generalised G-structure $\tilde{P}_G$, is the projection of the torsion T onto W_{int}^G. By definition this is independent of the choice of $\hat{D}$. It is the part of the torsion that cannot be changed by varying our choice of compatible connection.

The intrinsic torsion T_{int}^G is the obstruction to finding a connection which is simultaneously torsion-free and compatible with the generalised G-structure. Hence, if it vanishes we say that $\tilde{P}_G$ is an *integrable* or *torsion-free* generalised G-structure.

Intrinsic torsion for hypermultiplet structures

Let us calculate the intrinsic torsion for a $\mathrm{Spin}^*(12)$ structure. Decomposing W under $\mathrm{SU}(2) \times \mathrm{Spin}^*(12)$ we have[11]

$$W = \mathbf{56} + \mathbf{912} = 2(\mathbf{2}, \mathbf{12}) + (\mathbf{1}, \mathbf{32}) + (\mathbf{3}, \mathbf{32}) + (\mathbf{1}, \mathbf{352}) + (\mathbf{2}, \mathbf{220}), \tag{3.115}$$

while for the space of $\mathrm{Spin}^*(12)$ connections we have

$$K_{\mathrm{Spin}^*(12)} = \big((\mathbf{2}, \mathbf{12}) + (\mathbf{1}, \mathbf{32})\big) \times (\mathbf{1}, \mathbf{66}) = (\mathbf{2}, \mathbf{12}) + (\mathbf{2}, \mathbf{220}) + (\mathbf{1}, \mathbf{32}) + (\mathbf{1}, \mathbf{352}). \tag{3.116}$$

This implies $W_{\mathrm{Spin}^*(12)} \subseteq (\mathbf{2}, \mathbf{12}) + (\mathbf{1}, \mathbf{32}) + (\mathbf{1}, \mathbf{352}) + (\mathbf{2}, \mathbf{220})$. Using the explicit form of the map τ, we can show that this is actually an equality, hence

$$W_{\text{int}}^{\mathrm{Spin}^*(12)} = (\mathbf{2}, \mathbf{12}) + (\mathbf{3}, \mathbf{32}). \tag{3.117}$$

We will now show that the triplet of moment maps constrain the same representations.

[11] Since calculating intrinsic torsion reduces to linear algebra at a point in the manifold, in what follows we do not distinguish between vector bundles and their representations.

Since $\hat{D}$ is compatible with the $\mathrm{Spin}^*(12)$ structure, by definition $\hat{D}J_\alpha = 0$. Using (3.108) and (3.109), and integrating by parts to move $\hat{D}$ from V to J_α, we have

$$\begin{aligned}\mu_\alpha(V) &\propto \epsilon_{\alpha\beta\gamma}\int_M \operatorname{tr}\Big(J_\beta\big((V\cdot\hat{D})J_\gamma - [(\hat{D}\times_{\mathrm{ad}}V), J_\gamma] - [T(V), J_\gamma]\big)\Big) \\ &\propto \int_M \kappa\operatorname{tr}\big(J_\alpha T(V)\big) + \kappa\operatorname{tr}\big(J_\alpha(\hat{D}\times_{\mathrm{ad}}V)\big) \\ &\propto \int_M \kappa\operatorname{tr}\big(J_\alpha\, T_{\mathrm{int}}^{\mathrm{Spin}^*(12)}(V)\big) + \int_M \tfrac{1}{2}T_{\mathrm{int}}^{\mathrm{Spin}^*(12)}(J_\alpha\cdot V)\cdot\kappa^2,\end{aligned} \tag{3.118}$$

where the second term in the last line comes from the torsion of $\hat{D}$ when evaluating the total derivative in the integration by parts. We have also used the fact that the expression is independent of the choice of compatible connection $\hat{D}$ and so only depends on the intrinsic torsion $T_{\mathrm{int}}^{\mathrm{Spin}^*(12)}$. We see that the moment maps vanish if and only if the $(\mathbf{3},\mathbf{1})$ component of $T_{\mathrm{int}}^{\mathrm{Spin}^*(12)}(V)$ vanish for all V. Recall that V transforms in the $\mathbf{56} = (\mathbf{2},\mathbf{12}) + (\mathbf{1},\mathbf{32})$ representation. Given the decomposition (3.117), we see that the $(\mathbf{3},\mathbf{1})$ component of $T_{\mathrm{int}}^{\mathrm{Spin}^*(12)}(V)$ vanishes if and only if both the $(\mathbf{2},\mathbf{12})$ and $(\mathbf{3},\mathbf{32})$ components of the intrinsic torsion vanish. Thus the vanishing of the moment maps is equivalent to the existence of a torsion-free $\mathrm{Spin}^*(12)$ structure.

Intrinsic torsion for vector-multiplet structures

Repeating the analysis for vector-multiplet structures by decomposing under $\mathrm{E}_{6(2)}$, we have

$$W = \mathbf{56} + \mathbf{912} = \mathbf{1} + 2\cdot\mathbf{27} + \mathbf{78} + \mathbf{351} + \text{c.c.}, \tag{3.119}$$

while for the space of $\mathrm{E}_{6(2)}$ connections we have

$$K_{\mathrm{E}_{6(2)}} = (\mathbf{1} + \mathbf{27} + \text{c.c.}) \times \mathbf{78} = \mathbf{27} + \mathbf{78} + \mathbf{351} + \mathbf{1728} + \text{c.c.} \tag{3.120}$$

This implies $W_{\mathrm{E}_{6(2)}} \subseteq \mathbf{27} + \mathbf{78} + \mathbf{351} + \text{c.c.}$ Using the explicit form of the map τ, we can show again that this is actually an equality, hence

$$W_{\mathrm{int}}^{\mathrm{E}_{6(2)}} = \mathbf{1} + \mathbf{27} + \text{c.c.} \tag{3.121}$$

We will now show that the $L_K K = 0$ condition is equivalent to vanishing generalised intrinsic torsion. Using (3.108), (3.109) and $\hat{D}K = 0$, we have

$$L_K K = L_K^{\hat{D}} K - T(K)\cdot K = -T_{\mathrm{int}}^{\mathrm{E}_{6(2)}}(K)\cdot K. \tag{3.122}$$

Since K is a singlet under $\mathrm{E}_{6(2)}$ and $L_K K$ is a generalised vector transforming in the $\mathbf{56} = \mathbf{1}+\mathbf{27}+\text{c.c.}$ representation, this condition implies that the $\mathbf{1}+\mathbf{27}+\text{c.c.}$ components of $T_{\mathrm{int}}^{\mathrm{E}_{6(2)}}$ vanish. However, these are precisely the components in the intrinsic torsion (3.121). Thus the vanishing of $L_K K$ is equivalent to the existence of a torsion-free $\mathrm{E}_{6(2)}$ structure.

Intrinsic torsion for HV structures

It was shown in [143, 144] that solutions of the $\mathcal{N}=2$ Killing spinor equations are in one-to-one correspondence with torsion-free SU(6) structures. We now show that the full set of integrability conditions on compatible pairs of structures $\{J_\alpha, K\}$ are equivalent to vanishing SU(6) intrinsic torsion and hence to solutions of the $\mathcal{N}=2$ Killing spinor equations.

Explicitly we have, decomposing under $\mathrm{SU}(2)\times\mathrm{SU}(6)$,

$$\begin{aligned} W = \mathbf{56}+\mathbf{912} &= (\mathbf{1},\mathbf{1}) + 2(\mathbf{1},\mathbf{15}) + (\mathbf{1},\mathbf{21}) + (\mathbf{1},\mathbf{35}) + (\mathbf{1},\mathbf{105}) \\ &\quad + 3(\mathbf{2},\mathbf{6}) + (\mathbf{2},\mathbf{20}) + (\mathbf{2},\mathbf{84}) + (\mathbf{3},\mathbf{1}) + (\mathbf{3},\mathbf{15}) + \text{c.c.} \end{aligned} \tag{3.123}$$

From the analysis in [143] we have

$$\begin{aligned} W_{\text{int}}^{\mathrm{SU}(6)} &= (\mathbf{2},\mathbf{1})\times(S+J) + \text{c.c.} \\ &= (\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{1}) + 2(\mathbf{2},\mathbf{6}) + (\mathbf{1},\mathbf{15}) + (\mathbf{3},\mathbf{15}) + (\mathbf{2},\mathbf{20}) + \text{c.c.}, \end{aligned} \tag{3.124}$$

where $S+J=\mathbf{8}+\mathbf{56}=(\mathbf{2},\mathbf{1})+2(\mathbf{1},\mathbf{6})+(\mathbf{2},\mathbf{15})+(\mathbf{1},\mathbf{20})$ are the representations in which the Killing spinor equations transform. Note that we can also decompose the hyper- and vector-multiplet intrinsic torsions as

$$\begin{aligned} W_{\text{int}}^{\mathrm{Spin}^*(12)} &= (\mathbf{2},\mathbf{6}) + (\mathbf{3},\mathbf{1}) + (\mathbf{3},\mathbf{15}) + \text{c.c.}, \\ W_{\text{int}}^{\mathrm{E}_{6(2)}} &= (\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{6}) + (\mathbf{1},\mathbf{15}) + \text{c.c.} \end{aligned} \tag{3.125}$$

Since the $(\mathbf{2},\mathbf{20})$ is missing from these decompositions, it is immediately clear that having an integrable hypermultiplet structure J_α and a compatible integrable vector-multiplet structure K is not sufficient to imply we have an integrable SU(6) structure.

As we will now see, the missing components are set to zero by the extra conditions $L_X J_\alpha = 0$. As before, given an SU(6)-compatible generalised connection, from (3.108), (3.109) and $\hat{D}K = \hat{D}J_\alpha = 0$ we have

$$L_X J_\alpha = L_X^{\hat{D}} J_\alpha - [T(X), J_\alpha] = -[T_{\text{int}}^{\mathrm{SU}(6)}(X), J_\alpha]. \tag{3.126}$$

Since X is a singlet under SU(6) and $L_X J_\alpha$ transforms in the $\mathbf{133}$ representation, we see that $L_X J_\alpha$ indeed includes the missing $(\mathbf{2},\mathbf{20})$ component. In appendix C, we calculate which components of the intrinsic torsion appear in which of the three supersymmetry conditions (3.118), (3.122) and (3.126). The results are summarised in table 3.4.1.

We see that collectively the three integrability conditions on $\{J_\alpha, K\}$ are equivalent to solving the $\mathcal{N}=2$ Killing spinor equations. Since an SU(6)-compatible connection is a special case of both a $\mathrm{Spin}^*(12)$- and an $\mathrm{E}_{6(2)}$-compatible connection, this decomposition also provides a direct proof that there are indeed no unexpected kernels in the τ map in these two cases, and that $\mu_\alpha = 0$ and $L_K K$ are equivalent to the existence of a torsion-free

	$W_{\text{int}}^{\text{SU}(6)}$ component						
Integrability condition	$(\mathbf{1},\mathbf{1})$	$(\mathbf{3},\mathbf{1})$	$(\mathbf{2},\mathbf{6})$	$(\mathbf{2},\mathbf{6})'$	$(\mathbf{1},\mathbf{15})$	$(\mathbf{3},\mathbf{15})$	$(\mathbf{2},\mathbf{20})$
$\mu_\alpha = 0$		$\times$	$\times$			$\times$	
$L_K K = 0$	$\times$			$\times$	$\times$		
$L_X J_\alpha = 0$	$\times$	$\times$		$\times$		$\times$	$\times$

Table 3.2: The components of the generalised intrinsic torsion $W_{\text{int}}^{\text{SU}(6)}$ appearing in each of the $\mathcal{N}=2$ supersymmetry conditions.

$\text{Spin}^*(12)$ and $\text{E}_{6(2)}$ generalised structure respectively.

We also see that certain components of $W_{\text{int}}^{\text{SU}(6)}$ appear in multiple conditions. The $\mu_\alpha(V)$ and $L_K K$ conditions are complementary. However the $L_X J_\alpha$ condition shares components with each of the other conditions. The relation between $(\mathbf{1},\mathbf{1})$ components comes from taking L_X of the second compatibility condition in (3.35) and using $L_X X = 0$

$$\operatorname{tr}(J_\alpha L_X J_\beta + J_\beta L_X J_\alpha) = -\tfrac{1}{2}\mathrm{i}\, s(X, L_X \bar{X})\, \delta_{\alpha\beta}, \tag{3.127}$$

while the relation between $(\mathbf{2},\mathbf{6})'$ components comes from taking L_X of the first condition in (3.35)

$$(L_X J_\alpha)\cdot K + J_\alpha \cdot L_X K = 0. \tag{3.128}$$

The relation between the $(\mathbf{3},\mathbf{1})$ and $(\mathbf{3},\mathbf{15})$ components arises from evaluating the moment maps on X

$$\mu_\alpha(X) = -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\int_M \operatorname{tr}(J_\beta L_X J_\gamma). \tag{3.129}$$

Let us end this section by briefly noting how the conventional SU(3) intrinsic torsion, which vanishes for type II Calabi–Yau backgrounds, embeds into the generalised case. The combined SU(8) spinor (3.6) defines two different embeddings of $\text{Spin}(6) \simeq \text{SU}(4)_\pm \subset \text{SU}(8)$: one for type IIA and one for type IIB, corresponding to the decompositions $\mathbf{8} = \mathbf{4} + \bar{\mathbf{4}}$ and $\mathbf{8} = \mathbf{4} + \mathbf{4}$ respectively. There are hence two different embeddings of $\text{SU}(3)_\pm \subset \text{SU}(6)$, giving the embeddings of the torsion classes defined in [223] for type IIA

$$\begin{aligned} &\mathcal{W}_1 : \mathbf{1}_{\mathbb{C}} \subset (\mathbf{3},\mathbf{1}), \quad \mathcal{W}_2 : \mathbf{8}_{\mathbb{C}} \subset (\mathbf{3},\mathbf{15}), \quad \mathcal{W}_3 : \mathbf{6} \subset (\mathbf{2},\mathbf{20}), \\ &\mathcal{W}_4 : \mathbf{3} \subset (\mathbf{2},\mathbf{6})', \quad \mathcal{W}_5 : \mathbf{3} \subset (\mathbf{2},\mathbf{6}), \end{aligned} \tag{3.130}$$

and for type IIB

$$\begin{aligned} &\mathcal{W}_1 : \mathbf{1}_{\mathbb{C}} \subset (\mathbf{3},\mathbf{1}), \quad \mathcal{W}_2 : \mathbf{8}_{\mathbb{C}} \subset (\mathbf{2},\mathbf{20}), \quad \mathcal{W}_3 : \mathbf{6} \subset (\mathbf{3},\mathbf{15}), \\ &\mathcal{W}_4 : \mathbf{3} \subset (\mathbf{2},\mathbf{6}), \quad \mathcal{W}_5 : \mathbf{3} \subset (\mathbf{2},\mathbf{6})', \end{aligned} \tag{3.131}$$

which in each case is consistent with the analysis of section 3.3.4.

3.4.2 Supersymmetry conditions from gauged supergravity

As we have already noted, there is a natural physical interpretation of the spaces of hypermultiplet and vector-multiplet structures. We can view them as arising from a rewriting of the full ten- or eleven-dimensional theory as in [146] but with only eight supercharges manifest [123, 181, 193]. The local $\mathrm{SO}(9,1)$ Lorentz symmetry is broken and the degrees of freedom can be repackaged into $\mathcal{N}=2$, $D=4$ multiplets. However, since all modes are kept – there is no Kaluza–Klein truncation – the vector- and hypermultiplet spaces $\mathcal{A}_{\mathrm{V}}$ and $\mathcal{A}_{\mathrm{H}}$ become infinite dimensional. As previously argued for $\mathcal{N}=1$ backgrounds in $\mathrm{O}(6,6)$ generalised geometry in [195] and in $\mathrm{E}_{7(7)}$ generalised geometry in [136, 181, 182], the integrability conditions can be similarly interpreted in a four-dimensional language. The interactions of the four-dimensional theory are encoded in the gauging of isometries on $\mathcal{A}_{\mathrm{H}}$ and $\mathcal{A}_{\mathrm{V}}$, together with the concomitant moment maps, as summarised in [194]. From the form (3.65) of the hyper-Kähler moment maps, we see that we are gauging generalised diffeomorphisms. The general conditions, coming from the vanishing of the gaugino, hyperino and gravitino variations, for the four-dimensional theory to admit a supersymmetric $\mathcal{N}=2$ Minkowski vacuum have been analysed in [196, 197]. As we now show, these translate directly into the three integrability conditions for J_α and K.

Recall that the scalar components of the hypermultiplets describe a quaternionic-Kähler space. Let $\mathcal{A}_{\mathrm{H}}$ be the associated hyper-Kähler cone. Similarly, the scalar components of the vector multiplets describe a local special Kähler space. Let $\mathcal{A}_{\mathrm{V}}$ be the associated rigid special Kähler cone. The gauging is a product of an action of a group $\mathcal{G}_{\mathrm{H}}$ on the quaternionic-Kähler space and of a group $\mathcal{G}_{\mathrm{V}}$ on the local special Kähler space, that each preserve the corresponding structures. These can always be lifted to an action on $\mathcal{A}_{\mathrm{H}}$ that preserves the triplet of symplectic forms and commutes with the $\mathrm{SU}(2)$ action on the cone, and an action on $\mathcal{A}_{\mathrm{V}}$ that preserves the Kähler form and complex structure and commutes with the $\mathrm{U}(1)$ action on the cone. Following [197], the conditions for a Minkowski vacuum in a generic gauged $\mathcal{N}=2$ theory, lifted to $\mathcal{A}_{\mathrm{H}}$ and $\mathcal{A}_{\mathrm{V}}$, can be written as

$$\Theta_\Lambda^\lambda \mu_{\alpha,\lambda}=0, \qquad X^\Lambda \Theta_\Lambda^\lambda k_\lambda^u=0, \qquad \bar{X}^\Lambda \hat{\Theta}_\Lambda^{\hat{\lambda}} \hat{k}_{\hat{\lambda}}^i=0. \tag{3.132}$$

Here λ parametrises the Lie algebra $\mathfrak{g}_{\mathrm{H}}$ of $\mathcal{G}_{\mathrm{H}}$ while $\hat{\lambda}$ parametrises the Lie algebra $\mathfrak{g}_{\mathrm{V}}$ of $\mathcal{G}_{\mathrm{V}}$, and k_λ and $\hat{k}_{\hat{\lambda}}$ are the corresponding sets of vector fields generating the actions on $\mathcal{A}_{\mathrm{H}}$ and $\mathcal{A}_{\mathrm{V}}$ (see also appendix E.1). The label u is a coordinate index on $\mathcal{A}_{\mathrm{H}}$ and i is a holomorphic coordinate index on $\mathcal{A}_{\mathrm{V}}$, so that $\hat{k}_{\hat{\lambda}}$ is actually the holomorphic part of the real vector generating the action. The $\mu_{\alpha,\lambda}$ are a triplet of moment maps $\mu_\alpha\colon \mathcal{A}_{\mathrm{H}} \to \mathfrak{g}_{\mathrm{H}}^*$. As discussed in appendix E.1, the complex vector X^Λ is a particular non-zero holomorphic vector on $\mathcal{A}_{\mathrm{V}}$, written in flat coordinates, that defines the special Kähler geometry and also generates the $\mathbb{C}^*$ action on the cone. The indices Λ denote components in the natural flat coordinates on $\mathcal{A}_{\mathrm{V}}$. The matrices Θ_Λ^λ and $\hat{\Theta}_\Lambda^{\hat{\lambda}}$ are the corresponding embedding tensors [229, 230].

Let us now translate this formalism into the geometrical objects defined previously when

$\mathcal{A}_{\mathrm{H}}$ and $\mathcal{A}_{\mathrm{V}}$ are the infinite-dimensional spaces of hyper- and vector-multiplet structures. In this case, the gauging is by generalised diffeomorphisms $\mathcal{G}_{\mathrm{H}} = \mathcal{G}_{\mathrm{V}} = \mathrm{GDiff}$. Recall that we parametrised the Lie algebra $\mathfrak{gdiff}$ by sections $V \in \Gamma(E)$ even though there was actually a kernel in this map. Furthermore, from (3.29), we saw that generalised vectors defined flat coordinates on $\mathcal{A}_{\mathrm{V}}$. Thus we can identify the embedding tensors with the map

$$\Theta = \hat{\Theta} \colon \Gamma(E) \to \mathfrak{gdiff}. \tag{3.133}$$

The vectors k_λ and $\hat{k}_{\hat{\lambda}}$ generate the action of GDiff on $\mathcal{A}_{\mathrm{H}}$ and $\mathcal{A}_{\mathrm{V}}$, so we can view them as maps

$$k \colon \mathfrak{gdiff} \to \Gamma(T\mathcal{A}_{\mathrm{H}}), \qquad \hat{k} \colon \mathfrak{gdiff} \to \Gamma(T\mathcal{A}_{\mathrm{V}}). \tag{3.134}$$

Hence we can identify the composite maps $k \circ \Theta$ and $\hat{k} \circ \hat{\Theta}$, acting on an arbitrary generalised vector V, with

$$\begin{aligned} V^\Sigma \Theta_\Sigma^\lambda k_\lambda &= L_V J_\alpha, \\ V^\Sigma \hat{\Theta}_\Sigma^{\hat{\lambda}} \hat{k}_{\hat{\lambda}} &= L_V X. \end{aligned} \tag{3.135}$$

From appendix E.1, note that $\hat{k} \circ \hat{\Theta}$ is just the set of generators $\mathcal{X}_{\Lambda\Xi}{}^\Sigma$ acting on X. Thus, as first noted in [140], in the infinite-dimensional gauging, we can identify a generic combination of generators $V^\Lambda \mathcal{X}_{\Lambda\Xi}{}^\Sigma$ with the generalised Lie derivative L_V. Similarly we have

$$V^\Sigma \Theta_\Sigma^\lambda \mu_{\alpha,\lambda} = \mu_\alpha(V). \tag{3.136}$$

Finally, recall from the discussion in section 3.2.2 that our notation is consistent and the holomorphic vector field X^Λ is indeed the complexified vector-multiplet structure $X = K + \mathrm{i}\hat{K}$. Thus the three conditions (3.132) are precisely

$$\mu_\alpha(V) = 0 \quad \text{for all } V, \qquad L_X J_\alpha = 0, \qquad L_{\bar{X}} X = 0. \tag{3.137}$$

We see that the integrability conditions on the structures have a very simple interpretation in terms of the gauged supergravity. This analysis is useful when looking for integrability conditions in other situations, in particular the backgrounds in $D = 5$ and $D = 6$ with eight supercharges which we discuss in later sections.

3.4.3 Moduli spaces

In this section, we will discuss some simple aspects of the moduli spaces of H, V and ECY structures. In the Calabi–Yau case, these come from deformations of the complex and symplectic structures. For example in type IIA, the H-structure moduli space describes the complex moduli together with harmonic three-form potentials C, while the V-structure moduli space describes the Kähler moduli. The main point here is that the H and V moduli spaces appear as hyper-Kähler and symplectic quotients respectively, and so by construction describe quaternionic and special Kähler geometries as required by supersymmetry.

Moduli space of hypermultiplet structures

We have already seen that the differential conditions (3.66) that define integrable H structures can be viewed as the vanishing of a triplet of moment maps for the action of the generalised diffeomorphism group GDiff on the space $\mathcal{A}_{\mathrm{H}}$. Acting on the moment maps with the vector field $\rho_W \in \Gamma(T\mathcal{A}_{\mathrm{H}})$, corresponding to an element of $\mathfrak{gdiff}$ labelled by W, we have, using integration by parts and Leibniz,

$$
\begin{aligned}
\imath_{\rho_W}\delta\mu_\alpha(V) &= -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\int_M \operatorname{tr}\big[(L_W J_\beta)(L_V J_\gamma) + J_\beta L_V(L_W J_\gamma)\big] \\
&= -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\int_M \operatorname{tr}\big(J_\beta(L_{L_V W} J_\gamma)\big) \\
&= \mu_\alpha(L_V W),
\end{aligned}
\tag{3.138}
$$

where we have used (3.63) and the Leibniz property. However the Lie bracket on $\mathfrak{gdiff}$ is

$$
[L_V, L_W] = L_{L_V W} = L_{[\![V,W]\!]}, \tag{3.139}
$$

where $[\![V,W]\!]$ is the antisymmetric Courant bracket for $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ [136, 137]. Thus we see that the moment maps (3.65) are equivariant. Since any two structures that are related by a generalised diffeomorphism – a combination of diffeomorphism and gauge transformation — are physically equivalent, the moduli space of integrable structures is naturally a hyper-Kähler quotient, defined as

$$
\mathcal{M}_{\mathrm{H}} = \mathcal{A}_{\mathrm{H}} /\!\!/\!\!/ \mathrm{GDiff} = \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)/\mathrm{GDiff}. \tag{3.140}
$$

By construction $\mathcal{M}_{\mathrm{H}}$ is also hyper-Kähler.

The space of structures $\mathcal{A}_{\mathrm{H}}$ is actually a hyper-Kähler cone, and for the quotient space to also be a hyper-Kähler cone one needs to take the zero level set[12] of the moment maps, as we do, and ensure that the GDiff action commutes with the SU(2) action on the cone. We can check that this is indeed that case. Under the SU(2) action we have $\delta J_\alpha = \epsilon_{\alpha\beta\gamma}\theta_\beta J_\gamma$, or, in other words, the action is generated by a triplet of vectors $\xi^\alpha \in \Gamma(T\mathcal{A}_{\mathrm{H}})$ such that

$$
\xi^\alpha(J_\beta) = \epsilon_{\alpha\beta\gamma} J_\gamma. \tag{3.141}
$$

Acting on the section-valued functions J_α, we see the Lie bracket is given by

$$
\big[\rho_V, \xi^\alpha\big](J_\beta) = L_V(\epsilon_{\alpha\beta\gamma}J_\gamma) - \epsilon_{\alpha\beta\gamma} L_V J_\gamma = 0. \tag{3.142}
$$

Hence the action of GDiff does indeed commute with the SU(2) action. This means that $\mathcal{M}_{\mathrm{H}}$ is also a hyper-Kähler cone [208], and we identify the physical moduli space with the base of the cone $\mathcal{M}_{\mathrm{H}}/\mathbb{H}^*$. By construction, as required by supersymmetry, this space is

[12] More generally, one requires that the level set is invariant under the SU(2) action.

quaternionic-Kähler.

It is worth noting that the action of GDiff on $\mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)$ is not generally free. For example, in a type IIB Calabi–Yau background, the integrable structure J_α, given in (3.39), is invariant under symplectomorphisms. Thus we expect that the moduli space $\mathcal{M}_{\mathrm{H}}$ is not generically a manifold, but has a complicated structure as a union of hyper-Kähler spaces [231]. We could still try to calculate the dimension of $\mathcal{M}_{\mathrm{H}}$ in a neighbourhood by considering the linear deformation away from a point $\sigma \in \mathcal{A}_{\mathrm{H}}$ corresponding to an integrable structure J_α. The variation of the moment maps is just $\delta\mu_\alpha$, where δ is the exterior derivative on $\mathcal{A}_{\mathrm{H}}$, while the infinitesimal generalised diffeomorphisms are generated by L_V. We can identify $\mathfrak{gdiff}$ with $\Gamma(E)$ and $T_\sigma\mathcal{A}_{\mathrm{H}}$ with sections of a bundle $\mathrm{ad}\,\tilde{P}^{\perp}_{\mathrm{Spin}^*(12)}$ as in (3.20). We then have the exact sequence of maps

$$\Gamma(E) \xrightarrow{L_\bullet J_\alpha} \Gamma(\mathrm{ad}\,\tilde{P}^{\perp}_{\mathrm{Spin}^*(12)}) \xrightarrow{\delta\mu_\alpha} \Gamma(E^*) \otimes \mathbb{R}^3.$$

Again, this is complicated by the existence of fixed points. From our examples, it appears that generically the sequence is not elliptic, and hence the moduli space is not finite-dimensional.

Moduli space of vector-multiplet structures

For the vector-multiplet structures we need to understand the constraint $L_K K = 0$ on the space of structures $\mathcal{A}_{\mathrm{V}}$ and again mod out by generalised diffeomorphisms. It turns out that the integrability condition can again be interpreted as the vanishing of a moment map as we now describe. In fact, this reformulation is not specific to this infinite-dimensional set up, but applies to any flat, supersymmetric vacuum of gauged $\mathcal{N} = 2$, $D = 4$ supergravity, giving a new interpretation of the conditions derived in [196, 197].

We have argued that from a gauged supergravity perspective the condition $L_K K$ arises from a gauging of the generalised diffeomorphism group on $\mathcal{A}_{\mathrm{V}}$. As discussed in appendix E.1, there are a number of requirements of the action of the gauge group on $\mathcal{A}_{\mathrm{V}}$ for it to preserve the special Kähler structure. First it must leave the symplectic form invariant. Let $\rho_V \in \Gamma(T\mathcal{A}_{\mathrm{V}})$ be the vector field on $\mathcal{A}_{\mathrm{V}}$ generating the action of a generalised diffeomorphism parametrised by $V \in \Gamma(E)$. Recall that the structure K can be viewed as a coordinate on $\mathcal{A}_{\mathrm{V}}$, as given in (3.29), thus associating $T_K\mathcal{A}_{\mathrm{V}} \simeq \Gamma(E)$ we have

$$\rho_V = L_V K \in \Gamma(T\mathcal{A}_{\mathrm{V}}). \tag{3.143}$$

Given an arbitrary vector field $W \in \Gamma(T\mathcal{A}_{\mathrm{V}})$ we have, from (3.31), that

$$\imath_{\rho_V}\Omega(W) = \Omega(\rho_V, W) = \int_M s(L_V K, W). \tag{3.144}$$

Using (3.63) and the Leibniz property of L_V we have

$$\begin{aligned}
\imath_{\rho_V}\Omega(W) &= \tfrac{1}{2}\int_M s(L_V K, W) - s(K, L_V W) \\
&= -\tfrac{1}{2}\int_M s(W, L_V K) + s(K, L_V W) \\
&= \imath_W \delta\mu(V),
\end{aligned} \tag{3.145}$$

where δ is the exterior derivative on $\mathcal{A}_{\mathrm{V}}$ and $\mu(V)$ is a the moment map

$$\mu(V) \coloneqq -\tfrac{1}{2}\int_M s(K, L_V K). \tag{3.146}$$

Thus the action of GDiff preserves the symplectic structure on $\mathcal{A}_{\mathrm{V}}$.

Acting on the moment map by the vector field ρ_W, corresponding to an element of $\mathfrak{gdiff}$ labelled by W, we have

$$\begin{aligned}
\imath_{\rho_W}\delta\mu(V) &= -\tfrac{1}{2}\int_M s(L_W K, L_V K) + s(K, L_V L_W K) \\
&= -\tfrac{1}{2}\int_M s(K, L_{L_V W} K) \\
&= \mu(L_V W),
\end{aligned} \tag{3.147}$$

where we have used (3.63) and the Leibniz property. Thus from the Lie bracket (3.139) on $\mathfrak{gdiff}$ we see that the moment map (3.146) is equivariant. We also see, using (3.32) and (3.63), that the Hitchin functional H is invariant under the action of ρ_V, since

$$\mathcal{L}_{\rho_V} H = \rho_V(H) = \int_M (L_V K)^M \frac{\partial}{\partial K^M}\sqrt{q(K)} = \int_M L_V \sqrt{q(K)} = 0. \tag{3.148}$$

In addition, $\rho_V = L_V K$ is clearly linear in K and so maps flat coordinates to flat coordinates. This is enough to show that the GDiff action also preserves the complex structure. Finally recall that the coordinate $K^M(x)$ can also be regarded as the components of a vector field and that the $\mathbb{C}^*$ action on $\mathcal{A}_{\mathrm{V}}$ is generated by $X = K + \mathrm{i}\hat{K} = K - \mathrm{i}\mathcal{I}\cdot K \in \Gamma(T\mathcal{A}_{\mathrm{V}})$, where $\mathcal{I}$ is the complex structure on $\mathcal{A}_{\mathrm{V}}$. As a vector field we have $\mathcal{L}_{\rho_V} K = [\rho_V, K] = 0$ and so, since $\mathcal{L}_{\rho_V}\mathcal{I} = 0$, we have $[\rho_V, X] = 0$ and hence the action of GDiff on $\mathcal{A}_{\mathrm{V}}$ commutes with the $\mathbb{C}^*$ action. These means that this gauging satisfies all the conditions necessary to preserve the special Kähler structure.

We now show that the condition $L_K K = 0$ is actually equivalent to the vanishing of the moment map $\mu(V)$ for all V. To do this we first note some identities using (B.16), (B.30) and (3.63)

$$\begin{aligned}
\Omega(V, L_V V) = \tfrac{1}{4}\int_M \mathcal{L}_v(\imath_v\tau) + \tfrac{1}{3}\mathrm{d}(\omega^3) + \mathrm{d}(\imath_v\omega\wedge\sigma) - \mathrm{d}(\imath_v\sigma\wedge\omega) \equiv 0, \\
\Omega(U, L_V W) - \Omega(W, L_V U) = \int_M L_V s(U, W) \equiv 0.
\end{aligned} \tag{3.149}$$

Under the identification of $V^{\Lambda}\mathcal{X}_{\Lambda\Xi}{}^{\Sigma}$ with L_V, these are just the representation constraint (E.13) and the first constraint of (E.12). These imply

$$\mu(V) = -\tfrac{1}{2}\Omega(K, L_V K) = \Omega(V, L_K K) = \int_M s(V, L_K K), \tag{3.150}$$

and hence we have the alternative definition for the integrability of K:

Definition. An *integrable* or *torsion-free* $\mathrm{E}_{7(7)}$ vector-multiplet structure K is one satisfying

$$\mu(V) = 0 \quad \text{for all } V \in \Gamma(E), \tag{3.151}$$

where $\mu(V)$ is given by (3.146).

This reformulation is actually generic for any gauged $\mathcal{N} = 2$, $D = 4$ theory as we now show. Using (E.9) and (E.11), we see that the third condition of (3.132) can be rewritten as

$$\bar{X}^{\Lambda}\hat{\Theta}_{\Lambda}^{\hat{\lambda}}\hat{k}^{i}_{\hat{\lambda}}(\partial_i X^{\Gamma})\Omega_{\Sigma\Gamma} = \bar{X}^{\Lambda}\mathcal{X}_{\Lambda\Xi\Sigma}X^{\Xi} = \tfrac{1}{2}\mathcal{X}_{\Lambda\Xi\Sigma}X^{\Xi}\bar{X}^{\Sigma} = 2\hat{\Theta}_{\Lambda}^{\hat{\lambda}}\mu_{\hat{\lambda}}, \tag{3.152}$$

where we have used the identities (E.12), (E.13) and (E.14). We see that the condition $\bar{X}^{\Lambda}\hat{\Theta}_{\Lambda}^{\hat{\lambda}}\hat{k}^{i}_{\hat{\lambda}} = 0$ is generically equivalent to the vanishing of the moment map $\mu_{\hat{\lambda}} = 0$.

This reformulation gives a simple realisation of the moduli space of vector-multiplet structures. Since any two structures related by a generalised diffeomorphism are equivalent, it is naturally given by the symplectic quotient

$$\mathcal{M}_{\mathrm{V}} = \mathcal{A}_{\mathrm{V}} /\!\!/ \mathrm{GDiff} = \mu^{-1}(0)/\mathrm{GDiff}. \tag{3.153}$$

By construction $\mathcal{M}_{\mathrm{V}}$ is also special Kähler. In fact it is a cone over a local special Kähler space, as required by supersymmetry. As usual for symplectic quotients of Kähler spaces, we can also view $\mathcal{M}_{\mathrm{V}}$ as a quotient by the complexified group $\mathcal{M}_{\mathrm{V}} = \mathcal{A}_{\mathrm{V}}/\mathrm{GDiff}_{\mathbb{C}}$. As for the case of hypermultiplet structures, generically GDiff does not act freely on $\mu^{-1}(0)$ and hence $\mathcal{M}_{\mathrm{V}}$ is not necessarily a manifold, but rather has a stratified structure [232].

Moduli space of ECYs

Finally we consider the moduli space of ECYs. We first define the space of compatible HV structures, though without the restriction on the norms. Formally, this is

$$\mathcal{A} = \{(J_\alpha, K) \in \mathcal{A}_{\mathrm{H}} \times \mathcal{A}_{\mathrm{V}} : J_\alpha \cdot K = 0\}. \tag{3.154}$$

The moduli space of ECYs is then given by

$$\mathcal{M} = \{(J_\alpha, K) \in \mathcal{A} : \mu_\alpha = 0,\, \mu = 0,\, L_X J_\alpha = 0,\, \kappa^2 = -2\sqrt{q(K)}\}/\mathrm{GDiff}. \tag{3.155}$$

The reason for dropping the norm compatibility condition from the definition of $\mathcal{A}$ is that it then has a fibred structure as we now discuss. One can imagine first choosing K and then J_α subject to the condition $J_\alpha \cdot K = 0$, or vice versa. At each point $x \in M$, we can then view the coset $\mathrm{E}_{7(7)}/\mathrm{SU}(6)$ as a fibration in two different ways:

$$\begin{array}{ccc} \mathrm{Spin}^*(12)/\mathrm{SU}(6) & \longrightarrow & \mathrm{E}_{7(7)}/\mathrm{SU}(6) \\ & & \downarrow \\ & & \mathrm{E}_{7(7)}/\mathrm{Spin}^*(12) \end{array} \qquad \begin{array}{ccc} \mathrm{E}_{6(2)}/\mathrm{SU}(6) & \longrightarrow & \mathrm{E}_{7(7)}/\mathrm{SU}(6) \\ & & \downarrow \\ & & \mathrm{E}_{7(7)}/\mathrm{E}_{6(2)} \end{array} \tag{3.156}$$

In both cases the fibres admit the appropriate geometry. Thus we can use exactly the same construction as in sections 3.2.1 and 3.2.2 to define the corresponding infinite-dimensional spaces of structures as hyper-Kähler and special Kähler manifolds. If we label these $\mathcal{A}^J_{\mathrm{V}}$ for the space of V structures given a fixed H structure J_α, and $\mathcal{A}^K_{\mathrm{H}}$ for the space of H structures given a fixed V structure K, the space $\mathcal{A}$ then has two different fibrations:

$$\begin{array}{ccc} \mathcal{A}^J_{\mathrm{V}} & \longrightarrow & \mathcal{A} \\ & & \downarrow \\ & & \mathcal{A}_{\mathrm{H}} \end{array} \qquad \begin{array}{ccc} \mathcal{A}^K_{\mathrm{H}} & \longrightarrow & \mathcal{A} \\ & & \downarrow \\ & & \mathcal{A}_{\mathrm{V}} \end{array} \tag{3.157}$$

Even with this fibred structure on $\mathcal{A}$, the structure of the moduli space $\mathcal{M}$ appears to be very complicated. Nonetheless, $\mathcal{N} = 2$ supergravity implies that it should become a product of the hyper- and vector-multiplet moduli spaces. Let us now comment on how this might translate into conditions on our structures. The product structure suggests that, at least locally, the moduli space of hypermultiplet structures is independent of the choice of vector-multiplet structure, and vice versa. One is tempted to conjecture that

$$\mathcal{M} = \mathcal{M}^K_{\mathrm{H}} \times \mathcal{M}^J_{\mathrm{V}}, \tag{3.158}$$

with $\mathcal{M}^K_{\mathrm{H}}$ and $\mathcal{M}^J_{\mathrm{V}}$ given by the quotients

$$\mathcal{M}^K_{\mathrm{H}} = \mathcal{A}^K_{\mathrm{H}} /\!/\!/ \mathrm{GDiff}_K, \qquad \mathcal{M}^J_{\mathrm{V}} = \mathcal{A}^J_{\mathrm{V}} /\!/ \mathrm{GDiff}_J, \tag{3.159}$$

where $\mathrm{GDiff}_K \subset \mathrm{GDiff}$ is the subset of generalised diffeomorphisms preserving K and $\mathrm{GDiff}_J \subset \mathrm{GDiff}$ is the subset preserving J_α. The point here is that $\mathcal{A}^K_{\mathrm{H}}$ and $\mathcal{A}^J_{\mathrm{V}}$ admit moment maps for GDiff_K and GDiff_J respectively, in complete analogy to section 3.3. For this to work the spaces $\mathcal{M}^K_{\mathrm{H}}$ and $\mathcal{M}^J_{\mathrm{V}}$ must (locally) be independent of the choice of K and J_α respectively.

We end with a few further comments. First, using the results of [143], the integrability conditions on K and J_α are equivalent to the Killing spinor equations, and we identically satisfy the Bianchi identities by defining the structure in terms of the gauge potentials. We then recall that for warped backgrounds of the form (3.1), the Killing spinor equations together with the Bianchi identities imply the equations of motion [98, 105, 143, 233]. Consequently, since the equations of motion on M are elliptic, the moduli space $\mathcal{M}$ must always be finite dimensional.

The second point relates to the generalised metric G. Recall that this defines an $\mathrm{SU}(8) \subset \mathrm{E}_{7(7)} \times \mathbb{R}^+$ structure and encodes the bosonic fields of the supergravity theory, restricted to M, along with the warp factor Δ [135,137]. Since $\mathrm{SU}(6) \subset \mathrm{SU}(8)$, the HV structure $\{J_\alpha, K\}$ determines the generalised metric G. Given a Lie subalgebra $\mathfrak{g}$ we can decompose $\mathfrak{e}_{7(7)} \oplus \mathbb{R} = \mathfrak{g} \oplus \mathfrak{g}^\perp$. Decomposing into $\mathrm{SU}(2) \times \mathrm{SU}(6)$ representations we find

$$\begin{aligned}
{\mathfrak{spin}^*_{12}}^\perp &= (\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{6}) + (\mathbf{2},\overline{\mathbf{6}}) + (\mathbf{2},\mathbf{20}) + (\mathbf{3},\mathbf{1}),\\
{\mathfrak{e}_{6(2)}}^\perp &= 2(\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{15}) + (\mathbf{1},\overline{\mathbf{15}}) + (\mathbf{2},\mathbf{6}) + (\mathbf{2},\overline{\mathbf{6}}),\\
{\mathfrak{su}_8}^\perp &= (\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{15}) + (\mathbf{1},\overline{\mathbf{15}}) + (\mathbf{2},\mathbf{20}),\\
{\mathfrak{su}_6}^\perp &= 2(\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{15}) + (\mathbf{1},\overline{\mathbf{15}}) + (\mathbf{2},\mathbf{6}) + (\mathbf{2},\overline{\mathbf{6}}) + (\mathbf{2},\mathbf{20}) + (\mathbf{3},\mathbf{1}).
\end{aligned} \tag{3.160}$$

Thus the deformations of $\{J_\alpha, K\}$ that do not change the generalised metric G are those in the $(\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{6}) + (\mathbf{2},\overline{\mathbf{6}}) + (\mathbf{3},\mathbf{1})$ representations. The first and last are the $\mathrm{U}(1)$ and $\mathrm{SU}(2)$ symmetries acting on K and J_α respectively. It is easy to see that the moment maps vanish only for constant rotations. The remaining $(\mathbf{2},\mathbf{6}) + (\mathbf{2},\overline{\mathbf{6}})$ deformations correspond to deforming the Killing spinors for a fixed background. If such solutions exist, they imply that the background actually admits more supersymmetries than the $\mathcal{N} = 2$ our formalism picks out. We also note that these deformations appear in both the deformations of K and J_α, and are related through the constraint $J_\alpha \cdot K = 0$. Thus we conclude that if the background is honestly $\mathcal{N} = 2$, then, up to a global $\mathrm{SU}(2) \times \mathrm{U}(1)$ rotation, there is a unique structure $\{J_\alpha, K\}$ for each generalised metric G and, infinitesimally, we can consider independent J_α and K deformations. This gives some credence to the conjecture that the moduli space takes the form (3.158).

Finally we note that the conditions $L_X \bar{X} = L_X J_\alpha = 0$ imply that

$$L_X G = 0, \tag{3.161}$$

and so K and $\hat{K}$ are generalised Killing vectors. This means there is a combination of diffeomorphism and gauge transformations under which all the supergravity fields are invariant. Hence, locally, one can always choose a gauge in which $L_X = \mathcal{L}_v$, where v is the vector component of X. If the metric g has no conventional Killing vectors, then $v = 0$ and the integrability conditions involving X are equivalent to

$$L_X(\text{anything}) = 0, \tag{3.162}$$

independent of the choice of J_α, as we saw happen explicitly in a number of our examples. In this case, an alternative approach to calculating $\mathcal{M}$ is to solve (3.162) and the moment map conditions on J_α independently, impose the compatibility conditions, and then quotient by GDiff.

3.5 $E_{d(d)}$ structures for $D = 5, 6$ supersymmetric flux backgrounds

In this section we consider generic $D = 5, 6$ type II and M-theory flux backgrounds preserving eight supercharges. In complete analogy with the $D = 4$ case, they define a pair of integrable generalised structures, though now in $E_{6(6)}$ for $\mathcal{N} = 1$, $D = 5$ backgrounds and $E_{5(5)} \simeq \mathrm{Spin}(5, 5)$ for $\mathcal{N} = (1, 0)$, $D = 6$ backgrounds. In both cases there is a H structure naturally associated to hypermultiplet degrees of freedom. In $D = 5$ there is also a V structure, though now the space of structures admits a very special real geometry rather than a special Kähler geometry, in line with the requirements of $\mathcal{N} = 1$, $D = 5$ gauged supergravity. In $D = 6$ we find the second structure is naturally associated to $\mathcal{N} = (1, 0)$ tensor multiplets.

Since much of the analysis follows mutatis mutandis the $D = 4$ case, we will be relatively terse in summarising the constructions.

3.5.1 $E_{6(6)}$ hyper- and vector-multiplet structures

For compactifications to $D = 5$, the relevant generalised geometry [137, 138] has an action of $E_{6(6)} \times \mathbb{R}^+$. The generalised tangent bundle transforms in the $\mathbf{27'_1}$ representation and decomposes under the relevant $GL(d)$ group as (2.11) or (2.13), where the one-form density terms are not present. The adjoint bundle transforms in the $\mathbf{1_0} + \mathbf{78_0}$ representation and decomposes as in (2.14) or (2.16), where the doublet of six-forms and six-vectors are not present for type IIB. In both type II and M-theory, the spinors transform under the USp(8) subgroup of $E_{6(6)} \times \mathbb{R}^+$. For $\mathcal{N} = 1$ backgrounds in $D = 5$, the single Killing spinor is stabilised by a USp(6) subgroup.

Structures and invariant tensors

The $E_{6(6)}$ generalised G-structures are defined as follows.

Definition Let G be a subgroup of $E_{6(6)}$. We define

- a *hypermultiplet structure* is a generalised structure with $G = \mathrm{SU}^*(6)$
- a *vector-multiplet structure* is a generalised structure with $G = F_{4(4)}$
- an *HV structure* is a generalised structure with $G = \mathrm{USp}(6)$

As before, an H structure is defined by a triplet of sections J_α of a weighted adjoint bundle, as in (3.11), such that they transform in the $\mathbf{78_{3/2}}$ representation of $E_{6(6)} \times \mathbb{R}^+$ and define a highest weight $\mathfrak{su}_2$ subalgebra of $\mathfrak{e}_{6(6)}$. The algebra and norms of the J_α are the same as for the $D = 4$ case, given in (3.12) and (3.13). A V structure is defined by a generalised vector[13]

$$K \in \Gamma(E), \quad \text{such that} \quad c(K, K, K) \neq 0, \tag{3.163}$$

[13] There are two distinct $E_{6(6)}$ orbits preserving $F_{4(4)}$, distinguished by the sign of $c(K, K, K)$ [216].

where c is the $E_{6(6)}$ cubic invariant given in (B.29) and (B.58). Compatibility between the vector- and hypermultiplet structures implies that the common stabiliser group $SU^*(6) \cap F_{4(4)}$ of the pair $\{J_\alpha, K\}$ is USp(6). The necessary and sufficient conditions are

$$\begin{aligned} J_\alpha \cdot K &= 0, \\ \operatorname{tr}(J_\alpha J_\beta) &= -c(K,K,K)\,\delta_{\alpha\beta}, \end{aligned} \tag{3.164}$$

where $\cdot$ is the adjoint action. Note that the second condition implies K is in the orbit where $c(K,K,K) > 0$. They are equivalent to

$$J_+ \cdot K = 0, \qquad c(K,K,K) = \kappa^2, \tag{3.165}$$

respectively, where κ is the factor appearing in (3.12). If the HV structure is integrable, we again say it defines an ECY geometry since in M-theory it is the flux generalisation of a compactification on a Calabi–Yau three-fold.

As with the $D = 4$ case, the infinite-dimensional space of H structures $\mathcal{A}_{\mathrm{H}}$ is the space of smooth sections of a bundle over M with fibre $W = E_{6(6)} \times \mathbb{R}^+/SU^*(6)$. This fibre is a hyper-Kähler cone over a pseudo-Riemannian Wolf space [207]

$$W/\mathbb{H}^* = E_{6(6)}/(SU^*(6) \times SU(2)). \tag{3.166}$$

The hyper-Kähler geometry on $\mathcal{A}_{\mathrm{H}}$ is again inherited directly from the hyper-Kähler geometry on W. The details follow exactly the $D = 4$ case in section 3.2.1 upon exchanging the relevant groups.

The infinite-dimensional space of V structures

$$\mathcal{A}_{\mathrm{V}} = \{K \in \Gamma(E) : c(K,K,K) > 0\} \tag{3.167}$$

can also be viewed as the space of smooth sections of a bundle over M with fibre $P = E_{6(6)} \times \mathbb{R}^+/F_{4(4)}$. It admits a natural (rigid) very special real metric, which again is inherited from the very special real metric on the homogeneous-space fibres.[14] The Riemannian symmetric spaces that admit (local) very special real metrics were analysed in [236] and include the case $E_{6(-26)}/F_4$. Here we need a pseudo-Riemannian form based on $E_{6(6)}$, where the relevant space is again a prehomogeneous vector space [222]

$$P/\mathbb{R}^+ = E_{6(6)}/F_{4(4)}. \tag{3.168}$$

The geometry on $\mathcal{A}_{\mathrm{V}}$ can be constructed as follows. Consider a point $K \in \mathcal{A}_{\mathrm{V}}$. Given $u, v, w \in T_K\mathcal{A}_{\mathrm{V}} \simeq \Gamma(E)$, the fibre-wise cubic invariant c defines a cubic form on $\mathcal{A}_{\mathrm{V}}$ by

$$C(u,v,w) = \int_M c(u,v,w), \tag{3.169}$$

[14]For reviews of very special real geometry see for example [234, 235].

where, since sections of E are weighted objects, we have $c(u,v,w) \in \Gamma(\det T^*M)$ and hence it can be integrated over M. The metric on $\mathcal{A}_{\mathrm{V}}$ is then defined as the Hessian of $C(K,K,K)$

$$C_{MN} = \frac{\delta C}{\delta K^M \delta K^N}. \tag{3.170}$$

In these expressions we are using the flat coordinates on $\mathcal{A}_{\mathrm{V}}$ defined by the vector-space structure of $\Gamma(E)$. On any rigid very special real geometry there is a global $\mathbb{R}^+$ symmetry, such that the quotient space is, by definition, a local very special real geometry. On $\mathcal{A}_{\mathrm{V}}$, the action of $\mathbb{R}^+$ is constant rescaling of the invariant tensor K. As for the hypermultiplet structure, the $\mathbb{R}^+$ action is simply a physically irrelevant constant shift in the warp factor Δ. Modding out by the symmetry, the physical space of structures $\mathcal{A}_{\mathrm{V}}/\mathbb{R}^+$ is an infinite-dimensional local very special real space.

In analogy with the $D = 4$ discussion of [123, 181, 193], we can view $\mathcal{A}_{\mathrm{V}}/\mathbb{R}^+$ and the quaternionic-Kähler base of $\mathcal{A}_{\mathrm{H}}$ as the infinite-dimensional spaces of vector- and hypermultiplet degrees of freedom, coming from rewriting the full ten- or eleven-dimensional supergravity theory as a five-dimensional $\mathcal{N} = 1$ theory.

Integrability

The integrability conditions for the $\mathrm{E}_{6(6)}$ generalised G-structures again arise from gauging the generalised diffeomorphism group, and are almost identical to those in $D = 4$ given in (3.66), (3.67) and (3.69), namely

$$\mu_\alpha(V) = 0 \qquad \text{for all } V \in \Gamma(E), \tag{3.171}$$

$$L_K K = 0, \tag{3.172}$$

$$L_K J_\alpha = 0. \tag{3.173}$$

In each case they are equivalent to the structure admitting a torsion-free, compatible generalised connection: if the first condition holds, J_α defines a torsion-free $\mathrm{SU}^*(6)$ structure; if the second condition holds, K defines a torsion-free $\mathrm{F}_{4(4)}$ structure; if all three conditions are satisfied, $\{J_\alpha, K\}$ define a torsion-free $\mathrm{USp}(6)$ structure. In the latter case, using the results of [143], this implies that these conditions are equivalent to the existence of an $\mathcal{N} = 1$ Killing spinor. Again, the pair of compatible and integrable H and V structures is not enough to imply that the HV structure is integrable.

Note that the condition $L_K K = 0$ can be written in an equivalent form as follows. Using the results and notation of [137], we have

$$L_K K = L_K K - [\![K, K]\!] = \partial \times_E (K \times_N K) = \mathrm{d}K', \tag{3.174}$$

where K' is the exterior derivative $K' = \delta C \in \Gamma(E^* \otimes \det T^*M)$, on $\mathcal{A}_{\mathrm{V}}$, of the invariant

	$W_{\text{int}}^{\text{USp}(6)}$ component						
Integrability condition	$(\mathbf{1},\mathbf{1})$	$(\mathbf{3},\mathbf{1})$	$(\mathbf{2},\mathbf{6})$	$(\mathbf{2},\mathbf{6})'$	$(\mathbf{1},\mathbf{14})$	$(\mathbf{3},\mathbf{14})$	$(\mathbf{2},\mathbf{14}')$
$\mu_\alpha = 0$		$\times$	$\times$			$\times$	
$L_K K = 0$	$\times$			$\times$	$\times$		
$L_K J_\alpha = 0$	$\times$	$\times$		$\times$		$\times$	$\times$

Table 3.3: The components of the generalised intrinsic torsion $W_{\text{int}}^{\text{USp}(6)}$ appearing in each of the $\mathcal{N} = 1$, $D = 5$ supersymmetry conditions.

functional $C(K, K, K)$. The condition (3.172) is then simply

$$\mathrm{d}K' = 0. \tag{3.175}$$

For example in M-theory, if $K = v + \omega + \sigma$ then

$$K' = \imath_v\omega + \imath_v\sigma - \tfrac{1}{2}\omega\wedge\omega + j\omega\wedge\sigma - \tfrac{1}{4}j\sigma\wedge\omega \in \Gamma(T^*M \oplus \wedge^4 T^*M \oplus T^*M \otimes \wedge^6 T^*M), \tag{3.176}$$

and the conditions are simply

$$\mathrm{d}K' = \mathrm{d}(\imath_v\omega) + \mathrm{d}(\imath_v\sigma - \tfrac{1}{2}\omega\wedge\omega) = 0. \tag{3.177}$$

To see that these differential conditions constrain the generalised intrinsic torsion for the different generalised structures, we start by noting that for $\mathrm{E}_{6(6)} \times \mathbb{R}^+$ generalised geometry the space of generalised torsions is [137]

$$W = \mathbf{27} + \mathbf{351}'. \tag{3.178}$$

Repeating the analysis of section 3.4, we find, decomposing under $\mathrm{SU}(2) \times \mathrm{SU}^*(6)$,

$$W_{\text{int}}^{\mathrm{SU}^*(6)} = (\mathbf{2},\mathbf{6}) + (\mathbf{3},\mathbf{15}), \tag{3.179}$$

while decomposing under $\mathrm{F}_{4(4)}$

$$W_{\text{int}}^{\mathrm{F}_{4(4)}} = \mathbf{1} + \mathbf{26}. \tag{3.180}$$

The intrinsic torsion components of an HV or USp(6) structure, decomposed under $\mathrm{SU}(2) \times \mathrm{USp}(6)$, along with which integrability conditions they constrain, are summarised in table 3.5.1. We note that it is equal to $(\mathbf{2},\mathbf{1}) \times (S + J)$, where $S + J = \mathbf{8} + \mathbf{48}$ is the USp(8) representation in which the Killing spinor equations transform. From the results of [137], we see that, again, the Killing spinor equations are equivalent to the differential conditions for an ECY or integrable USp(6) structure.

As in the $D = 4$ case, the integrability conditions have a direct interpretation in terms of $D = 5$ gauged supergravity. Following [197], the conditions for a Minkowski vacuum in a generic gauged $\mathcal{N} = 1$ theory can be written as[15]

$$\Theta_I^{\lambda}\mu_{\alpha,\lambda} = 0, \qquad h^{\Lambda}\Theta_{\Lambda}^{\lambda}k_{\lambda}^{u} = 0, \qquad h^{\Lambda}\hat{\Theta}_{\Lambda}^{\hat{\lambda}}\hat{k}_{\hat{\lambda}}^{i} = 0. \tag{3.181}$$

The only difference compared with the $D = 4$ case is that the vector h^{Λ} is now the coordinate vector in the real special geometry on $\mathcal{A}_{\mathrm{V}}$, written in flat coordinates, which here we identify with K. The three conditions (3.181) then translate directly into the three integrability conditions (3.171)–(3.173).

We can again consider the moduli spaces of structures. The integrability conditions for the H structure are identical to those in $D = 4$, and again the moduli space is a hyper-Kähler quotient, exactly as discussed in section 3.4.3. The arguments leading to the identification of the moduli space of V structures are also similar to those of $D = 4$, and so we simply summarise the relevant observations and results.

As discussed in [238], rigid very special real geometry requires the existence of a flat torsion-free connection $\hat{\nabla}$ preserving a metric tensor C_{mn} that, with respect to the flat coordinates, can be written as the Hessian of a cubic function C. For us, the vector-space structure of $\Gamma(E)$ defines natural flat coordinates on $\mathcal{A}_{\mathrm{V}}$ and the cubic function is given by (3.169). The function is invariant under the action of generalised diffeomorphisms, and since $\rho_V = L_V K$ is linear in K, it maps flat coordinates to flat coordinates. Thus GDiff preserves the very special real structure. Furthermore, we observe that given an integrable structure K such that $L_K K = 0$, any other choice of structure related to K by the action of GDiff is automatically integrable too. This means that integrability of the structure is well defined under equivalence by GDiff, so that both the very special real structure and the integrability condition descend to the quotient space. Thus the moduli space of integrable vector-multiplet structures is

$$\mathcal{M}_{\mathrm{V}} = \{K \in \mathcal{A}_{\mathrm{V}} : L_K K = 0\}/\mathrm{GDiff}, \tag{3.182}$$

which, as the $\mathbb{R}^+$ action generated by K commutes with GDiff, is a rigid very special real space. As required by supersymmetry, it is a cone over a local very special real space. The moduli space of ECYs is again more complicated, though all the comments made in the $D = 4$ case also apply here.

Example: Calabi–Yau manifold in M-theory

Just for orientation, we consider the simplest example of a generalised USp(6) structure, namely M-theory on a six-dimensional Calabi–Yau manifold M. In fact, assuming M has only an SU(3) structure, supersymmetry implies that the metric is Calabi–Yau and that

[15] Note that the third condition comes from the term in W^{xAB} proportional to ϵ^{AB} [237], which was assumed to vanish in [197].

the warp factor Δ and four-form flux F vanish [239, 240]. The goal here is to see how these conditions arise from the integrability conditions on the H and V structures.

The untwisted H and V structures are encoded by Ω and ω respectively. We have

$$\begin{aligned}\tilde{J}_+ &= -\tfrac{1}{2}\kappa\Omega + \tfrac{1}{2}\kappa\Omega^\sharp,\\ \tilde{J}_3 &= \tfrac{1}{2}\kappa I - \tfrac{1}{16}\mathrm{i}\kappa\Omega\wedge\bar{\Omega} - \tfrac{1}{16}\mathrm{i}\kappa\Omega^\sharp\wedge\bar{\Omega}^\sharp,\end{aligned} \tag{3.183}$$

where I is the almost complex structure (A.4) and the $\mathrm{E}_{6(6)}$-invariant volume is $\kappa^2 = \mathrm{e}^{3\Delta}\,\mathrm{vol}_6$, while

$$\tilde{K} = -\mathrm{e}^{\Delta}\omega. \tag{3.184}$$

It is easy to check, using the expressions in appendix B, that J_α generate an $\mathfrak{su}_2$ algebra and that the structures satisfy the correct normalisation and compatibility conditions, given (A.2). As previously, the actual structure will include the three-form potential A via the adjoint action: $J_\alpha = \mathrm{e}^A\tilde{J}_\alpha\mathrm{e}^{-A}$ and $K = \mathrm{e}^A\tilde{K}$. In what follows it will be easiest to use the untwisted forms with the twisted Dorfman derivative in the differential conditions.

The hypermultiplet structure is integrable if the triplet of moment maps vanish. We start with μ_3. The moment map is a sum of terms that depend on arbitrary $\tilde{v}$, $\tilde{\omega}$ and $\tilde{\sigma}$. Considering each component in turn we find

$$\mu_3(\tilde{\sigma}) \propto \int_M \kappa^2(\bar{\Omega}^\sharp\wedge\Omega^\sharp)\lrcorner\mathrm{d}\tilde{\sigma} \propto \int_M \mathrm{e}^{3\Delta}\mathrm{d}\tilde{\sigma} \propto \int_M \mathrm{d}(\mathrm{e}^{3\Delta})\wedge\tilde{\sigma}, \tag{3.185}$$

$$\mu_3(\tilde{\omega}) \propto \int_M \kappa^2(\bar{\Omega}^\sharp\wedge\Omega^\sharp)\lrcorner(\tilde{\omega}\wedge F) \propto \int_M \mathrm{e}^{3\Delta}\tilde{\omega}\wedge F. \tag{3.186}$$

These imply $\mathrm{d}\Delta = F = 0$. Using the fact that Δ is constant, the $\tilde{v}$ component of μ_3 and the $\tilde{\omega}$ component of μ_+ simplify to

$$\begin{aligned}\mu_3(\tilde{v}) &\propto \int_M \kappa^2(\bar{\Omega}^\sharp\lrcorner\mathcal{L}_{\tilde{v}}\Omega - \Omega^\sharp\lrcorner\mathcal{L}_{\tilde{v}}\bar{\Omega}) \propto \int_M \mathrm{e}^{3\Delta}(\imath_{\tilde{v}}\bar{\Omega}\wedge\mathrm{d}\Omega - \imath_{\tilde{v}}\Omega\wedge\mathrm{d}\bar{\Omega}),\\ \mu_+(\tilde{\omega}) &\propto \int_M \mathrm{e}^{3\Delta}\Omega\wedge\mathrm{d}\tilde{\omega} \propto \int_M \mathrm{e}^{3\Delta}\mathrm{d}\Omega\wedge\tilde{\omega}.\end{aligned}$$

The first requires the $(3,1)$ component of $\mathrm{d}\Omega$ to vanish or, in the language of [223], the $\mathcal{W}_5$ component of the SU(3) torsion is set to zero, while the second vanishes if and only if $\mathrm{d}\Omega$ vanishes, that is, $\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_5 = 0$. Finally, the $\tilde{\sigma}$ component of μ_+ vanishes identically, while the $\tilde{v}$ term vanishes if F vanishes. Together, we see that the integrability of the hypermultiplet structure requires a constant warp factor, a vanishing four-form flux and that Ω is closed.

For the V structure we have

$$\hat{L}_{\tilde{K}}\tilde{K} = -\omega\wedge\mathrm{d}\omega = 0, \tag{3.187}$$

which requires the $\mathcal{W}_4$ component of the SU(3) torsion to vanish. Note that requiring K

to define a integrable $\mathrm{F}_{4(4)}$ structure is considerably weaker than the condition for ω to define an integrable symplectic structure. Finally, the $L_K J_\alpha = 0$ condition required for an integrable USp(6) structure is equivalent to

$$\hat{L}_K J_+ \propto j\Omega^\sharp \lrcorner j\mathrm{d}\omega - \tfrac{1}{3}\mathbb{1}\Omega^\sharp \lrcorner \mathrm{d}\omega - \mathrm{d}\omega \wedge \Omega = 0. \tag{3.188}$$

One can show this vanishes if and only if $\mathrm{d}\omega$ vanishes, that is $\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = 0$.

We have shown that for this restricted SU(3) ansatz, integrability of the generalised USp(6) structure requires M to be Calabi–Yau, that is $\mathrm{d}\omega = \mathrm{d}\Omega = 0$, with a constant warp factor and a vanishing four-form flux.

3.5.2 $\mathbf{E_{5(5)}}$ hyper- and tensor-multiplet structures

For compactifications to $D = 6$ the relevant generalised geometry [137, 138] has an action of $\mathrm{E}_{5(5)} \times \mathbb{R}^+ \simeq \mathrm{Spin}(5,5) \times \mathbb{R}^+$. The generalised tangent bundle transforms in the $\mathbf{16_1}$ representation and decomposes under the relevant $\mathrm{GL}(d)$ group as (2.11) or (2.13), where the doublet of five-forms are not present for type IIB and the one-form density terms are not present for type IIB or M-theory. The adjoint bundle transforms in the $\mathbf{1_0} + \mathbf{45_0}$ representation and decomposes as in (2.14) or (2.16), where the six-forms and six-vectors are not present for type IIB or M-theory. In both type II and M-theory, spinors transform under a $\mathrm{USp}(4) \times \mathrm{USp}(4) \simeq \mathrm{Spin}(5) \times \mathrm{Spin}(5)$ subgroup of $\mathrm{E}_{5(5)} \times \mathbb{R}^+$. For $\mathcal{N} = (1,0)$ backgrounds, the Killing spinor is stabilised by an $\mathrm{SU}(2) \times \mathrm{USp}(4)$ subgroup.

Structures and invariant tensors

The $\mathrm{E}_{5(5)}$ generalised G-structures are defined as follows.

Definition. Let G be a subgroup of $\mathrm{E}_{5(5)}$. We define

- a *hypermultiplet structure* is a generalised structure with $G = \mathrm{SU}(2) \times \mathrm{Spin}(1,5)$
- a *tensor-multiplet structure* is a generalised structure with $G = \mathrm{Spin}(4,5)$
- an *HT structure* is a generalised structure with $G = \mathrm{SU}(2) \times \mathrm{USp}(4)$

As before, the H structure is defined by a triplet of sections J_α of a weighted adjoint bundle, as in (3.11), such that they transform in the $\mathbf{45_2}$ representation of $\mathrm{E}_{5(5)} \times \mathbb{R}^+$ and define a highest weight $\mathfrak{su}_2$ subalgebra. The algebra and norms of the J_α are the same as for the $D = 4$ case, given in (3.12) and (3.13).

The T or tensor-multiplet structure is new. It is defined by choosing a section of the bundle N transforming in the $\mathbf{10_2}$ representation of $\mathrm{E}_{5(5)} \times \mathbb{R}^+$. For M-theory on a five-dimensional manifold M,

$$N \simeq T^*M \oplus \wedge^4 T^*M, \tag{3.189}$$

while for type IIB on a four-dimensional manifold M it is

$$N \simeq S \oplus \wedge^2 T^*M \oplus S \otimes \wedge^4 T^*M. \tag{3.190}$$

The invariant generalised tensor for a Spin(4, 5) structure is a section of N:

$$Q \in \Gamma(N) \quad \text{such that} \quad \eta(Q, Q) > 0, \tag{3.191}$$

where η is the SO(5, 5) metric given in (B.28) and (B.57).

A pair of compatible structures define an SU(2) × USp(4) structure and satisfy

$$\begin{aligned} J_\alpha \cdot Q &= 0, \\ \operatorname{tr}(J_\alpha J_\beta) &= -\eta(Q, Q)\, \delta_{\alpha\beta}, \end{aligned} \tag{3.192}$$

where $\cdot$ is the adjoint action. They are equivalent to

$$J_+ \cdot Q = 0, \tag{3.193}$$

and the normalisation condition

$$\eta(Q, Q) = \kappa^2, \tag{3.194}$$

respectively, where κ is the factor appearing in (3.12). If the HT structure is integrable, we again say it defines an ECY geometry since it preserves eight supercharges and is the analogue of the corresponding structures in $D = 4$ and $D = 5$. In this case, there is no example without flux that is a Calabi–Yau space so the nomenclature is somewhat misleading, although the simplest flux example discussed in section 3.5.2 does have an underlying Calabi–Yau two-fold.

As before, the infinite-dimensional space of H structures $\mathcal{A}_\mathrm{H}$ is the space of smooth sections of a bundle over M with fibre $W = \mathrm{E}_{5(5)} \times \mathbb{R}^+/(\mathrm{SU}(2) \times \mathrm{Spin}(1, 5))$. This fibre is a hyper-Kähler cone over a pseudo-Riemannian Wolf space [207][16]

$$W/\mathbb{H}^* = \mathrm{SO}(5, 5)/(\mathrm{SO}(4) \times \mathrm{SO}(1, 5)). \tag{3.195}$$

The hyper-Kähler geometry on $\mathcal{A}_\mathrm{H}$ is again inherited directly from the hyper-Kähler geometry of W. The details of this exactly follow the $D = 4$ case in section 3.2.1 upon exchanging the relevant groups.

The infinite-dimensional space of T structures

$$\mathcal{A}_\mathrm{T} = \{Q \in \Gamma(N) : \eta(Q, Q) > 0\} \tag{3.196}$$

can also be viewed as the space of smooth sections of a homogeneous-space bundle

[16] Recall $\mathrm{E}_{5(5)} \simeq \mathrm{Spin}(5, 5)$, $\mathrm{Spin}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\mathrm{USp}(2) \simeq \mathrm{SU}(2)$, and note we have not been careful here to keep track of any discrete group factors.

Z_{T} over M with fibre $P = \mathrm{E}_{5(5)} \times \mathbb{R}^+/\mathrm{Spin}(4,5) \simeq \mathrm{SO}(5,5) \times \mathbb{R}^+/\mathrm{SO}(4,5) \simeq \mathbb{R}^{5,5}$. It admits a natural flat metric, which again is inherited from the flat metric on the fibres P. In $\mathcal{N} = (1,0)$ gauged supergravity, the scalar fields in the tensor multiplets describe Riemannian geometries of the form $\mathrm{SO}(n,1)/\mathrm{SO}(n)$, where the cone over this space is just flat $\mathbb{R}^{n,1}$ [241]. Here our fibres P are isomorphic to $\mathbb{R}^{5,5}$ with a flat pseudo-Riemannian metric, with the base of the cone given by the hyperboloid

$$P/\mathbb{R}^+ = \mathrm{SO}(5,5)/\mathrm{SO}(4,5), \tag{3.197}$$

where the $\mathbb{R}^+$ action is just the overall scaling. The flat metric is given by the quadratic form on $\mathcal{A}_{\mathrm{T}}$

$$\Sigma(v,w) = \int_M \eta(v,w), \tag{3.198}$$

where $v, w \in \Gamma(T_Q\mathcal{A}_{\mathrm{T}}) \simeq \Gamma(N)$, and since sections of N are weighted objects, we have $\eta(v,w) \in \Gamma(\det T^*M)$ and hence it can be integrated over M. The flat metric on $\mathcal{A}_{\mathrm{T}}$ is simply Σ. On $\mathcal{A}_{\mathrm{T}}$, the action of $\mathbb{R}^+$ is constant rescaling of the invariant tensor Q. As for the hypermultiplet structure, the $\mathbb{R}^+$ action is simply a reparametrisation of the warp factor Δ. Modding out by the symmetry, the physical space of structures $\mathcal{A}_{\mathrm{T}}/\mathbb{R}^+$ is an infinite-dimensional hyperbolic space.

As in the discussion of [123, 181, 193], we view $\mathcal{A}_{\mathrm{T}}/\mathbb{R}^+$ and the quaternionic-Kähler base of $\mathcal{A}_{\mathrm{H}}$ as an infinite-dimensional spaces of tensor- and hypermultiplet degrees of freedom, coming from rewriting the full ten- or eleven-dimensional supergravity theory as a six-dimensional $\mathcal{N} = (1,0)$ theory.

Integrability

The integrability conditions again arise from gauging the generalised diffeomorphism group and, for the H structures, are identical to those in $D = 4, 5$ given in (3.66), namely

$$\mu_\alpha(V) = 0 \qquad \text{for all } V \in \Gamma(E), \tag{3.199}$$

which is equivalent to the structure admitting a torsion-free, compatible generalised connection.

The integrability condition for the T structure Q is new and does *not* require the generalised Lie derivative. Instead, it appears in much the same way as the integrability of the pure spinors $\Phi^\pm$ describing generalised complex structures in $\mathrm{O}(d,d) \times \mathbb{R}^+$ generalised geometry. Recall that the usual derivative operator ∂ embeds in E^* which transforms in the $\mathbf{16}^c_{-\mathbf{1}}$ representation of $\mathrm{Spin}(5,5)$. We can use the $\mathbf{16}^c_{-\mathbf{1}} \times \mathbf{10_2} \to \mathbf{16_1}$ action to form the projection $E^* \otimes N \to E$, given in (B.22) and (B.51). This means there is a natural action of d on Q which results in a generalised vector, and furthermore, in this case, it is covariant. We then have

Definition. An *integrable* or *torsion-free* tensor-multiplet structure Q is one satisfying

$$\mathrm{d}Q = 0, \tag{3.200}$$

or in other words Q is closed under the exterior derivative.

These conditions are equivalent to there being a torsion-free generalised connection compatible with the generalised $\mathrm{Spin}(4,5)$ structure defined by Q. We can also consider the integrability conditions for the HT generalised structure defined by the compatible pair $\{J_\alpha, Q\}$.

Definition. An ECY geometry admits an *integrable* or *torsion-free* HT structure $\{J_\alpha, Q\}$ such that J_α and Q are separately integrable. There are no further conditions.

In contrast to the case of compatible V and H structures, the existence of a pair of compatible and integrable H and T structures is enough to imply that the HT structure is integrable. These conditions are equivalent to there being a torsion-free generalised connection compatible with the generalised $\mathrm{SU}(2) \times \mathrm{USp}(4)$ structure defined by $\{J_\alpha, Q\}$. Using the results of [143], this implies that the conditions are equivalent to the existence of an $\mathcal{N} = (1,0)$ Killing spinor.

To see that these differential conditions constrain the appropriate generalised intrinsic torsion for the different generalised structures, we start by noting that for $\mathrm{E}_{5(5)} \times \mathbb{R}^+$ generalised geometry the space of generalised torsions is [137]

$$W = \mathbf{16}^c + \mathbf{144}^c. \tag{3.201}$$

Repeating the analysis of section 3.4, we find, decomposing under $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{Spin}(1,5)$ where the first factor is the $\mathrm{SU}(2)$ generated by J_α,

$$W_{\mathrm{int}}^{\mathrm{SU}(2)\times\mathrm{Spin}(1,5)} = (\mathbf{2}, \mathbf{1}, \mathbf{4}^c) + (\mathbf{3}, \mathbf{2}, \mathbf{4}), \tag{3.202}$$

while decomposing under $\mathrm{Spin}(4,5)$

$$W_{\mathrm{int}}^{\mathrm{Spin}(4,5)} = \mathbf{16}. \tag{3.203}$$

The intrinsic torsion components of an HT or $\mathrm{SU}(2)\times\mathrm{USp}(4)$ structure, decomposed under $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{USp}(4)$, along with which integrability conditions they constrain, are summarised in table 3.5.2. We note that the intrinsic torsion is equal to $(\mathbf{2}, \mathbf{1}, \mathbf{1})\times(S^- + J^-)$, where $S^- + J^- = (\mathbf{1}, \mathbf{4}) + (\mathbf{5}, \mathbf{4})$ are the $\mathrm{USp}(4) \times \mathrm{USp}(4)$ representations in which the Killing spinor equations transform for $\mathcal{N} = (1,0)$ supersymmetry [143]. Again, from the results of [137], the Killing spinor equations are equivalent to the differential conditions for an integrable $\mathrm{SU}(2) \times \mathrm{USp}(4)$ structure.

As in the $D = 4$ and $D = 5$ cases, the integrability conditions have a interpretation in terms of $D = 6$ gauged supergravity as we now sketch. The gauging of $D = 6$

	$W_{\text{int}}^{\text{SU}(2)\times\text{USp}(4)}$ component			
Integrability condition	$(\mathbf{1},\mathbf{2},\mathbf{4})$	$(\mathbf{2},\mathbf{1},\mathbf{4})$	$(\mathbf{2},\mathbf{1},\mathbf{4})'$	$(\mathbf{3},\mathbf{2},\mathbf{4})$
$\mu_\alpha = 0$		$\times$		$\times$
$\mathrm{d}Q = 0$	$\times$		$\times$	

Table 3.4: The components of the generalised intrinsic torsion $W_{\text{int}}^{\text{SU}(2)\times\text{USp}(4)}$ appearing in each of the $\mathcal{N}=(1,0)$, $D=6$ supersymmetry conditions.

supergravity coupled to tensor-, vector- and hypermultiplets using the embedding tensor formalism is discussed in the context of "magical supergravities" in [242]. The conditions for a supersymmetric Minkowski background coming from the vanishing of the gaugino variations read[17]

$$\Theta_\Lambda^\lambda \mu_{\alpha,\lambda} = 0, \qquad L^I \theta_I^\Lambda = 0. \tag{3.204}$$

The key difference compared with the $D=4$ case is that the vector L^I is now the coordinate vector on the flat tensor-multiplet space $\mathcal{A}_{\text{T}}$. Note that the first condition was previously discussed in [197]. In making the translation to the integrability conditions we note that L^I corresponds to Q, while the matrix θ_I^Λ is a map

$$\theta \colon \Gamma(N) \to \Gamma(E), \tag{3.205}$$

which we can identify with the action of the exterior derivative d discussed above, so that

$$L^I \theta_I = \mathrm{d}Q. \tag{3.206}$$

Hence the conditions (3.204) are precisely (3.199) and (3.200). Note that there are number of conditions on θ_I^Λ, as well as on the intertwiner between $\Gamma(N)$, $\Gamma(E)$ and $\Gamma(E^*)$, related to the tensor hierarchy and necessary for the supersymmetry algebra to close. It would be interesting to see how these are satisfied by the exterior derivative d in the infinite-dimensional case. The fact that the geometry on each fibre $\text{SO}(5,5)\times\mathbb{R}^+/\text{SO}(4,5)$ of the homogeneous-space bundle Z_{T} is a pseudo-Riemannian variant of that appearing in one of the magical supergravity theories suggests that the structure will essentially be inherited fibre-wise.

The moduli spaces in this case are much the same as the previous examples we have seen. The moduli space of H structures is again as discussed in section 3.4.3. The space of T structures $\mathcal{A}_{\text{T}}$ admits a flat geometry, defined by the metric Σ. Again, the vector-space structure of $\Gamma(N)$ defines natural flat coordinates on $\mathcal{A}_{\text{T}}$ and hence a flat connection that, by definition, preserves Σ. GDiff preserves the flat structure, and furthermore an

[17]Note we use a different index notation from [242] to match the notation used in $D=4$ and $D=5$. Also, the first term in (3.204) comes contracted with a matrix $m^{\Sigma\Lambda}$ in the gaugino variation, but using the fact that $m^2 \propto (L^I L_I)\mathbb{1}$ we see that this term can only vanish if the first term in (3.204) vanishes.

integrable structure Q remains integrable under the action of GDiff. Thus the moduli space of integrable tensor-multiplet structures is

$$\mathcal{M}_{\mathrm{T}} = \{Q \in \mathcal{A}_{\mathrm{T}} : \mathrm{d}Q = 0\}/\mathrm{GDiff}, \tag{3.207}$$

which is again flat. As required by supersymmetry, it is a cone over a hyperbolic space. As for the previous cases, generically GDiff does not act freely on $\mathcal{A}_{\mathrm{T}}$ and hence $\mathcal{M}_{\mathrm{T}}$ is not necessarily a manifold. The moduli space of ECYs is again more complicated, though all the comments made in the $D = 4$ case also apply here.

Example: NS5-branes on a hyper-Kähler space in type IIB

The standard NS5-brane solution is a warped product of six-dimensional Minkowski space with a flat four-dimensional transverse space and preserves sixteen supercharges [243]. Exchanging the flat transverse space for a four-dimensional hyper-Kähler space breaks supersymmetry further, leaving eight supercharges [77]. Thus, we expect it can be formulated as an integrable $\mathrm{SU}(2) \times \mathrm{USp}(4)$ structure within $\mathrm{E}_{5(5)} \times \mathbb{R}^+$ generalised geometry. The metric takes the standard form (3.1) with $D = 6$, and the four-dimensional space M admits an SU(2) structure, with a triplet of two-forms ω_α as in (A.19), and a canonical volume form $\frac{1}{2}\omega_\alpha \wedge \omega_\beta = \delta_{\alpha\beta}\,\mathrm{vol}_4$. The solution also has non-trivial NS-NS three-form flux H and dilaton ϕ, but the warp factor Δ is zero. The solution is supersymmetric if the SU(2) structure and three-form flux satisfy [77, 98]

$$\mathrm{d}(\mathrm{e}^{-2\phi}\omega_\alpha) = 0, \qquad \star H = -\mathrm{e}^{2\phi}\mathrm{d}(\mathrm{e}^{-2\phi}). \tag{3.208}$$

Recall that in type II theories there are two types of ten-dimensional spinors. The NS5-brane solutions are an example of a pure NS-NS geometry preserving eight supercharges where the preserved Killing spinors are all of one type: they have ∇^+ special holonomy in the language of [98]. As such they cannot be described by generalised complex structures [111]. For this reason it is interesting to see how they do appear in the $\mathrm{E}_{5(5)} \times \mathbb{R}^+$ generalised geometry. (Note that we described the same solution wrapped on $\mathbb{R}^2$ in $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ generalised geometry in terms of the wrapped M5-brane background of appendix A.5.)

Embedding in type IIB, the H structure is determined by the ω_α, such that the untwisted objects are

$$\tilde{J}_\alpha = -\tfrac{1}{2}\kappa I_\alpha + \tfrac{1}{2}\kappa u^i \omega_\alpha + \tfrac{1}{2}\kappa v^i \omega^\sharp_\alpha, \tag{3.209}$$

where $(I_\alpha)^m{}_n = -(\omega_\alpha)^m{}_n$ is the triplet of almost complex structures, $u^i = (-1, 0)$ and $v^i = (0, -1)$, and $\kappa^2 = \mathrm{e}^{-2\phi}\,\mathrm{vol}_4$ is the $\mathrm{E}_{5(5)}$-invariant volume. The untwisted T structure depends only on the volume form and dilaton through

$$\tilde{Q} = u^i + \mathrm{e}^{-2\phi} v^i\,\mathrm{vol}_4, \tag{3.210}$$

where u and v are the same as above. It is easy to check from the results of appendix B that the J_α generate an $\mathfrak{su}_2$ algebra, and that the normalisation and compatibility conditions are satisfied. The NS-NS three-form flux H embeds in the first component of F_3^i (see (2.23)). Thus, as previously, the actual structures will include the NS-NS two-form potential via the adjoint action: $J_\alpha = \mathrm{e}^{B^1}\tilde{J}_\alpha \mathrm{e}^{-B^1}$ and $Q = \mathrm{e}^{B^1}\tilde{Q}$. In what follows it will be easiest to use the untwisted forms with the twisted Dorfman derivative in the differential conditions.

For the moment maps the $\tilde{\lambda}^i$ terms give

$$\mu_\alpha(\tilde{\lambda}^i) \propto \int_M \mathrm{e}^{-2\phi}\omega_\alpha \wedge \mathrm{d}\tilde{\lambda}^1, \tag{3.211}$$

which vanishes for $\mathrm{d}(\mathrm{e}^{-2\phi}\omega_\alpha) = 0$, completely fixing the intrinsic torsion of the underlying SU(2) structure. Using this condition, the $\tilde{v}$ terms simplify to

$$\mu_\alpha(\tilde{v}) \propto \int_M \epsilon_{\alpha\beta\gamma}\mathrm{e}^{-2\phi}\mathrm{d}\phi \wedge \omega_\beta \wedge \imath_{\tilde{v}}\omega_\gamma - \mathrm{e}^{-2\phi}\omega_\alpha \wedge \imath_{\tilde{v}} H. \tag{3.212}$$

This vanishes for $\star H = -\mathrm{e}^{2\phi}\mathrm{d}(\mathrm{e}^{-2\phi})$. In terms of the untwisted objects, the integrability of the T structure is given by

$$\mathrm{d}_{F^i}\tilde{Q} = 0, \tag{3.213}$$

where the action of d_{F^i} on $\tilde{Q} \in \Gamma(N)$ is defined in (B.56). Using the explicit form of $\tilde{Q}$, we have

$$\mathrm{d}_{F^i}\tilde{Q} = \mathrm{d}u^i + \epsilon_{ij}u^i F^j \tag{3.214}$$

The one-form term vanishes as u^i has constant entries. The three-form term also vanishes as the contraction of u^i with F^j picks out $F^2 = F_3$, which is zero for the NS5-brane background.

Finally, note that we can embed the D5-brane solution in a similar way. The dilaton now appears as a warp factor Δ, so the $\mathrm{E}_{5(5)}$-invariant volume is $\kappa^2 = \mathrm{vol}_4$. We also take $u^i = (0,1)$ and $v^i = (-1,0)$, and drop the factor of $\mathrm{e}^{-2\phi}$ in $\tilde{Q}$. The moment maps then vanish if

$$\mathrm{d}(\mathrm{e}^{\phi}\omega_\alpha) = 0, \qquad F_3 = -2 \star \mathrm{d}(\mathrm{e}^{-\phi}). \tag{3.215}$$

The first of these is the correct differential condition for the SU(2) structure. The second is the correct three-form flux, coming from the dual of the seven-form flux due to the D5-brane [98]. The integrability of the T structure takes the same form as for the NS5-brane, but now the contraction of u^i with F^j picks out $F^1 = H$, which is zero for the D5-brane background.

3.6 Summary

In this chapter we have given a new geometrical interpretation of generic flux backgrounds in type II supergravity and M-theory, preserving eight supercharges in $D = 4, 5, 6$ Minkowski spacetime, as integrable G-structures in $E_{d(d)} \times \mathbb{R}^+$ generalised geometry. As in conventional geometry, integrability was defined as the existence of a generalised torsion-free connection that is compatible with the structure, or equivalently as the vanishing of the generalised intrinsic torsion, defining what we called an "exceptional Calabi–Yau" (ECY) space.

We found the differential conditions on the structures implied by integrability, and showed that they took a simple form in terms of the generalised Lie derivative or moment maps for the action of the generalised diffeomorphism group. As for Calabi–Yau backgrounds, supersymmetric solutions are described as the intersection of two separate structures that can be associated to hypermultiplet and vector-multiplet degrees of freedom in the corresponding gauged supergravity. We saw how the simple examples of Calabi–Yau, generalised Calabi–Yau and hyper-Kähler structures appear in our formalism, as well as various other simple supersymmetric flux backgrounds.

We saw that the spaces of hypermultiplet and vector-multiplet structures admit hyper-Kähler and special Kähler metrics respectively. The integrability conditions for each took the form of a moment map for the action of the generalised diffeomorphism group, so that the moduli spaces of structures is given by a hyper-Kähler or symplectic quotient.

Chapter 4

Exceptional Sasaki–Einstein backgrounds

In this chapter we analyse generic AdS flux backgrounds preserving eight supercharges in $D = 4$ and $D = 5$ dimensions using exceptional generalised geometry. We show that they are described by a pair of globally defined generalised structures, identical to those that appear for Minkowski flux backgrounds but with different integrability conditions. We give a number of explicit examples of such "exceptional Sasaki–Einstein" backgrounds in type IIB supergravity and M-theory. In particular, we give the complete analysis of the generic AdS_5 M-theory backgrounds. We also briefly discuss the structure of the moduli space of solutions. In all cases, one structure defines a "generalised Reeb vector" that generates a Killing symmetry of the background corresponding to the R-symmetry of the dual field theory, and in addition encodes the generic contact structures that appear in the $D = 4$ M-theory and $D = 5$ type IIB cases. Finally, we investigate the relation between generalised structures and quantities in the dual field theory, showing that the central charge and R-charge of BPS wrapped-brane states are both encoded by the generalised Reeb vector, as well as discussing how volume minimisation (the dual of a- and $\mathcal{F}$-maximisation) is encoded.

4.1 Introduction

Supersymmetric AdS backgrounds are of central importance to gauge/gravity duality. In the simplest examples, corresponding to branes at conical singularities where only a top-form field strength is non-zero, they describe familiar geometries [64], such as Sasaki–Einstein or weak-G_2 spaces. However, backgrounds with generic fluxes are much more complicated and at first glance have no simple geometrical description. Significant progress has been made analysing them using G-structures [97, 98, 104, 105], for example as means of classifying AdS_4 and AdS_5 solutions with eight supercharges in both type II theories [244] and M-theory [240,245]. More generally one can use generalised geometry [108–

110] to characterise the type II backgrounds, as for example in [130, 132, 246]. In both cases the geometry is defined by set of invariant tensors, typically only locally defined, satisfying some first-order differential equations that capture the lack of integrability of the structure in terms of the form-field flux. It is natural then to ask if there is a single notion of geometry that captures the known examples in terms of a global, integrable structure, perhaps also in a way adapted to the degrees of freedom of the dual theory.

As we saw in the previous chapter, the answer is to use $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ generalised geometry [135–138]: generic Minkowski flux backgrounds in $D = 4, 5, 6$ preserving eight supercharges can be formulated as "exceptional Calabi–Yau" geometries. In this chapter we will give the extension of this formalism for "exceptional Sasaki–Einstein" geometries, that is, generic type II and M-theory AdS backgrounds in $D = 4, 5$ preserving eight supercharges. The generalised structures are identical to those that appear for Minkowski backgrounds, however the integrability conditions are modified in a way that depends on the cosmological constant, and is equivalent to the presence of singlet intrinsic torsion for the corresponding generalised connection [145]. In each case the vector-multiplet structure is defined by an invariant generalised vector which is Killing: it generates a combination of diffeomorphisms and gauge transformations that leave the background invariant, corresponding in this case to the R-symmetry of the dual field theory. By analogy to the Sasaki–Einstein case we refer to it as the "generalised Reeb vector". The formalism also allows one to analyse the structure of the moduli space of backgrounds. In particular we find that the space of integrable hypermultiplet structures appears as a Kähler slice of a hyper-Kähler quotient of the original space of structures, in a way closely related to the "HK/QK correspondence" of Haydys [247]. This mirrors the analysis of gauged $D = 4, 5$ supergravity [198, 248] precisely because the structures can be thought of as describing a rewriting of the ten- or eleven-dimensional supergravity as a $D = 4, 5$ theory coupled to an infinite number of hyper- and vector-multiplets [181].

We analyse three explicit cases to show how known supersymmetric AdS flux backgrounds appear in our formalism. For $D = 5$ in type IIB, we consider the Sasaki–Einstein solutions, and also give the form of the generalised Reeb vector for the generic backgrounds in terms of spinor bilinears defined in [244]. For $D = 5$ in M-theory, we give a completely general analysis, showing how the structures are defined in terms of the bilinears of [240], and also that the integrability conditions are satisfied. Finally, for $D = 4$ in M-theory we again consider the Sasaki–Einstein solutions, and give the form of the generalised Reeb vector for the generic backgrounds in terms of bilinears of [245].

One striking feature that emerges is the role played by the generalised Reeb vector. It is already known that, remarkably, the generic $D = 5$ type IIB and $D = 4$ M-theory backgrounds admit contact structures [134, 245, 249], which encode both the central charge a (or free energy $\mathcal{F}$) of the theory and the R-charges of operators dual to wrapped branes. This structure appears very naturally in the exceptional Sasaki–Einstein description: it is simply the generalised Reeb vector. As we discuss, this also leads to a very natural conjecture, following the work of [250], for the generic notion of "volume minimisation" [251, 252], the gravity dual of a- and $\mathcal{F}$-maximisation in the field theory [253–255].

This chapter is organised as follows. We begin in section 4.2 by reviewing the generalised structures that appear for $D = 4, 5$ Minkowski backgrounds preserving eight supercharges, and then recall the integrability conditions on the structures. We then move onto the main result, namely the extension of the integrability conditions for AdS backgrounds. We leave the interpretation of the conditions and a discussion of the moduli spaces of integrable structures to section 4.3. We provide some concrete examples of supersymmetric AdS backgrounds in sections 4.4 and 4.5 and show they do indeed define integrable structures. In section 4.6, we comment on the relation between vector-multiplet structures and several field theory quantities, in particular the central charge and free energy, the dimension of operators dual to wrapped branes and the dual of a- and $\mathcal{F}$-maximisation.

4.2 Generalised structures for AdS

We begin by reviewing the generalised structures that define $D = 4, 5$ backgrounds preserving eight supercharges. These were defined in the previous chapter for Minkowski vacua, but are equally applicable to AdS. The only difference is in the integrability conditions, and one of the main results of this chapter is to give the conditions relevant to AdS. We provide some concrete examples, including the case of completely general fluxes in M-theory giving an AdS_5 vacuum. We leave the interpretation of the conditions and a discussion of the moduli spaces of integrable structures to section 4.3.

4.2.1 Hyper- and vector-multiplet structures in $\mathrm{E}_{d(d)}$ generalised geometry

We consider type II and M-theory solutions of the form $\mathrm{AdS}_D \times M$, where M is $(10 - D)$-dimensional for type II and $(11 - D)$-dimensional for M-theory. We assume the metric is a warped product

$$\mathrm{d}s^2 = \mathrm{e}^{2\Delta}\mathrm{d}s^2(\mathrm{AdS}_D) + \mathrm{d}s^2(M), \tag{4.1}$$

where Δ is a scalar function on M. We take m to be the inverse AdS radius, so that the Ricci tensor is normalised to $R_{\mu\nu} = -(D-1)m^2 g_{\mu\nu}$, where g is the metric on AdS_D, and the cosmological constant is $\Lambda = -\frac{1}{2}(D-1)(D-2)m^2$. As in the previous chapter, we allow generic fluxes compatible with the AdS symmetry of the external spacetime and use the string frame metric for type II solutions.

We showed in the previous chapter that a generic Minkowski background preserving eight supercharges is completely characterised by a pair of generalised G-structures in exceptional generalised geometry. These structures were first defined in [181], in the context of type II theories. The pairs of structures that appear for $\mathcal{N} = 2$, $D = 4$ and $\mathcal{N} = 1$, $D = 5$ backgrounds were named hypermultiplet and vector-multiplet structures,

	G_{frame}	H structure	V structure	HV structure
$D = 4$	$E_{7(7)} \times \mathbb{R}^+$	$\text{Spin}^*(12)$	$E_{6(2)}$	$SU(6)$
$D = 5$	$E_{6(6)} \times \mathbb{R}^+$	$SU^*(6)$	$F_{4(4)}$	$USp(6)$

Table 4.1: The generalised G-structures with $G \subset E_{7(7)}$ and $G \subset E_{6(6)}$ that define eight-supercharge backgrounds in $D = 4$ and $D = 5$ respectively.

or H and V structures for short, since they are associated to hyper- and vector-multiplet scalar degrees of freedom in D dimensions. The relevant structure groups defined by the H and V structures are summarised in table 4.2.1.

The hypermultiplet structure is defined by a triplet of sections of a weighted adjoint bundle

$$\text{H structure :} \qquad J_\alpha \in \Gamma\big(\text{ad}\,\tilde{F} \otimes (\det T^*M)^{1/2}\big) \qquad \alpha = 1, 2, 3, \tag{4.2}$$

which define a highest weight $\mathfrak{su}_2$ subalgebra of $\mathfrak{e}_{d(d)}$ and are normalised using the $\mathfrak{e}_{d(d)}$ Killing form such that

$$[J_\alpha, J_\beta] = 2\kappa\epsilon_{\alpha\beta\gamma}J_\gamma, \qquad \text{tr}(J_\alpha J_\beta) = -\kappa^2\delta_{\alpha\beta}. \tag{4.3}$$

Similarly, the vector-multiplet or V structure is defined by a section of the generalised tangent bundle E

$$\text{V structure :} \qquad K \in \Gamma(E), \tag{4.4}$$

which has a positive norm with respect to the $E_{7(7)}$ quartic invariant $q(K) > 0$ or the $E_{6(6)}$ cubic invariant $c(K) > 0$.[1] In $D = 4$, one can use the quartic invariant as a Hitchin function to define a second invariant tensor $\hat{K}$ and combine the two into a complex object

$$X = K + \mathrm{i}\hat{K}. \tag{4.5}$$

Explicitly, $\hat{K}$ is defined by the relation

$$s(V, \hat{K}) = 2q(K)^{-1/2}q(V, K, K, K). \tag{4.6}$$

for arbitrary $V \in \Gamma(E)$.

Finally the pair of structures $\{J_\alpha, K\}$ define an HV structure if they are compatible,

[1] Recall that for $E_{7(7)}$ there is a symmetric quartic invariant $q(V_1, V_2, V_3, V_4)$ and a symplectic invariant $s(V_1, V_2)$ and for $E_{6(6)}$ a symmetric cubic invariant $c(V_1, V_2, V_3)$. We use the shorthand $q(V) = q(V, V, V, V)$ and $c(V) = c(V, V, V)$.

that is, if they satisfy the conditions

$$\text{HV structure :} \qquad J_\alpha \cdot K = 0, \qquad \operatorname{tr}(J_\alpha J_\beta) = \begin{cases} -2\sqrt{q(K)}\delta_{\alpha\beta} & D = 4, \\ -c(K)\delta_{\alpha\beta} & D = 5, \end{cases} \tag{4.7}$$

where for $D = 4$ from (4.6) we also have $\sqrt{q(K)} = \frac{1}{2}s(K, \hat{K})$.

Given a pair of globally defined spinors on M, one can construct "untwisted" structures $\{\tilde{J}_\alpha, \tilde{K}\}$ in terms of spinor bilinears. The full structures include the potentials for the appropriate form fields and are given by the exponentiated adjoint action on the untwisted objects, thus in M-theory we have

$$J_\alpha = \mathrm{e}^{A+\tilde{A}} \tilde{J}_\alpha \mathrm{e}^{-A-\tilde{A}}, \qquad K = \mathrm{e}^{A+\tilde{A}} \tilde{K}, \tag{4.8}$$

where A is the three-form potential and $\tilde{A}$ is the dual six-form potential, and for type IIB

$$J_\alpha = \mathrm{e}^{B^i+C} \tilde{J}_\alpha \mathrm{e}^{-B^i-C}, \qquad K = \mathrm{e}^{B^i+C} \tilde{K}, \tag{4.9}$$

where B^i is the $\mathrm{SL}(2;\mathbb{R})$ doublet of two-form potentials and C is the four-form potential. In this case one also needs to include dressing by the IIB axion and dilaton, as described in appendix B. Since these transformations are in $\mathrm{E}_{d(d)}$, the algebra, normalisation and compatibility conditions (4.3) and (4.7) can be checked using either the twisted or untwisted objects.

4.2.2 Exceptional Sasaki–Einstein geometry

We now describe the integrability conditions on the HV structure for the case of a supersymmetric AdS background preserving eight supercharges. As discussed in [143, 145], the difference from the Minkowski case is that there is a constant singlet component of the generalised intrinsic torsion, resulting in a background with weak generalised holonomy. This leads to a simple modification of the Minkowski conditions given in the previous chapter.

Recall that the space of H structures has a natural hyper-Kähler cone geometry and admits a triplet of moment maps for the action of the generalised diffeomorphism group GDiff, that is, the diffeomorphism and gauge transformation symmetries of the underlying supergravity theory. Infinitesimal transformations are generated by the generalised Lie derivative L_V and so are parametrised by generalised vectors $V \in \Gamma(E)$. The moment maps for a given element in $\mathfrak{gdiff}$ parametrised by V are given by

$$\mu_\alpha(V) = -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma} \int_M \operatorname{tr}(J_\beta L_V J_\gamma). \tag{4.10}$$

For Minkowski backgrounds, supersymmetry implied that the moment maps vanished. For

AdS backgrounds they take a fixed non-zero value. Let us define the real functions

$$D = 4: \qquad \gamma(V) := 2\int_M q(K)^{-1/2} q(V, K, K, K), \tag{4.11}$$

$$D = 5: \qquad \gamma(V) := \int_M c(V, K, K). \tag{4.12}$$

Note that the first definition can also be written in terms of $\hat{K}$ using (4.6).

We can then define the exceptional generalised geometry analogue of a Sasaki–Einstein structure, corresponding to an AdS background with generic fluxes. We have

Definition. An *exceptional Sasaki–Einstein (ESE) structure* is an HV structure $\{J_\alpha, K\}$ satisfying

$$\mu_\alpha(V) = \lambda_\alpha \gamma(V) \qquad \text{for all } V \in \Gamma(E), \tag{4.13}$$

$$L_K K = 0, \tag{4.14}$$

$$L_K J_\alpha = \epsilon_{\alpha\beta\gamma} \lambda_\beta J_\gamma, \qquad L_{\hat{K}} J_\alpha = 0, \tag{4.15}$$

where $\gamma(V)$ is given by (4.11) and (4.12), and λ_α are real constants related to the inverse AdS radius by $|\lambda| = 2m$ for $D = 4$ and $|\lambda| = 3m$ for $D = 5$, where $|\lambda|^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$. The second condition in (4.15) is relevant only for $D = 4$.

The integrability condition for the vector-multiplet structure (4.14) is unchanged from the Minkowski case. As shown in appendix C, for $D = 4$ this is equivalent to $L_X \bar{X} = 0$, as $L_X X$ vanishes identically. The other two conditions now have right-hand sides, determined by the singlet torsion. Note that the third condition (4.15) simply states that the action of L_K is equivalent to an SU(2) rotation of the J_α. Note also that this condition is consistent with the moment map conditions when taking $V = K$ (and $V = \hat{K}$ in $D = 4$). As shown in appendix F, for ESE spaces, the second compatibility constraint in (4.7) is actually a consequence of the integrability conditions.

Recall that for $D = 4$ there is a global U(1) R-symmetry that acts on X, taking $X \to X' = \mathrm{e}^{\mathrm{i}\alpha} X$. Strictly, one should write the condition (4.15) replacing K with $\operatorname{Re} X'$ and $\hat{K}$ with $\operatorname{Im} X'$. However, the point is that one can always choose a gauge where the condition takes the form (4.15). In a similar way one can use the SU(2) global R-symmetry to set $\lambda_{1,2} = 0$. (The only unbroken part of the R-symmetry is then a U(1) preserving λ_3.) The conditions (4.10) can then be written as

$$\mu_3(V) = \lambda_3 \gamma(V), \qquad \mu_+(V) = 0, \qquad \text{for all } V \in \Gamma(E), \tag{4.16}$$

while the conditions (4.15) read

$$D = 4: \qquad L_K J_+ = \mathrm{i}\lambda_3 J_+, \qquad L_{\hat{K}} J_+ = 0, \tag{4.17}$$

$$D = 5: \qquad L_K J_+ = \mathrm{i}\lambda_3 J_+. \tag{4.18}$$

These are the forms we will use when checking the integrability for various examples.

We can immediately deduce various properties from the integrability conditions. The first is that the ESE space is generalised Einstein. Recall that the HV structure $\{J_\alpha, K\}$ determines the generalised metric G that encodes the supergravity degrees of freedom on M. Given a generalised metric one can construct a unique generalised Ricci tensor following [137]. Using the generalised intrinsic torsion of the ESE background, which we discuss in section 3.4, and the results of [145], we find that the generalised Ricci tensor satisfies the generalised Einstein equation[2]

$$R_{MN} = \frac{(D-1)(D-2)}{d_E} m^2 G_{MN}, \tag{4.19}$$

where M and N are indices running over the dual generalised tangent space E^* and d_E is the dimension of E.

Next we note that since $L_K K = 0$ and $L_K J_\alpha$ is equal to an SU(2) R-symmetry rotation, which simply rotates the Killing spinors but leaves the supergravity degrees of freedom unchanged, we can conclude that $L_K G = 0$ and so

$$L_K G = 0 \quad \Leftrightarrow \quad K \text{ is a generalised Killing vector}, \tag{4.20}$$

as is also the case for Minkowski backgrounds. Note that for the $D = 4$ solutions, $\hat{K}$ is also generalised Killing. Let us decompose K into vector and form components

$$K = \begin{cases} \xi + \omega + \sigma + \tau & \text{M theory}, \\ \xi + \lambda^i + \rho + \sigma^i + \tau & \text{type IIB}, \end{cases} \tag{4.21}$$

where ξ is the vector component. The generalised Killing vector condition in M-theory is equivalent to

$$\mathcal{L}_\xi g = 0, \qquad \mathcal{L}_\xi A - \mathrm{d}\omega = 0, \qquad \mathcal{L}_\xi \tilde{A} - \mathrm{d}\sigma + \tfrac{1}{2}\mathrm{d}\omega \wedge A = 0, \tag{4.22}$$

where A is the three-form potential and $\tilde{A}$ is the dual six-form potential. In type IIB it is equivalent to

$$\begin{aligned} \mathcal{L}_\xi g &= 0, \qquad & \mathcal{L}_\xi C &= \mathrm{d}\rho - \tfrac{1}{2}\epsilon_{ij}\mathrm{d}\lambda^i \wedge B^j, \\ \mathcal{L}_\xi B^i &= \mathrm{d}\lambda^i, \qquad & \mathcal{L}_\xi \tilde{B}^i &= \mathrm{d}\sigma^i + \tfrac{1}{2}\mathrm{d}\lambda^i \wedge C - \tfrac{1}{2}\mathrm{d}\rho \wedge B^i + \tfrac{1}{12}B^i \wedge \epsilon_{kl}B^k \wedge \mathrm{d}\lambda^l, \end{aligned} \tag{4.23}$$

where B^i is the SL(2; $\mathbb{R}$) doublet of two-form potentials, $\tilde{B}^i$ are their six-form duals and C is the four-form potential. In each case, the form components give compensating gauge transformations so that the field strengths ($F = \mathrm{d}A$ etc.) are invariant under the diffeomorphism generated by ξ. We immediately see that if $\xi = 0$ then all the

[2] We are using $R_{MN} = R^0_{MN} + \frac{1}{d_E}G_{MN}R$, where R^0 and R are the generalised tensors defined in [137].

form components are closed and hence L_K acting on *any* generalised tensor vanishes. However, this is in contradiction with the condition (4.15). Hence we conclude that ξ is non-zero and the solution admits a Killing vector that also preserves all the fluxes. Furthermore from (4.15) we see that it generates the unbroken $\mathrm{U}(1) \subset \mathrm{SU}(2)$ R-symmetry. On Sasaki–Einstein spaces this vector is known as the Reeb vector. Thus we are led to define

Definition. We call K the *generalised Reeb vector* of the exceptional Sasaki–Einstein geometry, noting that its vector component $\xi \in \Gamma(TM)$ is necessarily non-vanishing.

The fact that K is generalised Killing means that, in the untwisted frame where there are no potentials in the generalised metric, the corresponding "twisted" generalised Lie derivative must reduce to just a conventional Lie derivative, that is

$$\hat{L}_{\tilde{K}} = \mathcal{L}_\xi, \tag{4.24}$$

where ξ is the vector component of $\tilde{K}$ (and hence also of K). Acting on an arbitrary untwisted generalised tensor $\tilde{\alpha}$, the twisted generalised Lie derivative takes the form

$$\hat{L}_{\tilde{V}}\tilde{\alpha} = \mathcal{L}_v\tilde{\alpha} - \tilde{R}\cdot\tilde{\alpha}, \tag{4.25}$$

where $\tilde{R}$ is a tensor in the adjoint representation of $\mathrm{E}_{d(d)}$, $\tilde{R}\cdot\tilde{\alpha}$ is the adjoint action, v is the vector component of $\tilde{V}$, $\mathcal{L}_v$ is the conventional Lie derivative and

$$\tilde{R} = \begin{cases} \mathrm{d}\tilde{\omega} - \imath_{\tilde{v}}F + \mathrm{d}\tilde{\sigma} - \imath_{\tilde{v}}\tilde{F} + \tilde{\omega}\wedge F & \text{for M-theory,} \\ \mathrm{d}\tilde{\lambda}^i - \imath_{\tilde{v}}F^i + \mathrm{d}\tilde{\rho} - \imath_{\tilde{v}}F - \epsilon_{ij}\tilde{\lambda}^i\wedge F^j + \mathrm{d}\tilde{\sigma}^i + \tilde{\lambda}^i\wedge F - \tilde{\rho}\wedge F^i & \text{for type IIB.} \end{cases} \tag{4.26}$$

The condition (4.24) thus means that the corresponding tensor $\tilde{R}$ vanishes. The conditions (4.14) and (4.15) can then be written as

$$\mathcal{L}_\xi\tilde{J}_\alpha = \epsilon_{\alpha\beta\gamma}\lambda_\beta\tilde{J}_\gamma, \qquad \mathcal{L}_\xi\tilde{K} = 0. \tag{4.27}$$

In what follows it will sometimes be simpler when checking solutions to use these forms of the conditions.

Finally, we note that there is a consistency condition on K implied by the moment map conditions (4.13). Strictly, there is a kernel in the map $L_\bullet\colon \Gamma(E) \to \mathfrak{gdiff}$, meaning that two different generalised vectors can generate the same generalised diffeomorphism. In other words, we have $L_V = L_{V+\Delta}$, which holds if

$$\Delta = \begin{cases} \omega+\sigma+\tau & \text{with } \mathrm{d}\omega = \mathrm{d}\sigma = 0 \text{ in M-theory,} \\ \lambda^i+\rho+\sigma^i+\tau & \text{with } \mathrm{d}\lambda^i = \mathrm{d}\rho = \mathrm{d}\sigma^i = 0 \text{ in type IIB.} \end{cases} \tag{4.28}$$

Thus for the conditions (4.13) to make sense we need

$$\gamma(\Delta) = 0, \tag{4.29}$$

which is a differential condition on K. In fact it is implied by the conditions (4.14) and (4.15). Note first that these conditions are satisfied by both K and $K + \Delta$. As we have already mentioned, substituting (4.15) into the expression for the moment maps (4.10) gives

$$\mu_\alpha(K) = \lambda_\alpha \int_M \kappa^2 = \lambda_\alpha \gamma(K), \tag{4.30}$$

where the second equality follows from the second of the compatibility conditions (4.7). From the homogeneity of q and c, we note that upon taking the functional derivative, where M runs over all the components of the generalised vector, we have

$$V^M \frac{\delta\gamma(K)}{\delta K^M} = (D-2)\gamma(V). \tag{4.31}$$

Then note that, using $\mu_\alpha(K+\Delta) = \mu_\alpha(K)$ and (4.30), we have

$$0 = \Delta^M \frac{\delta\mu_\alpha(K)}{\delta K^M} = \lambda_\alpha (D-2)\gamma(\Delta), \tag{4.32}$$

and hence indeed $\gamma(\Delta) = 0$. Note that this derivation did not use the moment map conditions (4.13) themselves, only the conditions (4.14) and (4.15) involving L_K.

Finally, in the $D = 4$ case $\hat{K}$ is also generalised Killing. However, from the condition $\gamma(\Delta) = 0$ and (4.6), we have

$$\gamma(\tau) = \int_M s(\tau, \hat{K}) = 0, \tag{4.33}$$

for all τ for both type IIB and M-theory. From the form of the symplectic invariant given in (B.30) and (B.30), this implies that the vector component of $\hat{K}$ vanishes. Since $\hat{K}$ is Killing this means

$$L_{\hat{K}}(\text{anything}) = 0, \tag{4.34}$$

or in other words, $\hat{K}$ is in the kernel of the map $L_\bullet \colon \Gamma(E) \to \mathfrak{gdiff}$, satisfying the same conditions as Δ in (4.28). As such, it generates a trivial generalised diffeomorphism and hence the generalised metric is not invariant under a second symmetry; only K generates a non-trivial transformation.

4.2.3 Generalised intrinsic torsion

As conjectured in [143] and proven in [145], the Killing spinor equations for generic AdS flux backgrounds preserving eight supercharges are in one-to-one correspondence with HV structures with constant singlet generalised intrinsic torsion.[3] In each case the non-zero torsion was in the $(\mathbf{3},\mathbf{1})$ of $\mathrm{SU}(2)\times G$, where G is the HV structure group, which breaks the SU(2) R-symmetry to U(1). These were called spaces with weak generalised special holonomy, in analogy with conventional G-structures. This is in contrast to Minkowski backgrounds where all components of the intrinsic torsion vanished. Note that there are no singlets in the generalised intrinsic torsion for $D=6$, giving the standard result that there are no $\mathcal{N}=(1,0)$ AdS solutions in six dimensions.

In order to prove that our conditions (4.13), (4.14) and (4.15) are equivalent to the conditions for supersymmetry, we need to check that they indeed admit a constant non-zero singlet in the $(\mathbf{3},\mathbf{1})$ component of the intrinsic torsion. To do this we can simply repeat the calculations of section 3.4. One immediately notes that the $(\mathbf{3},\mathbf{1})$ component appears in the moment maps and $L_K J_\alpha$, but not $L_K K$. This explains why the $L_K K=0$ condition is unchanged from the Minkowski case. By definition, the right-hand side of (4.15) is a constant singlet in $(\mathbf{3},\mathbf{1})$ as it is a constant linear combination of J_α. Consistency with the moment maps then implies (4.13) for $V=K$. This proves that the integrability conditions are indeed equivalent to the Killing spinor equations.

4.3 Gauged supergravity and moduli spaces

4.3.1 Integrability conditions from gauged supergravity

As stressed in [181] and the previous chapter, the infinite-dimensional spaces $\mathcal{A}_{\mathrm{H}}$ and $\mathcal{A}_{\mathrm{V}}$ of hyper- and vector-multiplet structures correspond to a rewriting of the ten- or eleven-dimensional supergravity theory so that only eight supercharges are manifest [146]. The local Lorentz symmetry is broken and the fields of the theory can be reorganised into $\mathcal{N}=2$, $D=4$ or $\mathcal{N}=1$, $D=5$ multiplets without making a Kaluza–Klein truncation. One can then interpret the integrability conditions in terms of conventional gauged $D=4$ or $D=5$ supergravity with an infinite-dimensional gauging by GDiff. The general conditions for supersymmetric vacua have been given in [196,197,248], and we showed in the previous chapter that for Minkowski backgrounds these conditions are precisely the integrability conditions on the generalised structures.

Let us now briefly show that the same is true for the AdS backgrounds. Following [197], a generic gauged $\mathcal{N}=2$, $D=4$ theory admits an AdS vacua provided

$$\Theta_\Lambda^\lambda \mu_{\alpha,\lambda} = -\tfrac{1}{2}\mathrm{e}^{K^{\mathrm{v}}/2}\Omega_{\Lambda\Sigma}\,\mathrm{Im}(\hat{\mu}\bar{X}^\Sigma)a_\alpha, \qquad \bar{X}^\Lambda\hat{\Theta}_\Lambda^{\hat{\lambda}}\hat{k}^i_{\hat{\lambda}}=0, \qquad X^\Lambda\Theta_\Lambda^\lambda k^u_\lambda = c a_\alpha(\xi^\alpha)^u, \tag{4.35}$$

where $|\hat{\mu}|\propto m$, a_α is unit-norm vector parametrising S^2, K^{v} is the Kähler potential and $\Omega_{\Lambda\Sigma}$ the symplectic structure on the space of vector multiplets $\mathcal{A}_{\mathrm{V}}$. We have written the

[3] Strictly for $D=4$ only the $\mathcal{N}=1$ case was considered in [145]. However, combined with the comments about $\mathcal{N}=2$ in [143], the results of [145] are sufficient to prove that for $\mathcal{N}=2$ there is a constant singlet torsion transforming in a triplet of SU(2).

last condition not on the quaternionic-Kähler space, but on the corresponding hyper-Kähler cone. Any Killing vector preserving the quaternionic-Kähler structure on the base lifts to a vector that rotates the three complex structures on the cone. Thus $(\xi^\alpha)^u$ are the three vectors generating the $\mathfrak{su}_2$ action the cone, normalised such that $\xi^\alpha \cdot \xi^\beta = \delta^{\alpha\beta}$. There is a consistency condition between the first and third conditions that arises from the identity $k_\lambda \cdot \xi^\alpha = -2\mu_{\alpha,\lambda}$ [213, 215]. This is the same consistency we already noted for the integrability conditions (4.15) and (4.14). Contracting the third expression in (4.35) with ξ^α and the first expression with X^Λ, we find

$$c = \mathrm{e}^{K^{\mathrm{v}}/2}\Omega_{\Lambda\Sigma}X^\Lambda \operatorname{Im}(\hat{\mu}\bar{X}^\Sigma). \tag{4.36}$$

We can then choose $\hat{\mu}$ to be real using the U(1) action on X. Using the identifications between terms in the $\mathcal{N} = 2$ expressions and the H and V geometries discussed in section 3.4, we see that, using (4.6) and for real $\hat{\mu}$, the three conditions in (4.35) exactly match (4.13), (4.14) and (4.15) respectively. Explicitly we can identify

$$\begin{aligned} V^\Lambda\Theta_\Lambda^{\lambda}\mu_{\alpha,\lambda} &= \mu_\alpha(V), \\ \Omega_{\Lambda\Sigma}V^\Lambda \operatorname{Im}(\hat{\mu}\bar{X}^\Sigma)a_\alpha &= a_\alpha\Omega(V,\hat{K}) = a_\alpha\gamma(V), \\ \bar{X}^\Lambda\hat{\Theta}_\Lambda^{\hat{\lambda}}\hat{k}_{\hat{\lambda}} &= L_{\bar{X}}X, \end{aligned} \tag{4.37}$$

and $\mathrm{e}^{-K^{\mathrm{v}}} = \mathrm{i}\Omega(X,\bar{X})$. While acting on the section-valued functions J_α, we have

$$\begin{aligned} a_\beta\xi^\beta(J_\alpha) &= -\epsilon_{\alpha\beta\gamma}a_\beta J_\gamma, \\ X^\Lambda\Theta_\Lambda^{\lambda}k_\lambda(J_\alpha) &= L_X J_\alpha. \end{aligned} \tag{4.38}$$

It is straightforward to see that conditions in $D = 5$ can similarly be matched to the gauged supergravity expressions for AdS vacua given in [197].

4.3.2 Moduli spaces of ESE backgrounds

We now turn to analysing the structure of the moduli space of exceptional Sasaki–Einstein backgrounds satisfying the integrability conditions (4.13)–(4.15). Given the relation to gauged supergravity discussed above, we can use known results on the form of the moduli space of AdS vacua in these theories [198, 248]. For example, for $\mathcal{N} = 2$, $D = 4$ gauged supergravity, it was shown in [198] that the vector-multiplet moduli space is a real subspace of the local special Kähler manifold $\mathcal{A}_{\mathrm{V}}/\mathbb{C}^*$, while the hypermultiplet moduli space is a Kähler submanifold of the quaternionic manifold $\mathcal{A}_{\mathrm{H}}/\mathbb{H}^*$, at least in the so called "minimal" solution. More generally, the combined moduli space is no longer a product.

In fact, the situation here is more complicated because we have to impose the compatibility conditions (4.7) between the H and V structures. This means that even before imposing the integrability conditions, the space $\mathcal{A}$ of HV structures is not actually a product $\mathcal{A}_{\mathrm{V}} \times \mathcal{A}_{\mathrm{H}}$. Nonetheless, as we have described, if we drop the normalisation part of the compatibility condition, we can view $\mathcal{A}$ as a fibration of a hyper-Kähler cone space over a special Kähler space (or vice versa). The same structure arises for $D = 5$ but now we have a hyper-Kähler cone over a very special real manifold (or vice versa).

Focussing for definiteness on $D = 5$, though an analogous analysis applies to $D = 4$, we can use this fibration picture to analyse the form of the moduli space. Let us first fix a generalised Reeb vector $K \in \mathcal{A}_{\mathrm{V}}$ satisfying the integrability condition $L_K K = 0$. We can now consider the space of H structures $\mathcal{A}_{\mathrm{H}}^K \subset \mathcal{A}_{\mathrm{H}}$ compatible with the fixed K, that is

$$\mathcal{A}_{\mathrm{H}}^K = \{J_\alpha \in \mathcal{A}_{\mathrm{H}} : J_\alpha \cdot K = 0\}\,. \tag{4.39}$$

We can drop the normalisation condition $\kappa^2 = c(K)$ since, as we show in appendix F, it is a consequence of the supersymmetry conditions. At each point on the manifold M, the space of possible J_α is given by the hyper-Kähler cone

$$W = \mathrm{F}_{4(4)} \times \mathbb{R}^+/\mathrm{USp}(6), \qquad W/\mathbb{H}^* \text{ is a Wolf space}, \tag{4.40}$$

and in complete analogy to the construction of $\mathcal{A}_{\mathrm{H}}$ we find that the infinite-dimensional space $\mathcal{A}_{\mathrm{H}}^K$ is itself a hyper-Kähler cone. We are now left with imposing the remaining two supersymmetry conditions

$$\mu_\alpha(V) = \lambda_\alpha \gamma(V), \qquad L_K J_\alpha = \epsilon_{\alpha\beta\gamma}\lambda_\beta J_\gamma. \tag{4.41}$$

We would like to have geometrical interpretations of both conditions. Recall first that since $\mathcal{A}_{\mathrm{H}}^K$ is a hyper-Kähler cone it admits a free SU(2) action generated by a triple of vectors $\xi^\alpha \in \Gamma(T\mathcal{A}_{\mathrm{H}}^K)$. The action of GDiff is triholomorphic (it preserves all three symplectic structures) and is generated by a vector $\rho_V \in \Gamma(T\mathcal{A}_{\mathrm{H}}^K)$ for each $V \in E$. By definition, acting on the J_α we have

$$\rho_V(J_\alpha) = L_V J_\alpha, \qquad \xi^\alpha(J_\beta) = \epsilon_{\alpha\beta\gamma} J_\gamma. \tag{4.42}$$

Because of the "source" term $\lambda_\alpha\gamma(V)$ in the moment maps, only a subgroup $\mathrm{U}(1) \subset \mathrm{SU}(2)$ of transformations leave the moment map conditions invariant. This group is generated by $r = \lambda_\alpha \xi^\alpha$ and preserves one linear combination of complex structures $I = \lambda_\alpha I^\alpha$ on $\mathcal{A}_{\mathrm{H}}^K$. Restricting to $V = K$, the vector ρ_K generates a one-dimensional subgroup $G_K \subset \mathrm{GDiff}$ corresponding to the generalised diffeomorphisms generated by K. As we showed in section 3.4, these two actions commute.

We can now interpret the condition (4.15) as a vector equation

$$\rho_K - r = 0, \tag{4.43}$$

that is, it restricts us to points on $\mathcal{A}_{\mathrm{H}}^K$ that are fixed points of a combined action of G_K and U(1). (Note that generically we expect that fixed points will only exist for certain choices of K satisfying $L_K K = 0$.) We define

$$\mathcal{N}_{\mathrm{H}} = \big\{p \in \mathcal{A}_{\mathrm{H}}^K : \rho_K(p) - r(p) = 0\big\}. \tag{4.44}$$

Since both ρ_K and r preserve the complex structure I, both are real holomorphic vectors and hence $\mathcal{N}_{\mathrm{H}}$ is a Kähler subspace of $\mathcal{A}_{\mathrm{H}}^K$ with respect to I.

Let us now turn to the moment maps. We would like to view them as defining a hyper-Kähler quotient. Thought of as single map $\mu\colon \mathcal{A}_{\mathrm{H}}^K \to \mathfrak{gdiff}^* \times \mathbb{R}^3$, for AdS backgrounds, the level set defined by (4.13) is $\mu^{-1}(\Lambda_\alpha)$, where the element $\Lambda_\alpha \in \mathfrak{gdiff}^* \times \mathbb{R}^3$ is given by the functional derivative $\Lambda_\alpha = \lambda_\alpha \delta\gamma/\delta V$. But since $\gamma(V)$ depends on K we see that it is not invariant under the full generalised diffeomorphism group. A hyper-Kähler quotient is well defined only on a level set that is invariant under the action of the quotient group. However, we can define a subgroup of generalised diffeomorphisms $\mathrm{GDiff}_K \subset \mathrm{GDiff}$ as those that leave K invariant, that is the stabiliser group,

$$\mathrm{GDiff}_K = \{\Phi \in \mathrm{GDiff} : \Phi \cdot K = K\}, \tag{4.45}$$

so that infinitesimally, V parametrises an element of the corresponding algebra $\mathfrak{gdiff}_K$ if $L_V K = 0$. Since $L_K K = 0$ note that $G_K \subset \mathrm{GDiff}_K$. For a fixed K, any two H structures related by an element of GDiff_K are equivalent. If we restrict to the subgroup GDiff_K, then we can view the moment maps as a hyper-Kähler quotient.[4] Since the moment map conditions break the SU(2) action to U(1), although the quotient space is by definition hyper-Kähler, it is not a hyper-Kähler cone, that is, there is no longer an underlying quaternionic-Kähler space.

Combining the quotient with the fixed-point conditions (4.43) we then have two possibilities: either take a quotient and then impose (4.43) or impose (4.43) and then take a quotient. Doing the latter we note that the fixed-point condition already imposes that we are on a Kähler subspace, so there is no notion of a hyper-Kähler quotient. However, we show in appendix F that, restricting to GDiff_K on $\mathcal{N}_{\mathrm{H}}$, two of the moment map conditions are identically satisfied. Thus we are actually only taking a *symplectic* quotient with a moment map given by $\mu(V) = \lambda_\alpha \mu_\alpha(V)$. Thus we have the diagram

$$\begin{tikzcd} \mathcal{A}_{\mathrm{H}}^K \arrow[r, "\rho_K - r = 0"] \arrow[d, "\text{HK quotient}"'] & \mathcal{N}_{\mathrm{H}} \arrow[d, "\text{sympl. quotient}"] \\ \mathcal{M}'_{\mathrm{H}} \arrow[r, "r'=0"] & \mathcal{M}_{\mathrm{H}} \end{tikzcd} \tag{4.46}$$

[4] The one caveat is that the conditions (4.13) are satisfied for arbitrary V parametrising all of $\mathfrak{gdiff}$ not just V with $L_V K = 0$ parametrising $\mathfrak{gdiff}_K$. Thus we need to be sure the conditions arising from the moment maps with restricted V, together with the other supersymmetry conditions (4.14) and (4.15), are sufficient. Although we have not found a general proof, we can see this is true in a number of explicit examples. This is not surprising, since the moment maps only constrain a relatively small independent component $(\mathbf{2}, \mathbf{6})$ of the intrinsic torsion.

where $\mathcal{M}'_{\mathrm{H}} = \mathcal{A}^K_{\mathrm{H}} /\!\!/\!\!/ \mathrm{GDiff}_K$ is a hyper-Kähler manifold, and the final moduli space

$$\mathcal{M}_{\mathrm{H}} = \mathcal{N}_{\mathrm{H}} /\!\!/ \mathrm{GDiff}_K \quad \text{is Kähler.}$$

The vector r' in (4.46) generates the U(1) action on the quotient space $\mathcal{M}'_{\mathrm{H}}$. Since the action of ρ_K is modded out on the quotient space, it is trivial and so the condition becomes just $r' = 0$. However, since r' is still real holomorphic with respect to I, we see that going via $\mathcal{M}'_{\mathrm{H}}$, the space $\mathcal{M}_{\mathrm{H}}$ is again Kähler. One caveat to taking the hyper-Kähler quotient first is that there might be additional solutions to $r' = 0$. Since r is freely acting, we have $r' = 0$ whenever there is a generalised diffeomorphism such that $L_V J_\alpha = \epsilon_{\alpha\beta\gamma}\lambda_\beta J_\gamma$. However, since $L_V K = 0$ as $V \in \mathfrak{gdiff}_K$, we see that such V are generalised Killing vectors. Thus, provided K is the only generalised Killing vector, we can take either path in the diagram (4.46).

We can slightly refine the construction to make a connection to the "HK/QK correspondence" of Haydys [247], which physically is related to the c-map. This also helps the analysis in the case where there are fixed points. Given V satisfying $L_V K = 0$, acting on any generalised tensor α we have

$$[L_V, L_K]\alpha = L_{L_V K}\alpha = 0. \tag{4.47}$$

Thus G_K is in the centre of GDiff_K and as such is a normal subgroup. Thus we can define the quotient group $\mathrm{GDiff}^0_K = \mathrm{GDiff}_K / G_K$ and write GDiff_K as a semi-direct product

$$\mathrm{GDiff}_K = G_K \rtimes \mathrm{GDiff}^0_K. \tag{4.48}$$

We can then perform the hyper-Kähler quotient in two stages: first by the action of G_K and then by GDiff^0_K, as described in symplectic case, for example, in [256]. We can then add one more level to the diagram (4.46)

$$\begin{tikzcd}
\mathcal{A}^K_{\mathrm{H}} \arrow[r, "\rho_K - r = 0"] \arrow[d, "\bullet /\!\!/\!\!/ G_K"'] & \mathcal{N}_{\mathrm{H}} \arrow[d, "\bullet /\!\!/ G_K"] \\
\mathcal{P}_{\mathrm{H}} \arrow[r, "r' = 0"] \arrow[d, "\bullet /\!\!/\!\!/ \mathrm{GDiff}^0_K"'] & \mathcal{Q}_{\mathrm{H}} \arrow[d, "\bullet /\!\!/ \mathrm{GDiff}^0_K"] \\
\mathcal{M}'_{\mathrm{H}} \arrow[r, "r' = 0"] & \mathcal{M}_{\mathrm{H}}
\end{tikzcd} \tag{4.49}$$

Consider the path through the diagram with two commuting Abelian actions on $\mathcal{A}^K_{\mathrm{H}}$ given by G_K and $\mathrm{U}(1) \subset \mathrm{SU}(2)$, with the latter preserving only one linear combination of the three complex structures. This is exactly the set up that appears in the HK/QK correspondence [247]: the hyper-Kähler manifold is $\mathcal{P}_{\mathrm{H}}$ while the quaternionic-Kähler manifold is $\mathcal{A}^K_{\mathrm{H}}/\mathbb{H}^*$.

4.4 AdS_5 backgrounds as ESE spaces

We now discuss the structure of exceptional Sasaki–Einstein (ESE) backgrounds for AdS_5. The generic flux backgrounds for type IIB were analysed in [244], and for M-theory in [240]. Here we first show how the standard type IIB Sasaki–Einstein reduction with five-form flux embeds as an ESE background, and comment on how this extends to the generic case. We then give the ESE form of the generic M-theory background, showing explicitly how the integrability conditions reproduce those given in [240].

4.4.1 Sasaki–Einstein in type IIB

Backgrounds of the form $\mathrm{AdS}_5 \times M$, where the five-dimensional space M is Sasaki–Einstein and there is a non-trivial self-dual five-form flux, are supersymmetric solutions of type IIB supergravity preserving at least eight supercharges [257]. The metric is a product of the form (4.1) with $D = 5$ and a constant warp factor, which we take to be zero. Five-dimensional Sasaki–Einstein spaces admit a nowhere-vanishing vector field ξ, known as the Reeb vector and a pair of two-forms Ω and ω, that together define an $\mathrm{SU}(2) \subset \mathrm{GL}(5; \mathbb{R})$ structure (for a review see for example [258, 259]). They satisfy the algebraic conditions

$$\Omega \wedge \bar{\Omega} = 2\omega \wedge \omega, \qquad \imath_\xi \Omega = \imath_\xi \omega = 0, \qquad \imath_\xi \sigma = 1, \tag{4.50}$$

where σ is the one-form constructed from ξ by lowering the index with the metric (that is $\xi = \sigma^\sharp$). In addition one has the differential conditions

$$\mathrm{d}\sigma = 2m\omega, \qquad \mathrm{d}\Omega = 3\mathrm{i}m\sigma \wedge \Omega, \tag{4.51}$$

where m is the inverse AdS_5 radius, usually normalised to $m = 1$. Such a compactification is supersymmetric provided there is a five-form flux given by

$$\mathrm{d}C = F = 4m\,\mathrm{vol}_5, \tag{4.52}$$

where $\mathrm{vol}_5 = -\frac{1}{2}\sigma \wedge \omega \wedge \omega$.

Note that these conditions imply that the Reeb vector ξ is a Killing vector that preserves σ and ω, but rotates Ω by a phase

$$\mathcal{L}_\xi \sigma = \mathcal{L}_\xi \omega = \mathcal{L}_\xi g = 0, \qquad \mathcal{L}_\xi \Omega = 3\mathrm{i}m\Omega. \tag{4.53}$$

The rotation of Ω corresponds to the R-symmetry of the solution. In what follows we also need the (transverse) complex structure

$$I^m{}_n = -\omega^m{}_n = \tfrac{\mathrm{i}}{4}(\bar{\Omega}^{mp}\Omega_{np} - \Omega^{mp}\bar{\Omega}_{np}), \tag{4.54}$$

which satisfies $I^p{}_m \Omega_{pn} = \mathrm{i}\Omega_{mn}$.

The Sasaki–Einstein geometry defines an "untwisted" HV structure invariant under $\mathrm{SU}^*(6)$

$$\begin{aligned}\tilde{J}_+ &= \tfrac{1}{2}\kappa u^i\Omega + \tfrac{1}{2}\kappa v^i\Omega^\sharp,\\ \tilde{J}_3 &= \tfrac{1}{2}\kappa I + \tfrac{1}{2}\kappa\hat{\tau}^i{}_j + \tfrac{1}{8}\kappa\Omega^\sharp\wedge\bar{\Omega}^\sharp - \tfrac{1}{8}\kappa\Omega\wedge\bar{\Omega},\end{aligned} \tag{4.55}$$

where $u^i = (-\mathrm{i},1)^i$, $v^i = (-1,-\mathrm{i})^i$, $\hat{\tau}$ is given in terms of the second Pauli matrix $\hat{\tau} = -\mathrm{i}\sigma_2$, and the $\mathrm{E}_{6(6)}$-invariant volume is $\kappa^2 = \mathrm{vol}_5$. The V structure invariant under $\mathrm{F}_{4(4)}$ is given by

$$\tilde{K} = \xi - \sigma\wedge\omega. \tag{4.56}$$

Using the adjoint action and the $\mathfrak{e}_{6(6)}$ Killing form from appendix B, one can check that $\tilde{J}_\alpha$ satisfy the $\mathfrak{su}_2$ algebra and are correctly normalised as in (4.3), while using the cubic invariant from appendix B and the algebraic conditions (4.50), one can check that $\tilde{K}$ and $\tilde{J}_\alpha$ satisfy the compatibility conditions (4.7), so that together $\{J_\alpha, K\}$ define a $\mathrm{USp}(6)$ structure. The full "twisted" structures include the four-form potential C as in (4.9), however, in what follows, it will actually be easier to work with the untwisted structures and use the twisted generalised Lie derivative in the differential conditions.

Let us now see how the integrability conditions on σ, ω, Ω and F arise. We turn first to the moment map conditions (4.16). Let $\tilde{V}$ be an untwisted generalised vector. Using the untwisted $\tilde{K}$, we see that the function (4.12) takes the form

$$\gamma(\tilde{V}) = \tfrac{1}{3}\int_M \imath_{\tilde{v}}\sigma\,\mathrm{vol}_5 + \omega\wedge\tilde{\rho}, \tag{4.57}$$

where $\tilde{v}$ and $\tilde{\rho}$ are the vector and three-form components of $\tilde{V}$. As the moment map condition must hold for an arbitrary generalised vector, we can consider each component of $\tilde{V}$ in turn. We begin with the $\tilde{\rho}$ components of μ_3:

$$\begin{aligned}\mu_3(\tilde{\rho}) - \lambda_3\gamma(\tilde{\rho}) &= -\tfrac{1}{8}\int_M \kappa^2(\Omega^\sharp\wedge\bar{\Omega}^\sharp)\lrcorner\mathrm{d}\tilde{\rho} - \tfrac{1}{3}\lambda_3\int_M \tilde{\rho}\wedge\omega\\ &= \int_M \tfrac{1}{2}\mathrm{d}\tilde{\rho}\wedge\sigma - \tfrac{1}{3}\lambda_3\tilde{\rho}\wedge\omega,\end{aligned} \tag{4.58}$$

which vanishes for $\mathrm{d}\sigma = \tfrac{2}{3}\lambda_3\omega$. Next we consider the μ_+ condition, which gives

$$\begin{aligned}\mu_+(\tilde{V}) &\propto \int_M \kappa^2\Omega^\sharp\lrcorner\mathrm{d}(\tilde{\lambda}^1 + \mathrm{i}\tilde{\lambda}^2)\\ &\propto \int_M (\Omega^\sharp\lrcorner\mathrm{vol}_5)\wedge\mathrm{d}(\tilde{\lambda}^1 + \mathrm{i}\tilde{\lambda}^2)\\ &\propto \int_M \mathrm{d}(\sigma\wedge\Omega)\wedge(\tilde{\lambda}^1 + \mathrm{i}\tilde{\lambda}^2).\end{aligned} \tag{4.59}$$

Using $\mathrm{d}\sigma \propto \omega$ from the previous condition, this vanishes for $\sigma\wedge\mathrm{d}\Omega = 0$. Finally we have the $\tilde{v}$ components of μ_3:

$$\begin{aligned}\mu_3(\tilde{v}) - \lambda_3\gamma(\tilde{v}) &= \tfrac{1}{8}\int_M \mathrm{i}\mathcal{L}_{\tilde{v}}\Omega \wedge \sigma \wedge \bar{\Omega} - \mathrm{i}\mathcal{L}_{\tilde{v}}\bar{\Omega} \wedge \sigma \wedge \Omega - 4\imath_{\tilde{v}}F \wedge \sigma - \tfrac{1}{3}\lambda_3 \int_M \imath_{\tilde{v}}\sigma \operatorname{vol}_5 \\ &= \int_M \imath_{\tilde{v}}\sigma\big(\tfrac{1}{4}\mathrm{i}\,\mathrm{d}\Omega \wedge \bar{\Omega} - \tfrac{1}{2}F - \tfrac{1}{3}\lambda_3 \operatorname{vol}_5\big),\end{aligned} \tag{4.60}$$

where we have simplified using the previous conditions. Requiring that the expression above vanishes for all $\tilde{v}$ fixes the flux to $F = \frac{1}{2}\mathrm{i}\,\mathrm{d}\Omega \wedge \bar{\Omega} - \frac{2}{3}\lambda_3 \operatorname{vol}_5$.

For the vector-multiplet structure (4.14), using the expression for the twisted Dorfman derivative, we find

$$\hat{L}_{\tilde{K}}\tilde{K} = \mathcal{L}_\xi\xi + \mathcal{L}_\xi(-\sigma \wedge \omega) - \imath_\xi\big(\mathrm{d}(-\sigma \wedge \omega) - \imath_\xi F_5\big) = -\mathrm{d}\omega, \tag{4.61}$$

which vanishes if ω is closed. Finally, the condition (4.18) on $L_K J_\alpha$, combined with the conditions from the hyper- and vector-multiplet structures, fixes the remaining SU(2) torsion classes and the five-form flux in terms of the cosmological constant. Setting $\lambda_3 = 3m$, we have

$$\mathrm{d}\sigma = 2m\omega, \qquad \mathrm{d}\Omega = 3\mathrm{i}m\sigma \wedge \Omega, \qquad F = 4m \operatorname{vol}_5 . \tag{4.62}$$

We see that we reproduce the full set of Sasaki–Einstein integrability conditions (4.51).

In summary, we have shown that a background consisting of a five-dimensional manifold with an SU(2) structure, and generic five-form flux defines a generalised USp(6) HV structure. Furthermore, requiring that the HV structure is ESE implies that the SU(2) structure is Sasaki–Einstein and the five-form flux takes the correct supersymmetric value.

4.4.2 Generic fluxes in type IIB

Although we will not give the full analysis, let us makes some comments on the case of generic fluxes in type IIB, first considered in [244] and recently reformulated in terms of generalised connections in [142]. In this case, the Killing spinors defines a local U(1) structure and there are a large number of tensors that can be defined in terms of spinor bilinears. The H and V structures for generic backgrounds, as in the Sasaki–Einstein case, can again be written in terms of appropriate spinor bilinears. In particular, it is relatively easy to show that the untwisted V structure takes the form

$$\tilde{K} = \xi + \mathrm{e}^{2\Delta'}\lambda^i + \mathrm{e}^{4\Delta'}\rho, \tag{4.63}$$

where, in terms of the fermion bilinears of [244], we have[5]

$$\xi = K_5^\sharp, \qquad \lambda^1 = \mathrm{e}^{\phi/2}\,\mathrm{Re}\,K_3, \qquad \lambda^2 = A\mathrm{e}^{\phi/2}\,\mathrm{Re}\,K_3 + \mathrm{e}^{-\phi/2}\,\mathrm{Im}\,K_3, \qquad \rho = -\star V, \tag{4.64}$$

where ξ is again the Killing vector for the R-symmetry. As we pointed out in (4.24), the fact that K is a generalised Killing vector means that the generalised Lie derivative along $\tilde{K}$ reduces to a conventional Lie derivative along the Killing direction. For this to be true, the tensor $\tilde{R}$, defined in (4.26), must vanish. This follows from the differential conditions

$$\mathrm{d}(\mathrm{e}^{2\Delta'}K_3) = \mathrm{i}Q \wedge K_3 - \mathrm{e}^{2\Delta}P \wedge \bar{K}_3 - \mathrm{i}_{K_5}G, \tag{4.65}$$

$$\mathrm{d}(\mathrm{e}^{4\Delta'}\star V) = -\imath_\xi F + \tfrac{\mathrm{i}}{2}\mathrm{e}^{2\Delta}(G \wedge \bar{K}_3 - \bar{G} \wedge K_3), \tag{4.66}$$

where G is the complex three-form flux and the other forms are defined in [244]. These conditions are most easily derived directly from the Killing spinor equations.

Recall that there is also a complex bilinear two-form W satisfying

$$D(\mathrm{e}^{6\Delta'}W) + P \wedge \mathrm{e}^{6\Delta'}\bar{W} = \frac{f}{4m}G, \tag{4.67}$$

where f is a constant related to the five-form flux on M. This condition implies that $B^1 + \mathrm{i}B^2 = (4m/f)\mathrm{e}^{6\Delta'}W$ are potentials for the three-form flux G [134]. Using these potentials in (4.9), and the explicit forms of the bilinears given in [244], we then find that the full twisted V structure is given by[6]

$$K = \xi - \sigma \wedge \omega + \imath_\xi C, \tag{4.68}$$

where $\mathrm{d}\sigma = (8m^2/f)\omega$, C is the four-form potential for the five-form flux $F = \mathrm{d}C - \frac{1}{2}F^i \wedge B^j$. In the notation of [244], σ and ω are defined as

$$\sigma = \frac{4m}{f}\mathrm{e}^{4\Delta'}K_4, \qquad \omega = -\mathrm{e}^{4\Delta'}V. \tag{4.69}$$

We see that the form of K is identical to the Sasaki–Einstein case. Furthermore, in [134,249], it was shown that σ is a contact structure, even in the case of generic flux, and ξ is the corresponding Reeb vector. The corresponding contact volume is

$$\tfrac{1}{2}\sigma \wedge \mathrm{d}\sigma \wedge \mathrm{d}\sigma = -\frac{64m^4}{f^2}\mathrm{e}^{3\Delta'}\,\mathrm{vol}_5 = -\frac{64m^4}{f^2}c(K), \tag{4.70}$$

where vol_5 is the volume of M in the Einstein frame, and we see that it is the $\mathrm{E}_{6(6)}$-invariant volume up to a constant.

[5] Note that $\Delta' = \Delta - \frac{1}{4}\phi$ is the warp factor in the Einstein frame, corresponding to that used in [244].

[6] Note that this includes the dressing by the axion-dilaton degrees of freedom. There is a slight subtlety that here we first twist by the B^i potentials defined by W and then dress by the axion-dilaton, whereas previously the transformations were made in the opposite order. Thus strictly the potentials defined by W differ from those we have been using by the axion-dilaton dressing.

4.4.3 Generic fluxes in M-theory

We now consider the most general supersymmetric solutions of eleven-dimensional supergravity of the form $\mathrm{AdS}_5 \times M$, as first discussed in [240]. In this case, the internal six-dimensional space M has a local SU(2) structure characterised by tensor fields constructed as bilinears of the Killing spinor on M. The metric on M always admits a Killing vector corresponding to the R-symmetry of the dual $\mathcal{N} = 1$ superconformal field theory. As we will see, in this case, the embedding of the SU(2) structure into the H and V structures is fairly intricate.

Let us start by summarising the structure of the solution and the relevant spinor bilinears. The metric is a warped product of the form (4.1) with $D = 5$. Locally, the internal metric can be written as

$$\mathrm{d}s^2(M) = \mathrm{d}s^2_{\mathrm{SU}(2)} + \zeta_1^1 + \zeta_2^2, \tag{4.71}$$

where the SU(2) structure on $\mathrm{d}s^2_{\mathrm{SU}(2)}$ is captured by a complex two-form Ω and a real fundamental two-form ω. The volume form is given by

$$\mathrm{vol}_6 = \tfrac{1}{2}\omega \wedge \omega \wedge \zeta_1 \wedge \zeta_2 = \tfrac{1}{4}\Omega \wedge \bar{\Omega} \wedge \zeta_1 \wedge \zeta_2. \tag{4.72}$$

We also have an almost complex structure for $\mathrm{d}s^2_{\mathrm{SU}(2)}$ given by

$$I^m{}_n = -\omega^m{}_n = \tfrac{1}{4}\mathrm{i}(\bar{\Omega}^{mp}\Omega_{np} - \Omega^{mp}\bar{\Omega}_{np}). \tag{4.73}$$

The set of spinor bilinears defined in [240] are[7]

$$\begin{aligned}
\sin\theta &= \bar{\epsilon}^+\epsilon^-, & Y &= \omega - \sin\theta\,\zeta_1 \wedge \zeta_2 = -\mathrm{i}\bar{\epsilon}^+\gamma_{(2)}\epsilon^+, \\
\tilde{\zeta}_1 &= \cos\theta\,\zeta_1 = \bar{\epsilon}^+\gamma_{(1)}\epsilon^+, & Y' &= \zeta_1 \wedge \zeta_2 - \sin\theta\,\omega = \mathrm{i}\bar{\epsilon}^+\gamma_{(2)}\epsilon^-, \\
\tilde{\zeta}_2 &= \cos\theta\,\zeta_2 = \mathrm{i}\bar{\epsilon}^+\gamma_{(1)}\epsilon^-, & X &= -\Omega \wedge (\sin\theta\,\zeta_1 - \mathrm{i}\zeta_2) = \epsilon^{+\mathrm{T}}\gamma_{(3)}\epsilon^+, \\
\tilde{\Omega} &= \cos\theta\,\Omega = \epsilon^{+\mathrm{T}}\gamma_{(2)}\epsilon^-, & V &= \cos\theta\,\omega \wedge \zeta_2 = \bar{\epsilon}^+\gamma_{(3)}\epsilon^-,
\end{aligned} \tag{4.74}$$

where γ are gamma matrices for Cliff(6) in an orthonormal frame for M and the Killing spinor on M is split into ϵ^+ and ϵ^-, where $\epsilon^- \propto \gamma_7\epsilon^+$. In the following we will also need four other, related bilinears

$$\begin{aligned}
-\mathrm{i} \star X &= \epsilon^{+\mathrm{T}}\gamma_{(3)}\epsilon^-, & \star V &= \mathrm{i}\bar{\epsilon}^+\gamma_{(3)}\epsilon^+, \\
\tfrac{1}{3!}Y \wedge Y \wedge Y &= \mathrm{i}\bar{\epsilon}^+\gamma_{(6)}\epsilon^+, & Z &= -\star\tilde{\zeta}_1 = \mathrm{i}\bar{\epsilon}^+\gamma_{(5)}\epsilon^-.
\end{aligned} \tag{4.75}$$

The differential conditions on the SU(2) structure derived from the Killing spinor

[7]Note that, compared with [240], we have relabelled λ to Δ, ζ to θ and K_i to ζ_i. We have also absorbed an overall warp factor into $\mathrm{d}s^2(M)$.

equations are given in (B.9) – (B.16) of [240]: we reproduce those that we need here[8]

$$
\begin{aligned}
\mathrm{d}(\mathrm{e}^{3\Delta}\sin\theta) &= 2m\mathrm{e}^{2\Delta}\tilde{\zeta}_1, & \mathrm{d}(\mathrm{e}^{5\Delta}\tilde{\zeta}_2) &= \star F + 4m\mathrm{e}^{4\Delta}Y, \\
\mathrm{d}(\mathrm{e}^{3\Delta}X) &= 0, & \mathrm{d}(\mathrm{e}^{3\Delta}V) &= \mathrm{e}^{3\Delta}\sin\theta\, F + 2m\mathrm{e}^{2\Delta}\star Y'.
\end{aligned}
\tag{4.76}
$$

One can use the Killing spinor equations to derive additional identities for forms that were not considered in [240] (but are implied by the conditions therein). We find

$$
\mathrm{d}(\mathrm{e}^{\Delta}Y') = -\imath_\xi F, \qquad \mathrm{d}(\mathrm{e}^{\Delta}Z) = \mathrm{e}^{\Delta}Y'\wedge F,
\tag{4.77}
$$

where $\xi = \mathrm{e}^{\Delta}\tilde{\zeta}_2^\sharp$ is the Killing vector that preserves the full solution

$$
\mathcal{L}_\xi F = \mathcal{L}_\xi \Delta = \mathcal{L}_\xi g = 0,
\tag{4.78}
$$

and generates the U(1) R-symmetry. Since the R-symmetry maps $\epsilon^\pm$ to $\mathrm{e}^{\mathrm{i}\alpha}\epsilon^\pm$, Lie derivatives of the spinor bilinears vanish except for

$$
\mathcal{L}_\xi\tilde{\Omega} = 3\mathrm{i}m\tilde{\Omega}, \qquad \mathcal{L}_\xi X = 3\mathrm{i}mX,
\tag{4.79}
$$

as can be derived from the conditions in [240].

Embedding as a generalised structure

The untwisted HV structure is defined in terms of the spinor bilinears as follows. For the $\mathrm{SU}^*(6)$ structure we have

$$
\begin{aligned}
\tilde{J}_+ &= \tfrac{1}{2}\kappa\big(\tilde{\Omega}_R - \mathrm{i}\star X + \mathrm{i}\star X^\sharp\big), \\
\tilde{J}_3 &= -\tfrac{1}{2}\kappa Y_R + \tfrac{1}{2}\kappa\big(\tilde{\zeta}_1\wedge Y - \tilde{\zeta}_1^\sharp \wedge Y^\sharp\big) - \tfrac{1}{2}\kappa\big(\tfrac{1}{3!}Y\wedge Y\wedge Y + \tfrac{1}{3!}Y^\sharp\wedge Y^\sharp\wedge Y^\sharp\big),
\end{aligned}
\tag{4.80}
$$

where $\kappa^2 = \mathrm{e}^{3\Delta}\,\mathrm{vol}_6$ is the $\mathrm{E}_{6(6)}$-invariant volume and $\tilde{\Omega}_R$ and Y_R are sections of $TM\otimes T^*M$, constructed by raising the first index of the corresponding two-form with the metric, that is $(\tilde{\Omega}_R)^m{}_n = g^{mp}\tilde{\Omega}_{pn}$ and $(Y_R)^m{}_n = g^{mp}Y_{pn}$. The $\mathrm{F}_{4(4)}$ structure is given by the generalised Reeb vector

$$
\tilde{K} = \xi - \mathrm{e}^{\Delta}Y' + \mathrm{e}^{\Delta}Z.
\tag{4.81}
$$

Using the adjoint action, $\mathfrak{e}_{6(6)}$ Killing form and cubic invariant given in appendix B, one can check the J_α satisfy an $\mathfrak{su}_2$ algebra and that both structures are correctly normalised. To be sure that together they define an USp(6) structure we also need to check the first compatibility condition in (4.7), or equivalently $\tilde{J}_+\cdot\tilde{K} = 0$. Splitting into vector, two-form

[8] As mentioned, we have absorbed an overall warp factor into the metric on M, so that the powers of Δ appearing here are different to those in [240].

and five-form components, we find

$$
\begin{aligned}
\tilde{J}_+ \cdot \tilde{K}\big|_{TM} &\propto \tilde{\Omega}_R \cdot \tilde{\zeta}_2^\sharp - \mathrm{i}(\star X)^\sharp \lrcorner Y' = 0, \\
\tilde{J}_+ \cdot \tilde{K}\big|_{\wedge^2 T^*M} &\propto \tilde{\Omega}_R \cdot Y' + \mathrm{i}\tilde{\zeta}_2^\sharp \lrcorner (\star X) - \mathrm{i}(\star X)^\sharp \lrcorner Z = 0, \\
\tilde{J}_+ \cdot \tilde{K}\big|_{\wedge^5 T^*M} &\propto \tilde{\Omega}_R \cdot Z + \mathrm{i}(\star X) \wedge Y' = 0,
\end{aligned}
\tag{4.82}
$$

where we have used the expressions for the spinor bilinears in terms of the SU(2) structure to see that each term vanishes. The full structures will be twisted by the three-form gauge potential A as in (4.8). However, it is again actually easier to work with the untwisted structures and use the twisted generalised Lie derivative in the differential conditions.

Integrability

We now turn to the integrability conditions starting with the moment maps (4.16). Let $\tilde{V} = \tilde{v} + \tilde{\omega} + \tilde{\sigma}$ be an untwisted generalised vector. The function (4.12) then takes the form

$$
\gamma(\tilde{V}) = -\tfrac{1}{3} \int_M \mathrm{e}^{2\Delta} \big(\tilde{\zeta}_1 \wedge \tilde{\sigma} + \star Y' \wedge \tilde{\omega} - \imath_{\tilde{v}} Y' \wedge Z \big). \tag{4.83}
$$

We first consider μ_3. The moment map is a sum of terms that depend on arbitrary $\tilde{v}$, $\tilde{\omega}$ and $\tilde{\sigma}$, so we can consider each component in turn. The $\tilde{\sigma}$ component is

$$
\begin{aligned}
\mu_3(\tilde{\sigma}) - \lambda_3 \gamma(\tilde{\sigma}) &= \tfrac{1}{16}\mathrm{i} \int_M \kappa^2 \big(\star \bar{X}^\sharp \wedge \star X^\sharp \big) \lrcorner \mathrm{d}\tilde{\sigma} + \tfrac{1}{3}\lambda_3 \int_M \mathrm{e}^{2\Delta} \tilde{\zeta}_1 \wedge \tilde{\sigma} \\
&= \tfrac{1}{2} \int_M \mathrm{e}^{3\Delta} \sin\theta \, \mathrm{d}\tilde{\sigma} + \tfrac{1}{3}\lambda_3 \int_M \mathrm{e}^{2\Delta} \tilde{\zeta}_1 \wedge \tilde{\sigma} \\
&= -\tfrac{1}{2} \int_M \mathrm{d}(\mathrm{e}^{3\Delta} \sin\theta) \wedge \tilde{\sigma} + \tfrac{1}{3}\lambda_3 \int_M \mathrm{e}^{2\Delta} \tilde{\zeta}_1 \wedge \tilde{\sigma}.
\end{aligned}
\tag{4.84}
$$

Remembering that $\lambda_3 = 3m$, this vanishes for

$$
\mathrm{d}(\mathrm{e}^{3\Delta} \sin\theta) = 2m \mathrm{e}^{2\Delta} \tilde{\zeta}_1. \tag{4.85}
$$

This is the first differential condition in (4.76). The $\tilde{\omega}$ component is

$$
\begin{aligned}
\mu_3(\tilde{\omega}) - \lambda_3 \gamma(\tilde{\omega}) &= \tfrac{1}{16}\mathrm{i} \int_M \kappa^2 \Big(\mathrm{i}\big(\bar{\tilde{\Omega}}_R \cdot \star X^\sharp + \tilde{\Omega}_R \cdot \star \bar{X}^\sharp \big) \lrcorner \mathrm{d}\tilde{\omega} + \big(\star \bar{X}^\sharp \wedge \star X^\sharp \big) \lrcorner (\tilde{\omega} \wedge F) \Big) \\
&\quad + \tfrac{1}{3}\lambda_3 \int_M \mathrm{e}^{2\Delta} \star Y' \wedge \tilde{\omega} \\
&= -\tfrac{1}{2} \int_M \Big(\mathrm{e}^{3\Delta} V \wedge \mathrm{d}\tilde{\omega} - \sin\theta \mathrm{e}^{3\Delta} \tilde{\omega} \wedge F \Big) + \tfrac{1}{3}\lambda_3 \int_M \mathrm{e}^{2\Delta} \star Y' \wedge \tilde{\omega} \\
&= -\tfrac{1}{2} \int_M \Big(\mathrm{d}(\mathrm{e}^{3\Delta} V) \wedge \tilde{\omega} - \sin\theta \mathrm{e}^{3\Delta} \tilde{\omega} \wedge F \Big) + \tfrac{1}{3}\lambda_3 \int_M \mathrm{e}^{2\Delta} \star Y' \wedge \tilde{\omega}.
\end{aligned}
\tag{4.86}
$$

This vanishes for

$$\mathrm{d}(\mathrm{e}^{3\Delta}V) = \mathrm{e}^{3\Delta}\sin\theta F + 2m\mathrm{e}^{2\Delta} \star Y'. \tag{4.87}$$

This is the fourth differential condition in (4.76). The $\tilde{v}$ component is rather long but can be shown to vanish as a result of the differential conditions in (4.76). For the μ_+ moment map, the contribution from terms containing $\tilde{\sigma}$ vanishes without imposing any differential conditions. The contribution from the $\tilde{\omega}$ terms simplifies to

$$\mu_+(\tilde{\omega}) = -\tfrac{\mathrm{i}}{2}\int_M \mathrm{e}^{3\Delta}X \wedge \mathrm{d}\tilde{\omega} = -\tfrac{\mathrm{i}}{2}\int_M \mathrm{d}(\mathrm{e}^{3\Delta}X) \wedge \tilde{\omega}. \tag{4.88}$$

This vanishes after imposing the third differential condition in (4.76)

$$\mathrm{d}(\mathrm{e}^{3\Delta}X) = 0. \tag{4.89}$$

The $\tilde{v}$ component is again somewhat involved but can be shown to vanish as a result of the conditions in (4.76).

For the vector-multiplet structure we first use the condition (4.24), which, substituting for $\tilde{K}$ in (4.26), gives

$$\tilde{R} = -\mathrm{d}(\mathrm{e}^{\Delta}Y') - \imath_\xi F + \mathrm{d}(\mathrm{e}^{\Delta}Z) - \mathrm{e}^{\Delta}Y' \wedge F = 0, \tag{4.90}$$

which reproduces the two equations in (4.77). We then have

$$\hat{L}_{\tilde{K}}\tilde{K} = \mathcal{L}_\xi \tilde{K} = 0, \tag{4.91}$$

since the bilinears $\xi = \mathrm{e}^{\Delta}\zeta_2^\sharp$, Y' and Z are all invariant. Finally we have the condition (4.18) which, given (4.79), reads

$$\hat{L}_K \tilde{J}_+ = \mathcal{L}_\xi \tilde{J}_+ = 3\mathrm{i}m\tilde{J}_+, \tag{4.92}$$

in agreement with $\lambda_3 = 3m$.

In summary, we have shown that the most general AdS_5 solutions of eleven-dimensional supergravity do indeed define an exceptional Sasaki–Einstein space.

4.5 AdS_4 backgrounds as ESE spaces

We now discuss the structure of exceptional Sasaki–Einstein (ESE) backgrounds for AdS_4. We first show how the standard M-theory Sasaki–Einstein reduction with seven-form flux embeds as an ESE background, and comment on how this extends to the generic case, given in [245].

4.5.1 Sasaki–Einstein in M-theory

We now briefly discuss the structure of exceptional Sasaki–Einstein (ESE) backgrounds for AdS_4, focussing on the example of conventional Sasaki–Einstein geometry in M-theory. These are supersymmetric solutions preserving at least eight supercharges [64], and are dual to a three-dimensional superconformal field theory living on a stack of M2-branes placed at the tip of the corresponding Calabi–Yau cone.

The metric is a product of the form (4.1) with $D = 4$ and a constant warp factor, which we take to be zero. Seven-dimensional Sasaki–Einstein spaces admit a nowhere-vanishing vector field ξ, known as the Reeb vector, a complex three-form Ω and real two-form ω, which together define an $\mathrm{SU}(3) \subset \mathrm{GL}(7;\mathbb{R})$ structure. They satisfy the algebraic conditions

$$\tfrac{1}{8}\mathrm{i}\Omega \wedge \bar{\Omega} = \tfrac{1}{3!}\omega \wedge \omega \wedge \omega, \qquad \imath_\xi \Omega = \imath_\xi \omega = 0, \qquad \imath_\xi \sigma = 1, \tag{4.93}$$

where σ is the one-form constructed from ξ by lowering the index with the metric. In addition one has the differential conditions

$$\mathrm{d}\sigma = m\omega, \qquad \mathrm{d}\Omega = 2\mathrm{i}m\sigma \wedge \Omega, \tag{4.94}$$

where m is the inverse AdS_4 radius, usually normalised to $m = 2$. Such a compactification is supersymmetric provided there is a seven-form flux given by

$$\mathrm{d}\tilde{A} = \tilde{F} = -3m\,\mathrm{vol}_7, \tag{4.95}$$

where $\mathrm{vol}_7 = \frac{1}{3!}\sigma \wedge \omega \wedge \omega \wedge \omega$. (Recall that $\tilde{F}$ is the Hodge-dual of the four-form flux $\mathcal{F} = 6m\,\mathrm{vol}(\mathrm{AdS}_4)$ in eleven-dimensions.) These conditions imply that the Reeb vector ξ is a Killing vector that preserves σ and ω, but rotates Ω by a phase

$$\mathcal{L}_\xi \sigma = \mathcal{L}_\xi \omega = \mathcal{L}_\xi g = 0, \qquad \mathcal{L}_\xi \Omega = 2\mathrm{i}m\Omega. \tag{4.96}$$

The rotation of Ω corresponds to the R-symmetry of the $\mathcal{N} = 2$ solution. In what follows we also need the (transverse) complex structure

$$I^m{}_n = -\omega^m{}_n = \tfrac{1}{8}\mathrm{i}(\bar{\Omega}^{mpq}\Omega_{npq} - \Omega^{mpq}\bar{\Omega}_{npq}), \tag{4.97}$$

which satisfies $I^q{}_m \Omega_{qnp} = \mathrm{i}\Omega_{mnp}$. For simplicity of presentation, we assume that the four-form flux and warp factor vanish, though one can show that these also follow from the integrability conditions.

The HV structure defined by the $\mathrm{SU}(3)$ structure is actually the same as the example considered the previous chapter, namely a Calabi–Yau threefold times a circle. The difference between the two is in the differential conditions on the $\mathrm{SU}(3)$ invariant forms.

We have the untwisted tensors

$$\begin{aligned}
\tilde{J}_+ &= \tfrac{\kappa}{2}\Omega - \tfrac{\kappa}{2}\Omega^\sharp, \\
\tilde{J}_3 &= \tfrac{\kappa}{2}I - \tfrac{\kappa}{2}\tfrac{\mathrm{i}}{8}\Omega\wedge\bar{\Omega} - \tfrac{\kappa}{2}\tfrac{\mathrm{i}}{8}\Omega^\sharp\wedge\bar{\Omega}^\sharp,
\end{aligned} \tag{4.98}$$

where $\kappa^2 = \mathrm{vol}_7$ is the $\mathrm{E}_{7(7)}$-invariant volume and

$$\tilde{X} = \xi + \mathrm{i}\omega - \tfrac{1}{2}\sigma\wedge\omega\wedge\omega - \mathrm{i}\sigma\otimes\mathrm{vol}_7\,. \tag{4.99}$$

Using the adjoint action, the symplectic invariant and the $\mathfrak{e}_{7(7)}$ Killing form, one can check that $\tilde{J}_\alpha$ generate an $\mathfrak{su}_2$ algebra and that both structures are correctly normalised and are compatible, as in (4.3) and (4.7).

We now show how the integrability conditions on the SU(3) structure arise by requiring $\{J_\alpha, K\}$ to be ESE. Starting with the moment maps (4.16), we note that if $\tilde{V} = \tilde{v}+\tilde{\omega}+\tilde{\sigma}+\tilde{\tau}$ is an arbitrary untwisted generalised vector, then

$$\gamma(\tilde{V}) = \int_M s(\tilde{V},\tilde{\tilde{K}}) = -\tfrac{1}{4}\int_M (\imath_{\tilde{v}}\sigma\,\mathrm{vol}_7 + \tilde{\sigma}\wedge\omega). \tag{4.100}$$

Starting with μ_3, the terms that depend on $\tilde{\sigma}$ are

$$\begin{aligned}
\mu_3(\tilde{\sigma}) - \lambda_3\gamma(\tilde{\sigma}) &= -\tfrac{1}{16}\mathrm{i}\int_M \kappa^2(\bar{\Omega}^\sharp\wedge\Omega^\sharp)\lrcorner\mathrm{d}\tilde{\sigma}) + \tfrac{1}{4}\lambda_3\int_M \tilde{\sigma}\wedge\omega \\
&= -\tfrac{1}{2}\int_M \mathrm{d}\tilde{\sigma}\wedge\sigma + \tfrac{1}{4}\lambda_3\int_M \tilde{\sigma}\wedge\omega \\
&= -\tfrac{1}{2}\int_M \tilde{\sigma}\wedge\mathrm{d}\sigma + \tfrac{1}{4}\lambda_3\int_M \tilde{\sigma}\wedge\omega,
\end{aligned} \tag{4.101}$$

which vanishes for $\mathrm{d}\sigma = \tfrac{1}{2}\lambda_3\omega$. The μ_+ moment map is

$$\begin{aligned}
\mu_+(\tilde{V}) &= -\tfrac{1}{2}\mathrm{i}\int_M -\tfrac{1}{4}\kappa^2\,\mathrm{tr}\big(I\cdot(j\Omega^\sharp\lrcorner j\mathrm{d}\tilde{\omega})\big) + \tfrac{1}{24}\kappa^2(\omega^\sharp\wedge\omega^\sharp\wedge\omega^\sharp)\lrcorner(\mathrm{d}\tilde{\omega}\wedge\Omega) \\
&= -\tfrac{1}{8}\mathrm{i}\int_M 3\mathrm{i}\kappa^2\Omega^\sharp\lrcorner\mathrm{d}\tilde{\omega} + \sigma\wedge\mathrm{d}\tilde{\omega}\wedge\Omega \\
&= \tfrac{1}{2}\mathrm{i}\int_M \sigma\wedge\Omega\wedge\mathrm{d}\tilde{\omega},
\end{aligned} \tag{4.102}$$

which, using $\mathrm{d}\sigma\propto\omega$ from above, vanishes for $\sigma\wedge\mathrm{d}\Omega = 0$. In the language of [224], this fixes the torsion classes $\{\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_5\}$ to zero. Finally, the $\tilde{v}$ components of μ_3 are

$$\begin{aligned}
\mu_3(\tilde{v}) - \lambda_3\gamma(\tilde{v}) &= -\tfrac{1}{16}\mathrm{i}\int_M \kappa\bar{\Omega}^\sharp\lrcorner\mathcal{L}_{\tilde{v}}(\kappa\Omega) + \mathcal{L}_{\tilde{v}}(\kappa\Omega^\sharp)\lrcorner\kappa\bar{\Omega} - \kappa^2(\bar{\Omega}^\sharp\wedge\Omega^\sharp)\lrcorner\imath_{\tilde{v}}\tilde{F} \\
&\qquad - \tfrac{1}{4}\lambda_3\int_M \imath_{\tilde{v}}\sigma\,\mathrm{vol}_7 \\
&= \tfrac{1}{8}\int_M \big(\imath_{\tilde{v}}\sigma\,\mathrm{d}\Omega\wedge\bar{\Omega} + 4\,\imath_{\tilde{v}}\sigma\,\tilde{F}\big) - \tfrac{1}{4}\lambda_3\int_M \imath_{\tilde{v}}\sigma\,\mathrm{vol}_7,
\end{aligned} \tag{4.103}$$

where we have used the previous results to reach the final line. Requiring this to vanish fixes the flux to $\tilde{F} = \frac{1}{2}\lambda_3 \operatorname{vol}_7 - \frac{1}{4}\mathrm{d}\Omega \wedge \bar{\Omega}$.

For the vector-multiplet structure, using the expression for the twisted generalised Lie derivative (4.25) and (4.26), we find

$$\hat{L}_{\tilde{K}}\tilde{K} = \mathcal{L}_\xi \xi + \mathcal{L}_\xi(-\tfrac{1}{2}\sigma \wedge \omega \wedge \omega) - \imath_\xi\big(\mathrm{d}(-\tfrac{1}{2}\sigma \wedge \omega \wedge \omega) - \imath_\xi \tilde{F}\big) = -\mathrm{d}\omega \wedge \omega, \tag{4.104}$$

so that integrability implies $\mathrm{d}\omega \wedge \omega = 0$. In the language of [224], the torsion classes corresponding to $\{\mathcal{W}_4, E + \bar{E}, V_2, T_2\}$ must vanish. Finally, the conditions from (4.17) combined with those from the H and V structures fix the remaining SU(3) torsion classes to $S = 0$ and $E = \mathrm{i}\lambda_3$, so that, with $\lambda_3 = 2m$, we have

$$\mathrm{d}\sigma = m\omega, \qquad \mathrm{d}\Omega = 2\mathrm{i}m\sigma \wedge \Omega, \qquad \tilde{F} = -3m \operatorname{vol}_7 . \tag{4.105}$$

We see we reproduce the full set of Sasaki–Einstein integrability conditions.

In summary, we have shown that a background consisting of a seven-dimensional manifold with an SU(3) structure and generic seven-form flux defines a generalised SU(6) structure. Furthermore, requiring that the HV structure is ESE implies the manifold must be Sasaki–Einstein and the seven-form flux matches that of the standard supersymmetry-preserving solution.

4.5.2 Generic fluxes in M-theory

Although we will not give the full analysis, let us now discuss some aspects of how the previous analysis extends to the case of generic fluxes in M-theory, first considered in [245]. In this case, the Killing spinors define a local SU(2) structure. The H and V structures for generic backgrounds, as in the Sasaki–Einstein case, can be written in terms of appropriate spinor bilinears. Assuming the seven-form $\tilde{F}$ is non-zero, it is relatively straightforward to show that the complex untwisted V structure takes the form

$$\tilde{X} = \xi + \mathrm{e}^{3\Delta} Y + \mathrm{e}^{6\Delta} Z - \mathrm{i}\,\mathrm{e}^{9\Delta}\tau, \tag{4.106}$$

where, in terms of the fermion bilinears, using the notation of [245], we have

$$\xi = \mathrm{i}\bar{\chi}^c_+ \gamma^{(1)} \chi_-, \qquad Y = \mathrm{i}\bar{\chi}^c_+ \gamma_{(2)} \chi_-, \qquad Z = \star Y, \qquad \tau = \xi^\flat \otimes \operatorname{vol}_7 . \tag{4.107}$$

The tensors Y and Z are generically complex, but, as shown in [245], ξ is real, so there is no vector component in the imaginary part of X, consistent with the general argument given at the end of section 4.2.2. The generalised Lie derivative along the real part of $\tilde{X}$ generates the R-symmetry, and so must reduce to a conventional Lie derivative along ξ.

We indeed find that the tensor $\tilde{R}$, defined in (4.26), vanishes due to

$$\mathrm{d}(\mathrm{e}^{3\Delta}Y) = \imath_\xi F, \tag{4.108}$$

$$\mathrm{d}(\mathrm{e}^{6\Delta}Z) = \imath_\xi \tilde{F} - \mathrm{e}^{3\Delta}Y \wedge F, \tag{4.109}$$

where the first is given in [245] and the second can be derived from the Killing spinor equations.

Recall also that there is also a spinor bilinear three-form satisfying

$$\mathrm{d}\big(\mathrm{e}^{6\Delta}\,\mathrm{Im}(\bar{\chi}^c_+\gamma_{(3)}\chi_-)\big) = \frac{\tilde{f}}{3m}F. \tag{4.110}$$

Compared with the expression given in [245], we have reinstated the inverse AdS radius m (set to $m=2$ in [245]), and $\tilde{f}$ (denoted by m in [245]) parametrises the seven-form flux, namely $\tilde{F} = -\tilde{f}\,\mathrm{vol}_7$. We see $(3m/\tilde{f})\mathrm{e}^{6\Delta}\,\mathrm{Im}(\bar{\chi}^c_+\gamma_{(3)}\chi_-)$ is a potential for the four-form flux F. Using this potential in (4.8) and the explicit forms of the bilinears given in [245], we then find that the full twisted V structure is given by

$$X = \mathrm{e}^{\tilde{A}}\Big[\xi + \mathrm{i}\omega - \tfrac{1}{2}\sigma\wedge\omega\wedge\omega - \mathrm{i}\sigma\otimes\big(\tfrac{1}{3!}\sigma\wedge\omega\wedge\omega\wedge\omega\big)\Big], \tag{4.111}$$

where $\mathrm{d}\sigma = (3m^2/\tilde{f})\omega$. In particular, the real part is given by

$$K = \xi - \tfrac{1}{2}\sigma\wedge\omega\wedge\omega + \imath_\xi\tilde{A}. \tag{4.112}$$

We see that the form of X matches that of the Sasaki–Einstein case (4.99). It was shown in [245] that σ is a contact structure, even in the case of generic flux, and ξ is the corresponding Reeb vector. The corresponding contact volume is

$$\tfrac{1}{3!}\sigma\wedge\mathrm{d}\sigma\wedge\mathrm{d}\sigma\wedge\mathrm{d}\sigma = \frac{27m^6}{\tilde{f}^3}\left(\tfrac{3m^2}{\tilde{f}}\right)^3\mathrm{e}^{9\Delta}\,\mathrm{vol}_7 = \frac{27m^6}{\tilde{f}^3}2\sqrt{q(K)}, \tag{4.113}$$

where vol_7 is the volume of M. Again it is simply a constant times the $\mathrm{E}_{7(7)}$-invariant volume.

4.6 Central charges, BPS wrapped branes and volume minimisation

Of the many field theory properties that can be determined from the dual geometry, two of the most studied are the central charge a or free energy $\mathcal{F}$ of the theory and the conformal dimension of operators that arise from supersymmetric wrapped branes. The key point of this section is that they are all encoded, in a universal way, by the generalised Reeb vector K. This also leads to a conjecture as to how the dual description of a-maximisation in $D=4$ and $\mathcal{F}$-maximisation in $D=3$ appears.

We have considered three ESE geometries in this chapter: AdS_5 in type IIB and M-theory and AdS_4 in M-theory. The generic generalised Reeb vector in each case is given by

$$K = \begin{cases} \xi - \sigma \wedge \omega + \imath_\xi C, & \mathrm{AdS}_5 \text{ in type IIB}, \\ \xi - \mathrm{e}^{\Delta} Y' + \mathrm{e}^{\Delta} Z + \imath_\xi A - A \wedge \mathrm{e}^{\Delta} Y', & \mathrm{AdS}_5 \text{ in M-theory}, \\ \xi - \frac{1}{2}\sigma \wedge \omega \wedge \omega + \imath_\xi \tilde{A}, & \mathrm{AdS}_4 \text{ in M-theory}, \end{cases} \tag{4.114}$$

where in the last case we are assuming the seven-form flux $\tilde{F}$ is non-trivial and in the first that five-form flux F is non-trivial. Each K is a generalised Killing vector that generates the global R-symmetry of the dual field theory. It is a combination of diffeomorphism (parametrised by ξ) and gauge transformation (parametrised by the p-form components), under which the transformations of the metric g and gauge potentials vanish, as in (4.22) and (4.23). For AdS_5 in type IIB [134, 249] and AdS_4 in M-theory [245], the generic geometry admits a canonical contact structure σ. As we have already noted, it is striking that this structure is equivalent to specifying the generalised Reeb vector K, where the integrability arises from requiring that K is generalised Killing.

For AdS_5 solutions the central charge a of the dual field theory is given by [260]

$$a = \frac{\pi}{8m^3 G_5}, \tag{4.115}$$

where G_5 is the effective five-dimensional Newton's constant. Using the results of [249] and [261], one finds that for both the generic type IIB and M-theory background the inverse of G_5 is given by the integral of the $\mathrm{E}_{6(6)}$-invariant volume

$$G_5^{-1} \propto \int_M \mathrm{e}^{3\Delta} \,\mathrm{vol} = \int_M c(K). \tag{4.116}$$

As review in appendix G, quantising so we have N units of background flux and fixing this integer N in the expression for a reverses the dependence on the invariant volume. This leads to a universal expression for the central charge in terms of the generalised Reeb vector, applicable to both type IIB and M-theory

$$a^{-1} \propto \int_M c(K), \tag{4.117}$$

where in type IIB the constant of proportionality scales as N^{-2} and in M-theory as N^{-3}. Recall that for type IIB, $c(K)$ is proportional to the contact volume $\frac{1}{2}\sigma \wedge \mathrm{d}\sigma \wedge \mathrm{d}\sigma$.

A similar formula for the free energy of the field theory on a three-sphere can be derived for generic AdS_4 backgrounds following [245]. The real part of the free energy is equal to the gravitational free energy and is given by

$$\mathcal{F} = \frac{\pi}{2m^2 G_4}, \tag{4.118}$$

where the four-dimensional Newton's constant is given by the $E_{7(7)}$-invariant volume

$$G_4^{-1} \propto \int_M e^{2\Delta} \operatorname{vol}_7 = \int_M 2\sqrt{q(K)}. \tag{4.119}$$

Fixing the quantised background flux then gives, as in [255],

$$\mathcal{F}^{-2} \propto \int_M \sqrt{q(K)}, \tag{4.120}$$

where the constant of proportionality scales as N^{-3}. Again, $\sqrt{q(K)}$ is proportional to the contact volume, $\frac{1}{3!}\sigma \wedge d\sigma \wedge d\sigma \wedge d\sigma$. Although we have not considered type IIB AdS_4 backgrounds, we expect that the same formula for the free energy holds since $q(K)$ (and $c(K)$ in the AdS_5 case) are U-duality invariants.

Let us now discuss how the properties of chiral operators in the dual SCFT coming from wrapped branes are encoded by K. For definiteness, we will focus on AdS_5 in type IIB. A probe D3-brane wrapping a supersymmetric three-cycle Σ_3 in M_5 gives rise to a BPS particle in AdS_5. The particle appears as the excitation of a field that couples to a chiral primary operator $\mathcal{O}_3$, and thus the probe D3-brane corresponds to a BPS operator in the dual field theory. The (warped) volume of the wrapped D3-brane is then associated to the conformal dimension of the operator $\Delta(\mathcal{O}_3)$, which in turn is proportional to the R-charge. In order for the three-cycle to be supersymmetric, it must be calibrated by a (generalised) three-form calibration. There are many ways to find this calibration, including using spinor bilinears of the full ten-dimensional Killing spinors or checking the κ-symmetry conditions directly.

A similar story applies to probe M2-branes wrapping supersymmetric two-cycles in M_6 and probe M5-branes wrapping supersymmetric five-cycles in M_7, corresponding to chiral primary operators in the dual four- and three-dimensional SCFTs. For all three cases, the relevant calibration form is known and the conformal dimensions of the corresponding operators are given by

$$\begin{aligned}
&\text{D3-branes in AdS}_5 \text{ [134, 249]:} && \Delta(\mathcal{O}_3) = -\frac{T_{\mathrm{D3}}}{m}\int_{\Sigma_3} \sigma \wedge \omega, \\
&\text{M2-branes in AdS}_5 \text{ [261]:} && \Delta(\mathcal{O}_2) = \frac{T_{\mathrm{M2}}}{m}\int_{\Sigma_2} e^{\Delta} Y', \\
&\text{M5-branes in AdS}_4 \text{ [245]:} && \Delta(\mathcal{O}_5) = -\frac{T_{\mathrm{M5}}}{m}\int_{\Sigma_5} \tfrac{1}{2}\sigma \wedge \omega \wedge \omega,
\end{aligned} \tag{4.121}$$

where $T_\bullet$ is the tension of the brane wrapping the cycle. From (4.114) we see that the relevant calibration form appears in the generalised Reeb vector K, implying that the components of K are the (generalised) calibrations that define supersymmetric cycles. This is not surprising since K is defined as a bilinear of the Killing spinors and imposing that L_K reduces to $\mathcal{L}_\xi$ requires the components of K to satisfy equations that resemble generalised calibration conditions. For backgrounds with non-trivial fluxes, the calibration condition is equivalent to asking that the energy of the wrapped brane is minimised. This suggests that the generalised calibration should be given by the twisted K. Notice however that, for the branes we discussed above, most of the potentials have vanishing pull-back on the wrapped cycle and hence do not contribute to the conditions (4.121). We leave for future work a more detailed analysis of how calibrations appear in this language.

As we have seen, the generalised Reeb vector K encodes the central charge or free energy of the dual field theory. For some time, a classic problem in four-dimensional $\mathcal{N}=1$ SCFTs was to find the correct U(1) symmetry that gives the R-symmetry as the theory flows from the UV to the IR. A general procedure for determining this was given by Intriligator and Wecht [253], namely a-maximisation. For three-dimensional $\mathcal{N}=2$ theories the analogous procedure consists of maximising the free energy [254, 255]. (Both cases can also be thought of as minimising the coefficient τ_{RR} of the two-point function of the R-symmetry current [262].) The bulk version of this process is known as volume minimisation [251, 252], and was originally derived for Sasaki–Einstein backgrounds, but a version also appears to hold for the case of generic type IIB backgrounds [250]. The idea is to relax the supersymmetry conditions slightly and show that the resulting supergravity action depends only on the choice of Reeb vector, ξ. The actual supersymmetric background then appears after minimising over the possible choices of ξ.

This leads to a natural question: what is the dual of a-maximisation (or $\mathcal{F}$-maximisation) in our language? Comparing with [250–252] there is a very natural candidate for relaxing the supersymmetry conditions, namely simply to drop the normalisation conditions $\kappa^2 = c(K)$ in $D=5$ and $\kappa^2 = 2\sqrt{q(K)}$ in $D=4$, defining a notion of an "exceptional Sasaki structure". Following the analogous analysis to that given in appendix F, we find this requires that the moment map conditions are slightly modified, giving

Definition. An *exceptional Sasaki structure* is a pair $\{J_\alpha, K\}$ of H and V structures satisfying $J_\alpha \cdot K = 0$ and the integrability conditions

$$\mu_\alpha(V) = \lambda_\alpha \int_M \phi(V) \qquad \forall V \in \Gamma(E), \tag{4.122}$$

$$L_K K = 0, \tag{4.123}$$

$$L_K J_\alpha = \epsilon_{\alpha\beta\gamma}\lambda_\beta J_\gamma, \qquad L_{\hat{K}} J_\alpha = 0, \tag{4.124}$$

where $\phi(V)$ is given by

$$\phi(V) = \begin{cases} \kappa^2 q(V,K,K,K)/q(K), & \text{for } D=4 \\ \kappa^2 c(V,K,K)/c(K), & \text{for } D=5 \end{cases} \tag{4.125}$$

where $\operatorname{tr}(J_\alpha J_\beta) = -\kappa^2\delta_{\alpha\beta}$ and λ_α are real constants, as in the definition of an ESE structure. The condition $L_{\hat{K}} J_\alpha = 0$ is relevant only for $D=4$.

An interesting open question is whether in the $D = 5$ type IIB case this agrees with the notion of a generalised Sasaki structure defined in [250]. The natural conjecture is then that, over the space of such structures, the supergravity action restricted to the internal space M is given by

$$S_{\text{sugra}} \propto \int_M \sqrt{q(K)}, \qquad \text{and} \qquad S_{\text{sugra}} \propto \int_M c(K), \tag{4.126}$$

for $D = 4$ and $D = 5$ respectively, and so depends only on the generalised Reeb vector. Extremising over the space of K then selects the generalised Reeb vector that corresponds to the actual R-symmetry.

Motivation for this formulation comes from the fact, already noted in section 4.3.1, that the supersymmetry conditions for an ESE structure can be interpreted in terms of gauged $D = 4$ or $D = 5$ supergravity with infinite dimensional spaces of hyper- and vector-multiplets. Various authors have considered the dual of a- and $\mathcal{F}$-maximisation from the point of view of a conventional dual gauged $D = 5$ or $D = 4$ supergravity [263–265], and showed explicitly that they correspond to extremising over the space of possible R-symmetries either, in $D = 5$, the cubic function that determines the real special geometry of the vector multiplets [263, 264], or, in $D = 4$, the real function that determines the special Kähler geometry of the vector multiplets [265]. In our language, this corresponds to varying K and extremising the integral of either $c(K)$ or $\sqrt{q(K)}$, exactly as we conjecture above.

Showing that such a procedure works would provide the dual of a- and $\mathcal{F}$-maximisation not only for an arbitrary flux background, generalising the Sasaki–Einstein cases in IIB on AdS_5 and M-theory on AdS_4, but also for the generic M-theory AdS_5 background for which no notion of volume minimisation exists. It may also provide insight into exactly what space of solutions one is extremising over in the flux case.

4.7 Summary

In this chapter we have given a new geometrical interpretation of generic AdS flux backgrounds preserving eight supercharges within generalised geometry. These "exceptional Sasaki–Einstein" (ESE) geometries are the natural string generalisations of Sasaki–Einstein spaces in five and seven dimensions. The geometries always admit a "generalised Reeb vector" that generates an isometry of the background corresponding to the R-symmetry of the dual field theory. In the language of [145], ESE spaces are weak generalised holonomy spaces, and the cone over such a space has generalised special holonomy. We have included a number of examples of ESE spaces including conventional Sasaki–Einstein in five and seven dimensions, as well as the most general AdS_5 solutions in M-theory. We also discussed the structure of the moduli spaces of ESE spaces, pointing out an interesting connection to the "HK/QK correspondence" [247].

A particular advantage of the formalism is that the generalised H and V structures defining the background are associated to hypermultiplet and vector-multiplet degrees of freedom in the corresponding gauged supergravity, providing a natural translation between bulk and boundary properties. We showed for example that the V structure, which is defined by the generalised Reeb vector K, encodes the contact structure that appears in generic $D = 5$ IIB and $D = 4$ M-theory backgrounds [134, 245, 249]. Furthermore K determines the central charge in $D = 5$ and free energy in $D = 4$ of the dual theory, and is a calibration for BPS wrapped branes giving the dimension of the dual operators. In the examples with contact structures, this framework allows one to calculate properties of the field theory using the relation between the contact volume and the choice of Reeb vector [134, 245, 249]. The special role of K also led us, following [250], to a conjecture for the generic form of volume minimisation [251, 252].

Chapter 5

Marginal deformations of $d = 4$, $\mathcal{N} = 1$ SCFTs

In this chapter we apply exceptional generalised geometry to the study of exactly marginal deformations of $\mathcal{N} = 1$ SCFTs that are dual to generic AdS_5 flux backgrounds in type IIB or eleven-dimensional supergravity. In the gauge theory, marginal deformations are parametrised by the space of chiral primary operators of conformal dimension three, while exactly marginal deformations come from quotienting this space by the complexified global symmetry group. We show how the supergravity analysis gives a geometric interpretation of the gauge theory results. The marginal deformations arise from deformations of generalised structures that solve moment maps for the generalised diffeomorphism group and have the correct charge under the generalised Reeb vector, generating the R-symmetry. If this is the only symmetry of the background, all marginal deformations are exactly marginal. If the background possesses extra isometries, there are obstructions that come from fixed points of the moment maps. The exactly marginal deformations are then given by a further quotient by these extra isometries.

Our analysis holds for any $\mathcal{N} = 1$ AdS_5 flux background. Focussing on the particular case of type IIB Sasaki–Einstein backgrounds we recover the result that marginal deformations correspond to perturbing the solution by three-form flux at first order. In various explicit examples, we show that our expression for the three-form flux matches those in the literature and the obstruction conditions match the one-loop beta functions of the dual SCFT.

5.1 Introduction

The AdS/CFT correspondence allows the study of a wide class of superconformal field theories in four dimensions, many of which are realised as the world-volume theories of D3-branes at conical singularities of Calabi–Yau manifolds. The best known examples are $\mathcal{N} = 4$ super Yang–Mills and the Klebanov–Witten model, which are obtained by stacking

D3-branes in flat space or at the tip of the cone over $\mathrm{T}^{1,1}$ respectively.

An interesting feature of $\mathcal{N}=1$ superconformal field theories (SCFTs) is that they may admit exactly marginal deformations, namely deformations that preserve supersymmetry and conformal invariance. A given $\mathcal{N}=1$ SCFT can then be seen as a point on a "conformal manifold" in the space of operator couplings. The existence and dimension of the conformal manifold for a given theory can be determined using $\mathcal{N}=1$ supersymmetry and renormalisation group arguments [266–269]. For instance, $\mathcal{N}=4$ super Yang–Mills admits two exactly marginal deformations, the so-called β- and cubic deformations.[1] Even in this simple case, it is difficult to determine the precise geometry of the conformal manifold.

Using AdS/CFT, the same questions can be asked by studying deformations of the supergravity background dual to the given SCFT. For $\mathcal{N}=4$ super Yang–Mills, the supergravity dual of the full set of marginal deformations is known only perturbatively. In [270], the first-order perturbation was identified with the three-form fluxes of type IIB, and the corresponding linearised solution was given in [225]. The second-order solution, including the back-reacted dilaton and metric, was constructed in [271], which also identified an obstruction to the third-order solution, corresponding to the vanishing of the gauge theory beta functions. This required considerable effort, and it seems unlikely one can reconstruct the full solution from a perturbative analysis. More promisingly, using duality transformations, Lunin and Maldacena were able to build the full analytic supergravity dual of the β-deformation [93]. The same transformation applied to $\mathrm{T}^{1,1}$ or $\mathrm{Y}^{p,q}$ manifolds gives the gravity duals of the β-deformation of the Klebanov–Witten theory and more general $\mathcal{N}=1$ quiver gauge theories [93]. For the other marginal deformations of $\mathrm{Y}^{p,q}$ models, the identification of the gravity modes dual to them can be found in [272], but no finite-deformation gravity solutions are known.

The Lunin–Maldacena (LM) solution has a nice interpretation in generalised complex geometry [132, 133], a formalism that allows one to geometrise the NS-NS sector of supergravity [108, 109]. One considers a generalisation of the tangent bundle of the internal manifold, given by the sum of the tangent and cotangent bundles. The structure group of this generalised tangent bundle is the continuous T-duality group $\mathrm{O}(d,d)$. The transformation that generates the LM solution is then identified as a bi-vector deformation inside $\mathrm{O}(d,d)$ [132]. However, this is not the case for the other marginal deformation of $\mathcal{N}=4$. In order to capture all exactly marginal deformations, one is tempted to look at the full U-duality group. This requires considering exceptional or $\mathrm{E}_{d(d)}\times\mathbb{R}^+$ generalised geometry [135, 136], where the U-duality groups appear as the structure groups of even larger extended tangent bundles.

The relevant structures for AdS_5 compactifications are a hypermultiplet (or H) structure J_α and a vector-multiplet (or V) structure K. These structures are naturally associated with the hypermultiplet and vector-multiplet degrees of freedom of the five-dimensional

[1] There is also a third exactly marginal deformation that is simple changing the gauge coupling.

gauged supergravity on AdS_5, hence their names. Together they are invariant under a USp(6) subgroup of $E_{6(6)} \times \mathbb{R}^+$ and also admit a natural action of the USp(2) local symmetry of $\mathcal{N} = 1$ supergravity in five dimensions.[2] Although our specific examples will focus on type IIB geometries, the same analysis applies equally to generic $\mathcal{N} = 1$ AdS_5 solutions of type IIB or eleven-dimensional supergravity.

This generalised geometric description of the internal geometry translates naturally to quantities in the dual field theory, which is particularly useful when analysing marginal deformations. Indeed, since hypermultiplets and vector multiplets of the gauged supergravity correspond to chiral and vector multiplets of the dual SCFT [263], the deformations of the H and V structures map directly to superpotential and Kähler deformations of the dual SCFT. Using the properties of the $\mathcal{N} = 1$ superconformal algebra, Green et al. [269] showed that marginal deformations can only be chiral operators of (superfield) dimension three and that the set of exactly marginal deformations is obtained by quotienting the space of marginal couplings by the complexified global symmetry group. The main result of this chapter will be to reproduce these features from deformations of generic solutions on the supergravity side: the supersymmetric deformations must preserve the V structure but can deform the H structure. In addition, the exactly marginal deformations are a symplectic quotient of the marginal deformations by the isometry group of the internal manifold. This corresponds to the global symmetry group of the dual field theory.

The chapter is organized as follows: we begin in section 5.2 with a discussion of marginal deformations of $\mathcal{N} = 1$ SCFTs focussing on a number of classic examples that are dual to $\mathrm{AdS}_5 \times M$ type IIB backgrounds, where M is a Sasaki–Einstein manifold. In section 5.3, we review the reformulation of AdS_5 backgrounds in terms of exceptional generalised geometry. We then describe how the moduli space of generalised structures appears and outline how this reproduces the findings of [267–269]. For concreteness, in section 5.4 we specialise to type IIB Sasaki–Einstein backgrounds. We find the explicit linearised supersymmetric deformations corresponding to the operators in the chiral ring, matching the Kaluza–Klein analysis of [273], and recover the result that the supersymmetric deformations give rise to three-form flux perturbations [225]. In section 5.5, we give the explicit examples of S^5, $T^{1,1}$ and $Y^{p,q}$, and show that our expression for the three-form flux on S^5 matches the supergravity calculation of Aharony et al. [271], and reproduces the flux of the LM solution for generic Sasaki–Einstein manifolds.

5.2 Marginal deformations of $\mathcal{N} = 1$ SCFTs

Conformal field theories can be seen as fixed points of the renormalisation group flow where the beta functions for all couplings vanish. Generically, since there are as many beta functions as there are couplings, CFTs correspond to isolated points in the space of

[2]We use the nomenclature $\mathcal{N} = 1$ to denote backgrounds with eight supercharges in five dimensions, as this is the minimal amount of supersymmetry.

couplings. This is not the case for supersymmetric field theories, where non-renormalisation theorems force the beta functions for the gauge and superpotential couplings to be linear combinations of the anomalous dimensions of the fundamental fields [266]. If global symmetries are present before introducing the marginal deformations, the number of independent anomalous dimensions will be smaller than the number of couplings and not all beta functions will be independent. The theory then admits a manifold of conformal fixed points, $\mathcal{M}_c$. This is equivalent to saying that a given SCFT at a point $p \in \mathcal{M}_c$ admits exactly marginal deformations, namely deformations that preserve conformal invariance at the quantum level. The dimension of the conformal manifold is given by the difference between the number of classically marginal couplings and the number of independent beta functions. The two-point functions of the exactly marginal deformations at each point $p \in \mathcal{M}_c$ defines a natural metric on $\mathcal{M}_c$ called the Zamolodchikov metric.

Recently, developing the argument in [267], the authors of [269] proposed an alternative method to analyse the $\mathcal{N} = 1$ exactly marginal deformations of four-dimensional SCFTs, which does not use explicitly the beta functions for the superpotential couplings, but instead relies on the properties of the $\mathcal{N} = 1$ algebra. Take a four-dimensional $\mathcal{N} = 1$ SCFT at some point p in the conformal manifold, and consider all possible marginal deformations. These are of two types: "Kähler deformations" which are perturbations of the form $\int \mathrm{d}^4\theta\, V$ where V is a real primary superfield of mass dimension $\Delta = 2$, and "superpotential" deformations which have the form $\int \mathrm{d}^2\theta\, \mathcal{O}$ where $\mathcal{O}$ is a chiral primary superfield with $\Delta = 3$.[3] The results of [269] are that:

- there are no marginal Kähler deformations since they correspond to conserved currents;
- there is generically a set of marginal superpotential deformations $\mathcal{O}_i$, with the generic deformation $W = h^i\mathcal{O}_i$ parametrised by a set of complex couplings $\{h^i\}$;
- if the undeformed theory has no global symmetries other than the $\mathrm{U}(1)_\mathrm{R}$ R-symmetry, all marginal deformations are exactly marginal;
- however if the original SCFT has a global symmetry G that is broken by the generic deformation $W = h^i\mathcal{O}_i$, then the conformal manifold, near the original theory, is given by the quotient of the space of marginal couplings by the complexified broken global symmetry group
$$\mathcal{M}_c = \{h^i\}/G_\mathbb{C}, \tag{5.1}$$
where $\mathcal{M}_c$ is Kähler with the Zamolodchikov metric.

The reduction (5.1) can be viewed as a symplectic quotient for the real group G, where setting the moment maps to zero corresponds to solving the beta function equations for

[3] Here we give the mass dimension of the operator written as an $\mathcal{N} = 1$ superfield. In component notation, in both cases the contribution to the Lagrangian has dimension $\Delta = 4$.

the deformations. Note also that the vector space of couplings h^i (modulo $G_{\mathbb{C}}$) parametrise the tangent space $T_p\mathcal{M}_c$ at the particular SCFT $p \in \mathcal{M}_c$, and so define local coordinates on the conformal manifold near p. Thus, as written (5.1), is only a local definition.

More generally one can also consider operators $\mathcal{O} = A + \theta\psi + \theta^2 F_A$ that are chiral primary superfields of any dimension, modulo the relations imposed by the F-terms of the SCFT. The lowest components A form the chiral ring under multiplication $A'' = AA'$ subject to the F-term relations, whereas the θ^2-components satisfy $F_{A''} = AF_{A'} + A'F_A$, and hence transform as a derivation on the ring (specifically like a differential "$\mathrm{d}\,A$"). In what follows it will be useful to define the infinite-dimensional complex space of couplings $\{\gamma^i, \gamma'^i\}$ corresponding to deforming the Lagrangian by a term $\Delta = \gamma^i F_{A_i} + \gamma'^i A_i$ for generic chiral ring elements A_i and θ^2-components F_{A_i}. The γ^i terms are supersymmetric, while the γ'^i terms break supersymmetry, and generically neither are marginal. One of our results is that the supergravity analysis implies that there is a natural hyper-Kähler structure on this space, since the pair (γ^i, γ'^i) arise from the scalar components of a hypermultiplet in the bulk AdS space. More precisely, if there is a global symmetry G, one naturally considers the space defined by the hyper-Kähler quotient[4]

$$\widetilde{\mathcal{M}} = \{\gamma^i, \gamma'^i\} /\!/\!/ G. \tag{5.2}$$

The conformal manifold is then a finite-dimensional complex submanifold of $\widetilde{\mathcal{M}}$

$$\mathcal{M}_c \subset \widetilde{\mathcal{M}}, \tag{5.3}$$

with the A_i couplings γ'^i set to zero and only the exactly marginal γ^i coefficients (denoted h_i above) non-zero.

We now give three examples of SCFTs whose conformal manifolds have been analysed and whose gravity duals will be discussed in the rest of the chapter.

5.2.1 $\mathcal{N} = 4$ super Yang–Mills

The most studied example of a SCFT in four dimensions is $\mathcal{N} = 4$ super Yang–Mills. The fields of the theory are – besides gauge fields – six scalars and four fermions, all in the adjoint representation of the gauge group $\mathrm{SU}(N)$ and transforming non-trivially under the $\mathrm{SU}(4)$ R-symmetry. In $\mathcal{N} = 1$ notation, these fields arrange into a vector multiplet and three chiral superfields Φ^i. The theory has a superpotential

$$W_{\mathcal{N}=4} = \tfrac{1}{6} h \epsilon_{ijk} \operatorname{tr}(\Phi^i \Phi^j \Phi^k), \tag{5.4}$$

which is antisymmetric in the fields, and the coupling is fixed by $\mathcal{N} = 4$ supersymmetry to be equal to the gauge coupling, $h = \tau$. In this notation, only the $\mathrm{SU}(3) \times \mathrm{U}(1)$ subgroup of the R-symmetry is manifest.

[4] For more on this hyper-Kähler quotient see section 5.3.2.

The marginal deformations compatible with $\mathcal{N}=1$ supersymmetry are given by the chiral operators

$$W = \tfrac{1}{6}h\epsilon_{ijk}\,\mathrm{tr}(\Phi^i\Phi^j\Phi^k) + \tfrac{1}{6}f_{ijk}\,\mathrm{tr}(\Phi^i\Phi^j\Phi^k), \tag{5.5}$$

where f_{ijk} is a complex symmetric tensor of SU(3) and h is a priori different from the gauge coupling τ. In all there are eleven complex marginal deformations. The superpotential (5.5) breaks the global SU(3) symmetry, leaving the $\mathrm{U}(1)_\mathrm{R}$ symmetry of $\mathcal{N}=1$ theories. Therefore, the conformal manifold is

$$\mathcal{M}_\mathrm{c} = \{h, f_{ijk}\}/\mathrm{SU}(3)_\mathbb{C}, \tag{5.6}$$

with complex dimension $\dim(\mathcal{M}_\mathrm{c}) = 11 - 8 = 1 + 2$. The first deformation is an SU(4) singlet corresponding to changing both τ and h, the other two are true superpotential deformations.

The same conclusions can be reached by studying the beta functions of the deformed theory [266,271]. One can show that the beta function equations for the gauge coupling and the superpotential deformations are proportional to the matrix of anomalous dimensions. At one loop, this (or more precisely its traceless part) is

$$\gamma_i{}^j = \frac{N^2-4}{64N\pi^2}\left(f_{ikl}\bar{f}^{jkl} - \tfrac{1}{3}\delta_i{}^j f_{klm}\bar{f}^{klm}\right) = 0, \tag{5.7}$$

corresponding to the SU(3) moment maps, when we view (5.6) as a symplectic quotient. This equation imposes eight real conditions on f_{ijk}. One can remove another eight real degrees of freedom using an SU(3) rotation of the fields Φ^i. Together, these reduce the superpotential deformation to [266]

$$W = \tfrac{1}{6}h\epsilon_{ijk}\,\mathrm{tr}(\Phi^i\Phi^j\Phi^k) + f_\beta\,\mathrm{tr}(\Phi^1\Phi^2\Phi^3 + \Phi^3\Phi^2\Phi^1) + f_\lambda\,\mathrm{tr}\big((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3\big). \tag{5.8}$$

The coupling f_β is the so-called β-deformation,[5] and f_λ is often called the cubic deformation. As mentioned above, the first term in this expression is to be interpreted as changing h and τ together.

One can go beyond the one-loop analysis. The deformed theory has a discrete $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry, which forces the matrix of anomalous dimensions of the Φ^i to be proportional to the identity. One can then show that the beta function condition (at all loops) reduces to just one equation, thus again giving a three-dimensional manifold of exactly marginal deformations. Since this will be relevant for the gravity dual, we stress that the only obstruction to having exactly marginal deformations is the one-loop constraint (5.7).

[5] This term can also be written as $\mathrm{tr}(e^{\mathrm{i}\pi\beta}\Phi^1\Phi^2\Phi^3 - e^{-\mathrm{i}\pi\beta}\Phi^3\Phi^2\Phi^1)$ where β is complex [93].

5.2.2 Klebanov–Witten theory

The Klebanov–Witten theory is the four-dimensional SCFT that corresponds to the world-volume theory of N D3-branes at the conifold singularity [257]. This is an $\mathcal{N}=1$ $\mathrm{SU}(N)\times\mathrm{SU}(N)$ gauge theory with two sets of bi-fundamental chiral fields A_i and B_i $(i=1,2)$ transforming in the $(\boldsymbol{N},\overline{\boldsymbol{N}})$ and $(\overline{\boldsymbol{N}},\boldsymbol{N})$ respectively. The superpotential is

$$W = h\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\,\mathrm{tr}(A_\alpha B_{\dot\alpha}A_\beta B_{\dot\beta}), \tag{5.9}$$

and preserves an $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)_\mathrm{R}$ global symmetry, under which the chiral fields transform as $(\mathbf{2},\mathbf{1},\mathbf{1/2})$ and $(\mathbf{1},\mathbf{2},\mathbf{1/2})$ respectively. The R-charges of the fields A_i and B_i are such that the superpotential has the standard charge $+2$. The superpotential is not renormalisable, suggesting that the theory corresponds to an IR fixed point of an RG flow. Indeed, one can show that this theory appears as the IR fixed point of the RG flow generated by giving mass to the adjoint chiral multiplet in the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ super Yang–Mills [257].

Classically, the marginal deformations of the KW theory are given by the following chiral operators

$$\begin{aligned} W = {}& h\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\,\mathrm{tr}(A_\alpha B_{\dot\alpha}A_\beta B_{\dot\beta}) + f^{\alpha\beta,\dot\alpha\dot\beta}\,\mathrm{tr}(A_\alpha B_{\dot\alpha}A_\beta B_{\dot\beta}) \\ &+ \tau\big[\mathrm{tr}(W_1W_1)-\mathrm{tr}(W_2W_2)\big], \end{aligned} \tag{5.10}$$

where the tensor $f^{\alpha\beta,\dot\alpha\dot\beta}$ is symmetric in the indices $\alpha\beta$ and $\dot\alpha\dot\beta$, and therefore transforms in the $(\mathbf{3},\mathbf{3})$ of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ global symmetry group. The deformation τ does not break the global symmetry of the theory and corresponds to a shift in the difference of the gauge couplings $(1/g_1^2-1/g_2^2)$.

The exactly marginal deformations of the KW theory were found in [274]. Only three components of the $f^{\alpha\beta,\dot\alpha\dot\beta}$ term are exactly marginal, so we have five exactly marginal deformations in total. This is in agreement with the dimension of the conformal manifold, given by

$$\mathcal{M}_\mathrm{c} = \{h, f^{\alpha\beta,\dot\alpha\dot\beta},\tau\}/(\mathrm{SU}(2)\times\mathrm{SU}(2))_\mathbb{C}. \tag{5.11}$$

One reaches the same conclusions by studying the beta functions of the deformed theory [257]. These are equivalent to the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ moment maps, which take the form

$$\begin{aligned} \gamma^\alpha{}_\beta &= f^{\alpha\gamma\dot\alpha\dot\beta}\bar f_{\beta\gamma\dot\alpha\dot\beta} - \tfrac12\delta^\alpha{}_\beta f^{\tau\gamma\dot\alpha\dot\beta}\bar f_{\tau\gamma\dot\alpha\dot\beta} = 0, \\ \gamma^{\dot\alpha}{}_{\dot\beta} &= f^{\alpha\beta\dot\alpha\dot\gamma}\bar f_{\alpha\beta\dot\beta\dot\gamma} - \tfrac12\delta^{\dot\alpha}{}_{\dot\beta} f^{\alpha\beta\dot\tau\dot\gamma}\bar f_{\alpha\beta\dot\tau\dot\gamma} = 0. \end{aligned} \tag{5.12}$$

These remove six real degrees of freedom. We can also redefine the couplings using the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ symmetry to remove another six real degrees of freedom, leaving three complex parameters. The exactly marginal deformations are then given by

$$\begin{aligned} W = h\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\,\mathrm{tr}(A_\alpha B_{\dot\alpha} A_\beta B_{\dot\beta}) &+ \tau\big[\mathrm{tr}(W_1W_1) - \mathrm{tr}(W_2W_2)\big] \\ &+ f_\beta(A_1B_{\dot 1}A_2B_{\dot 2} + A_1B_{\dot 2}A_2B_{\dot 1}) + f_2(A_1B_{\dot 1}A_1B_{\dot 1} + A_2B_{\dot 2}A_2B_{\dot 2}) \\ &+ f_3(A_1B_{\dot 2}A_1B_{\dot 2} + A_2B_{\dot 1}A_2B_{\dot 1}). \end{aligned} \tag{5.13}$$

The deformation parametrised by f_β is the β-deformation for the KW theory, since it is the deformation that preserves the Cartan subgroup of the global symmetry group ($\mathrm{U}(1) \times \mathrm{U}(1)$ in this case).

5.2.3 $\mathrm{Y}^{p,q}$ gauge theories

The KW theory is the simplest example of an $\mathcal{N} = 1$ quiver gauge theory in four dimensions. A particularly interesting class of these theories arise as world-volume theories of D3-branes probing a Calabi–Yau three-fold with a toric singularity, where the singular Calabi–Yau spaces are cones over the infinite family of Sasaki–Einstein $\mathrm{Y}^{p,q}$ manifolds [275,276].[6] These theories have rather unusual properties, such as the possibility of irrational R-charges. The field theories dual to the infinite family of geometries were constructed in [277], which we review quickly.

The properties of the dual field theories can be read off from the associated quiver. The fields theories have $2p$ gauge groups with $4p + 2q$ bi-fundamental fields. Besides the $\mathrm{U}(1)_\mathrm{R}$, they have an $\mathrm{SU}(2) \times \mathrm{U}(1)_\mathrm{F}$ global symmetry. The $4p+2q$ fields split into doublets and singlets under SU(2): p doublets labelled U, q doublets labelled V, $p - q$ singlets labelled Z and $p + q$ singlets labelled Y. The general superpotential is

$$W = h\epsilon_{\alpha\beta}\left(\sum_{k=1}^{q}(U_k^\alpha V_k^\beta Y_{2k-1} + V_k^\alpha U_{k+1}^\beta Y_{2k}) + \sum_{j=q+1}^{p} Z_j U_{j+1}^\alpha Y_{2j-1} U_j^\beta\right), \tag{5.14}$$

where the α and β indices label the global SU(2). The R-charges of the fields are

$$\begin{aligned} r_U &= \tfrac{2}{3}pq^{-2}\big(2p - (4p^2 - 3q^2)^{1/2}\big), \\ r_V &= \tfrac{1}{3}q^{-1}\big(3q - 2p + (4p^2 - 3q^2)^{1/2}\big), \\ r_Y &= \tfrac{1}{3}q^{-2}\big(-4p^2 + 3q^2 + 2pq + (2p - q)(4p^2 - 3q^2)^{1/2}\big), \\ r_Z &= \tfrac{1}{3}q^{-2}\big(-4p^2 + 3q^2 - 2pq + (2p + q)(4p^2 - 3q^2)^{1/2}\big), \end{aligned} \tag{5.15}$$

while their charges under the additional $\mathrm{U}(1)_\mathrm{F}$ symmetry are respectively 0, 1, -1 and 1.

[6] The integer numbers p and q satisfy $0 \leq q \leq p$. Note that $\mathrm{Y}^{1,0} = \mathrm{T}^{1,1}$, the five-dimensional manifold in the KW theory.

The marginal deformations of these theories are given by [274]

$$W=(h\epsilon_{\alpha\beta}+f_{\alpha\beta})\left(\sum_{k=1}^{q}(U_k^\alpha V_k^\beta Y_{2k-1}+V_k^\alpha U_{k+1}^\beta Y_{2k})+\sum_{j=q+1}^{p}Z_jU_{j+1}^\alpha Y_{2j-1}U_j^\beta\right)\tau\,\mathcal{O}_{\text{gauge}}, \tag{5.16}$$

where $f_{\alpha\beta}$ is symmetric and $\mathcal{O}_{\text{gauge}}$ is an operator involving differences of gauge couplings. Note that W preserves $\mathrm{U}(1)_\mathrm{F}$, but the $f_{\alpha\beta}$ terms break the SU(2) to U(1). The SU(2) moment maps giving the beta functions are

$$\epsilon_{abc}f^b\bar{f}^c=0, \tag{5.17}$$

where $f_{\alpha\beta}=f^a(\sigma_a)_{\alpha\beta}$, which has the solution $f^a=r^a\mathrm{e}^{\mathrm{i}\phi}$. Modding out by the SU(2) action leaves a single deformation that is exactly marginal, namely the analogue of the β-deformation for the $\mathrm{Y}^{p,q}$ theories. As mentioned previously, the β-deformation breaks the global symmetry to its Cartan generators. Thus one can take f^3 non-zero, or equivalently

$$f_{11}=-f_{22}\equiv f_\beta. \tag{5.18}$$

Note that the counting is in agreement with the dimension of the conformal manifold, given by

$$\mathcal{M}_\mathrm{c}=\{h,f_{\alpha\beta},\tau\}/\mathrm{SU}(2)_\mathbb{C}=\{h,f_\beta,\tau\}. \tag{5.19}$$

Naively the quotient gives the wrong counting. However $f_{\alpha\beta}$ does not completely break SU(2) but instead preserves a U(1), meaning that the quotient removes only two complex degrees of freedom.

5.3 Deformations from exceptional generalised geometry

According to AdS/CFT, the supergravity dual of a given conformal field theory in four dimensions is a geometry of the form $\mathrm{AdS}_5\times M$, where the AdS_5 factor reflects the conformal invariance of the theory. The duals of exactly marginal deformations that preserve $\mathcal{N}=1$ supersymmetry are expected to be of the same form, but with a different geometry on the internal manifold. Generically, the solution will also have non-trivial fluxes and dilaton, if present. These solutions should be parametrically connected to the undeformed solution, so that the moduli space of exactly marginal deformations of the gauge theory is mapped to the moduli space of AdS_5 vacua.

Finding the full supergravity duals of exactly marginal deformations is not an easy task; few exact solutions are known, and those that are were found using solution-generating techniques based on dualities [93]. The idea we pursue is to exploit as much of the symmetry structure of supergravity as possible to look for the generic exactly marginal deformations. As we have outlined, this is most naturally done in the context of generalised geometry.

In this section, we outline the general results applicable to arbitrary AdS_5 supergravity backgrounds, whether constructed from type II or eleven-dimensional supergravity. In particular, we find the supergravity dual of the field theory results of [269]. In the following section, we discuss the specific case of type IIB compactifications on Sasaki–Einstein manifolds, giving considerably more detail.

5.3.1 Generalised structures and deformations

Consider a generic supersymmetric solution of the form $\mathrm{AdS}_5 \times M$, where M can be either five- or six-dimensional depending on whether we are compactifying type II or eleven-dimensional supergravity. We allow all fluxes that preserve the symmetry of AdS_5. We are looking for the duals of $\mathcal{N} = 1$ SCFTs in four dimensions and so the dual supergravity backgrounds preserve eight supercharges, that is $\mathcal{N} = 1$ in five dimensions. As we have seen, a background preserving eight supercharges is completely determined by specifying a pair of generalised structures: a "vector-multiplet structure" K and a "hypermultiplet structure" J_α, a triplet of objects labelled by $\alpha = 1, 2, 3$. Supersymmetry implies that the structures K and J_α satisfy three differential conditions, given in (4.13)–(4.15). The two of particular relevance to us are

$$\mu_\alpha(V) = \lambda_\alpha \int c(K, K, V) \qquad \forall\, V, \tag{5.20}$$

$$L_K J_\alpha = \epsilon_{\alpha\beta\gamma} \lambda_\beta J_\gamma, \tag{5.21}$$

where the triplet of functions $\mu_\alpha(V)$ are defined to be

$$\mu_\alpha(V) := -\tfrac{1}{2} \epsilon_{\alpha\beta\gamma} \int \operatorname{tr}(J_\beta\, L_V J_\gamma). \tag{5.22}$$

The third condition is

$$L_K K = 0. \tag{5.23}$$

The constants λ_α are related to the AdS_5 cosmological constant and can always be fixed to

$$\lambda_1 = \lambda_2 = 0, \qquad \lambda_3 = 3. \tag{5.24}$$

As we showed in (4.20), K is a "generalised Killing vector", that is L_K generates a generalised diffeomorphism that leaves the solution invariant, and this symmetry corresponds to the R-symmetry of the SCFT. In analogy to the Sasaki–Einstein case, we sometimes refer to K as the "generalised Reeb vector". In addition, the functions μ_α can be interpreted as a triplet of moment maps for the group of generalised diffeomorphisms acting on the space of J_α structures. As such we will often refer to (5.20) as the moment map conditions.

To find the marginal deformations of the $\mathcal{N} = 1$ SCFT we need to consider perturbations of the structures K and J_α that satisfy the supersymmetry conditions, expanded to first order in the perturbation. These are of two types,[7] which correspond to the two types of deformation in the SCFT. The easiest way to justify this identification is to note that, from the point of view of five-dimensional supergravity, fluctuations of K live in vector multiplets and those of J_α live in hypermultiplets. According to the AdS/CFT dictionary, vector multiplets and hypermultiplets correspond to real primary superfields and chiral primary superfields in the SCFT [263].

Let us first consider the Kähler deformations, where we hold J_α fixed and deform K. Looking at the moment maps (5.20), we see the left-hand side depends only on J_α and so does not change, but the right-hand side can vary, thus we must have

$$\int c(K, \delta K, V) = 0 \qquad \forall V. \tag{5.25}$$

The K tensor is invariant under an $\mathrm{F}_{4(4)} \subset \mathrm{E}_{6(6)}$ subgroup. Decomposing into $\mathrm{F}_{4(4)}$ representations, we find $\mathbf{27} = \mathbf{1} + \mathbf{26}$ and a singlet in the tensor product $\mathbf{26} \times \mathbf{26} = \mathbf{1} + \ldots$. Writing

$$\delta K = aK + K_{\mathbf{26}}, \qquad V = bK + V_{\mathbf{26}}, \tag{5.26}$$

the terms that form a singlet in the cubic invariant are

$$\int ab\, c(K, K, K) + \int c(K, K_{\mathbf{26}}, V_{\mathbf{26}}) = 0. \tag{5.27}$$

The first term is generically non-vanishing, so we must take $a = 0$ implying there is no singlet component in δK. We cannot simply scale K. For the second term, we know the F_4 Dynkin diagram has no symmetries, so the fundamental representation is equivalent to its dual. This means the singlet in $\mathbf{26} \times \mathbf{26}$ appears in the symmetric or the antisymmetric product. If the singlet were to appear in the antisymmetric product, $c(K, K_{\mathbf{26}}, V_{\mathbf{26}})$ would vanish identically as the cubic invariant is itself symmetric and $K_{\mathbf{26}}$ would be unconstrained. For F_4 the singlet appears in the symmetric product [278].[8] Thanks to Weyl's unitary trick, the real forms that have the same complexification as F_4 also admit an invariant symmetric product. This is the case for $\mathrm{F}_{4(4)}$. This means $K_{\mathbf{26}} \times V_{\mathbf{26}}$ is symmetric and is generically non-zero in $c(K, K_{\mathbf{26}}, V_{\mathbf{26}})$. Given that it must vanish for any $V_{\mathbf{26}}$, $K_{\mathbf{26}}$ must itself vanish. Together these mean $\delta K = 0$, so there are no deformations of K that satisfy the moment maps. This matches the field theory analysis that there are no deformations of Kähler type.

For the superpotential deformations we can solve (5.20) and (5.21) to first order in δJ_α. We do this in two steps. First we solve the linearised moment map conditions (5.20).

[7] There is actually a third type where both $\delta J_\alpha \neq 0$ and $\delta K \neq 0$, but in this case none of the supergravity fields are perturbed; instead it corresponds to a deformation of the Killing spinors, implying the background admits more than eight supersymmetries. For this reason it will not interest us here.

[8] One can find a basis for $\mathfrak{f}_4$ in terms of matrices in $\mathfrak{so}_{26}$ that stabilise a certain cubic polynomial in 26 dimensions [279]. This means the $\mathbf{26}$ representation is real and that $\mathfrak{f}_4$ inherits a symmetric bilinear from $\mathfrak{so}_{26}$.

This gives an infinite number of solutions which correspond to θ^2-components and fields in the chiral ring of the dual gauge theory; generically these are not marginal. Imposing the first-order generalised Lie derivative condition (5.21) will select a finite number of these modes that are massless in AdS_5 and correspond to the actual marginal deformations.

5.3.2 Exactly marginal deformations and fixed points

We now turn to how the supergravity structure encodes the SCFT result that all marginal deformations are exactly marginal unless there is an additional global symmetry group G. The key point, as we will see, is that the differential conditions (5.20) appear as moment maps for the generalised diffeomorphisms.

A priori, to see if the marginal deformations are exactly marginal one needs to satisfy the equations (5.20) and (5.21) not just to first order, but to all orders in the deformation. In general this is a complicated problem: typically there can be obstructions at higher order that mean not all marginal deformations are actually exactly marginal. For example, a detailed discussion of deformations of $\mathcal{N} = 4$ up to third order is given in [271].

However, viewing the conditions (5.20) as a triplet of moment maps provides an elegant supergravity dual of the field theory result that does not require detailed case-by-case calculations. We discussed the generic situation in section 4.3.2, which we now review. Moment maps arise when there is a group action preserving a symplectic or hyper-Kähler structure. Here the μ_α correspond to the action of generalised diffeomorphisms acting on the structure J_α. Thus to get physically distinct solutions we need to satisfy the moment map conditions (5.20) and then identify solutions that are related by a generalised diffeomorphisms. Formally this defines a subspace of hypermultiplet structures

$$\widetilde{\mathcal{M}} = \{J_\alpha : \mu_\alpha = \lambda_\alpha \gamma\}/\mathrm{GDiff}_K, \tag{5.28}$$

where γ is the function

$$\gamma(V) = \int c(K, K, V), \tag{5.29}$$

and GDiff_K is the subgroup of generalised diffeomorphisms that leave K invariant. In other words, we are considering the moduli space of solutions for J_α for fixed K. By construction (5.28) defines a hyper-Kähler quotient and hence $\widetilde{\mathcal{M}}$ is hyper-Kähler. The condition (5.21) then defines a Kähler subspace $\mathcal{M}_\mathrm{c} \subset \widetilde{\mathcal{M}}$. We can also consider first imposing (5.21) and then the moment maps (5.20). Let $\mathcal{A}_\mathrm{H}^K$ be the space of H-structures $J_\alpha(x)$ for fixed K. Imposing (5.21) defines a Kähler subspace $\mathcal{N}_\mathrm{H} \subset \mathcal{A}_\mathrm{H}^K$. The moment map conditions then take a symplectic quotient of $\mathcal{N}_\mathrm{H}$ rather than a hyper-Kähler quotient.

We then have the following picture

$$\begin{tikzcd}
\mathcal{A}^K_{\mathrm{H}} \arrow[r, "(5.21)"] \arrow[d, "\text{HK quotient (5.20)}"'] & \mathcal{N}_{\mathrm{H}} \arrow[d, "\text{sympl. quotient (5.20)}"] \\
\widetilde{\mathcal{M}} \arrow[r, "(5.21)"] & \mathcal{M}_{\mathrm{c}}
\end{tikzcd} \tag{5.30}$$

A nice property of moment map constructions is that generically there are no obstructions to the linearised problem: every first-order deformation around a given point $p \in \widetilde{\mathcal{M}}$ in the hyper-Kähler quotient (or alternatively $p \in \mathcal{M}_{\mathrm{c}}$ for the symplectic quotient) can be extended to an all-order solution. The only time this fails is if the symmetry group at p defining the moment map has fixed points. In our context this means there are generalised diffeomorphisms that leave the particular J_α and K structures invariant, so that one can find a V such that the generalised Lie derivatives vanish

$$L_V J_\alpha = L_V K = 0. \tag{5.31}$$

From equation (4.20) and the discussion preceding it, these imply $L_V G = 0$ so that V is a generalised Killing vector and the vector component of V is a Killing vector. These V generate isometries of the background (beyond the $\mathrm{U}(1)_{\mathrm{R}}$ R-symmetry), corresponding to the global symmetry group G of the dual field theory.[9] Thus we directly derive the result that every marginal deformation is exactly marginal in the absence of global symmetries.

Suppose now that the global symmetry group G is non-trivial. By construction, those V that generate G fall out of the linearised moment map conditions – they trivially solve the moment maps as $L_V J_\alpha = 0$. Thus to solve the full non-linear problem, one must somehow impose these additional conditions. It is a standard result in symplectic (or hyper-Kähler) quotients that the missing equations correspond to a quotient by the global group G on the space of linearised solutions. Suppose $\{\gamma^i, \gamma'^i\}$ are coordinates on the space of linearised deformations, corresponding to couplings of operators F_{A_i} and A_i. Imposing (5.21) then restricts to the marginal operators $\{h_i\} \subset \{\gamma^i, \gamma'^i\}$. By construction, there is a flat hyper-Kähler metric on $\{\gamma^i, \gamma'^i\}$ and a flat Kähler metric on $\{h_i\}$. In addition there is a linear action of G on each space that preserves these structures. The origin is a fixed point of G owing to the fact that we are expanding about a solution with a global symmetry. The moduli space of finite deformations then corresponds to a quotient of each

[9]For example, for $M = \mathrm{S}^5$ the isometry group is $\mathrm{SO}(6) \simeq \mathrm{SU}(4) \supset \mathrm{U}(1)_{\mathrm{R}} \times \mathrm{SU}(3)$, so V would give the Killing vectors that generate $\mathrm{SU}(3)$.

space by G (at least in the neighbourhood of the original solution). Thus we have

$$
\begin{tikzcd}
\{\gamma^i, \gamma'^i\} \arrow[r, "(5.21)"] \arrow[d, "\text{HK quotient by } G"'] & \{h_i\} \arrow[d, "\text{sympl. quotient by } G"] \\
\widetilde{\mathcal{M}} \arrow[r, "(5.21)"] & \mathcal{M}_c
\end{tikzcd}
\tag{5.32}
$$

This structure is discussed in little more detail in section 5.4.4. We see that we directly recover the field theory result (5.1) that the conformal manifold is given by $\mathcal{M}_c = \{h_i\}/\!\!/G = \{h_i\}/G_{\mathbb{C}}$.[10]

Note that interpreting the supersymmetry conditions in terms of moments maps nicely mirrors the field theory analysis of the moduli space of marginal deformations. Indeed imposing (5.21) and solving the linearised moment maps (5.20) is equivalent to restricting to chiral operators of dimension three that satisfy the F-term conditions. The further symplectic quotient by the isometry group G then corresponds to imposing the D-term constraints and modding out by gauge transformations.

5.4 The case of D3-branes at conical singularities

The results summarized in the previous section are completely general and apply to any AdS_5 flux background. To make the discussion more concrete we will focus on deformations of $\mathcal{N} = 1$ SCFTs that are realised on the world-volume of D3-branes at the tip of a Calabi–Yau cone over a Sasaki–Einstein (SE) manifold M.

Before turning to the generalized geometric description of the supergravity duals, we present their description in terms of "conventional" geometry.

5.4.1 The undeformed Sasaki–Einstein solution

In the ten-dimensional type IIB solution dual to the undeformed SCFT, the metric takes the form[11]

$$
\begin{aligned}
\mathrm{d}s^2_{10} &= e^{2\Delta}\mathrm{d}s^2(\mathbb{R}^{3,1}) + e^{-2\Delta}\mathrm{d}s^2(\mathrm{CY}) \\
&= r^2\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \frac{1}{r^2}\left(\mathrm{d}r^2 + r^2\mathrm{d}s^2(\mathrm{SE})\right) \\
&= \mathrm{d}s^2(\mathrm{AdS}_5) + \mathrm{d}s^2(\mathrm{SE}),
\end{aligned}
\tag{5.33}
$$

where the radial direction of the Calabi–Yau cone together with the four-dimensional warped space form AdS_5. In the second and third line we have used the explicit form of the warp factor for AdS_5, $e^\Delta = r$. The solution has constant dilaton, $e^\phi = 1$, and five-form

[10] The space of marginal operators $\{h_i\}$ is Kähler, so the symplectic and complexified quotients agree.

[11] In these conventions the radius of AdS_5 is $R = 1$, so the cosmological constant is $\Lambda = -6$.

flux given by

$$F_5 = 4(\mathrm{vol}_{\mathrm{AdS}} + \mathrm{vol}_5), \tag{5.34}$$

where vol_5 is the volume form on M. The metric on the Sasaki–Einstein manifold locally takes the form

$$\mathrm{d}s^2(\mathrm{SE}) = \sigma^2 + \mathrm{d}s^2(\mathrm{KE}), \tag{5.35}$$

where σ is called the contact form and the four-dimensional metric is Kähler–Einstein (KE), with symplectic two-form given by

$$\omega = \tfrac{1}{2}\mathrm{d}\sigma. \tag{5.36}$$

There is also a holomorphic (2,0)-form Ω, compatible with ω

$$\omega \wedge \Omega = 0, \qquad \omega \wedge \omega = \tfrac{1}{2}\Omega \wedge \bar{\Omega}, \tag{5.37}$$

satisfying

$$\mathrm{d}\Omega = 3\mathrm{i}\sigma \wedge \Omega. \tag{5.38}$$

The five-dimensional volume form is then $\mathrm{vol}_5 = -\frac{1}{2}\sigma \wedge \omega \wedge \omega$.[12] The forms σ, Ω and ω define an SU(2) structure on the Sasaki–Einstein manifold. The complex structure I for the Kähler–Einstein metric can be written as

$$I^m{}_n = -\omega^m{}_n = \tfrac{1}{4}\mathrm{i}(\bar{\Omega}^{mp}\Omega_{np} - \Omega^{mp}\bar{\Omega}_{np}), \tag{5.39}$$

which satisfies $I^p{}_m\Omega_{pn} = \mathrm{i}\Omega_{mn}$.

The R-symmetry of the field theory is realised in the dual geometry by the Reeb vector field ξ, satisfying

$$\imath_\xi \sigma = 1, \qquad \imath_\xi \mathrm{d}\sigma = 0. \tag{5.40}$$

Locally we can introduce a coordinate ψ such that

$$\sigma = \tfrac{1}{3}(\mathrm{d}\psi + \eta), \qquad \xi = 3\partial_\psi. \tag{5.41}$$

If a tensor X satisfies $\mathcal{L}_\xi X = \mathrm{i}qX$, we say it has charge q under the action of the Reeb vector. The objects defining the SU(2) structure on M have definite charge

$$\mathcal{L}_\xi \sigma = \mathcal{L}_\xi \omega = \mathcal{L}_\xi I = 0, \qquad \mathcal{L}_\xi \Omega = 3i\Omega. \tag{5.42}$$

The R-charge r is related to q by $q = 3r/2$. For example, Ω is charge +3 under the Reeb vector and has R-charge +2.

[12] These conventions are chosen to match [244].

The contact and Kähler structures allow a decomposition of the exterior derivative as

$$\mathrm{d} = \partial + \bar{\partial} + \sigma \wedge \mathcal{L}_\xi, \tag{5.43}$$

where $\bar{\partial}$ is the tangential Cauchy-Riemann operator, which satisfies [280, 281]

$$\bar{\partial}^2 = \partial^2 = 0, \qquad \partial\bar{\partial} + \bar{\partial}\partial = -2\omega \wedge \mathcal{L}_\xi. \tag{5.44}$$

For calculations, it is useful to introduce a frame such that the complex, symplectic and contact structure have the following form

$$\begin{aligned} \Omega &= (e^2 + \mathrm{i}e^5) \wedge (e^4 + \mathrm{i}e^3), \\ \omega &= e^2 \wedge e^5 + e^4 \wedge e^3, \\ \sigma &= e^1. \end{aligned} \tag{5.45}$$

If the SE manifold is “regular” the Reeb vector defines a U(1) fibration over a Kähler–Einstein base. This is the case for S^5 and $\mathrm{T}^{1,1}$, dual to $\mathcal{N} = 4$ SYM and the $\mathcal{N} = 1$ KW theory, where the base manifolds are respectively $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$. The $\mathrm{Y}^{p,q}$ spaces are generically not fibrations.

5.4.2 Embedding in exceptional generalised geometry

Let us quickly review the description of supersymmetric $\mathrm{AdS}_5 \times M$ solutions in $\mathrm{E}_{6(6)} \times \mathbb{R}^+$ generalised geometry following the presentation in chapter 4. Although we will focus on type IIB for definiteness, we stress that the construction is equally applicable to solutions of eleven-dimensional supergravity. In particular, one could apply our methods to the generic M-theory AdS_5 solution of [240], which we embedded in $\mathrm{E}_{6(6)} \times \mathbb{R}^+$ generalised geometry in the previous chapter.

The generalised structures K and J_α transform under $\mathrm{E}_{6(6)} \times \mathbb{R}^+$ as an element of the $\mathbf{27'}$ and a triplet of elements in the $\mathbf{78}$. The J_α form an SU(2) triplet under the $\mathrm{E}_{6(6)}$ adjoint action, corresponding to the R-symmetry of the $\mathcal{N} = 1$ supergravity

$$[J_\alpha, J_\beta] = 2\kappa\epsilon_{\alpha\beta\gamma}J_\gamma, \tag{5.46}$$

where κ^2 is the volume form on M for an unwarped solution with vanishing dilaton. The normalisations of K and J_α are fixed by

$$c(K, K, K) = \kappa^2, \qquad \mathrm{tr}(J_\alpha J_\beta) = -\kappa^2\delta_{\alpha\beta}, \tag{5.47}$$

where c is the cubic invariant of $\mathrm{E}_{6(6)}$, and tr is the trace in the adjoint representation (see (B.58) and (B.60)). The two structures are compatible, which means they satisfy

$$J_\alpha \cdot K = 0, \tag{5.48}$$

where $\cdot$ is the adjoint action on a generalised vector: $\mathbf{78} \times \mathbf{27'} \to \mathbf{27'}$ (see (B.39)).

The generalised structures K and J_α are combinations of the geometric structures on M built from bilinears of the $\mathcal{N} = 1$ Killing spinors [142]. For Sasaki–Einstein manifolds, these are the Reeb vector ξ, the symplectic form ω and the holomorphic two-form Ω. We gave the form of K and J_α in section 4.4.1, which we reproduce here[13]

$$\begin{aligned} K &= \xi - \sigma \wedge \omega, \\ J_+ &= \tfrac{1}{2}\kappa u^i (\Omega - \mathrm{i}\Omega^\sharp), \\ J_3 &= \tfrac{1}{2}\kappa\Big(I - \mathrm{i}\sigma_2 - \tfrac{1}{4}\Omega \wedge \bar{\Omega} + \tfrac{1}{4}\Omega^\sharp \wedge \Omega^\sharp\Big), \end{aligned} \tag{5.49}$$

where $J_+ = J_1 + \mathrm{i}J_2$, σ_2 is the second Pauli matrix and the $\mathrm{SL}(2;\mathbb{R})$ vector is $u^i = (-\mathrm{i}, 1)$. Note that K depends only on the Reeb vector and the contact structure, whereas J_α depends only on the complex structure of the Kähler–Einstein metric.

Supersymmetry conditions

For a supersymmetric compactification to AdS_5, the structures K and J_α must satisfy the differential conditions (5.20)–(5.23). We showed this to be the case in section 4.4.1: the first two reduce to (5.36), (5.38) and (5.42), thus fixing the constants λ_α as in (5.24), while condition (5.23) gives no extra equations. Note that since the deformations we are after leave the structure K invariant, the latter condition will play no role in the following. As we discussed in section 4.3.1, the supersymmetry conditions can be viewed as the internal counterpart of the supersymmetry conditions in five-dimensional gauged supergravity [197]: (5.20) comes from the gravitino and gaugino variations (as does (5.23)), while (5.21) is related to the hyperino variation (recall K is associated to the vector multiplets, while J_α is associated to the hypermultiplets).

The key ingredient in the supersymmetry conditions is the generalised Lie derivative L. This encodes the differential geometry of the background, unifying the diffeomorphisms and gauge symmetries of the supergravity. Given two generalised vectors V and V' the generalised Lie derivative is given by (B.45). This can be extended to an action on any generalised tensor. For example, the action on the adjoint representation is given in (B.46). One always has the choice to include the supergravity fluxes in the structures K and J_α or as a modification of the generalised Lie derivative. Here the latter option turns out to be more convenient. This defines a "twisted generalised Lie derivative" $\hat{L}$, which takes the same form as (B.45) but with the substitutions

$$\mathrm{d}\lambda^i \to \mathrm{d}\lambda^i - \imath_v F_3^i, \qquad \mathrm{d}\rho \to \mathrm{d}\rho - \imath_v F_5 - \epsilon_{ij}\lambda^i \wedge F_3^j. \tag{5.50}$$

In the remainder of this chapter, we will use exclusively the untwisted structures and the

[13] Note that we are using the "untwisted" structures but have dropped the tildes. In what follows, we will use the untwisted structures and the twisted Dorfman derivative.

twisted Dorfman derivative. In order to avoid cluttered notation, we drop the tildes from untwisted structures and the hat from the twisted Dorfman derivative.

As we emphasised in section 4.3.2, there is a natural hyper-Kähler geometry on the space of J_α structures. There is also an action of generalised diffeomorphisms taking one J_α into another. This action preserves the hyper-Kähler structure. The conditions (5.20) can then be viewed as moment maps for the action of the generalised diffeomorphisms. By construction the space $\widetilde{\mathcal{M}}$ of solutions to this condition in (5.28) is also hyper-Kähler. The generalised Lie derivative condition (5.21) takes a Kähler slice of this space. For the SE structure (5.49) and five-form flux given in (5.34) we have

$$L_K = \mathcal{L}_\xi, \tag{5.51}$$

and thus L_K generates the $\mathrm{U}(1)_\mathrm{R}$ symmetry. Recall from (4.24) that this is actually a general result: the slice taken by condition (5.21) essentially fixes the R-charge of J_+ to be $+3$, and J_3 to be zero.

5.4.3 Linearised deformations

The structures K and J_α lie in orbits of the $\mathrm{E}_{6(6)}$ action. The linearised deformations $\mathcal{A}$ are therefore elements in the adjoint of $\mathrm{E}_{6(6)}$, which take us from a given point in these orbits corresponding to the original solution (in the case of Sasaki–Einstein, this is (5.49)), to another point in the orbit corresponding to the structures of the deformed geometry. We have seen from the gauge theory that we expect the marginal deformations $\mathcal{A}$ to leave the structure K invariant, while deforming J_α. This implies

$$\delta K = \mathcal{A}\cdot K = 0\,, \qquad \delta J_\alpha = [\mathcal{A}, J_\alpha] \neq 0. \tag{5.52}$$

As we will discuss in more detail in appendix H, the deformations $\mathcal{A}$ are doublets under the SU(2) generated by J_α

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_-^{(r)} \\ \mathcal{A}_+^{(r-2)} \end{pmatrix}, \tag{5.53}$$

with $\mathcal{A}_- = [J_+, \mathcal{A}_+]$.[14] The signs $\pm$ denote the charge under J_3, $[J_3, \mathcal{A}_\pm] = \pm\mathrm{i}\mathcal{A}_\pm$, and r is the charge under the action of L_K corresponding to their R-charge

$$L_K \mathcal{A}_\pm^{(r)} = \tfrac{3}{2}\mathrm{i}r\mathcal{A}_\pm^{(r)}. \tag{5.54}$$

The difference in the R-charge of the two components follows from (5.21), (5.54) and the definition $\mathcal{A}_- = [J_+, \mathcal{A}_+]$.

We now need to find pairs of solutions for $\mathcal{A}_\pm$ satisfying the linearised supersymmetry conditions and, for definiteness, R-charge $r \geq 0$. In the next subsection, we start by

[14]Strictly speaking, this should be $\mathcal{A}_- = \kappa^{-1}[J_+, \mathcal{A}_+]$, but we have dropped the factors of κ for ease of presentation in this section.

first finding solutions to the linearised moment maps. We then have to mod out by the symmetry, identifying deformations that are related by diffeomorphisms or form-field gauge transformations as corresponding to the same physical deformation. This process corresponds to finding the bulk modes dual to the bosonic components of all chiral superfields: namely the chiral ring operators A_i (associated to $\mathcal{A}_-$) and the related supersymmetric deformations of the Lagrangian F_{A_i} (associated to $\mathcal{A}_+$). Then in the following subsection, we turn to finding the subset of marginal deformations. The technical details are discussed in appendix H. Here we outline the procedure and present the results.

The chiral ring

The linearised moment map equations are given by[15]

$$\delta\mu_\alpha(V) = \int \kappa \operatorname{tr}(J_\alpha, L_V \mathcal{A}) = 0 \qquad \forall V \in \mathbf{27'}, \tag{5.55}$$

where we are using the fact that the deformation leaves K invariant.

We start by looking for $\mathcal{A}_+$ that solve (5.55). The $\mathcal{A}_+$ deformations can be distinguished by which components of the $\mathrm{E}_{6(6)} \times \mathbb{R}^+$ adjoint are non-zero. They fall into two classes

$$\check{\mathcal{A}}_+ = B^i + \beta^i, \qquad \hat{\mathcal{A}}_+ = a^i{}_j, \tag{5.56}$$

where the first contains only two-forms and the corresponding bi-vectors, and the second contains only $\mathfrak{sl}_2$ entries.

As shown in appendix H.2, the two-form part of the $\check{\mathcal{A}}_+$ solutions to (5.55) consists of two independent terms

$$B^i = -\tfrac{1}{2}\mathrm{i}\bar{u}^i \left[f\bar{\Omega} + \tfrac{1}{2q(q-1)}\partial(\partial f \lrcorner \bar{\Omega}) + \tfrac{\mathrm{i}}{q}\sigma \wedge (\partial f \lrcorner \bar{\Omega}) \right] - \mathrm{i}\bar{u}^i \delta, \tag{5.57}$$

where Ω and σ are the holomorphic two-form and the contact form on the SE manifold, and the $\mathrm{SL}(2;\mathbb{R})$ vector is $u^i = (-\mathrm{i}, 1)$. The expression $\partial f \lrcorner \bar{\Omega}$ is equivalent to $(\partial f)^m \bar{\Omega}_{mn}$ in indices. The bi-vector part of the solution is obtained by raising indices with the SE metric. The term in the brackets is completely determined by a function f on the SE manifold satisfying

$$\bar{\partial} f = 0, \qquad \mathcal{L}_\xi f = \mathrm{i}qf. \tag{5.58}$$

Note that f is holomorphic with respect to $\bar{\partial}$ if and only if it is the restriction of a holomorphic function on the Calabi–Yau cone over the Sasaki–Einstein base [273]. The second term depends only on a primitive $(1,1)$-form δ on the KE base that is closed under both ∂ and $\bar{\partial}$

$$\delta \wedge \omega = 0, \qquad \partial\delta = \bar{\partial}\delta = 0. \tag{5.59}$$

[15] As we discuss in appendix H.2, the actual deformation is by $\mathcal{A} = \operatorname{Re} \mathcal{A}_+$ so that the deformed structures are real. This do not affect the discussion that follows.

Imposing that the deformation $\check{\mathcal{A}}_+$ has fixed R-charge $r-2$, and using (5.42), gives

$$\mathcal{L}_\xi f = \tfrac{3}{2}\mathrm{i}rf, \qquad \mathcal{L}_\xi \delta = \tfrac{3}{2}\mathrm{i}(r-2)\delta, \tag{5.60}$$

so that f is a homogeneous function on the Calabi–Yau cone of degree $\frac{3}{2}r$.

Let us now consider $\hat{\mathcal{A}}_+$. Its only non-zero components are $a^i{}_j \in \mathfrak{sl}_2$, which are again determined by a function $\tilde{f}$ on the manifold

$$\hat{\mathcal{A}}_+ = -\tfrac{1}{2}\tilde{f}\bar{u}^i\bar{u}_j, \tag{5.61}$$

where $\bar{u}_i = \epsilon_{ij}\bar{u}^j$ and the function $\tilde{f}$ is holomorphic

$$\bar{\partial}\tilde{f} = 0. \tag{5.62}$$

The deformations of fixed R-charge $r-2$ satisfy

$$\mathcal{L}_\xi \tilde{f} = \tfrac{3}{2}\mathrm{i}(r-2)\tilde{f}, \tag{5.63}$$

so that $\tilde{f}$ is a homogeneous function on the Calabi–Yau cone of degree $\frac{3}{2}(r-2)$.

For each solution $\mathcal{A}_+$, one can generate an independent solution $\mathcal{A}_-$ by acting with J_+. Indeed, any deformation of the form $\mathcal{A}_- = [J_+, \mathcal{A}_+]$ is automatically a solution of the moment maps, provided $\mathcal{A}_+$ is. The explicit form of these deformations for $\check{\mathcal{A}}_-$ and $\hat{\mathcal{A}}_-$ is given in (H.11) and (H.13). Thus the solutions of the linearised moment maps consist of an infinite set of deformations $\mathcal{A}_+$ labelled by their R-charge r, which are generated by the two holomorphic functions, f and $\tilde{f}$, and a (1,1)-form, δ, and another independent set of deformations $\mathcal{A}_-$ generated by f', $\tilde{f}'$ and δ'. Together these give the general solution to the deformation problem. Arranging these deformations as in (5.53), we find three types of multiplets, schematically,

$$\begin{pmatrix} \mathcal{A}_-^{(r)} \\ \mathcal{A}_+^{(r-2)} \end{pmatrix} \sim \begin{pmatrix} f' \\ f \end{pmatrix}, \begin{pmatrix} \tilde{f}' \\ \tilde{f} \end{pmatrix}, \begin{pmatrix} \delta' \\ \delta \end{pmatrix}, \tag{5.64}$$

with charge r given respectively by $r > 0$, $r \geq 2$ and $r = 2$.

Let us now identify what these solutions correspond to physically. For this it is convenient to compute the action of the linearised deformations on the bosonic fields of type II supergravity and then interpret the multiplets (5.64) in terms of Kaluza–Klein modes on the Sasaki–Einstein manifold. One way to read off the bosonic background is from the generalised metric G. This is defined in (H.39) and encodes the metric, dilaton, the NS-NS field B_2 and the R-R fields C_0, C_2 and C_4. As discussed in appendix H.2.1, the two-form and bi-vector deformations f and their partners f' at leading order generate NS-NS and R-R two-form potentials, and a combination of internal four-form potential and metric[16]

$$\begin{pmatrix} f' \\ f \end{pmatrix} \sim \begin{pmatrix} C_4 + g^a{}_a \\ C_2 - \mathrm{i}B_2 \end{pmatrix} \propto \begin{pmatrix} \frac{1}{2}f'\Omega\wedge\bar\Omega + \frac{\mathrm{i}}{2q}\Omega\wedge\sigma\wedge(\partial f'\lrcorner\bar\Omega) + \dots \\ f\bar\Omega + \frac{1}{2q(q-1)}\partial(\partial f\lrcorner\bar\Omega) + \frac{\mathrm{i}}{q}\sigma\wedge(\partial f\lrcorner\bar\Omega) \end{pmatrix}. \tag{5.65}$$

Similarly one can show that the holomorphic function $\tilde f$ and its partner $\tilde f'$ correspond to the axion-dilaton, and NS-NS and R-R two-form potentials

$$\begin{pmatrix} \tilde f' \\ \tilde f \end{pmatrix} \sim \begin{pmatrix} C_2 - \mathrm{i}B_2 \\ C_0 - \mathrm{i}\phi \end{pmatrix} \propto \begin{pmatrix} \tilde f'\Omega \\ \tilde f \end{pmatrix}. \tag{5.66}$$

Finally the two-form and bi-vector deformations δ and its partner δ' generate NS-NS and R-R two-form potentials and a component of the internal metric

$$\begin{pmatrix} \delta' \\ \delta \end{pmatrix} \sim \begin{pmatrix} g^m{}_n \\ C_2 - \mathrm{i}B_2 \end{pmatrix} \propto \begin{pmatrix} (j\Omega^\sharp\lrcorner j\delta' + j\delta'^\sharp\lrcorner j\Omega)^m{}_n \\ \delta \end{pmatrix}. \tag{5.67}$$

The Kaluza–Klein (KK) spectrum for a generic Sasaki–Einstein background was analysed in [273] by solving for eigenmodes of the Laplacian on the manifold. The states arrange into long and short multiplets of $\mathcal{N}=1$ supergravity in five dimensions. Our multiplets (5.65), (5.66) and (5.67) are indeed the short multiplets of [273].

In terms of the bulk five-dimensional supergravity, each $(\mathcal{A}_-^{(r)}, \mathcal{A}_+^{(r-2)})$ doublet of fixed R-charge corresponds to a different hypermultiplet. In the dual field theory the $\mathcal{A}_+^{(r-2)}$ piece corresponds to the θ^2-component of a chiral superfield while the $\mathcal{A}_-^{(r)}$ piece corresponds to the lowest component [263]. We then have the following mapping between supergravity and field theory multiplets

$$\begin{aligned} &\begin{pmatrix} f' \\ f \end{pmatrix} \sim \operatorname{tr}\mathcal{O}_f, && \text{superpotential deformations, } r>0, \\ &\begin{pmatrix} \tilde f' \\ \tilde f \end{pmatrix} \sim \operatorname{tr}W_\alpha W^\alpha \mathcal{O}_{\tilde f}, && \text{coupling deformations, } r\geq 2, \\ &\begin{pmatrix} \delta' \\ \delta \end{pmatrix} \sim \mathcal{O}_{\text{gauge}}, && \text{difference in gauge couplings, } r=2. \end{aligned} \tag{5.68}$$

For S^5 the first two sets of multiplets corresponds to the operators $\operatorname{tr}(\Phi^k)$ and $\operatorname{tr}(W_\alpha W^\alpha\Phi^k)$, where Φ denotes any of the three adjoint chiral superfields of $\mathcal{N}=4$ SYM, and the last multiplet is not present. For $\mathrm{T}^{1,1}$, one has $\operatorname{tr}(\mathcal{O}_f) = \operatorname{tr}(AB)^k$, $\operatorname{tr}(W_\alpha W^\alpha\mathcal{O}_{\tilde f}) = \operatorname{tr}\big[(W_A^2+W_B^2)(AB)^k\big]$ and $\mathcal{O}_{\text{gauge}} = \operatorname{tr}(W_A^2 - W_B^2)$ where A and B denote the two doublets

[16] The full form of the four-form potential and metric is given by (H.11) with $\bar\nu' = \frac{\mathrm{i}}{2q}\partial f'\lrcorner\bar\Omega$ and $\hat\omega' = \frac{1}{4q(q-1)}\partial(\partial f'\lrcorner\bar\Omega)$.

of bi-fundamental chiral superfields. In analogy with the $\mathrm{T}^{1,1}$ case, for a generic SE the operators $\mathcal{O}_f$ and $\mathcal{O}_{\tilde{f}}$ are products of chiral bi-fundamental superfields of the theory, while $\mathcal{O}_{\text{gauge}}$ corresponds to changing the relative couplings of the gauge groups.

The tower of deformations gives the space $\widetilde{\mathcal{M}}$ defined in (5.2). In particular, the $\mathcal{A}_- = (f', \tilde{f}', \delta') \sim A_i$ deformations parametrise the chiral ring, while $\mathcal{A}_+ = (f, \tilde{f}, \delta) \sim F_{A_i}$ parametrise the superpotential deformations.

Marginal deformations

The marginal deformations are a subspace of solutions in $\widetilde{\mathcal{M}}$ that also satisfy the second differential condition (5.21). At first order in the deformation, this is

$$[L_K \mathcal{A}, J_\alpha] = 0, \tag{5.69}$$

where we have used again the fact that the deformations leave K invariant. Since the commutators with J_α are non-zero, this condition amounts to the requirement

$$L_K \mathcal{A} = 0 \qquad \Rightarrow \qquad \mathcal{L}_\xi \mathcal{A} = 0. \tag{5.70}$$

In other words, the R-charge of $\mathcal{A}$ vanishes. Comparing with (5.64) we see that the $\mathcal{A}_-$ components always have positive R-charge and therefore are not solutions of (5.69). Thus marginal deformations can only be given by the $\mathcal{A}_+^{(r-2)}$ components with $r = 2$. This is consistent, because, as we have mentioned, the $\mathcal{A}_+$ components correspond to deforming the SCFT by θ^2 terms, which are supersymmetric, whereas the $\mathcal{A}_-$ terms correspond to the lowest component of a chiral superfield and so do not give supersymmetric deformations.

From (5.60) and (5.63) we see that the $\mathcal{A}_+^{(0)}$ components ($r = 2$) are[17]

$$f \text{ of degree 3}, \qquad \tilde{f} = \text{constant}, \qquad \delta \in \mathrm{H}^{1,1}_{\text{prim}}(M), \tag{5.71}$$

corresponding precisely to superpotential deformations with $\Delta = 3$, a change in the original superpotential (and at the same time of the sum of coupling constants), and a change in the relative gauge couplings respectively.

Linearised supergravity solution

We want now to compute the supergravity solutions at linear order. As discussed in detail in appendix H.2.1, this can be done by looking at the action of the marginal deformations $\check{\mathcal{A}}_+$ and $\hat{\mathcal{A}}_+$ on the generalised metric, which encodes the bosonic fields of type IIB supergravity. We first consider the effect of a $\check{\mathcal{A}}_+$ deformation to linear order. As already mentioned, such a deformation generates NS-NS and R-R two-form potentials, given by

$$C_2 - \mathrm{i}B_2 = -\mathrm{i}\big(f\bar{\Omega} + \tfrac{1}{12}\partial(\partial f \lrcorner \bar{\Omega}) + \tfrac{1}{3}\mathrm{i}\sigma \wedge (\partial f \lrcorner \bar{\Omega})\big) - 2i\delta. \tag{5.72}$$

[17] $\mathrm{H}^{1,1}_{\text{prim}}(M)$ denotes the cohomology of primitive $(1,1)$-forms.

Taking an exterior derivative, the complexified flux $G_3 = \mathrm{d}(C_2 - \mathrm{i}B_2)$ to leading order is

$$G_3 = -\tfrac{4}{3}\mathrm{i}\partial f \wedge \bar{\Omega} + 4f\sigma \wedge \bar{\Omega} - \tfrac{1}{3}\sigma \wedge \partial(\partial f \lrcorner \bar{\Omega}). \tag{5.73}$$

The (1,1)-form $\delta \in \mathrm{H}^2(M)$ is closed and therefore does not contribute to the flux. On the Calabi–Yau cone, it is well-known that superpotential deformations correspond to imaginary anti-self-dual (IASD) flux [225]. The G_3 here is the component of the IASD flux restricted to the Sasaki–Einstein space.

Now consider the effect of a marginal $\hat{\mathcal{A}}_+$ deformation to linear order. As we show in appendix H.2.2, such a deformation allows for non-zero, constant values of the axion and dilaton, given by

$$\tilde{f} = C_0 - \mathrm{i}\phi. \tag{5.74}$$

We stress that this calculation and the expressions for the leading-order corrections to the solution (5.73) for the NS-NS and R-R three-form flux and the axion-dilaton in (5.74) are valid for *any* Sasaki–Einstein background. One simply needs to plug in the expressions for the holomorphic form and contact structure of the given Sasaki–Einstein space. These objects are given in terms of a frame in (5.45). We will give the explicit form of the frame for the examples of S^5, $\mathrm{T}^{1,1}$ and the $\mathrm{Y}^{p,q}$ manifolds, and compare the flux with some known results in section 5.5.

5.4.4 Moment maps, fixed points and obstructions

The linearised analysis above has identified the supergravity perturbations dual to marginal chiral operators in the SCFT. However, this is not the end of the story. Really we would like to find the exactly marginal operators. In the gravity dual this means solving the supersymmetry equations not just to first order but to all orders. In general there are obstructions to solving the supersymmetry conditions to higher orders, and not all marginal deformations are exactly marginal [271]. As we saw in section 5.2, in the field theory these obstructions are related to global symmetries [269].

As we discussed in section 5.3.2, the fact that the supergravity conditions in exceptional generalised geometry appear as moment maps gives an elegant interpretation of the field theory result. This analysis was completely generic, equally applicable to type II and eleven-dimensional supergravity backgrounds. We will now give a few more details, using the Sasaki–Einstein case as a particular example.

The key point is that generically there are no obstructions to extending the linearised solution of a moment map to an all-orders solution. The only case when this fails is when one is expanding around a point where some of the symmetries defining the moment map have fixed points (see for instance [185]). Since here the moment maps are for the generalised diffeomorphisms, we see that there are obstructions only when the background is invariant under some subgroup G of diffeomorphisms and gauge transformations, called the stabiliser group. Such transformations correspond to additional global symmetries in the SCFT. Furthermore, one can use a linear analysis around the fixed point to show that the obstruction appears as a further symplectic quotient by the symmetry group G. This mirrors the field theory result that all marginal deformations are exactly marginal unless there is an enhanced global symmetry group and that the space of exactly marginal operators is a symplectic quotient of the space of marginal operators.

To see this in a little more detail let us start by reviewing how the conditions (5.20) appear as moment maps and how the obstruction appears. We will first consider $\widetilde{\mathcal{M}}$, the space of chiral ring elements and θ^2-components, and then at the end turn to the actual marginal deformations. As we stressed above, this discussion is completely generic and not restricted to Sasaki–Einstein spaces. One first considers the space $\mathcal{A}_{\mathrm{H}}^{K}$ of all possible hypermultiplet structures compatible with a fixed K, in other words

$$\mathcal{A}_{\mathrm{H}}^{K} = \{J_\alpha(x) : J_\alpha \cdot K = 0\}. \tag{5.75}$$

Since each point $p \in \mathcal{A}_{\mathrm{H}}^{K}$ is a choice of structure defined by a triplet of functions $J_\alpha(x)$ on M, the space $\mathcal{A}_{\mathrm{H}}^{K}$ is infinite dimensional. Nonetheless it is hyper-Kähler. A tangent vector v at the point p can be thought of as a small change in the structure

$$v_\alpha(x) = \delta J_\alpha(x) = [\mathcal{A}(x), J_\alpha(x)] \in T_p\mathcal{A}_{\mathrm{H}}^{K}, \tag{5.76}$$

where $\mathcal{A}(x)$ is some $\mathrm{E}_{6(6)} \times \mathbb{R}^+$ element. The hyper-Kähler structure is characterised by a triplet of closed symplectic forms, Ω_α. These symplectic structures Ω_α are defined such that, given a pair of tangent vectors $v, v' \in T_p\mathcal{A}_{\mathrm{H}}^{K}$, the three symplectic products are given by

$$\Omega_\alpha(v, v') = \epsilon_{\alpha\beta\gamma} \int \operatorname{tr}(v_\beta v'_\gamma) = 2 \int \kappa \operatorname{tr}\big([\mathcal{A}, \mathcal{A}'] J_\alpha\big). \tag{5.77}$$

The generalised diffeomorphism group acts on $J_\alpha(x)$ and hence on $\mathcal{A}_{\mathrm{H}}^{K}$. Furthermore its action leaves the symplectic forms Ω_α invariant. Infinitesimally, generalised diffeomorphisms are generated by the generalised Lie derivative so that $\delta J_\alpha = L_V J_\alpha \in T_p\mathcal{A}_{\mathrm{H}}^{K}$. Thus, just as vector fields parametrise the Lie algebra of conventional diffeomorphisms via the Lie derivative, one can view the generalised vectors V as parametrising the Lie algebra $\mathfrak{gdiff}$ of the generalised diffeomorphism group.[18] One can then show that the $\mu_\alpha(V)$ in (5.20) are precisely the moment maps for the action of the generalised diffeomorphism group on $\mathcal{A}_{\mathrm{H}}^{K}$. As written they are three functions on $\mathcal{A}_{\mathrm{H}}^{K} \times \mathfrak{gdiff}$ where J_α gives the point in $\mathcal{A}_{\mathrm{H}}^{K}$ and V parametrises the element of $\mathfrak{gdiff}$, but they can equally well be viewed as a single map $\mu \colon \mathcal{A}_{\mathrm{H}}^{K} \to \mathfrak{gdiff}^* \times \mathbb{R}^3$ where $\mathfrak{gdiff}^*$ is the dual of the Lie algebra. Solving the moment map conditions (5.20) and modding out by the generalised diffeomorphisms to obtain $\widetilde{\mathcal{M}}$ as in (5.28) is a hyper-Kähler quotient. As discussed in section 4.3.2, one subtlety is that,

[18] Note from (4.26) that shifting the form components λ^i and ρ of V by exact terms does not change L_V, furthermore it is independent of σ^i. Thus different generalised vectors can parametrise the same Lie algebra element.

in order to define a quotient, the right-hand side of the conditions $\lambda_\alpha \gamma$, given in (5.29) and which depends on K, must be invariant under the action of the group. Thus the quotient is really defined not for the full generalised diffeomorphism group, but rather the subgroup GDiff_K that leaves K invariant. Infinitesimally V parametrises an element of the corresponding Lie algebra $\mathfrak{gdiff}_K$ if $L_V K = 0$. Thus we have the quotient (5.28).

The linearised analysis of the last section first fixes a point $p \in \mathcal{A}_{\mathrm{H}}^K$ corresponding to the Sasaki–Einstein background satisfying the moment map conditions, and then finds deformations of the structure $\delta J_\alpha \in T_p \mathcal{A}_{\mathrm{H}}^K$ for which the variations of the moment maps $\delta\mu_\alpha(V)$ vanish for all V. If we view $\delta\mu_\alpha$ as a single map $\delta\mu \colon T_p\mathcal{A}_{\mathrm{H}}^K \to \mathfrak{gdiff}_K^* \times \mathbb{R}^3$, the linearised solutions live in the kernel. Suppose now that p is fixed under some subset of generalised diffeomorphisms, that is we have a stabiliser group $G \subset \mathrm{GDiff}_K$. The corresponding Lie subalgebra $\mathfrak{g} \subset \mathfrak{gdiff}_K$ is

$$\mathfrak{g} = \{V \in \mathfrak{gdiff}_K : L_V J_\alpha = 0\}. \tag{5.78}$$

At a generic point in $\mathcal{A}_{\mathrm{H}}^K$ satisfying the moment map conditions, all elements of GDiff_K act non-trivially and so the stabiliser group is trivial. Thus solving $\delta\mu_\alpha(V) = 0$ we get a constraint for every $V \in \mathfrak{gdiff}_K$. In contrast, at the point p, we miss those constraints corresponding to $V \in \mathfrak{g}$. Thus we see that the obstruction to extending the first-order deformation to all orders lies precisely in $\mathfrak{g}^* \times \mathbb{R}^3$, that is, it is the missing constraints. Put more formally,[19] the embedding $i \colon \mathfrak{g} \to \mathfrak{gdiff}_K$ induces a map $i^* \colon \mathfrak{gdiff}_K^* \to \mathfrak{g}^*$ on the dual spaces and, at p, we have an exact sequence

$$T_p\mathcal{A}_{\mathrm{H}}^K \xrightarrow{\delta\mu} \mathfrak{gdiff}_K^* \times \mathbb{R}^3 \xrightarrow{i^*} \mathfrak{g} \times \mathbb{R}^3 \ . \tag{5.79}$$

The map $\delta\mu$ is not onto and the obstruction is its cokernel $\mathfrak{g}^* \times \mathbb{R}^3$.

The standard argument for moment maps at fixed points actually goes further. Let $\mathcal{U}$ be the vector space of linearised solutions $\delta\mu_\alpha(V) = 0$ at p, up to gauge equivalence. For the Sasaki–Einstein case it is the space of solutions, dual to the couplings of the operators (A_i, F_{A_i}), given in (5.68). Formally $\mathcal{U}$ is defined as follows. Recall that the space of solutions is $\ker\delta\mu \subset T_p\mathcal{A}_{\mathrm{H}}^K$. The action of GDiff_K on $p \in \mathcal{A}_{\mathrm{H}}^K$ defines an orbit $O \subset \mathcal{A}_{\mathrm{H}}^K$, and modding out by the tangent space to the orbit T_pO at p corresponds to removing gauge equivalence, so that

$$\mathcal{U} = \ker\delta\mu / T_pO. \tag{5.80}$$

The moment map construction means that the hyper-Kähler structure on $T_p\mathcal{A}_{\mathrm{H}}^K$ descends to $\mathcal{U}$. By definition, the stabiliser group G acts linearly on $T_p\mathcal{A}_{\mathrm{H}}^K$ and this also descends to $\mathcal{U}$. Furthermore it preserves the hyper-Kähler structure. Thus we can actually define moment maps $\tilde{\mu}_\alpha$ for the action of G on $\mathcal{U}$. The standard argument is then that the space of unobstructed linear solutions can be identified with the hyper-Kähler quotient of $\mathcal{U}$ by

[19] See for example the note in section 5 of [185].

G, so near p we have

$$\widetilde{\mathcal{M}} = \mathcal{U} /\!\!/\!\!/ G \coloneqq \{\mathcal{A} \in \mathcal{U} : \tilde{\mu}_\alpha = 0\}/G, \tag{5.81}$$

just as in (5.2). The idea here is that if we move slightly away from p we are no longer at a fixed point and there are no missing constraints. Thus we really want to take the hyper-Kähler quotient in a small neighbourhood of $\mathcal{A}^K_{\mathrm{H}}$ near p. However we can use the tangent space $T_p\mathcal{A}^K_{\mathrm{H}}$ to approximate the neighbourhood. The moment map on $T_p\mathcal{A}^K_{\mathrm{H}}$ can be thought of in two steps: first we impose $\delta\mu_\alpha = 0$ at the origin and mod out by the corresponding gauge symmetries, reducing $T_p\mathcal{A}^K_{\mathrm{H}}$ to the space $\mathcal{U}$. However this misses the conditions coming from the stabiliser group G which leaves the origin invariant. Imposing these conditions takes a further hyper-Kähler quotient of $\mathcal{U}$ by G. Finally, note that since G acts linearly on $\mathcal{U}$, the obstruction moment maps $\tilde{\mu}_\alpha$ are quadratic in the deformation $\mathcal{A}$. This exactly matches the analysis in [271], where in solving the deformation to third-order the authors found a quadratic obstruction. What is striking is that we have been able to show how the obstructions appear for completely generic supersymmetric backgrounds.

This discussion has been somewhat abstract. Let us now focus on the simple case of S^5 to see how it works concretely. The full isometry group is $\mathrm{SO}(6) \simeq \mathrm{SU}(4)$. However, only an $\mathrm{SU}(3)$ subgroup preserves J_α and K, hence

for S^5 the stabiliser group is $G = \mathrm{SU}(3)$.

Rather than consider the full space of linearised solutions (5.68), for simplicity we will just focus on f and f', and furthermore assume both functions are degree three: $\mathcal{L}_\xi f = 3\mathrm{i}f$ and $\mathcal{L}_\xi f' = 3\mathrm{i}f'$. In terms of holomorphic functions on the cone $\mathbb{C}^3$, this implies both functions are cubic

$$f = f^{ijk} z_i z_j z_k, \qquad f' = f'^{ijk} z_i z_j z_k. \tag{5.82}$$

The coefficients (f^{ijk}, f'^{ijk}) parametrise a subspace in the space of linearised gauge-fixed solutions $\mathcal{U}$. Using the expressions (5.56) and (5.77) one can calculate the hyper-Kähler metric on the (f^{ijk}, f'^{ijk}) subspace. Alternatively, one notes that the hyper-Kähler structure on $\mathcal{A}^K_{\mathrm{H}}$ descends to a flat hyper-Kähler structure the subspace, parametrised by f^{ijk} and f'^{ijk} as quaternionic coordinates. We then find the three symplectic forms

$$\begin{aligned} \Omega_3 &= \tfrac{1}{2}\mathrm{i}\,\mathrm{d}f^{ijk} \wedge \mathrm{d}\bar{f}_{ijk} - \tfrac{1}{2}\mathrm{i}\,\mathrm{d}f'^{ijk} \wedge \mathrm{d}\bar{f}'_{ijk}, \\ \Omega_+ &= \mathrm{d}f^{ijk} \wedge \mathrm{d}\bar{f}'_{ijk}, \end{aligned} \tag{5.83}$$

where $\Omega_+ = \Omega_1 + \mathrm{i}\Omega_2$ and indices are raised and lowered using δ_{ij}. The $\mathrm{SU}(3)$ group acts infinitesimally as

$$\begin{aligned} \delta f^{ijk} &= a^{[i}{}_l f^{jk]l}, \\ \delta f'^{ijk} &= a^{[i}{}_l f'^{jk]l}, \end{aligned} \tag{5.84}$$

where $\operatorname{tr} a = 0$ and $a^\dagger = -a$. This action is generated by the vectors

$$\rho(a) = a^i{}_j \big(f^{jkl} \partial_{ikl} - \bar{f}_{ikl} \bar{\partial}^{jkl} + f'^{jkl} \partial'_{ikl} - \bar{f}'_{ikl} \bar{\partial}'^{jkl} \big), \tag{5.85}$$

where $\partial_{ijk} = \partial / \partial f^{ijk}$ and $\partial'_{ijk} = \partial / \partial f'^{ijk}$. It is then easy to solve for the (equivariant) moment maps $\tilde{\mu}_\alpha(a)$ satisfying $i_{\rho(a)} \Omega_\alpha = \mathrm{d}\tilde{\mu}_\alpha(a)$, to find

$$\begin{aligned} \tilde{\mu}_3(a) &= \tfrac{1}{2} \mathrm{i} a^i{}_j \big(f^{jkl} \bar{f}_{ikl} - f'^{jkl} \bar{f}'_{ikl} \big), \\ \tilde{\mu}_+(a) &= a^i{}_j f^{jkl} \bar{f}'_{ikl}. \end{aligned} \tag{5.86}$$

Solving the moment maps $\tilde{\mu}_\alpha(a) = 0$ for all $a^i{}_j$ gives

$$\begin{aligned} \tfrac{1}{2} \mathrm{i} \big(f^{ikl} \bar{f}_{jkl} - \tfrac{1}{3} \delta^i{}_j f^{klm} \bar{f}_{klm} - f'^{ikl} \bar{f}'_{jkl} + \tfrac{1}{3} \delta^i{}_j f'^{klm} \bar{f}'_{klm} \big) &= 0, \\ f^{ikl} \bar{f}'_{jkl} - \tfrac{1}{3} \delta^i{}_j f^{klm} \bar{f}'_{klm} &= 0. \end{aligned} \tag{5.87}$$

Imposing these conditions and modding out by SU(3) then gives the unobstructed deformations living in $\widetilde{\mathcal{M}}$. If we actually included all the modes in (f, f') we would find polynomials with arbitrary coefficients $f^{i_1 \cdots i_p}$ but the construction would be essentially the same. This also applies to the $(\tilde{f}, \tilde{f}')$ modes. Since $\mathrm{H}^2(\mathrm{S}^5) = 0$ there are no (δ, δ') solutions on S^5.

So far we have discussed how the existence of fixed points leads to obstructions in the construction of the space $\widetilde{\mathcal{M}}$. However ultimately we would like to find the unobstructed exactly marginal deformations $\mathcal{M}_\mathrm{c}$. Returning to the generic case, recall that the marginal deformations corresponded to a subspace given by the $\mathcal{A}^{(0)}_+$ components of the full set of deformations, satisfying the condition (5.70). (In the Sasaki–Einstein case these are given in (5.71).) Let us denote this subspace by $\mathcal{U}_\mathrm{c} \subset \mathcal{U}$. Since $L_K J_\alpha$ is a holomorphic vector on $\widetilde{\mathcal{M}}$ with respect to one of the complex structures (see section 4.3.2), $\mathcal{U}_\mathrm{c}$ is a Kähler subspace. Furthermore, taking the hyper-Kähler quotient by G and then restricting to the marginal deformations is the same as restricting to the marginal deformations and then taking a symplectic quotient by G using only the moment map $\lambda^\alpha \tilde{\mu}_\alpha$. In other words the diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[d] & \mathcal{U}_\mathrm{c} \arrow[d] \\ \widetilde{\mathcal{M}} = \mathcal{U}_\mathrm{c} /\!/\!/ G \arrow[r] & \mathcal{M}_\mathrm{c} = \mathcal{U}_\mathrm{c} /\!/ G \end{tikzcd} \tag{5.88}$$

commutes. This is because the action of L_K which enters the generalised Lie derivative condition (5.21) commutes with the action of L_V generating G.[20] Given $\mathcal{U}_\mathrm{c} /\!/ G = \mathcal{U}_\mathrm{c} / G_\mathbb{C}$, we see that we reproduce the field theory result (5.2).

It is simple to see how this works in the case of S^5. The marginal modes correspond to

[20] We have $[L_V, L_K] = L_{L_V K} = 0$ since by definition $L_V K = 0$ if V is in the stabiliser group G.

$f' = \tilde{f}' = 0$, while f is restricted to be degree three and $\tilde{f}$ constant (recall δ and δ' are absent on S^5). Since constant $\tilde{f}$ is invariant under SU(3), the moment map conditions $\tilde{\mu}_\alpha = 0$ on the marginal modes reduce to a single condition that comes from $\tilde{\mu}_3$ (given $\lambda_1 = \lambda_2 = 0$), namely

$$\tfrac{1}{2}\mathrm{i}\big(f^{ikl}\bar{f}_{jkl} - \tfrac{1}{3}\delta^i{}_j f^{klm}\bar{f}_{klm}\big) = 0, \tag{5.89}$$

since the $\tilde{\mu}_+$ moment map is satisfied identically as $f' = \tilde{f}' = 0$. Comparing with section 5.2, we see that we indeed reproduce the field theory result that the exactly marginal deformations are a symplectic quotient of the marginal deformations by the global symmetry group G.

5.5 Examples

In the previous section we derived the first-order supergravity solution dual to exactly marginal deformations on any Sasaki–Einstein background. We now apply this to the explicit examples of the supergravity backgrounds dual to $\mathcal{N} = 4$ super Yang–Mills, the $\mathcal{N} = 1$ Klebanov–Witten theory and $\mathcal{N} = 1$ $\mathrm{Y}^{p,q}$ gauge theories.

5.5.1 $\mathcal{N} = 4$ super Yang–Mills

The Sasaki–Einstein manifold that appears in the dual to $\mathcal{N} = 4$ SYM is S^5, whose four-dimensional Kähler–Einstein base is $\mathbb{CP}^2$. The metric on S^5 can be written as[21]

$$\mathrm{d}s^2(\mathrm{S}^5) = \mathrm{d}\alpha^2 + s_\alpha^2\mathrm{d}\theta^2 + c_\alpha^2\mathrm{d}\phi_1^2 + s_\alpha^2 c_\theta^2\mathrm{d}\phi_2^2 + s_\alpha^2 s_\theta^2\mathrm{d}\phi_3^2, \tag{5.90}$$

where the coordinates are related to the usual complex coordinates on $\mathbb{C}^3$, pulled back to S^5, by

$$z_1 = c_\alpha \mathrm{e}^{\mathrm{i}\phi_1}, \qquad z_2 = s_\alpha c_\theta \mathrm{e}^{\mathrm{i}\phi_2}, \qquad z_3 = s_\alpha s_\theta \mathrm{e}^{\mathrm{i}\phi_3}. \tag{5.91}$$

We can take the following frame for S^5

$$\begin{aligned} e^1 &= c_\alpha^2\mathrm{d}\phi_1 + c_\theta^2 s_\alpha^2\mathrm{d}\phi_2 + s_\alpha^2 s_\theta^2\mathrm{d}\phi_3, \\ e^2 + \mathrm{i}e^5 &= \mathrm{e}^{3\mathrm{i}\psi/2}(\mathrm{d}\alpha - \mathrm{i}c_\alpha s_\alpha\mathrm{d}\phi_1 + \mathrm{i}c_\alpha c_\theta^2 s_\alpha\mathrm{d}\phi_2 + \mathrm{i}c_\alpha s_\alpha s_\theta^2\mathrm{d}\phi_3,) \\ e^4 + \mathrm{i}e^3 &= \mathrm{e}^{3\mathrm{i}\psi/2}(s_\alpha\mathrm{d}\theta - \mathrm{i}c_\theta s_\alpha s_\theta\mathrm{d}\phi_2 + \mathrm{i}c_\theta s_\alpha s_\theta\mathrm{d}\phi_3), \end{aligned} \tag{5.92}$$

where $3\psi = \phi_1 + \phi_2 + \phi_3$. The complex, symplectic and contact structures are defined in terms of the frame in (5.45). One can check they satisfy the correct algebraic and differential relations (5.40)–(5.42).

The marginal deformations are given in terms of a function f which is of charge three under the Reeb vector and the restriction of a holomorphic function on $\mathbb{C}^3$. In our

[21] Here s_α and c_α are shorthand for $\sin\alpha$ and $\cos\alpha$, and similarly for θ.

parametrisation the Reeb vector field is

$$\xi = 3\partial_\psi = \partial_{\phi_1} + \partial_{\phi_2} + \partial_{\phi_3}, \tag{5.93}$$

and the coordinates z_i have charge $+1$

$$\mathcal{L}_\xi z_i = \mathrm{i} z_i. \tag{5.94}$$

Thus, f must be a cubic function of the z_i. An arbitrary cubic holomorphic function on $\mathbb{C}^3$ has ten complex degrees of freedom and can be written as

$$f = f^{ijk} z_i z_j z_k, \tag{5.95}$$

where f^{ijk} is a complex symmetric tensor of SU(3) with ten complex degrees of freedom. This is the same structure as the superpotential deformation (5.5). As mentioned before, not all components of f correspond to exactly marginal deformations because we still need to take into account the broken SU(3) global symmetry. This imposes the further constraint

$$f^{ikl}\bar{f}_{jkl} - \tfrac{1}{3}\delta^i{}_j f^{klm}\bar{f}_{klm} = 0, \tag{5.96}$$

which removes eight real degrees of freedom. We can also redefine the couplings using the SU(3) symmetry to remove another eight real degrees of freedom, leaving a two-complex dimensional space of exactly marginal deformations. Thus, there are two independent solutions

$$f_\beta \propto z_1 z_2 z_3, \tag{5.97}$$

and

$$f_\lambda \propto z_1^3 + z_2^3 + z_3^3, \tag{5.98}$$

corresponding to the β-deformation and the cubic deformation in (5.8).

The supergravity dual of the β-deformation was worked out in [93]. One can check that using our frame for S^5 and taking

$$f_\beta = -\tfrac{3}{2}\gamma z_1 z_2 z_3, \tag{5.99}$$

where γ is real, our expression (5.73) for the three-form fluxes reproduces those in the first-order β-deformed solution [93]. To generate the complex deformation of LM, we promote γ to $\gamma - \mathrm{i}\sigma$, where both γ and σ are real. This reproduces the LM fluxes with $\tau = \mathrm{i}$. The full complex deformation with general τ can be obtained using the $\mathrm{SL}(2;\mathbb{R})$ frame from [139].

Unlike the β-deformation, the supergravity dual of the cubic deformation is known only perturbatively. Aharony et. al have given an expression for the three-form flux for both the β and cubic deformations to first order [225]. Again, one can check that our expression reproduces this flux for both f_β and f_λ.

We saw that the marginal deformations (5.71) also allow for closed primitive $(1,1)$-forms that do not contribute to the flux. If such terms are not exact – if they are non-trivial in cohomology – they give additional marginal deformations. On $\mathbb{CP}^2$, the base of S^5, there are no closed primitive $(1,1)$-forms that are not exact, and so the marginal deformations are completely determined by the function f.

5.5.2 Klebanov–Witten theory

A similar analysis can be performed for deformations of the Klebanov–Witten theory. In this case the dual geometry is $\mathrm{T}^{1,1}$, the coset space $\mathrm{SU}(2)\times\mathrm{SU}(2)/\mathrm{U}(1)$ with the topology of $\mathrm{S}^2\times\mathrm{S}^3$. $\mathrm{T}^{1,1}$ can also be viewed as a $\mathrm{U}(1)$ fibration over $\mathbb{CP}^1\times\mathbb{CP}^1$ with metric [282]

$$\mathrm{d}s^2(\mathrm{T}^{1,1}) = \tfrac{1}{9}(\mathrm{d}\psi+\cos\theta_1\mathrm{d}\phi_1+\cos\theta_2\mathrm{d}\phi_2)^2+\tfrac{1}{6}\sum_{i=1,2}(\mathrm{d}\theta_i^2+\sin^2\theta_i\mathrm{d}\phi_i^2). \tag{5.100}$$

Each $\mathrm{SU}(2)$ acts on one $\mathbb{CP}^1$, and the $\mathrm{U}(1)$ acts as shifts of ψ. The Reeb vector field is

$$\xi = 3\partial_\psi. \tag{5.101}$$

As with S^5, a holomorphic function on the cone over $\mathrm{T}^{1,1}$ determines the marginal deformations. In this case, the cone is the conifold, defined by

$$z_1^2+z_2^2+z_3^2+z_4^2=0, \qquad z_i\in\mathbb{C}^4. \tag{5.102}$$

The conifold equation can also be written as

$$\det Z_{ij}=0, \tag{5.103}$$

where $Z_{ij}=\sigma^a_{ij}z_a$, $\sigma^a=(\boldsymbol{\sigma},\mathrm{i}\mathbb{1})$ and $\boldsymbol{\sigma}$ are the Pauli matrices. We can choose complex coordinates A_α and $B_{\dot\alpha}$ $(\alpha=1,2)$, corresponding to each $\mathbb{CP}^1$, which are dual to the chiral fields of the gauge theory

$$Z=\begin{pmatrix} z_3+\mathrm{i}z_4 & z_1-\mathrm{i}z_2 \\ z_1+\mathrm{i}z_2 & -z_3+\mathrm{i}z_4\end{pmatrix}=\begin{pmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2\end{pmatrix}. \tag{5.104}$$

The complex coordinates z_a can be parametrised by

$$\begin{aligned} z_1 &= \frac{1}{2}\left(\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\mathrm{e}^{\mathrm{i}(\psi-\phi_1-\phi_2)/2} - \mathrm{i}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\mathrm{e}^{\mathrm{i}(\psi+\phi_1+\phi_2)/2}\right), \\ z_2 &= \frac{1}{2\mathrm{i}}\left(\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\mathrm{e}^{\mathrm{i}(\psi-\phi_1-\phi_2)/2} + \mathrm{i}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\mathrm{e}^{\mathrm{i}(\psi+\phi_1+\phi_2)/2}\right), \\ z_3 &= \frac{1}{2}\left(\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\mathrm{e}^{\mathrm{i}(\psi+\phi_1-\phi_2)/2} + \mathrm{i}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\mathrm{e}^{\mathrm{i}(\psi-\phi_1+\phi_2)/2}\right), \\ z_4 &= -\frac{1}{2\mathrm{i}}\left(\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\mathrm{e}^{\mathrm{i}(\psi+\phi_1-\phi_2)/2} - \mathrm{i}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\mathrm{e}^{\mathrm{i}(\psi-\phi_1+\phi_2)/2}\right), \end{aligned} \tag{5.105}$$

from which we see they have charge 3/2 under the Reeb vector field

$$\mathcal{L}_\xi z_a = \tfrac{3}{2}\mathrm{i} z_a. \tag{5.106}$$

We can take the following frame for $\mathrm{T}^{1,1}$

$$\begin{aligned} e^1 &= \tfrac{1}{3}(\mathrm{d}\psi + \cos\theta_1\mathrm{d}\phi_1 + \cos\theta_2\mathrm{d}\phi_2), \\ e^2 + \mathrm{i}e^5 &= \tfrac{1}{\sqrt{6}}\mathrm{e}^{\mathrm{i}\psi/2}(\mathrm{i}\,\mathrm{d}\theta_1 + \sin\theta_1\mathrm{d}\phi_1), \\ e^4 + \mathrm{i}e^3 &= \tfrac{1}{\sqrt{6}}\mathrm{e}^{\mathrm{i}\psi/2}(\mathrm{i}\,\mathrm{d}\theta_2 + \sin\theta_2\mathrm{d}\phi_2). \end{aligned} \tag{5.107}$$

The complex, symplectic and contact structures are defined in terms of the frame in (5.45). One can check they satisfy the correct algebraic and differential relations (5.40)–(5.42).

The function f defining the marginal deformations is of weight three under the Reeb vector and a restriction of a holomorphic function on the conifold. Thus f must be a quadratic function of the z_a, namely

$$f = f^{ab}z_a z_b = f^{\alpha\beta,\dot\alpha\dot\beta}A_\alpha B_{\dot\alpha}A_\beta B_{\dot\beta}, \tag{5.108}$$

where f^{ab} is symmetric and traceless (by condition (5.102)), or analogously $f^{\alpha\beta,\dot\alpha\dot\beta}$ is symmetric in $\alpha\beta$ and $\dot\alpha\dot\beta$. These deformations are the $\mathrm{SU}(2)\times\mathrm{SU}(2)$-breaking deformations in (5.10) and generically give nine complex parameters. We remove six real degrees of freedom when solving the moment maps to account for the broken $\mathrm{SU}(2)\times\mathrm{SU}(2)$ symmetry. The moment maps are precisely the beta function conditions given in (5.12). We can also redefine the couplings using $\mathrm{SU}(2)\times\mathrm{SU}(2)$ rotations to remove another six real degrees of freedom, leaving a three-complex dimensional space of exactly marginal deformations labelled f_β, f_2 and f_3 in (5.13). We have

$$\begin{aligned} f_\beta &\propto z_1^2 + z_2^2 - z_3^2 - z_4^2, \\ f_2 &\propto z_3^2 - z_4^2, \\ f_3 &\propto z_1^2 - z_2^2. \end{aligned} \tag{5.109}$$

The first of these is the β-deformation for the KW theory. The supergravity dual of the

β-deformation was worked out in [93]. One can check that using our frame for $\mathrm{T}^{1,1}$ and taking

$$f = \tfrac{1}{3}\mathrm{i}\gamma(z_1^2 + z_2^2 - z_3^2 - z_4^2), \tag{5.110}$$

our expression (5.73) reproduces the three-form fluxes that appear in the first-order β-deformed solution [93]. To our knowledge, the fluxes for the other deformations were not known before.

Unlike $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$ admits a primitive, closed $(1,1)$-form that is not exact (specifically the difference of the Kähler forms on each $\mathbb{CP}^1$), giving one more exactly marginal deformation, corresponding to a shift of the B-field on the S^2. On the gauge theory side, this corresponds to the $\mathrm{SU}(2) \times \mathrm{SU}(2)$-invariant shift in the difference of the gauge couplings in (5.10). Together with h, coming from the superpotential itself, one finds a five-dimensional conformal manifold.

5.5.3 $\mathrm{Y}^{p,q}$ gauge theories

We can repeat the analysis of the Klebanov–Witten theory for the $\mathcal{N} = 1$ quiver gauge theories of section 5.2.3. The dual geometries are the family of Sasaki–Einstein spaces known as $\mathrm{Y}^{p,q}$, which have topology $\mathrm{S}^2 \times \mathrm{S}^3$ (recall $0 \leq q \leq p$ and $\mathrm{Y}^{1,0} = \mathrm{T}^{1,1}$). The metric is [276]

$$\begin{aligned} \mathrm{d}s^2(\mathrm{Y}^{p,q}) = {} & \tfrac{1}{6}(1-y)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi) + w(y)^{-1}q(y)^{-1}\mathrm{d}y^2 + \tfrac{1}{36}w(y)q(y)(\mathrm{d}\beta + \cos\theta\mathrm{d}\phi)^2 \\ & + \tfrac{1}{9}\big(\mathrm{d}\psi - \cos\theta\mathrm{d}\phi + y(\mathrm{d}\beta + \cos\theta\mathrm{d}\phi)\big)^2, \end{aligned} \tag{5.111}$$

where the functions $w(y)$ and $q(y)$ are

$$w(y) = \frac{2(a-y^2)}{1-y}, \qquad q(y) = \frac{a - 3y^2 + 2y^3}{a - y^2}, \tag{5.112}$$

and a is related to p and q by

$$a = \frac{1}{2} - \frac{p^2 - 3q^2}{4p^3}\sqrt{4p^2 - 3q^2}. \tag{5.113}$$

The Reeb vector field is

$$\xi = 3\partial_\psi. \tag{5.114}$$

As with S^5, a holomorphic function on the cone over $\mathrm{Y}^{p,q}$ determines the marginal deformations. The complex coordinates that define the cone for a generic $\mathrm{Y}^{p,q}$ are known but rather complicated [283]. However, we need only the coordinates that can contribute to a holomorphic function with charge $+3$ under the Reeb vector – fortunately there are only three such coordinates

$$
\begin{aligned}
b_1 &= \mathrm{e}^{\mathrm{i}(\psi-\phi)} \cos^2 \frac{\theta}{2} \prod_{i=1}^{3} (y-y_i)^{1/2}, \\
b_2 &= \mathrm{e}^{\mathrm{i}(\psi+\phi)} \sin^2 \frac{\theta}{2} \prod_{i=1}^{3} (y-y_i)^{1/2}, \\
b_3 &= \mathrm{e}^{\mathrm{i}\psi} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \prod_{i=1}^{3} (y-y_i)^{1/2}.
\end{aligned}
\tag{5.115}
$$

The y_i are the roots of a certain cubic equation and are given in terms of p and q as

$$
\begin{aligned}
y_1 &= \tfrac{1}{4} p^{-1} \big(2p - 3q - (4p^2 - 3q^2)^{1/2}\big), \\
y_2 &= \tfrac{1}{4} p^{-1} \big(2p + 3q - (4p^2 - 3q^2)^{1/2}\big), \\
y_3 &= \tfrac{3}{2} - y_1 - y_2.
\end{aligned}
\tag{5.116}
$$

The coordinates b_a actually have charge $+3$ under the Reeb vector

$$
\mathcal{L}_\xi b_a = 3\mathrm{i} b_a,
\tag{5.117}
$$

and so the holomorphic function that encodes the marginal deformations will be a linear function of the b_a.

We can take the following frame for any $\mathrm{Y}^{p,q}$

$$
\begin{aligned}
e^1 &= \tfrac{1}{3} \big(\mathrm{d}\psi - \cos\theta \mathrm{d}\phi + y(\mathrm{d}\beta + \cos\theta \mathrm{d}\phi)\big), \\
e^2 + \mathrm{i} e^5 &= \mathrm{e}^{\mathrm{i}\psi/2} \left(\frac{1-y}{6}\right)^{1/2} (\mathrm{d}\theta + \mathrm{i}\sin\theta \mathrm{d}\phi), \\
e^4 + \mathrm{i} e^3 &= \mathrm{e}^{\mathrm{i}\psi/2} w(y)^{-1/2} q(y)^{-1/2} \big(\mathrm{d}y + \tfrac{1}{6}\mathrm{i} w(y) q(y) (\mathrm{d}\beta + \cos\theta \mathrm{d}\phi)\big).
\end{aligned}
\tag{5.118}
$$

The complex, symplectic and contact structures are defined in terms of the frame in (5.45). One can check they satisfy the correct algebraic and differential relations (5.40)–(5.42).

The function f defining the marginal deformations is of weight three under the Reeb vector and a restriction of a holomorphic function on the cone. Thus f must be a linear combination of the b_a, namely

$$
f = f^a b_a.
\tag{5.119}
$$

These deformations are the SU(2)-breaking deformations in (5.16) and generically give three complex parameters. We remove two real degrees of freedom when solving the moment maps to account for the broken SU(2) symmetry (leaving a U(1) unbroken). The moment maps are precisely the beta function conditions given in (5.17). We can also redefine the couplings using SU(2) rotations to remove another two real degrees of freedom, leaving a one-complex-dimensional space of exactly marginal deformations. The single independent solution is

$$f_\beta \propto b_3. \tag{5.120}$$

This is the β-deformation for the quiver gauge theory. The supergravity dual of the β-deformation for $\mathrm{Y}^{p,q}$ was worked out in [93]. One can check that using the frame for $\mathrm{Y}^{p,q}$ given in (5.118) and taking (5.120), our expression (5.73) reproduces the three-form fluxes that appear in the first-order β-deformed solution [93]. Together with h and τ (dual respectively to the axion-dilaton and the B-field on the S^2), one finds a three-dimensional conformal manifold.

5.6 Summary

In this chapter we have used exceptional generalised geometry to analyse exactly marginal deformations of $d = 4$, $\mathcal{N} = 1$ SCFTs that are dual to AdS_5 backgrounds in type II or eleven-dimensional supergravity. In the gauge theory, marginal deformations are determined by imposing F-term conditions on operators of conformal dimension three and then quotienting by the complexified global symmetry group. We have shown that the supergravity analysis gives a geometric interpretation of this gauge theory result. The marginal deformations are obtained as solutions of moment maps for the generalised diffeomorphism group that have the correct charge under the Reeb vector, which generates the $\mathrm{U}(1)_\mathrm{R}$ symmetry. If this is the only symmetry of the background, all marginal deformations are exactly marginal. If the background possesses extra isometries, there are obstructions that come from fixed points of the moment maps. The exactly marginal deformations are then given by a further quotient by these extra isometries.

For the specific case of Sasaki–Einstein backgrounds in type IIB we showed how supersymmetric deformations can be understood as deformations of generalised structures which give rise to three-form flux perturbations at first order. Using explicit examples, we checked that our expression for the three-form flux matches those in the literature and the obstruction conditions match the one-loop beta functions of the dual SCFT.

Chapter 6

Marginal deformations of $d = 3$, $\mathcal{N} = 2$ SCFTs

In the previous chapter, we set out how to find marginal deformations of $d = 4$, $\mathcal{N} = 1$ superconformal field theories by considering the dual AdS_5 solutions in type IIB supergravity. Deformations of the field theory appeared as deformations of the hypermultiplet structure, and the marginal deformations were those that have the correct charge under the generalised Reeb vector. The exactly marginal deformations were selected following an analysis of the fixed points of the moment maps, and taking a further quotient by the stabiliser group, corresponding to the broken global symmetry group.

As we emphasised, this approach is completely general and applies to deformations of arbitrary backgrounds. In particular, our analysis also applies to $\mathcal{N} = 2$ AdS_4 solutions in M-theory. Deformations of these backgrounds are dual to marginal deformations of $d = 3$, $\mathcal{N} = 2$ superconformal field theories. In this chapter, we analyse theories that arise on a stack of M2-branes at a conical singularity. The backgrounds are of the form $\mathrm{AdS}_4 \times M$, where M is a seven-dimensional Sasaki–Einstein manifold. Again, we find a first-order expression for the four-form flux that is dual to marginal deformations of the field theory and compare with known results in the literature.

6.1 Introduction

The AdS/CFT correspondence relates supergravity on backgrounds with an AdS factor to the conformal field theory living on the boundary. Usually one considers the field theory to be superconformal, with supersymmetry allowing the calculation of protected quantities. One is then interested in deformations by operators that preserve the superconformal symmetry. Classically, such operators are known as marginal. If the operators also preserve the symmetry at the quantum level, they are known as exactly marginal deformations.

In the space of marginal couplings, the exactly marginal directions are said to define the conformal manifold $\mathcal{M}_{\mathrm{c}}$.

Exactly marginal deformations of the SCFT appear in the supergravity dual as a continuous family of AdS solutions. Aharony et al. performed a perturbative analysis of $\mathrm{AdS}^5 \times \mathrm{S}^5$ that identified the marginal deformations and found an obstruction at third order in the deformation reminiscent of the one-loop beta-function [271]. Later, Lunin and Maldacena proposed a method for generating AdS solutions from backgrounds possessing at least two U(1) isometries [93]. The new solutions are dual to exactly marginal deformations of the SCFT known as β-deformations. Unlike the perturbative approach, the solution-generating technique gives the full analytic supergravity backgrounds to all orders in the deformation. Ideally one would like to find the analytic solutions dual to the other marginal deformations.

The solution-generating technique of Lunin and Maldacena also applies to M-theory backgrounds with three U(1) isometries, where it has been used to find new AdS_4 solutions by deforming S^7, $\mathrm{Q}^{1,1,1}$, $\mathrm{M}^{1,1,1}$ and others [93, 284, 285]. Unlike $\mathrm{AdS}^5 \times \mathrm{S}^5$, there has not been a perturbative analysis of the marginal deformations of $\mathrm{AdS}_4 \times \mathrm{S}^7$, however there is some guidance from the dual field theory. The S^7 solution preserves $\mathcal{N} = 8$ supersymmetry, or 32 supercharges, and arises as the near-horizon limit of a stack of M2-branes in flat space. The dual three-dimensional CFT living on the branes has an SO(8) global symmetry coming from the eight directions transverse to the branes. Although the theory does not have a Lagrangian description, there has been a proposal for the number of exactly marginal deformations [267]. The couplings that preserve eight supercharges define a conformal manifold

$$\mathcal{M}_{\mathrm{c}} = \mathbf{35}/\mathrm{SU}(4)_{\mathbb{C}} = \mathbf{35}/\!\!/\mathrm{SU}(4), \tag{6.1}$$

where SU(4) is the broken global symmetry group, and $\mathbf{35}$ is the rank-four symmetric tensor of SU(4). From this we expect the exactly marginal deformations to be determined by 20 complex functions. The existence of a conformal manifold for $\mathcal{N} = 2$ Chern–Simons-matter theories was first found in [286–289] following explicit calculations, and the calculation was extended to an all-orders weak-coupling argument in [290].

The analysis of the conformal manifold was systematised in [269] for $\mathcal{N} = 1$, $d = 4$ SCFTs that may be strongly coupled; we reviewed the results of this work in section 5.2. The use of the superconformal algebra to constrain the allowed deformations generalises to $\mathcal{N} = 2$ theories in three dimensions as the multiplets are similar in structure to those of the four-dimensional theories. In this case, the marginal deformations of the superpotential are given by chiral primary superfields of dimension $\Delta = 2$. The other possible deformations come from real primary superfields of dimension $\Delta = 1$, but these are conserved currents and so there are no deformations. This mirrors the analysis in the $d = 4$ case: the conformal manifold, near to the undeformed theory, is again simply the quotient of the space of marginal couplings by the complexified broken global symmetry group.

In this chapter, we use the language of generalised geometry to find supersymmetric deformations of AdS_4 times Sasaki–Einstein backgrounds in M-theory. The deformations turn on a four-form flux perturbation that is dual to a marginal deformation in the field theory. Our analysis applies to any Sasaki–Einstein background preserving at least eight supercharges, including S^7 and the previously mentioned examples. We find the marginal deformations are encoded in a function of charge four under the Reeb vector that is holomorphic on the Calabi–Yau cone over the Sasaki–Einstein manifold. In particular, for S^7 we find the marginal deformations are defined by a quartic function of the complex coordinates z_i on $\mathbb{C}^4$. Such a quartic function generically has 35 complex degrees of freedom. The obstruction appears in our formalism as an extra symplectic quotient that reduces this to 20 complex degrees of freedom, agreeing with the counting from the dual field theory. We also carry out the same analysis for $\mathrm{Q}^{1,1,1}$ and $\mathrm{M}^{1,1,1}$.

We begin in section 6.2 by finding the algebraic form of the linearised deformation. We then examine the differential conditions imposed by integrability and give the four-form flux generated by the deformation. The expression for the flux is valid for any Sasaki-Einstein background and includes the linearised fluxes found using the solution-generating technique of Lunin and Maldacena as a special case. In section 6.3 we look at the examples of S^7, $\mathrm{Q}^{1,1,1}$ and $\mathrm{M}^{1,1,1}$ and find agreement with the known results.

6.2 Linearised deformations

Backgrounds of the form $\mathrm{AdS}_4 \times M$, where M is a seven-dimensional Sasaki–Einstein manifold, are supersymmetric solutions of eleven-dimensional supergravity preserving at least eight supercharges [64]. They are dual to the three-dimensional superconformal field theory living on a stack of M2-branes placed at the tip of the corresponding Calabi-Yau cone.[1] As we showed in section 4.5, these solutions can be formulated as SU(6) structures with singlet torsion within $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ generalised geometry. We now want to investigate the possible deformations of this structure that are still integrable. In other words, we look for deformations of the supergravity background that preserve eight supercharges. We expect these to be dual to exactly marginal deformations in the field theory.

J_α and K define $\mathrm{Spin}^*(12)$ and $\mathrm{E}_{6(2)}$ structures respectively and together they define an SU(6) structure. As marginal deformations of the field theory are dual to deformations of the hypermultiplets in supergravity, we should vary J_α while keeping K fixed. Thus we want

$$\delta K = \mathcal{A} \cdot K = 0, \qquad \delta J_\alpha = [\mathcal{A}, J_\alpha] \neq 0, \tag{6.2}$$

for some $\mathcal{A} \in \Gamma(\operatorname{ad} \tilde{F})$. As the deformations leave the $\mathrm{E}_{6(2)}$ structure invariant, at a point

[1] The most studied example of M-theory on $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$ is dual to the Chern–Simons-matter theory living on a stack of M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity [291]. Of these only S^7 and $\mathrm{S}^7/\mathbb{Z}_2$ are Sasaki–Einstein. The supersymmetry parameters transform in the $\mathbf{8}$ of SU(8). This breaks under SU(6) as $\mathbf{8} = \mathbf{6} + \mathbf{1} + \mathbf{1}$. Viewing S^7 as Sasaki–Einstein picks out the two supercharges that are singlets under the SU(6) structure. These are not the supercharges that are picked out in ABJM theory, which live in the $\mathbf{6}$ instead. Thus we can view S^7 and $\mathrm{S}^7/\mathbb{Z}_2$ as Sasaki–Einstein, but not further quotients.

on the internal manifold $\mathcal{A}$ parametrises an element of $E_{6(2)}/SU(6)$. The adjoint of $E_{6(2)}$ decomposes under $SU(6) \times SU(2)$ as

$$\mathbf{78} = (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{35}, \mathbf{1}) \oplus (\mathbf{20}, \mathbf{2}). \tag{6.3}$$

The first term corresponds to SU(2) rotations of the triplet J_α. The second term is the adjoint of SU(6), which leaves both J_α and K invariant by definition. Therefore, the deformations are in the $(\mathbf{20}, \mathbf{2})$ and form a doublet under the SU(2) defined by J_α. We are free to choose them to be eigenstates of J_3

$$[J_3, \mathcal{A}_\lambda] = \mathrm{i}\lambda\kappa\mathcal{A}_\lambda. \tag{6.4}$$

The non-zero eigenstates correspond to $\lambda = 0, 1, 2$. The $\lambda = \pm 2$ eigenstates correspond to $J_\pm$. The $\lambda = 0$ eigenstates are in SU(6), so will leave J_α and K invariant. The deformations we seek are the $\lambda = \pm 1$ eigenstates, which we refer to as $\mathcal{A}_\pm$. We note that we can find an eigenstate with eigenvalue $-\mathrm{i}\kappa$ from $\mathcal{A}_+$ by acting with J_+ thanks to the Jacobi identity

$$[J_3, \kappa^{-1}[J_\pm, \mathcal{A}_\pm]] = \mp\mathrm{i}\kappa[J_\pm, \mathcal{A}_\pm]. \tag{6.5}$$

Complex conjugation gives an eigenstate with the opposite eigenvalue. We notice that L_K commutes with the action of J_3 (as $L_K J_3 = 0$) so we can also label states by their R-charge. We organise the states into doublets

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_-^{(r)} \\ \mathcal{A}_+^{(r-2)} \end{pmatrix}, \qquad r \geq 0, \tag{6.6}$$

where doublets with $r \leq 0$ are related by complex conjugation.

To find the explicit form of the eigenstates, it is useful to note that the action of J_3 splits into separate actions on $\{a, \alpha\}$ and $\{r, \tilde{a}, \tilde{\alpha}\}$. We then organise the eigenstates as

$$\mathcal{A}_+ = a + \alpha, \qquad \mathcal{A}_- = \kappa^{-1}[J_+, \mathcal{A}_+] = r + \tilde{a} + \tilde{\alpha}. \tag{6.7}$$

Using this as a basis, the modes $\{\mathcal{A}_+, \mathcal{A}_-^*\}$ give the possible $+\mathrm{i}\kappa$ eigenstates. Using the algebraic conditions on ω, Ω and I, it is simple to show that the $\mathcal{A}_+$ eigenstate is

$$\mathcal{A}_+ = \left[\phi + f\bar{\Omega} + \sigma \wedge (\nu^\sharp \lrcorner \bar{\Omega})\right] + \left[\phi^\sharp + f\bar{\Omega}^\sharp + \xi \wedge (\bar{\Omega}^\sharp \lrcorner \nu)\right], \tag{6.8}$$

where ϕ is a $(1,2)$-form, f is a function, $\bar{\Omega}$ is the conjugate of the complex three-form, σ is the contact form and ν is a $(1,0)$-form. Notice that the components are related by $a_{mnp} = \alpha_{mnp}$, where we lower the indices of α using the undeformed metric.[2] The $\mathcal{A}_-$ mode in the same doublet as $\mathcal{A}_+$ is given by $\mathcal{A}_- = \kappa^{-1}[J_+, \mathcal{A}_+]$.

[2]The three-form components of the USp(8) Lie algebra embed in $E_{7(7)}$ as $a_{mnp} = -\alpha_{mnp}$, so the deformation is not in $\mathfrak{usp}_8$.

6.2.1 Supersymmetry conditions

We are looking for deformations of the Sasaki–Einstein background that preserve supersymmetry, so that the deformed K and J_α still define a torsion-free SU(6) structure. At linear order in the deformation, the supersymmetry conditions reduce to

$$\delta\mu_\alpha(V) = \int \kappa\,\mathrm{tr}(J_\alpha L_V \mathcal{A}) = 0 \qquad \forall\, V \in \mathbf{56}, \tag{6.9}$$

$$[L_K \mathcal{A}, J_\alpha] = 0. \tag{6.10}$$

As the deformed structures should be real, we take the deformation to be $\mathcal{A} = \mathrm{Re}\,\mathcal{A}_+$, where $\mathrm{Re}\,\mathcal{A}_+ = \frac{1}{2}(\mathcal{A}_+ + \mathcal{A}_+^*)$. In this section we give the derivation of the constraints that these equations impose on $\mathcal{A}_+$.

For what follows, it is useful to note that the contractions of the components of $\mathcal{A}_+$ with the volume form are

$$\begin{aligned}
\big(\xi \wedge (\bar{\Omega}^\sharp \lrcorner \nu)\big) \lrcorner\, \mathrm{vol}_7 &= \mathrm{i}\nu \wedge \bar{\Omega}, \\
f\bar{\Omega}^\sharp \lrcorner\, \mathrm{vol}_7 &= -\mathrm{i} f \sigma \wedge \bar{\Omega}, \\
\phi^\sharp \lrcorner\, \mathrm{vol}_7 &= \mathrm{i}\sigma \wedge \phi.
\end{aligned} \tag{6.11}$$

We also use the identities

$$\mathrm{vol}_7(\alpha \lrcorner a) = (\alpha \lrcorner\, \mathrm{vol}_7) \wedge a, \qquad \mathrm{vol}_7(\tilde{\alpha} \lrcorner \tilde{a}) = (\tilde{\alpha} \lrcorner\, \mathrm{vol}_7) \wedge \tilde{a}, \tag{6.12}$$

where a, α, $\tilde{a}$ and $\tilde{\alpha}$ are an arbitrary three-form, three-vector, six-form and six-vector respectively.

Moment map conditions

The calculation of the conditions follows closely that of appendix H.2. For example, the variation of μ_3 is of a similar form as that for the previous AdS$_5$ case:

$$\int \kappa\,\mathrm{tr}(J_3 L_V \mathcal{A}) = \int \kappa\,\mathrm{tr}(J_3[\mathrm{d}\omega, \mathcal{A}]) \propto \int \kappa\,\mathrm{tr}(\mathrm{d}\omega[J_3, \mathcal{A}]). \tag{6.13}$$

We then note that $[J_3, \mathcal{A}] \propto \mathcal{A}_+ - \mathcal{A}_-$, which gives

$$\int \kappa\,\mathrm{tr}(\mathrm{d}\omega[J_3, \mathcal{A}]) \propto \int \kappa^2\,\mathrm{Im}\,\alpha \lrcorner \mathrm{d}\omega \propto \int \mathrm{d}(\mathrm{Im}\,\alpha \lrcorner\, \mathrm{vol}_7) \wedge \omega, \tag{6.14}$$

where α is the three-vector component of $\mathcal{A}_+$. As this should vanish for arbitrary ω, we require

$$\mathrm{d}(\mathrm{Im}\,\alpha \lrcorner\, \mathrm{vol}_7) = 0. \tag{6.15}$$

Using the explicit form of $\mathcal{A}_+$ (6.8) and the contractions of its components with the volume (6.11), the conditions from $\delta\mu_3 = 0$ and $\delta\mu_+ = 0$ reduce to

$$\begin{aligned} \partial\phi + \bar{\partial}\bar{\phi} &= 0, \\ \partial\nu &= 0, \\ \bar{\partial}\phi + 4\mathrm{i}\nu\wedge\bar{\Omega} &= 0, \\ \bar{\partial}\nu - 2f\omega &= 0, \end{aligned} \qquad \begin{aligned} \mathcal{L}_\xi\nu + \partial f &= 0, \\ \bar{\partial} f &= 0, \\ \bar{\partial}(\nu\lrcorner\bar{\Omega}) + 6\mathrm{i}f\bar{\Omega} &= 0, \end{aligned} \tag{6.16}$$

where we have simplified some of the relations using

$$v\wedge\bar{\Omega} + \mathrm{i}\omega\wedge(v\lrcorner\bar{\Omega}) = 0, \tag{6.17}$$

where v is an arbitrary $(1,0)$-form with respect to I.

We now want to solve the system of differential equations to find the general form of the deformation. Following a method similar to that in appendix H.2, one can show that a solution is given by

$$\bar{\partial} f = 0, \qquad \mathcal{L}_\xi f = \mathrm{i}qf, \qquad \nu = \frac{\mathrm{i}}{q}\partial f, \qquad \phi = \frac{1}{2q(q-1)}\partial(\partial f\lrcorner\bar{\Omega}). \tag{6.18}$$

One can check this solves the system of equations using

$$\bar{\partial}(\partial f\lrcorner\bar{\Omega}) = -6qf\bar{\Omega}, \tag{6.19}$$

where f is a holomorphic with respect to $\bar{\partial}$ and has charge $+q$ under the Reeb vector field. One can also include in ϕ a $(1,2)$-form χ that is closed. The three-form component of the deformation is then

$$\mathcal{A}_+ = f\bar{\Omega} + \frac{\mathrm{i}}{q}\sigma\wedge(\partial f\lrcorner\bar{\Omega}) + \frac{1}{2q(q-1)}\partial(\partial f\lrcorner\bar{\Omega}) + \chi, \tag{6.20}$$

where $\bar{\partial} f = 0$, $\mathcal{L}_\xi f = \mathrm{i}qf$ and $\mathrm{d}\chi = 0$. The three-vector component is given by substituting (6.18) into (6.8) or by raising the indices of the three-form component using the undeformed metric.

Lie derivative along *K*

Now that we have satisfied the moment map conditions we must impose the Dorfman derivative condition. At first order this is given by (6.10). The commutators are non-zero for both J_+ and J_3 so that the condition reduces to $L_K\mathcal{A} = 0$. As K is a generalised Killing vector, the Dorfman derivative along K reduces to a Lie derivative along the Reeb vector field ξ, so the deformation condition is

$$\mathcal{L}_\xi\mathcal{A} = 0.$$

We see that the deformation must have charge zero under the Reeb vector field. From the explicit form $\mathcal{A}_+$, we see f is charge $+4$ and the closed term χ is charge zero. These are the conditions for a deformation to be marginal.

6.2.2 Fluxes

As for the previous type IIB case, the three-vector component of the deformation can be traded for a three-form deformation by considering its action on the generalised metric. The three-form potential induced by a first-order deformation by a three-form a and a three-vector α is

$$A_{mnp} = a_{mnp} + \alpha_{mnp} = 2a_{mnp},$$

where we lower the indices of the three-vector using the undeformed metric. Obviously, this procedure becomes more complicated at higher orders in the deformation due to contractions between the three-form and three-vector terms that can correct the metric, warp factor and fluxes.

The real deformation will generate a three-form potential of the form

$$A = 2\,\mathrm{Re}\Big(f\bar{\Omega} + \tfrac{1}{4}\mathrm{i}\sigma \wedge (\partial f \lrcorner \bar{\Omega}) + \tfrac{1}{24}\partial(\partial f \lrcorner \bar{\Omega}) + \chi\Big). \tag{6.21}$$

The flux due to A is

$$F = \mathrm{Re}\Big(\tfrac{3}{2}\partial f \wedge \bar{\Omega} - \tfrac{\mathrm{i}}{4}\sigma \wedge \partial(\partial f \lrcorner \bar{\Omega}) + 6\mathrm{i} f \sigma \wedge \bar{\Omega}\Big). \tag{6.22}$$

This flux is valid for the marginal deformations of *any* seven-dimensional Sasaki–Einstein background and, as we will discuss, it reproduces the first-order fluxes of the β-deformation of Lunin and Maldacena [93].

6.3 Examples

In the previous section we used the existence of a torsion-free generalised structure to derive the first-order fluxes that are dual to exactly marginal deformations for any Sasaki–Einstein background. We now give the explicit examples of the supergravity backgrounds where the internal space is S^7, $\mathrm{Q}^{1,1,1}$ or $\mathrm{M}^{1,1,1}$. In what follows, it proves useful to take an orthonormal frame on M in which the invariant objects defining the Sasaki–Einstein structure are

$$\Omega = (e^1 + \mathrm{i}e^2) \wedge (e^3 + \mathrm{i}e^4) \wedge (e^5 + \mathrm{i}e^6), \qquad \omega = e^{12} + e^{34} + e^{56}, \qquad \sigma = e^7. \tag{6.23}$$

6.3.1 $\mathbf{S^7}$

As an AdS_4 background in M-theory, the seven-sphere S^7 preserves 32 supercharges. When viewed as a Sasaki-Einstein manifold, we pick out eight of these supercharges – it is these supercharges that will be preserved by the first-order flux we have given. We can view S^7 as a U(1) fibration over $\mathbb{CP}^3$, a six-dimensional Kähler–Einstein base. The metric on S^7 can be written as[3] [93]

$$\mathrm{d}s^2(\mathrm{S}^7) = \mathrm{d}\theta^2 + s_\theta^2\mathrm{d}\alpha^2 + s_\theta^2 s_\alpha^2\mathrm{d}\beta^2 + c_\theta^2\mathrm{d}\phi_1^2 + s_\theta^2 c_\alpha^2\mathrm{d}\phi_2^2 + s_\theta^2 s_\alpha^2 c_\beta^2\mathrm{d}\phi_3^2 + s_\theta^2 s_\alpha^2 s_\beta^2\mathrm{d}\phi_4^2. \quad (6.24)$$

We can introduce an explicit frame in terms of the coordinates on S^7:

$$\begin{aligned}
e^1 + \mathrm{i}e^2 &= \mathrm{e}^{4\mathrm{i}\psi/3}(\mathrm{d}\theta - \mathrm{i}s_\theta c_\theta\mathrm{d}\phi_1 + \mathrm{i}s_\theta c_\theta c_\alpha^2\mathrm{d}\phi_2 + \mathrm{i}s_\theta c_\theta c_\beta^2 s_\alpha^2\mathrm{d}\phi_3 + \mathrm{i}s_\theta c_\theta s_\alpha^2 s_\beta^2\mathrm{d}\phi_4),\\
e^3 + \mathrm{i}e^4 &= \mathrm{e}^{4\mathrm{i}\psi/3}(s_\theta\mathrm{d}\alpha - \mathrm{i}s_\alpha c_\alpha s_\theta\mathrm{d}\phi_2 + \mathrm{i}s_\alpha c_\alpha s_\theta c_\beta^2\mathrm{d}\phi_3 + \mathrm{i}s_\alpha c_\alpha s_\theta s_\beta^2\mathrm{d}\phi_4),\\
e^5 + \mathrm{i}e^6 &= \mathrm{e}^{4\mathrm{i}\psi/3}(s_\alpha s_\theta\mathrm{d}\beta - \mathrm{i}s_\beta c_\beta s_\theta s_\alpha\mathrm{d}\phi_3 + \mathrm{i}s_\alpha s_\beta c_\beta s_\theta\mathrm{d}\phi_4),\\
e^7 &= c_\theta^2\mathrm{d}\phi_1 + s_\theta^2 c_\alpha^2\mathrm{d}\phi_3 + s_\alpha^2 s_\theta^2 c_\beta^2\mathrm{d}\phi_3 + s_\theta^2 s_\alpha^2 s_\beta^2\mathrm{d}\phi_4,
\end{aligned} \quad (6.25)$$

where $4\psi = \phi_1+\phi_2+\phi_3+\phi_4$. Using this frame, one can check that the complex, symplectic and contact structures given in (6.23) satisfy the algebraic and differential conditions (4.93), (4.94) and (4.96).

Up to closed three-forms, the marginal deformations are parametrised by a holomorphic function f that descends from the Calabi–Yau cone over S^7. The function f is of charge four under the Reeb vector. In our parametrisation, the Reeb vector field is

$$\xi = 4\partial_\psi = \partial_{\phi_1} + \partial_{\phi_2} + \partial_{\phi_3} + \partial_{\phi_4}. \quad (6.26)$$

The cone over S^7 is $\mathbb{C}^4$, and the coordinates on S^7 are related to the usual complex coordinates on $\mathbb{C}^4$ by

$$z_1 = c_\theta\mathrm{e}^{\mathrm{i}\phi_1}, \quad z_2 = s_\theta c_\alpha\mathrm{e}^{\mathrm{i}\phi_2}, \quad z_3 = s_\theta s_\alpha c_\beta\mathrm{e}^{\mathrm{i}\phi_3}, \quad z_4 = s_\theta s_\alpha s_\beta\mathrm{e}^{\mathrm{i}\phi_4}, \quad (6.27)$$

where the coordinates z_i have charge +1 under the Reeb vector field

$$\mathcal{L}_\xi z_i = \mathrm{i}z_i. \quad (6.28)$$

Thus f must be a quartic function of the z_i. The general form of such a function is

$$f = f^{ijkl}z_i z_j z_k z_l, \quad (6.29)$$

where f^{ijkl} is a complex symmetric tensor of SU(4). There are generically 35 complex degrees of freedom in such a symmetric rank-four tensor, corresponding to the the 35 marginal deformations previously discussed by Kol [267]. Requiring our first-order perturbation to extend to higher orders forces us to consider if there are fixed-point isometries at the S^7 point in the space of couplings. We can think of S^7 as a U(1) fibration over a $\mathbb{CP}^3$ base, where the SU(4) that acts on the base leaves the S^7 solution invariant. In

[3]We are using s_α and c_α as shorthand for $\sin\alpha$ and $\cos\alpha$, and similarly for β and θ.

other words, $S^7 = SU(4)/SU(3)$ where the action of SU(4) preserves the U(1) fibration. This is not true of the other presentations of S^7 as a homogeneous space. This means we have an SU(4)'s worth of fixed-point symmetries, where the marginal deformations defined by f generically break this SU(4). To account for this we construct a moment map for the SU(4) action on the space of couplings and perform a symplectic reduction. The deformations that survive are those that extend to higher orders, namely the exactly marginal deformations. These deformations satisfy

$$f^{iklm}\bar{f}_{jklm} - \tfrac{1}{4}\delta^i{}_j f^{klmn}\bar{f}_{klmn} = 0. \tag{6.30}$$

This removes 15 real degrees of freedom and we can use the SU(4) action to remove another 15 real degrees of freedom, leaving 20 complex parameters, in agreement with the counting given by Kol [267]. Recall that $H^3(S^7) = 0$ and so there are no marginal deformations due to closed (1, 2)-forms χ.

The β-deformed S^7 solution was first given in [93], which we reproduce in appendix I. Taking $f \propto i\gamma z_1 z_2 z_3 z_4$, where γ is real, and using our frame for S^7, one can check that our expression (6.22) reproduces the four-form flux of the first-order β-deformed S^7 solution. Notice that we can also take $f \propto \gamma z_1 z_2 z_3 z_4$, where we have dropped a factor of i compared with the LM solution. This will also solve the moment map conditions and thus is a marginal deformation, similar to the full complex β-deformation of $\mathcal{N} = 4$.

6.3.2 $Q^{1,1,1}$

As an AdS_4 background in M-theory, the Sasaki–Einstein manifold $Q^{1,1,1}$ preserves eight supercharges. Viewing $Q^{1,1,1}$ as a U(1) fibration over $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$, the metric[4] can be written as [292, 293]

$$ds^2(Q^{1,1,1}) = \tfrac{1}{16}\Big(d\psi + \sum_{i=1}^{3}\cos\theta_i d\phi_i\Big)^2 + \tfrac{1}{8}\sum_{i=1}^{3}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2). \tag{6.31}$$

We can introduce an explicit frame in terms of the coordinates on $Q^{1,1,1}$:

$$\begin{aligned} e^1 + ie^2 &= \tfrac{1}{2\sqrt{2}}e^{i\psi/3}(i\,d\theta_1 + \sin\theta_1 d\phi_1),\\ e^3 + ie^4 &= \tfrac{1}{2\sqrt{2}}e^{i\psi/3}(i\,d\theta_2 + \sin\theta_2 d\phi_2),\\ e^5 + ie^6 &= \tfrac{1}{2\sqrt{2}}e^{i\psi/3}(i\,d\theta_3 + \sin\theta_3 d\phi_3),\\ e^7 &= \tfrac{1}{4}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2 + \cos\theta_3 d\phi_3). \end{aligned} \tag{6.32}$$

Using this frame, one can check that the complex, symplectic and contact structures given in (6.23) satisfy the algebraic and differential conditions (4.93), (4.94) and (4.96).

Up to closed three-forms, the deformation is parametrised by a holomorphic function

[4] The metric has been scaled to ensure $R_{\mu\nu} = 6g_{\mu\nu}$.

f that descends from the Calabi–Yau cone over $\mathrm{Q}^{1,1,1}$. The deformations are marginal if f is of weight four under the Reeb vector. In our parametrisation, the Reeb vector is

$$\xi = 4\partial_\psi. \tag{6.33}$$

The cone over $\mathrm{Q}^{1,1,1}$ is described by an embedding in $\mathbb{C}^8$ using eight complex coordinates w_i that satisfy nine constraint equations. The explicit form of the coordinates is [294]

$$\begin{aligned}
w_1 &= \mathrm{e}^{\frac{\mathrm{i}}{2}(\psi+\phi_1+\phi_2+\phi_3)} c_{\theta_1/2} c_{\theta_2/2} c_{\theta_3/2}, \quad w_2 = \mathrm{e}^{\frac{\mathrm{i}}{2}(\psi-\phi_1-\phi_2-\phi_3)} s_{\theta_1/2} s_{\theta_2/2} s_{\theta_3/2}, \\
w_3 &= \mathrm{e}^{\frac{\mathrm{i}}{2}(\psi+\phi_1-\phi_2-\phi_3)} c_{\theta_1/2} s_{\theta_2/2} s_{\theta_3/2}, \quad w_4 = \mathrm{e}^{\frac{\mathrm{i}}{2}(\psi-\phi_1+\phi_2+\phi_3)} s_{\theta_1/2} c_{\theta_2/2} c_{\theta_3/2}, \\
w_5 &= \mathrm{e}^{\frac{\mathrm{i}}{2}(\psi+\phi_1+\phi_2-\phi_3)} c_{\theta_1/2} c_{\theta_2/2} s_{\theta_3/2}, \quad w_6 = \mathrm{e}^{\frac{\mathrm{i}}{2}(\psi-\phi_1+\phi_2-\phi_3)} s_{\theta_1/2} c_{\theta_2/2} s_{\theta_3/2}, \\
w_7 &= \mathrm{e}^{\frac{\mathrm{i}}{2}(\psi+\phi_1-\phi_2+\phi_3)} c_{\theta_1/2} s_{\theta_2/2} c_{\theta_3/2}, \quad w_8 = \mathrm{e}^{\frac{\mathrm{i}}{2}(\psi-\phi_1-\phi_2+\phi_3)} s_{\theta_1/2} s_{\theta_2/2} c_{\theta_3/2}.
\end{aligned} \tag{6.34}$$

The embedding coordinates w_i are charge $+2$ under the Reeb vector field, so the general form of the function f is

$$f = f^{ij} w_i w_j, \tag{6.35}$$

where f^{ij} is symmetric with complex entries. There are generically 36 complex degrees of freedom in such a symmetric rank-two tensor, but 9 of them will not contribute to f due to the constraints on the w_i. Thus there are 27 complex degrees of freedom corresponding to 27 marginal deformations. We can also use homogeneous coordinates A_a, $B_{\dot{a}}$ and $C_{\ddot{a}}$ that are related to the w_i by [295]

$$\begin{aligned}
&w_1 = A_1 B_2 C_1, \quad w_2 = A_2 B_1 C_2, \quad w_3 = A_1 B_1 C_2, \quad w_4 = A_2 B_2 C_1, \\
&w_5 = A_1 B_1 C_1, \quad w_6 = A_2 B_1 C_1, \quad w_7 = A_1 B_2 C_2, \quad w_8 = A_2 B_2 C_2.
\end{aligned} \tag{6.36}$$

We can then write the generic deformation as

$$f = f^{ab,\dot{a}\dot{b},\ddot{a}\ddot{b}} A_a B_{\dot{a}} C_{\ddot{a}} A_b B_{\dot{b}} C_{\ddot{b}}, \tag{6.37}$$

where $f^{ab,\dot{a}\dot{b},\ddot{a}\ddot{b}}$ is symmetric in (ab), $(\dot{b}\dot{b})$ and $(\ddot{a}\ddot{b})$. We can think of $\mathrm{Q}^{1,1,1}$ as a U(1) fibration over a $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ base, so there is an $\mathrm{SU}(2)^3$ isometry that leaves the solution invariant, and we have an $\mathrm{SU}(2)^3$ of fixed-point symmetries. Again, we want to take a symplectic reduction of the space of couplings by the action of $\mathrm{SU}(2)^3$. The moment map for the first SU(2) action is

$$\mu_{\mathrm{SU}(2)} = f^{ac,\dot{a}\dot{b},\ddot{a}\ddot{b}} \bar{f}_{bc,\dot{a}\dot{b},\ddot{a}\ddot{b}} - \tfrac{1}{2}\delta^a{}_b f^{cd,\dot{a}\dot{b},\ddot{a}\ddot{b}} \bar{f}_{cd,\dot{a}\dot{b},\ddot{a}\ddot{b}}, \tag{6.38}$$

and the others follow by swapping undotted for dotted or double-dotted indices. The conformal manifold of exactly marginal deformations that preserve eight supercharges is given by the symplectic reduction

$$\mathcal{M}_{\rm c} = \{f^{ab,\dot{a}\dot{b},\ddot{a}\ddot{b}}\}/\!\!/\mathrm{SU}(2)^3. \tag{6.39}$$

The three moment maps for SU(2) gives 9 real conditions on the $f^{ab,\dot{a}\dot{b},\ddot{a}\ddot{b}}$, and we can remove another 9 degrees of freedom using $\mathrm{SU}(2)^3$ rotations of the couplings. In addition, $\mathrm{H}^3(\mathrm{Q}^{1,1,1}) = 0$ and so there are no marginal deformations due to closed $(1,2)$-forms χ. Thus, the conformal manifold is $27 - 9 = 18$ complex dimensional.

The β-deformed $\mathrm{Q}^{1,1,1}$ solution was first given in [284, 285], which we reproduce in appendix I. Taking $f \propto \gamma w_1 w_2 = \gamma A_1 A_2 B_1 B_2 C_1 C_2$, where γ is real, and using our frame for $\mathrm{Q}^{1,1,1}$, one can check that our expression (6.22) reproduces the four-form flux of the first-order β-deformed solution.

6.3.3 $\mathbf{M^{1,1,1}}$

As an AdS_4 background in M-theory, the Sasaki–Einstein manifold $\mathrm{M}^{1,1,1}$ preserves eight supercharges. Following the presentation in [296], the metric on $\mathrm{M}^{1,1,1}$ can be written as

$$\begin{aligned}\mathrm{d}s^2(\mathrm{M}^{1,1,1}) = \tfrac{3}{4}\Big(&\mathrm{d}\mu^2 + \tfrac{1}{4}s_\mu^2 c_\mu^2(\mathrm{d}\psi + c_{\tilde\theta}\mathrm{d}\tilde\phi)^2 + \tfrac{1}{4}s_\mu^2(\mathrm{d}\tilde\theta^2 + s_{\tilde\theta}^2\mathrm{d}\tilde\phi^2)\Big)\\ &+ \tfrac{1}{8}(\mathrm{d}\theta^2 + s_\theta^2\mathrm{d}\phi^2) + \tfrac{1}{64}(\mathrm{d}\tau + \lambda + 2c_\theta\mathrm{d}\phi)^2,\end{aligned} \tag{6.40}$$

where $\lambda = \frac{1}{2}(1 + 3\cos 2\mu)\mathrm{d}\psi - 3\cos\tilde\theta\sin^2\mu\mathrm{d}\tilde\phi$.[5] We can introduce an explicit frame in terms of the coordinates on $\mathrm{M}^{1,1,1}$:

$$\begin{aligned} e^1 + \mathrm{i}e^2 &= \tfrac{\sqrt{3}}{2}\mathrm{e}^{\mathrm{i}\tau/6}\Big(\mathrm{d}\mu - \tfrac{1}{4}\mathrm{i}\sin 2\mu(\mathrm{d}\psi + \cos\tilde\theta\mathrm{d}\tilde\phi)\Big),\\ e^3 + \mathrm{i}e^4 &= \tfrac{\sqrt{3}}{4}\mathrm{e}^{\mathrm{i}\tau/6}\sin\mu(\mathrm{d}\tilde\theta + \mathrm{i}\sin\tilde\theta\mathrm{d}\tilde\phi),\\ e^5 + \mathrm{i}e^6 &= \tfrac{1}{2\sqrt{2}}\mathrm{e}^{\mathrm{i}\tau/6}(\mathrm{d}\theta - \mathrm{i}\sin\theta\mathrm{d}\phi),\\ e^7 &= \tfrac{1}{8}(\mathrm{d}\tau + \lambda + 2\cos\theta\mathrm{d}\phi).\end{aligned} \tag{6.41}$$

Using this frame, one can check that the complex, symplectic and contact structures given in (6.23) satisfy the algebraic and differential conditions (4.93), (4.94) and (4.96).

Up to closed three-forms, the deformation is parametrised by a holomorphic function f that descends from the Calabi–Yau cone over $\mathrm{M}^{1,1,1}$. The deformations are marginal if f is of weight four under the Reeb vector. In our parametrisation, the Reeb vector is

$$\xi = 8\partial_\tau. \tag{6.42}$$

The cone over $\mathrm{M}^{1,1,1}$ is described by an embedding in $\mathbb{C}^{30}$ [297]. Instead we use homogeneous coordinates U_i and V_a which are charge $+8/9$ and $+2/3$ respectively under the Reeb vector

[5] Note that the λ we use differs from that of [284, 296] by $2\mathrm{d}\psi$.

field [285], so the general form of the function f is

$$f = f^{ijk,ab} U_i U_j U_k V_a V_b, \tag{6.43}$$

where $f^{ijk,ab}$ is symmetric on (ijk) and (ab) with complex entries, transforming in the $(\mathbf{10}, \mathbf{3})$ of $\mathrm{SU}(3) \times \mathrm{SU}(2)$. There are generically 30 complex degrees of freedom in such a tensor, thus there are 30 complex degrees of freedom corresponding to 30 marginal deformations.

Again we consider if there are fixed-point isometries at the $\mathrm{M}^{1,1,1}$ point in the moduli space of couplings. $\mathrm{M}^{1,1,1}$ is a U(1) fibration over a $\mathbb{CP}^2 \times \mathbb{CP}^1$ base, so there is an $\mathrm{SU}(3) \times \mathrm{SU}(2)$ isometry that acts on the base, leaving the solution invariant. We can construct a moment map for the $\mathrm{SU}(3) \times \mathrm{SU}(2)$ action on the space of couplings and perform a symplectic reduction. The moment maps are

$$\begin{aligned}
\mu_{\mathrm{SU}(3)} &= f^{ikl,ab} \bar{f}_{jkl,ab} - \tfrac{1}{3} \delta^i{}_j f^{klm,ab} \bar{f}_{klm,ab}, \\
\mu_{\mathrm{SU}(2)} &= f^{ijk,ac} \bar{f}_{ikl,bc} - \tfrac{1}{2} \delta^a{}_b f^{ijk,cd} \bar{f}_{ijk,cd}.
\end{aligned}$$

The conformal manifold of exactly marginal deformations that preserve eight supercharges is given by the symplectic reduction

$$\mathcal{M}_{\mathrm{c}} = \{ f^{ijk,ab} \} /\!\!/ \mathrm{SU}(3) \times \mathrm{SU}(2). \tag{6.44}$$

The moment maps give $8+3$ real conditions on the $f^{ijk,ab}$, and we can remove another 11 degrees of freedom using rotations of the couplings. In addition, $\mathrm{H}^3(\mathrm{M}^{1,1,1}) = 0$ and so all global three-forms are trivial in cohomology [297]. This means there are no marginal deformations due to closed $(1,2)$-forms χ. Thus, the conformal manifold is $30 - 11 = 19$ complex dimensional.

The β-deformed $\mathrm{M}^{1,1,1}$ solution was first given in [284, 285], which we reproduce in appendix I. Taking $f \propto i\gamma \mathrm{e}^{\mathrm{i}\tau/2} \sin\theta \sin\tilde{\theta} \sin^2\mu \cos\mu$, where γ is real, and using our frame for $\mathrm{M}^{1,1,1}$, one can check that our expression (6.22) reproduces the four-form flux of the first-order β-deformed solution.

6.4 Summary

In this chapter we have used exceptional generalised geometry to analyse exactly marginal deformations of $d = 3$, $\mathcal{N} = 2$ SCFTs that are dual to AdS_4 backgrounds in eleven-dimensional supergravity. In the gauge theory, marginal deformations are determined by imposing F-term conditions on operators of conformal dimension two and then quotienting by the complexified global symmetry group. We have shown that the supergravity analysis gives a geometric interpretation of this gauge theory result. The marginal deformations are obtained as solutions of moment maps for the generalised diffeomorphism group that have the correct charge under the Reeb vector, which generates the $\mathrm{U}(1)_{\mathrm{R}}$ symmetry. If this is the only symmetry of the background, all marginal deformations are exactly marginal. If the background possesses extra isometries, there are obstructions that come from fixed points of the moment maps. The exactly marginal deformations are then given by a further quotient by these extra isometries.

For the specific case of Sasaki–Einstein backgrounds, we showed how supersymmetric deformations can be understood as deformations of generalised structures which give rise to four-form flux perturbations at first order. Using explicit examples, we checked that our expression for the four-form flux matches those in the literature.

Chapter 7

Discussion

In this thesis, we have presented the idea that generalised geometry provides a geometrical interpretation of generic flux backgrounds in type II supergravity and M-theory. We focussed on backgrounds preserving eight supercharges in $D = 4, 5, 6$ Minkowski or $D = 4, 5$ AdS spacetimes and showed they define integrable G-structures in $E_{d(d)} \times \mathbb{R}^+$ generalised geometry. As in conventional geometry, integrability is defined as the existence of a generalised torsion-free connection that is compatible with the structure, or equivalently as the vanishing of the generalised intrinsic torsion (or a non-vanishing singlet component in the AdS case). This led to us defining what we called "exceptional Calabi–Yau" (ECY) spaces and "exceptional Sasaki–Einstein" (ESE) spaces, which provide the natural flux generalisations of Calabi–Yau and Sasaki–Einstein spaces. For both ECY and ESE spaces, we found the differential conditions on the structures implied by integrability, and showed that they took a simple form in terms of the generalised Lie derivative or moment maps for the action of the generalised diffeomorphism group. As for Calabi–Yau backgrounds, supersymmetric solutions are described as the intersection of two separate structures. We also discussed the structure of the moduli spaces of ECY and ESE spaces, and pointed out an interesting connection to the "HK/QK correspondence" [247].

We saw how examples of ECY geometries are given by the simple examples of Calabi–Yau, generalised Calabi–Yau and hyper-Kähler spaces as well as various other supersymmetric flux backgrounds.

In the ESE case, we saw that such geometries always admit a "generalised Reeb vector" that generates an isometry of the background corresponding to the R-symmetry of the dual field theory. In the language of [145], ESE spaces are weak generalised holonomy spaces, and the cone over such a space has generalised special holonomy. We have included a number of examples of ESE spaces including conventional Sasaki–Einstein in five and seven dimensions, as well as the most general AdS_5 solutions in M-theory.

A particular advantage of the formalism is that the H and V structures defining the background are associated to hypermultiplet and vector-multiplet degrees of freedom in the corresponding gauged supergravity. This provides a natural translation between bulk and boundary properties. We showed for example that the V structure, which is defined by the generalised Reeb vector K, encodes the contact structure that appears in generic $D = 5$ type IIB and $D = 4$ M-theory backgrounds [134, 245, 249]. Furthermore, K determines the central charge in $D = 5$ and free energy in $D = 4$ of the dual theory, and is a calibration for BPS wrapped branes giving the dimension of the dual operators.

As we saw, a key application of our formalism is to the AdS/CFT correspondence and we took some first steps in this direction. We used $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ generalised geometry to analyse exactly marginal deformations of $\mathcal{N} = 1$ SCFTs that are dual to AdS_5 backgrounds in type II or eleven-dimensional supergravity. In the gauge theory, marginal deformations are determined by imposing F-term conditions on operators of conformal dimension three and then quotienting by the complexified global symmetry group. We showed that the supergravity analysis gives a geometric interpretation of this gauge theory result. The marginal deformations are obtained as solutions of moment maps for the generalised diffeomorphism group that have the correct charge under the Reeb vector, which generates the $\mathrm{U}(1)_\mathrm{R}$ symmetry. If this is the only symmetry of the background, all marginal deformations are exactly marginal. If the background possesses extra isometries, there are obstructions that come from fixed points of the moment maps. The exactly marginal deformations are then given by a further quotient by these extra isometries. For the specific case of Sasaki–Einstein backgrounds in type IIB we showed how supersymmetric deformations can be understood as deformations of generalised structures which give rise to three-form flux perturbations at first order. Using explicit examples, we showed that our expression for the three-form flux matches those in the literature and the obstruction conditions match the one-loop beta functions of the dual SCFT.

Finally, we extended our analysis to AdS_4 backgrounds in eleven-dimensional supergravity. We showed how deformations of generalised structures give rise to supersymmetry-preserving four-form flux perturbations at first order and how higher-order obstructions again come from fixed points of the moment maps. Using explicit examples, we showed that our expression for the four-form flux matches those in the literature.

7.1 Future directions

There are many directions for future study. The obvious extension is to find the analogous structures for backgrounds with different amounts of supersymmetry. In $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ generalised geometry the supersymmetry parameters transform under the maximal compact subgroup H_d. As shown in [143, 144], supersymmetric backgrounds preserving $\mathcal{N}$ supersymmetries are given by integrable G-structures where $G \subset \mathrm{H}_d$ is the stabiliser group of the $\mathcal{N}$ Killing spinors. Thus, for example, $D = 4$, $\mathcal{N} = 1$ backgrounds define an $\mathrm{SU}(7) \subset \mathrm{SU}(8)$ structure [136], which in M-theory would give the flux generalisation of a G_2 structure. This viewpoint should give insight into the moduli space of $\mathcal{N} = 1$ flux compactifications. We have also seen that the structures are naturally associated to multiplets in the

D-dimensional theory, and furthermore that the integrability conditions can be deduced from the standard form of the D-dimensional gauged supergravity. This should provide a relatively simple prescription for deriving the conditions for other examples.

We discussed some general properties of the moduli spaces, notably that they arise as hyper-Kähler and symplectic quotients and that the full moduli space has a fibred structure. However, for Calabi–Yau compactifications it is known that the moduli space splits and is simply a product of the hypermultiplet and vector-multiplet moduli spaces. It is still an open problem to understand how the moduli space of ECYs splits into a product of a hyper-Kähler space and a special Kähler space.

As we have seen, integrability of the H structure generically is captured by a moment map. Typically, the vanishing of a moment map is closely allied to a notion of stability (see for example [185]), which, if it exists, would here define integrable complex or symplectic structures (and their generalisations) under the action of some quaternionic version of the full generalised diffeomorphism group.

There is a natural question about reduction of structures, similar to that for generalised complex geometry [298]: how, given a generalised Killing vector, structures with eight supercharges on M define structures on a space of one dimension lower. Physically this would realise the r-map of [234]. In the AdS case, K is always a generalised Killing vector and the cone over an ESE space has generalised special holonomy. In the conventional Sasaki–Einstein case one can use the Reeb vector to define a symplectic reduction of the Calabi–Yau cone. Locally, this gives a four-dimensional geometry that is Kähler–Einstein. When one moves to generalised complex geometry, there is an analogous result using the theory of generalised quotients that the transverse space admits a generalised Hermitian structure [134]. It would be interesting to understand how this carries over to exceptional generalised geometry by developing a theory of generalised quotients.

Conventional generalised complex geometry is known to capture the A and B topological string models on backgrounds with H flux [115, 116, 119]. The geometries defined here should encode some extension to M-theory or with the inclusion of R-R flux. It was previously proposed [299, 300] that the relevant topological M-theory was related to Hitchin's formulation of G_2 structures [301], which combines both the A and B model. Here we have a slightly different picture with two candidate structures in M-theory. Note that in principle either the H structures or the V structures could be viewed as generalisations of the A and B models, with mirror symmetry mapping H (or V) structures in IIA to H (or V) structures in IIB. However, the integrability conditions on the V structure are considerably weaker – for example, for a generalised complex structure they do not imply that $\mathrm{d}\Phi^{\pm} = 0$. In this case it would appear one would need to choose a fixed background J_α and impose the $L_X J_\alpha$ condition. The hypermultiplet structure integrability, on the other hand, does imply $\mathrm{d}\Phi^{\pm} = 0$, and hence these give the natural candidates for generalisations of the topological string models. It would be particularly interesting to consider the quantisation of these models, as in [116] though now with a hyper-Kähler rather than symplectic space of structures.

In the AdS examples with contact structures, one can calculate properties of the field theory using the relation between the contact volume and the choice of Reeb vector [134, 245, 249]. The special role of K also led us, following [250], to a conjecture for generic form volume minimisation [251, 252]. It would be particularly interesting to see if we can extend these techniques to the case of $D = 5$ M-theory backgrounds using the generalised Reeb vector. Moreover, it should be possible to use generalised intrinsic torsion to show that the supergravity actions are given by the integral of the $\mathrm{E}_{d(d)}$-invariant volume, as in (4.126).

An important question both for phenomenology and the AdS/CFT correspondence is to identify the deformations of the structures. Our analysis holds for any $\mathcal{N} = 1$ AdS_5 background so it would be interesting to apply it to one of the few examples of non-Sasaki–Einstein backgrounds, such as the Pilch–Warner solution [302]. This is dual to a superconformal fixed point of $\mathcal{N} = 4$ super Yang–Mills deformed by a mass for one of the chiral superfields. More generally, one expects that the deformation problem is described by some underlying differential graded Lie algebra (DGLA) with cohomology classes capturing the first-order deformations and obstructions, as described for example in [303]. For H structures in IIA, this would be some generalisation of the Dolbeault complex. Such extensions appear in generalised complex geometry [109, 304, 305], but this would go further to include R-R degrees of freedom. In the generalised complex structure case, starting from a conventional complex structure, it is known that the extra deformations can be associated with gerbe and non-commutative deformations of the algebraic geometry [109, 115]. An open question is how to understand the corresponding R-R deformations when they exist. Furthermore, in the AdS context solving the deformation problem gives a way of finding the exactly marginal deformations in the dual field theory. One might hope that understanding the underlying DGLA structure may help identify the all-order supergravity backgrounds dual to the deformations; so far only the dual of the β-deformation has been obtained. With these in hand, one would be able to perform many non-trivial checks of the AdS/CFT correspondence, including calculating the metric on the conformal manifold.

There are also applications to phenomenology. Supersymmetric deformations of the geometry give rise to moduli fields in the low-energy effective action obtained after compactifying on the internal manifold. Determining the number and nature of moduli fields that arise in flux compactifications is difficult in general as we lose many of the mathematical tools used in Calabi–Yau compactifications. In our formalism, fluxes and geometry are both encoded by the generalised structure whose deformations will give all the moduli of the low-energy theory. Generalised geometry points to a new set of tools to understand these deformations, such as generalisations of cohomology and special holonomy.

Appendix A

Examples of $\mathcal{N} = 2$, $D = 4$ backgrounds

In this appendix, we shall summarise a number of simple $\mathcal{N} = 2$ backgrounds in both type II and M-theory, with and without fluxes. We use these to provide concrete examples of $E_{7(7)}$ structures in section 3.2.4 and to show how the usual supersymmetry conditions are recovered from integrability conditions in section 3.3.4.

A.1 Calabi–Yau manifolds in type II and SU(3) structures

Calabi–Yau manifolds admit a single covariantly constant spinor χ^+ defining an $SU(3) \subset Spin(6) \simeq SU(4)$ structure. In this case, the two SU(8) Killing spinors of (3.6) are given by [172]

$$\epsilon_1 = \begin{pmatrix} \chi^+ \\ 0 \end{pmatrix}, \qquad \epsilon_2 = \begin{pmatrix} 0 \\ \chi^- \end{pmatrix}, \tag{A.1}$$

Equivalently it admits a symplectic form ω and a holomorphic three-form Ω that are compatible, which translates to

$$\omega \wedge \Omega = 0, \qquad \tfrac{1}{3!}\omega \wedge \omega \wedge \omega = \tfrac{1}{8}\mathrm{i}\Omega \wedge \bar{\Omega}. \tag{A.2}$$

One can choose a frame $\{e^a\}$ for the metric on M where the invariant forms take the form

$$\omega = e^{12} + e^{34} + e^{56}, \qquad \Omega = (e^1 + \mathrm{i}e^2) \wedge (e^3 + \mathrm{i}e^4) \wedge (e^5 + \mathrm{i}e^6), \tag{A.3}$$

where $e^{ab} = e^a \wedge e^b$. Raising an index on ω defines an almost complex structure I on the six-dimensional space

$$I^m{}_n = -\omega^m{}_n = \tfrac{1}{8}\mathrm{i}(\bar{\Omega}^{mpq}\Omega_{npq} - \Omega^{mpq}\bar{\Omega}_{npq}), \qquad I^q{}_m\Omega_{qnp} = \mathrm{i}\Omega_{mnp}. \tag{A.4}$$

In the language of G-structures, ω and Ω define $Sp(6;\mathbb{R})$ and $SL(3;\mathbb{C})$ structures respectively.

The compatibility conditions (A.2) imply that the common subgroup is given by $\mathrm{Sp}(6;\mathbb{R})\cap \mathrm{SL}(3;\mathbb{C}) = \mathrm{SU}(3)$. The fact that χ is covariantly constant is equivalent to the integrability conditions

$$\mathrm{d}\omega = 0, \qquad \mathrm{d}\Omega = 0. \tag{A.5}$$

A.2 $\mathbf{CY_3 \times S^1}$ in M-theory

Let us also briefly note the form of the M-theory lift of the type IIA Calabi–Yau background. The seven-dimensional internal space is a product $M = M_{\mathrm{SU}(3)} \times \mathrm{S}^1$ with metric

$$\mathrm{d}s^2(M) = \mathrm{d}s^2(M_{\mathrm{SU}(3)}) + \zeta^2, \tag{A.6}$$

where $\zeta = \mathrm{d}y$, with y the coordinate on the M-theory circle, and $\mathrm{d}s^2(M_{\mathrm{SU}(3)})$ is the IIA Calabi–Yau metric on $M_{\mathrm{SU}(3)}$. The Killing spinors take the same form as (A.1) but are now viewed as complex Spin(7) spinors. They again determine an SU(3) structure, which can equivalently be defined by the triplet of forms $\{\omega, \Omega, \zeta\}$. If we raise an index to define the vector $\zeta^\sharp = \partial_y$, we have the compatibility conditions

$$\tfrac{1}{3!}\omega\wedge\omega\wedge\omega = \tfrac{1}{8}\mathrm{i}\Omega\wedge\bar{\Omega}, \qquad \omega\wedge\Omega = 0, \qquad \imath_{\zeta^\sharp}\omega = 0, \qquad \imath_{\zeta^\sharp}\Omega = 0, \tag{A.7}$$

and the integrability conditions

$$\mathrm{d}\omega = 0, \qquad \mathrm{d}\Omega = 0, \qquad \mathrm{d}\zeta = 0. \tag{A.8}$$

Note that they imply $\zeta^\sharp$ is a Killing vector.

A.3 Generalised Calabi–Yau metrics in type II and pure spinors

Returning to type II, we now consider the case where we include non-trivial $H = \mathrm{d}B$ flux and dilaton. For simplicity, the warp factor is taken to vanish. The two SU(8) Killing spinors of (3.6) are given by[1]

$$\epsilon_1 = \begin{pmatrix} \chi_1^+ \\ 0 \end{pmatrix}, \qquad \epsilon_2 = \begin{pmatrix} 0 \\ \chi_2^- \end{pmatrix}. \tag{A.9}$$

The background can then be characterised using $\mathrm{O}(d,d)\times\mathbb{R}^+$ generalised geometry following [111].

[1] As mentioned in [172], this is not the most general spinor ansatz. There are pure NS-NS, $\mathcal{N}=2$ backgrounds where the Killing spinors do not take the form (A.9), and hence are not described by generalised complex structures.

The generalised tangent bundle $E \simeq TM \oplus T^*M$ admits a natural $\mathrm{O}(d,d)$ metric η. The background is defined by two complex polyforms, taking $d = 6$,

$$\Phi^{\pm} \in \Gamma(\wedge^{\pm} T^*M), \tag{A.10}$$

which can then be viewed as sections of the positive and negative helicity $\mathrm{Spin}(6,6) \times \mathbb{R}^+$ spinor bundles, where the $\mathbb{R}^+$ factor acts by a simple rescaling. The generalised spinors are not generic but are "pure" meaning they are stabilised by an $\mathrm{SU}(3,3) \subset \mathrm{Spin}(6,6)$ subgroup. They also satisfy the consistency conditions

$$\langle \Phi^+, \bar{\Phi}^+ \rangle = \langle \Phi^-, \bar{\Phi}^- \rangle, \qquad \langle \Phi^+, V \cdot \Phi^- \rangle = \langle \bar{\Phi}^+, V \cdot \Phi^- \rangle = 0 \quad \forall V, \tag{A.11}$$

where, given $V = \xi + \lambda \in \Gamma(TM \oplus T^*M)$, one defines the Clifford action $V \cdot \Phi^{\pm} = V^A \Gamma_A \Phi^{\pm} = \imath_\xi \Phi^{\pm} + \lambda \wedge \Phi^{\pm}$. In addition, $\langle \cdot, \cdot \rangle$ is the $\mathrm{Spin}(6,6)$-invariant spinor bilinear, or Mukai pairing, given by

$$\langle \Psi, \Sigma \rangle = \sum_p (-)^{[(p+1)/2]} \Psi_{(p)} \wedge \Sigma_{(6-p)}, \tag{A.12}$$

where $\Psi_{(p)}$ denotes the p-form component of Ψ and $[p]$ is the integer part of p.

Each pure spinor defines an (almost) generalised complex structure $\mathcal{J}^{\pm} \in \Gamma(\mathrm{ad}\,\tilde{F})$, where $\mathrm{ad}\,\tilde{F} \simeq \Gamma((TM \otimes T^*M) \oplus \wedge^2 T^*M \oplus \wedge^2 TM)$ is the principal $\mathrm{O}(6,6)$ frame bundle for E. The generalised complex structures are given by

$$\mathcal{J}^{\pm A}{}_B = \mathrm{i} \frac{\langle \Phi^{\pm}, \Gamma^A{}_B \bar{\Phi}^{\pm} \rangle}{\langle \Phi^{\pm}, \bar{\Phi}^{\pm} \rangle}, \tag{A.13}$$

where Γ^A are $\mathrm{O}(6,6)$ gamma matrices with $A = 1, \ldots, 12$, and indices are raised and lowered using the $\mathrm{O}(6,6)$ metric. Note that acting on pure spinors it has the property

$$\tfrac{1}{4} \mathcal{J}^{\pm}_{AB} \Gamma^{AB} \Phi^{\pm} = 3\mathrm{i} \Phi^{\pm}. \tag{A.14}$$

The integrability conditions are

$$\mathrm{d}\Phi^+ = 0, \qquad \mathrm{d}\Phi^- = 0, \tag{A.15}$$

which define what is known as a generalised Calabi–Yau metric [109]. These conditions imply that each almost generalised complex structure is separately torsion-free. Each is also equivalent to the existence of a torsion-free generalised connection compatible with the $\mathrm{SU}(3,3)_{\pm}$ structure defined by $\Phi^{\pm}$.

A.4 D3-branes on HK $\times \mathbb{R}^2$ in type IIB

Let us now turn to three further flux examples. The first corresponds to D3-branes in type IIB at a point in a space $M = M_{\mathrm{SU}(2)} \times \mathbb{R}^2$, where $M_{\mathrm{SU}(2)}$ is a four-dimensional hyper-Kähler space. This is in the class of the solutions first given in [225–227] and analysed in detail for $M = M_{\mathrm{SU}(2)} \times \mathbb{R}^2$ in [228]. We have a conformal factor Δ and an R-R five-form flux F, and in general also an imaginary self-dual three-form flux. The metric on M takes the form

$$\mathrm{d}s^2 = \mathrm{d}\tilde{s}^2(M_{\mathrm{SU}(2)}) + \zeta_1^2 + \zeta_2^2, \tag{A.16}$$

where $\mathrm{d}\tilde{s}^2(M_{\mathrm{SU}(2)})$ is an SU(2)-structure metric on $M_{\mathrm{SU}(2)}$ and

$$\zeta_1 = \mathrm{e}^{-\Delta}\mathrm{d}x, \qquad \zeta_2 = \mathrm{e}^{-\Delta}\mathrm{d}y. \tag{A.17}$$

The type IIB axion-dilaton $\tau = C_0 + \mathrm{i}\,\mathrm{e}^{\phi}$ is constant, and for convenience we take $\tau = \mathrm{i}$.

The two SU(8) Killing spinors take the form

$$\epsilon_1 = \begin{pmatrix} \chi_1^+ \\ \mathrm{i}\chi_1^+ \end{pmatrix}, \qquad \epsilon_2 = \begin{pmatrix} -\mathrm{i}\chi_2^+ \\ \chi_2^+ \end{pmatrix}. \tag{A.18}$$

The two spinors χ_i^+ define a conventional SU(2) structure, which is simply the one defined by the hyper-Kähler geometry. It is determined by a triplet of symplectic forms ω_α and the pair of one-forms $\{\zeta_1, \zeta_2\}$. One can always choose a frame $\{e^a\}$ for the metric on M where these take the form

$$\begin{gathered} \omega_1 = e^{14} + e^{23}, \qquad \omega_2 = e^{13} - e^{24}, \qquad \omega_3 = e^{12} + e^{34}, \\ \zeta_1 = e^5, \qquad \zeta_2 = e^6. \end{gathered} \tag{A.19}$$

The corresponding triplet of complex structures is given by $(I_\alpha)^m{}_n = -(\omega_\alpha)^m{}_n$, such that for $\Omega = \omega_2 + \mathrm{i}\omega_1$, we have $(I_3)^p{}_m\Omega_{pn} = \mathrm{i}\Omega_{mn}$. The volume form on M is defined by

$$\tfrac{1}{2}\omega_\alpha \wedge \omega_\beta \wedge \zeta_1 \wedge \zeta_2 = \delta_{\alpha\beta}\,\mathrm{vol}_6\,. \tag{A.20}$$

If we include only five-form flux, the integrability conditions for the structure are

$$\mathrm{d}(\mathrm{e}^{\Delta}\zeta_i) = 0, \qquad \mathrm{d}(\mathrm{e}^{2\Delta}\omega_\alpha) = 0, \qquad \mathrm{d}\Delta = -\tfrac{1}{4}\star F, \tag{A.21}$$

where F is the component of the five-form flux on M and $\star$ is the six-dimensional Hodge duality operator, calculated using the metric (A.16). If one also includes a non-zero three-form flux on M, it has to have the form [228]

$$H + \mathrm{i}F_3 = \mathrm{d}\gamma_I(z) \wedge \tau_I, \tag{A.22}$$

where $\gamma_I(z)$ are analytic functions of $z = x + iy$ and τ_I are harmonic anti-self-dual two-forms on the hyper-Kähler space. The functions $\gamma_I(z)$ are constrained by a differential equation arising from the Bianchi identity for F.

A.5 Wrapped M5-branes on HK $\times\, \mathbb{R}^3$ in M-theory

For our final two examples, we consider the M-theory backgrounds corresponding to wrapped M5-branes in a seven-dimensional geometry that is a product of a four-dimensional hyper-Kähler space with $\mathbb{R}^3$. There are two possibilities: the branes can either wrap a Kähler two-cycle in the four-dimensional hyper-Kähler space or wrap an $\mathbb{R}^2$ plane in $\mathbb{R}^3$. In each case, the spacetime is a product $M = M_{\mathrm{SU}(2)} \times \mathbb{R}^3$ with the metric

$$\mathrm{d}s^2 = \mathrm{d}\tilde{s}^2(M_{\mathrm{SU}(2)}) + \zeta_1^2 + \zeta_2^2 + \zeta_3^2, \tag{A.23}$$

where $M_{\mathrm{SU}(2)}$ admits an SU(2) structure, $\mathrm{d}\tilde{s}^2(M_{\mathrm{SU}(2)})$ is the metric determined by the structure, ζ_i are one-forms, and there is a non-trivial four-form flux F. Crucially, because of the back-reaction of the wrapped brane, the SU(2) structure has torsion, in other words the metric is no longer hyper-Kähler.

One can choose a frame $\{e^a\}$ for the metric on M such that the forms determining the SU(2) structure are given by

$$\begin{gathered}\omega_1 = e^{14} + e^{23}, \qquad \omega_2 = e^{13} - e^{24}, \qquad \omega_3 = e^{12} + e^{34}, \\ \zeta_1 = e^5, \qquad \zeta_2 = e^6, \qquad \zeta_3 = e^7.\end{gathered} \tag{A.24}$$

The corresponding triplet of complex structures is given by $(I_\alpha)^m{}_n = -(\omega_\alpha)^m{}_n$, while the volume form on M is defined by

$$\tfrac{1}{2}\omega_\alpha \wedge \omega_\beta \wedge \zeta_1 \wedge \zeta_2 \wedge \zeta_3 = \delta_{\alpha\beta}\,\mathrm{vol}_7\,. \tag{A.25}$$

The integrability conditions differ in the two cases. Consider first the case where the M5-brane wraps a Kähler cycle, calibrated by ω_3, in the hyper-Kähler manifold. The metric takes the form [306, 307]

$$\mathrm{d}s^2 = \mathrm{d}\tilde{s}^2(M_{\mathrm{SU}(2)}) + \mathrm{e}^{-4\Delta}\big(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\big), \tag{A.26}$$

so that

$$\zeta_1 = \mathrm{e}^{-2\Delta}\mathrm{d}x, \qquad \zeta_2 = \mathrm{e}^{-2\Delta}\mathrm{d}y, \qquad \zeta_3 = \mathrm{e}^{-2\Delta}\mathrm{d}z. \tag{A.27}$$

The remaining conditions can then be written as

$$\begin{gathered}\mathrm{d}(\mathrm{e}^{\Delta}\omega_1) = \mathrm{d}(\mathrm{e}^{\Delta}\omega_2) = 0, \qquad \mathrm{d}(\mathrm{e}^{4\Delta}\omega_3) = \mathrm{e}^{4\Delta} \star F, \\ \mathrm{d}(\mathrm{e}^{4\Delta}\omega_3 \wedge \zeta_1 \wedge \zeta_2 \wedge \zeta_3) = 0,\end{gathered} \tag{A.28}$$

where $\star$ is the Hodge duality operator calculated using the metric (A.23). Note that the integrability conditions preserve the SO(3) symmetry between the ζ_α but break the symmetry between the ω_α.

For an M5-brane wrapping $\mathbb{R}^2$, the metric takes the form

$$\mathrm{d}s^2 = \mathrm{e}^{-4\Delta}\mathrm{d}\tilde{s}^2_{\mathrm{HK}}(M_{\mathrm{SU}(2)}) + \mathrm{e}^{2\Delta}\big(\mathrm{d}x^2 + \mathrm{d}y^2\big) + \mathrm{e}^{-4\Delta}\mathrm{d}z^2. \tag{A.29}$$

where $\mathrm{d}\tilde{s}^2_{\mathrm{HK}}(M_{\mathrm{SU}(2)})$ is a hyper-Kähler metric on $M_{\mathrm{SU}(2)}$, and

$$\zeta_1 = \mathrm{e}^{\Delta}\mathrm{d}x, \qquad \zeta_2 = \mathrm{e}^{\Delta}\mathrm{d}y, \qquad \zeta_3 = \mathrm{e}^{-2\Delta}\mathrm{d}z. \tag{A.30}$$

In addition

$$\begin{gathered} \mathrm{d}(\mathrm{e}^{4\Delta}\omega_1) = \mathrm{d}(\mathrm{e}^{4\Delta}\omega_2) = \mathrm{d}(\mathrm{e}^{4\Delta}\omega_3) = 0, \\ \mathrm{d}(\mathrm{e}^{4\Delta}\zeta_1 \wedge \zeta_2) = \mathrm{e}^{4\Delta} \star F, \qquad \mathrm{d}(\mathrm{e}^{4\Delta}\zeta_3 \wedge \mathrm{vol}_4) = 0, \end{gathered} \tag{A.31}$$

where $\frac{1}{2}\omega_\alpha \wedge \omega_\beta = \delta_{\alpha\beta}\,\mathrm{vol}_4$. Now the symmetry between the ζ_α is broken but that between the ω_α is preserved.

These examples are interesting because we have the same SU(2) structure in each case but very different integrability conditions. A seven-dimensional SU(2) structure in M-theory actually admits four independent globally defined spinors.[2] In the two examples, different pairs of spinors are picked out by the Killing spinor equations. When we turn to generalised geometry, we will see that these different choices give two very different embeddings of the SU(2) structure into the generalised structure. Note that, dimensionally reducing along ζ_3, these solutions also correspond to wrapped NS5-branes in type IIA. In the first case of branes wrapped on a Kähler cycle, the ten-dimensional Killing spinors actually take the form (A.9), and so these geometries are included in the class of $\mathrm{SU}(3) \times \mathrm{SU}(3)$ NS-NS backgrounds described in appendix A.3. However, when the brane is wrapped on $\mathbb{R}^2$, the Killing spinors take the form

$$\epsilon_1 = \begin{pmatrix} \chi_1^+ \\ 0 \end{pmatrix}, \qquad \epsilon_2 = \begin{pmatrix} \chi_2^+ \\ 0 \end{pmatrix}, \tag{A.32}$$

and, although the background is purely NS-NS, we see that it is not described by an torsion-free $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure, an exceptional case first noted in [111].

[2]This counting is reflected in the fact that compactifying M-theory on $\mathrm{K3} \times \mathrm{T}^3$ breaks half the supersymmetry.

Appendix B

$\mathbf{E}_{d(d)} \times \mathbb{R}^+$ representations

B.1 Notation

Our notation follows [138]. Wedge products and contractions are given by

$$\begin{aligned}
(v \wedge u)^{a_1 \dots a_{p+p'}} &:= \frac{(p+p')!}{p!p'!} v^{[a_1 \dots} u^{a_{p+1} \dots a_{p+p'}]}, \\
(\lambda \wedge \rho)_{a_1 \dots a_{q+q'}} &:= \frac{(q+q')!}{q!q'!} \lambda_{[a_1 \dots a_q} \rho_{a_{q+1} \dots a_{q+q'}]}, \\
(v \lrcorner \lambda)_{a_1 \dots a_{q-p}} &:= \frac{1}{p!} v^{b_1 \dots b_p} \lambda_{b_1 \dots b_p a_1 \dots a_{q-p}} \quad \text{if } p \leq q, \\
(v \lrcorner \lambda)^{a_1 \dots a_{p-q}} &:= \frac{1}{q!} v^{a_1 \dots a_{p-q} b_1 \dots b_q} \lambda_{b_1 \dots b_q} \quad \text{if } p \geq q, \\
(jv \lrcorner j\lambda)^a{}_b &:= \frac{1}{(p-1)!} v^{ac_1 \dots c_{p-1}} \lambda_{bc_1 \dots c_{p-1}}, \\
(j\lambda \wedge \rho)_{a, a_1 \dots a_d} &:= \frac{d!}{(q-1)!(d+1-q)!} \lambda_{a[a_1 \dots a_{q-1}} \rho_{a_q \dots a_d]}.
\end{aligned} \tag{B.1}$$

Given a basis $\{\hat{e}_a\}$ for TM and a dual basis $\{e^a\}$ for T^*M, there is a natural $\mathfrak{gl}_d$ action on tensors. For example, the action on a vector and a three-form is

$$(r \cdot v)^a = r^a{}_b v^b, \qquad (r \cdot \lambda)_{abc} = -r^d{}_a \lambda_{dbc} - r^d{}_b \lambda_{adc} - r^d{}_c \lambda_{abd}. \tag{B.2}$$

When writing the components of generalised tensors, we sometimes use the notation that $(\dots)_{(p)}$ and $(\dots)^{(q)}$ denote p-form and q-vector components respectively. For a p-form ρ, we denote by $\rho^\sharp$ the p-vector obtained by raising the indices of ρ using the conventional metric on the manifold.

We define the Hodge star as

$$\star\, e^{a_1 \dots a_p} = \frac{1}{q!} \epsilon^{a_1 \dots a_p}{}_{b_1 \dots b_q} e^{b_1 \dots b_q}. \tag{B.3}$$

With a Euclidean metric we have $\epsilon_{1 \dots d} = \epsilon^{1 \dots d} = 1$, so that $\star 1 = \mathrm{vol}$ and $\star\, \mathrm{vol} = 1$. With a mostly plus Minkowski metric we have $\epsilon_{0\dots d-1} = -\epsilon^{0\dots d-1} = 1$, so that $\star 1 = \mathrm{vol}$ and $\star\,\mathrm{vol} = -1$. In particular this choice implies

$$(\lambda^\sharp \lrcorner \rho)\,\mathrm{vol}_d = \rho \wedge \star\lambda. \tag{B.4}$$

B.2 $\mathbf{E}_{d(d)} \times \mathbb{R}^+$ for M-theory

We review from [138] a construction of $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ using the $\mathrm{GL}(d)$ subgroup appropriate to M-theory, including useful representations, tensor products and the generalised Lie derivative.

On a d-dimensional manifold M, the generalised tangent bundle is

$$E \simeq TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M \oplus (T^*M \otimes \wedge^7 T^*M). \tag{B.5}$$

We write sections of this bundle as

$$V = v + \omega + \sigma + \tau, \tag{B.6}$$

where $v \in \Gamma(TM)$, $\omega \in \Gamma(\wedge^2 T^*M)$, $\sigma \in \Gamma(\wedge^5 T^*M)$ and $\tau \in \Gamma(T^*M \otimes \wedge^7 T^*M)$. The adjoint bundle is

$$\mathrm{ad}\,\tilde{F} \simeq \mathbb{R} \oplus (TM \otimes T^*M) \oplus \wedge^3 T^*M \oplus \wedge^6 T^*M \oplus \wedge^3 TM \oplus \wedge^6 TM. \tag{B.7}$$

We write sections of the adjoint bundle as

$$R = l + r + a + \tilde{a} + \alpha + \tilde{\alpha}, \tag{B.8}$$

where $l \in \mathbb{R}$, $r \in \Gamma(\mathrm{End}\,TM)$, $a \in \Gamma(\wedge^3 T^*M)$ etc. We take $\{\hat{e}_a\}$ to be a basis for TM with a dual basis $\{e^a\}$ on T^*M so there is a natural $\mathfrak{gl}_d$ action on tensors. The $\mathfrak{e}_{d(d)}$ subalgebra is generated by setting $l = r^a{}_a/(9-d)$. This relation fixes the weight of generalised tensors under the $\mathbb{R}^+$ factor, so that a scalar of weight k is a section of $(\det T^*M)^{k/(9-d)}$

$$\mathbf{1}_k \in \Gamma\big((\det T^*M)^{k/(9-d)}\big). \tag{B.9}$$

We define the adjoint action of $R \in \Gamma(\mathrm{ad}\,\tilde{F})$ on $V \in \Gamma(E)$ to be $V' = R \cdot V$. The components of V' are

$$\begin{aligned} v' &= lv + r\cdot v + \alpha \lrcorner \omega - \tilde{\alpha} \lrcorner \sigma, \\ \omega' &= l\omega + r\cdot\omega + v \lrcorner a + \alpha \lrcorner \sigma + \tilde{\alpha} \lrcorner \tau, \\ \sigma' &= l\sigma + r\cdot\sigma + v \lrcorner \tilde{a} + a \wedge \omega + \alpha \lrcorner \tau, \\ \tau' &= l\tau + r\cdot\tau - j\tilde{a} \wedge \omega + ja \wedge \sigma. \end{aligned} \tag{B.10}$$

We define the adjoint action of R on R' to be $R'' = [R, R']$. The components of R'' are

$$\begin{aligned}
l'' &= \tfrac{1}{3}(\alpha \lrcorner a' - \alpha' \lrcorner a) + \tfrac{2}{3}(\tilde{\alpha}' \lrcorner \tilde{a} - \tilde{\alpha} \lrcorner \tilde{a}'), \\
r'' &= [r, r'] + j\alpha \lrcorner j a' - j\alpha' \lrcorner j a - \tfrac{1}{3}\mathbb{1}(\alpha \lrcorner a' - \alpha' \lrcorner a) \\
&\quad + j\tilde{\alpha}' \lrcorner j\tilde{a} - j\tilde{\alpha} \lrcorner j\tilde{a}' - \tfrac{2}{3}\mathbb{1}(\tilde{\alpha}' \lrcorner \tilde{a} - \tilde{\alpha} \lrcorner \tilde{a}'), \\
a'' &= r \cdot a' - r' \cdot a + \alpha' \lrcorner \tilde{a} - \alpha \lrcorner \tilde{a}', \\
\tilde{a}'' &= r \cdot \tilde{a}' - r' \cdot \tilde{a} - a \wedge a', \\
\alpha'' &= r \cdot \alpha' - r' \cdot \alpha + \tilde{\alpha}' \lrcorner a - \tilde{\alpha} \lrcorner a', \\
\tilde{\alpha}'' &= r \cdot \tilde{\alpha}' - r' \cdot \tilde{\alpha} - \alpha \wedge \alpha'.
\end{aligned} \tag{B.11}$$

The dual of the generalised tangent bundle is E^*. We embed the usual derivative operator in the one-form component of E^* via the map $T^*M \to E^*$. In coordinate indices M, one defines

$$\partial_M = \begin{cases} \partial_m & \text{for } M = m, \\ 0 & \text{otherwise.} \end{cases} \tag{B.12}$$

We then define a projection to the adjoint as

$$\times_{\text{ad}} \colon E^* \otimes E \to \operatorname{ad} \tilde{F}. \tag{B.13}$$

Explicitly, as a section of $\operatorname{ad} \tilde{F}$ we have

$$\partial \times_{\text{ad}} V = \partial \otimes v + \mathrm{d}\omega + \mathrm{d}\sigma. \tag{B.14}$$

The generalised Lie (or Dorfman) derivative is defined as

$$L_V W = V^B \partial_B W^A - (\partial \times_{\text{ad}} V)^A{}_B W^B. \tag{B.15}$$

This can be extended to act on tensors by using the adjoint action of $\partial \times_{\text{ad}} V \in \Gamma(\operatorname{ad} \tilde{F})$ in the second term. We will need explicit expressions for the Dorfman derivative of sections of E and $\operatorname{ad} \tilde{F}$. The Dorfman derivative acting on a generalised vector is

$$\begin{aligned}
L_V V' &= \mathcal{L}_v v' + (\mathcal{L}_v \omega' - \imath_{v'} \mathrm{d}\omega) + (\mathcal{L}_v \sigma' - \imath_{v'} \mathrm{d}\sigma - \omega' \wedge \mathrm{d}\omega) \\
&\quad + (\mathcal{L}_v \tau' - j\sigma' \wedge \mathrm{d}\omega - j\omega' \wedge \mathrm{d}\sigma).
\end{aligned} \tag{B.16}$$

The Dorfman derivative acting on a section of the adjoint bundle is

$$\begin{aligned}
L_V R &= (\mathcal{L}_v r + j\alpha \lrcorner j\mathrm{d}\omega - \tfrac{1}{3}\mathbb{1}\alpha \lrcorner \mathrm{d}\omega - j\tilde{\alpha} \lrcorner j\mathrm{d}\sigma + \tfrac{2}{3}\mathbb{1}\tilde{\alpha} \lrcorner \mathrm{d}\sigma) \\
&\quad + (\mathcal{L}_v a + r \cdot \mathrm{d}\omega - \alpha \lrcorner \mathrm{d}\sigma) + (\mathcal{L}_v \tilde{a} + r \cdot \mathrm{d}\sigma + \mathrm{d}\omega \wedge a) \\
&\quad + (\mathcal{L}_v \alpha - \tilde{\alpha} \lrcorner \mathrm{d}\omega) + (\mathcal{L}_v \tilde{\alpha}).
\end{aligned} \tag{B.17}$$

For $\mathrm{E}_{5(5)}$, we also need the vector bundle transforming in the $\mathbf{10_2}$ representation of

$\mathrm{Spin}(5,5) \times \mathbb{R}^+$. We define this bundle as

$$N \simeq T^*M \oplus \wedge^4 T^*M. \tag{B.18}$$

We write sections of this bundle as

$$Q = m + n, \tag{B.19}$$

where $m \in \Gamma(T^*M)$ and $n \in \Gamma(\wedge^4 T^*M)$. We define the adjoint action of $R \in \Gamma(\mathrm{ad}\,\tilde{F})$ on $Q \in \Gamma(N)$ to be $Q' = R \cdot Q$, with components

$$\begin{aligned} m' &= 2lm + r \cdot m - \alpha \lrcorner n, \\ n' &= 2ln + r \cdot n - a \wedge m. \end{aligned} \tag{B.20}$$

Using $\mathbf{16}^c \times \mathbf{10} \to \mathbf{16}$, we define a projection to E as

$$\times_E \colon E^* \otimes N \to E. \tag{B.21}$$

Explicitly, as a section of E, this allows us to define

$$\mathrm{d}Q \coloneqq \partial \times_E Q = \mathrm{d}m + \mathrm{d}n. \tag{B.22}$$

We define a patching of the bundle E such that on the overlaps of local patches $U_i \cap U_j$ we have

$$V_{(i)} = \mathrm{e}^{\mathrm{d}\Lambda_{(ij)} + \mathrm{d}\tilde{\Lambda}_{(ij)}} V_{(j)}, \tag{B.23}$$

where $\Lambda_{(ij)}$ and $\tilde{\Lambda}_{(ij)}$ are locally two- and five-forms respectively. This defines the gauge-invariant field strengths as

$$F = \mathrm{d}A, \qquad \tilde{F} = \mathrm{d}\tilde{A} - \tfrac{1}{2} A \wedge F. \tag{B.24}$$

The *twisted* Dorfman derivative $\hat{L}_{\tilde{V}}$ of an untwisted generalised tensor $\tilde{\mu}$ is defined as

$$\hat{L}_{\tilde{V}} \tilde{\mu} = \mathrm{e}^{-A-\tilde{A}} L_{\mathrm{e}^{A+\tilde{A}} \tilde{V}} (\mathrm{e}^{A+\tilde{A}} \tilde{\mu}). \tag{B.25}$$

The twisted Dorfman derivative $\hat{L}_{\tilde{V}}$ is given by the same expression as the usual Dorfman derivative with the substitutions

$$\mathrm{d}\omega \to \mathrm{d}\tilde{\omega} - \imath_{\tilde{v}} F, \qquad \mathrm{d}\sigma \to \mathrm{d}\tilde{\sigma} - \imath_{\tilde{v}} \tilde{F} + \tilde{\omega} \wedge F. \tag{B.26}$$

The projection $\partial \times_E Q$ also simplifies in a similar fashion allowing us to define

$$\mathrm{d}_F Q \coloneqq \mathrm{e}^{-A} \big(\partial \times_E (\mathrm{e}^A Q) \big) = \mathrm{d}m + \mathrm{d}n - F \wedge m. \tag{B.27}$$

The quadratic invariant for $E_{5(5)}$ is

$$\eta(Q,Q) = -m \wedge n. \tag{B.28}$$

The cubic invariant for $E_{6(6)}$ is

$$c(V,V,V) = -(\imath_v \omega \wedge \sigma + \tfrac{1}{3!}\omega \wedge \omega \wedge \omega). \tag{B.29}$$

The symplectic invariant for $E_{7(7)}$ is

$$s(V,V') = -\tfrac{1}{4}(\imath_v \tau' - \imath_{v'}\tau + \sigma \wedge \omega' - \sigma' \wedge \omega). \tag{B.30}$$

The $\mathfrak{e}_{d(d)}$ Killing form is

$$\operatorname{tr}(R,R') = \tfrac{1}{2}\Big(\tfrac{1}{9-d}\operatorname{tr}(r)\operatorname{tr}(r') + \operatorname{tr}(rr') + \alpha \lrcorner a' + \alpha' \lrcorner a - \tilde{\alpha} \lrcorner \tilde{a}' - \tilde{\alpha}' \lrcorner \tilde{a}\Big). \tag{B.31}$$

The form of the $E_{d(d)}$-invariant volume κ^2 depends on the compactification ansatz. For compactifications of the form

$$g_{11} = e^{2\Delta} g_{11-d} + g_d, \tag{B.32}$$

the invariant volume is

$$\kappa^2 = e^{(9-d)\Delta}\sqrt{g_d}. \tag{B.33}$$

B.3 $E_{d+1(d+1)} \times \mathbb{R}^+$ for type IIB

We provide details of the construction of $E_{d+1(d+1)} \times \mathbb{R}^+$ using the $GL(d) \times SL(2)$ subgroup appropriate to type IIB supergravity, including useful representations, tensor products and the generalised Lie derivative.

On a d-dimensional manifold M, the generalised tangent bundle is

$$\begin{aligned} E &\simeq TM \oplus T^*M \oplus (T^*M \oplus \wedge^3 T^*M \oplus \wedge^5 T^*M) \oplus \wedge^5 T^*M \oplus (T^*M \otimes \wedge^6 T^*M) \\ &\simeq TM \oplus (T^*M \otimes S) \oplus \wedge^3 T^*M \oplus (\wedge^5 T^*M \otimes S) \oplus (T^*M \otimes \wedge^6 T^*M), \end{aligned} \tag{B.34}$$

where S transforms as a doublet of $SL(2)$. We write sections of this bundle as

$$V = v + \lambda^i + \rho + \sigma^i + \tau, \tag{B.35}$$

where $v \in \Gamma(TM)$, $\lambda^i \in \Gamma(T^*M \otimes S)$, $\rho \in \Gamma(\wedge^3 T^*M)$, $\sigma \in \Gamma(\wedge^5 T^*M \otimes S)$ and $\tau \in \Gamma(T^*M \otimes \wedge^6 T^*M)$. The adjoint bundle is

$$\begin{aligned} \operatorname{ad}\tilde{F} = \mathbb{R} \oplus (TM \otimes T^*M) \oplus (S \otimes S^*)_0 \oplus (S \otimes \wedge^2 TM) \oplus (S \otimes \wedge^2 T^*M) \\ \oplus \wedge^4 TM \oplus \wedge^4 T^*M \oplus (S \otimes \wedge^6 TM) \oplus (S \otimes \wedge^6 T^*M), \end{aligned} \tag{B.36}$$

where the subscript on $(S \otimes S^*)_0$ denotes the traceless part. We write sections of the adjoint bundle as

$$R = l + r + a + \beta^i + B^i + \gamma + C + \tilde{\alpha}^i + \tilde{a}^i, \tag{B.37}$$

where $l \in \mathbb{R}$, $r \in \Gamma(\operatorname{End} TM)$, etc. We take $\{\hat{e}_a\}$ to be a basis for TM with a dual basis $\{e^a\}$ on T^*M so there is a natural $\mathfrak{gl}_d$ action on tensors.

The $\mathfrak{e}_{d+1(d+1)}$ subalgebra is generated by setting $l = r^a{}_a/(8-d)$. This fixes the weight of generalised tensors under the $\mathbb{R}^+$ factor, so that a scalar of weight k is a section of $(\det T^*M)^{k/(8-d)}$

$$\mathbf{1}_k \in \Gamma\big((\det T^*M)^{k/(8-d)}\big). \tag{B.38}$$

We define the adjoint action of $R \in \Gamma(\operatorname{ad}\tilde{F})$ on $V \in \Gamma(E)$ to be $V' = R \cdot V$. The components of V' are

$$\begin{aligned}
v' &= lv + r \cdot v + \gamma \lrcorner \rho + \epsilon_{ij}\beta^i \lrcorner \lambda^j + \epsilon_{ij}\tilde{\alpha}^i \lrcorner \sigma^j, \\
\lambda'^i &= l\lambda^i + r \cdot \lambda^i + a^i{}_j \lambda^j - \gamma \lrcorner \sigma^i + v \lrcorner B^i + \beta^i \lrcorner \rho - \tilde{\alpha}^i \lrcorner \tau, \\
\rho' &= l\rho + r \cdot \rho + v \lrcorner C + \epsilon_{ij}\beta^i \lrcorner \sigma^j + \epsilon_{ij}\lambda^i \wedge B^j + \gamma \lrcorner \tau, \\
\sigma'^i &= l\sigma^i + r \cdot \sigma^i + a^i{}_j \sigma^j - C \wedge \lambda^i + \rho \wedge B^i + \beta^i \lrcorner \tau + v \lrcorner \tilde{a}^i, \\
\tau' &= l\tau + r \cdot \tau + \epsilon_{ij} j\lambda^i \wedge \tilde{a}^j - j\rho \wedge C - \epsilon_{ij} j\sigma^i \wedge B^j.
\end{aligned} \tag{B.39}$$

We define the adjoint action of R on R' to be $R'' = [R, R']$. The components of R'' are

$$\begin{aligned}
l' &= \tfrac{1}{2}(\gamma \lrcorner C' - \gamma' \lrcorner C) + \tfrac{1}{4}\epsilon_{kl}(\beta^k \lrcorner B'^l - \beta'^k \lrcorner B^l) + \tfrac{3}{4}\epsilon_{ij}(\tilde{\alpha}^i \lrcorner \tilde{a}'^j - \tilde{\alpha}'^i \lrcorner \tilde{a}^j), \\
r'' &= (r \cdot r' - r' \cdot r) + \epsilon_{ij}(j\beta^i \lrcorner jB'^j - j\beta'^i \lrcorner jB^j) - \tfrac{1}{4}\mathbb{1}\epsilon_{kl}(\beta^k \lrcorner B'^l - \beta'^k \lrcorner B^l) \\
&\quad + (j\gamma \lrcorner jC' - j\gamma' \lrcorner jC) - \tfrac{1}{2}\mathbb{1}(\gamma \lrcorner C' - \gamma' \lrcorner C) \\
&\quad + \epsilon_{ij}(j\tilde{\alpha}^i \lrcorner j\tilde{a}'^j - j\tilde{\alpha}'^i \lrcorner j\tilde{a}^j) - \tfrac{3}{4}\epsilon_{ij}(\tilde{\alpha}^i \lrcorner \tilde{a}'^j - \tilde{\alpha}'^i \lrcorner \tilde{a}^j), \\
a''^i{}_j &= (a \cdot a' - a' \cdot a)^i{}_j + \epsilon_{jk}(\beta^i \lrcorner B'^k - \beta'^i \lrcorner B^k) - \tfrac{1}{2}\delta^i{}_j \epsilon_{kl}(\beta^k \lrcorner B'^l - \beta'^k \lrcorner B^l) \\
&\quad + \epsilon_{jk}(\tilde{\alpha}^i \lrcorner \tilde{a}'^k - \tilde{\alpha}'^i \lrcorner \tilde{a}^k) - \tfrac{1}{2}\delta^i{}_j \epsilon_{kl}(\tilde{\alpha}^k \lrcorner \tilde{a}'^l - \tilde{\alpha}'^k \lrcorner \tilde{a}^l), \\
\beta''^i &= (r \cdot \beta'^i - r' \cdot \beta^i) + (a \cdot \beta' - a' \cdot \beta)^i - (\gamma \lrcorner B'^i - \gamma' \lrcorner B^i) - (\tilde{\alpha}^i \lrcorner C' - \tilde{\alpha}'^i \lrcorner C), \\
B''^i &= (r \cdot B'^i - r' \cdot B^i) + (a \cdot B' - a' \cdot B)^i + (\beta^i \lrcorner C' - \beta'^i \lrcorner C) - (\gamma \lrcorner \tilde{a}'^i - \gamma' \lrcorner \tilde{a}^i), \\
\gamma'' &= (r \cdot \gamma' - r' \cdot \gamma) + \epsilon_{ij}\beta^i \wedge \beta'^j + \epsilon_{ij}(\tilde{\alpha}^i \lrcorner B'^j - \tilde{\alpha}'^i \lrcorner B^j), \\
C'' &= (r \cdot C' - r' \cdot C) - \epsilon_{ij} B^i \wedge B'^j + \epsilon_{ij}(\beta^i \lrcorner \tilde{a}'^j - \beta'^i \lrcorner \tilde{a}^j), \\
\tilde{\alpha}''^i &= (r \cdot \tilde{\alpha}'^i - r' \cdot \tilde{\alpha}^i) + (a \cdot \tilde{\alpha}' - a' \cdot \tilde{\alpha})^i - (\beta^i \wedge \gamma' - \beta'^i \wedge \gamma), \\
\tilde{a}''^i &= (r \cdot \tilde{a}'^i - r' \cdot \tilde{a}^i) + (a \cdot \tilde{a}' - a' \cdot \tilde{a})^i + (B^i \wedge C' - B'^i \wedge C).
\end{aligned} \tag{B.40}$$

The dual of the generalised tangent bundle is E^*. We embed the usual derivative operator in the one-form component of E^* via the map $T^*M \to E^*$. In coordinate indices

M, one defines

$$\partial_M = \begin{cases} \partial_m & \text{for } M = m, \\ 0 & \text{otherwise.} \end{cases} \tag{B.41}$$

We then define a projection to the adjoint as

$$\times_{\mathrm{ad}} \colon E^* \otimes E \to \mathrm{ad}\,\tilde{F}. \tag{B.42}$$

Explicitly, as a section of $\mathrm{ad}\,\tilde{F}$ we have

$$\partial \times_{\mathrm{ad}} V = \partial \otimes v + \mathrm{d}\lambda^i + \mathrm{d}\rho + \mathrm{d}\sigma^i. \tag{B.43}$$

The generalised Lie (or Dorfman) derivative is defined as

$$L_V W = V^B \partial_B W^A - (\partial \times_{\mathrm{ad}} V)^A{}_B W^B. \tag{B.44}$$

This can be extended to act on tensors by using the adjoint action of $\partial \times_{\mathrm{ad}} V \in \Gamma(\mathrm{ad}\,\tilde{F})$ in the second term. We will need explicit expressions for the Dorfman derivative of sections of E and $\mathrm{ad}\,\tilde{F}$. The Dorfman derivative acting on a generalised vector is

$$\begin{aligned} L_V V' = \mathcal{L}_v v' &+ (\mathcal{L}_v \lambda'^i - \imath_{v'} \mathrm{d}\lambda^i) + (\mathcal{L}_v \rho' - \imath_{v'} \mathrm{d}\rho + \epsilon_{ij} \mathrm{d}\lambda^i \wedge \lambda'^j) \\ &+ (\mathcal{L}_v \sigma'^i - \imath_{v'} \mathrm{d}\sigma^i + \mathrm{d}\rho \wedge \lambda'^i - \mathrm{d}\lambda^i \wedge \rho') \\ &+ (\mathcal{L}_v \tau' - \epsilon_{ij} j\lambda'^i \wedge \mathrm{d}\sigma^j + j\rho' \wedge \mathrm{d}\rho + \epsilon_{ij} j\sigma'^i \wedge \mathrm{d}\lambda^j). \end{aligned} \tag{B.45}$$

The Dorfman derivative acting on a section of the adjoint bundle is

$$\begin{aligned} L_V R = (\mathcal{L}_v l &+ \tfrac{1}{2}\gamma \lrcorner \mathrm{d}\rho + \tfrac{1}{4}\epsilon_{kl}\beta^k \lrcorner \mathrm{d}\lambda^l + \tfrac{3}{4}\epsilon_{kl}\tilde{\alpha}^k \lrcorner \mathrm{d}\sigma^l) \\ &+ (\mathcal{L}_v r + j\gamma \lrcorner j\mathrm{d}\rho - \tfrac{1}{2}\mathbb{1}\gamma \lrcorner \mathrm{d}\rho + \epsilon_{ij} j\beta^i \lrcorner j\mathrm{d}\lambda^j - \tfrac{1}{4}\mathbb{1}\epsilon_{kl}\beta^k \lrcorner \mathrm{d}\lambda^l \\ &\quad + \epsilon_{ij} j\tilde{\alpha}^i \lrcorner j\mathrm{d}\sigma^j - \tfrac{3}{4}\mathbb{1}\epsilon_{kl}\tilde{\alpha}^k \lrcorner \mathrm{d}\sigma^l) \\ &+ (\mathcal{L}_v a^i{}_j + \epsilon_{jk}\beta^i \lrcorner \mathrm{d}\lambda^k - \tfrac{1}{2}\delta^i{}_j \epsilon_{kl}\beta^k \lrcorner \mathrm{d}\lambda^l + \epsilon_{jk}\tilde{\alpha}^i \lrcorner \mathrm{d}\sigma^k - \tfrac{1}{2}\delta^i{}_j \epsilon_{kl}\tilde{\alpha}^k \lrcorner \mathrm{d}\sigma^l) \\ &+ (\mathcal{L}_v \beta^i - \gamma \lrcorner \mathrm{d}\lambda^i - \tilde{\alpha}^i \lrcorner \mathrm{d}\rho) \\ &+ (\mathcal{L}_v B^i + r \cdot \mathrm{d}\lambda^i + a^i{}_j \mathrm{d}\lambda^j + \beta^i \lrcorner \mathrm{d}\rho - \gamma \lrcorner \mathrm{d}\sigma^i) \\ &+ (\mathcal{L}_v \gamma + \epsilon_{ij}\tilde{\alpha}^i \lrcorner \mathrm{d}\lambda^j) \\ &+ (\mathcal{L}_v C + r \cdot \mathrm{d}\rho + \epsilon_{ij}\mathrm{d}\lambda^i \wedge B^j + \epsilon_{ij}\beta^i \lrcorner \mathrm{d}\sigma^j) + (\mathcal{L}_v \tilde{\alpha}^i) \\ &+ (\mathcal{L}_v \tilde{a}^i + r \cdot \mathrm{d}\sigma^i + a^i{}_j \mathrm{d}\sigma^j - \mathrm{d}\lambda^i \wedge C + B^i \wedge \mathrm{d}\rho). \end{aligned} \tag{B.46}$$

For $\mathrm{E}_{5(5)}$, we also need the vector bundle transforming in the $\mathbf{10_2}$ representation of $\mathrm{Spin}(5,5) \times \mathbb{R}^+$. We define this bundle as

$$N \simeq S \oplus \wedge^2 T^* M \oplus S \otimes \wedge^4 T^* M. \tag{B.47}$$

We write sections of this bundle as

$$Q = m^i + n + p^i, \tag{B.48}$$

where $m^i \in \Gamma(S)$, $n \in \Gamma(\wedge^2 T^*M)$ and $p^i \in \Gamma(S \otimes \wedge^4 T^*M)$. We define the adjoint action of $R \in \Gamma(\operatorname{ad}\tilde{F})$ on $Q \in \Gamma(N)$ to be $Q' = R \cdot Q$, with components

$$\begin{aligned}
m'^i &= 2lm^i + a^i{}_j m^j + \beta^i \lrcorner n - \gamma \lrcorner p^i,\\
n' &= 2ln + r \cdot n + \epsilon_{ij}\beta^i \lrcorner p^j + \epsilon_{ij} m^i B^j,\\
p'^i &= 2lp^i + r \cdot p^i + a^i{}_j p^j + B^i \wedge n - m^i C.
\end{aligned} \tag{B.49}$$

Using $\mathbf{16}^c \times \mathbf{10} \to \mathbf{16}$, we define a projection to E as

$$\times_E \colon E^* \otimes N \to E. \tag{B.50}$$

Explicitly, as a section of E, this allows us to define

$$\mathrm{d}Q \coloneqq \partial \times_E Q = \mathrm{d}m^i + \mathrm{d}n. \tag{B.51}$$

We define a patching of the bundle such that on the overlaps of local patches $U_i \cap U_j$ we have

$$V_{(i)} = \mathrm{e}^{\mathrm{d}\Lambda^i_{(ij)} + \mathrm{d}\tilde{\Lambda}_{(ij)}} V_{(j)}, \tag{B.52}$$

where $\Lambda^i_{(ij)}$ and $\tilde{\Lambda}_{(ij)}$ are locally one- and three-forms respectively. This defines the gauge-invariant field strengths as

$$F^i = \mathrm{d}B^i, \qquad F = \mathrm{d}C - \tfrac{1}{2}\epsilon_{ij} F^i \wedge B^j. \tag{B.53}$$

We embed the NS-NS and R-R three-form fluxes as $F^1_3 = H$ and $F^2_3 = F_3$.

The *twisted* Dorfman derivative $\hat{L}_V$ of an untwisted generalised tensor $\tilde{\mu}$ is defined by

$$\hat{L}_{\tilde{V}}\tilde{\mu} = \mathrm{e}^{-B^i - C} L_{\mathrm{e}^{B^i + C}\tilde{V}}(\mathrm{e}^{B^i + C}\tilde{\mu}). \tag{B.54}$$

The twisted Dorfman derivative $\hat{L}_V$ is given by the same expression as the usual Dorfman derivative but with the substitutions

$$\begin{aligned}
\mathrm{d}\lambda^i &\to \mathrm{d}\tilde{\lambda}^i - \imath_{\tilde{v}} F^i,\\
\mathrm{d}\rho &\to \mathrm{d}\tilde{\rho} - \imath_{\tilde{v}} F - \epsilon_{ij}\tilde{\lambda}^i \wedge F^j,\\
\mathrm{d}\sigma^i &\to \mathrm{d}\tilde{\sigma}^i + \tilde{\lambda}^i \wedge F - \tilde{\rho} \wedge F^i.
\end{aligned} \tag{B.55}$$

The projection $\partial \times_E Q$ also simplifies in a similar fashion allowing us to define

$$\mathrm{d}_{F^i} Q \coloneqq \mathrm{e}^{-B^i}\big(\partial \times_E (\mathrm{e}^{B^i} Q)\big) = \mathrm{d}m^i + \mathrm{d}n + \epsilon_{ij} m^i F^j. \tag{B.56}$$

The quadratic invariant for $E_{5(5)}$ is

$$\eta(Q,Q) = \epsilon_{ij} m^i p^j - \tfrac{1}{2} n \wedge n. \tag{B.57}$$

The cubic invariant for $E_{6(6)}$ is

$$c(V,V,V) = -\tfrac{1}{2}(\imath_v \rho \wedge \rho + \epsilon_{ij}\rho \wedge \lambda^i \wedge \lambda^j - 2\epsilon_{ij}\imath_v \lambda^i \sigma^j). \tag{B.58}$$

The symplectic invariant for $E_{7(7)}$ is

$$s(V,V') = -\tfrac{1}{4}\big((\imath_v \tau' - \imath_{v'}\tau) + \epsilon_{ij}(\lambda^i \wedge \sigma'^j - \lambda'^i \wedge \sigma^j) - \rho \wedge \rho'\big). \tag{B.59}$$

The $\mathfrak{e}_{d+1(d+1)}$ Killing form is

$$\begin{aligned} \operatorname{tr}(R,R') = \tfrac{1}{2}\Big(&\tfrac{1}{8-d}\operatorname{tr}(r)\operatorname{tr}(r') + \operatorname{tr}(rr') + \operatorname{tr}(aa') + \gamma \lrcorner C' + \gamma' \lrcorner C + \epsilon_{ij}(\beta^i \lrcorner B'^j + \beta'^i \lrcorner B^j) \\ &+ \epsilon_{ij}(\tilde{\alpha}^i \lrcorner \tilde{a}'^j + \tilde{\alpha}'^i \lrcorner \tilde{a}^j)\Big). \end{aligned} \tag{B.60}$$

The form of the $E_{d+1(d+1)}$-invariant volume κ^2 depends on the compactification ansatz. For compactifications of the form

$$g_{10} = e^{2\Delta} g_{10-d} + g_d, \tag{B.61}$$

the invariant volume includes a dilaton dependence and is given by

$$\kappa^2 = e^{-2\phi} e^{(8-d)\Delta}\sqrt{g_d}. \tag{B.62}$$

We can include non-zero axion C_0 and dilaton ϕ in our formalism using the SL(2) frame given in [139]. Let $\hat{f}^i{}_{\hat{i}}$ be an SL(2) frame written in terms of a parametrisation of SL(2)/SO(2) as

$$\hat{f}^i{}_{\hat{i}} = \begin{pmatrix} e^{\phi/2} & 0 \\ C_0 e^{\phi/2} & e^{-\phi/2} \end{pmatrix}. \tag{B.63}$$

Comparing with the split frame of [139], we see we can write a generalised vector as

$$V = v + e^{-\phi/2}\lambda^i + e^{-\phi}\rho + e^{-3\phi/2}\sigma^i, \tag{B.64}$$

where $\lambda^i = \hat{f}^i{}_{\hat{i}}\lambda^{\hat{i}}$ etc., and $\lambda^{\hat{i}}$ contains no explicit axion-dilaton dependence. Using this we can determine where the dilaton appears in the adjoint for $E_{d(d)}$ and Q for $E_{5(5)}$

$$\begin{aligned} R &= l + r + a^i{}_j + e^{\phi/2}\beta^i + e^{-\phi/2}B^i + e^{\phi}\gamma + e^{-\phi}C + e^{3\phi/2}\tilde{\alpha}^i + e^{-3\phi/2}\tilde{a}^i, \\ Q &= e^{-\phi/2}m^i + e^{-\phi}n + e^{-3\phi/2}p^i. \end{aligned} \tag{B.65}$$

Looking back to $\tilde{J}_\alpha$ and $\tilde{Q}$ for the NS5-brane solution in (3.209) and (3.210), we see they are indeed of this form. The various powers of the dilaton correspond to the exponentiated action of the adjoint element given by

$$l + r = \frac{\phi}{4}(-1 + \mathbb{1}). \tag{B.66}$$

Appendix C

Intrinsic torsion for SU(6)

Following [143, 144], we first calculate the intrinsic torsion space $W_{\text{int}}^{\text{SU}(6)}$ for generalised SU(6) structures. Decomposing under $\text{SU}(2) \times \text{SU}(6)$ the space of generalised torsions decomposes as

$$\begin{aligned} W = \mathbf{56} + \mathbf{912} = (\mathbf{1},\mathbf{1}) + 2(\mathbf{1},\mathbf{15}) + (\mathbf{1},\mathbf{21}) + (\mathbf{1},\mathbf{35}) + (\mathbf{1},\mathbf{105}) \\ + 3(\mathbf{2},\mathbf{6}) + (\mathbf{2},\mathbf{20}) + (\mathbf{2},\mathbf{84}) + (\mathbf{3},\mathbf{1}) + (\mathbf{3},\mathbf{15}) + \text{c.c.} \end{aligned} \tag{C.1}$$

The space of SU(6) connections is given by

$$\begin{aligned} K_{\text{SU}(6)} &= \big((\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{6}) + (\mathbf{1},\mathbf{15}) + \text{c.c.}\big) \times (\mathbf{1},\mathbf{35}) \\ &= (\mathbf{1},\mathbf{15}) + (\mathbf{1},\mathbf{21}) + (\mathbf{1},\mathbf{35}) + (\mathbf{1},\mathbf{105}) \\ &\quad + (\mathbf{1},\mathbf{384}) + (\mathbf{2},\mathbf{6}) + (\mathbf{2},\mathbf{84}) + (\mathbf{2},\mathbf{120}) + \text{c.c.} \end{aligned} \tag{C.2}$$

Thus we have

$$W_{\text{int}}^{\text{SU}(6)} \supseteq (\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{15}) + 2(\mathbf{2},\mathbf{6}) + (\mathbf{2},\mathbf{20}) + (\mathbf{3},\mathbf{1}) + (\mathbf{3},\mathbf{15}) + \text{c.c.}, \tag{C.3}$$

where equality holds if there are no unexpected kernels in the map $\tau\colon K_{\text{SU}(6)} \to W$. To see that this is indeed the case, we need the explicit map. In SU(8) indices, sections of $K_{\text{SU}(8)}$ are given by

$$\hat{\Sigma} = (\hat{\Sigma}_{\alpha\beta}{}^{\gamma}{}_{\delta}, \bar{\hat{\Sigma}}^{\alpha\beta\gamma}{}_{\delta},) \in (\mathbf{28} + \overline{\mathbf{28}}) \times \mathbf{63}, \tag{C.4}$$

where the elements are antisymmetric on α and β and traceless on contracting γ with δ. The space W decomposes as

$$W = \mathbf{56} + \mathbf{912} = \mathbf{28} + \mathbf{36} + \mathbf{420} + \text{c.c.}, \tag{C.5}$$

and the map τ is

$$\begin{aligned} \tau(\hat{\Sigma})_{\alpha\beta} &= \hat{\Sigma}_{\alpha\gamma}{}^{\gamma}{}_{\beta} && \in \mathbf{36} + \mathbf{28}, \\ \tau(\hat{\Sigma})_{\alpha\beta\gamma}{}^{\delta} &= 3\hat{\Sigma}^{0}_{[\alpha\beta}{}^{\delta}{}_{\gamma]} && \in \mathbf{420}, \end{aligned} \tag{C.6}$$

where the "0" superscript on $\hat{\Sigma}^0_{[\alpha\beta}{}^\delta{}_{\gamma]}$ means it is completely traceless. The **28** and **36** representations correspond to the symmetric and antisymmetric parts of $\tau(\hat{\Sigma})_{\alpha\beta}$. There are similar expressions for the conjugate representations in terms of $\bar{\hat{\Sigma}}$.

Turning to SU(6) connections, let Σ be a section of $K_{\mathrm{SU}(6)}$. We can split the spinor indices α into $a = 1, \ldots, 6$ and $i = 7, 8$ so that the non-zero components are

$$\begin{aligned}
\Sigma_{ab}{}^c{}_d &\in (\mathbf{1}, \mathbf{15}) \times (\mathbf{1}, \mathbf{35}), \\
\Sigma_{ai}{}^c{}_d = -\Sigma_{ia}{}^c{}_d &\in (\mathbf{2}, \mathbf{6}) \times (\mathbf{1}, \mathbf{35}), \\
\Sigma_{ij}{}^c{}_d &\in (\mathbf{1}, \mathbf{35}),
\end{aligned} \tag{C.7}$$

and similarly for the conjugate $\bar{\Sigma}$. We then find the non-zero components of $\tau(\Sigma)$ are

$$\begin{aligned}
\tau(\Sigma)_{ab} &= \Sigma_{ac}{}^c{}_b && \in (\mathbf{1}, \mathbf{15}) + (\mathbf{1}, \mathbf{21}), \\
\tau(\Sigma)_{ib} &= \Sigma_{ic}{}^c{}_b && \in (\mathbf{2}, \mathbf{6}), \\
\tau(\Sigma)_{abc}{}^d &= 3\Sigma_{[ab}{}^d{}_{c]} + \Sigma_{[a|e|}{}^e{}_b \delta^d_{c]} && \in (\mathbf{1}, \mathbf{105}) + (\mathbf{1}, \mathbf{15}), \\
\tau(\Sigma)_{abi}{}^c &= 2\Sigma_{i[a}{}^c{}_{b]} + \tfrac{2}{3}\Sigma_{ie}{}^e{}_{[a}\delta^c_{b]} && \in (\mathbf{2}, \mathbf{84}) + (\mathbf{2}, \mathbf{6}), \\
\tau(\Sigma)_{aij}{}^c &= \Sigma_{ij}{}^c{}_a && \in (\mathbf{1}, \mathbf{35}), \\
\tau(\Sigma)_{abi}{}^j &= \tfrac{1}{3}\Sigma_{[a|c|}{}^c{}_{b]}\delta^j_i && \in (\mathbf{1}, \mathbf{15}), \\
\tau(\Sigma)_{aij}{}^k &= -\tfrac{1}{3}\Sigma_{[i|c}{}^c{}_a\delta^k_{j]} && \in (\mathbf{2}, \mathbf{6}),
\end{aligned} \tag{C.8}$$

and hence W_{int} is indeed given by an equality in (C.3). Note in addition that

$$\tau(\Sigma)_{abi}{}^i - \tfrac{2}{3}\tau(\Sigma)_{[ab]} = 0, \qquad \tau(\Sigma)_{aij}{}^j + \tfrac{1}{6}\tau(\Sigma)_{ia} = 0. \tag{C.9}$$

We now turn to showing which components of the intrinsic torsion enter each of the integrability conditions on the pair $\{J_\alpha, X\}$. For this it is useful to have an expression for $T(V)$ for SU(6) connections. We first note that the compatible SU(6) connection $\hat{D}$ must also be an SU(8) connection and hence can be written as

$$\hat{D} = D + \hat{\Sigma}, \tag{C.10}$$

where $\hat{\Sigma} \in K_{\mathrm{SU}(8)}$ and D is a torsion-free SU(8) connection. (That such connections exist is central to the formulation of supergravity in terms of generalised geometry: they are the analogues of the Levi-Civita connection of conventional gravity [137, 138].) Since D is torsion-free, the torsion of $\hat{D}$ is given by

$$T = \tau(\hat{\Sigma}). \tag{C.11}$$

We can then calculate $T(V)$. Writing $V = (V^{\alpha\beta}, \bar{V}_{\alpha\beta})$ for the decomposition $\mathbf{56} = \mathbf{28} + \overline{\mathbf{28}}$ and $T(V) = (T(V)_0, T(V)^\alpha{}_\beta, T(V)^{\alpha\beta\gamma\delta})$ for the decomposition of the adjoint $\mathbf{1} + \mathbf{133} =$

$\mathbf{1}+\mathbf{63}+\mathbf{70}$, we define the adjoint action on a generalised vector W as

$$\begin{aligned}\big[T(V)\cdot W\big]^{\alpha\beta} &= T(V)_0 W^{\alpha\beta} + T(V)^{\alpha}{}_{\gamma}W^{\gamma\beta} + T(V)^{\beta}{}_{\gamma}W^{\alpha\gamma} + T(V)^{\alpha\beta\gamma\delta}\bar{W}_{\gamma\delta},\\ \big[\overline{T(V)\cdot W}\big]_{\alpha\beta} &= T(V)_0\bar{W}_{\alpha\beta} - T(V)^{\gamma}{}_{\alpha}\bar{W}_{\gamma\beta} - T(V)^{\gamma}{}_{\beta}\bar{W}_{\alpha\gamma} + \bar{T}(V)_{\alpha\beta\gamma\delta}W^{\gamma\delta}.\end{aligned} \tag{C.12}$$

From the form of the generalised Lie derivative in SU(8) indices given in appendix D of [126], we find

$$\begin{aligned}T(V)_0 &= \tfrac{1}{32}V^{\alpha\beta}\tau(\hat{\Sigma})_{\alpha\beta} + \text{c.c.},\\ T(V)^{\alpha}{}_{\beta} &= \tfrac{1}{32}V^{\gamma\gamma'}\Big(\tau(\hat{\Sigma})_{\gamma\gamma'\beta}{}^{\alpha} + \tfrac{5}{3}\tau(\hat{\Sigma})_{\beta\gamma}\delta^{\alpha}_{\gamma'} + \tfrac{1}{3}\tau(\hat{\Sigma})_{\gamma\beta}\delta^{\alpha}_{\gamma'} + \tfrac{1}{6}\tau(\hat{\Sigma})_{\gamma\gamma'}\delta^{\alpha}_{\beta}\Big) + \text{c.c.},\\ T(V)^{\alpha\beta\gamma\delta} &= -\tfrac{1}{8}V^{\epsilon\epsilon'}\Big(\bar{\tau}(\hat{\Sigma})^{[\alpha\beta\gamma}{}_{\epsilon}\delta^{\delta]}_{\epsilon'} - \bar{\tau}(\hat{\Sigma})^{[\alpha\beta}\delta^{\gamma}_{\epsilon}\delta^{\delta]}_{\epsilon'}\Big) - \star(\text{c.c.}),\end{aligned} \tag{C.13}$$

where $\star$(c.c.) is the Hodge dual of the conjugate expression.

We also have expressions for the structures X and J_α in terms of the spinor indices. For X the non-zero component is the singlet in the $\mathbf{28} = (\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{6}) + (\mathbf{1},\mathbf{15})$ representation

$$X^{\alpha\beta} = (T^{ij}, T^{ia}, T^{ab}) \propto (\epsilon^{ij}, 0, 0), \tag{C.14}$$

while for J_α it is the triplet in the $\mathbf{63} = (\mathbf{1},\mathbf{1})+(\mathbf{3},\mathbf{1})+(\mathbf{2},\mathbf{6})+(\mathbf{2},\overline{\mathbf{6}})+(\mathbf{1},\mathbf{35})$ representation

$$(J_\alpha)^{\alpha}{}_{\beta} = \big((J_\alpha)_0\delta^{i}{}_{j}, (J_\alpha)^{i}{}_{j}, (J_\alpha)^{i}{}_{a}, (J_\alpha)^{ia}, (J_\alpha)^{a}{}_{b}\big) \propto \big(0, (\sigma_\alpha)^{i}{}_{j}, 0, 0, 0\big), \tag{C.15}$$

where σ_α are the Pauli matrices. Substituting into the generalised Lie derivative in SU(8) indices, we find

$$L_X X \equiv 0 \qquad \text{identically}, \tag{C.16}$$

simply from the form of the X given in (C.14).

For the moment maps, since κ^2 has weight two, the condition (3.118) on the intrinsic torsion can be written as

$$\begin{aligned}\operatorname{tr}\big(J_\alpha T(V)\big) + T(J_\alpha\cdot V)_0 \propto{}& \tfrac{1}{2}V^{\gamma\gamma'}\sigma_\alpha{}^{j}{}_{i}\,\tau_{\gamma\gamma' j}{}^{i} + \tfrac{1}{6}V^{\gamma i}\sigma_\alpha{}^{j}{}_{i}(5\tau_{j\gamma} + \tau_{\gamma j})\\ &+ V^{\gamma i}\sigma_\alpha{}^{j}{}_{i}(\tau_{\gamma j} - \tau_{j\gamma}) + \text{c.c.},\end{aligned} \tag{C.17}$$

where we abbreviate $\tau(\hat{\Sigma})_{\alpha\beta}$ and $\tau(\hat{\Sigma})_{\alpha\beta\gamma}{}^{\delta}$ as $\tau_{\alpha\beta}$ and $\tau_{\alpha\beta\gamma}{}^{\delta}$. This vanishes for all V if and only if

$$\begin{aligned}\sigma_\alpha{}^{j}{}_{i}\,\tau_{abj}{}^{i} &= 0 \quad \in (\mathbf{3},\mathbf{15}),\\ (\tau_{aij}{}^{j} + \tfrac{1}{6}\tau_{ia}) - \tfrac{7}{6}\tau_{ai} &= 0 \quad \in (\mathbf{2},\mathbf{6}),\\ \tau_{(ij)} &= 0 \quad \in (\mathbf{3},\mathbf{1}).\end{aligned} \tag{C.18}$$

Note, comparing with (C.9), that the $(\mathbf{2},\mathbf{6})$ representation appearing in the second line is indeed independent of the $(\mathbf{2},\mathbf{6})$ component of the torsion generated by an SU(6) generalised connection.

The non-zero components of $T(X)$ are

$$\begin{aligned} T(X)_0 &\propto \epsilon^{ij}\tau_{ij}, & T(X)^i{}_j &\propto -2\epsilon^{ik}\tau_{(jk)} - \tfrac{1}{2}(\epsilon^{kl}\tau_{kl})\delta^i_j, \\ T(X)^i{}_a &\propto \epsilon^{kl}\tau_{akl}{}^i - \tfrac{1}{3}\epsilon^{ik}(5\tau_{ak} + \tau_{ka}), & T(X)^a{}_b &\propto \epsilon^{kl}\tau_{bkl}{}^a + \tfrac{1}{6}(\epsilon^{kl}\tau_{kl})\delta^a_b, \\ \bar{T}(X)^{abij} &\propto \epsilon^{ik}\bar{\tau}^{abj}{}_k - \epsilon^{jk}\bar{\tau}^{abi}{}_k + \tfrac{2}{3}\epsilon^{ij}\bar{\tau}^{[ab]}, & \bar{T}(X)^{abci} &\propto \epsilon^{ik}\bar{\tau}^{abc}{}_k, \end{aligned} \tag{C.19}$$

so the non-zero components of $T(X)\cdot\bar{X}$ are

$$\begin{aligned} \big(T(X)\cdot\bar{X}\big)_{ij} &\propto 4\tau_{[ij]} && \in (\mathbf{1},\mathbf{1}), \\ \big(T(X)\cdot\bar{X}\big)_{ia} &\propto 2(\tau_{aij}{}^j + \tfrac{1}{6}\tau_{ia}) + \tfrac{5}{3}\tau_{ai} && \in (\mathbf{2},\mathbf{6})', \\ \big(\overline{T(X)\cdot\bar{X}}\big)^{ab} &\propto -2(\bar{\tau}^{abi}{}_i - \tfrac{2}{3}\bar{\tau}^{ab}) && \in (\mathbf{1},\overline{\mathbf{15}}). \end{aligned} \tag{C.20}$$

Note again that the linear combination of torsions in the second and third lines are independent of those appearing in an SU(6) generalised connection, and further that the combination in the second line is different from the one in the second line of (C.18), and hence we denote it $(\mathbf{2},\mathbf{6})'$. Similarly, the non-zero components of $[T(X), J_\alpha]$ are

$$\begin{aligned} [T(X),J_\alpha]^i{}_j &\propto (\epsilon^{kl}\tau_{kl})\sigma_\alpha{}^i{}_j - 2\epsilon^{ik}\tau_{(lk)}\sigma_\alpha{}^l{}_j + 2\epsilon^{lk}\tau_{(jk)}\sigma_\alpha{}^i{}_l && \in (\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{1}), \\ [T(X),J_\alpha]^i{}_a &\propto \big(2(\tau_{aij}{}^j + \tfrac{1}{6}\tau_{ia}) + \tfrac{5}{3}\tau_{ai}\big)\sigma_\alpha{}^i{}_j\epsilon^{jk} && \in (\mathbf{2},\mathbf{6})', \\ [T(X),J_\alpha]^{abci} &\propto -\epsilon^{jk}\bar{\tau}^{abc}{}_k\sigma_\alpha{}^i{}_j && \in (\overline{\mathbf{2}},\overline{\mathbf{20}}), \\ [T(X),J_\alpha]^{abij} &\propto 2\bar{\tau}^{abj}{}_k\epsilon^{lk}\sigma_\alpha{}^i{}_l && \in (\mathbf{3},\overline{\mathbf{15}}). \end{aligned} \tag{C.21}$$

Note that the combination of torsions appearing in the second line is the same as the combination appearing in the second line of (C.20).

Appendix D

Moment maps and quotients

In this appendix, we briefly review the notion of moment maps, and symplectic and hyper-Kähler quotients, including the infinite-dimensional example of flat gauge connections on a Riemann surface due to Atiyah and Bott [186].

Consider a manifold Y with a symplectic form Ω that is closed, $\mathrm{d}\Omega = 0$. Suppose there is an action of a Lie group $\mathcal{G}$ on Y that preserves the symplectic structure – that is $\mathcal{G}$ acts on Y via symplectomorphisms. An element g in the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ induces a vector field ρ_g on Y. As the group $\mathcal{G}$ acts via symplectomorphisms, the Lie derivative of Ω with respect to ρ_g vanishes. Together with $\mathrm{d}\Omega = 0$, this implies $\mathrm{d}\imath_{\rho_g}\Omega = 0$ and so $\imath_{\rho_g}\Omega$ is closed. A moment map for the action of the group $\mathcal{G}$ on the manifold Y is a map $\mu\colon Y \times \mathfrak{g} \to \mathbb{R}$ such that, for all $g \in \mathfrak{g}$,

$$\mathrm{d}\mu(g) = \imath_{\rho_g}\Omega. \tag{D.1}$$

The moment map is defined up to an additive constant of integration. If $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$, one can also view μ as a map from Y to $\mathfrak{g}^*$. If $\mathcal{G}$ is non-Abelian one can fix the constant by requiring that the map is equivariant, that is, that μ commutes with the action of $\mathcal{G}$ on Y. Still viewing μ as a map from Y to $\mathfrak{g}^*$, one can then form the *symplectic quotient*

$$Y/\!\!/\mathcal{G} = \mu^{-1}(0)/\mathcal{G}. \tag{D.2}$$

This quotient space inherits a symplectic structure from Y and is a manifold if $\mathcal{G}$ acts freely on Y. (Generically the reduced space is not a manifold, but is a "stratified space".)

On a hyper-Kähler manifold Y, one can consider an action of $\mathcal{G}$ that preserves all three symplectic forms Ω_α. Instead of a single moment map, one can then consider a triplet of maps $\mu_\alpha\colon Y \to \mathfrak{g}^*$ satisfying

$$\mathrm{d}\mu_\alpha(g) = \imath_{\rho_g}\Omega_\alpha. \tag{D.3}$$

Choosing them to be equivariant, one can then define the *hyper-Kähler quotient* [308]

$$Y/\!\!/\!\!/\mathcal{G} = \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)/\mathcal{G}. \tag{D.4}$$

This space inherits a hyper-Kähler structure from Y, and the quotient is a manifold if $\mathcal{G}$ acts freely.

We can also consider the case where both the group and the symplectic space are infinite dimensional. A well-known example is the work of Atiyah–Bott [186]. Let G be a compact Lie group and P be a principal G-bundle over a compact Riemann surface Σ. The group of gauge transformations $\mathcal{G}$ is the set of G-equivariant diffeomorphisms of P. Infinitesimally it is generated by sections of the adjoint bundle $\operatorname{ad} P$, that is $\operatorname{Lie}(\mathcal{G}) = \Gamma(\operatorname{ad} P)$. Let Y be the infinite-dimensional space of connections on P. The curvature of a connection $A \in Y$ is

$$F = \mathrm{d}A + \tfrac{1}{2}[A, A]. \tag{D.5}$$

One can associate the tangent space $T_A Y$ at $A \in Y$ with the space of $\operatorname{ad} P$-valued one-forms $\Omega^1(\Sigma, \operatorname{ad} P)$. Given two elements $\alpha, \beta \in T_A Y$, one can define a symplectic product

$$\Omega(\alpha, \beta) = \int_\Sigma \operatorname{tr}(\alpha \wedge \beta), \tag{D.6}$$

where tr is a gauge-invariant inner product on $\mathfrak{g}$, for example the Killing form if $\mathfrak{g}$ is semi-simple. To see that Ω is non-degenerate note that, given a metric on Σ, we have

$$\Omega(\alpha, \star\alpha) = \int_\Sigma \operatorname{tr}(\alpha \wedge \star\alpha) = \|\alpha\|^2 \geq 0, \tag{D.7}$$

and so $\Omega(\alpha, \star\alpha) = 0$ if and only if $\alpha = 0$. Furthermore, any connection A can be written as $A = A^{(0)} + \alpha$ for some fixed connection $A^{(0)}$ and $\alpha \in \Omega^1(\Sigma, \operatorname{ad} P)$ (in other words Y is an affine space modelled on $\Omega^1(\Sigma, \operatorname{ad} P)$), meaning that in this parametrisation Ω is independent of A and hence, in particular, Ω is a closed two-form on Y.

The moment map for the $\mathcal{G}$-action on Y is $\mu = F$. To see this note that, given an element Λ of $\operatorname{Lie}(\mathcal{G}) \simeq \Gamma(\operatorname{ad} P)$, the induced vector field on Y is just the gauge transformation of A, namely

$$\rho_\Lambda = \mathrm{d}\Lambda + [A, \Lambda]. \tag{D.8}$$

Thus we have, for any $\alpha \in \Gamma(TY)$,

$$\begin{aligned} \imath_{\rho_\Lambda}\Omega(\alpha) = \Omega(\rho_\Lambda, \alpha) &= \int_\Sigma \operatorname{tr}\big[(\mathrm{d}\Lambda + [A, \Lambda]) \wedge \alpha\big] = \int_\Sigma \operatorname{tr}\big[\Lambda \wedge (\mathrm{d}\alpha + [A, \alpha])\big] \\ &= \imath_\alpha\Big(\delta \int_\Sigma \operatorname{tr} \Lambda F\Big), \end{aligned} \tag{D.9}$$

where δ is the exterior derivative on Y, that is, in coordinates, the functional derivative $\delta/\delta A_m(x)$. Viewed as a map $\mu \colon Y \to \operatorname{Lie}(\mathcal{G})^*$, we see that $\mu = F$.

This map is equivariant, and so we may form the symplectic reduction by quotienting by the space of gauge transformations $\mathcal{G}$

$$Y /\!\!/ \mathcal{G} = \mu^{-1}(0)/\mathcal{G}. \tag{D.10}$$

This is the moduli space of flat connections, that is $A \in Y$ such that $F = 0$ modulo gauge equivalence. The space of connections Y and the group of gauge transformations $\mathcal{G}$ are infinite dimensional, but the moduli space is actually finite dimensional.

Appendix E

Special geometries

E.1 Special Kähler geometry

There are a number of different ways to define rigid (or affine) special Kähler geometry [217, 218,309]. The most appropriate to our needs follows [218], stating that it is a $2n$-dimensional Kähler manifold $\mathcal{A}_\mathrm{V}$ with a *flat*, torsion-free connection $\hat{\nabla}$ satisfying

$$\hat{\nabla}_m \Omega_{np} = 0, \qquad \hat{\nabla}_{[m}\mathcal{I}^p{}_{n]} = 0, \tag{E.1}$$

where Ω is the Kähler form and $\mathcal{I}$ is the complex structure. Note that $\hat{\nabla}$ is not the Levi-Civita connection, since these conditions do not imply $\hat{\nabla}$ is metric compatible.

Locally, by the Poincaré Lemma, the condition on $\mathcal{I}$ can be integrated. The usual formulation is to note that, since $\hat{\nabla}$ is torsion-free, one also has $\hat{\nabla}_{[m}\delta^k{}_{n]} = 0$, thus locally there exists a complex vector field X such that

$$\hat{\nabla}_n X^m = \delta^m{}_n - \mathrm{i}\mathcal{I}^m{}_n. \tag{E.2}$$

Writing the real and imaginary parts as

$$X^m = x^m + \mathrm{i}\hat{x}^m, \tag{E.3}$$

so that $\nabla_n x^m = \delta^m{}_n$ and $\nabla_n \hat{x}^m = -\mathcal{I}^m{}_n$, one notes that the metric is given by $g_{mn} = \Omega_{mp}\mathcal{I}^p{}_n = -\Omega_{mp}\hat{\nabla}_n\hat{x}^p = -\hat{\nabla}_n(\Omega_{mp}\hat{x}^p)$. But since g_{mn} is symmetric, this means there exists a local real function H such that the metric is given by the Hessian

$$g_{mn} = -\hat{\nabla}_m\hat{\nabla}_n H, \tag{E.4}$$

and $\Omega_{mn}\hat{x}^n = \hat{\nabla}_m H = \partial_m H$. Note that in these conventions, following [218], H is equal to minus the Kähler potential.

The fact that $\hat{\nabla}$ is torsion-free and flat means one can always introduce real coordinates such that $\hat{\nabla}_m = \partial/\partial x^m$. This notation is consistent with (E.3) since the condition

$\nabla_n \operatorname{Re} X^m = \delta^m{}_n$ means that in flat coordinates we can always locally identify $\operatorname{Re} X^m$ with the coordinate x^m. It is conventional to use a different index notation x^Σ to distinguish flat coordinates (or equivalently Σ is the index for a flat frame). If one requires that the symplectic structure takes a standard form in the flat coordinates, then the choice of x^Σ is determined up to affine symplectic transformations

$$x'^\Sigma = P^\Sigma{}_\Xi x^\Xi + c^\Sigma, \tag{E.5}$$

where $P \in \mathrm{Sp}(2n;\mathbb{R})$ and c is constant. Note that in flat coordinates $g_{\Sigma\Xi} = -\partial_\Sigma\partial_\Xi H$. Since $\hat{\nabla}$ is not the Levi-Civita connection, one cannot introduce coordinates that are both flat and complex. However, one can go halfway and define so-called "special coordinates" X^I such that

$$X = X^\Sigma \frac{\partial}{\partial\lambda^\Sigma} = X^I \frac{\partial}{\partial x^I} - F_I \frac{\partial}{\partial y_I}, \tag{E.6}$$

where $x^\Sigma = (x^I, y_I)$ are flat Darboux coordinates (that is ones where $\Omega = \mathrm{d}x^I \wedge \mathrm{d}y_I$), implying that $x^I = \operatorname{Re} X^I$ and $y_I = -\operatorname{Re} F_I$. Furthermore, the condition (E.2) implies that there is a local holomorphic function $F(X^I)$, called the prepotential, such that $F_I = \partial F/\partial X^I$.

Again following [218], one can define a local (or projective) special Kähler manifold in terms of the complex cone over it, in analogy to the way a quaternionic-Kähler manifold defines a hyper-Kähler cone. Suppose $\mathcal{A}_\mathrm{V}$ is a rigid special Kähler manifold such that there is a globally defined holomorphic complex vector field X satisfying (E.2) that generates a $\mathbb{C}^*$ action that preserves the structure. Then the rigid Kähler structure on Y descends to a local special Kähler structure on the quotient space $\mathcal{A}_\mathrm{V}/\mathbb{C}^*$.[1] One can also show that, as a function of any set of flat coordinates, H is homogeneous of degree two. Furthermore, the Kähler potential K on $\mathcal{A}_\mathrm{V}/\mathbb{C}^*$ is given by

$$\mathrm{e}^{-K} = H = \tfrac{1}{4}\mathrm{i}\Omega(X, \bar{X}), \tag{E.7}$$

where we use the homogeneity of H to derive the last equality.

In gauged $\mathcal{N} = 2$ supergravity one identifies an action of a group $\mathcal{G}_\mathrm{V}$ on $\mathcal{A}_\mathrm{V}/\mathbb{C}^*$, which can be lifted to an action on $\mathcal{A}_\mathrm{V}$ that commutes with the $\mathbb{C}^*$ action. Supersymmetry requires that the action of $\mathcal{G}_\mathrm{V}$ preserves the special Kähler structure. If $\hat{k}_{\hat{\lambda}} \in \Gamma(T\mathcal{A}_\mathrm{V})$ is the vector field corresponding to the action of an element of the Lie algebra $\hat{\lambda} \in \mathfrak{g}_\mathrm{V}$, then one first requires

$$\mathcal{L}_{\hat{k}_{\hat{\lambda}}}\Omega = 0, \qquad \mathcal{L}_{\hat{k}_{\hat{\lambda}}}\mathcal{I} = 0, \tag{E.8}$$

or, in other words, that $\hat{k}_{\hat{\lambda}}$ is a real holomorphic Killing vector. In addition, it must map flat coordinates to flat coordinates by a symplectic rotation, equivalent to the condition,

[1] Strictly, the fermions provide an additional integral condition on the cohomology of the Kähler form on the quotient [217].

in components, that it is linear in x^Σ, that is

$$\hat{k}^\Sigma_{\hat{\lambda}} = p_{\hat{\lambda}}{}^\Sigma{}_\Xi x^\Xi, \tag{E.9}$$

where $p_{\hat{\lambda}} \in \mathfrak{sp}_n(\mathbb{R})$. It is easy to see that the corresponding moment map is given by

$$\mu_{\hat{\lambda}} = \tfrac{1}{2} p_{\hat{\lambda}\,\Sigma\Xi} x^\Sigma x^\Xi, \tag{E.10}$$

where $p_{\hat{\lambda}\Sigma\Xi} = p_{\hat{\lambda}}{}^\Lambda{}_\Xi \Omega_{\Lambda\Sigma}$. The particular gauging of the $\mathcal{N}=2$ theory is encoded in an embedding tensor $\hat{\Theta}^{\hat{\lambda}}_\Lambda$ [229, 230]. This can be used to define a set of (constant) generators in $\mathfrak{sp}_n(\mathbb{R})$

$$\mathcal{X}_{\Lambda\Xi}{}^\Sigma = \hat{\Theta}^{\hat{\lambda}}_\Lambda p_{\hat{\lambda}}{}^\Sigma{}_\Xi, \tag{E.11}$$

so that, by definition, they must satisfy [229]

$$\mathcal{X}_{\Lambda[\Xi\Sigma]} = 0, \qquad \mathcal{X}_{\Lambda_1\Gamma}{}^\Sigma \mathcal{X}_{\Lambda_2\Xi}{}^\Gamma - \mathcal{X}_{\Lambda_2\Gamma}{}^\Sigma \mathcal{X}_{\Lambda_1\Xi}{}^\Gamma = \mathcal{X}_{\Lambda_1\Lambda_2}{}^\Gamma \mathcal{X}_{\Gamma\Xi}{}^\Sigma, \tag{E.12}$$

where $\mathcal{X}_{\Lambda\Xi\Sigma} = \mathcal{X}_{\Lambda\Xi}{}^\Gamma \Omega_{\Gamma\Sigma}$. They also satisfy a "representation constraint"

$$\mathcal{X}_{(\Lambda\Xi\Sigma)} = 0. \tag{E.13}$$

Finally, we note that contracting the moment map (E.10) with the embedding tensor gives $\hat{\Theta}^{\hat{\lambda}}_\Lambda \mu_{\hat{\lambda}} = \frac{1}{2}\mathcal{X}_{\Lambda\Xi\Sigma} x^\Xi x^\Sigma$. Using the condition $\mathcal{X}_{\Lambda\Xi\Sigma} X^\Xi X^\Sigma = 0$ given in [230], which is a consequence of $\hat{k}_{\hat{\lambda}}$ being holomorphic, we have

$$\hat{\Theta}^{\hat{\lambda}}_\Lambda \mu_{\hat{\lambda}} = \tfrac{1}{4}\mathcal{X}_{\Lambda\Xi\Sigma} X^\Xi \bar{X}^\Sigma. \tag{E.14}$$

E.2 Hyper-Kähler geometry of Wolf spaces

A Wolf space is a symmetric quaternionic-Kähler space $W/\mathbb{H}^* = G'/(G \times \mathrm{SU}(2))$ (as always we are ignoring discrete factors). The Riemannian case was first studied by Wolf in [205] and classified by Alekseevsky in [206], while the pseudo-Riemannian case, of relevance here, was analysed by Alekseevsky and Cortés in [207]. It is known that every quaternionic-Kähler manifold admits a bundle over it whose structure group is SU(2) [310]. More importantly, there exists a tri-Sasaki structure on this bundle [311] and hence the cone over the SU(2) bundle is hyper-Kähler [208]. The geometry on this "Swann bundle" W for Wolf spaces has been explicitly constructed in [209].

We can construct the tri-Sasaki and hyper-Kähler structures as follows. The tri-Sasaki space over the Wolf space is simply the symmetric space $S = G'/G$. As for any symmetric space, given an element $k \in G'$ one can decompose the right-invariant one-form θ as

$$\theta = k\mathrm{d}k^{-1} = \pi + A, \tag{E.15}$$

where $\pi \in \mathfrak{g}' \ominus \mathfrak{g}$ and $A \in \mathfrak{g}$. The one-forms π descend to one-forms on S, while A transforms as a G-connection. Since S is the tri-Sasaki space over the Wolf space, G' contains an SU(2) factor whose centraliser is G. We can then define a triplet of maps $\hat{\jmath}_\alpha \colon G' \to \mathfrak{g}'$ as parametrising the orbit

$$\hat{\jmath}_\alpha(k) = k\hat{\jmath}_\alpha^{(0)}k^{-1}, \tag{E.16}$$

where $\hat{\jmath}_\alpha^{(0)}$ is some fixed set of $\mathfrak{su}_2 \subset \mathfrak{g}'$ generators, stabilised by G. We normalise such that $\hat{\jmath}_\alpha$ satisfy the algebra

$$[\hat{\jmath}_\alpha, \hat{\jmath}_\beta] = 2\epsilon_{\alpha\beta\gamma}\hat{\jmath}_\gamma. \tag{E.17}$$

By definition $\hat{\jmath}_\alpha(kg) = \hat{\jmath}_\alpha(k)$ for all $g \in G$. Thus $\hat{\jmath}_\alpha$ descend to a triplet of $\mathfrak{g}'$-valued functions on $S = G'/G$

$$\hat{\jmath}_\alpha \colon S \to \mathfrak{g}', \tag{E.18}$$

where, by definition, there is a one-to-one correspondence between points in S and points on the orbit in $\mathfrak{g}'$. The exterior derivative of $\hat{\jmath}_\alpha$ on S is

$$\begin{aligned} \mathrm{d}\hat{\jmath}_\alpha &= (\mathrm{d}k)k^{-1}\hat{\jmath}_\alpha + \hat{\jmath}_\alpha k\mathrm{d}k^{-1} \\ &= [\hat{\jmath}_\alpha, \theta] \\ &= [\hat{\jmath}_\alpha, \pi], \end{aligned} \tag{E.19}$$

where we have used $[\hat{\jmath}_\alpha, A] = 0$ as the $\hat{\jmath}_\alpha$ are stabilised by G.

The tri-Sasaki structure is defined by a triplet of one-forms whose derivatives give a triplet of symplectic forms on the base of the SU(2) fibration. Following the discussion in [312], the one-forms are given by

$$\begin{aligned} \hat{\eta}_\alpha &= -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\,\mathrm{tr}(\mathrm{d}\hat{\jmath}_\beta \cdot \hat{\jmath}_\gamma) \\ &= \mathrm{tr}(\pi \cdot \hat{\jmath}_\alpha), \end{aligned} \tag{E.20}$$

which are clearly the right-invariant forms projected onto the $\mathfrak{su}_2$ subalgebra.

Now consider the metric cone over the tri-Sasaki space $W = G' \times \mathbb{R}^+/G$, with cone coordinate r. The one-forms on the cone are inherited from those on the base as [312]

$$\eta_\alpha = r^2\hat{\eta}_\alpha. \tag{E.21}$$

From the definition of $\hat{\eta}_\alpha$ in terms of the $\hat{\jmath}_\alpha$, this can be viewed as taking the triplet of functions $j_\alpha \colon W \to \mathfrak{g}'$ on the cone to be

$$j_\alpha = r\hat{\jmath}_\alpha. \tag{E.22}$$

An exterior derivative gives the symplectic forms

$$\begin{aligned}\omega_\alpha &= \tfrac{1}{2}\mathrm{d}\eta_\alpha \\ &= \tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\,\mathrm{tr}(\mathrm{d}j_\beta \wedge \mathrm{d}j_\gamma).\end{aligned} \tag{E.23}$$

Note that the symplectic forms are manifestly closed. Given two vector fields $v, w \in \Gamma(W)$, if we define the triplet of functions $v_\alpha = \imath_v \mathrm{d}j_\alpha$, then

$$\omega_\alpha(v, w) = \epsilon_{\alpha\beta\gamma}\,\mathrm{tr}(v_\beta w_\gamma). \tag{E.24}$$

Any change in the functions j_α defining a point in W can be generated by the adjoint action of $a_v \in \mathfrak{g}'$, so we can also view vector fields as $v_\alpha = [a_v, j_\alpha]$. We then have

$$\begin{aligned}\omega_\alpha(v, w) &= \epsilon_{\alpha\beta\gamma}\,\mathrm{tr}\big([a_v, j_\beta][a_w, j_\gamma]\big) \\ &= 2\,\mathrm{tr}\big([a_v, a_w]j_\alpha\big).\end{aligned} \tag{E.25}$$

This is the analogue of the Kirillov–Kostant–Souriau symplectic structure on coadjoint orbits, as discussed in [209].

Appendix F

Two results on normalisations and the supersymmetry conditions

We first show that the $D = 5$ normalisation condition $\kappa^2 = c(K)$ is implied by the supersymmetry conditions for ESE spaces. Consider the set of generalised vectors of the form $V = fK$ where f is an arbitrary function. Using the standard form of the generalised Lie derivative given in [137], we have

$$L_{fK} J_\alpha = f L_K J_\alpha - \big[(\mathrm{d}f \times_{\mathrm{ad}} K), J_\alpha\big], \tag{F.1}$$

where $\times_{\mathrm{ad}}$ is the projection to the adjoint bundle $\times_{\mathrm{ad}} \colon E^* \otimes E \to \mathrm{ad}\,\tilde{F}$. Since $J_\alpha \cdot K = 0$, we have $\mathrm{tr}\big((\mathrm{d}f \times_{\mathrm{ad}} K) J_\alpha\big) = 0$ and hence

$$\begin{aligned} \epsilon_{\alpha\beta\gamma} \,\mathrm{tr}\big(J_\beta [\mathrm{d}f \times_{\mathrm{ad}} K, J_\gamma]\big) &= -\epsilon_{\alpha\beta\gamma} \,\mathrm{tr}\big((\mathrm{d}f \times_{\mathrm{ad}} K)[J_\beta, J_\gamma]\big) \\ &= -2\kappa \,\mathrm{tr}\big((\mathrm{d}f \times_{\mathrm{ad}} K) J_\alpha\big) \\ &= 0. \end{aligned} \tag{F.2}$$

Thus

$$\mu_\alpha(fK) = -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma} \int_M f \,\mathrm{tr}(J_\beta L_K J_\gamma) = \lambda_\alpha \int_M f\kappa^2, \tag{F.3}$$

where we have used the supersymmetry condition $L_K J_\alpha = \epsilon_{\alpha\beta\gamma}\lambda_\beta J_\gamma$. But we also have

$$\gamma(fK) = \int_M c(fK, K, K) = \int_M f c(K). \tag{F.4}$$

Hence the moment map conditions (4.13) imply that

$$\int_M f\kappa^2 = \int_M f c(K), \qquad \text{for all } f \tag{F.5}$$

which implies the normalisation condition $\kappa^2 = c(K)$. The analogous calculation in $D = 4$ shows that the normalisation condition $\kappa^2 = 2\sqrt{q(K)}$ is similarly a consequence of the integrability conditions.

Focussing again on $D = 5$, for definiteness we set $\lambda_{1,2} = 0$. We now show that for the action of GDiff_K, that is those generalised diffeomorphisms that preserve K, the moment map conditions $\mu_+(V) = 0$ are implied by the fixed-point conditions $L_K J_\alpha = \epsilon_{\alpha\beta\gamma}\lambda_\beta J_\gamma$, which read

$$L_K J_\pm = \pm\mathrm{i}\lambda_3 J_\pm, \qquad L_K J_3 = 0. \tag{F.6}$$

Acting on the first condition with L_V we have

$$\mathrm{i}\lambda_3\, L_V J_+ = L_V(L_K J_+) = L_{L_V K} J_+ + L_K(L_V J_+) = L_K(L_V J_+), \tag{F.7}$$

since we have $L_V K = 0$ for elements of the Lie algebra $\mathfrak{gdiff}_K$. Substituting into the μ_+ moment maps we have

$$\begin{aligned} \mu_+(V) &\coloneqq -\mathrm{i}\int_M \operatorname{tr}(J_3 L_V J_+) \\ &= -\lambda_3^{-1}\int_M \operatorname{tr}(J_3 L_K L_V J_+) = \lambda_3^{-1}\int_M \operatorname{tr}\big((L_K J_3)(L_V J_+)\big) = 0, \end{aligned} \tag{F.8}$$

where we have used the second condition in (F.6).

Appendix G

Flux quantisation, central charges and free energy

We briefly review the derivation of the central charge from [249] and [261]. The central charge a is given in terms of the effective five-dimensional Newton's constant as [260]

$$a = \frac{\pi}{8m^3 G_5}, \tag{G.1}$$

where G_5 in type IIB is given by

$$G_{5,\mathrm{IIB}}^{-1} = \frac{32\pi^2}{(2\pi\ell_s)^8 g_s^2} \int_M \mathrm{e}^{3\Delta'} \mathrm{vol}_5 = \frac{32\pi^2}{(2\pi\ell_s)^8 g_s^2} \int_M c(K), \tag{G.2}$$

while for M-theory it is given by

$$G_{5,\mathrm{M}}^{-1} = \frac{32\pi^2}{(2\pi\ell_{11})^9} \int_M \mathrm{e}^{3\Delta} \mathrm{vol}_6 = \frac{32\pi^2}{(2\pi\ell_{11})^9} \int_M c(K). \tag{G.3}$$

The corresponding flux quantisation conditions are

$$\begin{aligned} N &= \frac{1}{(2\pi\ell_s)^4 g_s} \int_M \mathrm{d}C \in \mathbb{Z} \qquad \text{type IIB}, \\ N_\Sigma &= \frac{1}{(2\pi\ell_{11})^3} \int_\Sigma \mathrm{d}A \in \mathbb{Z} \qquad \text{M-theory}, \end{aligned} \tag{G.4}$$

where Σ is any four-cycle in M. From the five-dimensional part of Einstein's equations we note that $\mathrm{d}C$ and $\mathrm{d}A$ must both scale as the inverse AdS radius m. Defining the dimensionless volumes

$$V_5 = m^5 \int_M \mathrm{e}^{3\Delta'} \mathrm{vol}_5, \qquad V_6 = m^6 \int_M \mathrm{e}^{3\Delta} \mathrm{vol}_6, \tag{G.5}$$

we expect the scaling dependence

$$N \sim \frac{1}{m^4 \ell_s^4 g_s} V_5, \qquad N_\Sigma \sim \frac{1}{m^3 \ell_{11}^3} V_6^{2/3}. \tag{G.6}$$

More generally, as in [249] and [261], one can solve explicitly for $\mathrm{d}C$ and $\mathrm{d}A$ in terms of the structure and find exact expressions for the flux quantisation. We also have

$$a_{\mathrm{IIB}} \sim \frac{1}{m^8 \ell_s^8 g_s^2} V_5, \qquad a_{\mathrm{IIB}} \sim \frac{1}{m^9 \ell_{11}^9} V_6. \tag{G.7}$$

Solving for m then gives

$$a_{\mathrm{IIB}} \sim \frac{N^2}{V_5}, \qquad a_{\mathrm{M}} \sim \frac{N_\Sigma^3}{V_6}, \tag{G.8}$$

and hence a^{-1} scales as $\int_M c(K)$ in both cases.

For M-theory AdS$_4$ backgrounds, we follow [245]. The free energy of the field theory is given by [313]

$$\mathcal{F} = \frac{\pi}{2m^2 G_4}, \tag{G.9}$$

where the effective four-dimensional Newton's constant is

$$G_{4,\mathrm{M}}^{-1} = \frac{32\pi^2}{(2\pi\ell_{11})^9} \int_M \mathrm{e}^{2\Delta} \operatorname{vol}_7 = \frac{32\pi^2}{(2\pi\ell_{11})^9} \int_M 2\sqrt{q(K)}. \tag{G.10}$$

The flux quantisation condition gives

$$N = \frac{1}{(2\pi\ell_{11})^6} \int_M \mathrm{d}\tilde{A} \in \mathbb{Z}. \tag{G.11}$$

Via the same scaling arguments as above, defining the dimensionless volume

$$V_7 = m^7 \int_M \mathrm{e}^{2\Delta} \operatorname{vol}_7, \tag{G.12}$$

we find (the exact relations are given in [245])

$$N \sim \frac{1}{m^6 \ell_{11}^6} V_7, \qquad \mathcal{F} \sim \frac{1}{m^9 \ell_{11}^9} V_7, \tag{G.13}$$

so that solving for m gives, as in [255],

$$\mathcal{F} \sim \frac{N^{3/2}}{V_7^{1/2}}, \tag{G.14}$$

and hence $\mathcal{F}^{-2}$ scales as $\int_M \sqrt{q(K)}$.

Appendix H

Supersymmetry conditions and deformations

In this appendix we give a detailed discussion of the deformations of the Sasaki–Einstein structure and of the derivation of the constraints from supersymmetry. We start with a brief description of the generalised structures and then move to their deformations and the conditions that supersymmetry imposes on them.

H.1 Embedding of the linearised deformations in generalised geometry

Here we will justify the choice of (5.56) for the linearised deformation. As already mentioned, K is left invariant by an $\mathrm{F}_{4(4)}$ subgroup of $\mathrm{E}_{6(6)}$ while the triplet J_α is left invariant by $\mathrm{SU}^*(6)$. Together J_α and K are invariant under a common USp(6) subgroup. We argued in section 5.3.1 that the dual of marginal deformations should leave K invariant, but modify the J_α. This means that at a point on the internal manifold they must be elements of the coset $\mathrm{F}_{4(4)} \times \mathbb{R}^+/\mathrm{USp}(6)$. The **52** (adjoint) representation of $\mathrm{F}_{4(4)}$ decomposes under $\mathrm{USp}(6) \times \mathrm{SU}(2)$ as

$$\mathbf{52} = (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{21}, \mathbf{1}) \oplus (\mathbf{14}, \mathbf{2}). \tag{H.1}$$

The first term corresponds to the triplet J_α and its action simply rotates the J_α among themselves. The second term is the adjoint of USp(6), which leaves both K and J_α invariant. Therefore, the deformations are in the $(\mathbf{14}, \mathbf{2})$ and form a doublet under the SU(2) defined by J_α. We can choose them to be eigenstates of J_3

$$[J_3, \mathcal{A}_{\pm\lambda}] = \pm \mathrm{i} \lambda \kappa \mathcal{A}_{\pm\lambda}. \tag{H.2}$$

The non-trivial eigenstates correspond to $\lambda = 0, 1, 2$. From the SU(2) algebra (5.46) we see that the eigenstates with $\lambda = 2$ are $J_\pm$ themselves. The eigenstates with eigenvalue zero are in USp(6), or in other words they leave J_α and K invariant, and we will therefore not consider them. To simplify notation we will call the $\lambda = \pm 1$ eigenstates $\mathcal{A}_\pm$. We note that we can generate an eigenstate with eigenvalue $-\mathrm{i}\kappa$ from $\mathcal{A}_+$ by acting with J_+, as the Jacobi identity implies

$$[J_3, \kappa^{-1}[J_\pm, \mathcal{A}_\pm]] = \mp \mathrm{i}\kappa [J_\pm, \mathcal{A}_\pm]. \tag{H.3}$$

We also note that complex conjugation also gives the eigenstate with opposite eigenvalue. Since $\hat{L}_K$ commutes with the action of J_3 we can also label states by their R-charge as in (5.54), so that we have doublets

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_-^{(r)} \\ \mathcal{A}_+^{(r-2)} \end{pmatrix}, \qquad r \geq 0. \tag{H.4}$$

We have chosen $r \geq 0$ for definiteness. Those doublets with $r \leq 0$ will be related by complex conjugation. (Note this convention leads to a slight over-counting for $0 \leq r \leq 2$, since the doublets with charge r have complex conjugates with charge $-r+2$. However, it is the most convenient form to adopt for out purposes.)

To compute the eigenstates with $\lambda = 1$ it helps to note that the $\mathrm{E}_{6(6)}$ action of J_3 acts separately on $\{B^i, \beta^i\}$, $a^i{}_j$ and $\{r, C, \gamma, l\}$ (see (B.40)). Using this we can organise the eigenstates as

$$\check{\mathcal{A}}_+ = B^i + \beta^i, \qquad \check{\mathcal{A}}_- = [J_+, \check{\mathcal{A}}_+] = r + C + \gamma + l, \tag{H.5}$$

$$\hat{\mathcal{A}}_+ = a^i{}_j, \qquad \hat{\mathcal{A}}_- = [J_+, \hat{\mathcal{A}}_+] = B'^i + \beta'^i. \tag{H.6}$$

As complex conjugation gives the eigenstate with opposite eigenvalue, using this basis, the modes $\{\check{\mathcal{A}}_+, \check{\mathcal{A}}_-^*, \hat{\mathcal{A}}_+, \hat{\mathcal{A}}_-^*\}$ fill out the possible $+\mathrm{i}\kappa$ eigenstates. In fact we will find that, with this basis, imposing $r \geq 0$ actual restricts to only $\check{\mathcal{A}}_+$ and $\hat{\mathcal{A}}_+$.

One can use the forms defining the SU(2) structure on a SE manifold – Ω, ω and σ – and the corresponding vectors to decompose the eigenstates. It is straightforward to verify that the eigenstate $\check{\mathcal{A}}_+$ is given by

$$\check{\mathcal{A}}_+ = -\tfrac{1}{2}\mathrm{i}\bar{u}^i\big[f\bar{\Omega} + 2(p\omega + \hat{\omega} + \sigma \wedge \bar{\nu})\big] - \tfrac{1}{2}\bar{u}^i\big[f\bar{\Omega}^\sharp - 2(p\omega^\sharp - \hat{\omega}^\sharp - \xi \wedge \bar{\upsilon})\big], \tag{H.7}$$

where $u^i = (-\mathrm{i}, 1)$, $\bar{\nu}$ is a (0,1)-form, $\bar{\upsilon}$ is a (1,0)-vector on the base, $\hat{\omega}$ is a primitive (1,1)-form on the base, and p and f are arbitrary complex functions on the SE manifold. The $\omega^\sharp$ and $\hat{\omega}^\sharp$ terms in the bi-vector are obtained from the two-forms by raising indices with the metric g^{mn}.

The requirement that the deformation leaves K invariant ($\check{\mathcal{A}}_+ \cdot K = 0$) translates to constraints on the components of $\check{\mathcal{A}}_+$, namely

$$\sigma \wedge \omega \wedge B^i = 0, \qquad \imath_\xi B^i = \beta^i \lrcorner (\sigma \wedge \omega), \tag{H.8}$$

which impose $p=0$ and $\bar{v}=\bar{\nu}^\sharp$. Thus the $\check{\mathcal{A}}_+$ deformation that leaves K invariant is

$$\check{\mathcal{A}}_+ = -\tfrac{1}{2}\mathrm{i}\bar{u}^i\big[f\bar{\Omega}+2(\hat{\omega}+\sigma\wedge\bar{\nu})\big]-\tfrac{1}{2}\bar{u}^i\big[f\bar{\Omega}^\sharp+2(\hat{\omega}+\xi\wedge\bar{\nu})\big], \tag{H.9}$$

where we have omitted the vector symbols $\sharp$ and it is understood that all terms in the bi-vector part are obtained by raising the GL(5) indices of the corresponding forms with the metric g^{mn}. Note that the two-form and bi-vector components are related by

$$B^i = -\epsilon^i{}_j(g\beta^j g), \tag{H.10}$$

where we lower the indices of the bi-vector with the undeformed metric g. The $\check{\mathcal{A}}_-$ mode in the same multiplet as $\check{\mathcal{A}}_+$ is given by $\check{\mathcal{A}}_- = \kappa^{-1}[J_+,\check{\mathcal{A}}_+]$ and has the following form

$$\begin{aligned}\check{\mathcal{A}}_- = \Big(&2\mathrm{i}f'\mathbb{1}_4 - \mathrm{i}f'\mathbb{1} + \mathrm{i}\big(j\Omega^\sharp\lrcorner j(\hat{\omega}'+\sigma\wedge\bar{\nu}') + j(\hat{\omega}'+\xi\wedge\bar{\nu}')\lrcorner j\Omega\big)\Big)\\ &+(\tfrac{1}{2}f'\Omega\wedge\bar{\Omega}+\Omega\wedge\sigma\wedge\bar{\nu}')+(\tfrac{1}{2}f'\Omega^\sharp\wedge\bar{\Omega}^\sharp+\Omega^\sharp\wedge\xi\wedge\bar{\nu}')+\mathrm{i}f',\end{aligned} \tag{H.11}$$

where we should regard f' as distinct from f.

Similarly, we can construct the $\hat{\mathcal{A}}_+$ deformation that leaves K invariant. It has only $a^i{}_j$ components, given by

$$\hat{\mathcal{A}}_+ = -\tfrac{1}{2}\tilde{f}\bar{u}^i\bar{u}_j. \tag{H.12}$$

The $\hat{\mathcal{A}}_-$ mode in the same multiplet as $\hat{\mathcal{A}}_+$ is given by $\hat{\mathcal{A}}_- = \kappa^{-1}[J_+,\hat{\mathcal{A}}_+]$ and has the following form

$$\hat{\mathcal{A}}_- = (-\tfrac{1}{2}\mathrm{i}\bar{u}^i\tilde{f}'\Omega)+(-\tfrac{1}{2}\bar{u}^i\tilde{f}'\Omega^\sharp), \tag{H.13}$$

where again we should regard $\tilde{f}'$ as distinct from $\tilde{f}$. We see this is of the form $B^i+\beta^i$ as expected from (H.6).

H.2 Supersymmetry conditions

We are interested in deformations of the Sasaki–Einstein background that preserve supersymmetry. This is equivalent to requiring that the deformed structures are integrable, that is the new J_α and K must satisfy (5.20) and (5.21). At linear order in the deformation these conditions reduce to

$$\delta\mu_\alpha(V) = \int \kappa\,\mathrm{tr}(J_\alpha, L_V\mathcal{A}) = 0 \qquad \forall\, V\in\mathbf{27'}, \tag{H.14}$$

$$[L_K\mathcal{A}, J_\alpha] = 0. \tag{H.15}$$

As we want the deformed structures to be real, we take the deformation to be $\mathcal{A}=\operatorname{Re}\mathcal{A}_+$, where $\operatorname{Re}\mathcal{A}_+ = \frac{1}{2}(\mathcal{A}_+ + \mathcal{A}_+^*)$. In this section we give the derivation of the constraints that these equations impose on the deformations $\check{\mathcal{A}}_+$. For the other deformations we give only the final results for the constraints, which can be derived in a similar fashion.

Moment map conditions

Let us first consider the deformation $\check{\mathcal{A}}_+$ and the conditions from $\delta\mu_3 = 0$. Given the form of J_3 (5.49), only the $a^i{}_j$, $r^m{}_n$, C_{mnpq} and γ^{mnpq} components of the generalised Lie derivative contribute. The relevant terms are

$$\begin{aligned} L_V \check{\mathcal{A}}_+ &= (\epsilon_{ij} j \beta^i \lrcorner j \mathrm{d}\lambda^j - \tfrac{1}{4}\mathbb{1}\epsilon_{kl}\beta^k \lrcorner \mathrm{d}\lambda^l) \\ &\quad + (\epsilon_{jk}\beta^i \lrcorner \mathrm{d}\lambda^k - \tfrac{1}{2}\delta^i{}_j \epsilon_{kl}\beta^k \lrcorner \mathrm{d}\lambda^l) + (\epsilon_{ij}\mathrm{d}\lambda^i \wedge B^i) \\ &= -[\mathrm{d}\lambda^i, \check{\mathcal{A}}_+], \end{aligned} \tag{H.16}$$

where B^i and β^i are the two-form and bi-vector components of $\check{\mathcal{A}}_+$. We use this and rearrange the trace to give

$$\int \kappa \operatorname{tr}(J_3, L_V \check{\mathcal{A}}_+) \propto \int \kappa \operatorname{tr}\big(J_3, [\mathrm{d}\lambda^i, \check{\mathcal{A}}_+]\big) \propto \int \kappa \operatorname{tr}\big(\mathrm{d}\lambda^i, [J_3, \check{\mathcal{A}}_+]\big), \tag{H.17}$$

with a similar expression for $\check{\mathcal{A}}^*_+$. Using that $\check{\mathcal{A}}_+$ is an eigenstate of J_3 with eigenvalue $+\mathrm{i}\kappa$ and the form of the trace (B.60), this simplifies to

$$\begin{aligned} \int \kappa \operatorname{tr}\big(\mathrm{d}\lambda^i, [J_3, \check{\mathcal{A}}_+]\big) &\propto \int \kappa^2 \epsilon_{ij}\beta^i \lrcorner \mathrm{d}\lambda^j \\ &\propto \int \epsilon_{ij}\mathrm{d}(\beta^i \lrcorner \operatorname{vol}_5) \wedge \lambda^j, \end{aligned} \tag{H.18}$$

where we have used $\operatorname{vol}_5(\beta^i \lrcorner \mathrm{d}\lambda^j) \propto (\beta^i \lrcorner \operatorname{vol}_5) \wedge \mathrm{d}\lambda^j$. When combined with the contribution from $L_V \check{\mathcal{A}}^*_+$, this should hold for arbitrary λ^j and so we require

$$\mathrm{d}\Big[\big(\beta^i - (\beta^i)^*\big) \lrcorner \operatorname{vol}_5\Big] = 0. \tag{H.19}$$

Using the explicit form of $\check{\mathcal{A}}_+$ (H.9), this condition gives

$$\begin{aligned} \partial(\bar{\nu} \lrcorner \Omega) &= 0, \\ \partial\hat{\omega} &= 0, \\ \partial f \wedge \bar{\Omega} + \tfrac{3}{2}\mathrm{i}\bar{\Omega} \wedge (\bar{\nu} \lrcorner \Omega) &= 2\bar{\partial}\hat{\omega} + \tfrac{1}{2}\bar{\Omega} \wedge \mathcal{L}_\xi(\bar{\nu} \lrcorner \Omega). \end{aligned} \tag{H.20}$$

The analysis of $\delta\mu_+$ follows from similar manipulations. For $\delta\mu_+$ there are terms that must vanish for arbitrary v and ρ. The terms in ρ give

$$\begin{aligned} 2\partial f &= \mathcal{L}_\xi(\bar{\nu} \lrcorner \Omega), \\ \bar{\partial} f &= 0, \\ \partial(\bar{\nu} \lrcorner \Omega) &= 0, \\ \bar{\partial}(\bar{\nu} \lrcorner \Omega) &= -4f\omega. \end{aligned} \tag{H.21}$$

The terms in v give

$$\begin{aligned}\bar\partial\bar\nu &= -2\mathrm{i}f\bar\Omega,\\ \bar\partial f &= 0,\\ 4\omega\wedge\bar\nu + 4\bar\partial\hat\omega + \tfrac{1}{2}\bar\Omega\wedge\mathcal{L}_\xi(\bar\nu\lrcorner\Omega) &= 2\mathrm{i}\bar\Omega\wedge(\bar\nu\lrcorner\Omega) + \bar\Omega\wedge\partial f.\end{aligned} \tag{H.22}$$

Taken together, the moment map conditions on the deformation $\check{\mathcal{A}}_+$ are

$$\partial(\bar\nu\lrcorner\Omega) = 0, \tag{H.23}$$

$$\partial\hat\omega = 0, \tag{H.24}$$

$$2\partial f = \mathcal{L}_\xi(\bar\nu\lrcorner\Omega), \tag{H.25}$$

$$\bar\partial f = 0, \tag{H.26}$$

$$\bar\partial(\bar\nu\lrcorner\Omega) = -4f\omega, \tag{H.27}$$

$$\bar\partial\bar\nu = -2\mathrm{i}f\bar\Omega, \tag{H.28}$$

$$\bar\partial\hat\omega = -3\omega\wedge\bar\nu. \tag{H.29}$$

Note that we have simplified some expressions using

$$4\omega\wedge\bar v = -\mathrm{i}\bar\Omega\wedge(\bar v\lrcorner\Omega), \qquad \omega\wedge(\bar v\lrcorner\Omega) = -\mathrm{i}\Omega\wedge\bar v, \tag{H.30}$$

where $\bar v$ is an arbitrary (0,1)-form with respect to I.

We want to solve the system (H.23)–(H.29) of differential equations to derive the form of the deformation. From (H.23) we know $\bar\nu\lrcorner\Omega$ is closed under ∂, and so it may be written as the sum of a ∂-closed term and a ∂-exact term. However, we also have $\mathrm{H}^{1,0}_\partial(M) = 0$ for a five-dimensional Sasaki–Einstein space M, and so only a ∂-exact term is needed. We make an ansatz

$$\bar\nu\lrcorner\Omega = -\frac{2\mathrm{i}}{q}\partial f, \tag{H.31}$$

where f has a well-defined scaling under ξ, $\mathcal{L}_\xi f = \mathrm{i}qf$, and q is non-zero.[1] Next (H.27) gives

$$\bar\partial(\bar\nu\lrcorner\Omega) = -\frac{2\mathrm{i}}{q}\bar\partial\partial f = 2\mathrm{i}\partial\bar\partial f - 4f\omega \equiv -4f\omega. \tag{H.32}$$

We can solve this by taking f to be holomorphic, which also solves (H.26). The ansatz for $\bar\nu\lrcorner\Omega$, together with the scaling under ξ and holomorphicity of f are enough to satisfy (H.25).

[1] If $q = 0$ and f is holomorphic, f is necessarily constant. But from (H.28), a constant f requires $\bar\Omega$ to be $\bar\partial$-exact, which is not true. The only solution to the differential conditions for constant$f$ is $f = 0$, and so we do not need to consider the case of $q = 0$

We can invert (H.31) and write $\bar{\nu}$ as

$$\bar{\nu} = \frac{\mathrm{i}}{2q} \partial f \lrcorner \bar{\Omega}. \tag{H.33}$$

Then (H.28) is automatically satisfied

$$\bar{\partial}\bar{\nu} = \frac{\mathrm{i}}{2q} \bar{\partial}(\partial f \lrcorner \bar{\Omega}) = \frac{\mathrm{i}}{2q}(-4qf\bar{\Omega}) \equiv -2\mathrm{i}f\bar{\Omega}, \tag{H.34}$$

where we have used $\bar{\partial}(\partial f \lrcorner \bar{\Omega}) = -4qf\bar{\Omega}$ for a holomorphic function f.[2]

If we take $\hat{\omega} = \frac{1}{4q(q-1)} \partial(\partial f \lrcorner \bar{\Omega}) + \delta$, (H.29) becomes

$$\begin{aligned}
\bar{\partial}\hat{\omega} &= \bar{\partial}\left(\tfrac{1}{4q(q-1)} \partial(\partial f \lrcorner \bar{\Omega}) + \delta\right) \\
&= \tfrac{1}{4q(q-1)} \left(-\partial\bar{\partial}(\partial f \lrcorner \bar{\Omega}) - 2\omega \wedge \mathcal{L}_\xi(\partial f \lrcorner \bar{\Omega})\right) + \bar{\partial}\delta \\
&= \tfrac{1}{q-1} \partial f \wedge \bar{\Omega} - \mathrm{i}\tfrac{q-3}{2q(q-1)} \omega \wedge (\partial f \lrcorner \bar{\Omega}) + \bar{\partial}\delta \\
&= -\tfrac{3\mathrm{i}}{2q} \omega \wedge (\partial f \lrcorner \bar{\Omega}) + \bar{\partial}\delta \\
&\equiv -3\omega \wedge \bar{\nu},
\end{aligned} \tag{H.35}$$

implying $\bar{\nu} = \frac{\mathrm{i}}{2q} \partial f \lrcorner \bar{\Omega}$, in agreement with above, and $\bar{\partial}\delta = 0$. Finally, (H.24) implies $\partial\delta = 0$.

Taken together, these determine the $\check{\mathcal{A}}_+$ solutions of the moment map equations. For example, the two-form component of $\check{\mathcal{A}}_+$ is

$$B^i = -\tfrac{1}{2}\mathrm{i}\bar{u}^i \left[f\bar{\Omega} + \tfrac{1}{2q(q-1)} \partial(\partial f \lrcorner \bar{\Omega}) + \tfrac{\mathrm{i}}{q} \sigma \wedge (\partial f \lrcorner \bar{\Omega}) \right] - \mathrm{i}\bar{u}^i \delta, \tag{H.36}$$

where f is holomorphic with respect to ∂ (and hence has charge $q \geq 0$ under the Reeb vector) and δ is ∂- and $\bar{\partial}$-closed (and hence has charge zero). The bi-vector component is determined from this using (H.10). Notice that f-dependent terms and δ are independent of each other, so we really have two eigenmodes within this expression. In fact, this solution to the moment map equations corresponds to the $\mathcal{A}_+^{(r-2)}$ modes with $r \geq 0$ labelled by f and δ in (5.64).

Consider now the deformations $\hat{\mathcal{A}}_+$ in (H.12). A similar analysis of the moment maps gives

$$\bar{\partial}\tilde{f} = 0, \tag{H.37}$$

so $\tilde{f}$ is holomorphic (and hence has charge $q \geq 0$ under the Reeb vector). This solution corresponds to the $\mathcal{A}_+^{(r-2)}$ modes with $r \geq 2$ labelled by $\tilde{f}$ in (5.64).

So far we have examined $\check{\mathcal{A}}_+$ and $\hat{\mathcal{A}}_+$, which correspond to the $\mathcal{A}_+^{(r-2)}$ modes in (5.64) and are parametrised by the holomorphic functions f and $\tilde{f}$, and a ∂- and $\bar{\partial}$-closed (1,1)-

[2] In general one has $\partial(\bar{\partial} f \lrcorner \Omega) = \frac{1}{2}(q^2 + 4q - \Delta_0) f\Omega$ and $\bar{\partial}(\partial f \lrcorner \bar{\Omega}) = \frac{1}{2}(q^2 - 4q - \Delta_0) f\bar{\Omega}$ for a function satisfying $\Delta f = \Delta_0 f$ and $\mathcal{L}_\xi f = \mathrm{i}qf$ [273].

form δ. Now we comment on the $\mathcal{A}_-^{(r)}$ modes, defined by $\mathcal{A}_-^{(r)} = \kappa^{-1}[J_+, \mathcal{A}_+^{(r-2)}]$. Naively, one might think we should solve the moment maps from scratch for an $\mathcal{A}_-^{(r)}$ deformation. For example, the deformation would be calculated using the generic form of $\check{\mathcal{A}}_-$, given by (H.11), and would then lead to differential conditions on the components of $\check{\mathcal{A}}_+$ from which $\check{\mathcal{A}}_-$ is generated. Fortunately, given a solution $\mathcal{A}_+$ to the deformed moment maps (H.14), one can show that $\mathcal{A}_- = \kappa^{-1}[J_+, \mathcal{A}_+]$ is automatically a solution too. The components of $\mathcal{A}_-$ are determined by $\mathcal{A}_+$ and the differential conditions on the components of $\mathcal{A}_-$ reduce to the differential conditions on $\mathcal{A}_+$ that we have already given. For example, we have seen that $\check{\mathcal{A}}_+$ is completely determined by a holomorphic function f and a ∂- and $\bar{\partial}$-closed (1,1)-form δ. As $\check{\mathcal{A}}_- = \kappa^{-1}[J_+, \check{\mathcal{A}}_+]$ is automatically a solution, it too is determined by a holomorphic function f' and a ∂- and $\bar{\partial}$-closed (1,1)-form δ'. Similarly $\hat{\mathcal{A}}_+$ will be determined by holomorphic function $\tilde{f}'$. Here, we should note, however, because of our slight over-counting, the $r = 2$ case with constant f' is actually the complex conjugate of the $r = 0$ case of $\check{\mathcal{A}}_+$.

Lie derivative along K

At first order in a generic deformation $\mathcal{A} \in \mathbf{78}$ of $E_{6(6)}$, the generalised Lie derivative condition is given by (5.69). It is straightforward to check that the commutators are non-zero for both J_+ and J_3, and so the condition reduces to $L_K\mathcal{A} = 0$. From (5.51), we know that the generalised Lie derivative along K reduces to the conventional Lie derivative along ξ, and so the deformation condition is simply

$$\mathcal{L}_\xi \mathcal{A} = 0. \tag{H.38}$$

We see that the deformation must have scaling dimension zero under the Reeb vector field. Using the explicit form of $\check{\mathcal{A}}_+$ and $\hat{\mathcal{A}}_+$, we find f is charge $+3$ and $\tilde{f}$ is charge zero (which together with $\bar{\partial}\tilde{f} = 0$ implies $\tilde{f}$ is constant). We also have δ is charge zero, which is consistent with it being ∂- and $\bar{\partial}$-closed. This agrees with (5.71). These are precisely the conditions for the deformations to be marginal.

H.2.1 Generalised metric

We have deformed the geometry by two-forms and bi-vectors, but the bosonic fields of type II supergravity do not include bi-vectors. As is typical in generalised complex geometry, acting on the bosonic fields, the bi-vector deformation can be traded for deformations by a gauge potential. We first construct the generalised metric and then give the dictionary for translating a bi-vector deformation into a two-form deformation.

A generalised metric defines a USp(8) structure. K and J_α together define a USp(6) structure and so also define a generalised metric, though reconstructing the metric from them may be complicated.[3] For this reason it proves simpler to construct the generalised

[3] For example, the conventional metric can be recovered from the three- and four-forms defining a G_2

metric from scratch. For a generalised vector V the generalised metric, in the untwisted basis, is

$$G(V,V) = v^m v_m + h_{ij}\lambda^i{}_m\lambda^{jm} + \tfrac{1}{3!}\rho_{m_1m_2m_3}\rho^{m_1m_2m_3} + \tfrac{1}{5!}h_{ij}\sigma^i{}_{m_1\ldots m_5}\sigma^{jm_1\ldots m_5}, \quad \text{(H.39)}$$

where h_{ij} is the standard metric on SL(2)/SO(2) and we have raised/lowered indices using the metric g_{mn}.[4]

The generalised metric defines a USp(8) structure and so should be left invariant by a USp(8) subgroup of $E_{6(6)} \times \mathbb{R}^+$. Using the adjoint action on $V \in \mathbf{27'}$, one can show that USp(8) is generated by elements of the $E_{6(6)} \times \mathbb{R}^+$ adjoint satisfying

$$\begin{aligned} l &= 0, & a_{ij} &= -a_{ji}, \\ r_{mn} &= -r_{nm}, & C_{mnpq} &= -\gamma_{mnpq}, \\ B^1{}_{mn} &= \beta^2{}_{mn}, & B^2{}_{mn} &= -\beta^1{}_{mn}. \end{aligned} \quad \text{(H.40)}$$

One can read off the new bosonic background by constructing the deformed generalised metric. The metric, axion-dilaton and four-form R-R potential receive corrections starting at second order. At first order, only the two-form potentials, B_2 and C_2, are corrected. If we consider a deformation by a two-form B^i and a bi-vector β^i, at first order the resulting two-form deformation is

$$B_2 = B^1 - g\beta^2 g, \qquad C_2 = B^2 + g\beta^1 g. \quad \text{(H.41)}$$

We see that the bi-vector can be traded for a two-form contribution. This will become more complicated at higher orders in the deformation due to terms from contractions of the bi-vector with the two-form.

As previously mentioned, this procedure is analogous to what is done when trading β-deformations in generalised complex geometry for metric and B-field deformations (see for example equations (3.3) and (3.4) in [314]).

Flux induced by deformation

Using (H.41) we have that our two-form deformation $\mathrm{Re}\,\check{\mathcal{A}}_+ = B^i + \beta^i$ will induce NS-NS and R-R two-form potentials given by

$$C_2 = 2B^2, \qquad B_2 = 2B^1. \quad \text{(H.42)}$$

The complexified potential is

$$C_2 - \mathrm{i}B_2 = -2\mathrm{i}(B^1 + \mathrm{i}B^2). \quad \text{(H.43)}$$

structure, but the relation between the two is not trivial.

[4] We have chosen $C_0 = \phi = 0$ for the backgrounds we consider, so h_{ij} is simply δ_{ij}.

Using the explicit form of $\check{\mathcal{A}}_+$ that solves the deformed moment maps (H.36), this is

$$C_2 - \mathrm{i}B_2 = -\mathrm{i}\left[f\bar{\Omega} + \tfrac{1}{2q(q-1)}\partial(\partial f \lrcorner \bar{\Omega}) + \tfrac{\mathrm{i}}{q}\sigma \wedge (\partial f \lrcorner \bar{\Omega}) + 2\delta\right], \tag{H.44}$$

where $\mathcal{L}_\xi f = \mathrm{i}qf$. From (H.38), this deformation will correspond to a marginal deformation if $q = 3$ and δ is d-closed. The complexified potential then simplifies to

$$C_2 - \mathrm{i}B_2 = -\mathrm{i}\left[f\bar{\Omega} + \tfrac{1}{12}\partial(\partial f \lrcorner \bar{\Omega}) + \tfrac{\mathrm{i}}{3}\sigma \wedge (\partial f \lrcorner \bar{\Omega}) + 2\delta\right]. \tag{H.45}$$

Taking an exterior derivative, the resulting complexified flux $G_3 = \mathrm{d}(C_2 - \mathrm{i}B_2)$ is

$$\begin{aligned} G_3 &= -\mathrm{i}\big(\partial f \wedge \bar{\Omega} + \tfrac{1}{12}\bar{\partial}\partial(\partial f \lrcorner \bar{\Omega}) + \mathrm{i}\tfrac{2}{3}\omega \wedge (\partial f \lrcorner \bar{\Omega}) - \mathrm{i}\tfrac{1}{3}\sigma \wedge (\partial + \bar{\partial})(\partial f \lrcorner \bar{\Omega})\big) \\ &= -\tfrac{4}{3}\mathrm{i}\partial f \wedge \bar{\Omega} + 4f\sigma \wedge \bar{\Omega} - \tfrac{1}{3}\sigma \wedge \partial(\partial f \lrcorner \bar{\Omega}), \end{aligned} \tag{H.46}$$

where we have used $\mathrm{d}\delta = 0$, $\omega \wedge (\partial f \lrcorner \bar{\Omega}) = \mathrm{i}\partial f \wedge \bar{\Omega}$ and $\bar{\partial}(\partial f \lrcorner \bar{\Omega}) = -12f\bar{\Omega}$. We stress once more that this flux is valid for marginal deformations of *any* Sasaki–Einstein structure and reproduces the first-order fluxes of the β-deformation of Lunin and Maldacena [93].

H.2.2 Marginal deformations and the axion-dilaton

Let us now consider the effect of an $\hat{\mathcal{A}}_+$ deformation. Such a deformation is marginal if $\tilde{f}$ is charge zero under ξ, which, when combined with $\bar{\partial}\tilde{f} = 0$, implies $\tilde{f}$ is simply a constant complex number. The physical effect of such a marginal deformation can be found from its action on the $\mathrm{SL}(2;\mathbb{R})$ doublets that appear in the generalised metric. For example, the undeformed generalised metric contains terms of the form

$$G(\lambda, \lambda) = \delta_{ij}\lambda^i \lrcorner \lambda^j + \dots. \tag{H.47}$$

To first order, the deformed generalised metric will then be

$$\begin{aligned} G(\lambda + \delta\lambda, \lambda + \delta\lambda) &= \delta_{ij}(\lambda^i + \delta\lambda^i) \lrcorner (\lambda^j + \delta\lambda^j) + \dots \\ &= (\delta_{ij} + 2m_{ij})\lambda^i \lrcorner \lambda^j + \dots, \end{aligned} \tag{H.48}$$

where

$$m_{ij} = \tfrac{1}{2}\begin{pmatrix} \operatorname{Im}\tilde{f} & -\operatorname{Re}\tilde{f} \\ -\operatorname{Re}\tilde{f} & -\operatorname{Im}\tilde{f} \end{pmatrix}, \tag{H.49}$$

which is simply the real part of (H.12). We now want to compare this to the form of the generalised metric when the axion-dilaton is included. From [139], we see this is

$$G(\lambda, \lambda) = h_{ij}\lambda^i \lrcorner \lambda^j + \dots, \tag{H.50}$$

where

$$h_{ij} = \mathrm{e}^{\phi} \begin{pmatrix} C_0^2 + \mathrm{e}^{-2\phi} & -C_0 \\ -C_0 & 1 \end{pmatrix} . \tag{H.51}$$

Expanding the fields to linear order, we find

$$h_{ij} = \delta_{ij} + \begin{pmatrix} -\phi & -C_0 \\ -C_0 & \phi \end{pmatrix} . \tag{H.52}$$

By comparing this expression with the deformed metric m_{ij}, we see we can encode a first-order change in the axion-dilaton by taking $\tilde{f} = C_0 - \mathrm{i}\phi$.

Appendix I

γ-deformed solutions

Here we summarise the results of the solution-generating technique of Lunin and Maldacena applied to AdS_4 solutions in M-theory [93]. We follow the general prescription laid out in [285]. The undeformed metric and four-form flux are assumed to have the form

$$\mathrm{d}s_{11}^2 = \tfrac{1}{4}\mathrm{d}s^2(\mathrm{AdS}_4) + \mathrm{d}s^2(M), \qquad F = \tfrac{3}{8}\,\mathrm{vol}_{\mathrm{AdS}}\,. \tag{I.1}$$

This is of the same form as the Sasaki-Einstein backgrounds we consider. Note that we have normalised the metric on the internal space M to give $R_{\mu\nu}(M) = 6g_{\mu\nu}(M)$.

First, we split the metric on M into a three-torus and a four-dimensional space M_4

$$\mathrm{d}s^2(M_7) = \mathrm{d}s^2(\mathrm{T}^3) + \mathrm{d}s^2(M_4). \tag{I.2}$$

The metric on the torus is then expressed as

$$\mathrm{d}s^2(\mathrm{T}^3) = \Delta^{1/3} M_{ab}\, D\varphi_a\, D\varphi_b, \tag{I.3}$$

where $D\varphi_a = \mathrm{d}\varphi_a + A_a$ and $\det M_{ab} = 1$. The particular form of the one-forms A_a depends on the undeformed metric. The eleven-dimensional solution obtained from the solution-generating technique is

$$\begin{aligned} \mathrm{d}s_{11}^2 &= G^{-1/3}\big(\tfrac{1}{4}\mathrm{d}s^2(\mathrm{AdS}_4) + \mathrm{d}s^2(M_4) + G\mathrm{d}s^2(\mathrm{T}^3)\big), \\ F &= \tfrac{3}{8}\,\mathrm{vol}_{\mathrm{AdS}} - 6\gamma\Delta^{1/2}\,\mathrm{vol}_4 - \gamma\mathrm{d}(G\Delta\, D\varphi_1 \wedge D\varphi_2 \wedge D\varphi_3), \end{aligned} \tag{I.4}$$

where $G = (1+\gamma^2\Delta)^{-1}$ and vol_4 is the volume form of $\mathrm{d}s^2(M_4)$. From this, we see the first-order contribution to the flux is

$$F = -6\gamma\Delta^{1/2}\,\mathrm{vol}_4 - \gamma\mathrm{d}(\Delta D\varphi_1 \wedge D\varphi_2 \wedge D\varphi_3). \tag{I.5}$$

This is the flux we will match our results to. To find the explicit form of this for a background, we need to specify φ_a, Δ, vol_4 and A_a. We now give these in our conventions.

I.1 S^7

The solution for S^7 is given in [93]. The angles parametrising the three-torus are

$$\varphi_1 = 3\psi - \phi_1 - \phi_2 - \phi_3, \quad \varphi_2 = 2\psi - \phi_1 - \phi_2, \quad \varphi_3 = \phi_1 - \psi. \tag{I.6}$$

Δ, vol_4 and the A_a are[1]

$$\begin{aligned}
\Delta &= s_\theta^4 s_\alpha^2 \big(c_\theta^2 c_\alpha^2 + s_\alpha^2 s_\beta^2 c_\beta^2 (c_\theta^2 + s_\theta^2 c_\alpha^2)\big),\\
\mathrm{vol}_4 &= -\Delta^{-1/2} s_\theta^5 c_\theta s_{2\alpha} s_\alpha^2 s_{2\beta} \mathrm{d}\theta \wedge \mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge \mathrm{d}\psi,\\
A_1 &= \frac{-4(1+2c_{2\beta})c_\theta^2 c_\alpha^2 + s_\alpha^2 s_{2\beta}^2 (c_\theta^2 + s_\theta^2 c_\alpha^2)}{4c_\theta^2 c_\alpha^2 + s_\alpha^2 s_{2\beta}^2 (c_\theta^2 + s_\theta^2 c_\alpha^2)} \mathrm{d}\psi,\\
A_2 &= 2\frac{-4c_\theta^2 c_\alpha^2 + s_\alpha^2 s_{2\beta}^2 (c_\theta^2 + s_\theta^2 c_\alpha^2)}{4c_\theta^2 c_\alpha^2 + s_\alpha^2 s_{2\beta}^2 (c_\theta^2 + s_\theta^2 c_\alpha^2)} \mathrm{d}\psi,\\
A_3 &= \left(1 - \frac{4s_\alpha^2 s_{2\beta}^2 s_\theta^2 c_\alpha^2}{4c_\theta^2 c_\alpha^2 + s_\alpha^2 s_{2\beta}^2 (c_\theta^2 + s_\theta^2 c_\alpha^2)}\right) \mathrm{d}\psi.
\end{aligned} \tag{I.7}$$

I.2 $\mathrm{Q}^{1,1,1}$

The solution for $\mathrm{Q}^{1,1,1}$ is given in [284, 285]. The angles parametrising the three-torus are

$$\varphi_1 = \phi_1, \quad \varphi_2 = \phi_2, \quad \varphi_3 = \phi_3. \tag{I.8}$$

Δ, vol_4 and the A_a are

$$\begin{aligned}
\Delta &= \frac{2c_{\theta_3}^2 s_{\theta_1}^2 s_{\theta_2}^2 + (2 - c_{2\theta_1} - c_{2\theta_2}) s_{\theta_3}^2}{2048},\\
\mathrm{vol}_4 &= 8^{-3/2} H^{-1/2} s_{\theta_1} s_{\theta_2} s_{\theta_3} \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 \wedge \mathrm{d}\theta_3 \wedge \mathrm{d}\psi,\\
A_1 &= \frac{8c_{\theta_1} s_{\theta_2}^2 s_{\theta_3}^2}{H} \mathrm{d}\psi,\\
A_2 &= \left(\frac{2 - c_{2\theta_1} - c_{2\theta_2}}{2s_{\theta_1}^2 c_{\theta_2}} + \frac{s_{\theta_2}^2 c_{\theta_3}^2}{c_{\theta_2} s_{\theta_3}^2}\right)^{-1} \mathrm{d}\psi,\\
A_3 &= \frac{8c_{\theta_3} s_{\theta_1}^2 s_{\theta_2}^2}{H} \mathrm{d}\psi.
\end{aligned} \tag{I.9}$$

The function H is

$$H = 5 - 3c_{2\theta_3} + c_{2\theta_1}(-3 + c_{2\theta_2} + c_{2\theta_3}) + c_{2\theta_2}(-3 + 2c_{\theta_1}^2 c_{2\theta_3}). \tag{I.10}$$

[1] Note that this corrects a typographical error in [93], where the term in the four-form flux coming from $\Delta^{1/2}\,\mathrm{vol}_4$ was written with $s_{2\alpha}^2$ instead of $s_{2\alpha} s_\alpha^2$.

I.3 $\mathrm{M}^{1,1,1}$

The solution for $\mathrm{M}^{1,1,1}$ is given in [284, 285]. The angles parametrising the three-torus are

$$\varphi_1 = \tilde{\phi}, \quad \varphi_2 = \phi, \quad \varphi_3 = \psi. \tag{I.11}$$

Δ, vol_4 and the A_a are[2]

$$\begin{aligned}
\Delta &= \tfrac{3}{262144} h \sin^2\mu, \\
\mathrm{vol}_4 &= -\tfrac{3\sqrt{3}}{16} h^{-1/2} \cos\mu \sin\theta \sin\tilde{\theta} \sin^2\mu \, \mathrm{d}\mu \wedge \mathrm{d}\tilde{\theta} \wedge \mathrm{d}\theta \wedge \mathrm{d}\tau, \\
A_1 &= -64 h^{-1} \cos\tilde{\theta} \cos^2\mu \sin^2\theta \, \mathrm{d}\tau, \\
A_2 &= 24 h^{-1} \cos\theta \sin^2\tilde{\theta} \sin^2 2\mu \, \mathrm{d}\tau, \\
A_3 &= 8 h^{-1} \sin^2\theta (3 + 5\cos 2\mu + 2\cos 2\tilde{\theta} \sin^2\mu) \mathrm{d}\tau.
\end{aligned} \tag{I.12}$$

The function h is

$$h = 8\sin^2\theta \cos 2\mu(\cos 2\tilde{\theta} + 7) - 6(\cos 2\theta + 3)\sin^2\tilde{\theta}\cos 4\mu + \cos 2\theta(\cos 2\tilde{\theta} - 33) - 13\cos 2\tilde{\theta} + 45$$

[2] Note that this is not the same deformation as [284].

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