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OPERATOR PRODUCTS IN 2-DIMENSIONAL CRITICAL THEORIES

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ABSTRACT

A noteworthy feature of certain conformally invariant 2-dimensional theories, such as the Ising and 3-state Potts models at the critical point, is the existence of "degenerate primary fields" associated with nullvectors of the Virasoro algebra. Such fields are endowed with a remarkably simple multiplication table under the operator product expansion, known as the fusion rules. In addition, correlation functions made up of these fields satisfy a system of linear homogeneous partial differential equations. We show here that these two properties are intimately related: for any n-point function, the number of conformally invariant solutions to the system of equations equals the number of times that the identity operator appears in the fusion of all n fields in the correlator. This theorem permits the calculation of some apparently intractable correlation functions.

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1. Introduction

Recently, Belavin, Polyakov and Zamolodchikov[1] (BPZ) and Friedan, Qiu and Shenker[2] (FQS) have initiated an ambitious program of study of the properties of 2-dimensional statistical mechanical systems at a second-order critical point, using the tools of conformal invariance.* A notable feature of 2-dimensional conformally invariant theories is the existence of "degenerate primary fields" $\phi_{pq}(z)$ and $\bar{\phi}_{pq}(\bar{z})$ associated with nullvectors of the Virasoro algebra. It is a remarkable fact that all possible scaling operators in a large class of interesting theories can either be expressed directly as bilinears

$$\Phi_{pqp'q'}(z,\bar{z}) = \phi_{pq}(z) \times \bar{\phi}_{p'q'}(\bar{z})$$

in degenerate primary fields, or can be obtained from these via conformal transformations. These theories include the Ising, tricritical Ising, 3-state Potts and tricritical 3-state Potts models, and (more generally[4]) the infinite sequence of q-state Potts models where q assumes the fractional values

$$q = 2 + 2\cos\frac{2\pi}{m+1}, \quad m = 3, 4, \cdots. \tag{1}$$

(Equation (1) is a realization of the "unitary series" of FQS.) Thus, for example, in the Ising model ($m = 3$), the spin density σ and the majorana fermion ψ can be thought of in this language as $\phi_{12}(z) \times \bar{\phi}_{12}(\bar{z})$ and $\phi_{21}(z) \times \bar{\phi}_{11}(\bar{z})$, respectively.

From a mathematical perspective, degenerate fields are nice objects to analyze, as they are endowed with a simple multiplication table under the operator product expansion, known as the "fusion rules." They also have the property that any correlation function containing $\phi_{pq}(z)$ or $\bar{\phi}_{pq}(\bar{z})$ must satisfy a linear

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* We shall assume that the reader is familiar with the salient results of Refs. 1 and 2, at least to the level of the skeleton review given in Ref. 5.

homogeneous partial differential equation ("BPZ equation") of order $p \times q$. It is therefore especially fruitful to study n-point functions consisting entirely of degenerate fields. These correlators must satisfy a large system of partial differential equations, and turn out, as a result, to be calculable.[3,5]

In Ref. 5, one of us (MPM) began a systematic investigation of the number of distinct solutions to the BPZ equations. Often the equations were found to be mutually inconsistent (*i.e.*, the corresponding correlator was forced to vanish identically), while in many other instances they admitted only one or two solutions. For the general n-point function

$$\begin{aligned} &< \Phi_{p_1 q_1 p'_1 q'_1}(z_1, \bar{z}_1) \cdots \Phi_{p_n q_n p'_n q'_n}(z_n, \bar{z}_n) >, \\ &\qquad\qquad \Phi_{p_i q_i p'_i q'_i}(z_i, \bar{z}_i) = \phi_{p_i q_i}(z_i) \times \bar{\phi}_{p'_i q'_i}(\bar{z}_i), \end{aligned} \tag{2}$$

the following counting rule was observed to hold:

*Let $\nu(\{p_i\}, \{q_i\})$ be the number of distinct ways that the identity operator ϕ_{11} appears, according to the fusion rules, in the operator product of all n ϕ_{pq}'s in the correlator (2). The number of independent conformally invariant solutions to the BPZ equations associated with (2) is then the product of ν with the analogous quantity $\bar{\nu}(\{p'_i\}, \{q'_i\})$ obtained from the $\bar{\phi}_{p'q'}$'s.**

A rigorous proof of this conjecture is made difficult by the fact that explicit forms of the BPZ equations are not known, save for small values of p and q. Nevertheless, the principal aim of the present paper is to elevate the conjecture, as much as possible, to a theorem.

⋆ *N.B.* By a "conformally invariant solution" we mean a function annihilated by the subalgebras $\{L_{-1}, L_0, L_1\}$ and $\{\bar{L}_{-1}, \bar{L}_0, \bar{L}_1\}$ of the Virasoro algebras. The precise meaning of the counting rule will be explained in Sec. 2, after a review of the fusion rules.

This result should be intuitive for particle physicists, as it is reminiscent of the familiar group-theoretic rules for counting the number of independent amplitudes of a certain type when the particles are endowed with a flavor symmetry. (A simple example is the process $\phi B_1 \to v B_2$, where ϕ, v, and $B_{1,2}$ are members of the pseudoscalar, pseudovector, and baryon octets, respectively. One finds eight independent such amplitudes, eight being the number of times the singlet representation $\mathbf{1}$ appears in the product $\mathbf{8} \otimes \mathbf{8} \otimes \mathbf{8} \otimes \mathbf{8}$.) The analogy leads one to view the fusion rules as defining the "Clebsch-Gordan series" of the Virasoro algebra, the "direct product" in this case being the operator product expansion. What is less intuitive is seeing the mechanism by which the BPZ equations *alone* conspire with conformal invariance to restrict the correlators to precisely the predicted number of solutions. Thus, if any extra requirements are imposed on the correlators (*e.g.*, single-valuedness[1,8]), then the equivalence will, in general, be spoiled, although the rule quoted above will still give an upper bound on the number of allowed solutions.

Our result has both calculational and physical import. From the calculational standpoint, it tells us that many n-point functions should be straightforward to determine explicitly, even when n is large. In particular, if the identity only appears once in the fusion of all the fields, the theorem implies that the BPZ equations have but one conformally invariant solution; as a consequence, they must necessarily be equivalent to a system of first-order equations which, although unknown *a priori*, can be obtained by use of the "reduction algorithm" decribed in Ref. 5.

Equally important, the correspondence that we shall establish between the fusion rules and the BPZ equations enables us to determine which of the BPZ equations must be taken into account when solving for a correlator, and which ones are actually redundant, and hence can be safely ignored. This knowledge is particularly valuable when some of the equations are of unmanageably high order (say, $p \times q > 10$). Not only are such equations difficult to manipulate for purposes of the reduction algorithm, but in addition, lacking explicit formulae, one is hard-pressed even to write them down! Fortunately, a typical correlator that one is liable to be interested in will be associated with a mix of low-order and high-order equations, and one can frequently bypass the latter entirely (see Sec. 5).

From a physical perspective, counting the number of solutions is important, as this number may be an indicator[5] of the presence of nontrivial symmetries in the theory, both at and away from T_c, such as the Kramers-Wannier duality. Discovering such symmetries is a particularly tantalizing prospect for the unitary models (1) with $m \geq 7$, which have only recently been discovered.[6,7]

The proof of the conjecture will proceed as follows. In Sec. 2 we review the fusion rules of BPZ, and discuss some of their basic properties. (This will allow us to illustrate precisely what is meant by "counting the identities.") In particular, we shall distinguish three successively more restrictive* versions of the fusion rules, which we call *unintersected*, *semi-intersected* and *fully-intersected*, the latter only being defined for certain "magic" values of c (Eq. (8) below) including—but not limited to—the values associated with the FQS unitary series (1).

The next two sections contain the heart of our proof. The main result of Sec. 3 is to establish a 1-to-1 correspondence between translation (L_{-1}) invariant solutions of the BPZ equations and the set of possible "fusion paths" obtained by sequentially fusing the fields in the correlator from left to right, according to the *unintersected* fusion rules. In Sec. 4 we examine the consequences of imposing the remaining requirements of conformal invariance, namely invariance

* By "more restrictive" we mean, for example, that any field contributing to the semi-intersected fusion of two fields necessarily contributes to their unintersected fusion, but not the other way around.

under dilatations (L_0) and "special conformal transformations" (L_1), on these solutions. We show that dilatation invariance restricts the allowed solutions to those whose corresponding fusion paths terminate in the identity operator, while special conformal invariance eliminates any solutions which violate *semi-intersected* fusion rules at any step along the path. We then specialize to magic values of c, and show how the so-called "reflected" BPZ equations that apply in these cases forbid all solutions save those consistent with *fully-intersected* fusion rules.

In short, we shall establish the following chain of 1-to-1 correspondences between solutions of the BPZ equations and sequential applications of the fusion rules:

L_{-1} invariant solutions	$\Longleftrightarrow$	unintersected fusion paths
L_{-1} and L_0 invariant solutions	$\Longleftrightarrow$	unintersected fusion paths terminating in the identity
L_{-1}, L_0 and L_1 invariant solutions	$\Longleftrightarrow$	semi-intersected fusion paths terminating in the identity
L_{-1}, L_0 and L_1 invariant solutions for magic values of c	$\Longleftrightarrow$	fully-intersected fusion paths terminating in the identity

The latter two mappings actually constitute our theorem for non-magic and magic values of c, respectively. In this way, we are reducing a difficult problem in analysis ("classify the solutions to a system of PDE's") to a trivial exercise in arithmetic ("perform a sequence of multiplications"). Note that, as we proceed down the chain, our system of equations grows, while our multiplication table shrinks.

These mappings turn out to be more than mere counting rules for determining the number of allowed solutions: we will see that they fully characterize the possible leading-order behavior of the correlators in the singular limits in which the various coordinates are pinched together in any desired fashion. The techniques of BPZ can then be used to generate the successively less singular terms in a systematic fashion (see Sec. 4c).

The calculational implications of our theorem mentioned earlier are the subject of Sec. 5. We shall work through in detail the example of a "typical" 5-point function in the tricritical 3-state Potts model, for which the associated BPZ equations range from fourth- to sixteenth-order. Using the simple methodology of Secs. 3-4, we will be able to demonstrate that, in this example, all the equations higher than sixth-order are redundant, and hence can be safely ignored from the outset. Finally, a variety of technical results required in the paper are established in the Appendix.

2. Fusion Rule Fundamentals

We begin by reviewing the fusion rules of BPZ. (Their origin will be reviewed in Sec. 3.) Adopting their notation, let us parametrize the scaling dimension Δ of a field by a complex number α, as follows:

$$\Delta(\alpha) = \frac{1}{24}(c-1) + \frac{1}{4}\alpha^2. \tag{3}$$

The primary field ϕ_α will be defined as having dimension $\Delta(\alpha)$.* In addition, we will distinguish a discrete set of primary fields labeled ϕ_{pq}, $p, q \in \mathbf{Z}$, with corresponding dimension

$$\Delta_{pq} = \frac{1}{24}(c-1) + \frac{1}{4}(\alpha_+ p + \alpha_- q)^2, \quad \alpha_\pm = \sqrt{\frac{1-c}{24}} \pm \sqrt{\frac{25-c}{24}}. \tag{4}$$

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* Throughout this paper, we shall focus exclusively on the analytic fields $\phi(z)$, since the analysis of the antianalytic fields $\bar{\phi}(\bar{z})$ is identical.

Note the important case

$$\Delta_{11} = 0,$$

implying that $\phi_{11}(z)$ can be viewed as the identity operator in the (analytic sector of the) theory. Likewise one finds

$$\Delta_{1,-1} = \Delta_{-1,1} = 1,$$

a result we shall make use of in Sec. 4.

When p and q are positive integers, ϕ_{pq} is a degenerate primary field for any value of c, associated with a nullvector of the Virasoro algebra (see Eq. (23) below). Let $\phi_{p_2q_2}$ be such a field. The fusion rules state:†

$$\begin{aligned}
&\phi_\alpha(z_1)\phi_{p_2q_2}(z_2) \\
&\quad = \sideset{}{'}\sum_{k=-(p_2-1)}^{p_2-1} \; \sideset{}{'}\sum_{l=-(q_2-1)}^{q_2-1} (z_1 - z_2)^{\Delta(\alpha+k\alpha_+ + l\alpha_-)-\Delta(\alpha)-\Delta_{p_2q_2}} \left[\phi_{\alpha+k\alpha_+ + l\alpha_-}(z_2)\right].
\end{aligned} \tag{5}$$

The primes on the summation symbols indicate that k and l only run over every other integer, so that there are $p_2 \times q_2$ contributions to the right-hand side. Following the notation of BPZ, we are using $[\phi(z_2)]$ as shorthand for the primary field $\phi(z_2)$, together with all its associated secondary fields $\phi^{(-k_1,\cdots,-k_n)}(z_2)$, multiplied by the powers of $z_1 - z_2$ dictated by dilatation invariance, *viz* :

$$\begin{aligned}
[\phi(z_2)] = \beta_0\phi(z_2) + \beta_{-1}(z_1 - z_2)\phi^{(-1)}(z_2) + \beta_{-2}(z_1 - z_2)^2\phi^{(-2)}(z_2) \\
+ \beta_{-1,-1}(z_1 - z_2)^2\phi^{(-1,-1)}(z_2) + \cdots,
\end{aligned} \tag{6}$$

where the β's are (*a priori* unspecified albeit determinable) numerical constants.

† The power of $z_1 - z_2$ in (5) is uniquely determined by requiring the left- and right-hand sides to scale in the same way under dilatations.

This formula is particularly elegant when $\phi_\alpha = \phi_{p_1q_1}$, in which case we find

$$(\mathbf{I}):\quad \phi_{p_1q_1}(z_1)\phi_{p_2q_2}(z_2) = \sum_{k=-(p_2-1)}^{p_2-1}{}' \sum_{l=-(q_2-1)}^{q_2-1}{}' (z_1-z_2)^{\Delta_{p_1+k,q_1+l}-\Delta_{p_1q_1}-\Delta_{p_2q_2}} \left[\phi_{p_1+k,q_1+l}(z_2)\right].$$

Of course, since operator products are defined only under time-ordering, it is irrelevant which field on the left-hand side of (I) is written first. If p_1 and q_1 are themselves positive integers, we could equally well have written*

$$(\mathbf{II}):\quad \phi_{p_1q_1}(z_1)\phi_{p_2q_2}(z_2) = \sum_{k=-(p_1-1)}^{p_1-1}{}' \sum_{l=-(q_1-1)}^{q_1-1}{}' (z_1-z_2)^{\Delta_{p_2+k,q_2+l}-\Delta_{p_1q_1}-\Delta_{p_2q_2}} \left[\phi_{p_2+k,q_2+l}(z_2)\right],$$

where now there are $p_1 \times q_1$ terms in the sum. A conformal family $[\phi_{pq}]$ can contribute to the operator product of $\phi_{p_1q_1}$ with $\phi_{p_2q_2}$ only if it appears on the right-hand sides of *both* (I) and (II). Thus, for example,

$$\phi_{21}\phi_{31} = \left([\phi_{01}] \oplus [\phi_{21}] \oplus [\phi_{41}]\right) \bigcap \left([\phi_{21}] \oplus [\phi_{41}]\right) = [\phi_{21}] \oplus [\phi_{41}], \tag{7}$$

and more generally

$$\phi_{p_1q_1}\phi_{p_2q_2} = \sum_{p=|p_1-p_2|+1}^{p_1+p_2-1} \sum_{q=|q_1-q_2|+1}^{q_1+q_2-1} [\phi_{pq}], \tag{8}$$

where the usual factors of $z_1 - z_2$ have been suppressed for compactness.

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* In (II) we have used the fact that $[\phi(z_1)] \equiv [\phi(z_2)]$, which follows trivially from the Taylor expansion $\phi^{(-k_1,\cdots,-k_n)}(z_1) = \phi^{(-k_1,\cdots,-k_n)}(z_2) + (z_1 - z_2)\frac{d}{dz_2}\phi^{(-k_1,\cdots,-k_n)}(z_2) + \cdots = \phi^{(-k_1,\cdots,-k_n)}(z_2) + (z_1 - z_2)\phi^{(-1,-k_1,\cdots,-k_n)}(z_2) + \cdots$.

We shall henceforth refer to the "naive" fusion rules (I) as *unintersected fusion rules*, and to expressions such as (7) and (8) that come from applying both (I) and (II) as *semi-intersected fusion rules.* Note that, if p_1, p_2, q_1 and q_2 are all positive integers, then the semi-intersected fusion of $\phi_{p_1q_1}$ and $\phi_{p_2q_2}$ contains only fields ϕ_{pq} such that p and q are themselves positive integers, a phenomenon referred to by BPZ as "truncation from below."

The fusion rules given above are valid for arbitrary values of c. Particularly interesting, however, are the discrete values

$$c = 1 - \frac{6(m'-m)^2}{mm'}, \quad 2 \le m < m', \tag{9}$$

where m and m' are relatively prime integers. (This set, which defines the "minimal models" of BPZ, includes as an important sub-sequence the unitary series (1), corresponding to the choice $m' = m+1$.) At these points, the theory is endowed with a *reflection symmetry*,[1,2] whereby

$$\Delta_{pq} = \Delta_{m-p,m'-q} = \frac{(m'p-mq)^2-(m'-m)^2}{4mm'}, \tag{10}$$

and hence

$$\phi_{pq} \equiv \phi_{m-p,m'-q}. \tag{11}$$

Likewise, the theory possesses *translation symmetry*,[1] whereby

$$\phi_{pq} \equiv \phi_{rm+p,rm'+q} \equiv \phi_{(r+1)m-p,(r+1)m'-q} \quad \forall r \in \mathbf{Z}. \tag{12}$$

Henceforth, we will refer to values of c satisfying (9) as *magic* values and those which do not as *non-magic* or *generic* values.

For magic values of c, the following can be shown:

Claim 1. The only primary fields ϕ_α that can be included in the theory in a consistent manner are the degenerate fields $\phi_\alpha = \phi_{pq}$ which satisfy $rm < p < (r+1)m$ and $rm' < q < (r+1)m'$ for some integer r.

The proof is given in the Appendix.

Thanks to this result and to Eq. (12), we can restrict our attention henceforth to the fields

$$\{\phi_{pq} : \quad 0 < p < m, \quad 0 < q < m'\} \tag{13}$$

when dealing with magic c. The reflection symmetry then implies two more distinct rewritings of the fusion rules, namely:

$$(\mathrm{III}): \quad \phi_{p_1q_1}(z_1)\phi_{m-p_2,m'-q_2}(z_2) = \sum_{k=-(m-p_2-1)}^{m-p_2-1}{}' \sum_{l=-(m'-q_2-1)}^{m'-q_2-1}{}' \left[\phi_{p_1+k,q_1+l}(z_2)\right],$$

and

$$(\mathrm{IV}): \quad \phi_{m-p_1,m'-q_1}(z_1)\phi_{p_2q_2}(z_2) = \sum_{k=-(m-p_1-1)}^{m-p_1-1}{}' \sum_{l=-(m'-q_1-1)}^{m'-q_1-1}{}' \left[\phi_{p_2+k,q_2+l}(z_2)\right].$$

Fields permitted in the operator product of $\phi_{p_1q_1}$ and $\phi_{p_2q_2}$ will then be restricted to the intersection of the right-hand sides of versions (I)-(IV) of the fusion rules.*

As an example, consider once again the operator product $\phi_{21}\phi_{31}$ when $m = 4$ and $m' = 5$ (the tricritical Ising model). Using $\phi_{21} \equiv \phi_{24}$ and $\phi_{41} \equiv \phi_{04}$, one

* There are actually four more conceivable versions of the fusion rules, obtained by applying the reflection symmetry (11) to all the fields in (I)-(IV), but these yield no new information.

now finds:

$$\begin{aligned}\phi_{21}\phi_{31} = &\big([\phi_{01}] \oplus [\phi_{21}] \oplus [\phi_{41}]\big) \bigcap \big([\phi_{21}] \oplus [\phi_{41}]\big) \\ &\bigcap \big([\phi_{2,-2}] \oplus [\phi_{20}] \oplus [\phi_{22}] \oplus [\phi_{24}]\big) \\ &\bigcap \big([\phi_{2,-2}] \oplus [\phi_{20}] \oplus [\phi_{22}] \oplus [\phi_{24}] \oplus [\phi_{4,-2}] \oplus [\phi_{40}] \oplus [\phi_{42}] \oplus [\phi_{44}]\big) \\ &= [\phi_{21}].\end{aligned} \tag{14}$$

We shall term an equation such as (14) obtained from **(I)**-**(IV)** a *fully-intersected fusion rule.*

We now have sufficient machinery to clarify the meaning of our principal theorem stated in the Introduction, wherein one is instructed to count the number of distinct ways that the identity operator ϕ_{11} appears in the fusion of all the fields in the correlator. Let us examine the 4-point function

$$< \phi_{21}\phi_{31}\phi_{21}\phi_{31} > \tag{15}$$

for generic values of c. The applicable fusion rules intended by the theorem are then the *semi-intersected* rules (8). Fusing the first and second pairs of fields in (15) yields

$$< \big([\phi_{21}] \oplus [\phi_{41}]\big)\big([\phi_{21}] \oplus [\phi_{41}]\big) > . \tag{16}$$

The identity then appears twice in the next (semi-intersected) fusion of the fields in (16): once in $\phi_{21}\phi_{21}$ and once in $\phi_{41}\phi_{41}$.† The theorem therefore predicts the existence of two independent conformally invariant solutions to the four BPZ equations associated with the four fields in (15).

† It is easy to see that ϕ_{11} appears in the semi-intersected fusion of ϕ_{pq} with $\phi_{p'q'}$ if and only if $p = p'$ and $q = q'$. As will become clear in the next section, multiple occurrences of the identity as we have in (16) can be distinguished from one another by their singularity structure in the z_i's.

Reassuringly, this counting rule is independent of the order in which one chooses to fuse the fields. For instance, we could have fused the fields in (15) iteratively from left to right, obtaining

$$\begin{aligned} < \phi_{21}\phi_{31}\phi_{21}\phi_{31} > &\sim < \big([\phi_{21}] \oplus [\phi_{41}]\big)\phi_{21}\phi_{31} > \\ &\sim < \big([\phi_{31}] \oplus [\phi_{31}] \oplus [\phi_{11}] \oplus [\phi_{51}]\big)\phi_{31} > \\ &\sim < [\phi_{11}] \oplus [\phi_{11}] \oplus \cdots >, \end{aligned}$$

with the same final result. This independence of order follows on general grounds from:

Claim 2. Semi-intersected (unlike unintersected) fusion rules are both commutative and associative.

(See Appendix.)

Continuing with our example, let us now specialize to the magic value of c corresponding to the tricritical Ising model ($m = 4$, $m' = 5$) discussed above. In this case it is the *fully-intersected* fusion rules that the theorem intends us to use. Fusing the first and second pairs of fields in (15) with the help of (14) gives

$$< [\phi_{21}][\phi_{21}] >,$$

so that the identity appears just once in the final fusion of the fields. There will therefore only be one conformally invariant solution to the *eight* BPZ equations (four as before, and four that result from reflecting each of the fields in (15) using (11)) for this magic value of c. As in the semi-intersected case, the counting can be shown to be independent of fusion order.

It might seem intuitive that there be fewer conformally invariant solutions to the BPZ equations at magic than at non-magic values of c, since the number of equations that need to be satisfied is twice as large in the former case, due to the reflection symmetry (11). However, this is not always true. For example, there is no solution corresponding to the 2-point function $< \phi_{21}\phi_{13} >$ for non-magic c,* since ϕ_{11} does not appear in the semi-intersected fusion of the two fields; but there does exist such a solution, namely $(z_1 - z_2)^{-1}$, in the special case of the Ising model ($m = 3$, $m' = 4$) thanks to the equivalence $\phi_{11} \equiv \phi_{23}$. The point is that, for magic c, the identity can be represented not only as ϕ_{11}, but also as $\phi_{m-1,m'-1}$.

Before proceeding to the proof of the principal theorem (Secs. 3-4), we should mention two little lemmas that will prove helpful. The first gives an explicit formula for fully-intersected fusions:

Claim 3. Let c assume one of the values (9) at which the reflection symmetry (11) is in effect. Then the intersected fusion rules that result just from (I) and (III) can be expressed in the form:

$$\phi_{p_1q_1}(z_1)\phi_{p_2q_2}(z_2) = \sum_{p=|p_1-p_2|+1}^{p_{\max}}{}' \sum_{q=|q_1-q_2|+1}^{q_{\max}}{}' \left[\phi_{pq}(z_2)\right], \tag{17}$$

where

$$p_{\max} = \min\left\{ p_1 + p_2,\, (m-p_1) + (m-p_2) \right\} - 1$$

and

$$q_{\max} = \min\left\{ q_1 + q_2,\, (m'-q_1) + (m'-q_2) \right\} - 1.$$

The proof of this claim is given in the Appendix.

From this one can draw two immediate conclusions:

1. Equation (17) is manifestly symmetric in the interchange $(p_1, q_1) \rightleftharpoons (p_2, q_2)$,

* See Sec. 4b below.

so it can likewise be interpreted as the intersected fusion rules resulting from **(II)** and **(IV)**. It follows that (17) actually expresses the fully intersected fusion rules implied by all four expressions **(I)**-**(IV)**. These assertions are easily checked in the example (14), where one finds

$$
\begin{aligned}
&\left([\phi_{01}] \oplus [\phi_{21}] \oplus [\phi_{41}]\right) \bigcap \left([\phi_{2,-2}] \oplus [\phi_{20}] \oplus [\phi_{22}] \oplus [\phi_{24}]\right) = \\
&\left([\phi_{21}] \oplus [\phi_{41}]\right) \bigcap \left([\phi_{2,-2}] \oplus [\phi_{20}] \oplus [\phi_{22}] \oplus [\phi_{24}] \oplus [\phi_{4,-2}] \oplus [\phi_{40}] \oplus [\phi_{42}] \oplus [\phi_{44}]\right) \\
&= [\phi_{21}].
\end{aligned}
\tag{18}
$$

2. The set of fields (13) form a closed set under (17). This is because both the minimum and the maximum values of p and q in the summation fall within these bounds (truncation from below *and* above).

We shall utilize these results in Sec. 4, where it will be crucial that **(I)** and **(III)** *alone* imply the fully-intersected fusion rules. We shall also be making repeated implicit use of the following lemma, which is likewise proved in the Appendix:

Claim 4. Suppose that a conformal family $[\phi_{pq}]$ appears in the operator product of $\phi_{p_1q_1}$ and $\phi_{p_2q_2}$. Then, in particular, the coefficient multiplying the primary field ϕ_{pq} [β_0 in Eq. (6)] must be nonzero.

It follows that the "naive" powers of $z_1 - z_2$ that appear in the fusion rules are indeed the most singular contributions from each conformal family in the limit $z_1 \to z_2$.

3. Unintersected fusion rules and the BPZ equations

Correlation functions of degenerate primary fields must satisfy two sorts of partial-differential equations. On the one hand, they must be invariant under the $SL(2,\mathbf{C})$ subgroup of the conformal transformations that preserves the "in" and "out" vacua. This can be expressed as invariance under the generators of translations, dilatations and "special conformal transformations," which take the respective forms

$$
\begin{aligned}
0 &= \hat{L}_{-1} < \phi_{p_1q_1}(z_1)\cdots\phi_{p_nq_n}(z_n) > \\
&\equiv \sum_{k=1}^{n} \frac{\partial}{\partial z_k} < \phi_{p_1q_1}(z_1)\cdots\phi_{p_nq_n}(z_n) >,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
0 &= \hat{L}_{0} < \phi_{p_1q_1}(z_1)\cdots\phi_{p_nq_n}(z_n) > \\
&\equiv \sum_{k=1}^{n} \left(z_k\frac{\partial}{\partial z_k} + \Delta_{p_kq_k}\right) < \phi_{p_1q_1}(z_1)\cdots\phi_{p_nq_n}(z_n) >,
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
0 &= \hat{L}_{1} < \phi_{p_1q_1}(z_1)\cdots\phi_{p_nq_n}(z_n) > \\
&\equiv \sum_{k=1}^{n} \left(z_k^2\frac{\partial}{\partial z_k} + 2z_k\Delta_{p_kq_k}\right) < \phi_{p_1q_1}(z_1)\cdots\phi_{p_nq_n}(z_n) > .
\end{aligned}
\tag{21}
$$

If a correlator satisfies (19)-(21), *we will say (somewhat loosely) that it is conformally invariant.* For future reference, we note the following commutation relations:

$$
[\hat{L}_{-1},\hat{L}_0] = \hat{L}_{-1}, \quad [\hat{L}_{-1},\hat{L}_1] = 2\hat{L}_0, \quad [\hat{L}_0,\hat{L}_1] = \hat{L}_1. \tag{22}
$$

On the other hand, each of the degenerate fields $\phi_{p_iq_i}$ with $p_i, q_i \geq 1$ is, by definition, associated with a nullvector of the Virasoro algebra at "level" $p_i \times q_i$, *viz* :*

$$0 = D_{p_i q_i} \left| \phi_{p_i q_i} \right\rangle = \left[a_1 (L_{-1})^{p_i q_i} + a_2 (L_{-1})^{p_i q_i - 2} L_{-2} + \cdots \right] \left| \phi_{p_i q_i} \right\rangle . \tag{23}$$

Correspondingly, the correlator satisfies the system of n BPZ equations†

$$\begin{aligned} 0 &= \hat{\mathcal{D}}^{(i)}_{p_i q_i} < \phi_{p_1 q_1} \cdots \phi_{p_n q_n} > \\ &= \left[a_1 (\hat{\mathcal{L}}^{(i)}_{-1})^{p_i q_i} + a_2 (\hat{\mathcal{L}}^{(i)}_{-1})^{p_i q_i - 2} \hat{\mathcal{L}}^{(i)}_{-2} + \cdots \right] < \phi_{p_1 q_1} \cdots \phi_{p_n q_n} >, \quad i = 1, \cdots, n, \end{aligned} \tag{24}$$

where the $\hat{\mathcal{L}}$'s are the first-order partial differential operators

$$\hat{\mathcal{L}}^{(i)}_{-k} = \sum_{j \neq i} \left(-(z_j - z_i)^{-k+1} \frac{\partial}{\partial z_j} - (1-k) \Delta_j (z_j - z_i)^{-k} \right). \tag{25}$$

Thus, for example, one can show that

$$\hat{\mathcal{D}}^{(i)}_{21} = (\hat{\mathcal{L}}^{(i)}_{-1})^2 - \frac{2}{3}(2\Delta_{21} + 1) \hat{\mathcal{L}}^{(i)}_{-2}. \tag{26}$$

Note that $\hat{\mathcal{D}}_{pq}$ will be of order $p \times q$, so long as the coefficient a_1 in (24) is nonzero.*

In the event that c assumes one of the magic values (9) at which the theory is endowed with reflection symmetry (11), correlators must, in addition to (24),

---

* The terms in (23) indicated by dots are of the form $a_l L^{k_1}_{-1} L^{k_2}_{-2} \cdots L^{k_m}_{-m}$, where $\sum n k_n = p_i \times q_i$. The a_l's are constants that depend on p_i, q_i and c; they are determined by the conditions $0 = L_1 D_{p_i q_i} \left| \phi_{p_i q_i} \right\rangle = L_2 D_{p_i q_i} \left| \phi_{p_i q_i} \right\rangle$ (see Ref. 1).

† The simplest way to derive (24) from (23) is to translate all the coordinates in the correlator by $-z_i$ so that the argument of $\phi_{p_i q_i}$ is 0, and then to follow the derivation given in Sec. 2 of Ref. 5.

* So far as we know, this is always true. However, if for a pathological choice of c, p and q, the coefficient a_1 turns out to vanish, then our counting rule still gives an *upper bound* on the number of solutions to the BPZ equations for correlators containing ϕ_{pq}.

satisfy the n "reflected" BPZ equations

$$0 = \hat{\mathcal{D}}^{(i)}_{m-p_i,m'-q_i} < \phi_{p_1q_1} \cdots \phi_{p_nq_n} >, \quad 1 \le i \le n. \tag{27}$$

In fact, thanks to the translation symmetry (12), we see that Eqs. (24) and (27) constitute but the lowest-order cases of the two infinite families of differential equations

$$\{\hat{\mathcal{D}}^{(i)}_{rm+p_i,rm'+q_i}\} \quad \text{and} \quad \{\hat{\mathcal{D}}^{(i)}_{(r+1)m-p_i,(r+1)m'-q_i}\}, \quad r = 0, 1, 2, \cdots, \tag{28}$$

that the correlator must obey. Fortunately, we shall show in Sec. 4e that the equations in (28) with $r > 0$ are irrelevant, in the sense that any solutions to the ones with $r = 0$ automatically satisfy the ones with $r > 0$.

In this section, we will concentrate on the subset

$$\{\hat{L}_{-1}, \hat{\mathcal{D}}^{(2)}_{p_2q_2}, \cdots, \hat{\mathcal{D}}^{(n)}_{p_nq_n}\} \tag{29}$$

of the full system of differential equations. We will show that the solutions to this subset are discrete in number, and in 1-to-1 correspondence with the set of all possible iterated fusions of the n fields in the correlator consistent with the unintersected fusion rules (I). A simple illustration of what is meant by this is the 3-point function $< \phi_{34}\phi_{21}\phi_{12} >$. Fusing the first two fields together via (I) gives ϕ_{24} and ϕ_{44}; fusing these, in turn, with ϕ_{12} yields the four fields ϕ_{23}, ϕ_{25}, ϕ_{43} and ϕ_{45}. And indeed, we will show that there exist four independent solutions to the system of equations $\{\hat{L}_{-1}, \hat{\mathcal{D}}^{(2)}_{21}, \hat{\mathcal{D}}^{(3)}_{21}\}$, which correspond in a well-defined way (via their singularity structure in $z_1 - z_2$ and $z_2 - z_3$) to these four sequential fusions. In Sec. 4 to follow, we shall examine the consequences of adjoining the "missing" equations $\hat{L}_0$, $\hat{L}_1$ and $\hat{\mathcal{D}}^{(1)}_{p_1q_1}$ (and, if applicable, the reflected BPZ equations as well) to (29).

We begin by considering, not correlation functions directly, but rather operator products

$$\phi_\alpha(z_1) \otimes \phi_{p_2q_2}(z_2) = \sum_{\alpha'} (z_1 - z_2)^{\Delta(\alpha') - \Delta(\alpha) - \Delta_{p_2q_2}} [\phi_{\alpha'}]. \tag{30}$$

Here p_2 and q_2 are positive integers, and, *a priori*, the sum runs over all conceivable values of $\Delta(\alpha')$. We can imagine forming correlators out of this expression, by sandwiching the fields with arbitrary strings of ϕ's (all under a time-ordering) and taking vacuum expectation values; in this event, both sides must satisfy the BPZ equation $\hat{\mathcal{D}}^{(2)}_{p_2q_2}$. Although this equation depends on all the coordinates and scaling dimensions involved in the correlator, it collapses to an ordinary differential equation in $z_1 - z_2$ as we let z_1 approach z_2. Demanding that the most singular term in $z_1 - z_2$ in the right-hand side of (30) satisfy this equation then yields a polynomial of degree $p_2 \times q_2$ in the variable $z_1 - z_2$, whose roots give the allowed values of $\Delta(\alpha')$ in terms of $\Delta(\alpha)$ and c.

To obtain these polynomials, note that

$$\hat{\mathcal{L}}^{(2)}_{-k} \quad \rightarrow \quad -(z_1 - z_2)^{-k+1} \frac{\partial}{\partial z_1} - (1-k)\Delta(\alpha)(z_1 - z_2)^{-k}$$

in this limit. Thus

$$\hat{\mathcal{L}}^{(2)}_{-k} \equiv [\gamma - (1-k)\Delta(\alpha)](z_1 - z_2)^{-k} \tag{31}$$

when acting on $(z_1 - z_2)^{-\gamma}$. (Note, in particular, that $\hat{\mathcal{L}}^{(2)}_{-k}$ increases γ by k.) As an example, the operator $\hat{\mathcal{D}}^{(2)}_{21}$ given by (26) yields the quadratic equation

$$0 = (\gamma + 1)\gamma - \frac{2}{3}(2\Delta_{21} + 1)(\gamma + \Delta(\alpha)), \quad \gamma = \Delta(\alpha) + \Delta_{21} - \Delta(\alpha'). \tag{32}$$

What we would like to establish first of all is the following correspondence:

Claim 5. The values of $\Delta(\alpha')$ consistent with $\hat{\mathcal{D}}^{(2)}_{p_2q_2}$ are precisely the $p_2 \times q_2$ values given by the unintersected fusion rules (**I**), *namely,*

$$\Delta(\alpha') \in \{\ \Delta(\alpha + k\alpha_+ + l\alpha_-) : \quad k = -p_2+1, -p_2+3, \cdots, p_2-1; \\ l = -q_2+1, -q_2+3, \cdots, q_2-1\ \}. \tag{33}$$

The simplest example of this is $\phi_{p_2q_2} = \phi_{11}$; in this case the most singular term in the fusion behaves like $(z_1 - z_2)^0$, which is indeed the unique (translationally invariant) function annihilated by $\hat{\mathcal{D}}^{(2)}_{11} \equiv \frac{\partial}{\partial z_1}$. Unfortunately, the claim is difficult to prove in general, because closed-form expressions for the $\hat{\mathcal{D}}_{pq}$'s are not known, except for low-lying values of p and q. Our approach is to proceed by induction from these special cases, which necessitates making some plausible assumptions about the nature of the operator product expansion in these conformal theories. This is done in the Appendix.

This correspondence is trivially recast as a constraint on the possible functional form of the 2-point function

$$G(z_1, z_2) =< \phi_\alpha(z_1)\phi_{p_2q_2}(z_2) >,$$

for which the subset (29) merely consists of $\hat{L}_{-1}$ and $\hat{\mathcal{D}}^{(2)}_{p_2q_2}$. Translational invariance ($\hat{L}_{-1}$) implies that

$$G(z_1, z_2) = G(z_1 - z_2),$$

while $\hat{\mathcal{D}}^{(2)}_{p_2q_2}$ restricts G further, to the $p_2 \times q_2$ functions

$$G(z_1, z_2) = (z_1 - z_2)^{\Delta(\alpha') - \Delta(\alpha) - \Delta_{p_2q_2}},$$

where $\Delta(\alpha')$ is in the set (33). Note that these are not necessarily *null* states of $\hat{L}_0$, as required by conformal invariance, but rather eigenstates with eigenvalue

$\Delta(\alpha')$; only if the field $\phi_{\alpha'}$ obtained in the fusion is the identity ϕ_{11}, for which

$$\Delta(\alpha') = \Delta_{11} = 0,$$

will G be annihilated by $\hat{L}_0$.

We would like to be able to repeat the above analysis for the n-point function

$$G(z_1, z_2, \cdots, z_n) = < \phi_\alpha(z_1)\phi_{p_2q_2}(z_2) \cdots \phi_{p_nq_n}(z_n) > . \tag{34}$$

This entails finding a suitable generalization of the concept of a "most singular term" to the case of a function of many variables. To this end, we introduce the notion of the *canonically ordered singularity structure* of a translationally invariant function G of $\{z_1, \cdots, z_n\}$, defined as follows: We rewrite G in terms of "nearest-neighbor" differences $z_i - z_{i+1}$, and treat the quantity $(z_i - z_{i+1})^{-1}$ as "more singular" than $(z_j - z_{j+1})^{-1}$ if $i < j$. This prescription is tantamount to examining G in the limit that

$$|z_1 - z_2| \ll |z_2 - z_3| \ll \cdots \ll |z_{n-1} - z_n| \to 0. \tag{35}$$

Thus, for example,

$$\begin{aligned}(z_1 - z_2)^{-\frac{1}{3}}(2z_1 - z_2 - z_3)^{\frac{1}{5}} &= (z_1 - z_2)^{-\frac{1}{3}}[2(z_1 - z_2) + (z_2 - z_3)]^{\frac{1}{5}} \\ &= (z_1 - z_2)^{-\frac{1}{3}}(z_2 - z_3)^{\frac{1}{5}}\left[1 + O\left(\frac{z_1 - z_2}{z_2 - z_3}\right)\right] \\ &= (z_1 - z_2)^{-\frac{1}{3}}(z_2 - z_3)^{\frac{1}{5}} + \text{less singular.}\end{aligned} \tag{36}$$

Of course, there are $n!$ conceivable orderings, and our results will not depend on which we adopt. From this point, through the end of Sec. 4, we will always measure whether one function is more or less singular than another with respect to the canonical ordering defined above.

This definition allows us to characterize the action of the first-order operators $\hat{\mathcal{L}}^{(i)}_{-k}$, $2 \le i \le n$, on the most singular term

$$(z_1 - z_2)^{-\gamma_1}(z_2 - z_3)^{-\gamma_2} \cdots (z_{n-1} - z_n)^{-\gamma_{n-1}} \tag{37}$$

of the n-point function (34). By straightforward manipulations one can show

$$\hat{\mathcal{L}}^{(i)}_{-k} \equiv [\gamma_{i-1} - (1-k)\Delta^{(i-1)}](z_{i-1} - z_i)^{-k} + \text{less singular}, \quad 2 \le i \le n, \tag{38}$$

when acting on (37), where we have introduced the quantities

$$\Delta^{(1)} = \Delta(\alpha) \tag{39}$$

and

$$\begin{aligned} \Delta^{(i)} &= \Delta^{(i-1)} + \Delta_{p_iq_i} - \gamma_{i-1} \\ &= \Delta_{p_iq_i} + \big(\Delta(\alpha) - \gamma_1\big) + \sum_{j=2}^{i-1} \big(\Delta_{p_jq_j} - \gamma_j\big); \quad i > 1. \end{aligned} \tag{40}$$

Comparing (38)-(40) to the 2-point case (31), we arrive at the following pleasing conclusion:

> $\hat{\mathcal{D}}^{(i)}_{p_iq_i}$ *implies the same polynomial equation, hence the same relation to the unintersected fusion rules, for the allowed values of* γ_{i-1} *in terms of* $\Delta^{(i-1)}$ *as we found in the 2-point case for* γ *in terms of* $\Delta(\alpha)$.

The physical meaning of the $\Delta^{(i)}$'s reveals itself if one rewrites the most singular term (37) as follows:

$$(z_1 - z_2)^{\Delta^{(2)} - \Delta(\alpha) - \Delta_{p_2q_2}}(z_2 - z_3)^{\Delta^{(3)} - \Delta^{(2)} - \Delta_{p_3q_3}} \cdots (z_{n-1} - z_n)^{\Delta^{(n)} - \Delta^{(n-1)} - \Delta_{p_nq_n}}. \tag{41}$$

So $\Delta^{(i)}$ can be interpreted as the dimension of the field obtained by performing $i-1$ sequential fusions of the fields in the correlator from left to right. Let us refer to a chain of $n-1$ iterated fusions culminating in a field of dimension $\Delta^{(n)}$ as a "fusion path." What the BPZ equations $\{\hat{\mathcal{D}}^{(2)}_{p_2q_2},\cdots,\hat{\mathcal{D}}^{(n)}_{p_nq_n}\}$ tell us, then, is that *each step* in the fusion path must be consistent with the unintersected fusion rules (**I**).

We have reached our stated goal of demonstrating a 1-to-1 correspondence between solutions to the subset of partial differential equations (29) and the set of all possible fusion paths allowed by the unintersected fusion rules (henceforth, "unintersected fusion paths"). In particular, the equivalence implies that the number of independent solutions will be given by $\sum_{i=2}^{n} p_i q_i$. In the following section we shall address the issue of deciding which of these solutions are actually conformally invariant.

4. Intersected fusion rules and conformal invariance

In this section we shall consider the consequences of adjoining the remaining partial differential equations to the subset (29). Specifically, we will establish the following:

(a) $\hat{L}_0$ invariance restricts the solutions to those for which the corresponding fusion path terminates in the identity.

(b) $\hat{L}_1$ invariance eliminates any solutions which, at any step in the corresponding fusion path, violate the *semi-intersected* fusion rules.

(c) For magic values of c, the reflected BPZ equations $\hat{\mathcal{D}}^{(i)}_{m-p_i,m'-q_i}$, $2\leq i\leq n$, restrict the solutions to those that obey *fully-intersected* fusion rules at each step.

(d) The equations $\hat{\mathcal{D}}^{(1)}_{p_1,q_1}$ and, if applicable, $\hat{\mathcal{D}}^{(1)}_{m-p_1,m'-q_1}$, impose no further restrictions on the allowed solutions.

(e) Likewise, for magic c, the "translated" equations $\hat{\mathcal{D}}^{(i)}_{rm+p_i,rm'+q_i}$ and $\hat{\mathcal{D}}^{(i)}_{(r+1)m-p_i,(r+1)m'-q_i}$, $r = 1, 2, \cdots$, are irrelevant.

a. Adjoining $\hat{L}_0$.

We start by considering an arbitrary solution G to the system of equations (29), with a most singular term G_{sing} proportional to (37). Applying (20) to (37) and using the definition (40), one finds

$$\hat{L}_0 G_{\text{sing}} = \Delta^{(n)} G_{\text{sing}}. \tag{42}$$

Equation (42) can actually be extended to the full function G,

$$\hat{L}_0 G = \Delta^{(n)} G, \tag{43}$$

provided that one chooses the appropriate linear combinations of solutions to (29). This follows from the fact that the BPZ equations $\hat{\mathcal{D}}^{(i)}_{p_iq_i}$ have uniform scaling behavior,

$$\left[\hat{L}_0, \hat{\mathcal{D}}^{(i)}_{p_iq_i}\right] = -p_iq_i\, \hat{\mathcal{D}}^{(i)}_{p_iq_i},$$

as a consequence of which their solutions can be chosen to be eigenstates of $\hat{L}_0$.

Recall that $\Delta^{(n)}$ can be interpreted as the dimension of the ultimate field obtained by fusing the operators in the n-point function (34) sequentially from left to right. Thus G will be dilatation $(\hat{L}_0)$ invariant if and only if the fusion path terminates in the identity operator ϕ_{11}, with dimension $\Delta^{(n)} = \Delta_{11} = 0$. This should come as no surprise: when one takes the operator product of n fields in this way, one is left at the end with a sum over single fields, weighted by c-number functions of the coordinates; our result merely reflects the fact that the identity operator is the only scaling field—primary or secondary—in a conformally invariant theory with a nonvanishing 1-point function. (All other fields are defined as having their expectation values subtracted off.)

b. Adjoining $\hat{L}_1$.

Conformal invariance requires that, in addition to translations and dilatations, correlators be invariant under the "special conformal transformations" generated by L_1; this condition is expressed by the differential equation (21). It turns out that the differential operator $\hat{L}_1$ has a remarkable role to play when viewed in terms of the correspondence between the BPZ equations and the set of all possible unintersected fusion paths:

Claim 6. $\hat{L}_1$ *is responsible for ensuring that the only solutions that survive are those consistent with the semi-intersected fusion rules, and vice versa.*

Before examining why this should be true, let us pause for some examples.

We focus first on the familiar example of the 2-point function

$$G(z_1, z_2) = < \phi_{p_1 q_1}(z_1) \phi_{p_2 q_2}(z_2) > \tag{44}$$

with p_2 and $q_2 > 0$. We have seen that $\hat{L}_{-1}$ and $\hat{D}^{(2)}_{p_2 q_2}$ restrict G to the form

$$G(z_1, z_2) = (z_1 - z_2)^{\Delta_{p_1+k, q_1+l} - \Delta_{p_1 q_1} - \Delta_{p_2 q_2}},$$

where $k = -p_2 + 1, -p_2 + 3, \cdots, p_2 - 1$ and $l = -q_2 + 1, -q_2 + 3, \cdots, q_2 - 1$.* And, as just shown, $\hat{L}_0$ invariance requires $\Delta_{p_1+k, q_1+l} = 0$, which, for generic c, means $p_1 + k = q_1 + l = 1$ (else G vanishes). This is possible if and only if $p_2 \geq |p_1|$ and $q_2 \geq |q_1|$ with $p_2 - p_1$ and $q_2 - q_1$ even; we shall assume that these conditions are met.

Consider the special case $p_1 = p_2$ and $q_1 = q_2$. The identity ϕ_{11} is then contained, not only in the *unintersected* fusion of the two fields, as given by (I), but also in their *semi-intersected* fusion, as constrained both by (I) and (II).

* As usual, we shall ignore $\hat{D}^{(1)}_{p_1 q_1}$.

Claim 6 therefore implies that $\hat{L}_1 G = 0$; that is, G is conformally invariant. In contrast, if either $p_2 > p_1$ or $q_2 > q_1$, then the identity no longer appears in the semi-intersected fusion of $\phi_{p_1q_1}$ and $\phi_{p_2q_2}$, and we therefore expect that $\hat{L}_1 G \neq 0$. Of course, it is easy to verify these assertions directly for 2-point functions. One finds:

$$\begin{aligned}\hat{L}_1 G &= \left(z_1^2\frac{\partial}{\partial z_1} + z_2^2\frac{\partial}{\partial z_2} + 2z_1\Delta_{p_1q_1} + 2z_2\Delta_{p_2q_2}\right)(z_1 - z_2)^{-\Delta_{p_1q_1}-\Delta_{p_2q_2}} \\ &= \left(\Delta_{p_1q_1} - \Delta_{p_2q_2}\right)(z_1 - z_2)^{1-\Delta_{p_1q_1}-\Delta_{p_2q_2}},\end{aligned} \tag{45}$$

so that G is indeed conformally invariant if and only if $\Delta_{p_1q_1} = \Delta_{p_2q_2}$, which, for generic values of c, is only possible when $p_1 = p_2$ and $q_1 = q_2$.

A more instructive example is the 4-point function

$$< \phi_{21}(z_1)\phi_{21}(z_2)\phi_{21}(z_3)\phi_{21}(z_4) > \tag{46}$$

in the Ising model, which corresponds to the choice $m = 3$ and $m' = 4$ in (9). One finds three independent solutions to the system of differential equations†

$$\{\hat{L}_{-1}, \hat{L}_0, \hat{D}_{21}^{(2)}, \hat{D}_{21}^{(3)}, \hat{D}_{21}^{(4)}\},$$

namely:

$$\begin{aligned} G_{\text{I}} &= \frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4} - \frac{1}{z_1 - z_3}\frac{1}{z_2 - z_4} + \frac{1}{z_1 - z_4}\frac{1}{z_2 - z_3}, \\ G_{\text{II}} &= G_{\text{I}} \int_0^x du\, (u^2 - u + 1)^{-2}[u(u-1)]^{\frac{2}{3}}, \end{aligned}$$

and

† Here, for purposes of illustration, we are ignoring, not only $\hat{D}_{21}^{(1)}$, but also the reflected equations $\hat{D}_{13}^{(i)}$, any one of which actually suffices to rule out G_{II} and G_{III}. The surviving solution G_{I} has the expected form for a 4-point function of a free fermion, which the Ising model is known to contain at T_c.

$$G_{\text{III}} = (z_1 - z_3)^{-1}(z_2 - z_4)^{-1}[\chi(\chi - 1)]^{\frac{2}{3}} \int_1^\theta du\, (u-1)^2 [u(u + \chi - u\chi)]^{-\frac{4}{3}},$$

where χ and θ are the quotients

$$\chi = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}, \quad \theta = \frac{z_1 - z_2}{z_1 - z_4}.$$

We know from the results of the previous section that G_{I}, G_{II} and G_{III} must correspond to the three possible unintersected fusion paths of the fields in the correlator terminating in the identity.* It is straightforward to work out their most singular terms, and to express these in such a way that the correspondence becomes manifest, to wit:

$$\begin{aligned} G_{\text{I}} &\sim (z_1 - z_2)^{-1}(z_2 - z_3)^{0}(z_3 - z_4)^{-1} \\ &= (z_1 - z_2)^{\Delta_{11} - \Delta_{21} - \Delta_{21}}(z_2 - z_3)^{\Delta_{21} - \Delta_{11} - \Delta_{21}}(z_3 - z_4)^{\Delta_{11} - \Delta_{21} - \Delta_{21}}, \end{aligned}$$

$$\begin{aligned} G_{\text{II}} &\sim (z_1 - z_2)^{\frac{2}{3}}(z_2 - z_3)^{-\frac{5}{3}}(z_3 - z_4)^{-1} \\ &= (z_1 - z_2)^{\Delta_{31} - \Delta_{21} - \Delta_{21}}(z_2 - z_3)^{\Delta_{21} - \Delta_{31} - \Delta_{21}}(z_3 - z_4)^{\Delta_{11} - \Delta_{21} - \Delta_{21}}, \end{aligned}$$

and

$$\begin{aligned} G_{\text{III}} &\sim (z_1 - z_2)^{-1}(z_2 - z_3)^{-\frac{1}{3}}(z_3 - z_4)^{-\frac{2}{3}} \\ &= (z_1 - z_2)^{\Delta_{11} - \Delta_{21} - \Delta_{21}}(z_2 - z_3)^{\Delta_{01} - \Delta_{11} - \Delta_{21}}(z_3 - z_4)^{\Delta_{11} - \Delta_{01} - \Delta_{21}}. \end{aligned}$$

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* *N.B.* We have, with the benefit of hindsight, already "diagonalized" these three solutions so that they correspond directly to the fusion paths, with no further need to form linear combinations. In other words, for each solution, the exponents of the successively less singular terms differ from those of the most singular term by integers (see, for example, Eq. (36)). Henceforth, we shall always assume that the solutions to the BPZ equations have been similarly diagonalized.

Note that each step in the fusion path for both $G_{\rm I}$ and $G_{\rm II}$ is actually consistent with the semi-intersected fusion rules; concomitantly, one can show that

$$\hat{L}_1 G_{\rm I} = \hat{L}_1 G_{\rm II} = 0,$$

so that $G_{\rm I}$ and $G_{\rm II}$ are conformally invariant. In contrast, the second fusion involved in $G_{\rm III}$, namely

$$\phi_{11}(z_2)\phi_{21}(z_3) \sim (z_2 - z_3)^{\Delta_{01}-\Delta_{11}-\Delta_{21}}\left[\phi_{01}(z_3)\right],$$

is prohibited by the semi-intersected fusion rules ("truncation from below"). According to Claim 6, $G_{\rm III}$ is therefore not expected to be conformally invariant, and indeed one finds

$$\hat{L}_1 G_{\rm III} \quad = \quad \frac{\left[(z_2 - z_3)(z_2 - z_4)(z_3 - z_4)\right]^{\frac{2}{3}}}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)}. \tag{47}$$

The mechanism by which $\hat{L}_1$ enforces the semi-intersected fusion rules is somewhat subtle. To understand how it works, we first need the following commutation relations:

Claim 7. Let G be any translation- and dilatation-invariant function:

$$0 = \hat{L}_{-1} G = \hat{L}_0 G.$$

Then, for all i,

$$\left[\hat{L}_1, \hat{\mathcal{D}}^{(i)}_{p_i q_i}\right] G = -2 z_i p_i q_i \, \hat{\mathcal{D}}^{(i)}_{p_i q_i} G. \tag{48}$$

The proof of this, which relies crucially on the Virasoro algebra, is given in the Appendix.

Now suppose that G is, in fact, a solution to the BPZ equations $\hat{\mathcal{D}}^{(2)}_{p_2q_2}$ through $\hat{\mathcal{D}}^{(n)}_{p_nq_n}$, so that it corresponds to an unintersected fusion path of the n fields in the correlator (34). It is easy to see using (48) that the function

$$\widetilde{G} \equiv \hat{L}_1 G,$$

if nonzero, will be a solution to the BPZ equations as well. In addition, $\widetilde{G}$ will be translation-invariant, since (22) implies

$$\hat{L}_{-1}\widetilde{G} = \left[\hat{L}_{-1}, \hat{L}_1\right] G = 2\hat{L}_0 G = 0. \tag{49}$$

It follows that $\widetilde{G}$, too, must correspond to an unintersected fusion path! Note that, unlike G, $\widetilde{G}$ will not be dilatation invariant, since

$$\hat{L}_0\widetilde{G} = \left[\hat{L}_0, \hat{L}_1\right] G = \hat{L}_1 G = \widetilde{G}. \tag{50}$$

Comparing (50) with (43), we can immediately conclude:

> *Either $\widetilde{G}$ vanishes, so that G is conformally invariant, or else the most singular term of $\widetilde{G}$ corresponds to an unintersected fusion path terminating in an operator of dimension $\Delta^{(n)} = 1$.*

As mentioned in Sec. 2, for generic values of c, the only such operators are $\phi_{1,-1}$ and $\phi_{-1,1}$. We shall also need to invoke the following fact about $\hat{L}_1$:*

> *The exponents $\{\gamma_1, \cdots, \gamma_{n-1}\}$ that characterize the most singular term (37) of G can differ by only integer amounts from the exponents $\{\tilde{\gamma}_1, \cdots, \tilde{\gamma}_{n-1}\}$ that appear in the most singular term of $\widetilde{G}$.*

In other words, $\tilde{\gamma}_i - \gamma_i \in \mathbf{Z}$ for all i.

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* This assumes that the G's have been "diagonalized" in the manner discussed above.

Let us reconsider our examples in light of this new-found knowledge. It is now clear why the 2-point function (44) must be conformally invariant when $p_1 = p_2$ and $q_1 = q_2$, for neither $\phi_{1,-1}$ nor $\phi_{-1,1}$ is contained in the unintersected fusion of the two fields. In contrast, suppose that (say) $p_2 = p_1 + 2$. Now $\phi_{-1,1}$ *is* contained in the unintersected fusion. The expression for $\hat{L}_1 G$ given by (45) will then be a solution to $\hat{\mathcal{D}}^{(2)}_{p_2 q_2}$, and, as a result, there is no reason to expect $\hat{L}_1 G$ to vanish.

As for the 4-point function (46), one can easily check that there exists a unique fusion path terminating in a field of dimension *unity*, namely:

$$
\begin{aligned}
&(z_1 - z_2)^{\Delta_{11} - \Delta_{21} - \Delta_{21}} (z_2 - z_3)^{\Delta_{01} - \Delta_{11} - \Delta_{21}} (z_3 - z_4)^{\Delta_{-1,1} - \Delta_{01} - \Delta_{21}} \\
&= (z_1 - z_2)^{-1} (z_2 - z_3)^{-\frac{1}{3}} (z_3 - z_4)^{\frac{1}{3}}.
\end{aligned}
\tag{51}
$$

For $G = G_{\mathrm{I}}$, G_{II} or G_{III}, we know that one of the following must be true: either $\widetilde{G} \equiv \hat{L}_1 G$ vanishes, or else $\widetilde{G}$ will be a solution to $\hat{\mathcal{D}}^{(2)}_{21}$, $\hat{\mathcal{D}}^{(3)}_{21}$ and $\hat{\mathcal{D}}^{(4)}_{21}$ whose most singular term will be given by (51). Let us compare the singularity structure of (51) to those given earlier for G_{I}, G_{II} and G_{III}. We see immediately that G_{I} and G_{II} must be annihilated by $\hat{L}_1$, as their exponents differ by *non-integral* amounts from those appearing in (51). However, this is not the case for G_{III}. We thus have no reason to expect $\hat{L}_1 G_{\mathrm{III}}$ to vanish, and have seen that it does not.†

It is clear that we are after the following general theorem:

Consider a fusion path terminating in the identity (like G above), as defined by the sequence of most-singular exponents $\{\gamma_1, \cdots, \gamma_{n-1}\}$. *Then these exponents can differ by integral amounts from those characterizing some fusion path that terminates in a field of dimension unity (like* $\widetilde{G}$

† In fact, it follows from Claim 8 in Sec. 5 that $\hat{L}_1 G_{\mathrm{III}}$ cannot vanish, since $\Delta^{(3)} = \Delta_{01} \neq \Delta_{21}$. It is straightforward to check explicitly that the function on the right-hand side of (47) has all the properties expected of it: it solves $\{\hat{L}_{-1}, \hat{\mathcal{D}}^{(2)}_{21}, \hat{\mathcal{D}}^{(3)}_{21}, \hat{\mathcal{D}}^{(4)}_{21}\}$, it satisfies $\hat{L}_0(\hat{L}_1 G_{\mathrm{III}}) = \hat{L}_1 G_{\mathrm{III}}$, and it has a most-singular term given by (51).

above) if and only if the first path violates the semi-intersected fusion rules at some step in the iteration.

This theorem turns out to be remarkably simple to prove. The key observation is that, for generic values of c, the only fields whose dimensions differ by integers from ϕ_{pq} are $\phi_{p,-q}$, $\phi_{-p,q}$ and $\phi_{-p,-q}$ [See Eq. (4)]. One finds:

$$\Delta_{-p,-q} = \Delta_{pq}, \quad \Delta_{p,-q} = \Delta_{-p,q} = \Delta_{pq} + pq.$$

Now consider an n-point function $< \phi_{p_1q_1} \cdots \phi_{p_nq_n} >$ where all the p_i and q_i are positive. By the property of "truncation from below" discussed in Sec. 2, any fusion path for this correlator consistent with the semi-intersected fusion rules at each step must lie entirely in the upper right-hand quadrant of the pq plane, and can therefore probe at most one of the four fields $\{\phi_{pq}, \phi_{p,-q}, \phi_{-p,q}, \phi_{-p,-q}\}$, namely, the one with both subscripts positive. This suffices to prove the "only if" direction of the theorem.

A single example will clarify the "if" direction. Consider the unintersected fusion path implied by the singularity structure

$$(z_1-z_2)^{\Delta_{32}-\Delta_{22}-\Delta_{21}}(z_2-z_3)^{\Delta_{25}-\Delta_{32}-\Delta_{66}}(z_3-z_4)^{\Delta_{23}-\Delta_{25}-\Delta_{13}}(z_4-z_5)^{\Delta_{11}-\Delta_{23}-\Delta_{23}}, \tag{52}$$

which we imagine as arising from the 5-point function

$$G = < \phi_{22}(z_1)\phi_{21}(z_2)\phi_{66}(z_3)\phi_{13}(z_4)\phi_{23}(z_5) > .$$

In this example, the semi-intersected fusion rules have been violated at the second step, in which we fused $\phi_{32}(z_2)$ with $\phi_{66}(z_3)$ to form $[\phi_{25}(z_3)]$. But now we *can* find another path whose exponents differ by integers from those of (52), namely

$$\begin{aligned}(z_1 - z_2)^{\Delta_{32}-\Delta_{22}-\Delta_{21}}(z_2 - z_3)^{\Delta_{-2,5}-\Delta_{32}-\Delta_{66}}& \\ \times \quad (z_3 - z_4)^{\Delta_{-2,3}-\Delta_{-2,5}-\Delta_{13}}(z_4 - z_5)^{\Delta_{-1,1}-\Delta_{-2,3}-\Delta_{23}}&,\end{aligned} \tag{53}$$

obtained from (52) simply by negating the p-indices on the $\Delta^{(i)}$'s from the first point of violation on. The key is that we were able to reach *both* ϕ_{25} and $\phi_{-2,5}$ by fusing ϕ_{32} with ϕ_{66} via the unintersected fusion rules—a phenomenon that is possible if and only if we violate the *semi*-intersected fusion rules in the process. Note that the two fusion paths (52) and (53) culminate in ϕ_{11} and $\phi_{-1,1}$, respectively, consistent with our earlier discussion.

The reader should note that we have rigorously only proved one direction of Claim 6, namely, that solutions corresponding to semi-intersected fusion paths are necessarily $\hat{L}_1$ invariant. In the case of a path that violates semi-intersected rules, we have only shown that there must exist a non-zero function $\tilde{G}$ whose singularity structure is consistent with the identification $\tilde{G} = \hat{L}_1 G$. It is conceivable, however, that for certain correlators at certain values of c, $\hat{L}_1 G$ actually vanishes, so that the correlator is "accidentally" conformally invariant. What implications does this gap in the proof have for the thesis of this paper?

At magic c, we shall see in the next subsection that the full set of BPZ equations alone (*i.e.*, without $\hat{L}_1$) imply fully-intersected, hence semi-intersected, fusion rules: this being the case, we do not need the incomplete direction of Claim 6. On the other hand, if for some n-point function Claim 6 is faulty for some non-magic values of c, then there would be more solutions to the equations than our thesis proposes.

c. Adjoining $\hat{\mathcal{D}}^{(2)}_{m-p_2,m'-q_2}$ through $\hat{\mathcal{D}}^{(n)}_{m-p_n,m'-q_n}$.

We now specialize to the magic values of c given by Eq. (9), and examine the consequences of adjoining the reflected BPZ equations

$$\{\hat{\mathcal{D}}^{(2)}_{m-p_2,m'-q_2},\cdots,\hat{\mathcal{D}}^{(n)}_{m-p_n,m'-q_n}\} \tag{54}$$

to the previously considered system of equations

$$\{\hat{L}_{-1},\hat{L}_0,\hat{\mathcal{D}}^{(2)}_{p_2,q_2},\cdots,\hat{\mathcal{D}}^{(n)}_{p_n,q_n}\} \tag{55}$$

(We will *not* need to assume $\hat{L}_1$ invariance *a priori*.) It is clear that the only fusion paths that can survive both sets of equations are those which, at each step, satisfy simultaneously the two versions of the fusion rules given by (I) and (III). We have seen* that this is actually equivalent to the *fully-intersected* fusion rules implied by all four versions (I)-(IV). In particular, as just shown, versions (I) and (II) suffice to establish $\hat{L}_1$ invariance, and hence conformal invariance, of the correlator.

In truth, this line of argument requires somewhat more care. To see this, consider adjoining just $\hat{D}^{(2)}_{m-p_2,m'-q_2}$ to the subset (55), and let us suppose that there does exist a fully-intersected fusion path terminating in the identity, with most-singular exponents

$$\{\gamma_1, \cdots, \gamma_{n-1}\}. \tag{56}$$

What we have actually proved up to the present can be phrased as follows: there must exist functions G_1 and G_2 that satisfy the systems of equations

$$\{\hat{L}_{-1}, \hat{L}_0, \hat{L}_1, \hat{D}^{(2)}_{p_2,q_2}, \hat{D}^{(3)}_{p_3,q_3}, \cdots, \hat{D}^{(n)}_{p_n,q_n}\}$$

and

$$\{\hat{L}_{-1}, \hat{L}_0, \hat{L}_1, \hat{D}^{(2)}_{m-p_2,m'-q_2}, \hat{D}^{(3)}_{p_3,q_3}, \cdots, \hat{D}^{(n)}_{p_n,q_n}\},$$

respectively, and whose most-singular terms each correspond to (56). However, what we would like to show is that the full functions G_1 and G_2—not just their most singular terms—are identical to one another: if not, (54) and (55) constitute an *overdetermined* system, with no nonvanishing solutions in common.

One quick way to establish this for *almost* every case is to normalize G_1 and G_2 so that the coefficients of the most singular terms are equal to one another,

* *Cf.* the first conclusion following Claim 3 in Sec. 2.

and then form the difference

$$G' = G_1 - G_2.$$

G' satisfies the smaller system of equations

$$\{\hat{L}_{-1}, \hat{L}_0, \hat{L}_1, \hat{\mathcal{D}}^{(3)}_{p_3,q_3}, \cdots, \hat{\mathcal{D}}^{(n)}_{p_n,q_n}\}.$$

Moreover, it is associated with most-singular exponents

$$\{\gamma'_1, \cdots, \gamma'_{n-1}\} \tag{57}$$

differing from (56) only by integers. An argument similar to that used in Sec. 4*b* shows that, with a few nettlesome exceptions,* this cannot occur; hence $G' = 0$. In this way we learn that we can safely adjoin $\hat{\mathcal{D}}^{(2)}_{m-p_2,m'-q_2}$, and similarly $\hat{\mathcal{D}}^{(3)}_{m-p_3,m'-q_3}$ through $\hat{\mathcal{D}}^{(n)}_{m-p_n,m'-q_n}$, to (55) in a consistent manner to all orders in the singularity expansion.

A more illuminating path to the same conclusion uses a key result of BPZ. They showed that conformal invariance suffices to determine recursively all the coefficients $\{\beta_{-1},\ \beta_{-2},\ \beta_{-1,-1}, \cdots\}$ which multiply the secondary fields in Eq. (6) in terms of the coefficient β_0 of the primary field of that conformal family. This implies that, once one knows the most singular piece of a conformally invariant solution to the BPZ equations, one can generate the less singular terms uniquely, order by order. Thus G_1 must indeed be equal to G_2.

* The isolated exceptions to this line of proof arise from the fact that, for certain integer values of m and m', one can find pairs of fields ϕ_{pq} and $\phi_{p'q'}$, each within the range (13), whose weights differ by integers.

d. Adjoining $\hat{\mathcal{D}}^{(1)}_{p_1,q_1}$ and/or $\hat{\mathcal{D}}^{(1)}_{m-p_1,m'-q_1}$.

We now examine the constraint imposed by the (hitherto neglected) BPZ equation $\hat{\mathcal{D}}^{(1)}_{p_1q_1}$ on the allowed form of the n-point function

$$G(z_1,\cdots,z_n) = < \phi_{p_1q_1}\cdots\phi_{p_nq_n} > .$$

The first task is to derive an expression for $\hat{\mathcal{L}}^{(1)}_{-k}$ analogous to that given in (38) for $\hat{\mathcal{L}}^{(i)}_{-k}$, $i \geq 2$. One easily finds:

$$\hat{\mathcal{L}}^{(1)}_{-k} \equiv [\gamma_1 - (1-k)\Delta_{p_2q_2}](z_2-z_1)^{-k} + \text{less singular}$$

when acting on the most singular term (37) of G. So $\hat{\mathcal{L}}^{(1)}_{-k}$ is equivalent to $\hat{\mathcal{L}}^{(2)}_{-k}$, with the substitution $\Delta_{p_1q_1} \leftrightarrow \Delta_{p_2q_2}$. The effect of $\hat{\mathcal{D}}^{(1)}_{p_1q_1}$ will therefore be to implement version (II) of the fusion rules at "step 1" of the corresponding fusion path, just as $\hat{\mathcal{D}}^{(2)}_{p_2q_2}$ enforces version (I).

It then follows from the results of Sec. 4*b* that any conformally invariant solution to the BPZ equations $\hat{\mathcal{D}}^{(2)}_{p_2q_2}$ through $\hat{\mathcal{D}}^{(n)}_{p_nq_n}$ will *automatically* be a solution to $\hat{\mathcal{D}}^{(1)}_{p_1q_1}$ as well, since, by definition, a semi-intersected fusion path satisfies both (I) and (II) at each step, including the first.†

Next, let us specialize to the magic values of c, and consider the effect of adjoining the remaining equation $\hat{\mathcal{D}}^{(1)}_{m-p_1,m'-q_1}$ to the grand system

$$\{\hat{L}_{-1},\hat{L}_0,\hat{L}_1,\hat{\mathcal{D}}^{(1)}_{p_1,q_1},\cdots,\hat{\mathcal{D}}^{(n)}_{p_n,q_n},\hat{\mathcal{D}}^{(2)}_{m-p_2,m'-q_2},\cdots,\hat{\mathcal{D}}^{(n)}_{m-p_n,m'-q_n}\}. \tag{58}$$

Just as $\hat{\mathcal{D}}^{(1)}_{p_1,q_1}$ enforces (II), so $\hat{\mathcal{D}}^{(1)}_{m-p_1,m'-q_1}$ implements version (IV) of the fusion rules at "step 1" in the fusion path. We know, however, that the solutions to

† Strictly speaking, this argument (as well as the following one concerning $\hat{\mathcal{D}}^{(1)}_{m-p_1,m'-q_1}$) only holds at the level of the most singular term of G, but its validity can easily be extended to all orders by the methods of Sec. 4*c*.

(58) correspond to fully-intersected fusion paths, which satisfy (I)-(IV) at each step. It follows that, like $\hat{\mathcal{D}}^{(1)}_{p_1,q_1}$, the reflected equation $\hat{\mathcal{D}}^{(1)}_{m-p_1,m'-q_1}$ imposes no further constraint on the allowed solutions to (58).

e. Adjoining $\hat{\mathcal{D}}^{(i)}_{rm+p_i,rm'+q_i}$ and $\hat{\mathcal{D}}^{(i)}_{(r+1)m-p_i,(r+1)m'-q_i}$, $r \geq 1$.

Finally, let us consider the translated BPZ equations (28) for $r \geq 1$ that apply at magic values of c. From a calculational point of view, it would be unfortunate if these equations needed to be taken into account; for instance, in the Ising model, the $r = 1$ equations associated with the magnetization operator σ are already of fifteenth and twentieth order. Fortunately, the translated equations can always safely be neglected.

To see this, note that the unintersected fusion (I) of $\phi_{p_1q_1}$ with $\phi_{rm+p_2,rm'+q_2}$ ($\phi_{(r+1)m-p_2,(r+1)m'-q_2}$) contains as a subset all the conformal families that contribute to the unintersected fusion of $\phi_{p_1q_1}$ with either $\phi_{p_2q_2}$ or $\phi_{m-p_2,m'-q_2}$, depending, respectively, on whether r is even (odd) or odd (even). The intersection of all such $r \geq 1$ fusions must then contain all the families on the right-hand side of Eq. (17). If follows that any fully-intersected fusion path will be consistent with the translated equations. The redundancy of the latter is then assured by the arguments of Sec. 4c.

This concludes the proof of our principal theorem, both for magic and non-magic values of c. In the following section, we shall take up the important calculational question of determining which of the BPZ equations are, like $\hat{\mathcal{D}}^{(1)}_{p_1q_1}$, $\hat{\mathcal{D}}^{(1)}_{m-p_1,m'-q_1}$, and the translated equations, redundant.

5. Calculational Implications

Consider an n-point function $< \phi_{p_1q_1} \cdots \phi_{p_nq_n} >$; for purposes of illustration, let us assume a magic value of c. This correlator must satisfy a total of $2n+3$ partial differential equations: three first-order equations that express conformal invariance, namely $\{\hat{L}_{-1}, \hat{L}_0, \hat{L}_1\}$, plus the $2n$ BPZ equations $\hat{\mathcal{D}}^{(i)}_{p_iq_i}$ and $\hat{\mathcal{D}}^{(i)}_{m-p_i,m'-q_i}$, $1 \le i \le n$, of order $p_i \times q_i$ and $(m-p_i) \times (m'-q_i)$, respectively. In practice, the most efficient method for solving such a system of linear homogenous equations of various orders is the so-called "reduction algorithm" described in Ref. 5. This algorithm entails repeatedly differentiating the lower-order equations and subtracting them from the higher-order ones, in such a way as to cancel the highest-order terms in the latter. The process continues until a "minimal system" of equations is obtained of the lowest possible order. At this point, the desired correlator can frequently be written down by inspection.

A serious impediment to this program occurs when some of the $(r=0)$ BPZ equations that one starts with are of unmanageably high order. For example, the field ϕ_{23} in the tricritical 3-state Potts model $(m=6,\ m'=7)$ is associated with differential operators of order six and sixteen, the latter resulting from the reflection symmetry $\phi_{23} \equiv \phi_{44}$. The sixth-order operator (equivalently, the nullvector D_{23} of the Virasoro algebra) is relatively straightforward to work out. However, the sixteenth-order operator involves a total of $P(16) = 231$ distinct terms, each of which (assuming we normalize the coefficient of $(\hat{\mathcal{L}}_{-1})^{16}$ to unity) is multiplied by a fraction typically consisting of upwards of 30 digits!

Clearly, from a calculational standpoint, it is crucial in cases such as this to know aforehand which *subsets* of the $2n+3$ equations suffice to generate the minimal system via the reduction algorithm. If fortune smiles, such knowledge would enable us to bypass the BPZ equations of high order from the outset.

We have already seen that some of the $2n+3$ equations associated with a correlator, namely $\hat{\mathcal{D}}^{(1)}_{p_1q_1}$ and $\hat{\mathcal{D}}^{(1)}_{m-p_1,m'-q_1}$, are redundant. Of course, the choice of which coordinate is labeled z_1 is but an artifact of our arbitrary ordering convention (35). It follows that any one pair $\{\hat{\mathcal{D}}^{(i)}_{p_i q_i},\ \hat{\mathcal{D}}^{(i)}_{m-p_i,m'-q_i}\}$, $1 \leq i \leq n$, can be excluded from the system. Surprisingly, it turns out that many more of the equations can similarly be bypassed. As we do not have any general theorems to offer in this regard, we will content ourselves here with an instructive example.

Let us work through the case of the 5-point function

$$< \phi_{15}(z_1)\phi_{23}(z_2)\phi_{51}(z_3)\phi_{33}(z_4)\phi_{41}(z_5) > \tag{59}$$

in the tricritical 3-state Potts model. It is easy to show using Eq. (17) that the identity operator (ϕ_{11} or ϕ_{56}) appears but once in the fully-intersected fusion of all the fields, implying a unique conformally invariant solution to the ten BPZ equations. As such, the correlator is a prime candidate—at least in principle—for the reduction algorithm: the minimal system ultimately obtained should consist entirely of first-order equations. In practice, however, one must contend with (in decreasing order of undesirability), not only the sixteenth-order operator $\hat{\mathcal{D}}^{(2)}_{44}$ mentioned above, but also the twelfth-order operators $\hat{\mathcal{D}}^{(4)}_{34}$ and $\hat{\mathcal{D}}^{(5)}_{26}$, the tenth-order operator $\hat{\mathcal{D}}^{(1)}_{52}$, and the ninth-order operator $\hat{\mathcal{D}}^{(4)}_{33}$. Fortunately, a little experimentation reveals that these can all be avoided. Let us see how.

It turns out, in this example, that the most convenient formulation of the singularity structure of the correlator is not given by the canonical ordering defined by Eq. (35); rather, it is the one defined by the limit

$$|z_2 - z_3| \ll |z_3 - z_1| \ll |z_1 - z_5| \ll |z_5 - z_4| \to 0. \tag{60}$$

If we wish to think in terms of fusing from left to right as we have grown accustomed, we ought, therefore, to rearrange the correlator as

$$< \phi_{23}(z_2)\phi_{51}(z_3)\phi_{15}(z_1)\phi_{41}(z_5)\phi_{33}(z_4) > .$$

Consistent with this new ordering, we can parametrize the most-singular term as

$$(z_2-z_3)^{\Delta^{(2)}-\Delta_{23}-\Delta_{51}}(z_3-z_1)^{\Delta^{(3)}-\Delta^{(2)}-\Delta_{15}}$$
$$\times \quad (z_1-z_5)^{\Delta^{(4)}-\Delta^{(3)}-\Delta_{41}}(z_5-z_4)^{\Delta^{(5)}-\Delta^{(4)}-\Delta_{33}},$$

where the $\Delta^{(i)}$'s that define the unique fully-intersected fusion path terminating in the identity are given by

$$\Delta^{(2)} = \Delta^{(3)} = \Delta_{43}, \quad \Delta^{(4)} = \Delta_{33}, \quad \Delta^{(5)} = \Delta_{11}. \tag{61}$$

The task before us is to find a manageable subset of the thirteen partial differential equations sufficient to imply (61). To begin with, we have seen that the condition $\Delta^{(5)} = \Delta_{11}$ is an immediate consequence of $\hat{L}_0$ invariance. It is also straightforward to establish the following general result:*

Claim 8. $\hat{L}_1$ invariance of the n-point function $G = <\phi_{\alpha_1}\cdots\phi_{\alpha_n}>$ forces $\Delta^{(n-1)} = \Delta(\alpha_n)$ whenever $\hat{L}_0 G = 0$.

So, in the case at hand, $\hat{L}_1$ accounts for $\Delta^{(4)} = \Delta_{33}$.

At this point there are several ways to proceed, of which the following turns out to be optimal. We know that the two BPZ equations $\hat{\mathcal{D}}^{(3)}_{51}$ and $\hat{\mathcal{D}}^{(3)}_{16}$ associated with $\phi_{51}(z_3)$ have the effect of enforcing the fully-intersected fusion rules at the first step in the fusion path. Since, by Eq. (17), $\phi_{23}\phi_{51} \sim [\phi_{43}]$, we obtain $\Delta^{(2)} = \Delta_{43}$ as desired. Continuing rightward, $\hat{\mathcal{D}}^{(1)}_{15}$ implies the "unintersected" result

$$\Delta^{(3)} \in \{\Delta_{4,-1}\,,\, \Delta_{41},\, \Delta_{43},\, \Delta_{45},\, \Delta_{47}\},$$

while $\hat{\mathcal{D}}^{(5)}_{41}$, in turn, excludes all of these values except Δ_{43}. (The other four possibilities cannot be fused with ϕ_{41} to produce a field of dimension $\Delta^{(4)} = \Delta_{33}$.)

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* The ordering prescription assumed in Claim 8 involves fusing from left to right, as usual. The claim is proved by demanding that the most singular term of the correlator be annihilated by $\hat{L}_1$. As such, it is a necessary, but by no means sufficient, condition for $\hat{L}_1$ invariance, as can be seen from Eq. (52).

In sum, the seven operators

$$\{\hat{L}_{-1},\, \hat{L}_0,\, \hat{L}_1,\, \hat{\mathcal{D}}_{51}^{(3)},\, \hat{\mathcal{D}}_{16}^{(3)},\, \hat{\mathcal{D}}_{15}^{(1)},\, \hat{\mathcal{D}}_{41}^{(5)}\}, \tag{62}$$

none of which is higher than sixth order,† are sufficient input into the reduction algorithm to produce the unique function associated with the correlator (59).

Of course, other ordering prescriptions would have led to different subsets of operators. Our choice of (60) was motivated by the desire to obtain a system of equations of the lowest possible order.

In many cases, a "compromise strategy" is advised, in which we exploit our prior knowledge of the leading singularity structure of the correlator in order to compensate for choosing an overly small subset of BPZ equations. Consider, as a simple example, the 4-point function

$$< \phi_{21}\phi_{p_2q_2}\phi_{p_3q_3}\phi_{p_4q_4} > .$$

Let us assume that the BPZ equations associated with the latter three fields are all of high order, but that the identity only appears once in the fusion of the fields. In such a case, the best approach is to solve the pared-down system

$$\{\hat{L}_{-1}, \hat{L}_0, \hat{L}_1, \hat{\mathcal{D}}_{21}^{(1)}\},$$

which, unlike the full system, has two independent solutions, expressible for all c as hypergeometric functions.[1] It is then a straightforward matter to find the unique linear combination of these solutions whose singularity structure corresponds to the known fusion path.

All in all, it is apparent that the correspondence between the BPZ equations and the fusion rules serves as much more than a counting rule for the number of solutions: it is an invaluable calculational aid as well.

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† In fact, we can reduce $\hat{\mathcal{D}}_{16}^{(3)}$ to a fifth-order equation *ab initio*, by multiplying $\hat{\mathcal{D}}_{51}^{(3)}$ by $\hat{\mathcal{L}}_{-1}^{(3)}$ and subtracting.

APPENDIX Technical details

Proof of Claim 1. The proof follows trivially from considering the product of ϕ_α with the identity operator $\phi_{11} \equiv \phi_{m-1,m'-1}$. One finds

$$\phi_\alpha\phi_{11} \ \cap \ \phi_\alpha\phi_{m-1,m'-1} \ = \ \text{empty set}$$

unless ϕ_α is as claimed.

Proof of Claim 2. Commutativity is obvious. Associativity follows from the observation that the multiplication law (8) is identical to the Clebsch-Gordan series for $SU(2) \times SU(2)$, where the first $SU(2)$ corresponds to the p-indices and the second to the q-indices, and each ϕ_{pq} is treated as having spin$\frac{p-1}{2}\times$spin$\frac{q-1}{2}$. Since $SU(2)$ tensor products are associative, so are semi-intersected fusion rules. (The same result holds for fully-intersected rules (17) as well, although in that case we do not have an elegant proof.)

Proof of Claim 3. The claim is obvious if one rewrites (I) and (III), respectively, as

$$\phi_{p_1q_1}\phi_{p_2q_2} = \sum_{p=p_1-p_2+1}^{p_1+p_2-1}{}' \ \sum_{q_1-q_2+1}^{q_1+q_2-1}{}' \ [\phi_{pq}] \tag{63}$$

and

$$\phi_{p_1q_1}\phi_{m-p_2,m'-q_2} = \sum_{p=p_2-p_1+1}^{2m-p_1-p_2-1}{}' \ \sum_{q_2-q_1+1}^{2m'-q_1-q_2-1}{}' \ [\phi_{pq}], \tag{64}$$

using the reflection symmetry (11). Equation (17) merely expresses the "naive" intersection of (63) and (64). The only potential pitfall here is if one of the fields has been wrongly excluded from this intersection, due to a failure to take (11) into account. This is only possible if, for some $\phi_{p_aq_a}$ and $\phi_{p_bq_b}$ appearing on the right-hand sides of (63) and (64), respectively, one has $p_a = m - p_b$ and $q_a = m' - q_b$. But this can happen only if both m and m' are even (recall that the summations only run over every other integer), contradicting our requirement in (9) that they be relatively prime.

Proof of Claim 4. Let ϕ_0, with weight Δ_0, be the "most singular" piece of a conformal family $[\phi]$ that appears in the operator product of $\phi_{p_1q_1}$ and $\phi_{p_2q_2}$. That is:

$$\begin{aligned}\phi_{p_1q_1}(z_1)\phi_{p_2q_2}(z_2) \sim (z_1 - z_2)^{\Delta_0 - \Delta_{p_1q_1} - \Delta_{p_2q_2}} \cdot \phi_0(z_2) \\ + \mathcal{O}\left((z_1 - z_2)^{\Delta_0 - \Delta_{p_1q_1} - \Delta_{p_2q_2}+1}\right),\end{aligned} \tag{65}$$

ignoring possible contributions from other conformal families. The claim is established by showing that ϕ_0 necessarily transforms like a primary field. Commuting both sides of (65) with L_n yields the consistency condition

$$\begin{aligned}&\Big(z_1^{n+1}\frac{\partial}{\partial z_1}+(n+1)\Delta_{p_1q_1}z_1^n + z_2^{n+1}\frac{\partial}{\partial z_2} + (n+1)\Delta_{p_2q_2}z_2^n\Big) \\ &\qquad \times \ \left\{(z_1 - z_2)^{\Delta_0 - \Delta_{p_1q_1} - \Delta_{p_2q_2}}\phi_0(z_2)\right\} \\ &\qquad = \ (z_1 - z_2)^{\Delta_0 - \Delta_{p_1q_1} - \Delta_{p_2q_2}}\left[L_n, \phi_0(z_2)\right] \quad + \quad \text{less singular.}\end{aligned} \tag{66}$$

The left-hand side of (66) can be rewritten as

$$(z_1 - z_2)^{\Delta_0 - \Delta_{p_1q_1} - \Delta_{p_2q_2}}\Big(z_2^{n+1}\frac{\partial}{\partial z_2} + (n+1)z_2^n\Delta_0\Big)\phi_0(z_2) \quad + \quad \text{less singular.}$$

Letting $z_1 \to z_2$ then forces

$$\left[L_n, \phi_0(z_2)\right] = \Big(z_2^{n+1}\frac{\partial}{\partial z_2} + (n+1)z_2^n\Delta_0\Big)\phi_0(z_2).$$

So ϕ_0 is indeed a primary field.

"Proof" of Claim 5. The claim is straightforward to verify explicitly for low-lying values of p_2 and q_2. Thus, when $(p_2, q_2) = (2,1)$, one can easily check that the choices $\Delta(\alpha') = \Delta(\alpha \pm \alpha_+)$ solve (32), and one can do likewise for $(p_2, q_2) = (1,2)$ and $(2,2)$. Eventually, however, we must resort to induction. In order to do so, we need to make an assumption such as the following:

For dense values of c and $\Delta(\alpha)$, there exist theories with the property that the operator product $\phi_\alpha \otimes \phi_{21}$ contains both conformal families allowed by the unintersected fusion rules, namely $[\phi_{\alpha\pm\alpha_+}]$; and likewise for $\phi_\alpha \otimes \phi_{12}$.

(*N.B.* We have seen that $\phi_\alpha \otimes \phi_{21}$ contains *at most* these two families, thanks to the restrictions imposed by $\hat{\mathcal{D}}^{(2)}_{21}$, but they need not both appear. Thus, for the particular value $\Delta(\alpha) = \Delta_{12}$, we find that $\phi_{12} \otimes \phi_{21}$ only contains $[\phi_{22}]$, the family $[\phi_{02}]$ being forbidden by $\hat{\mathcal{D}}^{(1)}_{12}$; similarly, when $c = \frac{1}{2}$, $\phi_{21} \otimes \phi_{21}$ only contains $[\phi_{11}]$, due to the reflected BPZ equations. In general, the constraint from $\hat{\mathcal{D}}^{(1)}$ will only occur at the *discrete* set $\Delta(\alpha) = \Delta_{pq}$ where $\hat{\mathcal{D}}^{(1)}$ applies; the constraints from the reflected BPZ equations will only occur at the discrete set of magic values of c.) This is the strongest assumption we need to make in this paper. Now consider, as an example of the induction process, the triple operator product

$$\big(\phi_\alpha(z_1) \otimes \phi_{21}(z_2)\big) \otimes \phi_{21}(z_3) = \phi_\alpha(z_1) \otimes \big(\phi_{21}(z_2) \otimes \phi_{21}(z_3)\big). \tag{67}$$

The left- and right-hand sides of (67) can be rewritten as

$$\big([\phi_{\alpha-\alpha_+}] \oplus [\phi_{\alpha+\alpha_+}]\big) \otimes \phi_{21} = [\phi_{\alpha-2\alpha_+}] \oplus [\phi_\alpha] \oplus [\phi_\alpha] \oplus [\phi_{\alpha+2\alpha_+}] \tag{68}$$

and

$$\phi_\alpha \otimes \big([\phi_{11}] \oplus [\phi_{31}]\big) = [\phi_\alpha] \oplus \big(\phi_\alpha \otimes [\phi_{31}]\big), \tag{69}$$

respectively, suppressing z dependence. Comparing (68) and (69), we conclude:

$$\phi_\alpha \otimes \phi_{31} = [\phi_{\alpha-2\alpha_+}] \oplus [\phi_\alpha] \oplus [\phi_{\alpha+2\alpha_+}]. \tag{70}$$

In this manner we have recovered the unintersected fusion rules for multiplication by ϕ_{31}, albeit with the crucial added knowledge that all three conformal families actually appear with nonzero coefficients. We can conclude that the roots of the cubic equation implied by $\hat{\mathcal{D}}_{31}^{(2)}$ must coincide with the values on the right-hand side of (70), else at least one of these three families would have to be absent (a contradiction). This establishes the claim for $\hat{\mathcal{D}}_{31}^{(2)}$. In a similar way, the induction process can be easily carried through for all $\hat{\mathcal{D}}_{pq}^{(2)}$, $p, q \geq 1$. By continuity, Claim 5 will then hold for *all* values of c and $\Delta(\alpha)$, not just the dense values assumed in the proof.

Proof of Claim 7. We begin by recalling[1] how one determines the coefficients a_i in the nullvector (23). For all $m \geq 1$ one requires

$$
\begin{aligned}
0 &= L_m D_{p_iq_i} \left|\phi_{p_iq_i}\right\rangle = \left[L_m, D_{p_iq_i}\right] \left|\phi_{p_iq_i}\right\rangle \\
&= \left[L_m, a_1 (L_{-1})^{p_iq_i} + a_2 (L_{-1})^{p_iq_i-2} L_{-2} + \cdots\right] \left|\phi_{p_iq_i}\right\rangle ,
\end{aligned}
\tag{71}
$$

where the L_m's satisfy the Virasoro algebra

$$
[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{72}
$$

(In practice, one need only check (71) for $m = 1$ and 2, since it will then necessarily be satisfied for $L_3 = [L_2, L_1]$, *etc.*) It is convenient for our purposes to rephrase the condition (71) slightly, as follows:

$$
\forall\, m \geq 1, \quad [L_m, D_{p_iq_i}] \left|\phi\right\rangle = 0 \quad \text{whenever} \quad L_0 \left|\phi\right\rangle = \Delta_{p_iq_i} \left|\phi\right\rangle . \tag{73}
$$

Now consider the first-order partial-differential operators $\hat{\mathcal{L}}_{-k}^{(i)}$, which make up the BPZ equations $\hat{\mathcal{D}}_{p_iq_i}^{(i)}$ in exactly the same way that the L_{-k}'s make up $D_{p_iq_i}$. We can extend their definition [Eq. (25)] to include the values $k \leq 0$ in addition to the usual case $k > 0$. Like the L_m's, the $\hat{\mathcal{L}}_m^{(i)}$'s can be shown to satisfy the Virasoro algebra, albeit with $c = 0$:

$$
[\hat{\mathcal{L}}_m^{(i)}, \hat{\mathcal{L}}_n^{(i)}] = (m-n)\hat{\mathcal{L}}_{m+n}^{(i)}. \tag{74}
$$

The key observation is that the maximal subalgebras $\{L_m : m \leq 1\}$ and $\{\hat{\mathcal{L}}_m^{(i)} : m \leq 1\}$ satisfy the *same* commutation relations as one another, since the central term in (72) never enters. This equivalence allows us to conclude from (73):

$$[\hat{\mathcal{L}}_1^{(i)}, \hat{\mathcal{D}}_{p_i q_i}^{(i)}]\, G(z_1, \cdots, z_n) = 0 \quad \text{whenever} \quad \hat{\mathcal{L}}_0^{(i)} G = \Delta_{p_i q_i} G. \tag{75}$$

(In contrast to (73), this will *not* be true for $\hat{\mathcal{L}}_m^{(i)}$ when $m > 1$.)

To prove our claim, we note that

$$\hat{\mathcal{L}}_0^{(i)} = -\hat{L}_0 + z_i \hat{L}_{-1} + \Delta_{p_i q_i}$$

and

$$\hat{\mathcal{L}}_1^{(i)} = -\hat{L}_1 + 2 z_i \hat{L}_0 - z_i^2 \hat{L}_{-1}.$$

We also need the commutation relations

$$\big[\hat{L}_{-1}, \hat{\mathcal{D}}_{p_i q_i}^{(i)}\big] = 0, \quad \big[\hat{L}_0, \hat{\mathcal{D}}_{p_i q_i}^{(i)}\big] = -p_i q_i\, \hat{\mathcal{D}}_{p_i q_i}^{(i)}, \quad \big[z_i, \hat{\mathcal{D}}_{p_i q_i}^{(i)}\big] = 0, \tag{76}$$

the latter following from the definition (25). Now suppose that G is translation ($\hat{L}_{-1}$) invariant. In that case, the condition $\hat{\mathcal{L}}_0^{(i)} G = \Delta_{p_i q_i} G$ invoked in (75) is equivalent to requiring $\hat{L}_0 G = 0$, *i.e.*, dilatation invariance. Equations (75) and (76) then imply

$$0 = \big[\hat{\mathcal{L}}_1^{(i)}, \hat{\mathcal{D}}_{p_i q_i}^{(i)}\big] G = \big[-\hat{L}_1, \hat{\mathcal{D}}_{p_i q_i}^{(i)}\big] G - 2 z_i p_i q_i\, \hat{\mathcal{D}}_{p_i q_i}^{(i)} G,$$

proving the claim.

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