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QUANTUM BEAMSTRAHLUNG FROM RIBBON PULSES*

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ABSTRACT

In this paper the high energy expansion introduced in our previous paper is used to describe to the problem of beamstrahlung due to an extended pulse with an elliptical cross section of arbitrary eccentricity. We show that the transverse geometry of the pulse enters in a remarkably simple scaling manner. This case is of interest because the radiative energy loss can be markedly reduced while simultaneously keeping a fixed luminosity if the beam pulses are very thin in one transverse direction, i.e., shaped like ribbons. Effects of other types of beam shaping are briefly discussed and the physics of the process is emphasized.

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1. Introduction and Review

An important parameter in the design of very high energy electron colliders is the fractional energy loss due to bremsstrahlung as one beam pulse passes through the other pulse.[1] Himel and Siegrist[2] treated this process, termed beamstrahlung,by adapting a quantum treatment of synchrotron radiation by an electron in a uniform magnetic field.[3–6] This adaptation necessarily involved several assumptions, in particular the approximation of the effects of the pulse by a uniform magnetic field in which the electron was orbiting as it radiated.[7] In fact, the electron sees the rapidly approaching pulse in the collider frame of reference as transverse, mutually orthogonal electric and magnetic fields of equal strengths whose spatial dependence is determined by the distribution of charges in the pulse.

A more general and physically transparent approach was developed in a previous paper and applied to extended pulses of circular cross-section.[8] The reader is referred there for a physical interpretation and an introduction to our notation.[9] *An extended target in this context is defined as having a length much longer than the smaller of the transverse or longitudinal coherence lengths described in Ref. 8.* The first one, the transverse coherence length $\ell_\perp$ ($= \ell_{coh}$ in Ref. 8) is defined as the length of path of the electron necessary to acquire a transverse momentum of $\sim m$ from the electric field. Since the width of the photon radiation pattern is also $\sim m$, the radiation can be coherent only from this length of the curving electron path. The second length, the longitudinal coherence length ℓ_z ($= \ell_{rad}$ in Ref. 8), is defined as the length of the target that the electron coherently scatters from during the radiation process. According to the uncertainty principle, it is the reciprocal of the minimum longitudinal momentum transferred to the target.

In Ref. 8 , it was shown that there are two scaling variables that completely characterize the behavior of beamstrahlung all the way from the classical to the extreme quantum limit. In this latter limit we confirmed the results of Himel and Siegrist.[2] Their limiting behavior has also been obtained recently by Jacob and Wu[10] using a similar approach as found in ref. 8 .

It is the purpose of this paper to extend the previous results to the study of more general pulse geometries.[11] As in the previous paper, only small disruption collisions will be treated analytically, although we will also comment on effects of large disruptions. Also, in this paper, we consider the process in which only one photon is radiated. In addition, we note a numerically small correction to our previous result for Dirac electrons which arises from our incorrect treatment of helicity flip processes, as pointed out by M. Bell and J. Bell.[12]

It has been suggested that the beamstrahlung loss can be reduced by forming pulses that are ribbon-shaped- i.e., ellipses of very large aspect ratio. Compared to pulses with an equal (but circular) cross section and containing the same number of electrons (positrons), the ribbon shape gives rise to weaker electromagnetic field strengths. Thus for collider operation at equal luminosity, there will be less energy loss to beamstrahlung if the pulses are ribbon shaped. Alternatively one can operate the collider at higher luminosity with no increase in the magnitude of the energy loss due to beamstrahlung .

This, of course, is well known and can be easily calculated for the classical domain. The object here is to study this effect in the quantum domain appropriate for higher energy colliders that are being considered beyond the SLC. It will be shown that the transverse geometry of the pulse, namely its aspect ratio, enters in the formulae for the spectrum and the fractional energy loss in a very simple manner. A simple scaling of the results given in our previous study allows one to completely summarize its effects.

We also present an analysis of the (numerically small) effect of smoothly shaping the front and back ends of the beam pulse. This permits a comparison with recent estimates of M. Jacob and T. T. Wu[13] of the energy radiated before and after the electron (positron) crosses a beam pulse and with an analysis of a gaussian shaped beam pulse by P. Chen.[14] Finally we comment on the effects of a smooth shaping of the sides of the beam pulse.

One interesting way to think about beamstrahlung is to view it as giving rise to a momentum spectrum in the incident particles. In other words, the incident particle is 'dressed' by the external field of the other pulse and it thereby acquires a structure function. This momentum spread is analogous to that used to describe the constituents in a hadron beam. However, in the electron case under discussion, it is possible to *adjust* this distribution through pulse shaping. This additional feature may prove useful in some experimental contexts.

Review:

In our previous paper[8] we found that there were two dimensionless variables that provided a convenient parametrization of the beamstrahlung process. The first scaling variable, y, is purely classical and is proportional to the square root of the luminosity per pulse,

$$y \equiv \frac{N\alpha}{mB} = \frac{\alpha}{m}\sqrt{\pi\,\mathcal{L}}\ . \tag{1.1}$$

The full luminosity is given by

$$Luminosity \equiv \mathcal{L} \times f = \frac{N^2}{A}\, f\ , \tag{1.2}$$

where N is the number of electrons (positrons) per pulse, which are assumed to be identical; f is the number of pulses per second; $\mathcal{L}$ is equal to the luminosity per pulse; and A is the cross sectional area of the pulses which can be written in terms of an equivalent cylinder radius B defined by $A = \pi B^2$.

The second scaling variable is inversely proportional to $\hbar$:

$$C = \frac{1}{4\gamma y} \left(\frac{\ell_0}{\hbar/mc} \right) , \tag{1.3}$$

where ℓ_0 is the length of the pulse and γmc^2 is the particle energy in the lab (CM) frame. Henceforth we will set $\hbar = c = 1$. C is a scaling variable in terms of which the fractional energy loss, δ, can be expressed for a wide range of physical parameters assuming only that the approximation of small disruption collisions is valid.

As described in Ref. 8 , we found for uniformly charged colliding pulses with cylindrical geometry that

$$\delta = \frac{2}{3} \alpha y \frac{F(C)}{C} = \delta_{classical} F(C) , \tag{1.4}$$

where the form factor $F(C) \to 1$ in the classical limit ($\hbar \to 0$; $C \to \infty$) and $F(C) \propto C^{4/3}$ for $C \to 0$ in the extreme quantum limit. The physical interpretation of C is the ratio of the transverse to the longitudinal coherence length. Characteristic values of the relevant parameters are shown in Table I for the SLC design and two notional extensions to higher energies. The notation will be defined fully in the text. It is characteristic of linear colliders (as compared with colliding ring designs) that the variable y assumes large values since, for constant luminosity, $\mathcal{L}$ must be large to compensate for small values of the pulse rate f. It follows that beamstrahlung is also much more important for linear colliders.

TABLE I—Circular Beam—G=1

	"SLC"	"SUPER"	" $\sqrt{\text{SUPER}}$"
N	5×10^{10}	3×10^{8}	10^{10}
γ	10^{5}	10^{7}	6.5×10^{5}
B	$10^{-4} cm$	$5 \times 10^{-8} cm$	$7 \times 10^{-6} cm$
ℓ_0	$10^{-1} cm$	$3 \times 10^{-5} cm$	$6 \times 10^{-2} cm$
L	$10^{+4} cm$	300 cm	$4 \times 10^{4} cm$
$y \equiv N\alpha/mb$	140	1.7×10^{3}	400
$\mathcal{L}$	$0.8 \times 10^{29} cm^{-2}$	$10^{31} cm^{-2}$	$0.6 \times 10^{30} cm^{-2}$
$C = m\ell_0/4\gamma y$	46	10^{-5}	1.5
$\ell_\perp \equiv L/2y$	35 cm	0.08 cm	50 cm
$\ell_{gr} \equiv L/N$	$2 \times 10^{-7} cm$ $\ll B$ dense	$10^{-6} cm$ $> B$ dilute	$4 \times 10^{-6} cm$ $\sim \frac{1}{2} B$
$\ell_z = 4\gamma^2/m$	1 cm $\ll \ell_\perp$	$10^{4} cm$ $\gg \ell_\perp$	65 cm $\sim \ell_\perp$
δ_{class}	0.015	$\sim 10^{6}$	1.3
$F(C)$	0.91	2×10^{-7}	0.30
δ	0.014	0.15	0.39

Classical Calculation for Elliptic Pulses:

We review first the classical beamstrahlung calculation in order to illustrate the effect of shaping the pulse with an elliptical rather than a circular cross section. As in Ref. 8 we work in the rest frame of the pulse which has the very long length $L = \ell_0 \gamma$. For a given area A and number of charges per unit length, N/L, the electrostatic field strengths are reduced by distorting the circle to an ellipse and thus so are the electron's acceleration and the resultant energy loss to radiation.

Neglecting end effects, the interior potential at the point b_x, b_y due to a uniformly charged cylindrical pulse of length L, and with an elliptical cross-section characterized by semi-major and semi-minor axes a_x and a_y can be written[15]

$$V(z, \vec{b}_\perp) = V_1 \left(\frac{b_x^2}{a_x} + \frac{b_y^2}{a_y} \right) \Theta(1 - \rho^2) \qquad V_1 = \frac{2N\alpha}{L(a_x + a_y)} \tag{1.5}$$

for $0 < z < L$, where

$$\rho \equiv \frac{b_x^2}{a_x^2} + \frac{b_y^2}{a_y^2} \ . \tag{1.6}$$

The cross-sectional area is given by $A = \pi a_x a_y$ and thus the equivalent radius of the circular cylinder is given by $B^2 = a_x a_y$.

The two components of the transverse electric field are given by

$$E_x = -2V_1 \frac{b_x}{a_x} \qquad E_y = -2V_1 \frac{b_y}{a_y} \tag{1.7}$$

from which we deduce two important results:

1. On any interior elliptical surface - i.e., $\rho =$ constant, the *magnitude* of the electric field is also constant,

$$(E_x^2 + E_y^2)_{ellipse} = 4 V_1^2 \rho^2 \quad . \tag{1.8}$$

2. Relative to a cylindrical pulse of the same linear charge density, N/L, and cross sectional area A, the field strength at the surface is reduced by the constant factor

$$\frac{1}{G} = \frac{2\sqrt{a_x a_y}}{a_x + a_y} \leq 1 \; . \tag{1.9}$$

This can be seen by comparing (1.8) with the corresponding fields for a cylinder of cross sectional area $A = \pi B^2$:

$$\begin{aligned} \vec{E}_{cyl} &= -\frac{2N\alpha}{LB^2}\vec{b} \\ (\vec{E}^2)_{cyl} &= \frac{4\pi N^2\alpha^2}{L^2 A}\left(\frac{\vec{b}}{B}\right)^2 \; , \end{aligned} \tag{1.10}$$

whereas for the ellipse with area A

$$(\vec{E}^2)_{ellipse} = \frac{4\pi N^2\alpha^2}{L^2 A}\frac{\rho^2}{G^2} \; . \tag{1.11}$$

In the region where the classical calculation is a valid approximation, the radiated energy is proportional to the square of the transverse acceleration, and thereby to the square of the corresponding field strengths. Denoting the fractional energy loss for a particle incident at an impact parameter (b_x, b_y) as

$\delta_{classical}(\overrightarrow{b}_{\perp})$, we obtain for small disruptions

$$\delta_{classical}(\overrightarrow{b}_{\perp}) = \frac{64}{3} \frac{\alpha^3 N^2 \gamma}{m^3 \ell_0 (a_x + a_y)^2} \left(\frac{b_x^2}{a_x^2} + \frac{b_y^2}{a_y^2} \right) . \tag{1.12}$$

Averaging over all impact parameters gives

$$\left\langle \frac{b_x^2}{a_x^2} \right\rangle_{ellipse} = \left\langle \frac{b_y^2}{a_y^2} \right\rangle_{ellipse} = \frac{1}{4} \tag{1.13}$$

so that these two contributions are equal independent of eccentricity. Thus, introducing (1.9) ,

$$\begin{aligned} \delta_{classical} &\equiv \left\langle \delta_{classical}(\overrightarrow{b}_{\perp}) \right\rangle \\ &= \frac{8}{3} \frac{\alpha^3 N^2 \gamma}{m^3 \ell_0 B^2} \frac{1}{G^2} = \delta^{cyl}_{classical} \frac{1}{G^2} \\ &= \frac{2}{3} \frac{y\alpha}{C} \frac{1}{G^2} . \end{aligned} \tag{1.14}$$

We find therefore that the classical fractional energy loss scales as $1/G^2$:

$$\delta^{ellipse}_{classical} = \frac{1}{G^2} \delta^{cyl}_{classical} . \tag{1.15}$$

Quantum Calculation:

As we saw in (1.11) , at the surface of the elliptical pulse, the magnitude of the field is constant but it is reduced by the factor $1/G$ relative to a circular pulse. In the general quantum case it is then not too surprising in view of the above behavior of the fields, that we find that δ is also reduced in this case and can be expressed by

$$\delta^{ellipse} = \delta^{ellipse}_{classical} F(CG) = \frac{2\alpha^2}{3m} \frac{\sqrt{\pi}\mathcal{L}}{G} \left[\frac{F(CG)}{CG} \right] = \frac{2}{3} \frac{\alpha y}{G} \frac{F(CG)}{(CG)} \ . \quad (1.16)$$

The differential spectrum similarly scales as

$$\frac{d\delta}{dx} = \frac{4\alpha y}{3G} \left[\ x \ R(u,x) \ \right] \ , \quad (1.17)$$

where the variable u depends only on the combination CG and x:

$$u^3 \equiv C^2 G^2 \left(\frac{1-x}{x} \right)^2 \ . \quad (1.18)$$

The function $R(u,x)$ will be given explicitly later. As found in Ref. 8 it peaks near the value $u \sim 1/2$ indicating that predominantly soft photons $(1 - x \to 0)$ are radiated in the classical region and hard ones $(x \to 0)$ in the quantum region. Note also that since $G > 1$, one moves closer to the classical domain for ribbon geometries. We turn now to a formal derivation of the new scaling law (1.16) which is the central result of this paper.

2. Formalism- The Phase

We shall repeat the derivation given in ref. 8 by calculating the emission of a photon during the scattering of a spin-zero electron from an elliptical pulse of N positrons. As formulated in ref. 8 and as corrected by Bell and Bell,[12] a simple correction extends this result to Dirac electrons. The general form of the matrix element of interest is

$$M = \left\langle \phi_f^{(-)} \middle| \vec{A} \cdot \vec{J} \middle| \phi_i^{(+)} \right\rangle , \tag{2.1}$$

where A is the photon field, $\vec{J}$ is the electron current and $\phi_f^{(-)}$ and $\phi_i^{(+)}$ are respectively the final (incoming) and initial (outgoing) scattering eigenstates of the electron in the static external field of the pulse. The calculation will be carried out in the rest frame of the pulse, and in this frame, the incident electron energy p is given by

$$p = 2m\gamma^2 . \tag{2.2}$$

and for generality, the incident momentum will be assumed to have a finite transverse component.

The solution to the Klein-Gordon equation in this frame will be written in the form

$$\phi(r) = \exp(i\Phi(r)) , \tag{2.3}$$

where the phase function Φ satisfies the equation

$$(E-V)^2 - m^2 = [\vec{\nabla}\Phi(r)]^2 - i[\vec{\nabla}^2\Phi(r)] . \tag{2.4}$$

As before, we must keep corrections of order $(1/p^2)$ relative to the leading term $\vec{p} \cdot \vec{r}$ in the phase.

Following each of the steps in Section 3 of Ref. 8 we obtain for the total phase of the matrix element (2.1)

$$\begin{aligned}\Phi_{tot} &= \Phi_i - \Phi_f - \vec{k} \cdot \vec{r} \\ &= -\vec{q} \cdot \vec{r} - \chi_0^{tot}(\vec{b}_\perp) - \frac{1}{p}\left[\chi_1^{tot}(z, \vec{b}_\perp) + i\chi_2^{tot}(z, \vec{b}_\perp)\right] ,\end{aligned} \tag{2.5}$$

where the zeroth order term is independent of z

$$\chi_0^{tot}(\vec{b}_\perp) = \int_{-\infty}^{\infty} dz' V(z', \vec{b}_\perp) , \tag{2.6}$$

while the first order terms still retain some z-dependence:

$$\chi_1^{tot}(z, \vec{b}_\perp) = \chi_1 + \frac{\tau_1}{x} . \tag{2.7}$$

The imaginary term χ_2^{tot} can be dropped, as was shown in Ref. 8 . The momentum fraction for the final state electron was introduced as $x = p^f/p^i$. The same approximations utilized in Ref. 8 apply here. The only new element is to evaluate χ_0^{tot}, χ_1 , and τ_1 for the potential (1.5) , corresponding to an elliptical charge distribution.

It is a simple matter to calculate that

$$\chi_0(z, \vec{b}_\perp) = V_1\left(\frac{b_x^2}{a_x} + \frac{b_y^2}{a_y}\right) z , \tag{2.8}$$

and

$$\chi_1(z, \vec{b}_\perp) = \frac{2}{3} V_1^2 \left(\frac{b_x^2}{a_x^2} + \frac{b_y^2}{a_y^2}\right) z^3 - V_1\left(\frac{b_x p_x^i}{a_x} + \frac{b_y p_y^i}{a_y}\right) z^2 . \tag{2.9}$$

For the final state with incoming wave boundary conditions, one finds

$$\tau_0(z, \vec{b}_\perp) = V_1 (\frac{b_x^2}{a_x} + \frac{b_y^2}{a_y}) (L - z) \ , \tag{2.10}$$

and

$$\tau_1(z, \vec{b}_\perp) = \frac{2}{3} V_1^2 (\frac{b_x^2}{a_x^2} + \frac{b_y^2}{a_y^2}) (L - z)^3 + V_1 (\frac{b_x p_x^f}{a_x} + \frac{b_y p_y^f}{a_y}) (L - z)^2 \ . \tag{2.11}$$

The elements of the *total phase* of the matrix element for the ribbon pulse follow directly. The leading order term is familiar,

$$\chi_0^{tot}(\vec{b}_\perp) = V_1 L (\frac{b_x^2}{a_x} + \frac{b_y^2}{a_y}) \ . \tag{2.12}$$

Recalling that the momentum fraction for the final state is $p_f \equiv x p_i$, and defining $p \equiv p_i$, the first order terms can be written

$$\begin{aligned} \chi_1^{tot}(z, \vec{b}_\perp) &= \frac{2}{3} V_1^2 (\frac{b_x^2}{a_x} + \frac{b_y^2}{a_y}) \left[z^3 + \frac{1}{x}(L - z)^3 \right] \\ &\quad + V_1 \left(\frac{b_x}{a_x} \xi_x(z) + \frac{b_y}{a_y} \xi_y(z) \right) , \end{aligned} \tag{2.13}$$

where for convenience, we now introduce the quantities

$$\begin{aligned} \eta(a, z) &= 1 + \frac{2V_1}{3pL\,a} \left[z^3 + \frac{1}{x} (L - z)^3 \right] \\ \vec{\xi}_\perp(z) &= -\vec{p}_\perp^{\,i} z^2 + \vec{p}_\perp^{\,f} \frac{(L - z)^2}{x} \ . \end{aligned} \tag{2.14}$$

The total phase can be rewritten in a more useful form as

$$\Phi_{tot} = -q_z z + \Phi_x + \Phi_y \tag{2.15}$$

where

$$-\Phi_x = q_x b_x + V_1 L \frac{b_x^2}{a_x} \eta(a_x, z) + V_1 \frac{b_x}{p a_x} \xi_x(z) \tag{2.16}$$

$$-\Phi_y = q_y b_y + V_1 L \frac{b_y^2}{a_y} \eta(a_y, z) + V_1 \frac{b_y}{p a_y} \xi_y(z) \quad . \tag{2.17}$$

For further manipulations, recall that

$$\begin{aligned} -q_z &= \frac{m^2 + \overrightarrow{p_\perp^f}^2}{2p^f} + \frac{\overrightarrow{k_\perp}^2}{2k} - \frac{m^2 + \overrightarrow{p_\perp^i}^2}{2p^i} \\ &= \frac{m^2(1-x)}{2xp} + \frac{k_\perp^2}{2(1-x)p} + \frac{\overrightarrow{p_\perp^f}^2}{2xp} - \frac{\overrightarrow{p_\perp^i}^2}{2p} \ , \end{aligned} \tag{2.18}$$

with $\overrightarrow{p_\perp^f} = \overrightarrow{q_\perp} - \overrightarrow{k_\perp} + \overrightarrow{p_\perp^i}$.

In ref. 8 , an analysis was given as to which particular corrections in $1/p$ actually were small corrections to the leading order and which were necessary to retain because the length of the target promoted them to leading order. We will make use of the results of this analysis in the following discussion.

3. Matrix Element- Stationary Phase

Neglecting certain normalization factors for the moment, the matrix element now achieves the form

$$M = \frac{e}{\pi} \int_0^L dz \int^A d^2b \;\; \vec{\epsilon} \cdot \vec{P}(z, \vec{b}_\perp) \;\; \exp[i\Phi_{tot}(z, \vec{b}_\perp)] \;\; , \tag{3.1}$$

where the factor $\vec{P}(z, \vec{b}_\perp)$ is the gauge invariant (to the order of this calculation in $1/p$) average of the initial and final momentum

$$\vec{P}(z, \vec{b}_\perp) = \frac{1}{2}\left[\vec{p}_i(loc; z, \vec{b}_\perp) + \vec{p}_f(loc; z, \vec{b}_\perp)\right] . \tag{3.2}$$

In component form, and to leading order in $1/p$, this is

$$\begin{aligned} P_z(z, \vec{b}_\perp) &= \frac{(1+x)}{2}p + \dots \\ P_x(z, \vec{b}_\perp) &= p_x^i - \frac{1}{2}k_x + \frac{1}{2}q_x - V_1 L\frac{b_x}{a_x}(1 - 2\frac{z}{L}) \\ P_y(z, \vec{b}_\perp) &= p_y^i - \frac{1}{2}k_y + \frac{1}{2}q_y - V_1 L\frac{b_y}{a_y}(1 - 2\frac{z}{L}) \end{aligned} \tag{3.3}$$

Gauge invariance ensures that we only need to know $\vec{P}$ to the above accuracy to compute the cross section to leading order.

The phase Φ_{tot} is at most quadratic in the transverse coordinates. The coefficients of the quadratic terms are very large hence the method of stationary phase will be used to evaluate the d^2b integral; to that end, introduce the stationary value of the coordinate $\vec{b}_\perp$ as (b_x^0, b_y^0).

The value of $\overrightarrow{b}_\perp$ at which the derivative of the phase vanishes is

$$2V_1 L\,\eta(a_x, z)\,\frac{b_x^0}{a_x} = -\left[q_x + \frac{V_1}{pa_x}\,\xi_x(z)\right] , \tag{3.4}$$

and

$$2V_1 L\,\eta(a_y, z)\,\frac{b_y^0}{a_y} = -\left[q_y + \frac{V_1}{pa_y}\,\xi_y(z)\right] . \tag{3.5}$$

The factors of η induce a z-dependence in both b_x^0 and b_y^0. These relations state, for example, that if the $x-$component of the momentum transfer to the pulse is fixed at q_x, and if the final electron momentum is to be p_x, then the b_x-coordinate of the electron orbit must have the value $b_x^0(z)$ where z is the point of emission of the photon. The term proportional to $p_\perp$ also induces a $z-$dependence which reflects the abrupt change in the trajectory as the photon is emitted. These properties reflect the curved classical trajectory and is the quantum source of the disruption parameter. Note that since the transverse coordinates are limited, because they must remain in the (elliptical) charge distribution to be counted in δ, the final transverse momenta ($q_\perp$ and $p_\perp$) cannot take on arbitrary values (otherwise the stationary point does not exist).

Expanding b_x and b_y around their stationary values, i.e. $\delta b_x \equiv b_x - b_x^0$ and $\delta b_y \equiv b_y - b_y^0$, allows the phase terms to be written as

$$\begin{aligned} \Phi_x(z, \overrightarrow{b}_\perp) &= \Phi_x(z, b_x^0) - \frac{V_1 L}{a_x}\eta(a_x, z)\delta b_x^2 \\ \Phi_y(z, \overrightarrow{b}_\perp) &= \Phi_y(z, b_y^0) - \frac{V_1 L}{a_y}\eta(a_y, z)\delta b_y^2 , \end{aligned} \tag{3.6}$$

To leading orders in $1/p$, the phase at the stationary point is

$$\begin{aligned}\Phi_{tot}(z) &\equiv \Phi_{tot}(z, b_x^0(z), b_y^0(z)) \\ &= -q_z z + \frac{q_x^2 a_x}{4V_1 L\eta(a_x, z)} + \frac{q_x \xi_x(z)}{2pL\eta(a_x, z)} \\ &\quad + \frac{q_y^2 a_y}{4V_1 L\eta(a_y, z)} + \frac{q_y \xi_y(z)}{2pL\eta(a_y, z)} \quad .\end{aligned} \tag{3.7}$$

The δb—integrals can now be performed and we achieve

$$M = -i\,\frac{eB}{V_1 L} \int_0^L \frac{dz}{\sqrt{\eta(a_x, z)\,\eta(a_y, z)}} \quad \vec{\epsilon} \cdot \vec{P}(z) \quad \exp[i\Phi_{tot}(z)] \quad . \tag{3.8}$$

We have assumed that both $V_1 L a_x \gg 1$ and $V_1 L a_y \gg 1$ in carrying out these integrals, so that edge effects could be neglected.

The square of the matrix element, summed over photon polarizations, is

$$\sum_{pol} M^* M = \frac{\alpha B^2}{(V_1 L)^2} \int_0^L \frac{dz_1 dz_2}{E(z_1, z_2)}\, S \;\exp[i(\Phi_{tot}(z_1) - \Phi_{tot}(z_2))] \quad , \tag{3.9}$$

where the polarization sum has been written as

$$S(Boson) = \sum_{pol} \vec{\epsilon} \cdot \vec{P}(z_1) \times \vec{\epsilon} \cdot \vec{P}(z_2) \ , \tag{3.10}$$

and the factor

$$E(z_1, z_2) = \sqrt{\eta(a_x, z_1)\eta(a_y, z_1)\ \eta(a_x, z_2)\eta(a_y, z_2)} \tag{3.11}$$

can be set equal to one to the accuracy that we are working in $1/p$. The photon polarization sum proceeds as in Ref. 8 .

We can now anticipate a remarkable feature of the result of our calculation by noting that if we expand (3.7) and retain only the terms of order $(1/p)$ as required, all dependence on the axes of the ellipse, a_x and a_y survives *only* in the leading term of order unity, and that term has no z−dependence. Since this part of the phase will cancel in the exponent of (3.9) , there will be no explicit dependence on the axes in the cross section. The entire effect of the pulse geometry will appear only in the overall normalization of the cross section and in the limits of the final phase space integrals.

Properties of the Phase:

The properties of the matrix element are largely determined by the z−dependence of the real part of the phase of the matrix element. First, recall from (2.18) that

$$-q_z = \frac{m^2(1-x)}{2xp} + \frac{k_\perp^2}{2(1-x)p} + \frac{\vec{p_\perp^f}^2}{2xp} - \frac{\vec{p_\perp^i}^2}{2p} , \tag{3.12}$$

where $\vec{p_\perp^f} = \vec{q_\perp} - \vec{k}_\perp + \vec{p_\perp^i}$ and $k = (1-x)p$. Consider the derivative of the phase with respect to z:

$$\begin{aligned} \frac{d\Phi_{tot}(z)}{dz} &= -q_z - \frac{q_x^2 a_x}{4V_1 L\eta(a_x,z)^2}\frac{d\eta(a_x,z)}{dz} + \frac{q_x}{2pL\eta(a_x,z)}\frac{d\xi_x(z)}{dz} \\ &\quad - \frac{q_y^2 a_y}{4V_1 L\eta(a_y,z)^2}\frac{d\eta(a_y,z)}{dz} + \frac{q_y}{2pL\eta(a_y,z)}\frac{d\xi_y(z)}{dz} , \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \frac{d\eta(a,z)}{dz} &= \frac{2V_1}{pLa}\left[z^2 - \frac{1}{x}(L-z)^2\right] \\ \frac{d\vec{\xi}_\perp(z)}{dz} &= -2\left[\vec{p_\perp^i} z + \vec{p_\perp^f}\frac{(L-z)}{x}\right] , \end{aligned} \tag{3.14}$$

and terms of order p^{-2} have been neglected.

Utilizing all of the above, one finds after some reduction that

$$\frac{d\Phi_{tot}(z)}{dz} = \frac{1}{2x(1-x)p}\left[m^2(1-x)^2 + \left(\vec{k}_\perp - (1-x)[\vec{p^i_\perp} + \frac{z}{L}\vec{q_\perp}]\right)^2\right] . \tag{3.15}$$

Note that as remarked earlier, *all* explicit dependence on the parameters of the ellipse, namely a_x and a_y, have *cancelled* to this order in $1/p$. Finally, the phase itself is achieved by integration.

In the evaluation of the ***absolute square*** of the matrix element, the relevant total phase will be the difference of the above phase evaluated at different z-values. In ref. 8 it was pointed out that this phase difference has the remarkable property that it depends only on the difference of z-coordinates and a 'natural' photon transverse momentum variable that rotates as the particle traverses the pulse following the classical (curved) path:[16]

$$[\Phi_{tot}(z_1) - \Phi_{tot}(z_2)] = sw + \frac{1}{3}\, r^3 w^3 , \tag{3.16}$$

where

$$\begin{aligned} w &= \frac{(z_1 - z_2)}{L} \\ s &\equiv \frac{L}{2x(1-x)p}\left[m^2(1-x)^2 + (\vec{k}'_\perp)^2\right] \\ r^3 &\equiv \frac{L(1-x)}{8xp}\,\vec{q_\perp}^2 . \end{aligned} \tag{3.17}$$

The (rotating) photon transverse momentum dependence is given by

$$\vec{k}'_\perp = \vec{k}_\perp - (1-x)\left[\vec{p^i_\perp}\,(1-\frac{Z}{L}) + (\vec{p^f_\perp} + \vec{k}_\perp)\,(\frac{Z}{L})\right] , \tag{3.18}$$

where $Z = \frac{1}{2}(z_1 + z_2)$ is the (average) point of photon emission. This clearly shows that the angular distribution of the photon tracks the classical path of the particle through the pulse.

Now to leading order, the momentum transfer is (see (3.4) and (3.5))

$$q_\perp^2 = (2V_1L)^2 \left(\frac{(b_x^0)^2}{a_x^2} + \frac{(b_y^0)^2}{a_y^2} \right) \quad , \tag{3.19}$$

for particles incident on the pulse with impact parameter $(b_x^0\,, b_y^0\,)$. The maximum value of the momentum transfer is thus achieved by those particles that hit the edge of the charge distribution, $q_\perp^2\,(max) \equiv (2V_1L)^2$.

In order to estimate the magnitude of r and s, note that they can be written to leading order in the form

$$\begin{aligned} s &= 2\frac{y}{G} \left[\left(CG\frac{1-x}{x} \right) \frac{m^2(1-x)^2 + (\vec{k'_\perp})^2}{2m^2(1-x)^2} \right] \\ r^3 &= \left(\frac{y}{G} \right)^3 \left[\left(CG\frac{1-x}{x} \right) \frac{q_\perp^2}{q_\perp^2\,(max)} \right] \quad , \end{aligned} \tag{3.20}$$

where we have introduced the two scaling variables that characterize the fractional energy loss and the factor G that describes the geometry:

$$\begin{aligned} C &\equiv \frac{m^2 L}{2py} = \frac{m\,\ell_0}{4\gamma y} \\ y &\equiv \frac{N\alpha}{m\,B} = \frac{V_1 L G}{m} \\ G &\equiv \frac{a_x + a_y}{2B} = \frac{a_x + a_y}{2\sqrt{a_x a_y}} \quad . \end{aligned} \tag{3.21}$$

Thus we see from (3.16) and the above that the natural scale for the variable w is set by y/G. Since y/G is large for interesting machines, the oscillations of the phase 'chops' the beam pulse into smaller coherent segments as will become clear shortly.

4. Spectrum and Cross Section

Final State Sum:

The square of the matrix element summed over the polarization and integrated over the transverse momentum of the photon is defined as

$$\int M * M \equiv \int \frac{d^2 k_\perp}{(2\pi)^2} \sum_{pol} M^* M \ . \tag{4.1}$$

Collecting the results of the earlier sections, our next task is to evaluate

$$\int M * M = \frac{\alpha B^2}{((1-x)V_1)^2} \int \frac{d^2 k_\perp}{(2\pi)^2} \int_0^L \frac{dz_1 dz_2}{L^2} D[w^2] \exp\left[i \left(sw + \frac{1}{3} r^3 w^3 \right) \right] \ , \tag{4.2}$$

where

$$D(w^2) = \left[(\vec{k}'_\perp)^2 - w^2 \frac{1}{4} (1-x)^2 q_\perp^2 \right] \ . \tag{4.3}$$

Since the parameters r and s are large, both of order y/G by (3.20) , the integral can be manipulated as in Ref. 8 . The result is

$$\int M * M = 2\pi \frac{\alpha B^2}{((1-x)V_1)^2} \int \frac{d^2 k'_\perp}{(2\pi)^2} D\left[-\frac{s}{r^3} \right] \frac{1}{r} Ai\left[\frac{s}{r} \right] \ , \tag{4.4}$$

where the standard definition of the Airy function [17] has been used. From the definition of r and s, D simplifies to

$$D\left[-\frac{s}{r^3} \right] = m^2 (1-x)^2 + 2(k'_\perp)^2 \ . \tag{4.5}$$

Now what follows is a succession of variable changes introduced in Ref. 8 to make this integral tractable. First define

$$s = r v , \tag{4.6}$$

and using (3.17) , the integration over $d^2 k'_\perp$ can be replaced by an integral over v . Paying attention to the limits of integration, and introducing the value of v at $(k'_\perp)^2 = 0$,

$$v_0 \equiv \frac{m^2 L(1-x)}{2xpr} , \tag{4.7}$$

one finds

$$\int M * M = 2\alpha \left(\frac{xpB}{LV_1}\right)^2 r \int_{v_0}^{\infty} dv \; [2v - v_0] \, Ai(v) . \tag{4.8}$$

Now the variables left to integrate are $d^2 q_\perp$, and the photon energy $p(1-x)$. Since the variable r^3 is linear in $q_\perp^2$, and since $q_\perp^2$ has a maximum value of $(2V_1 L)^2$, r^3 will also have an upper limit of

$$(r_{max})^3 \equiv \frac{V_1^2 L^3 (1-x)}{2xp} = \left(\frac{y}{G}\right)^3 CG \, \frac{1-x}{x} . \tag{4.9}$$

At this point note that the integration over $q_\perp^2$ can be transformed into an integral over the (initial) impact parameter b by using (3.19) . This will be done later to give additional physical insight into beamstrahlung in the classical and quantum regimes.

If we write $r = r_{max} t$, where $0 < t < 1$, then $v_0 = u/t$, where u has been introduced by

$$u^3 = C^2 G^2 \left(\frac{1-x}{x}\right)^2 . \tag{4.10}$$

These transformations of variables lead to

$$\begin{aligned} q_\perp^2 &= (2V_1L)^2\, t^3 \\ d^2q_\perp &= \pi(2V_1L)^2\, 3\, t^2 dt \ , \end{aligned} \tag{4.11}$$

and the partial cross section for fixed photon energy fraction $(1-x)$ becomes

$$\int \frac{d^2q_\perp}{(2\pi)^2} \int M * M = J \int_0^1 dt\ 3t^3 \int_{u/t}^{\infty} dv \left[2v - \frac{u}{t}\right] Ai(v) \ , \tag{4.12}$$

with

$$J = \frac{2\alpha}{\pi} (xpB)^2\, r_{max} \ . \tag{4.13}$$

The integrations can be interchanged, the dt integration performed, and the result is

$$\int \frac{d^2q_\perp}{(2\pi)^2} \int M * M = J \int_u^{\infty} dv \left[\frac{3}{2}v - u - \frac{u^4}{2v^3}\right] Ai(v) \ . \tag{4.14}$$

Spin Effects:

In ref. 8 , it was claimed that the spinor factors appropriate for Dirac electrons would lead to an additional multiplicative factor of $T(x)$ in the differential cross section, where $T(x) = (1 + x^2)/2x$. This result was wrong as was shown by Bell and Bell.[12] The error arose in an incorrect estimate of the helicity flip amplitude; instead of being small and of order (m/p), these terms are of order $(k/p) = 1 - x$. For hard photons this correction could be important, especially in the quantum regime. However, this does not turn out to be the case. If these matrix elements are evaluated correctly using the formulas given in ref. 8 for the spin structure, one finds instead of (4.8) ,

$$\sum_{helicity} \int M * M = 2\alpha \left(\frac{xpB}{LV_1}\right)^2 r \int_{v_0}^{\infty} dv \; [2v\,T(x) - v_0]\, Ai(v) \; . \tag{4.15}$$

The integral over transverse momentum then yields

$$\int \frac{d^2 q_\perp}{(2\pi)^2} \sum_{helicity} \int M * M = J \int_u^{\infty} dv \; \left[\frac{3}{2} v\, T(x)(1 - \frac{u^4}{v^4}) - u(1 - \frac{u^3}{v^3})\right] Ai(v) \; . \tag{4.16}$$

Note for soft photons, where $T(x \to 1) \to 1$, this is the same as (4.14) . In the hard photon limit, $T(x \to 0) \gg 1$ so that the second term in the integrand of (4.16) , which is the helicity flip contribution, can be neglected. Thus in this limit too the correction can be neglected. Its small numerical effect will be computed in the next section.

5. Final Results and Scaling Laws

The differential cross section for beamstrahlung is achieved by dividing by the normalization factors for the initial and final electron and photon wave functions, $[p^3 x(1-x)]$; the fractional power spectrum by then multiplying by an extra factor of $(1-x)$ and dividing by πB^2 , together with trivial numerical factors.

The final result for the power spectrum can be written

$$\frac{d\delta}{dx} = [\ 2\ x\ D\ R(u,x)\ ]\ \delta_{classical} \quad , \tag{5.1}$$

where $D = CG$ is the new scaling variable and the spin-spectrum function $R(u,x)$, also in a scaling form, is defined as

$$R(u,x) = \frac{3}{2} u^{1/2} \int_u^\infty dv \left[\frac{3}{2} v\, T(x)(1-\frac{u^4}{v^4}) - u(1-\frac{u^3}{v^3}) \right] Ai(v) \tag{5.2}$$

with $u^3 \equiv D^2 \left(\frac{1-x}{x}\right)^2$ and $T(x) = (1+x^2)/2x$.

The form factor described earlier is easily computed from the above results. Recalling that $\delta_{classical} = \frac{2}{3}\alpha\frac{y}{GD}$, and using the definition

$$F(D) \equiv \frac{\delta}{\delta_{classical}} \tag{5.3}$$

with $x = x(D) = [1+u^{3/2}/D]^{-1}$, it follows that

$$F(D) = 2 \int_0^1 dx\, x\, D\, R(u,x)\ . \tag{5.4}$$

Explicitly, the form factor for electrons is

$$F(D) = 3 \int_0^\infty \frac{du\, u^{1/2}}{[1+\frac{u^{3/2}}{D}]^3} R(u,x)\ , \tag{5.5}$$

where $R(u,x)$ is given by (5.2) , with x expressed as a function of u and D by

(4.10) . The normalization can be checked as shown in Ref. 8 , by taking the limit of large D and interchanging the order of integration.

For completeness, we present here a useful numerical approximation to the form factor modified (from that given in ref. 8) to account correctly for the helicity flip contributions:

$$F(D) = \left(1 + \frac{1}{b_1}\left[D^{-4/3} + b_2\, D^{-2/3}\,(1 + 0.20D)^{-1/3}\right]\right)^{-1} , \qquad (5.6)$$

where $b_1 = 0.83..$,and $b_2 = 1.67....$ The values of b_1 and b_2 have been computed analytically, and the latter adjusted slightly to improve the fit. Note that the value of b_2 is different from the value 2.0 given in ref. 8 ; this is the effect of the helicity flip. Bell and Bell[12] show that this correction is a maximum relative to the nonflip term in the transition region, and is approximately 8 % at $D \sim 0.1$. As noted earlier, it vanishes for $D \to 0$ and $D \to \infty$. The curves given in ref. 8 are altered very little by this correction.

6. Pulse Shaping

Thus far we have studied beam pulses with uniform charge distributions and sharp edges in both transverse and longitudinal dimensions. The resulting electric potential and fields are given by (1.5) , (1.6) , and (1.7) .

It is a direct and useful consequence of these assumptions that the phase Φ_{tot} in the matrix element (2.15) , (2.16) , and (2.17) is a quadratic form in the impact parameter $\vec{b}_\perp$. This makes it a simple matter to apply the stationary phase method as used in Section 3. We shall now relax these assumptions in order to study more general pulse shapes. Our approach uses the concept of pulse 'slicing' that was introduced in ref. 8 . The property of slicing rests upon the fact that the scaling variable D depends on the charge per unit length while y is linear in the total charge (see below). We can thus calculate the energy loss by adding up contributions from individual slices of the pulse. These contributions add independently (and incoherently) to the probability of radiating during the beam traversal so long as the length of each slice is large compared to the shorter of the two coherence lengths $\ell_\perp (= \ell_{coh})$ and $\ell_z (= \ell_{rad})$. Our discussion in Section 3 showed that a finite initial transverse momentum had no effect on the predicted beamstrahlung ; therefore so long as the disruption induced while passing through a thin slice is small, the following argument can be carried through.

First consider a uniform pulse of area $A = \pi a_x a_y$, geometric factor G , length L, and total charge N. Divide this pulse into I identical sub-pulses, with the same area and G factor, but with length $l = L/I$, and charge $n = N/I$. These quantities determine the value of δ_i through the scaling variables y and D:

$$y = \frac{N\alpha}{m}\sqrt{\frac{\pi}{A}} \qquad\qquad D = \frac{m^2 L}{2py} G \, . \tag{6.1}$$

The full fractional energy loss is then

$$\delta = \sum_i^I \delta_i = I \times \delta_i \; . \tag{6.2}$$

Fix the aspect ratio so that G is constant, but allow the area, length and charge of the slices to change to the values A_i, l_i, and charge n_i for the i^{th} slice, The resultant fractional energy loss will be denoted by Δ_i, and the ratio to (6.1), is

$$r_i = \frac{\Delta_i}{\delta_i} = \frac{n_i^2}{n^2} \times \frac{l\,A}{l_i A_i} \times \frac{F[CG(\frac{n\,l_i}{n_i\,l}\sqrt{\frac{A_i}{A}})]}{F[CG]} \; . \tag{6.3}$$

This relation exposes the geometric behavior of beamstrahlung. Assuming that the changes are such that the incident beam fills each slice, the full fractional energy loss from the modified beam pulse is then given by

$$\Delta = \sum_i^I \Delta_i \; , \tag{6.4}$$

or in a more useful form,

$$R = \frac{\Delta}{\delta} = \frac{1}{I} \sum_i^I r_i \; . \tag{6.5}$$

We will now demonstrate the effects of longitudinal pulse shaping by discussing some simple examples.

Longitudinal Shaping:

Assume that the transverse dimensions and the sub-pulse lengths are fixed and uniform; then if the variation in the longitudinal direction is sufficiently slow (over the relevant coherence length), the sum over slices in R can be written as

$$R = \int_{-L/2}^{L/2} \frac{dz}{L}\, r(z) \,, \tag{6.6}$$

where the variation of the charge density along the pulse gives rise to the z-dependence of $r(z)$. Similarly we introduce $n_i \to n(z)dz$, where $n(z)$ is the charge per unit length, and $l_i = dz$ with $L = Idz$. For such slow variations the transverse electric field can be expressed in terms of the local charge density as in (1.7) . Longitudinal fields contribute negligibly for large γ.

In terms of these quantities we have

$$N = \int dz\, n(z) \quad \text{and} \quad \frac{1}{12} \mathrm{L}^2\mathrm{N} = \int \mathrm{dz\, z^2\, n(z)} - \left(\int \mathrm{dz\, z\, n(z)}\right)^2 \tag{6.7}$$

and finally

$$R = \int \frac{dz}{L} \left(\frac{n(z)}{n_0}\right)^2 \frac{F(D\frac{n_0}{n(z)})}{F(D)} \,, \tag{6.8}$$

where $n_0 = N/L$. In the classical and quantum regimes, $F(D) \sim 1$ and $\sim b_1 D^{4/3}$ respectively, so that we can write

$$\begin{aligned} R_{classical} &= (\frac{L}{N})^2 \int \frac{dz}{L} (n(z))^2 \\ R_{QM} &= (\frac{L}{N})^{2/3} \int \frac{dz}{L} (n(z))^{2/3} \,. \end{aligned} \tag{6.9}$$

To compare with results from a uniform pulse with sharp edges we apply these formula to two simple longitudinal distributions.

1. Gaussian Pulse: The form of a Gaussian pulse that satisfies (6.7) is

$$n(z) = n_g \exp(-z^2/s^2) , \tag{6.10}$$

where

$$N = n_g s \sqrt{\pi} \qquad \text{and} \qquad \mathrm{L}^2 = 6\,\mathrm{s}^2 . \tag{6.11}$$

The results for the ratio R in the two extremes are

$$\begin{aligned} R_{classical} &= \left(\frac{3}{\pi}\right)^{1/2} \sim 0.98... \\ R_{QM} &= \left(\frac{9\pi}{16}\right)^{1/6} \sim 1.10... \end{aligned} \tag{6.12}$$

2. Power-Law Pulse: This smooth pulse of total length $2s$ is defined by

$$n(z) = n_p \,(1 - (|z|/s)^q)\, \theta(s - |z|) , \tag{6.13}$$

and then by (6.7)

$$N = n_p\, s \frac{2q}{1+q} \qquad \text{and} \qquad \mathrm{L}^2 = \mathrm{s}^2 \frac{4(1+\mathrm{q})}{3+\mathrm{q}} . \tag{6.14}$$

The results for the ratio R in the two extreme regimes are simple to evaluate for any value of q.[18] For the linear case, $q = 1$, these are

$$\begin{aligned} R_{classical} &= \frac{2}{3}\,(2)^{1/2} \sim 0.94... \\ R_{QM} &= \frac{6}{5}\left(\frac{1}{2}\right)^{1/6} \sim 1.07... \end{aligned} \tag{6.15}$$

This behavior is qualitatively the same as that found for the Gaussian pulse. We thus establish that longitudinal pulse shaping leads to relatively small corrections to the fractional energy loss. Jacob and Wu[13] find a similar small effect from the transition region immediately before and after the pulse.

Impact Parameter Density:

From our discussion of the classical beamstrahlung problem, we note that the fractional energy loss depends quadratically upon the impact parameter, as is given explicitly in (1.12) . This directly reflects the square of the strength of the transverse electric field. This dependence is well hidden in the quantum formalism; one may well ask how it arises and what does it become in the extreme quantum limit.

This question can be explored directly by modifying the argument given in section 5 and interchanging the order of performing the final state sum. First note that from (4.11) and the stationary phase point given by (3.4) , (3.5) and (3.19) we have

$$\begin{aligned} t^3 &= \frac{q_\perp^2}{(2V_1L)^2} = \frac{q_\perp^2}{q_\perp^2(max)} \\ &= \left(\frac{b_x^2}{a_x^2} + \frac{b_y^2}{a_y^2}\right) . \end{aligned} \tag{6.16}$$

Now using (4.12) , (4.14) , and the definition of the form factor, one simply repeats the calculation but performs the t integral last. The result is

$$F(D) = \int d^2b_\perp \, \frac{dF(D, b_\perp^2)}{d^2b_\perp} , \tag{6.17}$$

where $D \equiv C\,G$ and the impact parameter density is given by

$$\frac{dF(D, b_\perp^2)}{d^2b_\perp} = \frac{9}{2\pi B^2} \int\limits_0^\infty \frac{du\, u}{[1 + \frac{u^{3/2}}{D}]^3} \int\limits_{u/t}^\infty dv\, [2\,v\,t - u]\, Ai(v) , \tag{6.18}$$

with t given in terms of the impact parameter by (6.16) .

The classical and extreme quantum limits are now easily evaluated. In the former case we have

$$\frac{dF(\infty, b_\perp^2)}{d^2 b_\perp} = \frac{2}{\pi B^2} \left(\frac{b_x^2}{a_x^2} + \frac{b_y^2}{a_y^2} \right) , \tag{6.19}$$

whereas for the latter, an evaluation of the integrals in the limit of $D \ll 1$ yields

$$\frac{dF(D, b_\perp^2)}{d^2 b_\perp} = \frac{F_0}{\pi B^2} \frac{4}{3} \left[D^4 \left(\frac{b_x^2}{a_x^2} + \frac{b_y^2}{a_y^2} \right) \right]^{1/3} , \tag{6.20}$$

with

$$\begin{aligned} F_0 &= \frac{27}{16\pi} 3^{1/6} \Gamma(\frac{2}{3}) \\ &= 0.87351... \end{aligned} \tag{6.21}$$

The above result is for scalar electrons; for the spin one-half case, $F_0 = 0.83..$ Note that in the extreme quantum limit, small values of the impact parameter contribute a relatively larger fraction of the total beamstrahlung than in the classical limit.

Transverse Shaping:

In the classical limit of beamstrahlung , it is straightforward to calculate the effect of shaping the pulse in its transverse dimensions. Consider first a pulse with a circular cross section and assume for illustrative purposes that the (normalized) charge density $d(b)$ is a gaussian distribution with mean square radius B^2 ,

$$d(b) = \frac{1}{\pi B^2} \exp[-b^2/B^2] , \tag{6.22}$$

which produces the electric field

$$\vec{E}(b) = \frac{2\vec{b}}{b^2} \left(1 - \exp[-b^2/B^2]\right) . \tag{6.23}$$

The fractional energy loss for an incident electron at impact parameter b is

$$\delta_{classical}(b) = \frac{4\lambda}{b^2} \left(1 - \exp[-b^2/B^2]\right)^2 , \tag{6.24}$$

where λ contains irrelevant parameters. Averaging over the distribution of impact parameters for an incident pulse of shape (6.22) gives

$$\delta_{classical} = \frac{4\lambda}{B^2} \ln(4/3) . \tag{6.25}$$

For comparison a sharp-edged pulse of the same mean square radius is described by a charge density

$$d_0(b) = \frac{1}{2\pi B^2} \theta(\sqrt{2}B - b) \tag{6.26}$$

which produces the electric field

$$\vec{E}(b) = \frac{1}{B^2} \vec{b} . \tag{6.27}$$

The corresponding fractional energy loss is, with the same parameter λ as in

(6.24) ,

$$\delta^0_{classical}(b) = \frac{\lambda}{B^4}\, b^2 \ , \tag{6.28}$$

which averaged over (6.26) gives

$$\delta^0_{classical} = \frac{\lambda}{B^2} \ . \tag{6.29}$$

Comparing (6.25) with (6.29) we see that the gaussian smoothing of the charge distribution increases the radiative loss by $\sim$ 15%. The effect of pulse shaping is presumably still smaller in the quantum limit since the small impact parameters contribute a larger fraction of the energy loss as we saw in the preceding section.

For a thin ribbon pulse, the effect of smoothing the charge density in the thin dimension is even smaller. In this situation, we compare the energy loss for two flat ribbons with charge distributions having the same mean square thickness:

$$d(b) = \frac{4}{\pi B}\,(1 + \frac{b_y^2}{B^2})^{-2} \qquad d_0(b) = \frac{1}{\sqrt{3}B}\,\theta(\sqrt{3}B - b_y) \ . \tag{6.30}$$

One finds that the radiative energy losses for finite impact parameter in the large, or x, direction. are equal to within 2% for these two cases. Finally we note in passing that the quantum calculation in Section 4 can also be carried out completely for a potential that contains both linear and quadratic terms in the impact parameter.

Large Disruption:

Large disruption collisions present a much more difficult calculational problem than small disruption ones because the interaction can no longer be described simply in terms of an electron scattering and radiating in a fixed static field. A large disruption analysis requires treating the mutually distorting interactions of both beams, which therefore becomes a complex many body problem.

The disruption, $\delta b/b$, is proportional to the length of a beam pulse as shown in Ref. 8 , eqn (2.26). We can, therefore, approach the study of large disruptions by slicing the pulse so that the small disruption formalism can be applied to each individual slice. In all examples of physical interest the electron trajectory makes a very small angle with respect to the direction of the pulse length, i.e., $\frac{db}{dz} \lesssim |eE|L/p \sim y/\gamma^2 \ll 1$ and the methods of Section 3 can be applied.

In particular, we see from the forms of (6.19) and (6.20) that the beamstrahlung depends quadratically on the ratio of the electron's impact parameter to the size of the bunch in the classical regime, but the power drops to 2/3 in the extreme quantum limit. This indicates that corrections due to large disruptions, which will alter the averages of these ratios, will be considerably smaller in the quantum than in the classical regime. In addition, if we consider ribbon-like elliptic pulses of large aspect ratio, it is only the thin direction of the pulse that is disrupted. This can be seen be integrating the equation of motion

$$p\,\overrightarrow{\ddot{b}} = e\,\overrightarrow{E}(b) \qquad (6.31)$$

under the assumption of small disruption. We find as in Eqns (2.24-26) of Ref. 8

$$\frac{\delta b_x}{b_x} \equiv \frac{b_x(L) - b_x(0)}{b_x(0)} = \frac{y\ell_0}{2\gamma G a_x}$$
$$\frac{\delta b_y}{b_y} \equiv \frac{b_y(L) - b_y(0)}{b_y(0)} = \frac{y\ell_0}{2\gamma G a_y} \, . \tag{6.32}$$

In the limit of very thin pulses with $a_x \ll a_x$, we can approximate G by

$$G \equiv \frac{a_x + a_y}{2\sqrt{a_x a_y}} \sim \sqrt{\frac{a_x}{a_y}} = \frac{B}{2a_y} \gg 1 \tag{6.33}$$

and

$$\frac{\delta b_x}{b_x} = \frac{y\ell_0}{\gamma B}\frac{a_y}{a_x} \qquad\qquad \frac{\delta b_y}{b_y} = \frac{y\ell_0}{\gamma B} \, . \tag{6.34}$$

This equation shows that the fractional disruption in the direction of the large axis of the ellipse is much smaller than in the thin direction. In fact for parameters of interest it can be neglected entirely.

7. Summary

At fixed luminosity per pulse $\mathcal{L}$, the scaling variable y is determined, since $y = \frac{\alpha}{m}\sqrt{\pi\mathcal{L}} = 495\sqrt{\mathcal{L}\times 10^{-30}cm^2}$. Thus the fractional energy loss becomes

$$\delta = \frac{2\alpha^2}{3m}\,\frac{\sqrt{\pi\mathcal{L}}}{G}\left[\frac{F(D)}{D}\right] , \tag{7.1}$$

where $D = CG = \frac{m\ell_0}{4\gamma y}G$. At fixed G, the maximum of δ occurs at the peak of the bracketed ratio, which occurs at $D \sim 0.20$ with a value $F(D)/D \sim 0.275$. This maximum is quite wide. Thus in order to minimize the fractional energy loss, one is forced far into the classical regime of large D or the quantum regime of small D.

The behavior of δ in the various regimes can be characterized by:

$$\begin{aligned}
&\delta(D \gg 1) \sim 2.4\sqrt{\mathcal{L}\times 10^{-30}cm^2}\left(\frac{1}{CG^2}\right) && \text{(classical limit)}\\
&\delta(D \sim 0.2) \sim 0.66\sqrt{\mathcal{L}\times 10^{-30}cm^2}\left(\frac{1}{G}\right) && \text{(peak)}\\
&\delta(D \ll 1) \sim 2.0\sqrt{\mathcal{L}\times 10^{-30}cm^2}\left(\frac{C}{G^2}\right)^{1/3} && \text{(quantum limit)} .
\end{aligned} \tag{7.2}$$

The differential spectra also has a scaling form,

$$\frac{d\delta}{dx} = \frac{4\alpy}{3G}\ [\ x\ R(u,x)\ ]\ , \tag{7.3}$$

where u depends only on the combination $D = CG$, and of course x through (4.10) .

TABLE II—Elliptical Shaping

	$G=1$			$G=5$		
	C	$\delta_{\text{classical}}$	δ	$D=CG$	$\delta_{\text{classical}}$	δ
SLC	46	0.015	0.014	230	0.6×10^{-3}	0.6×10^{-3}
SUPER	10^{-5}	10^{6}	0.15	5×10^{-5}	3×10^{4}	0.05
$\sqrt{\text{SUPER}}$	1.5	1.3	0.39	7.5	0.05	0.03

As an illustration of the effects of pulse shaping, consider the examples shown in Table II which compares a circular pulse with an elliptical one with aspect ratio $a_x/a_y = 100$. All the other pulse parameters, including $\mathcal{L}$, are the same as given in Table I. The effect of adopting a transverse ribbon shape is quite dramatic and the reduction of beamstrahlung in all the regimes is sizeable. The fact that such effects can be completely described by our scaling laws in terms of the quantities D, G and y is an unexpected simplification. Furthermore, tapering of the front and back ends and of the sides of the beam pulse was shown to have a small effect on these calculations. We note in conclusion that possible near term colliders are characterized by design parameters that place them in the transition between the classical and the quantum regimes, with $D \sim 1.5 - 7.5$. Thus a clear understanding of this region is important.

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16. This property is crucial since the phase itself is very large, of order $N\alpha$, which is $\sim 10^8$ for our case, and few approximations can be tolerated.

17. See p. 447 of M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, U.S. Government Printing Office (1964), Washington, D.C.

18. For the power-law case with general q, the classical limit is

$$R_{classical} = \sqrt{\frac{1+q}{3+q}}\,\frac{2(1+q)}{1+2q}\ .$$

The result in the quantum regime is

$$R_{QM} = \sqrt{\frac{1+q}{q}}\left(\frac{(3+q)}{q}\right)^{1/6}\frac{\Gamma(1+\frac{1}{q})\,\Gamma(\frac{5}{3})}{\Gamma(\frac{1}{q}+\frac{5}{3})}\ .$$

For large q both of these go to one as required. For $q = 1$ they are given in the text.