Concepts of Hyperbolicity and Relativistic Continuum Mechanics

Robert Beig

Institut für Theoretische Physik der Universität Wien, Boltzmanngasse 5, 1090 Wien, Austria
robert.beig@univie.ac.at

Abstract. After a short introduction to the characteristic geometry underlying weakly hyperbolic systems of partial differential equations we review the notion of symmetric hyperbolicity of first-order systems and that of regular hyperbolicity of second-order systems. Numerous examples are provided, mainly taken from nonrelativistic and relativistic continuum mechanics.

1 Introduction

The notion of hyperbolicity of a partial differential equation (PDE), or a system of PDE's, is central for the field theories of mathematical physics. It is closely related to the well-posedness of the Cauchy problem and to the causal structure underlying these theories. In standard theories describing relativistic fields in vacuo this causal structure is that given by the spacetime metric, a second-order symmetric tensor of Lorentzian signature. If matter is included, things become both more complicated and more subtle. In fact, the awareness of some of those complications predates Relativity by centuries. An example is afforded by the phenomenon, already studied by Huygens, of birefringence in crystal optics[1].

There is currently an increase of attention in the field of Relativity, due in part to demands from Numerical Relativity, devoted to certain notions of hyperbolicity applied to the Einstein equations (for an excellent review see [18]). There the main challenge, not discussed in the present notes at all, comes from the fact that, already in vacuum, the Einstein equations by themselves, i.e. prior to the imposition of any gauge conditions, are not hyperbolic. The main burden, then, is to find a "hyperbolic reduction" turning the Einstein equations, or a subset thereof, into a hyperbolic system appropriate for the purpose at hand. However the complications in the causal structure one finds in continuum mechanics, which are our main focus here, are absent in the Einstein vacuum case – at least for the reductions proposed so far. Of course, these complications do come into play ultimately once matter-couplings are included.

[1] For a fascinating account of the history of the associated mathematics see [20].

R. Beig: *Concepts of Hyperbolicity and Relativistic Continuum Mechanics*, Lect. Notes Phys. **692**, 101–116 (2006)
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These notes attempt an elementary introduction to some notions of hyperbolicity and the "characteristic geometry" associated with or underlying these notions. The section following this one is devoted to the general notion of a hyperbolic polynomial, which in our case of course arises as the characteristic polynomial of a PDE. It is interesting that this notion is on one hand restrictive enough to encode essentially all the required features of a theory in order to be "causal" – on the other hand flexible enough to account for an amazing variety of phenomena – relativistic or nonrelativistic – ranging from gravitational radiation to water waves or phonons in a crystal. We devote a significant fraction of Sect. 2 to examples, which at least in their nonrelativistic guise all appear in the standard literature such as [13], though not perhaps from the unified viewpoint pursued here. Some of these examples are not fully worked out, but perhaps the interested reader is encouraged to fill in more details, possibly using some of the cited literature. We hope that some workers in Relativity, even if they have little interest in continuum mechanics for its own sake, find these examples useful for their understanding of the notion of hyperbolicity. While hyperbolicity of the characteristic polynomial of a theory is important, it is not in general sufficient for the well-posedness of the initial value problem for that theory. Well-posedness is the subject of our Sect. 3. We recall the notion of a symmetric hyperbolic system of a system of 1st order PDE's, which is indeed sufficient for well-posedness. A similar role for 2nd order equations is played by a class of systems, which were to some extent implicit in the literature, and for which an elaborate theory has been recently developed in [10,11]. These systems are called regular hyperbolic. They encompass many second order systems arising in physics one would like to qualify as being hyperbolic – such as the Einstein equations in the harmonic gauge. If applicable, the notion of regular hyperbolicity is particularly natural for systems of 2nd order derivable from an action principle, as is the case for many problems of continuum mechanics. We show the fact, obvious for symmetric hyperbolic systems and easy-to-see although not completely trivial for regular hyperbolic ones, that these systems are special cases of weakly hyperbolic systems, i.e. ones the determinant of whose principal symbol is a hyperbolic polynomial. We also touch the question of whether a system of the latter type can be reduced to one of the former type by increasing the number of dependent variables. Throughout this section our treatment will be informal in the sense of ignoring specific differentiability requirements. We also do not touch questions of global well-posedness.

2 Hyperbolic Polynomials

The PDE's we are interested in are of the form

$$M_{AB}^{\mu_1 \ldots \mu_l}(x, f, \partial f, \ldots, \partial^{(l-1)} f)\partial_{\mu_1} \ldots \partial_{\mu_l} f^B + \text{lower order terms} = 0 \,. \quad (1)$$

Here $A, B = 1, ..., m$ and $\mu_i = 1, \ldots, n$. Relevant equations of this form are the Euler equations for a barotropic fluid (for $n = 4, l = 1, m = 4$), the Einstein equations (for $n = 4, l = 2, m = 10$) or the equations governing an ideal elastic solid (for $n = 4, l = 2, m = 3$). The Maxwell equations, in the form they are originally written down, are not of this form, but a suitable subset of them is, as we will discuss later.

The principal symbol of the PDE (1) is defined as

$$M_{AB}(k) = M_{AB}^{\mu_1 \ldots \mu_l} k_{\mu_1} \ldots k_{\mu_l}, \quad k_\mu \in (\mathbb{R}^n)^* \tag{2}$$

We here suppress the dependence on x and on f. The characteristic polynomial $P(k)$ is defined by $P(k) = det\, M_{AB}(k)$, where the determinant is taken with respect to some volume form on f-space $\subset \mathbb{R}^m$. The polynomial $P(k)$ is homogenous of degree $p = m \cdot l$. A homogenous polynomial of degree $p > 0$ is called hyperbolic with respect to $\xi_\mu \in (\mathbb{R}^n)^*$ if $P(\xi) \neq 0$ and the map $\lambda \mapsto P(\eta + \lambda\xi)$, itself a polynomial of degree p, has only real roots $\lambda_i, i = 1, \cdots, p$ for all $\eta \in (\mathbb{R}^n)^*$. The roots $\lambda_i(\xi, \eta)$ need not be distinct. If, for all η with $\eta \wedge \xi \neq 0$, $\lambda_i(\xi, \eta) \neq \lambda_j(\xi, \eta)$ for $i \neq j$, P is called strictly hyperbolic[2]. We write $\mathcal{C}^*$ for the set of $k \in (\mathbb{R}^n)^* \setminus \{0\}$, where P vanishes. It is sometimes called the cone of characteristic conormals.

It is clear that a product of hyperbolic polynomials is hyperbolic. Also, if a hyperbolic polynomial can be factorized into polynomials of lower degree (in which case it is called reducible), these factors are also hyperbolic. There is a wealth of information which can be inferred about a polynomial $P(k)$ if it is hyperbolic. Before explaining some of this, we look at a few examples for hyperbolic polynomials.

Example 1. $P(k) = (a, k) = a^\mu k_\mu$ for some nonzero $a^\mu \in \mathbb{R}^n$. The set $\mathcal{C}^*$ is a punctured hyperplane $\subset (\mathbb{R}^n)^*$.

Clearly $P(k)$ is hyperbolic with respect to any ξ_μ such that $a^\mu \xi_\mu \neq 0$. The polynomial $P(k) = (a_1, k)(a_2, k)(a_3, k)$, with a_1, a_2, a_3 linearly independent $\in \mathbb{R}^3$, arises in the problem of finding, for a three dimensional positive definite metric, a coordinate system in which the metric is diagonal (see [14]) – which shows that hyperbolic problems can also arise in purely Riemannian contexts.

Example 2. $P(k) = \gamma^{\mu\nu} k_\mu k_\nu$, where $\gamma^{\mu\nu}$ is a (contravariant) metric of Lorentzian signature $(-, +, \ldots, +)$. The set $\mathcal{C}^*$ is the two-sheeted Minkowski light cone.

When $n = 2$, $P(k)$ is hyperbolic with respect to any non-null ξ_μ, when $n > 2$, $P(k)$ is hyperbolic with respect to any ξ_μ with $\gamma^{\mu\nu}\xi_\mu\xi_\nu < 0$, i.e. ξ is timelike with respect to $\gamma^{\mu\nu}$. Checking that $P(k)$ is hyperbolic according to our definition is equivalent to the so-called reverse Cauchy-Schwarz

[2] This case is not general enough for the purposes of physics. Furthermore there exist physically relevant cases of non-strictly hyperbolic polynomials which are stable, in the sense that they possess open neighbourhoods in the set of hyperbolic polynomials just containing non-strictly hyperbolic ones [26, 28].

inequality for two covectors one of which is timelike or null with respect to $\gamma^{\mu\nu}$ (which is the mathematical rationale behind the twin "paradox" of Relativity). Surprisingly there are similar inequalities for general hyperbolic polynomials (see [19]) which play a role in diverse fields of mathematics [5].

Example 2 is of course the most familiar one. If it arises from nonrelativistic field theory, the quantity $\gamma^{\mu\nu}$ currently runs under the name of the "Unruh or acoustic metric" [4] (see also [12]) in the Relativity community. It is not an elementary object of the theory, but is built as follows: Take first the Galilean metric $h^{\mu\nu}$, a symmetric tensor with signature $(0, +, \ldots, +)$ together with a nonzero covector τ_μ satisfying $h^{\mu\nu}\tau_\nu = 0$: these are the absolute elements . Then pick a 4-vector u^μ normalized so that $u^\mu \tau_\mu = 1$ and define $\gamma^{\mu\nu} = h^{\mu\nu} - c^{-2} u^\mu u^\nu$. This describes waves propagating isotropically at phase velocity c in the rest system, defined by u^μ, of a material medium. The relativistic version of the above is as follows: Start with the spacetime metric $g^{\mu\nu}$ and define $\gamma^{\mu\nu} = g^{\mu\nu} + (1 - c^{-2}) u^\mu u^\nu$, where u^μ is normalized by taking $\tau_\mu = -g_{\mu\nu} u^\nu$, with $g_{\mu\nu}$ the covariant spacetime metric defined by $g_{\mu\nu} g^{\nu\lambda} = \delta_\mu{}^\lambda$. Note: if there are metrics $\gamma_1^{\mu\nu}$,$\gamma_2^{\mu\nu}$ with $c_2 < c_1$, then the "faster" cone lies inside the slower one. We will come back to this point later.

Example 3. $P(k) = s^{\mu\nu} k_\mu k_\nu$, where $s^{\mu\nu}$ has signature $(-, +, ..., +, 0, ..., 0)$, is hyperbolic with respect to any ξ such that $s^{\mu\nu}\xi_\mu \xi_\nu < 0$.

Here is a case occurring in the real world. Let $g_{\mu\nu}$ be a Lorentz metric on $\mathbb{R}^4$, u^μ a normalized timelike vector, i.e. $g_{\mu\nu} u^\mu u^\nu = -1$, $F_{\mu\nu} = F_{[\mu\nu]}$ nonzero with $F_{\mu\nu} u^\nu = 0$. The quadratic form $s^{\mu\nu} = -e u^\mu u^\nu + 1/2\ F_{\rho\sigma} F^{\rho\sigma} g^{\mu\nu} - F^\mu{}_\rho F^{\nu\rho}$, with $e > 0$, has signature $(-, +, +, 0)$. The characteristic cone $\mathcal{C}^*$ of $P(k) = 0$ consists of two hyperplanes punctured at the origin. When e is interpreted as $e =$"energy density + pressure" and $F_{\mu\nu}$ as the frozen-in magnetic field of an ideally conducting plasma, then $P(k)$ describes the Alfvén modes of relativistic magnetohydrodynamics [45] [2].

Example 4. Let $n = 4$, $\varepsilon_{\mu\nu\lambda\rho}$ some volume form on $\mathbb{R}^4$ and $m^{\mu\nu\lambda\rho} = m^{[\mu\nu][\lambda\rho]}$. With $G^{\mu\nu\rho\sigma} = \varepsilon_{\alpha\beta\delta\epsilon}\, \varepsilon_{\kappa\phi\psi\omega}\, m^{\alpha\beta\kappa(\mu} m^{\nu|\delta\phi|\rho} m^{\sigma)\epsilon\psi\omega}$ we define P by $P(k) = G^{\mu\nu\rho\sigma} k_\mu k_\nu k_\rho k_\sigma$.

As a special case take $m^{\mu\nu\lambda\rho}$ of the form $m^{\mu\nu\lambda\rho} = h^{\lambda[\mu} h^{\nu]\rho} - e^{\lambda[\mu} u^{\nu]} u^\rho + e^{\rho[\mu} u^{\nu]} u^\lambda$, where the symmetric tensors $h^{\mu\nu}$ and $s^{\mu\nu}$, both of signature $(0, +++)$, satisfy $h^{\mu\nu}\tau_\nu = e^{\mu\nu}\tau_\nu = 0$ for $u^\mu \tau_\mu \neq 0$: this is the situation encountered in crystal optics with the nonzero eigenvalues of $e^{\mu\nu}$ relative to $h^{\mu\nu}$ being essentially the dielectric constants. The crystal is optically biaxial or triaxial, depending on the number of mutually different eigenvalues. The 4th order polynomial $P(k)$ turns out to be hyperbolic with respect to all ξ_μ in some neighbourhood of $\xi_\mu = \tau_\mu$, and the associated characteristic cone is the Fresnel surface (see e.g. [25]). For an optically isotropic medium or in vacuo $P(k)$ is reducible, in fact the square of a quadratic polynomial of the type of Example 2. We leave the details as an exercise. More general conditions on $m^{\mu\nu\lambda\rho}$ in order for $P(k)$ to be hyperbolic can be inferred from [32].

The quartic polynomial $P(k)$, as defined above, comes from a generalized ("pre-metric") version of electrodynamics (see [24]), as follows: Let $F_{\mu\nu}$ be the electromagnetic field strength and write $H^{\mu\nu} = m^{\mu\nu\lambda\rho} F_{\lambda\rho}$ for the electromagnetic excitation. The premetric Maxwell equations then take the form

$$\partial_{[\mu}(\varepsilon_{\nu\lambda]\rho\sigma} H^{\rho\sigma}) = J_{\mu\nu\lambda}, \qquad \partial_{[\mu} F_{\nu\lambda]} = 0, \tag{3}$$

where $J_{\mu\nu\lambda}$ is the charge three form[3]. The (3) reduce to the standard ones in vacuo when $m^{\mu\nu\lambda\rho} \sim g^{\lambda[\mu} g^{\nu]\rho}$ with $g^{\mu\nu}$ the metric of spacetime. If one sets $m^{\mu\nu\lambda\rho} = \gamma^{\lambda[\mu}\gamma^{\nu]\rho}$, with $\gamma^{\mu\nu} = h^{\mu\nu} - c^{-1} u^\mu u^\nu$, $h^{\mu\nu}$ the Galilean metric and u^μ a constant vector field s.th. $u^\mu \tau_\mu = 1$, one has the Maxwell equations in a "Galilean" (not Galilean-invariant) version with u^μ describing the rest system of the aether (see [43]). One then looks at hypersurfaces along which singularities can propagate. The result is that the conormal n_μ of such surfaces has to satisfy $P(n) = 0$. Put differently, one can look at the 8 x 6 – principal symbol of the Maxwell equations: then $P(k) = 0$ is exactly the condition for this principal symbol to have nontrivial kernel. If one considered an appropriately chosen subset amongst (3), the evolution equations, one would obtain an equation of the form (1), whose characteristic polynomial contains $P(k)$ as a factor. We will treat the vacuum case of this later.

Our last and most complicated example comes from elasticity [6]:

Example 5. Take $n = 4, l = 2, m = 3$ in (1) with

$$M^{\mu\nu}_{AB} = -G_{AB} u^\mu u^\nu + C^{\mu\nu}_{AB}\ , \tag{4}$$

where $G_{AB} = G_{(AB)}$ and $C^{\mu\nu}_{AB} = C^{\nu\mu}_{BA}$ and $C^{\mu\nu}_{AB}\tau_\nu = 0$ for some covector τ satisfying $(u,\tau) = u^\mu \tau_\mu = 1$. The theory is intrinsically quasilinear: all quantities entering (4) are functions of f and ∂f and in general also of x. For example f^A is required to have maximal rank, and u^μ satisfies $u^\mu(\partial_\mu f^A) = 0$. Furthermore $C^{\mu\nu}_{AB} = C_{ADBE}(\partial_\rho f^D)(\partial_\sigma f^E) h^{\rho\mu} h^{\sigma\nu}$, with $h^{\mu\nu}, \tau_\mu$ being, in the nonrelativistic case, the absolute Galilean objects, or, in the relativistic case, $h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ and $\tau_\mu = -g_{\mu\nu} u^\nu$.

There are the following basic constitutive assumptions.

$$G_{AB} \text{ is positive definite}, C^{\mu\nu}_{AB} m^A m^B \eta_\mu \eta_\nu > 0 \text{ for } m \neq 0,\ \eta \wedge \tau \neq 0 \tag{5}$$

Defining the linear map $(\mathbf{M})^A{}_B$ by $(\mathbf{M})^A{}_B(k) = -(u,k)^2 \delta^A{}_B + G^{AD} C^{\mu\nu}_{DB} k_\mu k_\nu$, the polynomial $P(k)$ can, by general linear algebra, be written as

$$6\ P(k) = (tr\mathbf{M})^3 - 3\ (tr\mathbf{M^2})(tr\mathbf{M}) + 2\ tr\mathbf{M^3}\ . \tag{6}$$

It will follow from a more general result, to be shown below, that the 6th order polynomial $P(k)$ is hyperbolic with respect to ξ_μ in some neighbourhood of

[3] These equations play a certain role in current searches for violations of Lorentz invariance in electrodynamics [30]

τ_μ. In the special case of an isotropic solid, the "elasticity tensor" C_{ABDE} has to be of the form

$$C_{ABDE} = l\ G_{AB}G_{DE} + 2m\ G_{D(A}G_{B)E}\ , \tag{7}$$

and the second of (5) is satisfied iff $c_2^2 = m > 0$, $c_1^2 = l + 2m > 0$. The polynomial $P(k)$ turns out to reduce to the form

$$P(k) \sim (\gamma_1^{\mu\nu} k_\mu k_\nu)(\gamma_2^{\rho\sigma} k_\rho k_\sigma)^2 \tag{8}$$

with $\gamma_{1,2}^{\mu\nu} = h^{\mu\nu} - c_{1,2}^{-2}\ u^\mu u^\nu$. The quantities c_1 and c_2 are the phase velocities of pressure and shear waves respectively. If the medium is elastically anisotropic, such as a crystal, one can start by classifying possible fourth-rank tensors C_{ABDE} according to the symmetry group of the crystal lattice, allow for dislocations, etc. The richness of possible structure of $\mathcal{C}^*$ and the corresponding range of captured physical phenomena – studied by theoreticians and experimentalists – is enormous.

This ends our list of examples. We now turn to some general properties of hyperbolic polynomials and their physical interpretation. It is clear from the definition that $\mathcal{C}^*$ has codimension 1: since $P(\eta+\lambda\xi)$ has to have at least one complex root for each η, and the roots are all real, there is at least one real root. And since there are no more than $p \cdot l$ different roots, $\mathcal{C}^*$ can not have larger codimension. It is then known from real algebraic geometry that $\mathcal{C}^*$ consists of smooth hypersurfaces outside a set of at least codimension 2 (see [9]). The roots $\lambda_i(\xi,\eta)$ can for fixed ξ be assumed to be ordered according to $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p$ for all η. The set of points $k = \eta + \lambda_i(\xi,\eta)\xi$ is called the i'th sheet of $\mathcal{C}^*$. The hypersurface $\mathcal{C}^*$ has to be smooth at all points k lying on a line intersecting p different sheets[4]. In particular all sheets are everywhere smooth when P is strictly hyperbolic.

Next recall that the defining property of a hyperbolic polynomial refers to a particular covector ξ. That covector however is not unique. It is contained in a unique connected, open, convex, positive cone $\Gamma^*(\xi)$ of covectors ξ' sharing with ξ the property that $P(\xi') \neq 0$ and $P(\eta+\lambda\xi')$ has only real zeros $\lambda_i(\xi',\eta)$ [19]. Note that $\Gamma^*(\xi) = -\Gamma^*(-\xi)$. Furthermore $\partial\Gamma^*(\xi) \subset \mathcal{C}^*$, and $\Gamma^*(\xi)$ is that connected component of the complement of $\mathcal{C}^*$ containing ξ. Not all points of $\partial\Gamma^*(\xi)$ have to be smooth points of $\mathcal{C}^*$.

The roots $\lambda_i(\xi,\eta), i = 1,\ldots,p$, due to the homogeneity of P, are homogenous in ξ of order -1 and positively homogenous of order 1 as a function of η. They also satisfy $\lambda_i(\xi,\eta) = -\lambda_{p+1-i}(\xi,\eta)$. At regular points of $\mathcal{C}^*$, i.e. when the gradient of P at $\eta + \lambda_i\xi$ is non-zero, $\lambda_i(\xi,\eta)$ is a smooth function of its arguments due to the implicit function theorem. Next choose a vector $X \in \mathbb{R}^n$ so that $(X,\xi') > 0$ for all $\xi' \in \Gamma^*(\xi)$. We call such a vector

[4]The reason is that a polynomial of order p in one real variable, if it has p different zeros, has non-vanishing derivative at each zero, so k is a non-critical point of P.

"causal". We now look at the intersection S of the hyperplane $(X, \xi') = 1$ with $\mathcal{C}^*$. Note $(X, \xi') = 1$ is transversal to $\mathcal{C}^*$ at smooth points of $\mathcal{C}^*$, so S is smooth there also. Note also that S may be empty, as in Example 1. For reasons explained below, S is often called "slowness surface". To describe S more concretely, we pick some $\tau \in \Gamma^*(\xi)$ with $(X, \tau) = 1$. The pair X, τ constitutes a "rest frame". Using it, we can decompose every covector k as $k_\mu = \tau_\mu + k^\perp_\mu$ where $k^\perp$ is tangential to the hyperplane, i.e. $(X, k^\perp) = 0$. This k lies on $\mathcal{C}^*$ iff $\lambda_i(\tau, k^\perp) = 1$ for some i. Thus S consists of the sheets $\lambda_J(\tau, k^\perp) = 1$, viewed as $(n-2)$-surfaces in $k^\perp \in \mathbb{R}^{n-1}$. Here J runs through some subset of the i's parametrizing λ_i from above. Clearly, as J increases, these sheets form a nested family of not necessarily compact surfaces[5]. The innermost of these surfaces is nothing but the intersection of $\partial\Gamma^*(\xi)$ with the hyperplane $(X, \xi') = 1$ and is hence convex. We call $\mathcal{C}^*(\xi)$ those components of $\mathcal{C}^*$ which consist of half-rays connecting the origin with the points of S. In the examples 2, 4, 5 the set S and $\mathcal{C}^*(\xi)$ consist of at most 1, respectively 2 and 3 sheets. For the last-mentioned case, see [15]. Not all the occurring sheets are compact. It is possible for example for $P(k)$ to be an irreducible hyperbolic polynomial with some sheets of S compact and others non-compact: this is the case e.g. for the acoustic modes in magnetohydrodynamics [13].

We now explain the name "slowness surface". Consider the hyperplane in $\mathbb{R}^n$ given by $(x, k) = 0$ for fixed $k \in \mathcal{C}^*$, i.e. the wave front of the plane wave associated with k. To measure the "speed" at which this wave front moves, decompose observers with tangent V which "move with this wave front", i.e. such that $(V, k) = 0$, according to $V = X + v^\perp$. It follows that $(v^\perp, k^\perp) = -1$. Thus, if there is a natural "spatial" metric h mapping elements $l^\perp$ into elements $w^\perp = h \circ l^\perp$ orthogonal to τ, one can define the "phase velocity" $v^\perp_{\mathrm{ph}} = -\|k^\perp\|^{-2}\, h \circ k^\perp$. Thus, the smaller $k^\perp$ is, the larger the phase velocity. Of course the equation $(v^\perp, k^\perp) = -1$ does not define $v^\perp$ uniquely. But there is a "correct" choice for V tangential to the wave front, called "ray or group velocity", which is independent of any spatial metric, and which is defined at least when k is a smooth point of $\mathcal{C}^*$: this V is given by the conormal to $\mathcal{C}^* \subset (\mathbb{R}^n)^*$ at k, which by duality is a vector $\in \mathbb{R}^n$. If k is in addition a non-critical point, this ray velocity V^μ is $\sim \partial/\partial k_\mu P(k)$, which satisfies $k_\mu V^\mu = 0$ by the positive homogeneity of $P(k)$. The spatial group velocity in the frame X, τ can then be written as which is also the textbook expression.

$$(v^\perp_{\mathrm{gr}})^\mu(k^\perp) = \left(\tau_\lambda \frac{\partial P}{\partial k_\lambda}\right)^{-1} \left(\delta^\mu{}_\nu - X^\mu \tau_\nu\right) \left.\frac{\partial P}{\partial k_\nu}\right|_{k=\tau+k^\perp}, \tag{9}$$

We should add a cautionary remark here. Although the differential topology of the slowness surface is independent of the choice of X satisfying $(X, \xi') > 0$ for all $\xi' \in \Gamma^*(\xi)$, its detailed appearance, and physical quantities such as phase velocity, group velocity or angle between two rays do

[5] In particular, when sheets seem to pass through each other, the two sides are counted as belonging to different sheets.

of course depend on the choice of rest system X, τ and a notion of spatial metric with respect to that observer. Of course there will be, for any particular physical theory, a singled-out class of rest systems, e.g. τ_μ can be the absolute object in a Galilean spacetime or be of the form $\tau_\mu = -g_{\mu\nu}X^\nu$ in a relativistic theory. Or the slowness surface can have more symmetry (say symmetry with respect to reflection at the origin) in some rest system than in others, as is the case with crystal optics or elasticity. For a careful discussion of these issues, in the more specialized context of "ray-optical structures" on a Lorentzian spacetime, consult [34].

We now come back to the "ray" concept. If k is a smooth critical point of $\mathcal{C}^*$, finding the map $k \mapsto V(k)$ is already a nontrivial problem in algebraic geometry [35]. If k is not a smooth point of $\mathcal{C}^*$, there is no unique assignment of a group velocity to k. Still well-defined is the set $\mathcal{C}$ of all $V \neq 0$ satisfying

$$(V, k) = 0 \qquad \text{where} \qquad P(k) = 0\,, \tag{10}$$

called the dual or ray cone. Loosely speaking, each sheet of the ray cone corresponds to a spherical wave front tangent to (or "supported by") the planar wave fronts defined by the different points k in some corresponding sheet of $\mathcal{C}^*$ [13]. There holds $(\mathcal{C}^*)^* = \mathcal{C}$. The dual cone is again an algebraic cone, which, except in degenerate cases, is again the zero-set of a single homogenous polynomial. The structure of this dual cone, in particular its singularity structure which can be very complicated, is another difficult matter of real algebraic geometry. For example the degree of its defining polynomial is in general much higher than that of $\mathcal{C}^*$ (see [37], [21]). This "dual" polynomial need not be hyperbolic: in order to be hyperbolic it would have to have a central sheet which is convex, which is not the case for some of the examples one finds in the literature. In our examples from above the situation is as follows: In our Example 1 the dual cone $\mathcal{C}^*$ consist of the two half-lines $\{\alpha a^\mu | \alpha > 0\}$ and $\{\alpha a^\mu | \alpha < 0\}$. The cone dual to the quadratic cone $g^{\mu\nu}k_\mu k_\nu = 0$ in Example 2 is given by $g_{\mu\nu}V^\mu V^\nu = 0$ with $g_{\mu\nu}g^{\nu\lambda} = \delta_\mu{}^\lambda$. For a nonrelativistic acoustic cone $\gamma^{\mu\nu} = h^{\mu\nu} - (1/c^2)u^\mu u^\nu$ we obtain for the ray cone $\gamma_{\mu\nu} = h_{\mu\nu} - c^2\tau_\mu\tau_\nu$, where $h_{\mu\nu}$ is the unique tensor defined by $h_{\mu\nu}u^\nu = 0$ and $h_{\mu\nu}h^{\nu\lambda} = \delta_\mu{}^\lambda - \tau_\mu u^\lambda$. If one has two sound cones, as in isotropic elasticity, it is the faster ray cone which lies outside. In Example 3 $\mathcal{C}$ is given as a subset of vectors X^μ in a linear space T, which is the annihilator of the null space of $s^{\mu\nu}$, namely where this subset is given by $s_{\mu\nu}X^\mu X^\nu = 0$, where $s_{\mu\nu}$ is the inverse of $s^{\mu\nu}$ on T. In the magnetohydrodynamic example the preceding statement corresponds to the fact that Alfvén waves "travel along the direction of the magnetic field". For Example 4 the ray cone $\mathcal{C}$ is a 4th order cone of the same type as $\mathcal{C}^*$, a fact already known by Ampère in the case of crystal optics and shown generally in [36]. For anisotropic elasticity the structure of the ray cone does not seem to be fully known, except for a general upper bound on its degree, namely 150 on grounds of general algebraic geometry (see [15], [37], [21]) and detailed studies for certain specific crystal symmetries – which give rise to a beautiful variety of acoustic phenomena [44][6].

3 Initial Value Problem

We now come to the issue of posing an initial value problem for hyperbolic equations of the form of (1). This requires two things: firstly a notion of "spacelike" initial value surface, secondly a notion of domain of dependance. Not surprisingly these notions can be formulated purely in terms of the characteristic polynomial. A hypersurface Σ in $\mathbb{R}^n$ will be called spacelike, if it has a conormal n_μ lying everywhere in $\Gamma^*(\xi)$ for some ξ. If the equation (1) is nonlinear, every property concerning the characteristic polynomial has to refer to the data of some reference field f_0, i.e. the value of f_0 on Σ and those of its derivatives up to order $l-1$. It is then the case that Σ is spacelike also for any sufficiently near-by data. The reason is that $\xi' \in \Gamma^*(\xi)$ can be characterized by $\lambda_1(\xi, \xi') > 0$, and the eigenvalues λ_i, being zeros of a polynomial having real roots only, depend continuously on the coefficients of this polynomial [1]. A point x in $\mathbb{R}^n$ is said to lie in the domain of dependence of Σ if each causal curve (i.e. each curve whose tangent vector X satisfies $(X, \xi') \neq 0$ for all $\xi' \in \Gamma^*(\xi)$) through x which is inextendible intersects Σ exactly once. The Cauchy problem for (1) is said to be well-posed if, for the above data, there is a unique solution in some domain of dependence of Σ and, secondly, if this solution depends in some appropriate sense continuously on the data. The question then is whether well-posedness holds under the above conditions. The answer is affirmative when (1) is linear with constant coefficients and the lower-order terms are absent. Then the initial value problem can be solved "explicitly" by using a fundamental solution ("Green function" in the physics literature) – which in turn can be obtained e.g. by the Fourier transform. By a refined version of a well-known argument in physics texts concerning the wave equation in Minkowski space (see e.g. [3]), one can show that the fundamental solution is supported in $\Gamma(\xi)$, which is the closure of the set of causal vectors just described. The set $\Gamma(\xi)$ is a closed, convex cone, dual to $\Gamma^*(\xi)$. If the outermost component of the cone $\mathcal{C}(\xi)$ dual to $\mathcal{C}^*(\xi)$ is convex, its closure is the same as $\Gamma(\xi)$, otherwise its convex closure is the same as $\Gamma(\xi)$. If one is interested in finer details than just wellposedness, even the linear, constant-coefficient case becomes very nontrivial. An example is the question of the existence of "lacunas", i.e. regions in $\Gamma(\xi)$ where the fundamental solution vanishes. For isotropic elasticity mentioned in Example 4, when $c_2 < c_1$ (which is the experimentally relevant case), the fundamental solution vanishes inside the inner shear cone determined by c_2. (Note that "inner" and "outer" are interchanged under transition between

[6] There are computer codes designed for algebraic elimination, which might be worth applying to this problem [23].

normal and ray cone.) For anisotropic elasticity this issue, or the somewhat related question of the detailed time decay, already presents great difficulties (see [15, 39]). The existence of lacunas for general, linear hyperbolic systems with constant coefficients was studied in [3].

The problem now is that many field equations in physics give rise to variable coefficients, to various forms of lower-order terms and-or nonlinearities. But if one has a system of PDE's with hyperbolic characteristic polynomial (such systems are often called "weakly hyperbolic") , which in addition has a well posed initial value problem, a perturbation of the coefficients will in general destroy the latter property (see e.g. [31]). It is thus hard to get any further without additional assumptions. One such assumption is that of having a symmetric hyperbolic system. This is given by a system of the form of (1) with $l = 1$. It is furthermore assumed that

$$M_{AB}^{\ \ \mu} = M_{(AB)}^{\ \ \ \ \mu} \tag{11}$$

and that there exists ξ_μ so that

$$M_{AB}(\xi) = M_{AB}^{\ \ \mu}\ \xi_\mu \quad \text{is positive definite} . \tag{12}$$

The symmetric hyperbolic system has a characteristic polynomial which is hyperbolic with respect to ξ. To see this one simply observes that the equation

$$det(M_{AB}^{\ \ \mu}(\eta_\mu + \lambda\xi_\mu)) = 0 \tag{13}$$

characterizes eigenvalues of the quadratic form $M_{AB}(\eta)$ relative to the metric $M_{AB}(\xi)$ – and these eigenvalues have to be real. There is then, for quasilinear symmetric hyperbolic systems, a rigorous existence statement along the lines informally outlined at the beginning of this section [29]. The uniqueness part uses the concept of "lens-shaped domains" (see e.g. [18]) which is essentially equivalent to that of domain of dependence above.

Several field theories of physical importance naturally give rise to a symmetric hyperbolic system. An example is afforded by the hydrodynamics of a perfect fluid both nonrelativistically and relativistically[7]. The most prominent examples are perhaps the Maxwell equations in vacuo and the vacuum Bianchi identities in the Einstein theory. For the latter this was first observed in [17]. For completeness we outline a proof for the well-known Maxwell case following [45]. We have that

$$\nabla^\nu F_{\mu\nu} = 0, \qquad \nabla_{[\mu} F_{\nu\lambda]} = 0 \tag{14}$$

with ∇_μ being the covariant derivative with respect to $g_{\mu\nu}$, a Lorentz metric on $\mathbb{R}^4$. These are 8 equations for the 6 unknowns $F_{\mu\nu}$. Next pick a timelike vector field u^μ with $u^2 = -1$ and define electric and magnetic fields by

$$E_\mu = F_{\mu\nu} u^\nu, \qquad B_{\mu\nu\lambda} = 3F_{[\mu\nu} u_{\lambda]} , \tag{15}$$

[7] For an elegant treatment of the latter, see [16]

so that

$$F_{\mu\nu} = -2E_{[\mu}u_{\nu]} - B_{\mu\nu\lambda}u^{\lambda} \ . \tag{16}$$

We assume for simplicity that u^μ is covariant constant, otherwise the ensuing equations contain zero'th order terms which are of no concern to us. The operator

$$\nabla_{\mu\nu} = 2u_{[\mu}\nabla_{\nu]} \tag{17}$$

contains derivatives only in directions orthogonal to u^μ. Using Eq.'s (14) we find the evolution equations

$$3\nabla_{[\mu\nu}E_{\lambda]} = -u^\rho\nabla_\rho B_{\mu\nu\lambda}, \qquad \nabla^{\lambda\rho}B_{\nu\lambda\rho} = 2u^\rho\nabla_\rho E_\nu \ . \tag{18}$$

Taking now $u^\lambda\nabla_\rho$ of (15), we rewrite the evolution equations in the form

$$W_{\mu\nu}^{\mu'\nu'\lambda}\nabla_\lambda F_{\mu'\nu'} = 0 \ . \tag{19}$$

Now take the positive definite metric

$$w^{\mu\nu} = 2u^\mu u^\nu + g^{\mu\nu} \ . \tag{20}$$

Consider now the positive definite metric $a^{\mu\nu\lambda\rho} = 2w^{\rho[\mu}w^{\nu]\lambda}$ on the space of 2-forms and use it to raise indices in $W_{\mu\nu}^{\mu'\nu'\lambda}$: One obtains quantities $W^{\mu\nu\mu'\nu'\lambda}$ satisfying

$$W^{\mu\nu\mu'\nu'\lambda} = W^{\mu'\nu'\mu\nu\lambda}, \qquad W^{\mu\nu\mu'\nu'\lambda}u_\lambda \sim a^{\mu\nu\mu'\nu'} \ . \tag{21}$$

Thus the (18) are symmetric hyperbolic with respect to u_μ. For the characteristic polynomial one finds

$$P(k) \sim (u^\mu k_\mu)^2(g^{\rho\nu}k_\rho k_\nu)^2 \ . \tag{22}$$

We now turn to 2nd order equations. Let us assume that the quantities $M_{AB}^{\mu\nu}$ of (1) satisfy

$$M_{AB}^{\mu\nu} = M_{BA}^{\nu\mu} \ . \tag{23}$$

This is necessarily the case when (1) comes from a variational principle, because then

$$M_{AB}^{\mu\nu} \sim \frac{\partial^2\mathcal{L}}{(\partial\partial_\mu f^A)(\partial\partial_\nu f^B)} \ , \tag{24}$$

where $\mathcal{L} = \mathcal{L}(x, f, \partial f)$. Of course, since only the quantities $M_{AB}^{(\mu\nu)}$ enter the differential equation (1), one might as well have assumed the stronger conditions $M_{AB}^{\mu\nu} = M_{AB}^{\nu\mu} = M_{BA}^{\mu\nu}$. This is not usually done in continuum mechanics. The reason is that, while the PDE (and hence the characteristic polynomial) is unaffected by the above symmetrization, other physical quantities in the theory, like the stress, depend on the unsymmetrized object – and such objects typically enter, if not the equation, then the natural boundary conditions for the equation on the surface say of an elastic body (see [8]). A related reason is that the symmetrized object would hide other symmetries – present in some situations – which are more fundamental such as invariances under isometries. For example the object $C_{A(B|D|E)}$, with C_{ABDE} the elasticity tensor of (7), is not symmetric in (AB) and (DE). But the latter symmetry is important for understanding the solutions of the linearized equations of motion when the spacetime has Killing vectors. The work [10] also uses the unsymmetrized form of $M^{\mu\nu}_{AB}$, the reason being that in this approach one is only interested in properties of $M^{\mu\nu}_{AB}$ which do not change when a total divergence is added to the Lagrangian, and the stronger symmetry, if present, would in general be destroyed by such an addition. Next it is assumed that there exists a pair X^μ, ξ_ν satisfying

$$M^{\mu\nu}_{AB}\xi_\mu\xi_\nu \text{ is negative definite} \tag{25}$$

and

$$M^{\mu\nu}_{AB}(m^A\eta_\mu)(m^B\eta_\nu) > 0 \text{ for all } m^A\eta_\mu \neq 0 \text{ with } (X,\eta) = 0\,. \tag{26}$$

The conditions (25,26) essentially state that the PDE is the sum of a "timelike part" and an "elliptic part", the latter obeying the Legendre-Hadamard condition of the calculus of variations [22]. If the equation (1) has $l = 2$ and satisfies (23,25,26), the system is called regular hyperbolic with respect to ξ. We now check that every regular hyperbolic system with respect to ξ is weakly hyperbolic with respect to ξ. The characteristic condition reads

$$det(M^{\mu\nu}_{AB}(\eta_\mu + \lambda\xi_\mu)(\eta_\nu + \lambda\xi_\nu)) = 0 \tag{27}$$

The covector η in (27) can be decomposed as $\eta = \frac{(X,\eta)}{(X,\xi)}\xi + l$ where l satisfies $(X,l) = 0$. Thus we can after redefining λ assume that η in (27) has $(X,\eta) = 0$. Defining $G_{AB} = -M_{AB}(\xi) = -M^{\mu\nu}_{AB}\xi_\mu\xi_\nu$, $V_{AB} = M_{AB}(\eta)$, $Q_{AB} = M^{\mu\nu}_{(AB)}\xi_\mu\xi_\nu$, consider the eigenvalue problem

$$\mathcal{D}\hat{f} = \lambda\mathcal{E}\hat{f}\,, \tag{28}$$

in

$$\hat{f} = \begin{pmatrix} u^A \\ v^B \end{pmatrix}$$

where the quadratic forms $\mathcal{D}$, $\mathcal{E}$ are given by

$$\mathcal{D} = \begin{pmatrix} 0 & V_{AB} \\ V_{AB} & 2Q_{AB} \end{pmatrix}$$

and

$$\mathcal{E} = \begin{pmatrix} V_{AB} & 0 \\ 0 & G_{AB} \end{pmatrix}.$$

Since $\mathcal{E}$ is positive definite, all eigenvalues λ are real. But (28) for $\hat{f} \neq 0$ is equivalent to

$$(-G_{AB}\lambda^2 + 2Q_{AB}\lambda + V_{AB})v^B = 0 \ , \tag{29}$$

for $v^A \neq 0$ which in turn is equivalent to (27). This proves our assertion that regular hyperbolic systems are weakly hyperbolic. (Note that every "timelike vector" X in the sense of (26) is causal, i.e. $(X, \xi) \neq 0$ for all $\xi \in \Gamma^*(\xi)$, but not conversely.) We can now come back to Example 5. The leading-order coefficients $M^{\mu\nu}_{AB}$ in (4) clearly belong to a regular hyperbolic system, when we choose the vector $X^\mu \sim u^\mu$. It then follows from the preceding result that the polynomial in (6) is indeed a hyperbolic polynomial.

As with symmetric hyperbolic systems, it turns out that there is, for regular hyperbolic systems, a local existence theorem [27] along the lines sketched at the beginning of this section. The appropriate domain of dependence theorem is proved in [10].

One may ask the question if it is possible to convert a regular hyperbolic system into an equivalent symmetric hyperbolic one by introducing first derivatives as additional dependent variables (at the price of course of having to solve constraints for the initial data). (This was the approach we originally followed for elasticity in [6], since we were unaware that there was already an existence theorem which applied, namely [27]). If the condition (12) is provisionally ignored, it turns out this is possible provided that $M^{\mu\nu}_{AB}$ is of the form of (4) for some pair u^μ, τ_ν, i.e. certain cross-terms vanish[8]. But the positivity condition (12) will not always be satisfied. (Essentially this requires the "rank-one positivity" condition (26) to be replaced by the stronger rank-two positivity: $M^{\mu\nu}_{AB} m^A{}_\mu m^B{}_\nu > 0$ for all $m^A{}_\mu \neq 0$ with $X^\mu m^A{}_\mu = 0$.) In the case of isotropic elasticity it was shown in [6] that one can add to $M^{\mu\nu}_{AB}$ a term of the form $\Lambda^{\mu\nu}_{AB}$, which has the symmetries $\Lambda^{\mu\nu}_{AB} = \Lambda^{[\mu\nu]}_{[AB]}$, so that both the field equations and the requirement (23) remains unchanged, but at the same time condition (12) is valid. However it is an algebraic fact that such a trick does not always work (see [38, 42]).

Finally let us mention the notion of strong hyperbolicity, which is intermediate between weak hyperbolicity and symmetric or regular hyperbolicity in the first or second order case respectively. This notion, which involves the tool of pseudodifferential reduction [40, 41], also gives wellposedness but has greater flexibility, see [33] for applications to the Einstein equations. It would be interesting to see if the chain "weakly hyperbolic – strongly hyperbolic – symmetric or regular hyperbolic" has an analogue for PDE's of order greater than 2.

Acknowledgements

I am indebted to Domenico Giulini for carefully reading the manuscript and suggesting several corrections and improvements. I also would like to

[8]In [7] I claimed this to be possible even without these cross-terms vanishing. I now see I have no proof of this assertion.

thank Helmuth Urbantke for several very helpful conversations and Piotr T. Chruściel and Helmut Rumpf for useful comments. This work was supported by Fonds zur Förderung der Wissenschaftlichen Forschung (Projekt P16745-N02)

References

1. D. Alekseevsky, A. Kriegl, M. Losik, P.W. Michor: Choosing roots of polynomials smoothly. Israel J. Math. **105**, 203–233 (1998)
2. A.M. Anile: *Relativistic Fluids and Magneto-Fluids* (Cambride University Press, Cambridge 1989)
3. M.F. Atiyah , R. Bott, L. Gårding: Lacunas for hyperbolic differential operators with constant coefficients I; II. Acta Math. **124**, 109–189 (1970); **131**, 145–206 (1973)
4. S. Barcelo, S. Liberati, S. Sonego, M. Visser: Causal structure of acoustic spacetimes (2004) `gr-qc/04080221`
5. H.H. Bauschke, O. Güler, A.S. Lewis, H.S. Sendov: Hyperbolic polynomials and convex analysis. Can. J. Math. **53**, 470–488 (2001)
6. R. Beig, B.G. Schmidt: Relativistic elasticity. Class. Quantum Grav. **20**, 889–904 (2003)
7. R. Beig: Nonrelativistic and relativistic continuum mechanics. To appear in the Proceedings of the Seventh Hungarian Relativity Workshop, Sarospatak, Hungary. `gr-qc/0403073`
8. R. Beig, M. Wernig-Pichler: The Cauchy problem for relativistic elastic bodies with natural boundary conditions. In preparation (2004)
9. R. Benedetti, J.-J. Risler: *Real Algebraic and Semialgebraic Sets* (Hermann, Paris 1990)
10. D. Christodoulou: *The Action Principle and Partial Differential Equations* (Princeton University Press, Princeton 2000)
11. D. Christodoulou: On hyperbolicity. Contemporary Mathematics **283**, 17–28 (2000)
12. P.T. Chruściel: Black holes. In *Conformal Structure of Spacetime*, ed by J. Frauendiener and H. Friedrich, Lecture Notes in Physics **604** (Springer, Heidelberg 2002)
13. R. Courant, D. Hilbert: *Methods of Mathematical Physics, Volume II* (Interscience Publishers, New York 1962)
14. D.M. DeTurck, D. Yang: Existence of elastic deformations with prescribed principal strains and triply orthogonal systems. Duke Math. Journal **51**, 243–260 (1984)
15. G.F.D. Duff: The Cauchy problem for elastic waves in an anisotropic medium. Transactions Roy. Soc. A **252**, 249–273 (1960)
16. J. Frauendiener: A note on the relativistic Euler equations. Class. Quantum Grav. **20**, L193–L196 (2003)
17. H. Friedrich: On the regular and the asymptotic characteristic initial value problem for Einstein's vacuum field equations. Proc. Roy. Soc. A **375**, 169–184 (1985)

18. H. Friedrich, A. Rendall: The Cauchy problem for the Einstein equations. In: *Einstein's Field Equations and their Physical Interpretation*, ed by B. G. Schmidt, Lecture Notes in Physics **540** (Springer, Heidelberg 2000)
19. L. Gårding: An inequality for hyperbolic polynomials. J. of Mathematics and Mechanics **8**, 957–965 (1959)
20. L. Gårding: History of the mathematics of double refraction. Arch. Hist. Ex. Sci. **40**, 355–385 (1989)
21. I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky: *Discriminants, Resultants and Multidemensional Determinants* (Birkhäuser, Boston 1994)
22. M. Giaquinta, S. Hildebrandt: *Calculus of Variations II* (Springer, Berlin 1996)
23. G.-M. Greuel, G. Pfister, H. Schönemann: SINGULAR 2.0. A computer algebra system for polynomial computations. Centre for Computer Algebra, University of Kaiserslautern (2001) `http://www.singular.uni-kl.de`
24. F.W. Hehl, Y.N. Obukhov: *Foundations of Classical Electrodynamics – Charge, Flux, and Metric* (Birkhäuser, Basel 2003)
25. G. Herglotz: *Vorlesungen über die Mechanik der Kontinua*. Göttingen lectures of 1926 and 1931, elaborated by R. B. Guenther and H. Schwerdtfeger (Teubner, Leipzig 1988)
26. L. Hörmander: Hyperbolic systems with double characteristics. Comm. Pure Appl. Math. **46**, 89–106 (1993)
27. T.J.R. Hughes, T. Kato, J.E. Marsden: Well-posed quasilinear second-order hyperbolic systems with applications to Nonlinear Elastodynamics and General Relativity. Arch. Rat. Mech. Anal. **63**, 273–294 (1977)
28. F. John: Algebraic conditions for hyperbolicity of systems of partial differential equations. Comm.Pure Appl.Math. **31**, 89–106 (1978)
29. T. Kato: The Cauchy problem for quasi-linear symmetric hyperbolic systems. Arch. Rat. Mech. Anal. **58**, 181–205 (1975)
30. A. Kostelecký, M. Mewes: Signals for Lorentz violation in electrodynamics. Phys. Rev. D **66**, 056005–056028 (2002)
31. H.-O. Kreiss, O. Ortiz: Some mathematical and numerical questions connected with first and second order time dependent systems of partial differential equations. In: *The Conformal Structure of Space-Time*, ed by J. Frauendiener and H. Friedrich), Lecture Notes in Physics **604** (2002)
32. C. Lämmerzahl, F. W. Hehl: Riemannian cone from vanishing birefringence in premetric electrodynamics. (2004) `gr-qc/0409072`
33. G. Nagy, O.E. Ortiz, O.A. Reula: Strongly hyperbolic second order Einstein's evolution equations. Phys. Rev. D **70**, 044012(15) (2004)
34. V. Perlick: *Ray Optics, Fermat's Principle, and Applications to General Relativity*, Lecture Notes in Physics **m61** (Springer, Heidelberg 2000)
35. J. Rauch: Group velocity at smooth points of hyperbolic characteristic varieties. Astérisque **284**, 265–269 (2003)
36. G.F. Rubilar: Linear pre-metric electrodynamics and deduction of the light-cone. Ann. Phys. (Leipzig) **11**, 717–782 (2002)
37. G. Salmon: *Lectures Introductory to the Modern Higher Algebra* (Dublin 1885; reprinted by Chelsea Publ., New York 1969)
38. D. Serre: Formes quadratiques et calcul des variations. J. Math. Pures et Appl. **62**, 177–196 (1983)
39. M. Stoth: Decay estimates for solutions of linear elasticity for anisotropic media. Math. Meth. Appl. Sci. **19**, 15–31 (1996)

40. M.E. Taylor: *Pseudodifferential Operators and Nonlinear PDE* (Birkhäuser, Boston 1991)
41. M.E. Taylor: *Pseudodifferential Operators* (Princeton University Press, Princeton 1981)
42. F.J. Terpstra: Die Darstellung biquadratischer Formen als Summen von Quadraten mit Anwendung auf die Variationsrechnung. Math. Ann. **116**, 166–180 (1938)
43. A. Trautman: Comparison of Newtonian and relativistic theories of space-time. In: *Perspectives in Geometry and Relativity*, ed by B. Hoffmann, (Indiana University Press 1966)
44. J.P. Wolfe: *Imaging Phonons* (Cambridge University Press, Cambridge 1998)
45. A.Ç. Zenginoglŭ: Ideal magnetohydrodynamics in curved spacetime. Masters Thesis, University of Vienna (2003)