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# D-branes, $B$ Fields and Deformation Quantization

*I.M. BURBAN*

*Bogoliybov Institute for Theoretical Physics, Kyiv 03143, Ukraine*
E-mail: *mmtpitp@gluk.org*

The worldvolume geometry of flat and curved Dp-branes embedded in flat and curved background spaces in the zero slope limit of Seiberg and Witten is studied.

## 1 Introduction

Dp branes in the Type $II$ superstring theory with nonzero NS-NS $B$ fields have the interesting features. When $B$ field is switched off the system of the D0-D6 BPS branes is nonsupersymmetric. But it becomes supersymmetric when a suitable constant $B$ field is turned on.

In the absence of the $B$ field the system of D0-D4 branes is supersymmetric. But the presence of the $B$ field changes the properties of supersymmetry. The D0-D4 system of branes remaines supersymmetric only if the $B$ field is anti-self-dual [1].

An identification of the Dp-brane charges with K-theory classes holds in the case of vanishing $B$ field. In the presence of a $B$ field the arguments of [2] have to be modified. The point is that a gauge field in the presence of a $B$ field is rather a connection over a noncommutative algebra than over a vector bundle. Therefore it is natural to suspect that Dp-brane charges must be identified with K-theory classes of some noncommutative algebra. It is the principal property of Dp-branes with switched $B$ fields is following: theirs worldvolume geometry is noncommutative.

The noncommutative geometry studies geometric spaces (and their generalizations) using noncommutative algebras of functions on them. The noncommutative torus is one of the most important examples of the manifolds in noncommutative geometry. The noncommutative geometry of the worldsheet plays an important role in the study of the string theory. These problems have attached much attention [3, 4, 5, 6].

But most of them were dealing with the case of a constant $B$ field in the flat background. Connes, Douglas, Schwarz [3] have shown that the matrix theory of $M$ theory compactified on a $T^2$ with a background three form potential, $C_{-12}$ is related to gauge theory on a noncommutative torus. Douglas and Hull [4] have studied Dp-branes on $T^2$ with the constant NS-NS two form field, $B$, and have shown that the effective worldvolume theory will be noncommutative gauge theory on the noncommutative torus.

When a Dp-brane is placed into a background which carries a non-vanishimg constant $B$ field the algebra of functions on its classical worldvolume is deformed. The involving of this constant $B$ field background can be described by replacing the ordinary product of functions on the worldvolume of the Dp-branes by the Moyal product, which is associative and noncommutative. This case corresponds to the embedding of a flat Dp-brane into a flat background.

In the zero slope limit $\alpha' \to 0$ of Seiberg and Witten this case is extended to the one in which $\omega = B + F$ is such that $d\omega = 0$. Here $F$ is the strength of some $U(1)$ gauge field on a Dp-brane. In this case the ordinary product of functions on the worldwolume of a Dp-brane is replaced by the Kontsevich star product which is also associative and noncommutative. This case corresponds to the embedding of a curved Dp-brane into a flat background.

There are several attempts to extend this consideration to open strings in a general background. In the terminology of Dp-branes it corresponds to the embedding of the curved Dp-branes in the curved backgrounds. The last corresponds to the case $d\omega = dB = H \neq 0$. The ordinary product of the algebra of functions on a Dp-brane is replaced by the Kontsevich star product. But in this case it is noncommutative and nonassociative. The algebra of functions on it defines "a noncommutative and nonassociative manifold".

In this article we shall study the noncommutative and homotopy associative algebras of functions on Dp-branes defining on them noncommutative and homotopy associative structures of the manifold. We shall clarify the role of their K-theory classes in the labeling of unequivalent unstable Dp-brane configurations.

## 2 Open string description of Dp-brane in parallelizable backgrounds

The bosonic part of the action for a fundamental open string ending on a Dp-brane in the background of a NS-NS $B$ field is

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma g_{ab}(X)dX^a \wedge \star dX^b + \frac{i}{4\pi\alpha'}\int_\Sigma B_{ab}(X)dX^a\wedge dX^b + \frac{i}{2\pi\alpha'}\int_{\partial\Sigma} ds(\partial_s X^a A_a(X)), \quad (1)$$

or

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma g_{ab}(X)dX^a \wedge \star dX^b + \frac{i}{4\pi\alpha'}\int_\Sigma (B_{ab}(X)+F_{ab}(X))dX^a\wedge dX^b, \quad (2)$$

where $F(X) = dA(X)$. The action (2) of the open string is invariant under both gauge transformations for the one-form gauge field $A \to A + d\Lambda$, $B \to B$ and for the two-form gauge field $B \to B + d\Lambda$, $A \to A - \Lambda$.

From now on we will consider the Dp-brane in the weakly curved backgrounds [7]. We shall restrict ourselves to the case of maximal branes and assume that $\Sigma$ has the topology of the disk.

In order to use (2) for calculation of the correlation functions it is useful to introduce Riemann normal coordinates at the origin in which we have [7]

$$g_{ab}(x) = g_{ab} - \frac{1}{3}R_{abcd}x^c x^d + \cdots, \quad (3)$$

$$B_{ab}(x) = B_{ab} + \frac{1}{3}H_{abc}x^c + \frac{1}{4}\nabla_d H_{abc}x^c x^d + \cdots. \quad (4)$$

With the help of (3) and (4) the action (2) can be represented in approximation in small curved devivation from the flat closed string background

$$S_B = S_0 + S_1, \quad (5)$$

$$S_0 = \frac{1}{2}g_{ab}\int_\Sigma dX^a\wedge\star dX^b + i\int_\Sigma (B_{ab}+F_{ab}(X))dX^a\wedge dX^b, \quad (6)$$

$$S_1 = \frac{i}{6}H_{abc}\int_\Sigma X^a dX^b \wedge X^c. \quad (7)$$

Let us denote $\omega(x) = B_{ab} + F_{ab}(x)$ and consider

$$S_0 + S_1 = \frac{1}{2}g_{ab}\int_\Sigma dX^a\wedge\star dX^b + i\int_\Sigma \omega_{ab}dX^a\wedge dX^b + \frac{i}{6}H_{abc}\int_\Sigma X^a dX^b\wedge X^c. \quad (8)$$

The simplest way to prove the noncommutativity of a Dp-brane is to quantize the open string ending on it. In the zero slope limit $\alpha' \to 0$ [1] the closed string metric $g$ scales to zero and since $d\omega = 0$ we can use the Cattaneo and Felder path integral representation of the Kontsevich deformation quantization product [8] to calculate of the correlation functions corresponding to $S_0$. If we choose $n$ functions $f_1, \dots, f_n$ positioned at ordered points $\tau_1, \dots, \tau_n$ on the boundary $\partial\Sigma$ of the string worldsheet, then path integral

$$\int [dX] e^{-S_0(X)} f_1(X(\tau_1)) \cdots f_n(X(\tau_n)) \tag{9}$$

defines the $n$-point correlation functions corresponding $S_0$ [6, 9]

$$\langle f_1 \cdots f_n \rangle = \int V(B) dx (f_1 * \cdots * f_n), \tag{10}$$

$V(B) = \sqrt{\det B}$ and $*$ is the Moyal star product (we put $F(x) = 0$)

$$f \star g = fg + \frac{i}{2}\alpha^{ab}\partial_a f \partial_b g - \frac{i}{8}\alpha^{ac}\alpha^{bd}\partial_a\partial_b f \partial_c\partial_d g + \mathcal{O}\left(\alpha^3\right), \qquad \alpha = B^{-1}.$$

Analogously, the path integral

$$\int [dX] e^{-S_0(X) - S_1(X)} f_1(X(\tau_1)) \cdots f_n(X(\tau_n)) \simeq \int [dX] e^{-S_0(X)} [1 + \nu + \mu], \tag{11}$$

$$\nu = -\frac{i}{6} H_{abc} x^c \int_\Sigma d\zeta^a \wedge d\zeta^b, \tag{12}$$

$$\mu = -\frac{i}{6} H_{abc} \int_\Sigma \zeta^a d\zeta^b \wedge d\zeta^c, \tag{13}$$

defines $n$-point correlation functions corresponding to $S_0 + S_1$ [7]. The integral (11) as a result of the path integration is decomposed into three parts. The first part gives the nonperturbed correlation function

$$\int f_1 * \cdots * f_n, \tag{14}$$

the second ones coming from the two-vertex $\nu$ gives

$$V + \sum_{i<j} V_{ij}, \tag{15}$$

$$V = \frac{1}{3} H_{abc}\theta_{bc} \int x^a * (f_1 * \cdots * f_n), \tag{16}$$

$$V_{ij} = \frac{i}{6} H_{abc}\theta^{a\tilde{a}}\theta^{a\tilde{b}} \int x^c * (f_1 * \cdots * \partial_{\tilde{a}} f_i * \cdots * \partial_{\tilde{b}} f_j * \cdots * f_n). \tag{17}$$

The third part coming from the three-vertex $\mu$ is given by the expression

$$\sum_{i<j<k} S\left(\frac{\tau_{ji}}{\tau_{ik}}\right) W_{i<j<k}, \tag{18}$$

$$W_{ijk} = -\frac{1}{12} H_{abc}\theta^{a\tilde{a}}\theta^{b\tilde{b}}\theta^{c\tilde{c}} \int f_1 * \cdots * \partial_{\tilde{a}} f_i * \cdots * \partial_{\tilde{b}} f_j * \cdots * \partial_{\tilde{c}} f_k * \cdots * f_n, \tag{19}$$

$S(x) = 1 - 2L(x)$, and $L(x)$ is the Rogers dilogarithm [10].

It is useful to change the notation and represent functions as operators, the $*$ product as the operator multiplication and integral $\int$ as Tr:

$$x^a \to X^a, \quad f_i \to F_i, \quad \int V(\omega) \to \mathrm{Tr}, \quad \theta^{a\tilde{a}}\partial_{\tilde{a}} f \to -i[X^a, F], \quad \theta^{ab} \to -i[X^a, X^b].$$

In these notations the formulas (16), (17), (19) are represented as

$$V = -\frac{2i}{3} H_{abc} \operatorname{Tr}\left(X^a X^b X^c F_1 \cdots F_n\right), \tag{20}$$

$$V_{ij} = -\frac{i}{6} H_{abc} \operatorname{Tr}\left(X^c F_1 \cdots [X^a, F_i] \cdots [X^b, F_j] \cdots F_n\right), \tag{21}$$

$$W_{ijk} = -\frac{i}{12} H_{abc} \operatorname{Tr}\left(F_1 \cdots [X^a, F_i] \cdots [X^b, F_j] \cdots [X^c, F_k] \cdots F_n\right). \tag{22}$$

The foregoing construction can be generalized to the case where $\omega_{ab} = B_{ab} + F_{ab}(x)$. As has shown in [7] to this end it is necessary to replace

$$W_{ijk} \to \mathbf{W}_{ijk} = W_{ijk} - \frac{1}{n}(W_{ij} - W_{ik} + W_{jk}), \tag{23}$$

$$\begin{aligned} W_{ij} = {} & \frac{i}{24} H_{abc} \operatorname{Tr}\left(F_1 \cdots [[X^a, X^b], F_i] \cdots [X^c, F_j] \cdots F_n\right) \\ & - \frac{i}{24} H_{abc} \operatorname{Tr}\left(F_1 \cdots [X^a, F_i] \cdots [[X^b, X^c], F_j] \cdots F_n\right). \end{aligned} \tag{24}$$

The correct generalization $\mathbf{V}$ of $V(F_1, \dots, F_n)$ [7] together with (23) gives the final result for the $n$-point correlation function in this case

$$\mathbf{V} = \sum_{i,j,k} S_{ijk} \mathbf{W}_{ijk}. \tag{25}$$

## 3 Nonassociative algebra of functions on worldvolume of Dp-brane

According to [8] the generalization of the symplectic form $\omega_{ab} = B_{ab}$ to the one $\tilde{\omega}_{ab}(x) = \omega_{ab} + F_{ab}(x)$ gives in the zero slope limit [1] the correlation functions

$$\langle f_1(X(\tau_1)) \cdots f_n(X(\tau_n)) \rangle = \int V(\omega) d^{n+1}x f_1 * \cdots * f_n, \tag{26}$$

where now $*$ is the Kontsevich star product (because $d\omega(x) = 0$) [11]

$$(f * g)(x) = \exp\left[\frac{i}{2}\alpha^{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial y^b}\right] f(x) g(y)|_{x=y} \tag{27}$$

or

$$\begin{aligned} f * g = {} & fg + \frac{i}{2}\alpha^{ab}\partial_a f \partial_b g - \frac{1}{8}\alpha^{ac}\alpha^{bd}\partial_a\partial_b f \partial_c \partial_d g \\ & - \frac{1}{12}\alpha^{ad}\partial^d \alpha^{bc}(\partial_a \partial_b f \partial_c g - \partial_b f \partial_a \partial_c g) + \mathcal{O}\left(\alpha^3\right). \end{aligned} \tag{28}$$

Hence,

$$(f * g) * h - f * (g * h) = \frac{1}{6}\left(\alpha^{il}\partial_l \alpha^{jk} + \alpha^{jl}\partial_l \alpha^{ki} + \alpha^{kl}\partial_l \alpha^{ij}\right)\partial_i f \partial_j g \partial_k h + \mathcal{O}\left(\alpha^3\right). \tag{29}$$

If $\alpha$ is invertible and $d\left(\alpha^{-1}\right) = d\omega = 0$, from (29) it follows the associativity of the $*$ product.

As it was shown in the previous section the $*$ product (28) with $\tilde{\omega}_{ab}(x) = B_{ab} + \frac{1}{3}H_{abc}x^c$ gives the correlation functions defined by (26). But in this case the product becomes nonassociative (because $d\tilde{\alpha} = H \neq 0$). It is denoted by $\bullet$. The associator of the product $\bullet$ of functions is defined by the equation

$$(f \bullet g) \bullet h - f \bullet (g \bullet h) = \frac{1}{6}\alpha^{ia}\alpha^{jb}\alpha^{kc}H_{abc}\partial_i f \partial_j g \partial_k h + \cdots . \tag{30}$$

Because $H_{abc} \neq 0$ the product $\bullet$ is not associative. The two products $*$ and $\bullet$ given by the Kontsevich expansion (28) in terms of $\omega_{ab}$ and $\tilde{\omega}_{ab}(x) = \omega_{ab}(x) + \frac{1}{3}H_{abc}x^c$, respectively, are connected between themselves by the relation

$$f \bullet g = f * g + \frac{i}{12}H_{abc}\left\{x^c, [x^a, f]_* * [x^b, g]_*\right\}_*, \tag{31}$$

or in the operator form

$$F \bullet G = FG - \frac{i}{12}H_{abc}\left\{X^c[X^a, F][X^b, G]\right\}. \tag{32}$$

One can find more general relation

$$f_1 \bullet (f_2 \bullet \cdots \bullet (f_{n-1} \bullet f_n) \cdots) = f_1 * f_2 * \cdots * f_n + \sum_{i<j} V_{ij}. \tag{33}$$

## 4 Homotopy associative structure of algebra of functions on Dp-brane

With the help of the methods of the preceding section we can obtain the exact expressions of the first correlation functions of the model. The two-point correlation function is given by

$$P_2(f_1, f_2) = \int f_1 f_2 \left(1 + \frac{1}{3}H_{abc}x^a\theta^{bc}\right). \tag{34}$$

The three-point correlation function is written

$$\begin{aligned} P_3(f_1, f_2, f_3) = &\int f_1 * f_2 * f_3 + \frac{1}{3}B_{bc}K^{abc}\int f_1 * y_a * f_2 * f_3 \\ &- \frac{i}{6}K^{abc}\int \left(-\partial_a f_1 * y_c * \partial_b f_2 * f_3 - \partial_a f_1 * y_c * f_2 * \partial_b f_3 + f_1 * y_c * \partial_a * f_2 \partial_b * f_3\right), \end{aligned} \tag{35}$$

where $K^{abc} = \theta^{a\tilde{a}}\theta^{b\tilde{b}}\theta^{c\tilde{c}}$ and $y_a = B_{ab}x^b$. Every $n$-point correlation function can be represented in the operator form $P_n[F_1, \ldots, F_n]$. It depends on the $n-3$ conformal moduli of the insertion points $\tau_1, \ldots, \tau_n$ of the functions $F_1, \ldots, F_n$. The one-point correlation function

$$P_1[F] = \mathrm{Tr}\,(F) + \frac{2i}{3}H_{abc}\,\mathrm{Tr}\left(X^a X^b X^c F\right) \tag{36}$$

defines operator $P[F] := P_1[F]$. With the help of the operator $P$ one can represent the correlation functions

$$\begin{aligned} &P_1[F_1] = P[O_1(F_1)(\tau_1)], \qquad P_2[F_1, F_2] = P[O_2(F_1, F_2)(\tau_1, \tau_2)], \\ &P_3[F_1, F_2, F_3] = P[O_3(F_1, F_2, F_3)(\tau_1\tau_2\tau_3)], \\ &P_4[F_1F_2F_3F_4] = P[O_4(F_1, F_2, F_3, F_4)(\tau_1, \tau_2, \tau_3, \tau_4)], \end{aligned}$$

where

$$O_1[F_1] = F_1, \tag{37}$$

$$O_2[F_1, F_2](\tau_1, \tau_2) = F_1 \bullet F_2, \tag{38}$$

$$O_3[F_1,F_2,F_3](\tau_i) = L(1-x)(F_1\bullet F_2)\bullet F_3 + L(x)F_1\bullet(F_2\bullet F_3), \qquad x=\frac{\tau_{21}}{\tau_{31}}, \tag{39}$$

$$\begin{aligned} O_4[F_1,\dots,F_4](\tau_i) &= L\left[\left(1-\frac{x}{y}\right)\left(1-\frac{1-y}{1-x}\right)\right](F_1\bullet F_2)\bullet(F_3\bullet F_4) \\ &+ L\left[\left(1-\frac{x}{y}\right)\left(\frac{1-x}{1-y}\right)\right]((F_1\bullet F_2)\bullet F_3)\bullet F_4 + L\left[\frac{x}{y}\left(\frac{1-y}{1-x}\right)\right]F_1\bullet(F_2\bullet(F_3\bullet F_4)) \\ &+ L\left[\frac{x}{y}(1-y)\right](F_1(\bullet F_2\bullet F_3))\bullet F_4 + L\left[x\left(\frac{1-y}{1-x}\right)\right]F_1((\bullet F_2\bullet F_3)\bullet F_4), \\ x &= \frac{\tau_{21}}{\tau_{41}}, \qquad y=\frac{\tau_{31}}{\tau_{41}}. \end{aligned} \tag{40}$$

The function $Q_5[F_1,F_2,F_3,F_4,F_5](x,y,z)$ is written by means of the sum of product of functions $F_1$, $F_2$, $F_3$, $F_4$, $F_5$ of 14 terms corresponding the different ways to insert parenthesis:

$$\begin{aligned} &\{(((F_1\bullet F_2)\bullet F_3)\bullet F_4)\bullet F_5,\ ((F_1\bullet F_2)\bullet F_3)(\bullet F_4\bullet F_5),\ (F_1\bullet F_2)((\bullet F_3\bullet F_4)\bullet F_5), \\ &((F_1\bullet F_2)\bullet(F_3\bullet F_4))\bullet F_5,\ (F_1\bullet(F_2\bullet F_3))\bullet F_4)\bullet F_5,\ (F_1\bullet(F_2\bullet F_3))\bullet(F_4\bullet F_5), \\ &(F_1\bullet((F_2\bullet F_3)\bullet F_4))\bullet F_5,\ F_1\bullet(((F_2\bullet F_3)\bullet F_4)\bullet F_5),\ (F_1\bullet(F_2\bullet(F_3\bullet F_4))\bullet F_5, \\ &F_1\bullet((F_2\bullet(F_3\bullet F_4))\bullet F_5),\ F_1\bullet(F_2((\bullet F_3\bullet F_4)\bullet F_5)),\ (F_1\bullet F_2)\bullet(F_3\bullet(F_4\bullet F_5)), \\ &F_1\bullet(F_2\bullet(F_3\bullet(F_4\bullet F_5))),\ F_1\bullet((F_2\bullet F_3)\bullet(F_4\bullet F_5))\}. \end{aligned} \tag{41}$$

For correlation functions of the higher order this procedure can be continued. There exists conjecture that every correlation function $P_n[F_1,\dots,F_n]$ can be represented in the form

$$P_n[F_1,\dots,F_n](\tau_i) = P[O_n(F_1,\dots,F_n)(\tau_i)]. \tag{42}$$

The functions $O_n(F_1,\dots,F_n)(\tau_i)$ define mappings of the algebra of functions on the Dp-brane onto itself. For example, the role of the mappings $O_n$ we can see, for example, in the case $O_3$. The homotopy properties of the mapping

$$O_3[F_1,F_2,F_3](x) : [0,1]\times C^\infty(M)^{\times 3}\to C^\infty M \tag{43}$$

is ensured by the equation (39) and by the properties of the Rogers dilogarithm $L(x)$:

$$L(x)+L(1-x)=0, \qquad L(0)=0, \qquad L(1)=1. \tag{44}$$

The mapping (43) connecting of the products $(F_1\bullet F_2)\bullet F_3$ and $F\bullet(F_3\bullet F_3)$, corresponding by two different ways to stay the parenthesis in the product $F_1\bullet F_2\bullet F_3$ is the homotopy equivalence. The mapping $O_3(m)$ defines the $A_3$ homotopy associative structure on the nonassociative algebra $C^\infty(M)$. It is obvious that homotopies $O_n[F_1,\dots,F_n]$ play the same role for higher product as $O_3[F_1,F_2,F_3]$ for $F_1\bullet F_2\bullet F_3$.

The concepts of the homotopy spaces and the strong homotopy algebras are due to Stasheff [12], where it is shown that a topological space has homotopy type of a loop space if and only if it is strong homotopy associative one. The strong homotopy algebras has been found at number of the unexpected places: in the topological conformal field theory, in Morse theory. The "nonassociative manifold" is defined by means of a strong homotopy algebra of functions on them.

## 5 Charges of Dp-branes

Soon after Polshinski's identification of Dp-branes as nonperturbative objects in the perturbative string theory that carry R-R charge, Witten [2] suggest that the D-branes charges should take the values in a K-theory of the spacetime. The groups $K^0(M)$, $K^1(M)$ are associated with Dp-branes in $IIA$, $IIB$ string theory, respectively. The presence of the $B$ field introduces the corrections in evaluation of the charges. The charges of the Dp-branes in the topological case are dependent on the cohomolgy class $[H] \in H^3(M, Z)$ of the strength $H = dB$.

Let $K^j(M, \mathcal{E}_H) = K_j(C_0(M, \mathcal{E}_H))$, $j = 0, 1$, denotes $K_\bullet$-groups of $C^*$-algebra $C_0(M, \mathcal{E}_H)$ generated by the continuous sections vanishing on the infinity of the unique local trivial gauge bundle $\mathcal{E}_H$ whose structure depends on Dixmier–Douady invariant $[H]$.

It is destinguished three cases:

1. $[H] = 0$, $(B = 0, H = 0)$ [2].

   $\mathcal{E}_H$ is the gauge bundle with fibre $C^n$ and gauge group $\operatorname{Aut} C^n = U(n)$.

2. $[H] = 0$, $(B \neq 0,\ H = 0)$ [15].

   $\mathcal{E}_H$ is the gauge bundle with fibre $M_n(C)$, the matrix algebra of $n \times n$ dimension and the gauge group $\operatorname{Aut} M_n(C) = SU(n)/Z_n$. In this case $C_0(M, \mathcal{E}_H)$ is called by the Azumaya algebra.

3. $[H] \neq 0$, $(B \neq 0,\ H \neq 0)$ [15].

   $\mathcal{E}_H$ is the gauge bundle with fibre $\mathcal{K}$, algebra of compact operators in a Hilbert space and the gauge group $\operatorname{Aut} \mathcal{K} = \lim\limits_{n\to\infty} SU(n)/Z_m$. In this case $C_0(\mathcal{K}, \mathcal{E}_H)$ is called by the Rosenberg algebra.

In the bosonic string theory the physical interpretation of K-theory classes is less clear then in type $II$ superstring theory, since the branes carry no conserved charges and, likely, are unstable. Conjecture is that these K-theory classes of the algebra of functions on the Dp-branes label unequivalent unstable Dp-brane configurations.


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