$\overline{SL(3,R)} \times T^6$ AS A NUCLEAR COLLECTIVE MOTION GROUP

Piero Truini
Dipartimento di Fisica - I.N.F.N.
v. Dodecaneso, 33 - 16146 Genova, Italia

INTRODUCTION. The study of the collective motions of the nuclei and their description via appropriate collective coordinates is an old problem in quantum mechanics. One of the first succesful nuclear models of this type is the Bohr-Mottelson model, with collective coordinates represented by nuclear rotational motion, [1]. A new developement to the subject was given by the pioneering work of Arima and Iachello, [2], who used an algebraic method and established one of the most succesful application of group theory to Physics: the interacting boson model (I.B.M.). Group theoretically the I.B.M. is a realization of broken $U(6)$ symmetry. The intervention of group theory and its powerful techniques considerably simplifies the problem of finding an orthonormal basis which diagonalizes the Hamiltonian $\mathcal{H}$ of the model, produces analytical solutions to the eigenvalue problem for $\mathcal{H}$, and gives an overall view in which the limit cases appear as substructures (subgroups or contractions of groups).

The I.B.M., based on the group $U(6)$, represents a macroscopic viewpoint of the nuclei. The microscopic explanation of the I.B.M., [3], led to new approaches based on symmetry and to new groups.

Considerable progress has been made in the last few years in clarifying the conceptual basis and define, from a micriscopical point of view, the right collective variables, [4,5,6]. This paper presents a new group theoretical approach based on the group $\overline{SL(3,R)} \times T^6$, that is also denoted by $CM(3)$, the semidirect product of the covering group of $SL(3,R)$ and the abelian group T^6 ($\cong R^6$).

The group $SL(3,R)$ was introduced in nuclear physics by Tomonaga, [7], and independently by Gell-Mann et al., [8]. It is the group of rotations and volume-preserving deformations in 3-space, thus fitting an intuitive description of the nuclear matter, which is nearly incompressible. It was noted by Biedenharn and Vanagas, [9], that:

1) $SL(3,R)$ has a covering $\overline{SL(3,R)}$ which allows the existence of half-integer spin representations;
2) using the Wigner-Inõnu contraction $\overline{SL(3,R)} \to T^5 \times SU(2)$, in agreement with the heavy nuclei rotational bands;
3) $\overline{SL(3,R)}$ has 5 collective degrees of freedom and 3 rotational intrinsic ones (vortex spin). In order to enlarge the scope of the model one can add the quadrupole degrees of freedom plus, necessarily for group theoretical reasons, a monopole degree of freedom and obtain the group $\overline{Sl(3,R)} \times T^6 = CM(3)$;
4) $CM(3)$ has a structure analogous to the Poincaré group $\mathcal{P}$: they are both semidirect products ($\mathcal{P} = \overline{SL(2,C)} \times T^4$, $CM(3) = \overline{SL(3,R)} \times T^6$); they both have two invariants (mass and spin for $\mathcal{P}$, volume and vortex-spin for $CM(3)$); the rest frame for $\mathcal{P}$ corresponds to the spherical mass-quadrupole tensor, [9], and the boosts to the deformation operators of $CM(3)$; finally the intrinsic spin for $\mathcal{P}$, namely the

angular momentum in the rest frame, corresponds to the vortex-spin for $CM(3)$, the angular momentum in the spherical frame.

Vortex spin is fundamental in the study of the rotational properties of the nuclei: absence of vortex spin implies irrotational collective motion, small moments of inertia, spectra typical of vibrational degrees of freedom; non-vanishing vortex spin and large moments of inertia correspond to rigid motion.

The new approach that I present, whose details will appear in a forthcoming paper, [10], singles out the relevance of vortex spin. The analogy with the Poincaré group is exploited in the study of the unitary (infinite-dimensional) representations of $CM(3)$, whose carrier space is the Hilbert space of the wave funtions of the nucleus on the mass-quadrupole tensor, viewed as the space of physical variables. The mathematical technique used in this study is the same used for studying the unitary irreducible representations of the Poincaré group and is based on the induced representations for a semidirect product and the imprimitivity theorem, [11].

UNITARY IRREDUCIBLE REPRESENTATIONS OF CM(3). We start by considering the explicit realization of the algebra $s\ell(3,R)$ containing half-integer spin representations given in [12]. Let us recall this construction. It is an infinite dimensional representation of the algebra, written in terms of a pair of boson operators $a_i (i = 1,2)$ obeying the canonical commutation relations

$$[a_i, a_j^*] = \delta_{ij}$$

The boson operators are defined on the space $D = \text{span}\,(|j,m>)$ where

$$|j,m> = [(j+m)!(j-m)!]^{-1/2} a_1^{*j+m} a_2^{*j-m} |0> \tag{1}$$

with $a_i|0>= 0$, where $j = 0, \frac{1}{2}, 1, \ldots$ and $-j \leq m \leq j$. In D a natural scalar product can be defined:

$$< j,m|j',m' >$$

using the representation (1), the commutation relations between the boson operators and defining $< 0,0 >= 1$. We denote by K the Hilbert space that is the completion of D with respect to this scalar product.

The following operators are defined on D:

$$J_+ = a_1^* a_2, \quad J_- = a_2^* a_1, \quad J_0 = \tfrac{1}{2}(a_1^* a_1 - a_2^* a_2)$$

$$T_2^M = \left[\frac{3!}{(2+M)!(2-M)!}\right]^{1/2} (N+1)^{-1/2} \times \left[(a_1^*)^{2+M}(a_2^*)^{2-M} + (-a_2)^{2+M}(a_1)^{2-M}\right](N+1)^{-1/2}$$

for $-2 \leq M \leq 2$, M integer; $N = a_1^* a_1 + a_2^* a_2$.

The hermitian combinations of these operators generate $s\ell(3,R)$.

Following the theory of Flato, Simon, Snellmann and Sternheimer, [13], one can show that this representation of the algebra can be exponentiated to a unique representation on K of the universal covering group $\overline{SL(3,R)}$ of $SL(3,R)$. Let us denote by V such a representation. We use V in order to build a representation U of $\overline{SL(3,R)}$

isomorphic to the representation U^L of $\overline{SL(3,R)}$ induced by a unitary representation L of the subgroup $SU(2)$, the vortex-spin rotational group. Omitting the details that will appear in [10], the representation U on a suitably defined Hilbert space $\mathcal{H}$ reads:

$$g \to U_g : \quad (U_g f)(q) = V(g) f(\delta(g^{-1})q)$$

$$f \in \mathcal{H}, \quad g \in \overline{SL(3,R)}, \quad q \in X \cong \overline{SL(3,R)}/SU(2)$$

where δ denotes the covering homomorphism.

The representation U has the necessary features for describing $\overline{SL(3,R)}$ as a symmetry group of a physical system: the carrier space of U is the Hilbert space of functions defined on the space of the physical variables, which can be regarded as the space of the positive symmetric matrices $[q]_{ij} = \sum_{n=1}^{A} x_{in} x_{jn}$, where x_{in} are the coordinates of the A nucleons; the representation U is written in an explicitly covariant form.

We now have all the ingredients for building the irreducible unitary representations of CM(3). In fact, appealing to a general result of the theory of the unitary representations of a semidirect product, [11], we have that the irreducible unitary representations of CM(3) are in one to one correspondence with the induced irreducible imprimitivity systems of $\overline{SL(3,R)}$ based on T^6; they are in turn in one to one correspondence with the irreducible unitary representations of the vortex-spin $SU(2)$. We do not enter into details here but this result assures that *any* irreducible unitary representation of CM(3), based on the orbit X, can be written as:

$$\left(U_{g,p} f\right)(q) = e^{iTr(pq)} \left(U_g f\right)(q)$$

where $p \in \hat{T^6}$ the dual space of T^6.

CONCLUSIONS AND OPEN PROBLEMS. We have classified, up to isomorphisms, all the unitary irreducible representations of $CM(3)$ on the space of the physical variables, and we have expressed them in an explicitly covariant form. The vortex spin plays a crucial role in the theory. It naturally appears as an intrinsic degree of freedom, just like the spin of the elementary particles in the case of the Poincaré group. The fact that the covering group of $SL(3,R)$ is a subgroup of $CM(3)$ allows to get half-integer vortex-spin representations.

One open problem is to find a bigger group containing the half-integer vortex-spin irreducible representations of CM(3). $Sp(6,R)$, which contains as subgroup $SL(3,R)$, would not work since it can only contain integer vortex-spins. If one could find such a group one could extend the success of $Sp(6,R)$, [14]. Another problem is to carry on the analysis of the model, write an Hamiltonian and compare the spectrum with the experimental data. We shall face these problems in the subsequent paper [10].

REFERENCES.

1. A.Bohr, B.R.Mottelson: Nuclear Structure, Benjamin, New York, 1969.
2. A.Arima, F.Iachello: Ann. Phys. (New York) **99**, 253 (1976); **111**, 201 (1978); **123**, 468 (1979).
3. J.N.Ginocchio: Ann. Phys., **126**, 234 (1980).

4. L.Weaver, L.C.Biedenharn, R.Y.Cusson: Ann. Phys. **77**, 250 (1973).
5. V.V.Vanagas: Sov. J. Part. Nucl. **11**, 172 (1980).
6. D.J.Rowe: proceedings of the XVII Intern. Colloq. on Group Theor. Methods in Physics, St. Adele, Canada, June 27-July 2, 1988.
7. S.Tomonaga: Prog. Theor. Phys. Jap. **13**, 467 (1955).
8. R.F. Dashen, M.Gell-Mann: Phys. Lett. **17**, 275 (1965).
9. L.C.Biedenharn, V.Vanagas: invited contribution to the workshop on Microscopic Models in Nuclear Structure Physics, 1988.
10. L.C.Biedenharn, G.Cassinelli, P.Truini: in preparation.
11. V.S.Varadarajan: "Geometry of Quantum Theory" II ed., New York N.Y., Spinger Verlag 1985.
12. L.C.Biedenharn, R.Y.Cusson, M.Y. Han, O.L.Weaver: Phys Lett 42B, N.2, 257-260 (1972).
13. M.Flato, J.Simon, H.Snellman, D.Sternheimer: Ann. Scient. de l'École Norm. Sup. 5, 423-434 (1972).
14. D.J.Rowe: Rep. Prog. Phys. **48**, 1419 (1985).