DERIVATIONS OF SEMI-BASIC FORMS AND SYMMETRIES OF SECOND-ORDER EQUATIONS

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The physical background of this paper concerns the equations of Newtonian mechanics: we consider (autonomous) 2nd-order differential equations, represented by the vector field $\Gamma = v^i\,\partial/\partial q^i + f^i(q,v)\,\partial/\partial v^i$ on TM. Associated to Γ, there are two important sets of 2nd-order linear PDE's, namely

$$\Gamma^2(\mu^i) - \Gamma(\mu^j)\frac{\partial f^i}{\partial v^j} - \mu^j\,\frac{\partial f^i}{\partial q^j} = 0\,, \tag{1}$$

$$\Gamma^2(\alpha_i) + \Gamma\left(\frac{\partial f^j}{\partial v^i}\,\alpha_j\right) - \frac{\partial f^j}{\partial q^i}\,\alpha_j = 0\,. \tag{2}$$

The first is the equation for symmetries of Γ, (2) is the adjoint of (1). Geometrical objects on TM associated to any solution of (1) and (2) are respectively: a symmetry vector field $Y = \mu^i\,\partial/\partial q^i + \Gamma(\mu^i)\,\partial/\partial v^i$ and a 1-form of type $\alpha = \alpha_i\,dv^i + \Gamma(\alpha_i)\,dq^i$ which we call an adjoint symmetry of Γ. Their intrinsic characterization could be described as follows (see [1] and [2]). Consider on TM the following sets:

$$\begin{aligned} \mathcal{X}_\Gamma &= \{X \in \mathcal{X}(TM) \mid S([\Gamma, X]) = 0\}\,, \\ \mathcal{X}^*_\Gamma &= \{\alpha \in \mathcal{X}^*(TM) \mid \mathcal{L}_\Gamma(S(\alpha)) = \alpha\}\,, \end{aligned}$$

where $S = \partial/\partial v^i \otimes dq^i$ is the type (1,1) tensor field defining the vertical endomorphism on TM. Then, $Y \in \mathcal{X}(TM)$ is a symmetry of Γ iff $Y \in \mathcal{X}_\Gamma$ & $\mathcal{L}_\Gamma Y \in \mathcal{X}_\Gamma$, whereas α is an adjoint symmetry of Γ iff $\alpha \in \mathcal{X}^*_\Gamma$ & $\mathcal{L}_\Gamma\alpha \in \mathcal{X}^*_\Gamma$.

Observe that $\mathcal{X}_\Gamma$ and $\mathcal{X}^*_\Gamma$ are modules over $C^\infty(TM)$ for the product $F * Y = FY + \Gamma(F)\,S(Y)$ and $F * \alpha = F\,\alpha + \Gamma(F)\,S(\alpha)$. Having recognized the interest of these sets and their duality, it is natural to think of an extension of $\mathcal{X}^*_\Gamma$, i.e. to look for specific classes of differential forms on TM which constitute a realization of the abstract algebra of forms on the module $\mathcal{X}_\Gamma$. It turns out (see [2]) that an appropriate generalization is obtained as follows:

$$\Lambda^p_\Gamma = \{\omega \in \Lambda^p(TM) \,|\, S \lrcorner \omega \text{ is a form } \& \ \mathcal{L}_\Gamma(S \lrcorner \omega) = \omega\},$$

which is a module over $C^\infty(TM)$ via the product $F * \omega = F\omega + \Gamma(F)\, S \lrcorner \omega$ and can be given the structure of a graded algebra via the rule $\omega \overset{\circ}{\wedge} \rho = (S \lrcorner \omega) \wedge \rho + \omega \wedge (S \lrcorner \rho)$. In [2] we further investigated the elementary derivations of Λ_Γ and also paid attention to (1,1) tensor fields preserving $\mathcal{X}_\Gamma$ and $\mathcal{X}^*_\Gamma$, which are characterized by

$$\mathcal{T}^{1,1}_\Gamma = \left\{R \in \mathcal{T}^{1,1}(TM) \,|\, S \circ R = R \circ S \ \ \& \ \ S \circ \mathcal{L}_\Gamma R = 0\right\}.$$

REMARKS: Constructing this new kind of calculus, with all commutator and bracket relations one normally expects, was rather laborious, so that proofs were omitted. On the other hand, for all Γ-related objects introduced above, it is obvious that really only one set of coefficients is important (the others following by Lie derivation), which suggests that calculations and proofs might be economized by concentrating first on the semi-basic forms $S \lrcorner \omega$. A paper on the full theory of derivations of semi-basic forms, which is needed for that purpose, is in preparation [3]. In what follows, we briefly report on its main results and the connection with the theory developed in [2]. Note for a start that one could think of derivations of semi-basic forms as being those derivations of $\Lambda(TM)$ which preserve the subset $\Lambda_0(TM)$ of semi-basic forms. It turns out that this does not provide the right approach for our needs. Instead, we study the theory of derivations of $\Lambda(\tau)$: the graded algebra of differential forms along the projection $\tau : TM \longrightarrow M$ (isomorphic with $\Lambda_0(TM)$), whose elements act $C^\infty(TM)$-multilinearly on $\mathcal{X}(\tau)$, the set of vector fields along τ.

Local coordinate expressions for elements of $\mathcal{X}(\tau)$ and e.g. $\Lambda^1(\tau)$ read:

$$X = X^i(q,v)\left(\frac{\partial}{\partial q^i} \circ \tau\right), \qquad \alpha = \alpha_i(q,v)\,(dq^i \circ \tau).$$

Along with the algebra $\Lambda(\tau)$, we will further need vector-valued forms along τ, i.e. elements of $V(\tau) = \Lambda(\tau) \otimes \mathcal{X}(\tau)$.

Definition: A derivation D of degree r on $\Lambda(\tau)$ is a map $D : \ \Lambda(\tau) \longrightarrow \Lambda(\tau)$ with the properties

$$\begin{aligned}
&D(\Lambda^p(\tau)) \subset \Lambda^{p+r}(\tau)\\
&D(\alpha + \lambda\beta) = D\alpha + \lambda\, D\beta, \qquad \lambda \in \mathbf{R}\\
&D(\alpha \wedge \beta) = D\alpha \wedge \beta + (-1)^{pr}\,\alpha \wedge D\beta, \quad \alpha \in \Lambda^p(\tau)
\end{aligned}$$

Every theory of derivations necessarily will follow the pattern of the pioneering work of Frölicher and Nijenhuis [4]. The fundamental exterior derivative of our model, denoted by d^V, is the derivation corresponding to d_S in the identification of $\Lambda(\tau)$ with $\Lambda_0(TM)$. Its action on functions, for example, is given by $d^V F = (\partial F/\partial v^i)\,(dq^i \circ \tau)$. We then might say that D is of type i_* if it vanishes on functions and of type d^V_* if it commutes with d^V. Examples of such derivations are, respectively, for $L \in V^r(\tau)$, the derivation i_L of degree r-1, defined for $\omega \in \Lambda^p(\tau)$ by

$$i_L\omega(X_1,\ldots,X_{p+r-1}) = \frac{1}{(p-1)!r!}\sum_\sigma \mathrm{sgn}(\sigma)\,\omega\big(L(X_{\sigma(1)},\ldots),X_{\sigma(r+1)},\ldots\big)$$

and the derivation d_L^V of degree r, defined by $d_L^V = [i_L, d^V] = i_L \circ d^V - (-1)^{r-1} d^V \circ i_L$. One can show that every D has a unique decomposition $D = D_1 + D_2$ with D_1 of type i_* and D_2 of type d_*^V. Also: D_1 is i_{L_1} for some $L_1 \in V(\tau)$. At this point, however, the standard pattern breaks down: D_2 is not necessarily of the form $d_{L_2}^V$. This is caused primarily by the fact that every d_L^V vanishes on $C^\infty(M)$, a property not necessarily shared by a general D_2. As a result, we need a way for extending the ordinary d on $C^\infty(M)$ to a corresponding action on $C^\infty(TM)$ and this is most easily achieved by introducing a connection on TM. Such a connection induces a map $\xi^H : \mathcal{X}(\tau) \longrightarrow \mathcal{X}(TM)$. Putting $H_i = \xi^H\left(\frac{\partial}{\partial q^i}\circ\tau\right) = \frac{\partial}{\partial q^i} - \Gamma_i^j \frac{\partial}{\partial v^j}$, where Γ_i^j are the connection coefficients, we can introduce the derivation d^H of $\Lambda(\tau)$, which e.g. on $C^\infty(TM)$ is given by $d^H F = H_i(F)\,(dq^i \circ \tau)$.

With this extra tool at our disposal, we now arrive at a satisfactory classification of all derivations of $\Lambda(\tau)$. First, we will slightly change the terminology: from now on a derivation is said to be of type d_*^V, if it commutes with d^V and vanishes on $C^\infty(M)$, which amounts to saying that we are dealing with a derivation of the form d_L^V. Likewise, a derivation is said to be of type d_*^H, if it is of the form $d_L^H = [i_L, d^H]$ for some L. Our main classification result then can be expressed as follows.

Theorem: Every derivation D of $\Lambda(\tau)$ of degree r has a unique decomposition in the form $D = i_{L_1} + d_{L_2}^V + d_{L_3}^H$, with $L_1 \in V^{r+1}(\tau)$, $L_2, L_3 \in V^r(\tau)$.

Some other interesting features of this theory are reflected by the commutator properties $[d^H, d^V] = d_T^V$ and $\frac{1}{2}[d^H, d^H] = i_P + d_R^V$, with $T, R \in V^2(\tau)$ and $P \in V^3(\tau)$. Here, T corresponds to the torsion tensor of the connection and R is the curvature. In addition, we have the following kind of generalized Bianchi identities: $[d^H, d_T^V] = d_P^V$, $[d^H, d_R^V] = d_P^H$. It is interesting to know that vanishing of the torsion T actually means that the connection is one that can be associated to a given second-order field. Observe finally that $V(\tau)$ acquires a graded Lie algebra structure via the property $[d_{L_1}^V, d_{L_2}^V] = d_{[L_1,L_2]}^V$.

Understanding the relationship between this theory of derivations and the calculus referred to at the beginning involves two steps. The first step consists of prolonging forms and fields along $\tau : TM \longrightarrow M$ to corresponding objects along $\tau_{21} : T^2M \longrightarrow TM$, which we denote by the same symbol with upperindex 1. One can prove the following properties of such prolongations: for $G \in C^\infty(TM)$, $X \in \mathcal{X}(\tau)$, $\alpha, \beta \in \Lambda(\tau)$, $L \in V(\tau)$, we have that $S \lrcorner \alpha^1$ is a form and

$$\begin{aligned}
(G\,X)^1 &= (\tau_{21}^* G)\,X^1 + G^1\,S(X^1),\\
(G\,\alpha)^1 &= (\tau_{21}^* G)\,\alpha^1 + G^1\,(S \lrcorner \alpha^1),\\
(G\,L)^1 &= (\tau_{21}^* G)\,L^1 + G^1\,(S \circ L^1),\\
(\alpha \wedge \beta)^1 &= \alpha^1 \wedge (S \lrcorner \beta^1) + (S \lrcorner \alpha^1) \wedge \beta^1.
\end{aligned}$$

These properties are very similar to the relations characterizing the module structure of $\mathcal{X}_\Gamma$, Λ_Γ and the $\mathring{\wedge}$ product.

For the second step, note that a 2nd-order equation in normal form can be interpreted as a section $\gamma : TM \longrightarrow T^2M$. Composing the prolongations of step 1 with γ then turns these into genuine geometric objects on TM. For example: the composition of $X^1 : T^2M \to T(TM)$ with γ yields a vector field $X^1 \circ \gamma \in \mathcal{X}(TM)$. The net result is that we obtain isomorphisms $I_\Gamma : \Lambda(\tau) \longrightarrow \Lambda_\Gamma, \quad \alpha \longmapsto \alpha^1 \circ \gamma$ and $J_\Gamma : V(\tau) \longrightarrow V_\Gamma, \quad L \longmapsto L^1 \circ \gamma$. The theory of derivations of $\Lambda(\tau)$ (and $V(\tau)$) thus translates to a corresponding theory of derivations of Λ_Γ (and V_Γ) via a rule of the form $D_\Gamma = I_\Gamma \circ D \circ I_\Gamma{}^{-1}$.

CONCLUSIONS: The unpublished proofs of [2] are now redundant: they follow from the theory of derivations of $\Lambda(\tau)$. Moreover, we have obtained a much more complete picture. For example: we now know of V_Γ^p, whereas previously we had only considered $\mathcal{T}_\Gamma^{(1,1)} = V_\Gamma^1$. By far the most important conclusion is that we know something is missing if we only consider d^V: there is need for a connection and the corresponding horizontal derivation d^H. Such a connection does not impose extra prerequisites in this context, since a 2nd-order field Γ comes equipped with one.

As a final point of interest, let us come to a new interpretation of the equations for symmetries and adjoint symmetries. The connection determined by Γ leads to a generalized notion of covariant derivative ∇, with the following action on functions and vector fields along τ: $\nabla F = \Gamma(F)$ for $F \in C^\infty(TM)$, and

$$\nabla X = (\Gamma(\mu^i) + \Gamma^i_k \mu^k) \left(\frac{\partial}{\partial q^i} \circ \tau\right) \quad \text{for} \quad X = \mu^i \left(\frac{\partial}{\partial q^i} \circ \tau\right).$$

Then by duality we have

$$\nabla \alpha = (\Gamma(\alpha_i) - \Gamma^k_i \alpha_k)\,(dq^i \circ \tau) \quad \text{for} \quad \alpha = \alpha_i\,(dq^i \circ \tau) \in \Lambda^1(\tau)$$

and ∇ further extends to a derivation of $\Lambda(\tau)$ (and $V(\tau)$). We next find the following interesting commutators: $[d^V, \nabla] = d^H$, $[d^H, \nabla] = i_R - d^V_\Phi$. The vector-valued 1-form Φ thus defined helps to re-interprete geometrically the equations for symmetries and adjoint symmetries, from the point of view of the calculus of forms and fields along τ. Indeed, Eq.(1) now reads $\nabla\nabla X + \Phi(X) = 0$ and may be called the generalized Jacobi equation. Eq.(2) similarly becomes $\nabla\nabla\alpha + \Phi(\alpha) = 0$.

References

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