RESTRICTIONS OF UNITARY REPRESENTATIONS TO SUBGROUPS AND ERGODIC THEORY: GROUP EXTENSIONS AND GROUP COHOMOLOGY

by

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PREFACE

These notes are divided into two rather distinct parts, the first of which concerns the restriction of unitary representations of a group to one of its subgroups, and the connection of this with ergodic theory, while the second part concerns group extensions and the connection of this with unitary ray representations. Some background concerning representation theory is assumed and the reader should consult relevant portions of Mackey's notes [33], and survey article [34], Dixmier's book [9], and Chapter I of [3]. The square brackets refer to the common bibliography for both Part I and Part II at the end.

PART I. RESTRICTIONS OF UNITARY REPRESENTATIONS TO SUBGROUPS AND ERGODIC THEORY

1. INTRODUCTION

This first part concerns the general question of what happens when one takes a unitary representation of a locally compact group G, say an irreducible one, and restricts it to a subgroup H of G. One source of interest in this problem is ergodic theory as we will indicate below, but we believe a thorough study of this type of question will shed much light not only on representation theory as such but will produce much useful information concerning the structure of locally compact groups and their subgroups. Gelfand and Fomin [13] were perhaps the first to realize the relevance of this kind of problem concerning unitary representations for ergodic theory. They showed how one could study geodesic flows on surfaces of constant negative curvature by looking at unitary representations of the group $SL_2(R)$. This approach was extended by Parasyuk [43], Mautner [35], Green [1], Auslander and Green [2], and the author [39]. Part of these notes are an exposition of some of the results in [39] without proofs, and the reader is referred to this paper for further details. We shall also discuss some related results which will appear shortly. Chapter I of [3] contains an exposition of some aspects of the theory of unitary representations which we shall use as a general source both for Part I and Part II.

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We shall suppose for the moment in order to illustrate our approach that G is a Lie group with Lie algebra $\mathcal{G}$. Let π be a continuous unitary representation of G and let X be an element of $\mathcal{G}$. Then $x(t) = \exp(tX)$ is a one-parameter subgroup of G where $\exp$ denotes the exponential map of $\mathcal{G}$ into G, and so $\pi(x(t))$ is a one-parameter group of unitary operators. It has an infinitesimal generator, or in other words, there is a unique (usually unbounded) self adjoint operator A such that $\pi(\exp(tX)) = \exp(itA)$ where the second $\exp$ is understood in the usual way for unbounded operators. We write $d\pi(X) = iA$, but we shall not enter into more discussion concerning the definition and properties of these operators since this is discussed in other lectures in this Rencontres.

The problem that concerns us specifically is to determine for a given group G and given $X \in \mathcal{G}$, the various possibilities for the unitary type of the operator $A = id\pi(X)$. The object is to get results that hold for fixed G and X, and for an arbitrary representation. One might hope to be able to say that the spectrum of A is limited to a very few possibilities or that one can put limitations on the possible eigenvalues of A. If G is for instance the real line R, so that $\mathcal{G}$ is one-dimensional, then any self adjoint operator A defines a unitary representation of R by $\pi(t) = \exp(itA)$. In this case we can extract no information concerning A, and in fact the same situation holds for any vector group G. Our results will concern exactly the opposite case, namely when G is semi-simple.

This same problem can be viewed slightly differently; suppose that $H = \{\exp(tX)\}$, $X \in \mathcal{G}$ is a one-parameter subgroup of G and suppose that $\pi(\exp(tX)) = \exp(itA)$ is a representation of H. We assume that this representation can be extended to a representation of G, and then ask what conclusions concerning the operator A can be drawn from this fact. Clearly whether one starts with a representation of G and restricts to H, or whether one starts with a representation of H and assumes that it extends to G comes to exactly the same thing. It is only a matter of emphasis.

Such problems are relevant in physics for if G is some postulated symmetry group of a quantum mechanical system, one has associated a unitary representation π of G on the Hilbert space associated to the system. (This is not quite true, but rather one has a ray representation of G; at our present heuristic level this doesn't matter, and in fact can be gotten around by well known methods to be treated in Part II of these lectures.) In any case, the operators $A = -id\pi(X)$ for various X in $\mathcal{G}$ have in many cases natural physical interpretations such as energy, momentum, angular momentum, and so on. It is an obvious question to ask what one can conclude about the spectrum or unitary type of these operators, based solely on the fact that they form part of the infinitesimal generators of a unitary representation of a larger group of some specified algebraic structure.

We have spoken about the restrictions of representations of a group G to one-parameter subgroups H. One can raise the same kinds of questions for larger subgroups of G. Some of the theorems below make sense in this generality and we shall state them in that form.

2. STATEMENT OF RESULTS

We turn now to the statement of our results, the first of which concerns as a special case the study of possible eigenvalues for infinitesimal generators as discussed above. In order to formulate the theorem it is convenient to introduce the following definition. Let G be a locally compact group and H a subgroup; we shall say that H has property E (in G) if for every representation π of G, and for every vector v in the Hilbert space of this representation such that $\pi(h)v = v$ for every $h \in H$, we have $\pi(g)v = v$ for every $g \in G$. In other words, if we have a representation π of G and are looking for invariant vectors for the restriction of the representation to the subgroup H, the condition says that we have only the obvious ones, namely the G-invariant vectors. This definition singles out a property which a subgroup H may or may not have which will be quite relevant for ergodic theory. If H is a one-parameter group $\{\exp(tX)\}$ corresponding to an element of the Lie algebra, then the condition essentially forbids the infinitesimal generator $A = -id\pi(X)$ from having 0 as an eigenvalue, unless of course the representation π of G has G-invariant vectors.

We want also to single out a slightly stronger property that a subgroup may have. More precisely, we say that H has property WM if for every representation π of G and every finite dimensional subspace V of the Hilbert space of the representation such that $\pi(h)V \subset V$ for every $h \in H$, we must have $\pi(g)v = v$ for every $g \in G$ and every $v \in V$. This condition, if satisfied, forbids the restriction of any representation π of G to H to have any finite dimensional subrepresentations other than the obvious ones. If again H is a one-parameter subgroup of a Lie group, the condition forbids the infinitesimal generator $A = id\pi(X)$ from having any eigenvalues. The terminology E and WM is motivated by ergodic theory and as we shall see later, whenever these conditions are satisfied, one may infer results asserting that certain group actions are ergodic (E) or weakly mixing (WM).

We note that a subgroup H has property E or WM (in G) if and only if its closure $\overline{H}$ does, and so it would suffice to consider closed subgroups. It is easy to see that a proper compact subgroup H of a group G can never have property E be examining the representation π of G induced by the trivial one-dimensional representation of K. Furthermore, if G is abelian, no proper closed subgroup can have property E. Our goal is at least in some cases to characterize those subgroups of a given group which have one or both of these properties, and from the two examples above, we see that we are going to have to assume that G is sufficiently non-commutative, and that H is sufficiently non-compact.

Semi-simple Lie groups are certainly one of the most important classes of groups, and they are in a sense as non-commutative as possible. The main result below will characterize those subgroups of such a group which have properties E and WM. This result is contained in [39] and we refer the reader to this paper for a more detailed discussion.

If G is a semi-simple Lie group, let G^* be its adjoint group, that is, G/Z, where Z is the center of G. It is well known that G^* is the product $\Pi_{i=1}^{n} G_i^*$ of a finite number of simple Lie groups, each having center reduced to the identity element. This is the global version of the decomposition of the Lie algebra of G into a sum of simple ideals. Let p_i denote the projection of G onto G_i^*. We shall say that a subgroup H of G is <u>totally non-compact</u> if $p_i(H)$ has non-compact closure in G_i^* for each i. Intuitively this says that H sticks out non-compactly in the adjoint group of each simple factor of G. If G is simple with finite center, the condition is simply that the closure of H be non-compact.

<u>Theorem 1</u>

For a semi-simple group G and a subgroup H of G, the following are equivalent

(a) H is totally non-compact.

(b) H has property E.

(c) H has property WM.

Thus for a totally non-compact subgroup H of G and any representation π of G which has no G-invariant vectors, we can conclude that the restriction of π to H has no invariant vectors, or for that matter no finite dimensional invariant subspaces. Even if H only partially satisfies the non-compactness conditions we can still extract information. For instance if $H \subset G_1 \times G_2$ and if the projection of H into the first factor G_1 is totally non-compact, then one can conclude that any finite dimensional subspace for H is left fixed pointwise by G_1. This follows by a detailed analysis of the proof in [39]. If we specialize the theorem to the case of a one-parameter subgroup $H = \{\exp(tX)\}$, $X \in \mathcal{G}$, we can conclude that the operator $A = -id\pi(X)$ has no eigenvalues provided H is totally non-compact.

This result overlaps with the O'Raifeartaigh theorem [42], and gives a stronger conclusion under much stronger hypotheses. To check that a one-parameter group is totally non-compact is in any given situation, a rather routine matter. The result above contains as special cases the results of Gelfand and Fomin, Parasyuk and Mautner mentioned above.

For one-parameter groups we can in fact get much more information concerning the infinitesimal generator $A = -id\pi(X)$ when $X \in \mathcal{G}$ generates a totally non-compact one-parameter subgroup. The unitary type of A is in fact limited to a rather small number of possibilities. We introduce the Hilbert space $H^+(n)$ which is to consist of all square integrable functions on the interval $(0,\infty)$, with Lebesgue measure, taking values in a standard n-dimensional Hilbert space H^n. Here n is an integer or $+\infty$. Let $H^-(n)$ and $H(n)$ denote the similar spaces of functions on $(-\infty,0)$, and on $(-\infty,\infty)$, and let $M^{\pm}(n^{\pm})(M(n))$ denote the unbounded self adjoint operator on $H^{\pm}(n)(H(n))$ which is multiplication by the scalar function $f(x) = x$. We note that $H^{\pm}(n^{\pm})$ is a non-negative (respectively a non-positive) operator.

Theorem 2

Let G be semi-simple and let $H = \{\exp(tX)\}$ be a totally non-compact one-parameter subgroup, and π be a representation of G with no G-invariant vectors, and let $A = -id\pi(X)$. Then if we write $A = A^+ + A^-$ where A^+ and A^- are the positive and negative parts of A, there exists $n^{\pm}$ such that $A^{\pm}$ is unitarily equivalent to $M^{\pm}(n^{\pm})$.

The proof of this is contained in [39] and the reader is referred to that paper for the details. We also note that if we have a single element g of G such that its powers g^n form a totally non-compact subgroup, we can obtain an entirely analogous result for the unitary type of the operator $\pi(g)$ (see [39]).

The result above for one-parameter subgroups is best possible in that all choices of n^+ and n^- occur, and if one considers a subgroup for which the hypothesis fails, then one can find a representation for which the conclusion fails. In fact, for $G = SL_2(R)$ and for a one-parameter subgroup generated by a nilpotent matrix all possible choices of n^+ and n^- occur. The situation for irreducible representations of $SL_2(R)$ is quite interesting; for the principal and complementary series, $n^+ = n^- = 1$, and for one discrete series, $n^+ = 1$, and $n^- = 0$, while $n^+ = 0$, and $n^- = 1$ holds for the other discrete series. The one representation of the principal series which is not irreducible decomposes into two irreducible summands which behave like discrete series for $n^{\pm}$.

For higher dimensional semi-simple groups, the situation becomes a bit simpler. More precisely, if we exclude any group G which has a simple factor locally isomorphic to $SL_2(R)$, the only possible choices for n^+ and n^- are either 0 or ∞. This fact is implicit in the argument contained in [39]. Thus in this case $A = -id\pi(X)$ for a totally non-compact one-parameter subgroup is up to unitary equivalence, one of three types, $M^+(\infty)$, $M^-(\infty)$ or $M(\infty)$. Since changing X into $-X$ or replacing π by its contragradient representation will interchange M^+ and M^-, we really have only two distinct cases, which we can classify as one sided spectrum or two sided spectrum.

We can raise the question of when every totally non-compact one-parameter group has two sided spectrum for every representation. S. Scull in a dissertation in progress has shown that this is true for $SL_n(R)$ for $n \geq 3$. B. Kostant has proved that this is also true whenever the Weyl group of the maximal compact subgroup

K of G contains the element -1. In this connection we should also remark that not every element of $Sp_n(R)$ has one sided spectrum as we shall see in a later section of these notes.

Finally we should like to indicate one application of these results to qauntum physics. Let P denote the Poincaré group, and let us assume that P is a subgroup a larger symmetry group G of unknown origin. We shall assume that G is semi-simple, and that π is a representation of G on a Hilbert space. To extract physical information one would restrict π to P and decompose it, and Theorems 1 and 2 above supply information about what this decomposition can look like. Indeed let G_1 be the largest normal subgroup of G containing P and let us assume that there are no G_1 invariant vectors since such vectors cannot be of interest. Now let X be the element of the Lie algebra of P corresponding to translation in time so that $A = -id\pi(X)$ is the energy. The spectrum of A controls to some extent the representations of P that can occur since in an irreducible representation of mass m, the energy operator has spectrum $[m,\infty)$.

Theorems 1 and 2 and the remarks following them give us the following result.

Theorem 3

Under the above hypotheses, A is unitarily equivalent either to $M^+(\infty)$, $M^-(\infty)$ or $M(\infty)$, as defined in Theorem 2.

Proof. It is a simple algebraic matter to verify that the one-parameter subgroup generated by X is totally non-compact in G_1, and that G_1 has no factors locally isomorphic to $SL_2(R)$ so the result follows. (The result concerning multiplicities it should be noted, is obvious on other grounds once one has the spectrum of A.)

This result says that in the decomposition of π on P, we must find representations corresponding to arbitrarily small mass or zero mass or imaginary mass. For a survey of the representations of P, see the article of O'Raifeartaigh in this volume.

As we have said before, we shall not enter into the details of the proofs of Theorems 1 and 2. There is, however, one important fact, Lemma 4.2 of [39] used in the proof of Theorem 1, for which we now have an alternate argument. This lemma says that a one-parameter subgroup of the universal covering group of $SL_2(R)$ has property E. The argument in [39] is based on infinitesimal methods due to T. Sherman. The alternate argument is global in nature and has in addition the property that it works for $SL_2(k)$, where k is a p-adic field, and also for the covering groups of this group defined in [40]. To carry this out however, one needs the algebraic analysis of the covering groups of $SL_2(R)$ and $SL_2(k)$ contained in

[40]. We will carry out the proof for $SL_2(R)$ with the understanding that if one redefines the meaning of the symbols that we will introduce, the argument will carry over word for word to the general case.

We may clearly assume that the one-parameter subgroup under consideration is $x(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. We let $y(t)$ be the transpose of the matrix $x(t)$, and we define $w(t) = x(t)y(-t^{-1})x(t)$ and $h(t) = w(t)w(-1)$. It may be verified that $h(t)$ is a diagonal matrix with entries t and t^{-1}. Let us suppose that π is a unitary representation of $SL_2(R)$ and that v is a vector of unit length such that $\pi(x(t))v = v$ for all t. An easy calculation shows that $(\pi(w(t))v,v) = (\pi(y(-t^{-1}))v,v)$ and if we let $|t| \to \infty$, we see that $\lim(\pi(w(t))v,v) = 1$. We let $u = \pi(w(-1))^{-1}v$ so that $(u,u) = 1$, and note that $(\pi(h(t))u,v) \to (v,v) = 1$ as $|t| \to \infty$. We then write $\pi(h(t))u = a(t)v + s(t)$ where $s(t)$ is orthogonal to v, and note that $|a(t)|^2 + |s(t)|^2 = 1$. Since $(\pi(h(t))u,v) = a(t)$ we see that $|s(t)| \to 1$ as $|t| \to \infty$, or equivalently, $\pi(h(t))u \to v$. It follows immediately that $\pi(h(s^{-1}t))u \to v$ for any $s \neq 0$ and then that $\pi(h(s))v = v$. Since the one-parameter group $h(t)$, $t > 0$, has property E (this is Mautner's lemma [35]), it follows that $\pi(g)v = v$ for $g \in SL_2(R)$. This completes the proof of the lemma in question.

3. APPLICATIONS TO ERGODIC THEORY

We shall now turn our attention to the applications of these results to ergodic theory and defer for a later section a treatment of some more questions concerning the restriction of representations to subgroups. These final results have little direct connection with ergodic theory whereas Theorems 1 and 2 have a very direct connection.

Let us first introduce the setting in which we are going to study ergodic theory. Let M be a Borel space, that is, a set equipped with σ-field of subsets, called the Borel sets, and let G be a locally compact group, separable in the sense of the second axiom of countability. We shall suppose that G acts on M as a transformation group so that we have specified a map f of $G \times M \to M$, written $f(g,m) = g \cdot m$ such that for fixed $g \in G$, the map $m \to g \cdot m$ is a bijective map of M onto itself and such that the function that associates to each g in G, this bijective map, is a homomorphism of G into the group of all such maps of M into itself. We say that G is a Borel transformation group if the map $f(g,m) = g \cdot m$ is a Borel map from $G \times M$ into M where G is given the σ-field of sets generated by the open sets and where $G \times M$ is given the product Borel structure. A function f is a Borel function if $f^{-1}(E)$ is a Borel set in the domain for every Borel set in the range. Thus not only is $m \to g \cdot m$ a Borel automorphism of M, but this Borel automorphism varies "smoothly" with g. The reader should consult Chapter I of [3] for further details.

For any Borel set E of M, $g \cdot E = \{g \cdot m | m \in E\}$ is a Borel set called the transform of E by g. If μ is a measure on M, we define the transform $g \cdot \mu$ of μ by g using the formula $g \cdot \mu(E) = \mu(g^{-1} \cdot E)$. We say that μ is G-invariant (or simply invariant) if $g \cdot \mu = \mu$ for all $g \in G$. Unfortunately, many interesting measures arising in practice fail to have this property, but possess instead the weaker property of quasi-invariance. A measure μ is quasi-invariant if μ and $g \cdot \mu$ are equivalent in the sense of absolute continuity, or more concretely, $\mu(E) = 0 \Leftrightarrow \mu(g \cdot E)$ for all $g \in G$.

If G is the additive group of the integers, then specifying an action of G on M is the same thing as specifying the Borel automorphism h or M corresponding to the group element one. If $h \cdot \mu = \mu$ for some measure or if $h \cdot \mu$ is equivalent to μ then μ is invariant (or respectively quasi-invariant). The case of a single measure preserving automorphism of a finite measure space is the classical setting for ergodic theory (see [15]). If G is the real line, an action of G consists in giving a one-parameter group $h(t)$ of Borel automorphisms of M subject to the joint measurability condition. This condition is readily verifiable in cases of interest and indeed in general it is a condition that permits us to work with actions of groups that are not discrete groups.

If M is a compact manifold and if X is a C^∞ vector field on M, then the usual existence theorems for ordinary differential equations provide us with a one-parameter group $h(t)$ of diffeomorphisms of M such that $\dot{h}(t) = X(h(t))$ and such that $(t,m) \to h(t)(m)$ is a C^∞ map and hence certainly Borel. Such a flow may or may not leave invariant a measure, but if for instance X is of Hamiltonian type, then Liouville's theorem provides an invariant measure. Since this subject is discussed in Kostant's article in this volume we will not go into more details here (see also [4]).

Before proceeding in our general context of which we have seen several examples above, we must impose a regularity condition on M of a technical nature; more precisely, we shall assume that M is an analytic Borel space. The reader is referred to [3] for further exposition concerning this condition; in any case it is a condition that is satisfied in all reasonable examples. Suppose that μ is a quasi-invariant measure on M for G. (In fact this is not just a property of μ, but rather a property of the set of all measures equivalent to μ, so we may speak of a quasi-invariant measure class.) One says that G acts ergodically on M, with respect to the measure μ, or that μ is an ergodic measure if whenever we have $g \cdot E = E$ for all $g \in G$, and some Borel set E, then $\mu(E) = 0$ or $\mu(M - E) = 0$. In other words, the only invariant Borel sets under the action are null sets or their complements. It also says that the action is indecomposable in that we cannot write $M = M_1 \cup M_2$ where M_1 and M_2 are disjoint invariant Borel sets of positive measure.

A rather natural modification of this definition consists in assuming that whenever $\mu(g \cdot E \Delta E) = 0$ for all $g \in G$ and some Borel set E, then $\mu(E) = 0$ or $\mu(M - E) = 0$. Here $g \cdot E \Delta E$ denotes the symmetric difference of the two sets, that is, the points in one but not the other of the two sets. A set with $\mu(g \cdot E \Delta E) = 0$ might be called almost-invariant, and one would be asserting that any such set is a null set or the complement of a null set. The second definition of ergodicity is clearly more restrictive than the first, and if G is countable they can easily be seen to be equivalent. For a general locally compact group it is a non-trivial result of Mackey (see [32]) that the two conditions are equivalent.

Suppose now that μ is a finite invariant measure for an action of the integers. This as we have seen is specified by a single measure preserving transformation, and if the action is ergodic one has the Birkhoff ergodic theorem [15]. For $f \in L_1(\mu)$,

$$\lim_{n\to\infty} \frac{1}{n+1} \Sigma_{i=0}^{n} f(u^i(x)) = \int f d\mu$$

for almost all x. (There is a similar statement for ergodic actions of the real line.) If we interpret this formula in its classical context of statistical mechanics where $u(x)$ is the evolution of a state x after unit time, then the left-hand side is the time average of a function f (a dynamical variable) and the right-hand side is the phase average of the same dynamical variable. The equality of these two averages is to hold for almost all initial states of the system. In fact it is not difficult to see that the validity of such a formula is equivalent to ergodicity. The question of equality of time averages and phase averages has a long history in statistical mechanics, and the ergodic theorem just reduces the question to the problem of showing that certain actions are ergodic (see [24]).

Not only for this reason, but for many others, one of the fundamental questions in ergodic theory is to supply sufficient conditions for an action or a class of actions to be ergodic. Our object here is to review a general method one has available for doing this by means of unitary representations and to apply the results of the previous section. We have remarked before that the present technique was initiated by Gelfand and Fomin for geodesic flows, although the observation that ergodic theory and unitary representations are closely connected goes back to Koopman.

We shall assume now that we are dealing with actions of a group G on a space M with a finite invariant measure μ, and we may assume without loss of generality that $\mu(M) = 1$. One can then define an associated unitary representation π of G on the Hilbert space $L_2(M,\mu)$. More precisely for $f \in L_2$, and $g \in G$, we define $(\pi(g)f)(x) = f(g^{-1} \cdot x)$. It is easy to verify that for each $g \in G$, $\pi(g)$ is a unitary operator, and using joint measurability of the action, one can show that π is a continuous unitary representation of G. The key observation is that one can detect ergodicity of an action merely by looking at π. Heuristically, the non-existence of invariant sets is equivalent to the non-existence of invariant measurable functions, and since the space has finite measure this is equivalent to the non-existence of square summable invariant functions.

Lemma 3.1

The action of G on M is ergodic if and only if $\pi(g)f = f$ for all $g \in G$ implies that f is a constant almost everywhere, and hence a constant in $L_2(M,\mu)$.

There is a somewhat stronger notion than ergodicity which is often useful, namely weak mixing. For this one must define the Cartesian square of an action of G on M. We notice that G acts on $M \times M$ by $g \cdot (m,n) = (g \cdot m, g \cdot n)$ and that the product measure $\mu \times \mu$ is invariant. One says that the action of G on M is weakly mixing [15] if the action of G on $M \times M$ is ergodic. This condition trivially implies ergodicity, and if it is satisfied, every Cartesian power of the action is weakly mixing and ergodic. Weak mixing can also be detected by looking at the unitary representation π (see [15] and [39]).

Lemma 3.2

The action of G on M is weakly mixing if and only if any finite dimensional subspace V of $L_2(M,\mu)$ invariant under π consists of constant functions (and hence is one-dimensional).

These lemmas serve to motivate the definitions of properties E and WM in Section 2 above since these definitions concern invariant vectors and finite dimensional subspaces of representations of a group. Finally, if G is the real line R or the integers Z, an action may possess the yet stronger property of strong mixing. To motivate this we note that the ergodic theorem implies that for any pair of measurable sets A and B

$$\lim_{t\to\infty} \mu(h(t)A \cap B) = \mu(A)\mu(B)$$

in the sense of Cesaro limits. Here $h(t)$ is the action defined for $t \in R$ or $t \in Z$. One says that the action is strongly mixing if the above limit exists in the usual sense [15]. One can find a sufficient condition in terms of the representation π for this to be the case; indeed by Stone's theorem [25], there exists a projection valued measure P on the Borel sets of the dual of G (R or the circle group T) corresponding to π. We shall say that P is absolutely continuous if $P(E) = 0$ if E is a Lebesgue null set. See [15] for the following.

Lemma 3.3

If G is as above, then an action of G is strongly mixing provided that the subrepresentation of π on the orthogonal complement of the constant function has an absolutely continuous projection valued measure.

It is clear that strong mixing implies weak mixing and that Theorem 2 of Section 2 is exactly the sort of result that will enable us to establish that certain actions are strongly mixing.

More specifically the theorems from Section 2 will be applied in the following manner. Suppose that we have an action of a group H on M with a finite invariant measure, and suppose that G is a larger group containing H which acts on M preserving μ such that the action of H on M determined by the fact that H is a subgroup of G coincides with the given action of H on M. In other words, we are assuming that the given action of H on M may be "embedded" in the action of a larger group G. We shall assume that the larger group G acts ergodically, and then ask if we can conclude from this that H also necessarily acts ergodically. Equivalently we can start with an action of G on M known to be ergodic by some method, and pick a subgroup H of G, and ask if H also acts ergodically. Properties E and WM from Section 2 are immediately relevant to this situation.

Theorem 4

Suppose that H has property E (respectively WM) in G, and that G acts ergodically on M with a finite invariant measure. Then H is also ergodic (respectively weakly mixing).

Proof. We consider the representation π of G on $L^2(M,\mu)$. If f is an invariant function for H then by property E, it is an invariant function for G, and by ergodicity of G, f is a constant. Hence H acts ergodically; by the same argument we can conclude weak mixing if H has property WM.

The following follows immediately using Theorem 1.

Corollary. If G is semi-simple and acts ergodically on M as above, and if H is totally non-compact in G, then H is ergodic and weakly mixing.

If H is a subgroup of G isomorphic to the real line (or the integers) we can also obtain results concerning strong mixing using Theorem 2.

Theorem 5

Let G be semi-simple and let $H \subset G$ be totally non-compact, and isomorphic to the real line (or the integers). If G acts ergodically on M as above, then the action of H is strongly mixing.

Although the hypothesis of the above theorems, as far as H is concerned, may seem rather special, their interest lies in the observation that this hypothesis of embedability in a larger group G is satisfied in many cases. Indeed it is satisfied for some classical flows, that seemingly have no connection with group theory. One way that an action of a group G can be seen to be ergodic is if it is transitive; that is if m and n are given points of M, there is an element $g \in G$ such that $g \cdot m = n$. If we put $\Gamma = \{g | g \cdot m = m\}$ for a fixed m, one can identify the coset space G/Γ with M by means of the map $g\Gamma \to g \cdot m$. With our hypotheses on M, one can conclude that this is a Borel isomorphism and that Γ is closed so that we may as well assume that M is G/Γ. (See [30] and [3] for an exposition of the details of this reasoning.) The action of G on G/Γ is then given concretely by $g \cdot (h\Gamma) = gh\Gamma$. Our assumption that M has a finite G-invariant measure means that G/Γ also possesses such a measure, and this places rather severe restrictions on what Γ can be (see [7] for instance). A transitive action is immediately seen to be ergodic by the first of the two definitions of ergodicity above.

One of the simplest examples of an action of the real line which is embedable in a transitive action is that of a rotation on a torus. Let $T^n = \{(z_1, \cdots, z_n),\ z_i$ complex numbers with $|z_i| = 1\}$ be an n-torus. We pick real numbers $a_1, \cdots, a_n$ and let R act on T^n by $t \cdot (z_1, \cdots, z_n) = (\exp(ita_n)z_1, \cdots, \exp(ita_n)z_n)$. This action is well known to be ergodic if and only if the a_i are rationally independent [15]. We observe, however, that this action can be embedded in a transitive action of R^n since $T^n = R^n/Z^n$ where Z^n is a lattice. The classical proof of the result quoted above is based in its essence on this observation. Leon Green [1] has established a beautiful generalization of this result to nil manifolds which again uses exactly the same ideas (see also [2]). Gelfand and Fomin [13] observed that geodesic flows on surfaces of constant negative curvature are embedded naturally in transitive actions of the group $G = SL_2(R)$; Parasyuk [43] observed the same for horocycle flow, and Mautner did the same for geodesic flow on certain higher dimensional manifolds [35]. The proofs of ergodicity are all based on the same idea and the following general result subsumes all such results concerning semi-simple groups, and follows immediately from Theorems 4 and 5.

<u>Theorem 6</u>

Let G be a semi-simple Lie group and let Γ be a subgroup such that G/Γ has a finite invariant measure, and let H be totally non-compact. Then the natural action of H on G/Γ is ergodic and weakly mixing. If H is the line (or the integers), then the action is strongly mixing.

Many examples of subgroups Γ satisfying this condition are known; for instance $SL_n(Z)$, the subgroup of $SL_n(R)$ consisting of matrices with integral entries (see [8]).

We shall close this brief discussion of one aspect of ergodic theory with a duality theorem of sorts which was noticed independently by G. W. Mackey and the author (see [39]). Above we were dealing with two subgroups H and Γ of G but there was an assumed asymmetry since we let H act on G/Γ. We can just as well let Γ operate on $H\backslash G = \{Hg \mid g \in G\}$ by $\gamma \cdot Hg = Hg\gamma^{-1}$. The following fact holds for any pair of closed subgroups Γ and H of G.

Lemma 3.4

The action of H on G/Γ is ergodic if and only if the action of Γ on $H\backslash G$ is ergodic.

In general a coset space G/Γ or $H\backslash G$ has no invariant measure, much less a finite one, but it always has a unique quasi-invariant measure class [28], and it is with respect to this measure class that the above lemma applies.

As an application of this, let $G = SL_n(R)$ and let $\Gamma = SL_n(Z)$, the subgroup of matrices with integral entries, and let H denote the subgroup of G consisting of matrices with first column $(1,0,\cdots,0)$. Then G/Γ has finite volume and H is totally non-compact if $n \geq 2$. Since H is ergodic on G/Γ, the duality principle says that Γ is ergodic on $H\backslash G$. The space $H\backslash G$ is easy to identify and is in fact R^n minus the origin with Lebesgue measure, and the action of G on this space is the natural linear action. Since a single point is a Lebesgue null set, $H\backslash G$ is measure theoretically the same as R^n and we have the following result.

Theorem 7

The linear action of $\Gamma = SL_n(Z)$ on R^n with Lebesgue measure is ergodic. Moreover, the same is true for any Γ such that $SL_n(R)/\Gamma$ has a finite invariant measure.

This easily stated result does not appear to be amenable to any direct approach.

4. MORE ON RESTRICTIONS

In Section 2 we studied certain aspects of the general problem of restricting a representation of a semi-simple Lie group to a subgroup and examining how it decomposes. These results are of immediate interest in ergodic theory and the connection was discussed in Section 3. In this final section we want to discuss some additional questions concerning the restriction of representations to subgroups. Theorem 2 for instance concerned the restriction of representations to one-parameter subgroups, and it is natural to raise similar questions concerning the restriction of representations to more general subgroups H, particularly general abelian subgroups. From the general version of Stone's theorem for abelian groups H [25], we know that any unitary representation of H leads to a projection valued measure on the Borel subsets of the dual group $\hat{H}$. Together with an appropriate multiplicity function (see [33]) this projection valued measure determines the representation. We are interested in the equivalence class of this projection valued measure P and in particular we would like to compare it to Haar measure on $\hat{H}$. We will say that P is absolutely continuous (with respect to Haar measure) if $P(E) = 0$ for any Haar null set $E \subset \hat{H}$, and that P is equivalent to Haar measure if $P(E) = 0$ if and only if E is a Haar null set. Theorem 2 above says in particular that if G is semi-simple and if H is a totally non-compact one-parameter subgroup of G and if π is a representation of G with no G-invariant vectors, then the projection valued measure on $\hat{H}$ associated with the restriction of π to H is absolutely continuous. It is natural to raise the question of when other abelian subgroups have this property. We note that non-compactness is not an issue since the conclusion above is trivially satisfied when H is a compact subgroup of any group G. If one knows such results about the projection valued measure for a vector subgroup, one may immediately conclude results concerning eigenvalues of any operator corresponding to an element of the enveloping algebra of the Lie algebra of H, a question of some interest in physics.

Let us now suppose that G is simple, and we write the Iwasawa decomposition $G = KAN$. Any abelian subgroup is in some vague sense made up of a part from K, a part from A, and a part from N, and we shall consider the three cases separately. We have already noted that the question posed is trivial for compact subgroups, and we turn to subgroups of A. Since A is abelian, we may as well consider the case $H = A$. T. Sherman [48] has observed at least in a special case that the answer is affirmative, and the same holds in general one can easily see.

Theorem 8

If G is simple and if π is a unitary representation of G with no G-invariant vectors, then the projection valued measure associated with the restriction of π to A is equivalent to Haar measure on $\hat{A}$. Moreover, the multiplicity is uniform.

We now turn to the consideration of subgroups of N, and here all we have at present is a counterexample. We consider the symplectic group $Sp_n(R)$ of real $2n \times 2n$ matrices preserving a non-degenerate skew bilinear form. Since the fundamental group of this group is the integers, there is a unique double covering group G of $Sp_n(R)$. (This is Weil's metaplectic group [51].) It is easy to verify that N for this group contains a normal abelian subgroup V isomorphic to the vector space of real $n \times n$ symmetric matrices. In fact N is the semi-direct product of V with a group T isomorphic to all strictly triangular $n \times n$ matrices (that is, all entries above the diagonal are 0 and all diagonal entries are one). Weil [51] has constructed a representation π of G called the metaplectic representation (see also Shale [46]). One may compute the projection valued measure on $\hat{V}$ associated with the restriction of π to V quite easily. In fact if one identifies V with $\hat{V}$ by means of the bilinear form $(a,b) = Tr(ab)$, this projection valued measure on $V = \hat{V}$ is concentrated on the set of positive definite matrices of rank one. Thus, if $n > 1$, this is a Haar null set, and gives an example where P is singular with respect to Haar measure. This example also shows that many one-parameter subgroups of V have one sided spectrum and hence that the phenomenon noticed for $SL_2(R)$ persists in higher dimensions.

There is another point worth noticing about this situation; let $\pi(k)$ denote the kth tensor power of the representation π with itself. Then it is quite easy to check that the projection valued measure associated with the restriction of $\pi(k)$ to V is concentrated on the set of positive definite matrices of rank equal to the minimum of k and n. Thus $\pi(n)$ is the first tensor power to have absolutely continuous spectrum. Since it is virtually obvious that any discrete series representation of G has a projection valued measure which is absolutely continuous, any connection between discrete series and the tensor powers of π analogous to the situation for $n = 1$, is likely to involve large tensor powers.

We notice that the condition above for abelian groups H that the projection valued measure on the dual group $\hat{H}$ associated to a representation be absolutely continuous, can be rephrased so as to make sense for any subgroup H. The condition can be readily seen to be equivalent to the condition that the representation of H should be unitarily equivalent to a subrepresentation of the direct sum of the regular representation with itself infinitely many times. This makes sense for any H, and we shall say following [33], that a representation of H satisfying this condition is quasi-contained in the regular representation. (If as often happens, the regular representation is equivalent to the infinite direct sum of itself, the condition is simply that the given representation is a subrepresentation of the regular representation.)

We do not have any general theorems concerning this situation, but there is one case of special interest. If G is semi-simple with Iwasawa decomposition $G = KAN$, we let M be the centralizer of A in K, and define $B = MAN$. One knows that B is a group [6], and in some sense it is one of the most important subgroups of G. The principal series representations of G consist simply of the representations of G induced by the finite dimensional representations of B [6], and B plays a key role in the structure of G. When G is $SL_n(R)$, then B is simply the subgroup of triangular matrices.

Our interest here will be the study of restrictions of representations of G to B in the spirit indicated above, and for this we shall need to know something about the structure of the regular representation of B. When $G = SL_2(R)$, the regular representation of B is rather striking. It is known that B has in addition to its obvious one-dimensional representations, exactly four other irreducible representations, all infinite dimensional [33], say π_i, i = 1, 2, 3, 4. A simple calculation will show that the regular representation π of B is the direct sum of the π_i each taken infinitely often, $\pi = \infty(\pi_1 \oplus \pi_2 \oplus \pi_3 \oplus \pi_4)$. Thus π is the discrete direct sum of irreducible representations, with in fact only a finite number of distinct types entering into the decomposition. It is easy to see that the same is true for $G = SL_2(C)$.

Is this an accident or do we find the same phenomenon for other semi-simple groups? A calculation for $G = SL_n(R)$ for $n \geq 3$ reveals that the regular representation of B decomposes as a continuous direct integral and contains no irreducible summands. On the other hand for $G = Sp_n(R)$, one can find exactly 4^n irreducible infinite dimensional representations of B such that the regular representation is the discrete direct sum of these representations, each taken infinitely often. We shall now state a general criterion which will appear in a forthcoming paper.

As usual we consider the normalizer M_1 of A in K, and note that $W = M_1/M$ is a finite group, called the Weyl group, which acts as a group of automorphisms of A, and hence also on the Lie algebra of A. The group W may or may not contain the automorphism $a \to a^{-1}$ of A (or equivalently the map $Y \to -Y$ of the Lie algebra of A). If it does we shall say that -1 belongs to W.

Theorem 9

If G is semi-simple, then the regular representation of its subgroup B either decomposes as a discrete direct sum of irreducible representations of B (with a finite or countable number of inequivalent summands) or it decomposes as a continuous direct integral with no irreducible summands. The first possibility occurs if and only if -1 belongs to W.

If $-1 \in W$, we may think of the finite or countably infinite number of irreducible representations which are summands of the regular representation as "discrete series", but we prefer to call them generic representations since they are in a sense those irreducible representations of B which are in "general position". One may not conclude as in the semi-simple case that the matrix entries are square integrable functions on B since B is not unimodular. Finally we note that Harish Chandra [21] has given a necessary and sufficient condition that the group G have a discrete series, that is, there are irreducible representations which are summands of the regular representation. It is interesting to note that his condition implies that $-1 \in W$ (and in fact is almost equivalent to it) and hence if G has a discrete series so does B.

Once we have this kind of control over at least some of the representations of B, it is natural to raise the question of what the restriction of a representation π of G to B looks like. This technique is exceedingly fruitful for $G = SL_2$, and Stein in his lectures in this volume uses a similar technique except with B replaced by an even larger subgroup.

One may ask if it is true that any representation π of a simple group G which has no G-invariant vectors has its restriction to B quasi-contained in the regular representation of B. This is true for all the series of representations constructed by Harish-Chandra [20] which are used to obtain the Plancherel formula. For discrete series this follows from the observation that any representation contained in the regular representation of G, has its restriction to any subgroup H quasi-contained in the regular representation of H. The general case follows from known facts concerning induced representations. Unfortunately the answer to the above question is negative in general, and the counterexample is our friend the metaplectic representation π of the double covering G of $Sp_n(R)$, $n \geq 2$. We have $B = MAN$ and N contains a normal subgroup isomorphic to the vector space of symmetric $n \times n$ matrices. If the restriction of π to B is quasi-contained in the regular representation of B, it follows by the comment above that its further restriction of V is quasi-contained in the regular representation of V which we know is false. Again we do not know what the best possible theorems are in the general case.

PART II. GROUP EXTENSIONS AND GROUP COHOMOLOGY

5. STRUCTURE OF LOCALLY COMPACT GROUPS

In this second part we shall take up a rather different aspect of group representations, and indeed here the major considerations will concern more the structure of locally compact groups. The motivation for the study of group extensions comes from the phenomenon of ray or projective representations of groups; however, to treat these questions properly, we feel it is better to first widen the problem, and then come back to the original questions using the general techniques which we shall develop.

We shall suppose that G and A are topological groups with A abelian, and that G operates on A is a topological transformation group of automorphisms. More precisely, we are given a continuous map of $G \times A \to A$ written $(g,a) \to g \cdot a$ such that for fixed $g \in G$, the map $a \to g \cdot a$ is an automorphism of A, which we denote by $p(g)$, and further that p is a homomorphism of G into the group of automorphisms of A. The hypothesis of joint continuity of $g \cdot a$ assures not only that $p(g)$ is an automorphism of the topological group A, but that $p(g)$ varies "smoothly" with g. If G and A satisfy the above, we say that A is a topological G-module or simply that A is a G-module [37].

This definition includes a wide variety of examples. If for instance A is a Hilbert space with its norm topology, and if π is a continuous unitary representation of G on A in the usual sense, then one may verify that $(g \cdot a) = \pi(g)(a)$ defines A as a G-module. If π is again a representation of G on a Hilbert space, and if A is some group of unitary operators on this Hilbert space such that $\pi(g)a\pi(g)^{-1} \in A$ $\forall a \in A$, then $g \cdot a = \pi(g)a\pi(g)^{-1}$ defines for each $g \in G$ an automorphism of A. It may be verified that A, equipped with the strong operator topology, is a G-module. Finally, if A is any topological group $g \cdot a = a$ defines A as a G-module. Such modules will be called trivial topological G-modules.

A group extension of a given G by a given topological G-module will be first of all an exact sequence of groups

$$1 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$$

where i is an injection of A into E, and π is a surjection of E onto G, and where the kernel of π is exactly the range of i. We assume not only that i is continuous, but also that it is a homeomorphism onto its range, and we assume that π is continuous and open. This means that A and i(A) can be identified not only as groups, but as topological groups where i(A) has the relative topology from E, and that E/i(A) and G may be identified as topological groups, the first of these having the quotient topology. Finally, we impose an algebraic assumption to take account of the action of G on A. We note that if $g \in G$ and if g' is an element of E with $\pi(g') = g$, then $a \to i^{-1}(g'i(a)(g')^{-1})$ is an automorphism of A which depends only on g, and not on the choice of g'. We demand that this automorphism be the given automorphism $a \to g \cdot a$ in the definition of A as a G-module. We note that whenever we have an extension of G by A, then by the above A becomes a G-module, the joint continuity of the map $G \times A \to A$ following from the axioms for a topological group. This observation is one of the main motivations for defining G-modules as we did by imposing the condition of joint continuity. The reader is referred to [37] and [38] for more details.

One of the simplest examples of a topological group extension is the extension of the circle T by the integers Z (viewed as a trivial module), defined by the real line R, namely

$$1 \to Z \to R \to T \to 1 .$$

Another example which is of more significance particularly in quantum physics is as follows. Let H be a Hilbert space, and let U(H) be the group of all unitary operators on H with the strong operator topology. Then the circle group T viewed as scalar multiples of the identity operator is a normal subgroup, and

$U(H)/T = PU(H)$ is called the projective unitary group. Then

$$1 \to T \to U(H) \to PU(H) \to 1$$

is an extension of $PU(H)$ by T, T being trivial module. Indeed if $P(H)$ is the set of one-dimensional subspaces of H, then any $a \in U(H)$ defines a collineation on $P(H)$ by $r \to a \cdot r$ where $a \cdot r$ is the transform of $r \in P(H)$ under a. A classic theorem of Wigner (see [52] or [5]) says that except for anti-unitary transformations, these are the only maps of $P(H)$ onto itself which preserve the function $f(r_1,r_2) = |(u_1,u_2)|$ which is defined for $r_i \in P(H)$ by picking unit vectors $u_i \in r_i$. We observe that the projective transformations corresponding to a and $b \in U(H)$ agree if and only if $a = tb$ with $t \in T$. Thus $U(H)/T = PU(H)$ is isomorphic naturally to a group of projective transformations.

If H is the Hilbert space associated with a quantum mechanical system, and if G is a symmetry group of this physical system, the axioms of quantum mechanics say that we have a homomorphism of G into $PU(H)$ (except for those symmetries which we would want to be anti-unitary, but this will not change anything essential in this heuristic discussion). Such a homomorphism is precisely what is known as a projective or ray unitary representation [5]. For the moment let us assume that $G \subset PU(H)$; then if p is the projection from $U(H)$ onto $PU(H)$, we let $G' = p^{-1}(G)$, and then

$$1 \to T \to G' \to G \to 1$$

becomes a group extension of G by T. Even when we do not want to identify G as a subgroup of $PU(H)$, we shall see that we can still construct a group extension

$$1 \to T \to E \to G \to 1$$

where E has a homomorphism into $U(H)$, or in other words, a unitary representation. The fact that projective representations can be viewed as ordinary representations of a suitable group extension is a well known and fundamental fact.

In these notes we want to present a brief outline of a systematic theory of group extensions and more generally of a theory of group cohomology which is intimately related to the initial problem. We refer the reader to [37] and [38] for more details and to the references cited there, in particular the pioneering work of G. W. Mackey [29], [31] who originated this point of view concerning group extensions. A large part of the contents of these notes will be the subject of a forthcoming paper of the author, and we will try to summarize the major new points involved. These results extend and generalize those in [38] and [39].

One of the most important problems is to classify the set of all extensions of a given group G by a given topological G-module A. Two extensions are said to be equivalent if there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & A & \to & E & \to & G & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & A & \to & E' & \to & G & \to & 1 \end{array}$$

of continuous maps where the end vertical maps are the identity maps and where the middle vertical is an isomorphism of topological groups. It should be noted that it is not sufficient to assume that E and E' are isomorphic as topological groups to have equivalent extensions, but rather there must be a particular isomorphism which respects the data of a group extension. One of the first facts is that the set of equivalence classes of extensions of G by A forms a group $\mathrm{Ext}(G,A)$ by means of the Baer product (cf. [16]), and at least in many cases this group is given as a two-dimensional cohomology group $H^2(G,A)$. It turns out to be useful to study the other cohomology groups $H^n(G,A)$, both to gain a better understanding of extensions, and also to have at hand general methods of computing $\mathrm{Ext}(G,A)$ in many specific situations.

6. G-MODULES

After this introduction we shall now proceed to some of the details. We will henceforth assume that G is locally compact and separable in the sense of the second axiom of countability. (Local compactness seems to be essential for this treatment, although we hope in the future to be able to dispense with it; separability is an assumption of a more technical nature used to avoid certain pathologies.) We shall also assume that A is separable, metrizable and moreover metrizable by some complete metric. Following Bourbaki, one might call such groups polonais, and we denote the family of all such groups by P. Since we will always be dealing with G-modules, we consider all polonais G-modules which we denote by $P(G)$. We note that $P(G)$ contains all separable locally compact G-modules A. Group extensions were studied in the case of locally compact A in [38] and [39], and one of the key points in the present treatment is that we now enlarge the category of modules to $P(G)$. In addition to including many important and interesting examples which were excluded before, we also achieve more technical versatility in that the larger category will contain cohomologically trivial modules, will enable us to define induced modules in a natural way, and will allow us to construct resolutions without going outside the category.

If $A,B \in P(G)$ a G-homomorphism f of A into B is simply a continuous intertwining homomorphism, that is, one satisfying $f(g \cdot a) = g \cdot f(a)$. We note that P (resp. $P(G)$) is closed under the operations of countable Cartesian products, closed subgroups (closed submodules), and quotient groups (quotient modules). In addition, if we have a sequence of elements of P

$$1 \to A' \xrightarrow{i} A \xrightarrow{\pi} A'' \to 1$$

which is exact in the sense of Section 5, then one can show that $A \in P$ if and only if A' and A'' are in P. The same is clearly true for $P(G)$ if the homomorphisms in the sequence above are G-equivariant. Morever, if A, A', A'' are in P, and it is only assumed that i and π are continuous, it follows by classical closed graph theorems (cf. [3]) that i is a homeomorphism and that π is open, and hence that the sequence is exact in the sense of Section 5.

In addition P is closed under the following construction which might be described as a sort of direct integral. This construction will be of paramount importance to us. Let (M,u) be a σ-finite measure space such that the measure algebra of (M,μ) is separable [14]. This means that we may as well assume that M is $[0,1]$ with Lebesgue measure together with a countable number of atoms. Now let $A \in P$, and define $U(M,A)$ to be the group of all measurable functions from M to A modulo the group of functions equal to 1 (the identity in A) almost everywhere. An element of $U(M,A)$ is then an equivalence class of measurable functions, all of which are equal to each other almost everywhere. (A function f is measurable if $f^{-1}(0)$ is measurable in M for every open 0 in A.) It is clear that $U(M,A)$ is a group under pointwise multiplication.

We topologize $U(M,A)$ by the topology of convergence in measure; more precisely let ρ_1 be a bounded metric on A, which always exists, and let ν be a finite measure on M equivalent to μ. We define a metric on $U(M,A)$ by

$$\rho(f,g) = \int \rho_1(f(x),g(x))d\nu(x)$$

which is always finite since ν is finite and ρ_1 is bounded.

<u>Lemma 6.1</u>

$U(M,A)$ with ρ as defined above is in P, and the topology is independent of the choice of ρ_1 and ν.

If $A = T$ is the circle group, $U(M,T)$ has a natural interpretation; namely let H be $L_2(M)$, the space of square integrable functions on M, and let $f \in U(M,T)$. Then f defines a unitary operator on H by multiplication by F, $(U(f)h)(x) = f(x)h(x)$. Clearly $U(f) = U(g)$ if and only if $f = g$ in $U(M,T)$ and so $U(M,T)$ may be viewed as a group of unitary operators on H. It may be verified that the topology on $U(M,T)$ introduced above is exactly the strong operator topology when we view $U(M,T)$ as operators. If $A = R$ is the real line, $U(M,R)$ is a topological vector space; in fact a Frechet space, although it is not locally convex. Finally, if $A \in P(G)$, we can define an action of G on $U(M,A)$ by means of the formula $(g \cdot F)(m) = g \cdot (F(m))$. This may be thought of as a direct integral of copies of A. If M is an atomic measure space, the construction does give the Cartesian product of copies of A. In analogy with direct integrals of representations [33] one might hope to find a reasonable definition of a measurable map of M into $P(G)$, and then define a direct integral where the "fiber" $A(m)$ over $m \in M$ is allowed to vary instead of remaining constant as above. Since we have found no use for this kind of construction as yet, we shall not proceed any further.

The group $U(M,A)$ for $A \in P$ has many interesting properties, and one of the most important for our purposes is a "law of exponents". We let $M = M_1 \times M_2$, and then intuitively a function of two variables on M into A can be thought of as a function of one variable (say m_1) into the space of functions of the second variable m_2 into A. Such a correspondence holds exactly and indeed follows from a version of the Fubini theorem.

Lemma 6.2

There is a canonical isomorphism of $U(M_1 \times M_2,A)$ onto $U(M_1,U(M_2,A))$ as topological groups.

A most important special case of the construction of $U(M,A)$ is when $M = G$ is a locally compact (separable) group with Haar measure. In this case $U(M,A)$ will be denoted by $I(A)$ and we note that $I(A)$ is itself a G-module for any $A \in P$. In fact we simply let G act by translation: $(g \cdot F)(s) = F(g^{-1}s)$. If in addition $A \in P(G)$ so that G also operates on A we can embed A into $I(A)$ by the map f defined by $(f(a))(s) = s^{-1} \cdot a$.

Lemma 6.3

If $A \in P$, then $I(A)$ is in $P(G)$. Moreover, f is an equivariant isomorphism of A onto a closed submodule of $I(A)$ so that

$$1 \to A \to I(A) \to U(A) \to 1$$

is exact where $U(A)$ is the quotient module.

It is clear that $I(A)$ is in some sense the regular representation of G with coefficients in A. In the case of a finite group, it is known that $I(A)$ is cohomologically trivial in that $H^n(G,A) = 0$ for $n \geq 1$ [45]. The fact that this can also be proved in the present context will be of vital importance. Lemma 6.3 above would then assert that any A may be embedded in a cohomologically trivial module, and this fact will allow us to use many techniques from homological algebra. It should also be noted that $I(A)$ is almost never locally compact. Finally, once we have defined the regular representation, it is but a short step to the notion of induced representations. If H is a closed subgroup of G, and if $A \in P(H)$, we define $I_H^G(A)$, the induced module, as a submodule of $I(A)$. More precisely, $I_H^G(A) = \{f | f \in I(A),\ f(gs) = s^{-1}f(g)$ for almost all pairs (s,g) in $H \times G$ where Haar measure is understood$\}$. We have engaged in the usual abuse of notation and have regarded elements of $I(A)$ as functions instead of equivalence classes of functions, but this poses no problem. It is easy to show that $I_H^G(A)$ is a closed submodule of $I(A)$ and hence is in $P(G)$. All of the expected properties of induced representations such as inducing in stages hold in our context, but we shall defer these details to our forthcoming paper.

7. GROUP EXTENSIONS

Having discussed the G-modules which will enter into our theory, we turn to a more explicit discussion of group extensions and group cohomology. In complete analogy with the case of discrete groups [11], we shall introduce cohomology groups $H^n(G,A)$, $n \geq 0$ for $A \in P(G)$. These groups have simple interpretations in low dimensions; namely $H^0(G,A) = A^G$, the G-fixed points in $A = \{a | g \cdot a = a$ for all $g \in G\}$. For $n = 1$, and a trivial G-module, $H^1(G,A)$ will be the continuous homomorphisms of G into A (while for a general module we will have equivalence classes of continuous crossed homomorphisms). For $n = 2$, $H^2(G,A)$ will be $\mathrm{Ext}(G,A)$, the group of topological group extensions of G by A.

By way of introduction to the cohomology we shall begin with a discussion of how one may parameterize the group extensions of G by A using cocycles. Let

$$1 \to A \to E \overset{\pi}{\to} G \to 1$$

be a given extension. The identity element of the group $\mathrm{Ext}(G,A)$ is the semi-direct product of G and A, and in the special case when A is a trivial G-module, the direct product of G and A. This extension is characterized by the property that one may find a continuous homomorphism f of G back into E such that $(\pi \circ f)(g) = g$. The idea behind the following is to compare a general extension of G by A to the semi-direct product. It is natural to consider a map f of G to E such that $\pi \circ f$ is the identity map, and compute the defect of f from being a homomorphism. Since we are dealing not with abstract groups, but with topological groups, it would not be sensible to choose any arbitrary map f. Ideally one would want to look for a continuous map f of G into E satisfying the above, however, it is simply a fact of life that such a continuous map does not always exist. Indeed even in the case

$$1 \to Z \to R \to Z \to 1$$

such a map does not exist, and in general the existence of such a continuous map for a general extension would imply that E viewed as a principal fiber bundle with base G and fiber A would be a trivial bundle and so in particular $E = G \times A$ as topological space. Mackey has shown how to resolve this, and following him we observe that one may always find a Borel map f of G into E satisfying $(\pi \circ f)(s) = s$ for $s \in G$, (see [9]). Other choices of an appropriate map may be considered such as those continuous at the identity element of G or those continuous in a neighborhood of the identity element of G, but we believe that the choice of a Borel map f leads to a theory which is in general more satisfactory.

Once we have selected such a Borel function f (or cross section as it is sometimes called) we note that $a(g,h) = f(g)f(h)f(gh)^{-1}$ is a Borel function from $G \times G$ into the subgroup $i(A)$ of E. We view it as a function from $G \times G$ into A and we notice that it is a Borel function, and as a consequence of the associative law in E satisfied the "cocycle identity",

$$a(s,t)a(st,r) = (s \cdot a(t,r))a(st,r)$$

for all $s, t, r, G \times G \times G$. We denote by $Z^2(G,A)$ the group of all Borel maps of $G \times G$ into A satisfying this identity, and call such functions 2-cocycles. The group structure is understood to be multiplication of such functions pointwise. We have associated now to each element of $\mathrm{Ext}(G,A)$ an element of $Z^2(G,A)$, but this depends on the selection of a Borel cross section f of G into E. If we replace f by any other Borel cross section f', the cocycle a changes, but it changes only by multiplication by a 2-cocycle of the form $(s \cdot b(t))b(s)b(st)^{-1}$ for some Borel function b of G into A. We call such functions 2-coboundaries, and denote the group of such by $B^2(G,A)$, and notice the very important fact that to each extension in $\mathrm{Ext}(G,A)$ we can associate a unique element of the quotient group $Z^2(G,A)/B^2(G,A)$ which is independent of any choices. This quotient group is denoted by $H^2(G,A)$, the two dimensional cohomology group of G with coefficients in the topological G-module A.

The map of $\mathrm{Ext}(G,A)$ into $H^2(G,A)$ may be verified to be a homomorphism of groups, and moreover may be seen to be injective. If A is locally compact Mackey [30] has shown that this map is surjective as well. We are able to show (see below) that this is also true for any $A \in P(G)$. This construction gives a parameterization of $\mathrm{Ext}(G,A)$ in terms of a cohomology group and also motivates the introduction of the general cohomology groups $H^n(G,A)$.

If $A \in P(G)$, we define a complex of groups $C^n(G,A)$, $n \geq 0$, where $C^n(G,A)$ is the set of all Borel functions from $G \times \cdots \times G$ (n factors) into A, and we define a coboundary operator δ_n from $C^n(G,A)$ into $C^{n+1}(G,A)$ by the classical formula [11],

$$\begin{aligned}(\delta_n f)(s_1,\cdots,s_{n+1}) &= s_1 \cdot f(s_2,s_3,\cdots,s_{n+1}) \\ &- f(s_1 s_2,s_3,\cdots,s_{n+1}) \cdots \pm f(s_1,\cdots,s_n s_{n+1}) \\ &\mp f(s_1,\cdots,s_n)\end{aligned}$$

where we are writing A additively. The verification that $\delta_n f$ is a Borel function if f is a Borel function is routine [38], as is also the formula $\delta_{n+1}\delta_n = 0$. We define $Z^n(G,A)$ to be the kernel of δ_n and $B^n(G,A)$ to be the range of δ_{n-1}, and $H^n(G,A)$ to be the quotient group Z^n/B^n. For $n = 2$, this gives the group $H^2(G,A)$ as defined above, so everything is compatible. For $n = 0$, a function of zero variables is by convention an element of A, and δ_0 is given by $\delta_0(a)(s) = s \cdot a - a$. Thus $B^0 = 0$ and $Z^0 = H^0(G,A) = A^G$, the G-fixed points in A. For $n = 1$, and a trivial G-module, $B^1 = 0$, and $Z^1 = H^1(G,A) = \{f \mid f(st) = f(s) + f(t),$ f Borel$\}$. By a classical theorem of Banach, every such Borel homomorphism is automatically continuous (cf. [3]), so $H^1(G,A)$ is the group of continuous homomorphisms of G into A. If G acts on A, $B^1 \neq 0$, and Z^1 consists of Borel crossed homomorphisms of G into A or functions satisfying $f(st) = s \cdot f(t) + f(s)$. Such a function is by the same theorem of Banach continuous so $H^1(G,A)$ consists of classes of continuous crossed homomorphisms of G into A where a class consists of a coset of $B^1(G,A) = \{f(s) = s \cdot a - a$ for some $a \in A\}$.

Elements of $Z^2(G,T)$ arise naturally in the study of unitary ray representations [31]. Let p be a continuous homomorphism of G into PU(H), the projective unitary group of a Hilbert space H as defined previously. We can find a Borel cross section f of PU(H) back into U(H) by general theorems as above, and then $f(p(s))f(p(t))f(p(st))^{-1} = a(s,t)$ can be seen to define an element of $Z^2(G,T)$, and hence an element of $H^2(G,T)$. It is clear that the element of $H^2(G,T)$ is zero if and only if we may find a continuous unitary represnetation π of G on H which "induces" p [31]. Thus an analysis of $H^2(G,T)$ is crucial for an understanding of when a ray representation "is" in fact an honest unitary representation. Even if the element of $H^2(G,T)$ is non zero we can still construct according to Mackey's theorem a group extension of G by T

$$1 \to T \to E \to G \to 1$$

and one may verify that E possesses an "honest" unitary representation on H which is of the form $t \to t \cdot 1$ on T and which "induces" the given projective or ray representation of G. This makes explicit our earlier comment that ray representations may be interpreted as ordinary representations of a group extension.

8. COHOMOLOGY GROUPS

The introduction of cohomology groups $H^n(G,A)$ is grantedly very *ad hoc*. First of all we selected a particular class of functions (Borel functions) which happened to give us what we wanted in low dimensions, and moreover we selected a perhaps somewhat artificial definition of δ_n. One's doubts are further compounded by the observation that the constructions of Section 6 suggest a somewhat different definition of the groups $H^n(G,A)$.

We defined $C^n(G,A)$ to be all Borel functions from $G\times\cdots\times G = G^n$ into A, but one is led to consider the possibility of replacing $C^n(G,A)$ by $U(G^n,A)$, the group of equivalence classes modulo null functions of measurable functions from G^n (Haar measure) into A, and we denote this group by $\underline{C}^n(G,A)$. It is not difficult to verify that δ_n as above is a well defined map from $\underline{C}^n$ to $\underline{C}^{n+1}$, and hence that we get cohomology groups $\underline{H}^n(G,A) = \underline{Z}^n(G,A)/\underline{B}^n(G,A)$ where $\underline{Z}^n$ is the kernel of δ_n and $\underline{B}^n$ is the range of δ_{n-1}. The cocycles in dimension zero consist of the kernel of δ_0 or the elements a of A such that $s \cdot a = a$ for almost all s in G. It is not hard to see that this implies that $s \cdot a = a$ for all $s \in G$, and hence $\underline{H}^0(G,A) = A^G$. If A is a trivial G-module, then the cocycles in dimension one are exactly the equivalence classes of functions f from G to A such that $f(st) = f(s) + f(t)$ for almost all pairs s and t. Similarly, in dimension two we look at functions which satisfy the cocycle identity above for almost all triples (s,t,r). A result of Mackey in [36] suggests that such an approach is not as outlandish as it first appears.

Motivated by the above, together with the possibility of a wide variety of other choices of cohomology groups we ask if we can somehow find a set of reasonable axioms which any cohomology theory should in principle satisfy, and then prove that there is up to isomorphism only one way of satisfying these axioms. We shall show that this is the case, and moreover that the groups H^n and $\underline{H}^n$ defined above by cocycles do satisfy these axioms. We then will know not only that these two definitions of cohomology groups agree, but also that any other attempt to define cohomology groups satisfying the axioms below must necessarily lead to the same groups.

(a) Our first axiom is of a general algebraic nature. We assume given for each $A \in P(G)$, G fixed, and for each $n \geq 0$, an abelian group denoted by $H^n(G,A)$ such that these are "functors of cohomological type". More precisely, we assume that for any G-homomorphism f of A into B, we have induced homomorphisms f^n of $H^n(G,A)$ into $H^n(G,B)$ such that the law of composition is satisfied: $(gf)^n = g^n f^n$ when g is a G-homomorphism of B into C. Moreover, $1^n = 1$ where 1 denotes the identity homomorphism of A into A, and we assume that for any short exact sequence

$$1 \to A' \to A \to A'' \to 1$$

in $P(G)$, we have natural coboundary operators $\partial_n: H^n(G,A'') \to H^{n+1}(G,A')$ such that the infinite long sequence

$$0 \to H^0(G,A') \to H^0(G,A) \to H^0(G,A'') \to H^1(G,A') \to \cdots \to H^n(G,A)$$
$$\to H^n(G,A'') \to H^{n+1}(G,A'') \to H^{n+1}(G,A) \to \cdots$$

is exact (see [38] and [45]).

(b) The second axiom demands $H^0(G,A) = A^G$ for any $A \in P(G)$.

(c) The third axiom is a vanishing axiom which is motivated by the cohomology of finite groups; namely we demand $H^n(G,I(A)) = (0)$ for $n \geq 1$, and every polonais group, where $I(A)$ is the "regular representation" as defined in Section 6.

Axiom (c) is of course the really crucial one; it asserts that certain modules are cohomologically trivial and although there is a great deal of motivation for it from the cohomology of abstract groups, it does represent a definite choice. One could conceivably select some other class of modules and assume them to be cohomologically trivial, and this would lead to a unicity theorem for some possibly different cohomology theory. Our defense here is that the groups defined by cochains above do satisfy this vanishing axiom, and that the groups $I(A)$ do seem to play a natural role in analysis and group representations.

The following unicity theorem follows immediately from Lemma 6.3 and standard methods of homological algebra.

Theorem 10

If $H_i^n(G,A)$, $i = 1, 2$ are two assignments of cohomology groups defined for $A \in P(G)$ for a fixed G which satisfy Axioms (a), (b), and (c) above, there are canonical isomorphisms of $H_1^n(G,A)$ onto $H_2^n(G,A)$ for all n and all A.

One of our major results is that the groups $H^n(G,A)$ and $\underline{H}^n(G,A)$ defined above by Borel cochains, and equivalence classes of measurable cochains do satisfy these axioms.

Theorem 11

The groups $H^n(G,A)$ and $\underline{H}^n(G,A)$ satisfy Axioms (a), (b), and (c) and hence are isomorphic. More precisely, the map which attaches to each Borel cochain in $C^n(G,A)$, its equivalence class in $\underline{C}^n(G,A)$, induces this isomorphism on cohomology.

The verification of Axioms (a) and (b) is routine in both cases (see [38]); however, the verification of Axiom (c) is non-trivial. In fact for $n = 1$, this verification is for all intents and purposes equivalent to Mackey's general version of the Stone-von Neumann theorem in [26]. A close examination of Mackey's argument in [26] reveals that what is essentially being proved is that $H^1(G,I(T)) = 0$. (Actually one wants to replace T by a unitary group $U(H)$ on a Hilbert space, and this would lead us into non-abelian cohomology (see [45]). The essential analytic details however are the same as when H is one-dimensional so that $U(H) = T$.) Theorem 11 is proved first for $n = 1$, and then the general case is reduced to this case by an induction argument. The argument follows in spirit the argument for abstract groups where in fact the result is trivial; however, there are non-trivial analytical complications concerning null sets in our case.

In view of Theorem 11 we shall henceforth use the notations $H^n(G,A)$ and $\underline{H}^n(G,A)$ interchangeably; our choice of notation will serve to emphasize that we are interested in a particular facet of these groups which may be evident from one of the definitions, but not the other. We note in particular that such results as the above are not approachable if one stays within the category of locally compact G-modules, and that essential use is made of non-locally compact modules.

We have remarked before that we have a natural notion of induced modules which gives us for each $A \in P(H)$, a module $I_H^G(A) \in P(G)$ where H is a closed subgroup of G. A very useful tool for finite groups is Shapiro's lemma [45] which relates the cohomology of A with that of the induced module.

Theorem 12

There are canonical isomorphisms $H^n(H,A) \approx H^n(G,I_H^G(A))$ for all $A \in P(H)$ and all n.

The proof is obtained by noting that both sides of the above as functors on $P(H)$ satisfy Axioms (a), (b), and (c), and then one applies Theorem 10. We note that for $H = (e)$, this is simply the vanishing theorem. Also we note for $n = 1$, that this theorem is essentially Mackey's imprimitivity theorem [27].

9. ADDITIONAL PROPERTIES

We shall now discuss some additional properties of these cohomology groups, and in particular nail down the connection with group extensions. For $n = 0$, we have already seen that $H^0(G,A) = A^G$ and that $n = 1$, $H^1(G,A)$ is the group of continuous crossed homomorphisms of G into A modulo principal ones, and if A is a trivial G-module, it is simply the group of all continuous homomorphisms of G into A. In Section 7 we constructed an injective homomorphism of the group $\mathrm{Ext}(G,A)$ of equivalence classes of topological group extensions into $H^2(G,A)$. For A locally compact, Mackey has shown that this map is onto, but his argument [30] does not extend since it makes essential use of the Haar measure on A. We have an alternate argument which works in general and which we outline below.

If $a \in H^2(G,A)$, we embed A into $I(A)$ by Lemma 6.3 and let a' be the image of the class a in $H^2(G,I(A))$ under the map given in Axiom (a). Since $H^2(G,I(A)) = 0$, $a' = 0$, and so there is clearly an extension of G by $I(A)$ corresponding to a', namely the semi-direct product $I(A) \cdot G$. We wish to construct an extension of G by A corresponding to $a \in H^2(G,A)$ and on general principles we would expect this extension, if it exists, as a subgroup of $I(A) \cdot G$. In fact if we pick a cocycle in the class a, we can immediately construct a subgroup E' of $I(A) \cdot G$ and then prove that it has all the required properties. (This particular construction is virtually forced on us, again by general principles.) Thus $\mathrm{Ext}(G,A) \simeq H^2(G,A)$.

The higher cohomology groups have as yet no direct interpretation, however, we certainly do expect $H^3(G,A)$ to contain obstructions to the construction of non-abelian extensions as in [3], Chapter IV.

When the cohomology groups are constructed via equivalence classes of measurable cochains, another interesting and significant property emerges. Namely, since $\underline{C}^n(G,A)$ is a polonais group, and since it may be readily checked that the coboundary operators δ_n are continuous, it follows that $\underline{Z}^n(G,A)$ is closed and hence in P. Thus $\underline{H}^n(G,A)$ is the quotient of a group in P by a subgroup and, hence when given the quotient topology, is itself a topological group. There is no *a priori* reason for $\underline{B}^n(G,A)$ to be closed, and it is an unpleasant fact of life that it is not always closed so that $\underline{H}^n(G,A)$ may not even be Hausdorff. The closure of the identity element in such a group is a closed subgroup, and upon dividing by it, we obtain a Hausdorff group which in the case of $\underline{H}^n(G,A)$ is simply

$\underline{Z}^n(G,A)$ divided by the closure of $\underline{B}^n(G,A)$. This quotient group will again be polonais, and $\underline{H}^n(G,A)$ will satisfy all the axioms of a polonais group except with "metric" replaced by "pseudo-metric". Thus $\underline{H}^n(G,A)$ is in a class of groups one might reluctantly call pseudo-polonais.

In any case, the fact that $\underline{H}^n(G,A)$ and hence $H^n(G,A)$ have a natural and more or less reasonable topology will be quite important for us. In fact we can strengthen Axiom (a) above and prove that the groups $\underline{H}^n(G,A)$ are functors of cohomological type taking values in the category of topological groups. Moreover, if $n = 1$, and if A is a trivial G module, $H^1(G,A)$ being continuous homomorphisms of G into A has a natural Hausdorff topology, namely that of convergence on compact sets. It may be verified that the topology on $\underline{H}^1(G,A)$ coincides with this topology. In [39] a great deal of effort was devoted to constructing a topology for $H^2(G,A)$ for various G and A by rather *ad hoc* methods. It is not hard to show that this topology coincides with the one above on $\underline{H}^2(G,A)$ whenever the former exists. Details of this will appear in our subsequent paper.

One reason for seeking a topology on $H^n(G,A)$ (aside from the esthetic one of expecting a topological object when one starts with topological data) is so that one can hope to make sense out of the spectral sequence for the cohomology of a group extension (cf. [19]). If H is a closed normal subgroup of G, the analogy with finite groups leads us to hope for a spectral sequence $E_r^{p,q}$ converging to $H^*(G,A)$ with $E_2^{p,q} = H^p(G/H,H^q(H,A))$ (see [19]). We observe that for this to begin to make sense, we must have $H^q(H,A) \in P(G/H)$, and in particular it must have a topology. We can show that there is always a spectral sequence of this type, and moreover that if $H^q(H,A)$ happens to be Hausdorff then the $E_2^{p,q}$ term is given by the expected formula. The existence of such a spectral sequence is quite important since it is an almost indispensable tool in making all but the simplest calculations of our cohomology groups. The reader is referred to [38] and [39] for examples in the case when A is locally compact.

We shall close this section with one final result concerning direct integrals of G-modules. Recall from Section 6 that if $A \in P(G)$, the group $U(X,A)$ had a natural structure as G-module which we called the direct integral. Since Cartesian products are a special case of this, and since cohomology commutes with products, we may ask if the same is true for integrals and we have the following result.

Theorem 13

If $\underline{H}^n(G,A)$ is Hausdorff, we have an isomorphism of topological groups

$$\underline{H}^n(G,U(X,A)) \simeq U(X,\underline{H}^n(G,A)) .$$

The content of this result is that a cocycle with values in a direct integral module $U(X,A)$ may be represented as a direct integral of cocycles. If $n = 1$

with trivial action, the side condition is satisfied and since one cocycles are homomorphisms, this result essentially gives us a new proof of the existence of direct integral decompositions of unitary representations.

10. EXAMPLES AND APPLICATIONS

We want to conclude with some examples, some computations, and some applications of the general theory above.

Suppose that $G = G_1 \times G_2$ and suppose for simplicity that A is a trivial G-module. Then either as a consequence of the spectral sequence above, or as a result of explicit computations (cf. [31]), we may obtain a structure theorem for $H^2(G,A)$ as follows:

$$H^2(G,A) \approx H^2(G_1,A) \oplus H^2(G_2,A) \oplus H^1(G_1,H^1(G_2,A)) .$$

The first two terms are easy enough to understand and represent the contributions of the factors G_1 and G_2 to the cohomology of G, while the final term is a cross-term representing the interaction of G_1 and G_2. This enables us for instance to immediately compute $H^2(R^n,T)$, $H^2(Z^n,T)$, and $H^2(T^n,T)$ by induction on n. Indeed it is easy to verify that $H^2(R,T) = H^2(Z,T) = H^2(T,T) = 0$ by looking at the possible group extensions in these three cases. Since $R^n = R^{n-1} \times R$, and so on, it follows readily by induction that $H^2(R^n,T)$ is isomorphic to a vector space V of dimension $n(n - 1)/2$, and that $H^2(Z^n,T)$ is isomorphic to a torus S again of dimension $n(n - 1)/2$, and that $H^2(T^n,T) = 0$. Moreover the topology defined above on the groups $\underline{H}^2(R^n,T)$ and $\underline{H}^2(Z^n,T)$ coincides with the usual topology on the vector V and torus S. The isomorphism can also be implemented quite explicitly since one may show that each class in $H^2(R^n,T)$ contains a unique skew symmetric continuous bilinear function, and one may identify $H^2(R^n,T)$ with the group of such functions which is a vector group of dimension exactly $n(n - 1)/2$. A similar but slightly more involved statement holds for $H^2(Z^n,T)$.

If G is a semi-simple Lie group and if A is a trivial locally compact G-module, it is classical [47] that $H^2(G,A) \simeq H^1(\pi_1(G),A)$ where $\pi_1(G)$ is the usual fundamental group of G. Furthermore $\underline{H}^2(G,A)$ is Hausdorff in its natural topology and this topology coincides with the compact open topology on $H^1(\pi_1(G),A)$ $= \mathrm{Hom}(\pi_1(G),A)$ which in this simple case is simply the topology of pointwise convergence. This result also holds for any trivial G-module in $P(G)$ and moreover a similar result holds for a much broader class of groups G if one is willing to suitably redefine and generalize the notion of the fundamental group $\pi_1(G)$ of G (see [40]).

Using the spectral sequence of the previous section one may compute $H^2(G,T)$ when G is a semi-direct product of a semi-simple group and say a vector group. One may verify in this case known results for the inhomogeneous Lorentz group, and similar kinds of groups. We refer the reader to [38] and [39] for more details.

Another application of this material, and especially of our results concerning non-locally compact G-modules concerns the following situation. Let $A = T^n$ be a finite or infinite dimensional torus where $n = 1, 2, \cdots, \infty$, and suppose that

$$1 \to A \to E \to G \to 1$$

is a group extension of G by A where E is locally compact and abelian. It is a trivial and well known consequence of the duality theory of locally compact abelian groups that such an extension splits (that is, represents the identity element of $\mathrm{Ext}(G,A)$) and so $E \simeq A \oplus G$ is a direct sum of A and G as topological groups.

With this result in mind for Cartesian products of circles, it is natural to ask if a similar result holds for direct integrals of the circle group and the answer is affirmative.

Theorem 14

If $1 \to U(X,T) \to E \to G \to 1$ is an extension of G by $U(X,T)$ with E abelian and G locally compact, then the extension is split so that $E = U(X,T) + G$ as topological groups.

The idea of the proof is quite simple; one may verify that $\underline{H}^2(G,T)$ is Hausdorff and so Theorem 13 is applicable. After some extra argument using the fact that E is assumed to be abelian, the problem is thrown back using Theorem 13 to case when $U(X,T) = T$ where the result is known. Theorem 14 is found to be quite useful in settling certain questions concerning the structure of non-locally compact topological groups. Moreover exactly the same technique allows us to establish the following result.

Theorem 15

If $H^2(G,T) = (0)$ then any extension $1 \to U(X,T) \to E \to G \to 1$ with G locally compact and $U(X,T)$ a trivial G-module splits.

This final result leads to a very useful theorem concerning automorphism groups of von Neumann algebras which will have some applications in quantum field theory. Suppose that B is a von Neumann algebra of operators on a separable Hilbert space and that G is a locally compact group. We suppose given a homomorphism f of G into the group of inner *-automorphisms of B satisfying the continuity requirements set down in [22]. Thus for each $g \in G$, we have a unitary operator $u(g)$ in B such that $f(g)(b) = u(g)b\,u(g)^{-1}$ for all $b \in B$. The question we raise is whether one can choose the operators $u(g)$ so that they form a continuous unitary representation of G. This question is relevant in quantum field theory when for instance B is some algebra of observables and G is some symmetry group of the physical system. If B is the algebra of all bounded operators on Hilbert space, a moment's reflection will show that we are raising exactly the question of when a projective or ray representation of G can be converted into an ordinary representation since the group of *-inner automorphisms of B is PU(H). It follows from our general discussion of group extensions that we can do this for projective representations if $H^2(G,T) = 0$, or equivalently if every group extension of G by the circle group splits as a product. The theorem to follow asserts that the same is true in the general context described above.

Theorem 16

If f is any homomorphism of G into the group of *-inner automorphisms of a von Neumann algebra B on a separable Hilbert space, continuous in the sense described in [22], and if $H^2(G,T) = 0$, then there is a unitary representation π of G with $\pi(g) \in B$ such that $f(g)(b) = \pi(g)b\pi(g)^{-1}$ for $b \in B$.

The proof is almost immediate for the map f immediately gives rise to a cohomology class a in $H^2(G,W)$ where W is the group of unitary operators in the center of B, such that $a = 0$ if and only if a representation π as described in the theorem exists. However, by the structure theory of von Neumann algebras W is of the form U(X,T) and the result follows by Theorem 15.

When G = R, Kadison in [22] established a special case of this. Recently R. Kallman [23] has obtained a far more general result. For the Poincaré group, another case of physical interest, L. Michel has already obtained the above result by rather different methods [36].

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