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Black hole thermodynamics with the cosmological constant as independent variable: Bridge between the enthalpy and the Euclidean path integral approaches

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ABSTRACT

Viewing the cosmological constant $\Lambda < 0$ as an independent variable, we consider the thermodynamics of the Schwarzschild black hole in an anti-de Sitter (AdS) background. For this system, there is one approach which regards the enthalpy as the master thermodynamic variable and makes sense if one considers the vacuum pressure due to the cosmological constant acting in the volume inside the horizon and the outer size of the system is not restricted. From this approach a first law of thermodynamics emerges naturally. There is yet another approach based on the Euclidean action principle and its path integral that puts the black hole inside a cavity, defines a quasilocal energy at the cavity's boundary, and from which a first law of thermodynamics in a different version also emerges naturally. The first approach has affinities with critical phenomena in condensed matter physics and the second approach is an ingredient necessary for the construction of quantum gravity. The bridge between the two approaches is carried out rigorously, putting thus the enthalpic thermodynamics with Λ as independent variable on the same footing as the quasilocal energy approach.

1. Introduction

In recent years, a new direction in gravitational and black hole thermodynamics appeared. It is based on treatment of the cosmological constant $\Lambda < 0$ as a thermodynamic variable and leads to a number of interesting physical consequences in which the thermodynamic potential enthalpy emerges naturally and a close analogy with van der Waals forces and critical phenomena in condensed matter physics can be carried out. The works and reviews on the subject can be found in [1–7]. These nontrivial features arise in cases like the Schwarzschild and Reissner–Nordström black holes in an anti-de Sitter (AdS) background [4]. A main feature is the first law of thermodynamics in which the term $d\Lambda$ is taken into account with Λ acting as a vacuum pressure P. Indeed P takes the value $P = -\frac{\Lambda}{8\pi}$ and is conjugate to a thermodynamic volume V given by $V = \frac{4}{3}\pi r_+^3$, r_+ being the horizon radius. Substantiation of this approach is based on precise derivations as well as heuristic arguments that take into account the volume inaccessible to an observer due to the existence of a horizon, whereas the outer size of the black hole thermodynamic system is not restricted in this approach.

On the other hand, there exists a well-defined gravitational black hole thermodynamics based on the Euclidean action principle and path integral approach which is recognized as an ingredient necessary for the construction of a quantum gravity [8–10]. This approach was put on a firm basis taking into account the finiteness of the system as an essential ingredient, i.e., the black hole is constrained to lie inside a cavity. At the cavity's boundary a temperature T is prescribed, defining thus a canonical ensemble [11–15], or if electric charge is present a grand canonical ensemble should be used [16,17]. Also at the cavity's boundary one can define a quasilocal energy and from it extract the first law of thermodynamics. In this approach the horizon turns into a bolt [9] and the volume under the horizon becomes totally irrelevant, whereas the outer size of the black hole thermodynamic system is required to have a well defined boundary.

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One approach uses the enthaply as the master thermodynamic variable, makes explicit the importance of the volume inside of the black hole horizon, and the outer size is not constrained. The other approach uses the quasilocal energy as the master thermodynamic variable, cuts the inside of the black hole horizon as unimportant, and puts a boundary in the outer region. Clearly, the two approaches look dual to each other. Thus, the existence of these two dual independent thermodynamic formulations with different concepts for the same system deserves a careful study and comparison.

The aim of the present work is to make the bridge between these two formulations and thus put the thermodynamics with Λ as an independent variable with the enthalpy as the master thermodynamic variable approach on the same footing as the quasilocal energy canonical ensemble approach. We consider a Schwarzschild–AdS black hole and give a rigorous derivation of the corresponding first law of thermodynamics in enthalpic terms from the first law of the finite size gravitational thermodynamics. In doing so, we substantiate, why and how the thermodynamic volume $\frac{4}{3}\pi r_+^3$ appears.

2. The spacetime

Let us consider the Schwarzschild–AdS spacetime black hole with metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

with

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}, \tag{2}$$

where M is the ADM mass and Λ is the cosmological constant. The cosmological constant is negative, $\Lambda < 0$ so that spacetime is asymptotically AdS. Equation $f(r) = 0$ has one root r_+ that corresponds to the event horizon, i.e., $1 - \frac{2M}{r_+} - \frac{\Lambda r_+^2}{3} = 0$, and so the mass M is given in terms of r_+ and Λ as

$$M = \frac{r_+}{2}\left(1 - \frac{\Lambda r_+^2}{3}\right). \tag{3}$$

The metric function (2) in terms of r_+ is then

$$f(r) = 1 - \frac{r_+}{r} + \frac{\Lambda}{3}\left(\frac{r_+^3}{r} - r^2\right). \tag{4}$$

3. Thermodynamics in a cavity in the Euclidean action quasilocal energy approach

Let us consider a black hole in the canonical ensemble, so the temperature T is fixed on a sphere with area A and radius R, with $A = 4\pi R^2$. The corresponding thermodynamics formalism was initiated in [11] for a Schwarzschild black hole in a cavity (see also [12,13]), extended in [14] where quasilocal energy was defined, and applied for an AdS black hole in a cavity [15] (for the electric charge extensions see [16,17]). In this approach one uses the quasilocal energy E as the master thermodynamic variable.

The quasilocal energy at a sphere with radius R, can be envisaged as a function of the mass M, the radius R, and Λ, i.e., $E = E(M, R, \Lambda)$. We are thus treating the cosmological constant as an independent variable that can change its value thermodynamically. Following [14,15] we can write $E = E(M, R, \Lambda)$ as

$$E(M, R, \Lambda) = E_0(R, \Lambda) - R\sqrt{f(M, R, \Lambda)}, \tag{5}$$

where from Eq. (2)

$$f(M, R, \Lambda) = 1 - \frac{2M}{R} - \frac{\Lambda R^2}{3}. \tag{6}$$

and $E_0(R, \Lambda)$ is an appropriate reference spacetime with respect to which the energy is calculated. Its meaning consists in subtracting from the energy $E(M, R, \Lambda)$ of the relevant spacetime the energy $E_0(R, \Lambda)$ of a reference spacetime with the same boundary data.

The quasilocal energy at a sphere with radius R, can be envisaged also as a function of the radius r_+, the radius R, and Λ, i.e., $E(r_+, R, \Lambda)$, and written as

$$E(r_+, R, \Lambda) = E_0(R, \Lambda) - R\sqrt{f(r_+, R, \Lambda)}, \tag{7}$$

where from Eq. (4)

$$f(r_+, R, \Lambda) = 1 - \frac{r_+}{R} + \frac{\Lambda}{3}\left(\frac{r_+^3}{R} - R^2\right), \tag{8}$$

and again $E_0(R, \Lambda)$ is an appropriate reference spacetime with respect to which the energy is calculated.

By construction

$$f(r_+, R, \Lambda) = f(M, R, \Lambda) \equiv f(R). \tag{9}$$

So in what follows we use $f(R)$ to simplify the notation.

The Bekenstein–Hawking entropy calculated in this formalism is [11–17]

$$S = \pi r_+^2. \tag{10}$$

So, clearly, from Eq. (10) we see that in Eq. (7) we can exchange r_+ for S straightforwardly. This exchange is useful in some situations.

4. First law of thermodynamics I: $E(M, R, \Lambda)$ (or $E(M, A, \Lambda)$)

Seeing the quasilocal energy as $E(M, R, \Lambda)$, or equivalently as $E(M, A, \Lambda)$ since $A = 4\pi R^2$, allows us to write formally the first law of thermodynamics as

$$dE = \mu dM - p dA - l d\Lambda, \tag{11}$$

where μ, p, and l are defined by

$$\mu = \left(\frac{\partial E}{\partial M}\right)_{A,\Lambda}, \tag{12}$$

$$p = -\left(\frac{\partial E}{\partial A}\right)_{M,\Lambda}, \tag{13}$$

$$l = -\left(\frac{\partial E}{\partial \Lambda}\right)_{M,A}. \tag{14}$$

The quantity μ is the thermodynamic variable conjugate to M at the sphere of radius R, p is the surface pressure at the sphere of radius R, and l is the thermodynamic variable conjugate to Λ at the sphere of radius R, it has units of volume. Explicitly, from Eqs. (5)–(6) and Eqs. (12)–(14) we have

$$\mu = \frac{1}{\sqrt{f(R)}}, \tag{15}$$

$$8\pi R p = \frac{1 - \frac{M}{R} - \frac{2\Lambda R^2}{3}}{\sqrt{f(R)}} - \left(\frac{\partial E_0}{\partial R}\right)_\Lambda, \tag{16}$$

$$l = -\frac{R^3}{6\sqrt{f(R)}} - \left(\frac{\partial E_0}{\partial \Lambda}\right)_R. \tag{17}$$

All three quantities depend on the redshift factor $f(R)$, and p and l depend on E_0.

5. First law of thermodynamics II: $E(r_+, R, \Lambda)$ (or $E(S, A, \Lambda)$)

Seeing the quasilocal energy as $E(r_+, R, \Lambda)$, or equivalently as $E(S, A, \Lambda)$ since $S = 4\pi r_+^2$ from Eq. (10) and $A = 4\pi R^2$, we can write formally the first law of thermodynamics as

$$dE = TdS - pdA - \lambda d\Lambda\,, \tag{18}$$

where

$$T = \left(\frac{\partial E}{\partial S}\right)_{A,\Lambda}\,, \tag{19}$$

$$p = -\left(\frac{\partial E}{\partial A}\right)_{r_+,\Lambda}\,, \tag{20}$$

$$\lambda = -\left(\frac{\partial E}{\partial \Lambda}\right)_{r_+,A}\,. \tag{21}$$

T is the temperature at the sphere of radius R, p is the surface pressure at the sphere of radius R, and λ is the thermodynamic quantity conjugate to Λ at the sphere of radius R, with units of volume. Explicitly, using $S = 4\pi r_+^2$, see Eq. (10), we have from Eq. (19) that $T = \left(\frac{\partial E}{\partial S}\right)_{A,\Lambda} = \frac{1}{2\pi r_+}\left(\frac{\partial E}{\partial r_+}\right)_{R,\Lambda}$. Then, using Eqs. (7)–(8), we find

$$T = \frac{\mathcal{T}}{\sqrt{f(R)}}\,, \qquad \mathcal{T} = \frac{1-\Lambda r_+^2}{4\pi r_+}\,. \tag{22}$$

Using $A = 4\pi R^2$, we have that Eq. (20) can be written as $p = -\left(\frac{\partial E}{\partial A}\right)_{r_+,\Lambda} = -\frac{1}{8\pi R}\left(\frac{\partial E}{\partial R}\right)_{r_+,\Lambda}$, and so from Eqs. (7)–(8), we find

$$8\pi R p = \frac{1 - \frac{r_+}{2R} + \frac{\Lambda}{3}\left(\frac{r_+^3}{2R} - 2R^2\right)}{\sqrt{f(R)}} - \left(\frac{\partial E_0}{\partial R}\right)_{\Lambda}\,. \tag{23}$$

Finally, from Eq. (21) and using Eqs. (7)–(8), we find

$$\lambda = \frac{r_+^3 - R^3}{6\sqrt{f(R)}} - \left(\frac{\partial E_0}{\partial \Lambda}\right)_R\,. \tag{24}$$

All three quantities depend on the redshift factor $f(R)$. In particular Eq. (22) means that the temperature T at R is given by the Tolman formula, i.e., it is given by some temperature $\mathcal{T}$ redshifted to R, where $\mathcal{T}$ can be considered the Hawking temperature. Note that p and λ depend on E_0.

6. First law of thermodynamics III: enthalpy $H(S, P)$

We are now in a position to find a thermodynamics description with the enthalpy as the master thermodynamic variable.

For that we equate Eqs. (11) and (18). But first note that Eqs. (16) and (23) give the same surface pressure p as it is seen from Eq. (3), i.e., $1 - \frac{M}{R} - \frac{2\Lambda R^2}{3} = 1 - \frac{r_+}{2R} + \frac{\Lambda}{3}\left(\frac{r_+^3}{2R} - 2R^2\right)$. Then equating Eqs. (11) and (18) the two terms $p\,dA$ cancel out and one finds

$$\mu\, dM = TdS + (l-\lambda)\, d\Lambda\,. \tag{25}$$

Eq. (15) gives $\mu = \frac{1}{\sqrt{f(R)}}$, and from Eq. (22) one gets $T = \frac{\mathcal{T}}{\sqrt{f(R)}}$. Also, from Eqs. (17) and (24), one has $l - \lambda = -\left[\left(\frac{\partial E}{\partial \Lambda}\right)_{M,A} - \left(\frac{\partial E}{\partial \Lambda}\right)_{r_+,A}\right] = -\frac{r_+^3}{6\sqrt{f(R)}}$. The dependence on E_0 has now disappeared. Collecting these expressions into Eq. (25) one finds,

$$dM = \mathcal{T}dS + VdP\,, \tag{26}$$

where

$$V = \frac{4}{3}\pi r_+^3\,, \tag{27}$$

and

$$P = -\frac{\Lambda}{8\pi}\,. \tag{28}$$

The quantity V formally coincides with the volume of a region in a flat spacetime inside a radius r_+, and usually it is referred to as the thermodynamic volume. The quantity P is positive since Λ is negative, and has the meaning of vacuum pressure. It is clear from Eq. (26) that the mass M can be identified with the enthalpy H, $M = H$ and so,

$$dH = \mathcal{T}dS + VdP\,, \tag{29}$$

so that $H = H(S, P)$. Thus, as we have just showed, the enthalpy approach is directly connected to the quasilocal energy path integral Euclidean approach and vice-versa.

7. Conclusions

We showed that the first law of black hole thermodynamics in the form of Eq. (26) with Λ as a thermodynamic variable and the enthalpy H as the master thermodynamic variable does indeed follow from the black hole gravitational finite-size thermodynamics based on the quasilocal energy Euclidean action principle. In this sense, in making this bridge, we substantiated the approach discussed in [1–7]. We explained the identification of the mass M with the enthalpy H. We clarified why the first law of thermodynamics, Eq. (26), does not depend on the reference point, i.e., the term E_0 disappears. As it follows from the process of derivation, we have not appealed to what is hidden under the horizon. Nonetheless, we demonstrated why and how a thermodynamic volume V emerges consistently in the first law with the expression for this volume being $V = \frac{4}{3}\pi r_+^3$. It is also worth noting that the original set of variables included the surface pressure p related to the boundary and we traced how p is replaced by the completely different bulk vacuum pressure P.

It would be interesting to generalize the present results to the electric charge and rotating cases.

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