STUDY OF COSMIC ISSUES IN MODIFIED GAUSS-BONNET THEORIES

By

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CERTIFICATE

I certify that the research work presented in this thesis is the original work of **Miss Ayesha Ikram** D/O **Muhammad Ikram Bhatti** and is carried out under my supervision. I endorse its evaluation for the award of **Ph.D.** degree through the official procedure of **University of the Punjab**.

Prof. Dr. Muhammad Sharif
(**Supervisor**)

DECLARATION

I, **Miss Ayesha Ikram** D/O **Muhammad Ikram Bhatti**, hereby declare that the matter printed in this thesis is my original work. This thesis does not contain any material that has been submitted for the award of any other degree in any university and to the best of my knowledge, neither does this thesis contain any material published or written previously by any other person, except due reference is made in the text of this thesis. Most of the contents have been appeared as my research papers.

Ayesha Ikram

DEDICATED

To

My Grandfather,

Parents

&

Sister

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Abstract

This thesis is devoted to study some interesting cosmic issues in the context of modified Gauss-Bonnet theories. Firstly, we explore the instability ranges of a spherically symmetric anisotropic collapsing fluid under expansion-free condition in $f(\mathcal{G})$ gravity. We apply the first order perturbation scheme to the metric components as well as fluid variables and construct the corresponding field equations for both static as well as perturbed configurations using viable power-law $f(\mathcal{G})$ model. We establish dynamical equations using contracted Bianchi identities to discuss the dynamical instability in both Newtonian and post-Newtonian regimes. It is found that instability ranges depend on energy density, anisotropic pressures and Gauss-Bonnet terms but independent of adiabatic index for expansion-free collapsing fluid.

Secondly, we generalize $f(\mathcal{G})$ gravity by introducing non-minimal coupling between Gauss-Bonnet invariant and trace of the energy-momentum tensor named as $f(\mathcal{G},T)$ gravity and explore energy conditions for two reconstructed models in the background of homogeneous and isotropic universe. It is found that the massive test particles move along geodesic trajectories due to the presence of extra force originated from non-zero divergence of the energy-momentum tensor. The energy bounds are expressed in terms of deceleration, jerk and snap cosmological parameters. We study energy conditions for reconstructed models corresponding to de Sitter and power-law cosmological background using pressureless fluid and obtain feasible constraints on free parameters.

Thirdly, we discuss stability of the Einstein static universe against homogeneous as well as inhomogeneous scalar perturbations in $f(\mathcal{G},T)$ gravity. We investigate stability regions for particular $f(\mathcal{G},T)$ models corresponding to zero as well as non-zero covariant divergence of the energy-momentum tensor. The graphical analysis shows that stable Einstein universe exists for both spatially closed as well as open universe

models against homogeneous and inhomogeneous perturbations for appropriate choice of parameters.

Finally, we analyze stability of some cosmic evolutionary models against linear perturbations in Hubble parameter and energy density of matter distribution in $f(\mathcal{G},T)$ gravity. We establish the field equations for both general and particular $f(\mathcal{G},T)$ forms in the context of FRW universe model. We apply the reconstruction technique and found that this theory describes the de Sitter universe, power-law solutions as well as phantom/non-phantom eras cosmological backgrounds. We also discuss stability of de Sitter and power-law reconstructed $f(\mathcal{G},T)$ models and find stable results against linear perturbations.

Abbreviations

In this thesis, the metric signatures will be $(+,-,-,-)$ and Greek indices $(\alpha, \beta, \gamma, ...)$ will vary from 0 to 3, if different it will be mentioned. We shall use the following list of abbreviations.

DE:	Dark Energy
DEC:	Dominant Energy Condition
EU:	Einstein Universe
EoS:	Equation of State
FRW:	Friedmann-Robertson-Walker
GB:	Gauss-Bonnet
GR:	General Relativity
ΛCDM:	Λ Cold Dark Matter
N:	Newtonian
NEC:	Null Energy Condition
pN:	post-Newtonian
SEC:	Strong Energy Condition
WEC:	Weak Energy Condition

Introduction

In the last few decades, cosmology has made tremendous progress in theoretical modeling as well as in high-precision observations. A diverse set of observational data accumulated from supernovae type Ia, cosmic microwave background, baryon acoustic oscillations etc have confirmed accelerated expansion of the universe. This cosmic expansion is considered as a consequence of some mysterious form of energy with anti-gravitational characteristics dubbed as DE. In GR, cosmological constant (Λ) is the simplest candidate to comprehend the nature of DE. There are two main attempts to address this issue. The first approach is obtained by modifying matter part of the Einstein-Hilbert action such as K-essence, quintessence, Chaplygin gas and its extensions etc. The second type modifies the geometric sector of GR leading to modified theories such as scalar-tensor theories, $f(R)$ gravity (R is the Ricci scalar), $f(R,T)$ gravity (T is the trace of energy-momentum tensor), GB gravity and its modifications, etc.

Gauss-Bonnet invariant ($\mathcal{G}$) is a four-dimensional topological term and free from spin-2 ghost instabilities. Nojiri and Odintsov [1] introduced the modified GB gravity or $f(\mathcal{G})$ gravity by adding generic function $f(\mathcal{G})$ in the action of GR. This theory successfully describes salient characteristics of late-time cosmology as well as effectively demonstrates transition from decelerated to accelerated phases of the universe [2]. De

Felice and Tsujikawa [3] formulated explicit forms of $f(\mathcal{G})$ models and constructed conditions under which these models are cosmological viable. The same authors [4] also found that these models are consistent with solar system constraints for a wide range of model parameters. This theory avoids finite-time future singularities, investigates dynamics of gravitational collapse as well as examines traversability of wormholes [5].

The non-minimal coupled modified theories also cover a broad spectrum of applications in cosmology. Bertolami et al. [6] introduced this coupling between geometry and matter such that scalar curvature admits direct interaction with matter Lagrangian density ($\mathcal{L}_m$). Mohseni [7] extended this idea to quadratic curvature invariant ($\mathcal{G}$). In these theories, the most interesting fact is the non-geodesic motion of test particles due to the presence of an extra force originated from this direct interaction. The non-minimally coupled $f(R,\mathcal{L}_m)$ theory explains the current cosmic expansion, explores the presence of dark matter in galaxies or cluster of galaxies, demonstrates the natural conditions for preheating in inflationary models and investigates the existence of physically realistic wormholes [8].

Gravitational collapse has a fundamental importance in structure formation of the universe which takes place when the state of hydrostatic equilibrium of massive stellar objects is destroyed. The existence of any static stellar object is significant only when it remains stable against small perturbations. The perturbation technique is helpful to measure stability criteria of self-gravitating objects. Chandrasekhar [9] was the pioneer to explore dynamical instability for spherical star filled with isotropic matter distribution using adiabatic index (Γ) also known as stiffness parameter. Herrera et al. [10] analyzed these ranges in the presence of dissipative effects for spherically symmetric spacetime and concluded that instability of matter distribution increases due to heat flow at N approximation. Sharif and Kausar [11] discussed stability of a collapsing spherical object in $f(R)$ gravity under expansion-free condition and found that instability ranges defined on external boundary as well as on internal vacuum cavity depend on both $f(R)$ model parameters and matter variables but independent of Λ. Sharif and Azam [12] explored the role of pressure anisotropy on the instability ranges for cylindrical as well as planar geometries in the presence of expansion-free condition. Sharif and Bhatti [13] investigated dynamical instability in the presence of electromagnetism under the same condition for cylindrical and planar geometries.

Energy conditions provide an interesting way to judge the physically realistic matter configuration. These fundamental constraints namely null, weak, strong and dominant energy conditions give the positivity of energy density as well as its dominance over pressure. These conditions have remarkable importance in various fascinating phenomena like Hawking-Penrose singularity theorem, positive mass theorem and validity of second law of black hole thermodynamics [14]. Banijamali et al. [15] explored the energy conditions in $f(\mathcal{G})$ gravity with non-minimal coupling to $\mathcal{L}_m$ and assured the positivity of WEC. Sharif and Waheed [16] derived these conditions in generalized second order scalar-tensor gravity using specific power-law form for scalar field. Sharif and Zubair [17] studied energy bounds for two particular $f(R,T,R_{\alpha\beta}T^{\alpha\beta})$ models as well as analyzed the validity of Dolgov-Kowasaki instability for two specific $f(R,T)$ models. This has also been investigated in other modified theories of gravity like $f(R)$ gravity, Brans-Dicke theory, $f(\mathcal{G})$ gravity, generalized teleparallel theory [18].

Emergent universe scenario has a significant importance in cosmology to avoid the existence of big-bang singularity. In this scenario, the cosmic initial state is EU rather than a big-bang singularity and then evolves to cosmic inflationary epoch [19]. Einstein universe is demonstrated by closed static FRW cosmological model filled with prefect fluid in the presence of Λ. The stability of EU against any kind of perturbation is the most crucial aspect of successful emergent universe. Eddington [20] discovered that EU is unstable against small spatially homogeneous and isotropic scalar perturbations in the presence of normal matter. Gibbons [21] concluded that EU maximizes the entropy against conformal metric perturbations if and only if the relation $c_s > 1/\sqrt{5}$ (c_s is the speed of sound) holds or EU is stable against Jean's instability. Barrow et al. [22] investigated stability of EU filled with perfect fluid and found stable solutions against small inhomogeneous vector and tensor perturbations. They also observed that EU is always neutrally stable under adiabatic scalar density inhomogeneities as long as the inequality $5c_s^2 > 1$ is satisfied and unstable otherwise. Thus, the emergent universe mechanism fails in the context of GR since homogenous scalar perturbations destroy the stability of initial static state of the universe.

The existence and stability of EU has widely been discussed in braneworld, Einstein-Cartan theory, loop quantum cosmology, Brans-Dicke theory, non-minimal kinetic coupled gravity etc [23]. Böhmer et al. [24] investigated its stability for specific forms of $f(R)$ models and concluded that $f(R)$ gravity stabilizes the EU against scalar homogeneous perturbations. Goswami et al. [25] found that EU is neutrally stable against vector as well as tensor perturbations while for scalar perturbations, the stable EU exists until the inequality $c_s^2 > (\sqrt{5}-1)/6$ holds for a particular $f(R)$ model. Böhmer and Lobo [26] determined stable regions for all values of EoS parameter against scalar homogeneous perturbations in $f(\mathcal{G})$ gravity. The stable regions are obtained against homogeneous as well as inhomogeneous scalar perturbations in hybrid metric-Palatini gravity [27]. Huang et al. [28] investigated stable regions against both homogeneous as well as inhomogeneous scalar perturbations in $f(\mathcal{G})$ gravity and concluded that stable EU exists only for homogeneous perturbations in closed universe.

The reconstruction technique in modified theories is a successful approach to reproduce the cosmic history. In this scheme, any known cosmic evolution is considered and used the modified field equations to get the particular form of Lagrangian density which reproduces the corresponding background. The stability of these reconstructed cosmic models is an interesting field of research in cosmology. In this stability criteria, usually energy density and Hubble parameter are perturbed upto first order to investigate the cosmic background stability as time evolves. Nojiri et al. [29] established the reconstruction technique in terms of e-folding to reproduce the background evolution corresponding to ΛCDM cosmology, phantom/non-phantom eras, late-time cosmic acceleration crossing the phantom-divide line, transient phantom era and oscillating universe in $f(R)$ gravity. Elizalde et al. [30] found that modified GB gravity with and without scalar field can efficiently describe the ΛCDM cosmological background in the absence of Λ. Sáez-Gómez [31] investigated the cosmological solutions and analyzed their stability in $f(R)$ Hořava-Lifshitz gravity.

Myrzakulov et al. [32] studied cosmological models in $f(\mathcal{G})$ gravity and concluded that ΛCDM model, inflationary epoch as well as DE phase can be discussed in this theory. Jamil and his collaborators [33] reconstructed the well-known cosmic evolutionary models in $f(R,T)$ gravity. de la Cruz-Dombriz and Sáez-Gómez [34] analyzed the stability of ΛCDM, de Sitter as well as power-law solutions in $f(R,\mathcal{G})$

gravity. Sharif and Zubair [35] illustrated that $f(R,T)$ gravity can reproduce the possible phase transition from decelerated to accelerated phases, phantom/non-phantom epochs as well as ΛCDM cosmological model and analyzed the stability of reconstructed de Sitter as well as power-law solutions.

This thesis is devoted to study some interesting cosmological aspects in modified GB theories of gravity. In this context, we discuss instability of expansion-free sphere, energy conditions, existence of stable EU and stability of some reconstructed cosmological models. This thesis is arranged as follows.

- Chapter **One** provides basic concepts and definitions related to this thesis.

- Chapter **Two** analyzes the dynamical instability of an expansion-free collapsing sphere in $f(\mathcal{G})$ gravity. We consider N and pN approximations to discuss both dynamical equations as well as instability ranges for a power-law $f(\mathcal{G})$ model.

- Chapter **Three** investigates the effects of non-minimal coupling between $\mathcal{G}$ and matter referred as $f(\mathcal{G},T)$ gravity. The energy conditions are constructed as well as expressed in terms of cosmological parameters for de Sitter and power-law solutions.

- Chapter **Four** explores the existence of stable EU against homogeneous as well as inhomogeneous scalar perturbations.

- Chapter **Five** presents cosmological reconstruction in $f(\mathcal{G},T)$ gravity. The stability of reconstructed models is also discussed against linear perturbations.

- Chapter **Six** summarizes the results and also suggests some interesting issues for future research.

Chapter 1

Preliminaries

This chapter deals with some important concepts that are useful for the better understanding of research work.

1.1 Modified Gauss-Bonnet Theories

The Einstein-Hilbert action for GB gravity (also dubbed as Einstein GB gravity) is given as [36]

$$\mathcal{S}=\int d^{\mathrm{D}}x\sqrt{-g}\left(\frac{R+\bar{\varsigma}\mathcal{G}}{2\kappa_{\mathrm{D}}^{2}}+\mathcal{L}_{(m)}\right), \tag{1.1.1}$$

where κ_{D}^2, $\bar{\varsigma}$ and g represent the coupling constant in D dimensions, coupling coefficient of $\mathcal{G}$ and determinant of metric tensor ($g_{\alpha\beta}$), respectively. The GB invariant is defined as

$$\mathcal{G}=R^{2}-4R_{\alpha\beta}R^{\alpha\beta}+R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}, \tag{1.1.2}$$

where $R_{\alpha\beta}$ and $R_{\alpha\beta\mu\nu}$ are the Ricci and Riemann tensors, respectively. Due to its quadratic nature, it is non-linear in second derivative of $g_{\alpha\beta}$ while it is four-dimensional topological invariant. This topological invariance does not imply that $\mathcal{G}$ is a constant quantity or vanishes in four dimensions rather than the integral

$\int d^4x\sqrt{-g}\mathcal{G}$ turns out to be zero. Varying the action (1.1.1) with respect to $g_{\alpha\beta}$, we obtain second order partial differential equations as

$$R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R-\frac{1}{2}g_{\alpha\beta}\bar{\varsigma}\mathcal{G}+2\bar{\varsigma}[RR_{\alpha\beta}-2R_{\alpha\mu}R^{\mu}_{\beta}-2R^{\mu\nu}R_{\alpha\mu\beta\nu}+R_{\alpha}^{\ \mu\nu\gamma}R_{\beta\mu\nu\gamma}]=\kappa^2_{\mathrm{D}}T_{\alpha\beta},$$

where $T_{\alpha\beta}$ is the energy-momentum tensor. Einstein GB gravity is considered as the direct generalization of GR in higher dimensions while in four dimensions, GR is recovered since the contribution of $\mathcal{G}$ disappears from the field equations. In order to discuss the effects of $\mathcal{G}$ in four dimensions, one needs to modify GB gravity either by coupling with scalar field $(\hat{\phi})$ or by introducing generic function $f(\mathcal{G})$ in the Einstein-Hilbert action.

1.1.1 Scalar-Gauss-Bonnet Gravity

The action for GB gravity with scalar coupling as an alternative model for DE is of the form [37]

$$\mathcal{S}=\int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2}-\frac{\epsilon}{2}\partial_\alpha\hat{\phi}\partial^\alpha\hat{\phi}-\mathbb{U}(\hat{\phi})+\varpi(\hat{\phi})\mathcal{G}\right), \tag{1.1.3}$$

where $\epsilon=\pm1$, $\mathbb{U}(\hat{\phi})$ and $\varpi(\hat{\phi})$ are arbitrary functions of scalar field. Variation in the above action with respect to $g_{\alpha\beta}$ yields the field equations as follows

$$\begin{aligned}
&\frac{1}{\kappa^2}\left(R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R\right)-\epsilon\left(\frac{1}{2}\partial_\alpha\hat{\phi}\partial_\beta\hat{\phi}-\frac{1}{4}g_{\alpha\beta}\partial^\mu\hat{\phi}\partial_\mu\hat{\phi}\right)-\frac{1}{2}g_{\alpha\beta}\left(\varpi(\hat{\phi})\mathcal{G}-\mathbb{U}(\hat{\phi})\right)\\
+\;&2RR_{\alpha\beta}\varpi(\hat{\phi})-8R_{\alpha\mu}R^{\mu}_{\beta}\varpi(\hat{\phi})+2R_{\alpha}^{\ \mu\nu\gamma}R_{\beta\mu\nu\gamma}-2\nabla_\alpha\nabla_\beta[R\varpi(\hat{\phi})]-4\nabla^2[R_{\alpha\beta}\varpi(\hat{\phi})]\\
+\;&4\nabla_\mu\nabla_\alpha[R^{\mu}_{\beta}\varpi(\hat{\phi})]+4\nabla_\mu\nabla_\beta[R^{\mu}_{\alpha}\varpi(\hat{\phi})]-4g_{\alpha\beta}\nabla_\mu\nabla_\nu[R^{\mu\nu}\varpi(\hat{\phi})]\\
-\;&4\nabla^\mu\nabla^\nu[R_{\alpha\mu\nu\beta}\varpi(\hat{\phi})]+2g_{\alpha\beta}\nabla^2[R\varpi(\hat{\phi})]=0,
\end{aligned}$$

where $\nabla^2 = \nabla_\alpha \nabla^\alpha$ (∇_α is a covariant derivative). On the other hand, variation with respect to $\hat{\phi}$ gives

$$\epsilon \nabla^2 \hat{\phi} + \varpi_{\hat{\phi}}(\hat{\phi})\mathcal{G} - \mathbb{U}_{\hat{\phi}}(\hat{\phi}) = 0,$$

where subscript $\hat{\phi}$ denotes derivative with respect to scalar field. The coupling between $\mathcal{G}$ and $\hat{\phi}$ naturally appears in the low-energy effective action of string theory which elegantly describes the non-singular cosmological solutions [38].

1.1.2 $f(\mathcal{G})$ Gravity

Without using scalar field, the dynamics of $\mathcal{G}$ in four dimensions can also be discussed by adding the generic function $f(\mathcal{G})$ in the action as [1]

$$\mathcal{S} = \int d^4x \sqrt{-g} \left(\frac{R + f(\mathcal{G})}{2\kappa^2} + \mathcal{L}_m \right). \tag{1.1.4}$$

Varying this action with respect to $g_{\alpha\beta}$, we obtain modified field equations as follows

$$\begin{aligned} & \kappa^2 T_{\alpha\beta} - R_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta} R + \frac{1}{2} g_{\alpha\beta} f(\mathcal{G}) - 2RR_{\alpha\beta} f_{\mathcal{G}}(\mathcal{G}) + 4R_{\alpha\mu} R^{\mu}_{\beta} f_{\mathcal{G}}(\mathcal{G}) \\ - \; & 2R_{\alpha}^{\ \mu\nu\gamma} R_{\beta\mu\nu\gamma} f_{\mathcal{G}}(\mathcal{G}) - 4R_{\alpha\mu\nu\beta} R^{\mu\nu} f_{\mathcal{G}}(\mathcal{G}) + 2R \nabla_\alpha \nabla_\beta f_{\mathcal{G}}(\mathcal{G}) - 2g_{\alpha\beta} R \nabla^2 f_{\mathcal{G}}(\mathcal{G}) \\ - \; & 4R^{\mu}_{\alpha} \nabla_\mu \nabla_\beta f_{\mathcal{G}}(\mathcal{G}) - 4R^{\mu}_{\beta} \nabla_\mu \nabla_\alpha f_{\mathcal{G}}(\mathcal{G}) + 4R_{\alpha\beta} \nabla^2 f_{\mathcal{G}}(\mathcal{G}) + 4g_{\alpha\beta} R^{\mu\nu} \nabla_\mu \nabla_\nu f_{\mathcal{G}}(\mathcal{G}) \\ - \; & 4R_{\alpha\mu\beta\nu} \nabla^\mu \nabla^\nu f_{\mathcal{G}}(\mathcal{G}) = 0, \end{aligned} \tag{1.1.5}$$

where $f_{\mathcal{G}}(\mathcal{G})$ is the derivative of $f(\mathcal{G})$ with respect to $\mathcal{G}$. It can be easily observed that GB contribution disappears from the field equations for $f(\mathcal{G}) = \mathcal{G}$ in four dimensions. Equation (1.1.5) can be expressed identical to Einstein field equations as

$$G_{\alpha\beta} = \kappa^2 T^{\text{eff}}_{\alpha\beta} = \kappa^2 (T_{\alpha\beta} + T^{\mathcal{G}}_{\alpha\beta}), \tag{1.1.6}$$

where $G_{\alpha\beta}=R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R$ is the Einstein tensor while $f(\mathcal{G})$ contribution is given by

$$\begin{aligned}\kappa^2T^{\mathcal{G}}_{\alpha\beta} &= \frac{1}{2}g_{\alpha\beta}f(\mathcal{G})-2RR_{\alpha\beta}f_{\mathcal{G}}(\mathcal{G})+4R_{\beta\mu}R^{\mu}_{\alpha}f_{\mathcal{G}}(\mathcal{G})-2R_{\alpha\mu\nu\gamma}R_{\beta}^{\ \mu\nu\gamma}f_{\mathcal{G}}(\mathcal{G})\\ &- 4R_{\alpha\mu\nu\beta}R^{\mu\nu}f_{\mathcal{G}}(\mathcal{G})+2R\nabla_{\alpha}\nabla_{\beta}f_{\mathcal{G}}(\mathcal{G})-4R^{\mu}_{\beta}\nabla_{\alpha}\nabla_{\mu}f_{\mathcal{G}}(\mathcal{G})-4R^{\mu}_{\alpha}\nabla_{\beta}\nabla_{\mu}f_{\mathcal{G}}(\mathcal{G})\\ &- 2g_{\alpha\beta}R\nabla^2f_{\mathcal{G}}(\mathcal{G})+4R_{\alpha\beta}\nabla^2f_{\mathcal{G}}(\mathcal{G})+4g_{\alpha\beta}R^{\mu\nu}\nabla_{\mu}\nabla_{\nu}f_{\mathcal{G}}(\mathcal{G})\\ &- 4R_{\alpha\mu\beta\nu}\nabla^{\mu}\nabla^{\nu}f_{\mathcal{G}}(\mathcal{G}),\end{aligned}$$

Unlike GR, the field equations are fourth order differential equations while the energy-momentum tensor remains conserved ($T^{\alpha\beta}_{\ \ ;\alpha}=0$) in this modified theory. The trace of Eq.(1.1.5) is

$$\kappa^2T+R+2f(\mathcal{G})-2\mathcal{G}f_{\mathcal{G}}(\mathcal{G})-2R\nabla^2f_{\mathcal{G}}(\mathcal{G})+4R_{\alpha\beta}\nabla^{\alpha}\nabla^{\beta}f_{\mathcal{G}}(\mathcal{G})=0,$$

where $T=g_{\alpha\beta}T^{\alpha\beta}$. The viability of any $f(\mathcal{G})$ model depends on the regularities of generic function and its derivatives along with the condition $f_{\mathcal{G}\mathcal{G}}(\mathcal{G})<0$ while this inequality is reversed for metric signatures $(-,+,+,+)$ for all $\mathcal{G}$ [3, 39]. We remark that the variation of action (1.1.3) with respect to ϕ under the substitutions

$$\mathbb{U}(\hat{\phi})=\frac{1}{2\kappa^2}\left[-f(\hat{\phi})+\hat{\phi}f_{\hat{\phi}}(\hat{\phi})\right],\quad \varpi(\hat{\phi})=\frac{\hat{\phi}f_{\hat{\phi}}(\hat{\phi})}{2\kappa^2},\quad \epsilon=0,$$

yield $\hat{\phi}=\mathcal{G}$. The $f(\mathcal{G})$ gravity is recovered when inserting this result back into the action (1.1.3) under the above transformations.

1.1.3 $f(\mathcal{G},T)$ Gravity

The non-minimally matter coupling in the action (1.1.4) of $f(\mathcal{G})$ gravity as [40]

$$\mathcal{S}=\int d^4x\sqrt{-g}\left(\frac{R+f(\mathcal{G},T)}{2\kappa^2}+\mathcal{L}_m\right). \tag{1.1.7}$$

Varying this action with respect to $g_{\alpha\beta}$, we obtain

$$\begin{aligned}0=\delta\mathcal{S} &= \int d^4x\sqrt{-g}\left(-\frac{1}{2}g_{\alpha\beta}[R+f(\mathcal{G},T)]\delta g^{\alpha\beta}+\delta R+f_{\mathcal{G}}(\mathcal{G},T)\delta\mathcal{G}\right.\\ &+ \left. f_T(\mathcal{G},T)\delta T-\kappa^2T_{\alpha\beta}\delta g^{\alpha\beta}\right), \qquad (1.1.8)\end{aligned}$$

where we have used the following relations

$$\delta\sqrt{-g}=-\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta},\quad \delta f(\mathcal{G},T)=f_{\mathcal{G}}(\mathcal{G},T)\delta\mathcal{G}+f_T(\mathcal{G},T)\delta T. \qquad (1.1.9)$$

The subscripts $\mathcal{G}$ and T denote derivatives of $f(\mathcal{G},T)$ with respect to $\mathcal{G}$ and T, respectively while the energy-momentum tensor is defined as [41]

$$T_{\alpha\beta}=-\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\alpha\beta}}. \qquad (1.1.10)$$

The variation of $\mathcal{G}$ and T are as follows

$$\begin{aligned}\delta\mathcal{G} &= 2R\delta R-4\delta(R_{\alpha\beta}R^{\alpha\beta})+\delta(R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}),\\ \delta T &= (T_{\alpha\beta}+\Theta_{\alpha\beta})\delta g^{\alpha\beta},\quad \Theta_{\alpha\beta}=g^{\mu\nu}\frac{\delta T_{\mu\nu}}{\delta g^{\alpha\beta}}, \qquad (1.1.11)\end{aligned}$$

Using these relations, the action (1.1.8) takes the form

$$\begin{aligned}0=\delta\mathcal{S} &= \int d^4x\sqrt{-g}\left(-\frac{1}{2}g_{\alpha\beta}[R+f(\mathcal{G},T)]\delta g^{\alpha\beta}+\delta R+f_{\mathcal{G}}(\mathcal{G},T)[2R\delta R\right.\\ &- 4\delta(R_{\alpha\beta}R^{\alpha\beta})+\delta(R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu})]+(T_{\alpha\beta}+\Theta_{\alpha\beta})f_T(\mathcal{G},T)\delta g^{\alpha\beta}\\ &- \left.\kappa^2T_{\alpha\beta}\delta g^{\alpha\beta}\right), \qquad (1.1.12)\end{aligned}$$

where the variations of $R^{\mu}{}_{\alpha\beta\nu}$, $R_{\alpha\eta}$ and R provide the following expressions

$$\begin{aligned}\delta R^{\mu}{}_{\alpha\beta\nu} &= \nabla_\beta(\delta\Gamma^{\mu}_{\nu\alpha})-\nabla_\nu(\delta\Gamma^{\mu}_{\beta\alpha})=(g_{\alpha\gamma}\nabla_{[\nu}\nabla_{\beta]}+g_{\gamma[\beta}\nabla_{\nu]}\nabla_\alpha)\delta g^{\mu\gamma}+\nabla_{[\nu}\nabla^{\mu}\delta g_{\beta]\alpha},\\ \delta R_{\alpha\nu} &= \delta R^{\mu}{}_{\alpha\mu\nu},\quad \delta R=(R_{\alpha\beta}+g_{\alpha\beta}\nabla^2-\nabla_\alpha\nabla_\beta)\delta g^{\alpha\beta}.\end{aligned}$$

We consider the following integrals used in the action (1.1.12)

$$\begin{aligned}
&\int d^4x\sqrt{-g}f_{\mathcal{G}}(\mathcal{G},T)\delta(R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}) \\
= \;& 2\int d^4x\sqrt{-g}\delta g^{\alpha\beta}f_{\mathcal{G}}(\mathcal{G},T)R_{\alpha}^{\;\mu\nu\gamma}R_{\beta\mu\nu\gamma}+2\int d^4x\sqrt{-g}f_{\mathcal{G}}(\mathcal{G},T)R_{\alpha}^{\;\beta\mu\nu}\delta R^{\alpha}{}_{\beta\mu\nu} \\
= \;& 2\int d^4x\sqrt{-g}\delta g^{\alpha\beta}[f_{\mathcal{G}}(\mathcal{G},T)R_{\alpha}^{\;\mu\nu\gamma}R_{\beta\mu\nu\gamma}+2\nabla^{(\mu}\nabla^{\nu)}[f_{\mathcal{G}}(\mathcal{G},T)R_{\alpha\mu\beta\nu}] \\
= \;& \int d^4x\sqrt{-g}\delta g^{\alpha\beta}[2f_{\mathcal{G}}(\mathcal{G},T)R_{\alpha}^{\;\mu\nu\gamma}R_{\beta\mu\nu\gamma}+4f_{\mathcal{G}}(\mathcal{G},T)\nabla^2R_{\alpha\beta}-4f_{\mathcal{G}}(\mathcal{G},T)R_{\alpha\mu}R^{\mu}_{\beta} \\
- \;& 2f_{\mathcal{G}}(\mathcal{G},T)\nabla_{\alpha}\nabla_{\beta}R+4R_{\alpha\mu\beta\nu}\nabla^{\mu}\nabla^{\nu}f_{\mathcal{G}}(\mathcal{G},T)+4f_{\mathcal{G}}(\mathcal{G},T)R_{\alpha\mu\beta\nu}R^{\mu\nu}], \qquad (1.1.13) \\
&\int d^4x\sqrt{-g}f_{\mathcal{G}}(\mathcal{G},T)\delta(R_{\alpha\beta}R^{\alpha\beta}) \\
= \;& \int d^4x\sqrt{-g}f_{\mathcal{G}}(\mathcal{G},T)[2R_{\alpha\mu}R^{\mu}_{\beta}\delta g^{\alpha\beta}+2R^{\alpha\beta}\delta R_{\alpha\beta}] \\
= \;& \int d^4x\sqrt{-g}\delta g^{\alpha\beta}[2f_{\mathcal{G}}(\mathcal{G},T)R_{\alpha\mu}R^{\mu}_{\beta}-2\nabla^{\mu}\nabla_{(\alpha}[f_{\mathcal{G}}(\mathcal{G},T)R_{\beta)\mu}]+\nabla^2[f_{\mathcal{G}}(\mathcal{G},T)R_{\alpha\beta}] \\
+ \;& g_{\alpha\beta}\nabla^{\mu}\nabla^{\nu}[f_{\mathcal{G}}(\mathcal{G},T)R_{\mu\nu}]] \\
= \;& \int d^4x\sqrt{-g}\delta g^{\alpha\beta}[2f_{\mathcal{G}}(\mathcal{G},T)R_{\alpha\mu}R^{\mu}_{\beta}-R_{\beta\mu}\nabla^{\mu}\nabla_{\alpha}f_{\mathcal{G}}(\mathcal{G},T)-f_{\mathcal{G}}(\mathcal{G},T)\nabla_{\alpha}\nabla_{\beta}R \\
- \;& 2f_{\mathcal{G}}(\mathcal{G},T)R_{\mu\alpha}R^{\mu}_{\beta}+2f_{\mathcal{G}}(\mathcal{G},T)R_{\alpha\mu\beta\nu}R^{\mu\nu}-R_{\alpha\mu}\nabla^{\mu}\nabla_{\beta}f_{\mathcal{G}}(\mathcal{G},T)+f_{\mathcal{G}}(\mathcal{G},T)\nabla^2R_{\alpha\beta} \\
+ \;& g_{\alpha\beta}R_{\mu\nu}\nabla^{\mu}\nabla^{\nu}f_{\mathcal{G}}(\mathcal{G},T)+\frac{1}{2}g_{\alpha\beta}f_{\mathcal{G}}(\mathcal{G},T)\nabla^2R+R_{\alpha\beta}\nabla^2f_{\mathcal{G}}(\mathcal{G},T)], \qquad (1.1.14) \\
&\int d^4x\sqrt{-g}[1+2Rf_{\mathcal{G}}(\mathcal{G},T)]\delta R \\
= \;& \int d^4x\sqrt{-g}\delta g^{\alpha\beta}(R_{\alpha\beta}+g_{\alpha\beta}\nabla^2-\nabla_{\alpha}\nabla_{\beta})[1+2Rf_{\mathcal{G}}(\mathcal{G},T)] \\
= \;& \int d^4x\sqrt{-g}\delta g^{\alpha\beta}[R_{\alpha\beta}+2RR_{\alpha\beta}f_{\mathcal{G}}(\mathcal{G},T)+2Rg_{\alpha\beta}\nabla^2f_{\mathcal{G}}(\mathcal{G},T)-2R\nabla_{\alpha}\nabla_{\beta}f_{\mathcal{G}}(\mathcal{G},T) \\
+ \;& 2g_{\alpha\beta}f_{\mathcal{G}}(\mathcal{G},T)\nabla^2R-2f_{\mathcal{G}}(\mathcal{G},T)\nabla_{\alpha}\nabla_{\beta}R], \qquad (1.1.15)
\end{aligned}$$

where we have used the following identities

$$\begin{aligned}
\nabla^{\alpha}\nabla^{\beta}R_{\alpha\beta} &= \frac{1}{2}\nabla^2R, \quad \nabla_{\mu}\nabla_{\alpha}R^{\mu}_{\beta}=\frac{1}{2}\nabla_{\alpha}\nabla_{\beta}R+R_{\mu\alpha}R^{\mu}_{\beta}-R_{\alpha\mu\beta\nu}R^{\mu\nu}, \\
\nabla^{\mu}\nabla^{\nu}R_{\alpha\mu\beta\nu} &= \nabla^2R_{\alpha\beta}-\nabla_{\mu}\nabla_{\alpha}R^{\mu}_{\beta}.
\end{aligned}$$

Substituting Eqs.(1.1.13)-(1.1.15) in (1.1.12), we obtain the field equations of $f(\mathcal{G},T)$ gravity as follows

$$\begin{aligned}
G_{\alpha\beta} &= \kappa^2 T_{\alpha\beta}+\frac{1}{2}g_{\alpha\beta}f(\mathcal{G},T)-(T_{\alpha\beta}+\Theta_{\alpha\beta})f_T(\mathcal{G},T)-(2RR_{\alpha\beta}-4R^{\mu}_{\alpha}R_{\mu\beta}\\
&- 4R_{\alpha\mu\beta\nu}R^{\mu\nu}+2R_{\alpha}^{\ \mu\nu\gamma}R_{\beta\mu\nu\gamma})f_{\mathcal{G}}(\mathcal{G},T)-(2Rg_{\alpha\beta}\nabla^2-2R\nabla_\alpha\nabla_\beta\\
&- 4g_{\alpha\beta}R^{\mu\nu}\nabla_\mu\nabla_\nu-4R_{\alpha\beta}\nabla^2+4R^{\mu}_{\alpha}\nabla_\beta\nabla_\mu+4R^{\mu}_{\beta}\nabla_\alpha\nabla_\mu\\
&+ 4R_{\alpha\mu\beta\nu}\nabla^\mu\nabla^\nu)f_{\mathcal{G}}(\mathcal{G},T). \qquad (1.1.16)
\end{aligned}$$

In the absence of curvature-matter coupling, this equation reduces to the field equations for $f(\mathcal{G})$ gravity given in Eq.(1.1.5) while the Einstein field equations are recovered when $f(\mathcal{G},T)=0$. The trace of the above equation is given by

$$\begin{aligned}
&R+\kappa^2T+2f(\mathcal{G},T)-(T+\Theta)f_T(\mathcal{G},T)-2\mathcal{G}f_{\mathcal{G}}(\mathcal{G},T)-2R\nabla^2 f_{\mathcal{G}}(\mathcal{G},T)\\
+\;& 4R^{\alpha\beta}\nabla_\alpha\nabla_\beta f_{\mathcal{G}}(\mathcal{G},T)=0, \quad \Theta=\Theta^{\alpha}_{\alpha}.
\end{aligned}$$

Taking the covariant derivative of Eq.(1.1.16), we obtain

$$\begin{aligned}
\nabla^\alpha T_{\alpha\beta} &= \frac{f_T(\mathcal{G},T)}{\kappa^2-f_T(\mathcal{G},T)}\left[(T_{\alpha\beta}+\Theta_{\alpha\beta})\nabla^\alpha(\ln f_T(\mathcal{G},T))+\nabla^\alpha\Theta_{\alpha\beta}\right.\\
&- \left.\frac{1}{2}g_{\alpha\beta}\nabla^\alpha T\right], \qquad (1.1.17)
\end{aligned}$$

which shows that covariant divergence of the energy-momentum tensor is non-zero in this theory.

We assume that matter configuration depends on the components of $g_{\alpha\beta}$ rather than its derivatives so that Eq.(1.1.10) takes the form

$$T_{\alpha\beta}=g_{\alpha\beta}\mathcal{L}_m-2\frac{\partial\mathcal{L}_m}{\partial g^{\alpha\beta}}. \qquad (1.1.18)$$

Differentiating with respect to $g_{\alpha\beta}$, we obtain

$$\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}=\frac{\delta g_{\alpha\beta}}{\delta g^{\mu\nu}}\mathcal{L}_m+g_{\alpha\beta}\frac{\partial\mathcal{L}_m}{\partial g^{\mu\nu}}-2\frac{\partial^2\mathcal{L}_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}. \qquad (1.1.19)$$

Using the relations
$$\frac{\delta g_{\alpha\beta}}{\delta g^{\mu\nu}} = -g_{\alpha\gamma}g_{\beta\varsigma}\delta^{\gamma\varsigma}_{\mu\nu}, \quad \delta^{\gamma\varsigma}_{\mu\nu} = \frac{\delta g^{\gamma\varsigma}}{\delta g^{\mu\nu}},$$
where $\delta^{\gamma\varsigma}_{\mu\nu}$ is the generalized Kronecker symbol. Substituting Eq.(1.1.19) in (1.1.11), we obtain a useful expression for $\Theta_{\alpha\beta}$ as
$$\Theta_{\alpha\beta} = -2T_{\alpha\beta} + g_{\alpha\beta}\mathcal{L}_m - 2g^{\mu\nu}\frac{\partial^2\mathcal{L}_m}{\partial g^{\alpha\beta}\partial g^{\mu\nu}}. \tag{1.1.20}$$
This shows that once the value of $\mathcal{L}_m$ is known, one can find the expression for tensor $\Theta_{\alpha\beta}$. Harko et al. [42] discussed various forms of $\mathcal{L}_m$ corresponding to perfect fluid, massless scalar and electromagnetic fields.

1.2 FRW Model and Some Cosmological Measures

The central premise of modern cosmology is that the universe is spatially homogeneous and isotropic at least at large scales (cosmological principle). The FRW universe model is considered as standard model of cosmology as it describes homogeneity as well as isotropy of the universe. This basic cosmological model was first established by Alexander Friedmann in 1922 and later in 1935, Howard Percy Robertson and Arthur Geoffrey Walker proposed its modified form. The line element for FRW universe model is given by
$$ds^2 = dt^2 - a^2(t)\left(\frac{1}{1-\mathcal{K}r^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right), \tag{1.2.1}$$
where $a(t)$ and $\mathcal{K}$ represent the scale factor depending on the cosmic time (t) and spatial curvature parameter, respectively. The parameter $\mathcal{K}$ corresponds to open $(\mathcal{K}=-1)$, flat $(\mathcal{K}=0)$ and closed $(\mathcal{K}=1)$ geometries of the universe.

Here, we briefly discuss some important parameters that are helpful to explore characteristics of the universe.

1.2.1 Scale Factor

The scale factor describes the cosmic evolution by measuring its size. It is dimensionless positive function which evaluates distance between galaxies through cosmological red-shift. Different forms of $a(t)$ are helpful to characterize distinct eras of the universe. The expanding behavior of the universe in early times is identified by its exponential form ($a(t) = a_0 e^{Ht}$, a_0 is constant at t_0 and H is the Hubble parameter) while radiation, matter and DE dominated eras are described by $a(t) = a_0 t^{\lambda}$ for $\lambda = 1/2,\ 2/3$ and greater than 1, respectively.

1.2.2 Hubble Parameter

In 1929, Edwin Hubble presented his noteworthy law about the expanding universe. He observed that nearby galaxies are moving apart from each other, thus exhibits larger red-shifts at larger distances. He established a proportionality relationship between the recessional velocity V of two galaxies and the distance d between them known as Hubble law given by

$$V = Hd,$$

where H is a proportionality factor depending on time dubbed as Hubble parameter. It measures expansion rate of the universe and is defined as

$$H(t) = \frac{\dot{a}}{a}, \qquad (1.2.2)$$

where dot represents derivative with respect to time.

1.2.3 Deceleration, Jerk and Snap Parameters

A dimensionless cosmological quantity which measures cosmic expansion is known as deceleration parameter and is given by

$$q = -\frac{a\ddot{a}}{\dot{a}^2}. \tag{1.2.3}$$

Its negative value identifies accelerated expansion of the universe while positive value determines decelerating rate. The third and fourth order time rate of $a(t)$ yield [43]

$$j = \frac{a^2\dddot{a}}{\dot{a}^3}, \quad s = \frac{a^3 a^{(iv)}}{\dot{a}^4}, \tag{1.2.4}$$

where j and s represent the jerk and snap parameters, respectively.

1.2.4 Equation of State Parameter

It is a dimensionless parameter describing a relationship between matter variables. In cosmological background, EoS provides a linear relationship between pressure (P) and energy density (ρ) of matter configuration as

$$P = \omega\rho, \tag{1.2.5}$$

where ω is an EoS parameter. For $\omega = 1,\ 1/3$ and 0, it corresponds to stiff matter, radiation and matter dominated eras while negative values of ω realize the concept of cosmic acceleration. This DE energy dominated era further categorizes into phantom and non-phantom phases for $\omega < -1$ and $\omega > -1$, respectively while $\omega = -1$ specifies the phantom divide line.

1.3 Einstein Universe

In 1917, Einstein tried to find static solution of his well-known field equations to study the homogeneous and isotropic universe but unfortunately, his field equations have no static solution. At that stage, he introduced Λ to obtain static solutions which was considered as the most favorable choice to discuss static cosmos. After about more than a decade, Hubble discovered cosmic expansion and in 1930, EU is found unstable against homogeneous and isotropic scalar perturbations [20]. Consequently, EU was no longer considered a realistic model of homogenous and isotropic universe. A renewed motivation to discuss EU comes from the emergent universe scenario. In this scenario, cosmic inflationary epoch emerges from a small static state and this initial state is the Einstein static universe. Modified theories of gravity have become fascinating approach to find stable static solutions.

The success of emergent universe is based on the stable EU against any kind of perturbations to avoid the primordial singularity. These perturbations are usually categorized into scalar, vector and tensor homogeneous as well as inhomogeneous perturbations which are governed by energy density, vorticity vector and traceless shear tensor, respectively. The stability criterion for static universe is determined by the linearized perturbed equation of motion of the system. The solution of corresponding differential equation contains frequencies of perturbations. In this context, stability of perturbed system depends upon the behavior of frequency. For real values of frequency, the exponential growth of the perturbations takes place providing the unstable EU while purely complex frequencies lead to the existence of stable EU.

1.4 Dynamical Instability

The evolution and stability of stellar objects during naturally occurring fascinating phenomenon of gravitational collapse have become the subject of great interest not only in GR but also in modified gravitational theories. This collapsing mechanism plays a primal role in the structure formation. During collapse, stellar objects pass through various stages which are described by dynamical equations. These differential equations explore kinetics of a collapsing system, i.e., they provide information about the evolution of the system parameters.

Stellar objects usually possess thermal as well as hydrostatic equilibrium. It is a captivating issue to analyze what happens when the equilibrium phases of these objects are perturbed? Will the disturbance be flourished (unstable state) or will it diminish (stable state). Since there are two types of equilibrium, therefore one needs to consider the following two types of stability [44].

- Secular stability: what happens when the state of thermal equilibrium is disturbed?
- Dynamical stability: what happens when the state of hydrostatic equilibrium is perturbed?

The stability criterion under hydrostatic equilibrium state can be established by linearizing the field as well as dynamical equations against perturbations. In this perturbation technique, initially the system is in static equilibrium, i.e., all metric components as well as fluid variables possess only radial dependence but as time passes, these quantities also have temporal dependence. If these perturbations reduce gradually and ultimately vanish such that the system reattains its initial state, then it is dynamically stable otherwise unstable. In case of collapsing fluid, it is in stable state if it returns back to the state of hydrostatic equilibrium otherwise, it becomes dynamically unstable.

1.4.1 Expansion-free Condition

A congruence is a set of curves in an open subset of spacetime such that through each point in the open region, there passes one and only one curve. If all the curves in this region are timelike, null and spacelike, then it is timelike, null and spacelike congruences, respectively. Consider the timelike geodesic congruence along with the corresponding timelike vector field v_α. A scalar quantity which measures the fractional rate at which volume changes in unit time is known as expansion scalar defined as

$$\vartheta = \nabla_\alpha v^\alpha = v^\alpha_{\ ,\alpha} + \Gamma^\alpha_{\alpha\beta} v^\beta. \tag{1.4.1}$$

The timelike geodesics move farther for positive values of ϑ which lead to the diverging congruence and hence the universe is expanding. If the fractional rate is negative, the geodesics get closer to each other which indicates decelerating phase of the universe. A fascinating phenomenon of cavity formulation has been observed inside the matter configuration for $\vartheta = 0$ dubbed as expansion-free condition.

The evolution of stellar objects under expansion-free condition has a notable importance in astrophysics. Skripkin [45] observed this interesting phenomenon of cavity formation inside non-dissipative isotropic matter distribution. Under this condition, the fluid evolves without being compressed. In these collapsing models, the innermost shell of matter configuration moves away from its center and thus initiates the formation of cavity within fluid configuration. The expansion-free condition effectively describes the cosmic voids which are underdense enormous regions occupying approximately 40% of the universe. They can be distinguished as minivoids and macrovoids on the basis of their sizes. The basic concept about cosmic voids is that they are spherical vacuum cavities enclosed by the fluid configuration but actually voids are neither empty nor spherical [46].

1.4.2 Adiabatic Index

In stability analysis, the perturbed pressure $\bar{P}$ can be expressed in terms of perturbed configuration of energy density $\bar{\rho}$ as [47]

$$\bar{P} = \Gamma \frac{P_0}{\rho_0 + P_0} \bar{\rho}. \tag{1.4.2}$$

This determines the variation between matter contents via Γ which measures the stiffness of collapsing matter distribution. It indicates that stability/instability ranges of stellar objects under expansion-free condition do not depend on Γ. In our analysis, we assume Γ as a constant quantity to discuss instability ranges of collapsing fluid in $f(\mathcal{G})$ gravity.

1.5 Energy Conditions

General relativity proposes a convincing approach in which curvature of spacetime is affected by matter contents defined by the energy-momentum tensor. This metric theory does not provide information about what kind of matter configurations are physically reasonable. For this reason, some constraints known as energy conditions must be imposed on the energy-momentum tensor such that it characterizes physically realistic matter distribution. These conditions are the coordinate invariant which incorporate the common features shared by almost every matter field as well as ruled out non-physical solutions of the field equations. In GR, these conditions play an important role in establishing various fascinating results.

We first discuss energy constraints in GR and then extend to modified gravitational theories. Raychaudhuri equations play a primal role in formulating energy conditions with the requirement that not only the energy density is positive but also gravity is attractive. These equations describe the temporal evolution of ϑ for the congruences of non-spacelike geodesics. For the congruence of timelike geodesics, the Raychaudhuri equation is given by [48]

$$\frac{d\vartheta}{d\tau}+\frac{1}{3}\vartheta^2-\omega^{\alpha\beta}\omega_{\alpha\beta}+\sigma^{\alpha\beta}\sigma_{\alpha\beta}+R_{\alpha\beta}v^\alpha v^\beta=0, \tag{1.5.1}$$

where $\omega_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ represent vorticity (measures the rotation of curves) and shear (measures the distortion of volume) tensors associated to the congruence, respectively. Raychaudhuri equation for a congruence of null geodesics is given as follows

$$\frac{d\vartheta}{d\tau}+\frac{1}{2}\vartheta^2-\omega^{\alpha\beta}\omega_{\alpha\beta}+\sigma^{\alpha\beta}\sigma_{\alpha\beta}+R_{\alpha\beta}l^\alpha l^\beta=0, \tag{1.5.2}$$

where l^α is the corresponding null vector field. For non-geodesic (non-spacelike) congruences, the temporal variation of ϑ changes in the presence of acceleration term as [49]

$$\frac{d\vartheta}{d\tau}+\frac{1}{3}\vartheta^2-\omega^{\alpha\beta}\omega_{\alpha\beta}+\sigma^{\alpha\beta}\sigma_{\alpha\beta}-\nabla_\alpha(v^\beta\nabla_\beta v^\alpha)+R_{\alpha\beta}v^\alpha v^\beta=0, \tag{1.5.3}$$

where the fifth term represents the divergence of four-acceleration which appears due to the non-gravitational force (pressure gradient). Using the condition of attractive nature of gravity ($\vartheta<0$ implies converging congruence of geodesics) and neglecting the quadratic terms, Raychaudhuri equations for non-spacelike geodesic congruences reduce to

$$R_{\alpha\beta}v^{\alpha}v^{\beta} \geq 0, \quad R_{\alpha\beta}l^{\alpha}l^{\beta} \geq 0.$$

In GR, these inequalities can be expressed in terms of energy-momentum tensor as

$$\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right)v^{\alpha}v^{\beta} \geq 0, \quad \left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right)l^{\alpha}l^{\beta} \geq 0.$$

The four fundamental energy bounds for perfect fluid matter configuration are given as follows

- **NEC**: $T_{\alpha\beta}l^{\alpha}l^{\beta} \geq 0 \quad \Rightarrow \quad \rho + P \geq 0,$
- **WEC**: $T_{\alpha\beta}v^{\alpha}v^{\beta} \geq 0 \quad \Rightarrow \quad \rho \geq 0, \quad \rho + P \geq 0,$
- **DEC**: $T_{\alpha\beta}v^{\alpha}v^{\beta} \geq 0, \quad T_{\alpha\beta}T^{\beta}_{\gamma}v^{\alpha}v^{\gamma} \leq 0 \quad \Rightarrow \quad \rho \geq 0, \quad \rho \pm P \geq 0,$
- **SEC**: $\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right)v^{\alpha}v^{\beta} \geq 0 \quad \Rightarrow \quad \rho + 3P \geq 0, \quad \rho + P \geq 0,$

for all null and timelike vectors.

Raychaudhuri equations possess purely geometric nature such that they work for all alternative theories of gravity. In these modified gravitational frameworks, the physical motivation of focussing of geodesic congruences along with attractive nature of gravity remains preserved. The energy conditions can be constructed with the assumption that the total cosmic matter configuration behaves like a perfect fluid. These constraints are simply obtained by replacing ρ and P with effective energy density and effective pressure, respectively as

- **NEC**: $\rho_{\text{eff}} + P_{\text{eff}} \geq 0,$
- **WEC**: $\rho_{\text{eff}} \geq 0, \quad \rho_{\text{eff}} + P_{\text{eff}} \geq 0,$

- **DEC**: $\rho_{\text{eff}} \geq 0, \quad \rho_{\text{eff}} \pm P_{\text{eff}} \geq 0,$

- **SEC**: $\rho_{\text{eff}} + 3P_{\text{eff}} \geq 0, \quad \rho_{\text{eff}} + P_{\text{eff}} \geq 0.$

It is interesting to mention here that NEC is the weakest energy bound whose violation leads to the violation of all remaining energy constraints. Furthermore, the violation of WEC ensures the violation of DEC while the validity of SEC cannot be judged from the positivity of all other energy conditions.

Chapter 2

Dynamical Instability of Expansion-free Spherical Star

This chapter is devoted to investigate dynamical instability of anisotropic spherically symmetric collapsing star under expansion-free condition in $f(\mathcal{G})$ gravity. We apply perturbation scheme to metric components as well as matter variables and formulate the field equations for power-law $f(\mathcal{G})$ model in both static and perturbed configurations. We construct dynamical equations using contracted Bianchi identities to analyze the dynamical instability in both N and pN regimes. The results of this chapter have been published [50].

The scheme of this chapter is as follows. Section **2.1** covers some important equations corresponding to the dynamics of collapsing star. In section **2.2**, we apply the first order perturbation scheme to construct static and perturbed configurations of field as well as dynamical equations using power-law $f(\mathcal{G})$ model. The instability dynamics for N and pN regimes under expansion-free condition are investigated in section **2.3**.

2.1 Field and Dynamical Equations

The matter distribution for collapsing fluid in comoving coordinate system is encapsulated by three-dimensional spherical hypersurface ($\Sigma_{(e)}$) which separates the four-dimensional spacetime into interior and exterior regions. The spherical symmetric spacetime interior to $\Sigma_{(e)}$ is

$$ds^2 = A^2(t,r)dt^2 - B^2(t,r)dr^2 - C^2(t,r)(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.1.1}$$

where the metric coefficients are positive functions in which $A(t,r)$ and $B(t,r)$ are dimensionless while $C(t,r)$ has dimension of length. The line element outside to $\Sigma_{(e)}$ is of the form [51]

$$ds_+^2 = \left(1 - \frac{2M}{r}\right) d\hat{\tau}^2 + 2drd\hat{\tau} - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.1.2}$$

where M and $\hat{\tau}$ represent total mass of the fluid and retarded time, respectively. The energy-momentum tensor for non-dissipative anisotropic fluid is considered in the interior region given by

$$T_{\alpha\beta} = (\rho + P_t)v_\alpha v_\beta - P_t g_{\alpha\beta} + (P_r - P_t)k_\alpha k_\beta, \tag{2.1.3}$$

where P_r and P_t are the radial and tangential pressures of the matter configuration, respectively. The quantities v_α (four-velocity) and k_α (unit four-vector in radial direction) satisfy the relations $v_\alpha v^\alpha = 1,\ k_\alpha k^\alpha = -1,\ v_\alpha k^\alpha = 0$ and are defined as

$$v^\alpha = A^{-1}\delta_0^\alpha, \quad k^\alpha = B^{-1}\delta_1^\alpha. \tag{2.1.4}$$

The GB invariant (1.1.2) for the interior metric (2.1.1) takes the form

$$\mathcal{G} = \frac{8}{ABC^2}\left[\left(1 + \frac{\dot{C}^2}{A^2} - \frac{C'^2}{B^2}\right)\left[\frac{A}{B}\left(\frac{A'B'}{AB} - \frac{A''}{A}\right) - \frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB} - \frac{\ddot{B}}{B}\right)\right]\right.$$

$$
\begin{aligned}
&+ \quad \frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-2\frac{\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\\
&\times \quad \left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)+\frac{2}{AB}\left(\dot{C}'^2-\ddot{C}C''\right)\\
&- \quad \frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)\Bigg], \qquad (2.1.5)
\end{aligned}
$$

where prime represents derivative with respect to r. Using Eqs.(2.1.1), (2.1.3) and (2.1.4) in (1.1.5), we obtain the fourth-order field equations as follows

$$
\begin{aligned}
\kappa^2\rho A^2 &= -\frac{1}{2}A^2 f(\mathcal{G})+\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left[\left(\frac{A}{C}\right)^2+\frac{4A}{BC^2}\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right.\right.\\
&- \left.\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}f_{\mathcal{G}}(\mathcal{G})-\frac{4\dot{B}}{BC^2}\dot{f}_{\mathcal{G}}(\mathcal{G})-\frac{4A^2B'}{B^3C^2}f'_{\mathcal{G}}(\mathcal{G})+\left.\frac{4A^2}{B^2C^2}f''_{\mathcal{G}}(\mathcal{G})\right]\\
&+ \left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)\left(\frac{2A^2}{BC}-\frac{8\dot{C}}{BC^2}\dot{f}_{\mathcal{G}}(\mathcal{G})+\frac{8A^2C'}{B^3C^2}f'_{\mathcal{G}}(\mathcal{G})\right)+\frac{4Af_{\mathcal{G}}(\mathcal{G})}{BC^2}\\
&\times \left[\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\right.\\
&\times \left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)\\
&+ \left.\frac{2}{AB}\left(\dot{C'}^2-\ddot{C}C''\right)\right], \qquad (2.1.6)\\
0 &= \frac{2}{C}\left[\frac{2}{C}\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left(\frac{A'}{A}\dot{f}_{\mathcal{G}}(\mathcal{G})+\frac{\dot{B}}{B}f'_{\mathcal{G}}(\mathcal{G})-\dot{f}'_{\mathcal{G}}(\mathcal{G})\right)\right.\\
&- \left.\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)\left(1-\frac{4\dot{C}}{A^2C}\dot{f}_{\mathcal{G}}(\mathcal{G})+\frac{4C'}{B^2C}f'_{\mathcal{G}}(\mathcal{G})\right)\right], \qquad (2.1.7)\\
\kappa^2B^2P_r &= \frac{1}{2}B^2f(\mathcal{G})-\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left[\frac{B^2}{C^2}+\frac{4B}{AC^2}\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right.\right.\\
&- \left.\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}f_{\mathcal{G}}(\mathcal{G})+\frac{4B^2\dot{A}}{A^3C^2}\dot{f}_{\mathcal{G}}(\mathcal{G})+\frac{4A'}{AC^2}f'_{\mathcal{G}}(\mathcal{G})-\left.\frac{4B^2}{A^2C^2}\ddot{f}_{\mathcal{G}}(\mathcal{G})\right]\\
&+ \left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)\left(\frac{2B^2}{AC}-\frac{8B^2\dot{C}}{A^3C^2}\dot{f}_{\mathcal{G}}(\mathcal{G})+\frac{8C'}{AC^2}f'_{\mathcal{G}}(\mathcal{G})\right)-\frac{4Bf_{\mathcal{G}}(\mathcal{G})}{AC^2}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left[\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\right.\\
&\times \left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)\\
&+ \left.\frac{2}{AB}\left(\dot{C'}^2-\ddot{C}C''\right)\right], \qquad (2.1.8)
\end{aligned}
$$

$$
\begin{aligned}
\kappa^2C^2P_t &= \frac{C}{A}\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)-\frac{C}{B}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)+\frac{C^2}{AB}\\
&\times \left[\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)-\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right]+\frac{1}{2}C^2f(\mathcal{G})-\frac{4f_{\mathcal{G}}(\mathcal{G})}{AB}\\
&\times \left[\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)-\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}\right.\\
&+ \frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\\
&\times \left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)\\
&+ \left.\frac{2}{AB}\left(\dot{C'}^2-\ddot{C}C''\right)\right]-\frac{4C}{A^3B}\left[\dot{B}\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)+\dot{A}\right.\\
&\times \left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)-\frac{2A'}{B}\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)+\dot{C}\left\{\frac{B}{A}\right.\\
&\times \left.\left.\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)-\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right\}\right]\dot{f}_{\mathcal{G}}(\mathcal{G})-\frac{4C}{AB^3}\left[B'\left(\frac{A'C'}{B^2}\right.\right.\\
&+ \left.\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)+A'\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)-\frac{2\dot{B}}{A}\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}\right.\\
&- \left.\left.\dot{C}'\right)-C'\left\{\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)-\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right\}\right]f'_{\mathcal{G}}(\mathcal{G})-\frac{8C}{A^2B^2}\\
&\times \left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)\dot{f}'_{\mathcal{G}}(\mathcal{G})+\frac{4C}{AB^2}\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)f''_{\mathcal{G}}(\mathcal{G})\\
&+ \frac{4C}{A^2B}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)\ddot{f}_{\mathcal{G}}(\mathcal{G}). \qquad (2.1.9)
\end{aligned}
$$

The dynamical equations play a significant role to examine interesting characteristics of collapsing stars. These equations are established through contracted Bianchi identities

$$G^{\alpha\beta}{}_{;\beta}v_\alpha = 0, \quad G^{\alpha\beta}{}_{;\beta}k_\alpha = 0,$$

as follows

$$\frac{\dot{\rho}}{A} + (\rho + P_r)\frac{\dot{B}}{AB} + 2(\rho + P_t)\frac{\dot{C}}{AC} + \Xi_1 = 0, \tag{2.1.10}$$

$$\frac{P_r'}{B} + (\rho + P_r)\frac{A'}{AB} + 2(P_r - P_t)\frac{C'}{BC} - \Xi_2 = 0, \tag{2.1.11}$$

where the expressions for Ξ_1 and Ξ_2 are provided in Appendix **A**. The expansion scalar (1.4.1) for the line element (2.1.1) is given by

$$\vartheta = \frac{1}{A}\left(\frac{\dot{B}}{B} + \frac{2\dot{C}}{C}\right). \tag{2.1.12}$$

Misner-Sharp mass function describes total energy of the spherical collapsing fluid with radius C as [52]

$$m(t,r) = \frac{C}{2}\left(1 + g^{\alpha\beta}C_{,\alpha}C_{,\beta}\right).$$

For the interior spacetime, it takes the form

$$m(t,r) = \frac{C}{2}\left(1 + \frac{\dot{C}^2}{A^2} - \frac{C'^2}{B^2}\right). \tag{2.1.13}$$

The proper time and radial differential operators are defined as

$$D_t = \frac{1}{A}\frac{\partial}{\partial t}, \quad D_C = \frac{1}{C'}\frac{\partial}{\partial r}. \tag{2.1.14}$$

The proper time derivative of $C(t,r)$ gives collapsing fluid velocity as

$$U(t,r) = D_tC = \frac{\dot{C}}{A}, \tag{2.1.15}$$

which is always negative in case of collapse. Taking into account this velocity, Eq.(2.1.13) gives

$$E=\frac{C'}{B}=\left(1-\frac{2m}{C}+U^2\right)^{\frac{1}{2}}. \tag{2.1.16}$$

Taking the proper radial derivative of mass function, we obtain

$$D_C m = \frac{1}{2}\kappa^2 C^2\left[\rho+\frac{T^{\mathcal{G}}_{00}}{A^2}-\frac{T^{\mathcal{G}}_{01}}{AB}\left(\frac{U}{E}\right)\right], \tag{2.1.17}$$

which shows the effect of energy density and GB terms on mass between consecutive spherical surfaces which would decrease due to anti-gravitational nature of DE. Integrating this equation with respect to C, we obtain

$$m=\frac{1}{2}\kappa^2\int_0^C\left[C^2\left\{\rho+\frac{T^{\mathcal{G}}_{00}}{A^2}-\frac{T^{\mathcal{G}}_{01}}{AB}\left(\frac{U}{E}\right)\right\}\right]dC. \tag{2.1.18}$$

Darmois junction conditions are used for the smooth matching of interior and exterior spacetimes over hypersurface [53]. These matching conditions involve continuity of the line elements and extrinsic curvatures leading to

$$M=m(t,r),\quad P_r=-\frac{T^{\mathcal{G}}_{01}}{AB}-\frac{T^{\mathcal{G}}_{11}}{B^2}, \tag{2.1.19}$$

on the boundary surface $\Sigma_{(e)}$. It is well-known that expansion-free condition forms a vacuum cavity inside the collapsing matter configuration. Such models require an additional boundary surface $\Sigma_{(i)}$ between the cavity and fluid distribution. We consider that the Minkowski line element is present inside the cavity and its smooth matching to the collapsing fluid over $\Sigma_{(i)}$ yields

$$m(t,r)=0,\quad P_r=-\frac{T^{\mathcal{G}}_{01}}{AB}-\frac{T^{\mathcal{G}}_{11}}{B^2}.$$

2.2 $f(\mathcal{G})$ Model and Perturbation Scheme

In this section, we consider a particular $f(\mathcal{G})$ model and employ the perturbation technique to the field as well as dynamical equations to examine the evolution of collapsing matter distribution. The power-law $f(\mathcal{G})$ model is given by [54]

$$f(\mathcal{G})=\chi\mathcal{G}^{n},\quad n>0,\neq 1, \tag{2.2.1}$$

where χ is an arbitrary constant. For the first order perturbation in metric components as well as matter variables, we consider

$$\begin{aligned}
A(t,r) &= A_0(r)+\wp\mathcal{T}(t)\bar{a}(r), &(2.2.2)\\
B(t,r) &= B_0(r)+\wp\mathcal{T}(t)\bar{b}(r), &(2.2.3)\\
C(t,r) &= C_0(r)+\wp\mathcal{T}(t)\bar{c}(r), &(2.2.4)\\
\rho(t,r) &= \rho_0(r)+\wp\bar{\rho}(t,r), &(2.2.5)\\
P_r(t,r) &= P_{r0}(r)+\wp\bar{P}_r(t,r), &(2.2.6)\\
P_t(t,r) &= P_{t_0}(r)+\wp\bar{P}_t(t,r), &(2.2.7)\\
m(t,r) &= m_0(r)+\wp\bar{m}(t,r), &(2.2.8)\\
\vartheta(t,r) &= \wp\bar{\vartheta}(t,r), &(2.2.9)
\end{aligned}$$

where subscript zero and $\mathcal{T}(t)$ represent static part of the corresponding quantities and arbitrary temporal function, respectively whereas $0<\wp\ll 1$ is a perturbation parameter. The expressions for $\mathcal{G}$ and $f(\mathcal{G})$ model are given by

$$\begin{aligned}
\mathcal{G}(t,r) &= \mathcal{G}_0(r)+\wp\mathcal{T}(t)\bar{g}(r), &(2.2.10)\\
f(\mathcal{G}(t,r)) &= \chi\mathcal{G}_0^{n}+\wp\chi n\mathcal{T}(t)\bar{g}(r)\mathcal{G}_0^{n-1}, &(2.2.11)\\
f_{\mathcal{G}}(t,r) &= \chi n\mathcal{G}_0^{n-1}+\wp\chi n(n-1)\mathcal{T}(t)\bar{g}(r)\mathcal{G}_0^{n-2}. &(2.2.12)
\end{aligned}$$

Choosing $C_0(r)=r$ as Schwarzschild coordinate, we obtain static configuration of $\mathcal{G}$ as well as field equations as

$$\begin{aligned}
\mathcal{G}_0 &= \frac{8}{r^2A_0B_0^2}\left[\left(1-\frac{1}{B_0^2}\right)\left(\frac{A_0'B_0'}{B_0}-A_0''\right)-\frac{2A_0'B_0'}{B_0^3}\right], && (2.2.13)\\
\kappa^2\rho_0 &= \frac{1}{r}\left(\frac{1}{r}-\frac{1}{rB_0^2}+\frac{2B_0'}{B_0^3}\right)-\frac{1}{2}\chi\mathcal{G}_0^n-4\chi n\frac{A_0''}{r^2A_0B_0^2}\left(1-\frac{1}{B_0^2}\right)\mathcal{G}_0^{n-1}\\
&- 4\chi n\frac{A_0B_0'}{r^2B_0^3}\left(1-\frac{3}{B_0^2}\right)\left(\frac{\mathcal{G}_0^{n-1}}{A_0}\right)'+4\chi n\frac{(n-1)}{r^2B_0^2}\left(1-\frac{1}{B_0^2}\right)\\
&\times \left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)', && (2.2.14)\\
\kappa^2P_{r0} &= \frac{1}{r^2B_0^2}\left(1-B_0^2+2r\frac{A_0'}{A_0}\right)+\frac{1}{2}\chi\mathcal{G}_0^n-\frac{4\chi n}{r^2A_0B_0^2}\left[\frac{A_0'B_0'}{B_0}\left(1-\frac{3}{B_0^2}\right)\right.\\
&- \left.A_0''\left(1-\frac{1}{B_0^2}\right)\right]\mathcal{G}_0^{n-1}-4\chi n(n-1)\frac{A_0'}{r^2A_0B_0^2}\left(1-\frac{3}{B_0^2}\right)\mathcal{G}_0^{n-2}\mathcal{G}_0',\\
& && (2.2.15)\\
\kappa^2P_{t0} &= \left(\frac{A_0''}{A_0}-\frac{A_0'B_0'}{A_0B_0}\right)\left(\frac{1}{B_0^2}-\frac{4\chi n}{r^2B_0^2}\mathcal{G}_0^{n-1}\right)+\frac{1}{rB_0^2}\left(\frac{A_0'}{A_0}-\frac{B_0'}{B_0}\right)+\frac{1}{2}\chi\mathcal{G}_0^n\\
&+ 4\chi n\frac{1}{B_0^4}\left(\frac{A_0''}{A_0}-\frac{3A_0'B_0'}{A_0B_0}\right)\left(\frac{1}{r}\mathcal{G}_0^{n-1}\right)'+4\chi n(n-1)\frac{A_0'}{rA_0B_0^4}\\
&\times \left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)'. && (2.2.16)
\end{aligned}$$

Using Eqs.(2.2.2)-(2.2.12), the perturbed configuration of $\mathcal{G}$ and the field equations are provided in Eqs.(A3)-(A7) of Appendix **A**. In static background, the first dynamical equation (2.1.10) is trivially satisfied while the second equation (2.1.11) takes the form

$$\begin{aligned}
&P_{r0}'+(\rho_0+P_{r0})\frac{A_0'}{A_0}+\frac{2}{r}(P_{r0}-P_{t0})-4\chi n\mathcal{G}_0^{n-1}\frac{1}{\kappa^2}\left[\frac{1}{8}\mathcal{G}_0'-\frac{2}{r^3B_0^2}\right.\\
\times\ &\left(1-\frac{1}{B_0^2}\right)\frac{A_0''}{A_0}+\frac{1}{r^2B_0^2}\left(1-\frac{1}{B_0^2}\right)\frac{A_0'''}{A_0}-\frac{1}{r^2B_0^2}\left(3-\frac{7}{B_0^2}\right)\frac{A_0''B_0'}{A_0B_0}\\
-\ &\frac{1}{r^2B_0^2}\left(1-\frac{1}{B_0^2}\right)\frac{A_0'A_0''}{A_0^2}-\frac{1}{r^2B_0^2}\left(1-\frac{3}{B_0^2}\right)\frac{A_0'B_0''}{A_0B_0}+\frac{2}{r^3B_0^2}\left(1-\frac{3}{B_0^2}\right)\\
\times\ &\left.\frac{A_0'B_0'}{A_0B_0}+\frac{1}{r^2B_0^2}\left(1-\frac{3}{B_0^2}\right)\frac{A_0'^2B_0'}{A_0^2B_0}+\frac{3}{r^2B_0^2}\left(1-\frac{5}{B_0^2}\right)\frac{A_0'B_0'^2}{A_0B_0^2}\right]=0. \quad (2.2.17)
\end{aligned}$$

The perturbed configuration of dynamical equations (2.1.10) and (2.1.11) are

$$\begin{aligned}
&\dot{\bar{\rho}}+\left[(\rho_0+P_{r_0})\frac{\bar{b}}{B_0}+2(\rho_0+P_{t_0})\frac{\bar{c}}{r}+\Xi_{1p}\right]\dot{\mathcal{T}}-4\chi n\mathcal{G}_0^{n-1}\\
\times\quad&\frac{1}{\kappa^2}\left[\frac{\bar{b}}{r^2A_0^2B_0}\left(1-\frac{1}{B_0^2}\right)-\frac{2\bar{c}B_0'}{r^2A_0^2B_0^3}\right]\dddot{\mathcal{T}}=0, \qquad (2.2.18)\\
&\frac{1}{B_0}\left[\bar{P}_r'+(\bar{\rho}+\bar{P}_r)\frac{A_0'}{A_0}+\frac{2}{r}(\bar{P}_r-\bar{P}_t)+\left\{(\rho_0+P_{r_0})\left(\frac{\bar{a}}{A_0}\right)'\right.\right.\\
+\quad&\left.\left.2(P_{r_0}-P_{t_0})\left(\frac{\bar{c}}{r}\right)'\right\}\mathcal{T}\right]-\frac{\bar{b}}{B_0^2}\left[P_{r0}'+(\rho_0+P_{r_0})\frac{A_0'}{A_0}+\frac{2}{r}(P_{r_0}-P_{t_0})\right]\mathcal{T}\\
-\quad&\Xi_{2p}=0, \qquad (2.2.19)
\end{aligned}$$

where Ξ_{1p} and Ξ_{2p} are given in Appendix **A**. Integrating Eq.(2.2.18) with respect to time, we obtain

$$\begin{aligned}
&\bar{\rho}+\left[(\rho_0+P_{r_0})\frac{\bar{b}}{B_0}+2(\rho_0+P_{t_0})\frac{\bar{c}}{r}+\Xi_{1p}\right]\mathcal{T}-4\chi n\mathcal{G}_0^{n-1}\\
\times\quad&\frac{1}{\kappa^2}\left[\frac{\bar{b}}{r^2A_0^2B_0}\left(1-\frac{1}{B_0^2}\right)-\frac{2\bar{c}B_0'}{r^2A_0^2B_0^3}\right]\ddot{\mathcal{T}}=0. \qquad (2.2.20)
\end{aligned}$$

The perturbed expansion scalar is given by

$$\bar{\vartheta}=\frac{1}{A_0}\left(\frac{\bar{b}}{B_0}+\frac{2\bar{c}}{r}\right)\dot{\mathcal{T}}. \qquad (2.2.21)$$

The expressions for static as well as non-static configurations of mass function are

$$m_0=\frac{1}{2}r\left(1-\frac{1}{B_0^2}\right),\quad \bar{m}=-\frac{1}{B_0^2}\left[\frac{\bar{c}}{2}(1-B_0^2)+r\left(\bar{c}'-\frac{\bar{b}}{B_0}\right)\right]\mathcal{T}. \qquad (2.2.22)$$

Under perturbation technique, the static background of second junction condition in Eq.(2.1.19) is formulated using Eq.(2.2.15) while the perturbed configuration is obtained from Eqs.(A5) and (A6). Substituting the corresponding static and perturbed matching conditions in Eq.(A6), we have

$$\ddot{\mathcal{T}}(t)+u(r)\dot{\mathcal{T}}(t)+v(r)\mathcal{T}(t)=0, \qquad (2.2.23)$$

on the boundary surface $\Sigma_{(e)}$. The quantities $u(r)$ and $v(r)$ have the following expressions

$$\begin{aligned}
u(r) &= 2\chi n(n-1)\mathcal{G}_0^{n-2}\frac{A_0}{r\bar{c}B_0}\left[\left\{(n-2)\frac{\bar{g}\mathcal{G}_0'}{\mathcal{G}_0}+A_0\left(\frac{\bar{g}}{A_0}\right)'\right\}\left(1-\frac{1}{B_0^2}\right)\right.\\
&- \left.\frac{\bar{b}}{B_0}\left(1-\frac{3}{B_0^2}\right)\mathcal{G}_0'-\frac{2A_0}{B_0^2}\left(\frac{\bar{c}}{A_0}\right)'\mathcal{G}_0'\right],\\
v(r) &= \frac{A_0^2}{\bar{c}}\left[\frac{1}{r}\left(\frac{\bar{b}}{B_0}-\frac{\bar{c}}{r}\right)\left(1-\frac{1}{B_0^2}\right)-\frac{2}{rB_0^2}\left(\bar{c}'-\frac{\bar{b}}{B_0}\right)+\frac{A_0'}{A_0B_0^2}\left(\frac{\bar{c}}{r}\right.\right.\\
&+ \left.\left.\frac{\bar{a}}{A_0}-\frac{2\bar{b}}{B_0}\right)-\frac{1}{A_0B_0^2}\left(\bar{a}'-2\bar{b}\frac{A_0'}{B_0}+\bar{c}'A_0'\right)\right].
\end{aligned}$$

Assume that $u(r)$ and $v(r)$ are positive functions for the instability regions. The solution of the above differential equation (2.2.23) is

$$\mathcal{T}(t)=-\exp(\Omega_{\Sigma(e)}t),\quad \Omega_{\Sigma(e)}=\frac{-u+\sqrt{u^2-4v}}{2}, \tag{2.2.24}$$

which shows that the system initiates collapsing at $t=-\infty$ with $\mathcal{T}(-\infty)=0$. At this stage, the system is in static state but it continues collapsing as t increases.

2.3 Newtonian and Post-Newtonian Regimes

In this section, we discuss dynamical instability in N and pN regimes under expansion-free condition. For this purpose, we convert the second dynamical equation into centimeter-gram-second (c.g.s.) units from relativistic units in the background of static configuration. Using Eq.(2.2.21), the expansion-free condition yields

$$\frac{\bar{b}}{B_0}=-\frac{2\bar{c}}{r}. \tag{2.3.1}$$

Taking into account Eqs.(2.2.14) and (2.2.15) along with static configuration of m from Eq.(2.2.22), it follows that

$$\frac{A_0'}{A_0} = \frac{r}{r-2m_0}\left[\frac{\kappa^2r^3}{2}(\rho_0+P_{r0})+\frac{m_0}{r^2}\left\{r^2-2\chi n(n-1)(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right\}\right.$$

$$
\begin{aligned}
&- \quad 4\chi n(n-1)(r-2m_0)\frac{m_0}{r}\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)'\Big]\left[r^2-2\chi n(n-1)(6m_0-2r)\right.\\
&\times \quad \left.\mathcal{G}_0^{n-2}\mathcal{G}_0'\right]^{-1}. \qquad (2.3.2)
\end{aligned}
$$

From Eq.(2.2.16), we obtain the expression for $\frac{A_0''}{A_0}$ as

$$
\begin{aligned}
\frac{A_0''}{A_0} &= \frac{r^4}{r-2m_0}\left[\kappa^2P_{t_0}-\frac{1}{2}\chi\mathcal{G}_0^n-\frac{m_0}{r^3}-\left\{\frac{m_0}{r^5}\left(r^3-8\chi n(r-3m_0)\mathcal{G}_0^{n-1}\right.\right.\right.\\
&+ \quad \left.12\chi n(n-1)r(r-2m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)+\frac{1}{r^3}(r-2m_0)\left(r+4\chi n(n-1)\right.\\
&\times \quad \left.\left.\left.(r-2m_0)\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)'\right)\right\}\frac{A_0'}{A_0}\right]\left[r^3+8\chi nm_0\mathcal{G}_0^{n-1}+4\chi n(n-1)\right.\\
&\times \quad \left.r(r-2m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right]^{-1}. \qquad (2.3.3)
\end{aligned}
$$

Similarly, one can obtain the value of $\frac{A_0'''}{A_0}$ by taking radial derivative of Eq.(2.2.14). Substituting Eqs.(2.2.22), (2.3.2), (2.3.3) and the value of $\frac{A_0'''}{A_0}$ in Eq.(2.2.17), we have

$$
\begin{aligned}
P_{r_0}' &= -\left[\frac{1}{r^6}\left\{r^6(\rho_0+P_{r_0})-8\chi nm_0(2r-m_0)\mathcal{G}_0^{n-1}-8\chi n(n-1)rm_0\right.\right.\\
&\times \quad (r-3m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\}\left\{\frac{r}{r-2m_0}\left[\frac{\kappa^2r^3}{2}(\rho_0+P_{r_0})+\frac{m_0}{r^2}\left\{r^2-2\chi n\right.\right.\right.\\
&\times \quad \left.(n-1)(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right\}-4\chi n(n-1)(r-2m_0)\frac{m_0}{r}\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)'\Big]\\
&\times \quad \left.\left[r^2-2\chi n(n-1)(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right]^{-1}\right\}+\frac{8m_0}{r^4}\chi n(n-1)\\
&\times \quad (r-2m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\left\{\frac{r^4}{r-2m_0}\left[\kappa^2P_{t_0}-\frac{1}{2}\chi\mathcal{G}_0^n-\frac{m_0}{r^3}-\left\{\frac{m_0}{r^5}\right.\right.\right.\\
&\times \quad \left(r^3-8\chi n(r-3m_0)\mathcal{G}_0^{n-1}+12\chi n(n-1)r(r-2m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)\\
&+ \quad \left.\frac{1}{r^3}(r-2m_0)\left(r+4\chi n(n-1)(r-2m_0)\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)'\right)\right\}\left\{\frac{r}{r-2m_0}\right.\\
&\times \quad \left[\frac{\kappa^2r^3}{2}(\rho_0+P_{r_0})+\frac{m_0}{r^2}\left\{r^2-2\chi n(n-1)(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right\}-4\chi n\right.\\
&\times \quad \left.(n-1)(r-2m_0)\frac{m_0}{r}\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)'\right]\left[r^2-2\chi n(n-1)(6m_0-2r)\right.\\
&\times \quad \left.\left.\left.\mathcal{G}_0^{n-2}\mathcal{G}_0'\right]^{-1}\right\}\right]\left[r^3+8\chi nm_0\mathcal{G}_0^{n-1}+4\chi n(n-1)r(r-2m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right]^{-1}\Big\}\\
&+ \quad \frac{2}{r}(P_{r_0}-P_{t_0})+\kappa^2\rho_0'+\frac{4m_0(2r-m_0)}{r^4(r-2m_0)}-\frac{8m_0}{r^4}\chi n(n-1)(r-2m_0)
\end{aligned}
$$

$$
\begin{aligned}
&\times \quad \Big\{(n-2)\mathcal{G}_0'(\mathcal{G}_0^{n-3}\mathcal{G}_0')' + (\mathcal{G}_0^{n-2}\mathcal{G}_0'')' + (n-2)r^2\mathcal{G}_0^{n-3}\mathcal{G}_0'\left(\frac{\mathcal{G}_0'}{r^2}\right)' \\
&- \quad \frac{2}{r}\mathcal{G}_0^{n-2}\mathcal{G}_0''\Big\} + \frac{8m_0}{r^6}\chi n(n-1)(r-3m_0)\left\{(n-2)r\mathcal{G}_0^{n-3}\mathcal{G}_0'^2\right. \\
&- \quad \left.\frac{(2r-m_0)}{(r-2m_0)}\mathcal{G}_0^{n-2}\mathcal{G}_0' - 2\mathcal{G}_0^{n-2}\mathcal{G}_0'\right\} + \frac{8m_0}{r^5}\chi n(n-1)(2r-7m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'' \\
&+ \quad \frac{24m_0^2(2r-5m_0)}{r^6(r-2m_0)}\chi n(n-1)\mathcal{G}_0^{n-2}\mathcal{G}_0' + 8\chi n(n-1)(n-2) \\
&\times \quad \frac{m_0}{r^5}(r-4m_0)\mathcal{G}_0^{n-3}\mathcal{G}_0'^2\Big]\,. \qquad (2.3.4)
\end{aligned}
$$

In c.g.s. units, this equation becomes

$$
\begin{aligned}
P_{r_0}' \quad &= \quad -\Big[\frac{1}{r^6}G\left\{r^6(\rho_0+\mathcal{C}^{-2}P_{r_0}) - 8\chi n m_0(2r-\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-1} - 8\chi n\right. \\
&\times \quad \left.(n-1)rm_0(r-3\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right\}\Big\{\frac{r}{(r-2\mathcal{C}^{-2}Gm_0)} \\
&\times \quad \Big[\frac{\kappa^2 r^3}{2}(\rho_0+\mathcal{C}^{-2}P_{r_0}) + \frac{m_0}{r^2}\left\{r^2+4\chi n(n-1)(r-3\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-1}\mathcal{G}_0'\right\} \\
&- \quad 4\chi n(n-1)(r-2\mathcal{C}^{-2}Gm_0)\frac{m_0}{r}(\mathcal{G}_0^{n-2}\mathcal{G}_0')'\Big]\big[r^2+4\chi n(n-1)(r \\
&- \quad 3\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\big]^{-1}\Big\} + \frac{8Gm_0}{r^4}\chi n(n-1)(r-2\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-2}\mathcal{G}_0' \\
&\times \quad \Big\{\frac{1}{(r-2\mathcal{C}^{-2}Gm_0)}\Big[\kappa^2 P_{t_0}r^4\mathcal{C}^{-2} - rm_0 - \frac{1}{2\mathcal{C}^{-2}G}r^4\chi\mathcal{G}_0^n - \Big\{\frac{m_0}{r}\big(r^3 \\
&- \quad 8\chi n(r-3\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-1} + 12\chi n(n-1)(r^2-2r\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\big) \\
&+ \quad \frac{r(r-2\mathcal{C}^{-2}Gm_0)}{\mathcal{C}^{-2}G}\left(r+4\chi n(n-1)(r-2\mathcal{C}^{-2}Gm_0)(\mathcal{G}_0^{n-1}\mathcal{G}_0')'\right)\Big\} \\
&\times \quad \Big\{\frac{r\mathcal{C}^{-2}G}{(r-2\mathcal{C}^{-2}Gm_0)}\Big[\frac{\kappa^2 r^3}{2}(\rho_0+\mathcal{C}^{-2}P_{r_0}) + \frac{m_0}{r^2}\big\{r^2+4\chi n(n-1) \\
&\times \quad (r-3\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-1}\mathcal{G}_0'\big\} - 4\chi n(n-1)(r-2\mathcal{C}^{-2}Gm_0)\frac{m_0}{r}(\mathcal{G}_0^{n-2}\mathcal{G}_0')'\Big] \\
&\times \quad \big[r^2+4\chi n(n-1)(r-3\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\big]^{-1}\Big\}\Big]\big[r^3+8\chi n m_0\mathcal{C}^{-2}G\mathcal{G}_0^{n-1} \\
&+ \quad 4r\chi n(n-1)(r-2\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\big]^{-1}\Big\} + \frac{2}{r}(P_{r_0}-P_{t_0}) + \frac{\kappa^2\rho_0'}{\mathcal{C}^{-2}} \\
&+ \quad \frac{4m_0(2r-\mathcal{C}^{-2}Gm_0)}{r\mathcal{C}^{-2}(r-\mathcal{C}^{-2}Gm_0)} - 8\frac{m_0}{r^4\mathcal{C}^{-2}}\chi n(n-1)(r-2\mathcal{C}^{-2}Gm_0)
\end{aligned}
$$

$$
\begin{aligned}
\times & \left\{(n-2)\mathcal{G}_0'(\mathcal{G}_0^{n-3}\mathcal{G}_0')' + (\mathcal{G}_0^{n-2}\mathcal{G}_0'')' + (n-2)r^2\mathcal{G}_0^{n-3}\mathcal{G}_0'\left(\frac{\mathcal{G}_0'}{r^2}\right)'\right.\\
- & \left.\frac{2}{r}\mathcal{G}_0^{n-2}\mathcal{G}_0''\right\} + 8\chi n(n-1)(r-3\mathcal{C}^{-2}Gm_0)\frac{m_0}{r^6\mathcal{C}^{-2}}\left\{(n-2)r\mathcal{G}_0^{n-3}\mathcal{G}_0'^2\right.\\
- & \left.\frac{(2r-\mathcal{C}^{-2}Gm_0)}{(r-2\mathcal{C}^{-2}Gm_0)}\mathcal{G}_0^{n-2}\mathcal{G}_0' - 2\mathcal{G}_0^{n-2}\mathcal{G}_0'\right\} + \frac{8m_0}{r^5\mathcal{C}^{-2}}\chi n(n-1)\\
\times & (2r-7\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'' + 24\chi n(n-1)m_0^2\frac{G(2r-5\mathcal{C}^{-2}Gm_0)}{r^6(r-2\mathcal{C}^{-2}Gm_0)}\\
\times & \left.\mathcal{G}_0^{n-2}\mathcal{G}_0' + \frac{8m_0}{r^5\mathcal{C}^{-2}}\chi n(n-1)(n-2)\mathcal{G}_0^{n-3}\mathcal{G}_0'^2(r-4\mathcal{C}^{-2}Gm_0)\right], \qquad (2.3.5)
\end{aligned}
$$

where G and $\mathcal{C}$ represent the gravitational constant and speed of light, respectively. Expanding the above equation upto order $\mathcal{C}^{-4}$ and separating out terms of order $\mathcal{C}^0$, $\mathcal{C}^{-2}$ and $\mathcal{C}^{-4}$ which are related to N, pN and parameterized pN regimes, respectively. Using Eq.(2.3.1) in (2.2.20), it follows that

$$
\bar{\rho} - \frac{2\bar{c}}{r}(P_{r0} - P_{t0})\mathcal{T} + \Xi_3 = 0, \qquad (2.3.6)
$$

where Ξ_3 is mentioned in Appendix **A**. Substituting the value of $\bar{\rho}$ from Eq.(2.3.6) in Eq.(1.4.2) (satisfied by radial pressure), we obtain

$$
\bar{P}_r = \Gamma\frac{P_{r0}}{\rho_0 + P_{r0}}\left[\frac{2\bar{c}}{r}(P_{r0} - P_{t0})\mathcal{T} - \Xi_3\right].
$$

In view of Eqs.(1.4.2), (2.3.2) and (2.3.6), it is found that the terms $\bar{\rho}\frac{A_0'}{A_0}$ as well as $\bar{P}_r$ lie in parameterized pN regime.

In order to examine instability ranges for N and pN approximations, we discard such terms belonging to parameterized pN regime. The metric coefficients for pN approximation in c.g.s units are [55]

$$
A_0 = 1 - \frac{m_0 G}{r\mathcal{C}^2} - \frac{1}{6}r^2(n-1)\chi\mathcal{G}_0^n, \quad B_0 = 1 + \frac{m_0 G}{r\mathcal{C}^2}. \qquad (2.3.7)
$$

Substituting the value of $\bar{P}_t$ from Eq.(A7) in (2.2.19) and applying the expansion-free condition along with arbitrary chosen radial functions as $\bar{a} = \bar{a}_0 r$, $\bar{c} = \bar{c}_0 r$ and $\bar{g} = \bar{g}_0 r$

in the resulting equation, we obtain

$$
\begin{aligned}
&(\rho_0+P_{r_0})\left(\frac{\bar{a}_0}{A_0}\right)\left\{1-r\frac{A_0'}{A_0}\right\}+2\bar{c}_0P_{r_0}'+\frac{4\bar{c}_0}{r}(P_{r_0}-P_{t_0})+\frac{4\bar{c}_0}{r}P_{t_0}\\
-&\quad\frac{2}{rB_0^2}\left\{\left(\frac{6\bar{c}_0}{r}-\frac{\bar{a}_0}{A_0}\right)\left(r\frac{A_0''}{A_0}+\frac{B_0'}{B_0}+\frac{A_0'}{A_0}\right)+\frac{\bar{a}_0}{rA_0}\right\}-\chi\mathcal{G}_0^n\left\{\frac{n\bar{g}_0}{G_0}+\frac{2\bar{c}_0}{r}\right\}\\
+&\quad 4\chi n\mathcal{G}_0^{n-1}\left[\frac{\bar{a}_0}{A_0B_0^2}\left[\frac{1}{r}\left(\frac{A_0'''}{A_0}-\frac{A_0''}{rA_0}\right)\left(1-\frac{1}{B_0^2}\right)+\frac{1}{r^2A_0}\left(\frac{B_0''}{B_0}-\frac{2B_0'}{rB_0}\right)\right.\right.\\
\times&\quad\left.\left(1-\frac{3}{B_0^2}\right)\right]-\frac{2\bar{c}_0}{r^2B_0^2}\left(\frac{A_0'''}{A_0}-\frac{2A_0''}{rA_0}\right)\left(2-\frac{5}{B_0^2}\right)-r(n-1)\frac{\bar{g}_0}{\mathcal{G}_0}\\
\times&\quad\left\{\frac{1}{r^2B_0^2}\left(1-\frac{1}{B_0^2}\right)\left(\frac{A_0'''}{A_0}-\frac{2A_0''}{rA_0}\right)+\frac{1}{8}(\mathcal{G}_0'+\bar{g}_0)\right\}+\frac{2}{r^3}\left[\frac{\bar{a}_0}{A_0B_0^2}\right.\\
\times&\quad\left(1-\frac{3}{B_0^2}\right)\frac{B_0'}{B_0}+\frac{rA_0''}{A_0B_0^2}\left[\left(1-\frac{1}{B_0^2}\right)\left\{\frac{\bar{a}_0}{A_0}-(n-1)\frac{\bar{g}_0}{\mathcal{G}_0}\right\}-\frac{4\bar{c}_0}{r}\right.\\
\times&\quad\left.\left.\left(1-\frac{3}{2B_0^2}\right)\right]+r(n-1)\frac{\mathcal{G}_0'}{B_0\mathcal{G}_0}\left[\frac{3\bar{a}_0B_0'}{A_0B_0}+r\left(\frac{\bar{a}_0}{A_0}-\frac{10\bar{c}_0}{r}\right)\frac{A_0''}{A_0}\right]\\
-&\quad\frac{r(n-1)A_0''\bar{g}_0}{B_0^4\mathcal{G}_0^{n-1}}\left(r\mathcal{G}_0^{n-2}\right)'-\frac{r(n-1)}{B_0^4\mathcal{G}_0^{n-1}}\left[\frac{\bar{a}_0}{A_0}\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)'+\frac{A_0'}{A_0}\left\{r(\mathcal{G}_0^{n-2}\mathcal{G}_0')'\right.\right.\\
\times&\quad\left.\left.\left.\left.\left(\frac{10\bar{c}_0}{r}-\frac{\bar{a}_0}{A_0}\right)+\bar{g}_0(\mathcal{G}_0^{n-2})'+(n-2)\bar{g}_0\left(r\mathcal{G}_0^{n-3}\mathcal{G}_0'\right)'\right\}\right]\right]\right]+\left[\frac{2\bar{c}_0}{rA_0^2}\right.\\
-&\quad 4\chi n\mathcal{G}_0^{n-1}\left[-\frac{2\bar{c}_0}{A_0}\left\{\frac{1}{rA_0B_0^2}\left(\frac{B_0''}{B_0}-\frac{2B_0'}{rB_0}\right)-\frac{2}{r^2A_0}\left(\frac{1}{r}+\frac{A_0'}{A_0}\right)\right.\right.\\
\times&\quad\left.\left(1-\frac{1}{B_0^2}\right)\right\}-\frac{2\bar{c}_0}{r^3A_0^2}\left(1-\frac{1}{B_0^2}\right)+\frac{B_0'}{rA_0^2B_0^3}\left\{\frac{2\bar{c}_0}{r}-\bar{c}_0(n-1)\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right.\\
-&\quad\left.\left.\left.(n-1)\frac{\bar{g}_0}{\mathcal{G}_0}\right\}-\frac{\bar{c}_0(n-1)}{rA_0B_0^2}\left\{\frac{2}{r}-\frac{(n-2)\mathcal{G}_0'}{A_0\mathcal{G}_0}\right\}\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right]\right]\Omega^2(r)=0. \qquad (2.3.8)
\end{aligned}
$$

Using Eqs.(2.2.22), (2.3.2), (2.3.3) and (2.3.7) with $G=\mathcal{C}=1$ for pN approximation in the above equation, we have

$$
\begin{aligned}
&\frac{\bar{a}_0}{r}(\rho_0+P_{r_0})(r+m_0+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n)+2\bar{c}_0P_{r_0}'+\frac{4\bar{c}_0}{r}(P_{r_0}-P_{t_0})\\
+&\quad\frac{4\bar{c}_0}{r}P_{t_0}-\frac{2\bar{a}_0}{r^4}(r-2m_0)(r+m_0+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n)-\chi n\mathcal{G}_0^{n-1}\\
\times&\quad(n\bar{g}_0+\frac{2\bar{c}_0}{r}\mathcal{G}_0)-4\chi n\mathcal{G}_0^{n-1}\left[\frac{r(n-1)\bar{g}_0}{8\mathcal{G}_0}(\mathcal{G}_0'+\bar{g}_0)+\frac{2\bar{a}_0(n-1)}{r^4\mathcal{G}_0^{n-1}}\right.\\
\times&\quad\left.(r-4m_0)(r+m_0+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n)(\mathcal{G}_0^{n-1}\mathcal{G}_0')'\right]+\left[\frac{2\bar{c}_0}{r^2}(r+2m_0\right.
\end{aligned}
$$

$$
\begin{aligned}
+ & \quad \frac{1}{3}r^3(n-1)\chi\mathcal{G}_0^n) - 4\chi n\mathcal{G}_0^{n-1}\left\{\frac{4\bar{c}_0 m_0}{r^5}(r+2m_0+\frac{1}{3}r^3(n-1)\chi\mathcal{G}_0^n)\right. \\
- & \quad \frac{\bar{c}_0}{r^4}(n-1)(r-2m_0)(r+m_0+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n)\left(2-(n-2)\right. \\
\times & \quad \left.\left.\left.(r+m_0+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n)\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right)\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right\}\right]\Omega^2(r) + [-\bar{a}_0(\rho_0+P_{r_0})(r+m_0 \\
+ & \quad \frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n) - \frac{2}{r^3}(r-2m_0)\left\{6\bar{c}_0 - \bar{a}_0(r+m_0+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n)\right\} \\
- & \quad \frac{8}{r^3}\chi n(n-1)(r-4m_0)\left\{\left[10\bar{c}_0 - \bar{a}_0(r+2m_0+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n)\right]\right. \\
\times & \quad \left.(\mathcal{G}_0^{n-2}\mathcal{G}_0')' + \bar{g}_0(\mathcal{G}_0^{n-2})' + \bar{g}_0(n-2)(r\mathcal{G}_0^{n-3}\mathcal{G}_0')'\right\} - \frac{32m_0\bar{c}_0}{r^4}\chi n\mathcal{G}_0^{n-1} \\
\times & \quad \left.(r+2m_0+\frac{1}{3}r^3(n-1)\chi\mathcal{G}_0^n)\Omega^2(r)\right]\Delta_1 + \left[-\frac{2}{r^3}(r-2m_0)\right. \\
\times & \quad \left\{6\bar{c}_0 - \bar{a}_0(r+m_0+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n)\right\} - 4\chi n\mathcal{G}_0^{n-1}\left[\frac{2\bar{a}_0 m_0}{r^5}\left(1-\frac{2m_0}{r}\right)\right. \\
\times & \quad (r+m_0+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n) + \frac{4\bar{c}_0}{r^5}(r-2m_0)(3r-10m_0) - 4(n-1) \\
\times & \quad (r-2m_0)\frac{m_0\bar{g}_0}{r^4\mathcal{G}_0} - \frac{2}{r^3}\left[\frac{2m_0}{r}(r-2m_0)\left\{\frac{\bar{a}_0}{r}(r+m_0+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n)\right.\right. \\
- & \quad \left.(n-1)\frac{\bar{g}_0}{\mathcal{G}_0}\right\} + (n-1)(r-m_0)\left\{\bar{a}_0(r+m_0+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n) - 10\bar{c}_0\right\} \\
\times & \quad \frac{\mathcal{G}_0'}{\mathcal{G}_0} - \frac{2\bar{c}_0}{r}(r-3m_0)(r-2m_0) + \frac{(n-1)\bar{g}_0}{r\mathcal{G}_0^{n-1}}(4-4m_0)(r-m_0-\frac{1}{6}r^3 \\
\times & \quad \left.\left.\left.(n-1)\chi\mathcal{G}_0^n)(r\mathcal{G}_0^{n-2})'\right]\right]\right]\Delta_2 - \left[4\chi n\mathcal{G}_0^{n-1}\left\{\frac{2\bar{a}_0 m_0}{r^4}(r-m_0)(r+m_0\right.\right. \\
+ & \quad \frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n) + \frac{2\bar{c}_0}{r^4}(r-2m_0)(3r-10m_0) - 2(n-1)(r-2m_0) \\
\times & \quad \left.\left.\frac{m_0\bar{g}_0}{r^3\mathcal{G}_0}\right\}\right]\Delta_3 + \left[-\frac{2}{r^3}(r-2m_0)\left\{6\bar{c}_0 - \bar{a}_0(r+m_0+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n)\right\}\right. \\
+ & \quad 4\chi n\mathcal{G}_0^{n-1}\left[\frac{4\bar{a}_0}{r^6}(r-3m_0)(r-2m_0)(r+2m_0+\frac{1}{3}r^3(n-1)\chi\mathcal{G}_0^n)\right. \\
+ & \quad \frac{2}{r^3}\left\{-\frac{4\bar{a}_0}{r^3}(r-2m_0)(r-3m_0)(r+m_0+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n) + \frac{3\bar{a}_0}{r}\right. \\
\times & \quad \left.\left.(r-m_0)(r+m_0+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n)\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right\}\right] + \left[-4\chi n\mathcal{G}_0^{n-1}\left\{\frac{1}{r^3}(r-2m_0)\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
\times &\quad (r+2m_0+\frac{1}{3}r^3(n-1)\chi\mathcal{G}_0^n)\left(\frac{6\bar{c}_0}{r}-\frac{(n-1)}{\mathcal{G}_0}\left\{\bar{c}_0\mathcal{G}_0'+\bar{g}_0\right\}\right)\Big\}\Big]\,\Omega^2(r)\Big]\\
\times &\quad \left\{-\frac{m_0}{r^3}(r+2m_0)\right\}+\frac{m_0}{r^4}(2r+7m_0)\left[\frac{8}{r^4}\chi n\mathcal{G}_0^{n-1}(r-2m_0)\right.\\
\times &\quad \left.(r+2m_0+\frac{1}{3}r^3(n-1)\chi\mathcal{G}_0^n)\left\{r\bar{c}_0\Omega^2(r)-\frac{\bar{a}_0}{r}(r-3m_0)\right\}\right]=0, \qquad (2.3.9)
\end{aligned}
$$

where Δ_i's $(i=1,2,3)$ are given in Appendix **A**. This equation shows that collapsing fluid evolves without being compressed being independent of Γ while instability region completely depends on physical variables and some arbitrary constants with considered $f(\mathcal{G})$ model. Thus, the chosen gravity model reflects the compatibility with physical results under the expansion-free condition. For N approximation, we consider $\rho_0 \gg P_{r_0},\ \rho_0 \gg P_{t_0}$ and drop all terms belonging to the order of pN and parameterized pN in Eq.(2.3.9) as follows

$$
\begin{aligned}
&\quad 2\bar{c}_0|P_{r_0}'|+\frac{4\bar{c}_0}{r}(P_{r_0}-P_{t_0})+\frac{4\bar{c}_0}{r}(P_{t_0}+\frac{3m_0}{r^3})-4\chi n\mathcal{G}_0^{n-1}\\
\times &\quad \left\{\frac{1}{4}(n\bar{g}_0+\frac{2\bar{c}_0}{r}\mathcal{G}_0)+r(n-1)(\mathcal{G}_0'+\bar{g}_0)\frac{\bar{g}_0}{8\mathcal{G}_0}\right\}\\
+ &\quad \left[\frac{\bar{a}_0}{r}(\rho_0+P_{r_0})-\frac{2\bar{a}_0}{r^4}(r-3m_0)-4\chi n\mathcal{G}_0^{n-1}\left\{\frac{2\bar{a}_0(n-1)}{r^4\mathcal{G}_0^{n-1}}(r-4m_0)\right.\right.\\
\times &\quad \left.\left.(\mathcal{G}_0^{n-1}\mathcal{G}_0')'-2(n-1)(r-2m_0)\Omega^2(r)\frac{\bar{c}_0\mathcal{G}_0'}{r^4\mathcal{G}_0}-\frac{2\bar{a}_0m_0}{r^5}\left(\frac{4}{r}-\frac{3\mathcal{G}_0'}{\mathcal{G}_0}\right)\right\}\right]\\
\times &\quad (r+m_0+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n)-24\chi n\mathcal{G}_0^{n-1}\frac{\bar{a}_0m_0^2}{r^6}(r+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n)\\
\times &\quad \left(\frac{4}{r}-\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right)+\left[\frac{2\bar{c}_0^2}{r^2}\Omega^2(r)-4\chi n\mathcal{G}_0^{n-1}\left\{\frac{m_0}{r^5}\left(\frac{8\bar{a}_0}{r}-3\bar{c}_0\Omega^2(r)\right)\right.\right.\\
+ &\quad \left.\left.\frac{m_0}{\mathcal{G}_0r^4}(n-1)(\bar{c}_0\mathcal{G}_0'+\bar{g}_0)\Omega^2(r)\right\}\right](r+2m_0+\frac{1}{3}r^3(n-1)\chi\mathcal{G}_0^n)\\
+ &\quad 8\chi n\mathcal{G}_0^{n-1}\frac{m_0^2}{r^6}\left(7\frac{\bar{a}_0}{r}+3\bar{c}_0\Omega^2\right)(r+\frac{1}{3}r^3(n-1)\chi\mathcal{G}_0^n)\\
- &\quad 4\chi n\mathcal{G}_0^{n-1}(n-1)(n-2)(r-2m_0)(r+m_0+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n)\\
\times &\quad (r+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_0^n)\frac{\bar{c}_0}{r^4}\left(\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right)^2\Omega^2(r)=0. \qquad (2.3.10)
\end{aligned}
$$

The positivity of this equation is required to explore the instability ranges for collapsing star. For this purpose, we assume that all dynamical quantities and constants are positive while $P'_{r0} < 0$ represents decreasing behavior of radial pressure during collapse. Furthermore, the following bounds must be satisfied

$$P_{r_0} > P_{t_0}, \quad n < 1, \quad r > 4m_0 \quad \frac{4}{3} > \frac{r\mathcal{G}'_0}{\mathcal{G}_0} > 0.$$

Under these conditions, the system is unstable at N approximation. For $\mathcal{G}_0(r) = \mathcal{G}_c=$ constant along with $\bar{g}_0 = 0$, Eq.(2.3.10) takes the form

$$\begin{aligned}
& 2\bar{c}_0|P'_{r_0}| + \frac{4\bar{c}_0}{r}(P_{r_0} - P_{t_0}) - \frac{4\bar{c}_0}{r}(P_{t_0} + \frac{3m_0}{r^3}) - \frac{2\bar{c}_0}{r}\chi n\mathcal{G}_c^n \\
+ & \left[\frac{\bar{a}_0}{r}(\rho_0 + P_{r_0}) - \frac{2\bar{a}_0}{r^4}(r - 3m_0) + \frac{32\bar{a}_0 m_0}{r^6}\chi n\mathcal{G}_c^{n-1}\right] \\
\times & (r + m_0 + \frac{1}{6}r^3(n-1)\chi\mathcal{G}_c^n) - \frac{96\bar{a}_0 m_0^2}{r^6}\chi n\mathcal{G}_c^{n-1}(1 + \frac{1}{6}r^2(n-1)\chi\mathcal{G}_c^n) \\
+ & \left[\frac{2\bar{c}_0^2}{r^2}\Omega^2(r) - 4\chi n\mathcal{G}_c^{n-1}\left(\frac{8\bar{a}_0 m_0}{r^6} - \frac{3\bar{c}_0 m_0}{r^5}\Omega^2(r)\right)\right](r + 2m_0 + \frac{1}{3}r^3(n-1) \\
\times & \chi\mathcal{G}_c^n) + 8\chi n\mathcal{G}_c^{n-1}\frac{m_0^2}{r^5}\left(7\frac{\bar{a}_0}{r} + 3\bar{c}_0\Omega^2\right)(1 + \frac{1}{3}r^2(n-1)\chi\mathcal{G}_c^n) = 0. \qquad (2.3.11)
\end{aligned}$$

Substituting the value of m_0 for constant $\mathcal{G}_0$ from Eq.(2.1.18) in the above expression, we have

$$\begin{aligned}
& 2\bar{c}_0|P'_{r_0}| + \frac{4\bar{c}_0}{r}(P_{r_0} - P_{t_0}) + \frac{4\bar{c}_0}{r}\left(P_{t_0} + \frac{3}{r^3}\left\{\frac{\kappa^2}{2}\int_{r_{\Sigma(i)}}^{r} r^2\rho_0 dr + \frac{1}{12}\chi\mathcal{G}_c^n\right.\right. \\
\times & \left.\left.(r^3 - r_{\Sigma(i)}^3)\right\}\right) - 2\chi n\mathcal{G}_c^n\frac{\bar{c}_0}{r} + \left[\frac{\bar{a}_0}{r}(\rho_0 + P_{r_0}) - \frac{2\bar{a}_0}{r^4}\left(r - 3\left\{\frac{\kappa^2}{2}\int_{r_{\Sigma(i)}}^{r} r^2\rho_0 dr\right.\right.\right. \\
+ & \left.\left.\frac{1}{12}\chi\mathcal{G}_c^n(r^3 - r_{\Sigma(i)}^3)\right\}\right) + \frac{32\bar{a}_0}{r^6}\chi n\mathcal{G}_c^{n-1}\left\{\frac{\kappa^2}{2}\int_{r_{\Sigma(i)}}^{r} r^2\rho_0 dr + \frac{1}{12}\chi\mathcal{G}_c^n\right. \\
\times & \left.\left.(r^3 - r_{\Sigma(i)}^3)\right\}\right]\left(r + \left\{\frac{\kappa^2}{2}\int_{r_{\Sigma(i)}}^{r} r^2\rho_0 dr + \frac{1}{12}\chi\mathcal{G}_c^n(r^3 - r_{\Sigma(i)}^3)\right\} + \frac{1}{6}r^3(n-1)\right. \\
\times & \left.\chi\mathcal{G}_c^n\right) - \frac{96\bar{a}_0}{r^6}\chi n\mathcal{G}_c^{n-1}(1 + \frac{1}{6}r^2(n-1)\chi\mathcal{G}_c^n)\left\{\frac{\kappa^2}{2}\int_{r_{\Sigma(i)}}^{r} r^2\rho_0 dr + \frac{1}{12}\chi\mathcal{G}_c^n\right.
\end{aligned}$$

$$\begin{aligned}
&\times \quad (r^3-r^3_{\Sigma(i)})\Big\}^2+\left[\frac{2\bar{c}_0^2}{r^2}\Omega^2(r)-4\chi n\mathcal{G}_c^{n-1}\left(\frac{8\bar{a}_0}{r^6}-\frac{3\bar{c}_0}{r^5}\Omega^2(r)\right)\right.\\
&\times \quad \left.\left\{\frac{\kappa^2}{2}\int_{r_{\Sigma(i)}}^{r}r^2\rho_0dr+\frac{1}{12}\chi\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\right\}\right]\left(r+2\left\{\frac{\kappa^2}{2}\int_{r_{\Sigma(i)}}^{r}r^2\rho_0dr\right.\right.\\
&+ \quad \left.\left.\frac{1}{12}\chi\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\right\}+\frac{1}{3}r^3(n-1)\chi\mathcal{G}_c^n\right)+\frac{8}{r^5}\chi n\mathcal{G}_c^{n-1}\left(7\frac{\bar{a}_0}{r}+3\bar{c}_0\Omega^2\right)\\
&\times \quad (1+\frac{1}{3}r^2(n-1)\chi\mathcal{G}_c^n)\left\{\frac{\kappa^2}{2}\int_{r_{\Sigma(i)}}^{r}r^2\rho_0dr+\frac{1}{12}\chi\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\right\}^2=0. \qquad (2.3.12)
\end{aligned}$$

In order to discuss the effect of energy density, we take the power-law form as $\rho_0=\zeta r^l$, where ζ and l are positive and arbitrary constants, respectively. Using this form in the above equation, we have

$$\begin{aligned}
&2\bar{c}_0|P'_{r_0}|+\frac{4\bar{c}_0}{r}(P_{r_0}-P_{t_0})+\frac{4\bar{c}_0}{r}\left(P_{t_0}+\frac{3}{r^3}\left\{\frac{\kappa^2\zeta}{2(l+3)}(r^{l+3}-r^{l+3}_{\Sigma(i)})\right.\right.\\
&+ \quad \left.\left.\frac{1}{12}\chi\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\right\}\right)-2\chi n\mathcal{G}_c^n\frac{\bar{c}_0}{r}+\left[\frac{\bar{a}_0}{r}(\rho_0+P_{r_0})-\frac{2\bar{a}_0}{r^4}\right.\\
&\times \quad \left(r-3\left\{\frac{\kappa^2\zeta}{2(l+3)}(r^{l+3}-r^{l+3}_{\Sigma(i)})+\frac{1}{12}\chi\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\right\}\right)+\frac{32\bar{a}_0}{r^6}\chi n\mathcal{G}_c^{n-1}\\
&\times \quad \left.\left\{\frac{\kappa^2\zeta}{2(l+3)}(r^{l+3}-r^{l+3}_{\Sigma(i)})+\frac{1}{12}\chi\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\right\}\right]\left(r+\left\{\frac{\kappa^2\zeta}{2(l+3)}\right.\right.\\
&\times \quad \left.\left.(r^{l+3}-r^{l+3}_{\Sigma(i)})+\frac{1}{12}\chi\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\right\}+\frac{1}{6}r^3(n-1)\chi\mathcal{G}_c^n\right)-\frac{96\bar{a}_0}{r^6}\chi n\mathcal{G}_c^{n-1}\\
&\times \quad (1+\frac{1}{6}r^2(n-1)\chi\mathcal{G}_c^n)\left\{\frac{\kappa^2\zeta}{2(l+3)}(r^{l+3}-r^{l+3}_{\Sigma(i)})+\frac{1}{12}\chi\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\right\}^2\\
&+ \quad \left[\frac{2\bar{c}_0^2}{r^2}\Omega^2(r)-4\chi n\mathcal{G}_c^{n-1}\left(\frac{8\bar{a}_0}{r^6}-\frac{3\bar{c}_0}{r^5}\Omega^2(r)\right)\left\{\frac{\kappa^2\zeta}{2(l+3)}(r^{l+3}-r^{l+3}_{\Sigma(i)})\right.\right.\\
&+ \quad \left.\left.\frac{1}{12}\chi\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\right\}\right]\left(r+2\left\{\frac{\kappa^2\zeta}{2(l+3)}(r^{l+3}-r^{l+3}_{\Sigma(i)})+\frac{1}{12}\chi\mathcal{G}_c^n\right.\right.\\
&\times \quad \left.\left.(r^3-r^3_{\Sigma(i)})\right\}+\frac{1}{3}r^3(n-1)\chi\mathcal{G}_c^n\right)+\frac{8}{r^5}\chi n\mathcal{G}_c^{n-1}\left(7\frac{\bar{a}_0}{r}+3\bar{c}_0\Omega^2\right)\\
&\times \quad (1+\frac{1}{3}r^2(n-1)\chi\mathcal{G}_c^n)\left\{\frac{\kappa^2\zeta}{2(l+3)}(r^{l+3}-r^{l+3}_{\Sigma(i)})\right.\\
&+ \quad \left.\frac{1}{12}\chi\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\right\}^2=0, \qquad (2.3.13)
\end{aligned}$$

provided that $l \neq -3$. During collapse, the expansion-free matter distribution approaches to instability ranges for

$$P_{r_0} > P_{t_0}, \quad r^{l+3} > r_{\Sigma(i)}^{l+3}, \quad r > r_{\Sigma(i)}, \tag{2.3.14}$$

where the last inequality ensures the existence of Minkowskian cavity in the center of anisotropic spherical fluid under expansion-free condition in $f(\mathcal{G})$ theory of gravity.

Chapter 3

Energy Constraints in $f(\mathcal{G}, T)$ Gravity

This chapter deals with the non-minimal curvature-matter coupling in $f(\mathcal{G}, T)$ gravity to investigate energy conditions in the background of FRW universe model. We construct the field equations as well as explore the effects of this non-minimal coupling on the motion of test particles. We express energy constraints in terms of deceleration, jerk and snap parameters. We apply the reconstruction technique using de Sitter and power-law cosmological solutions and analyze the energy bounds for the reconstructed models. Results of this chapter have been published [40].

The format of this chapter is as follows. In the next section, we formulate the field equations, divergence of the energy-momentum tensor as well as equation of motion for test particles in the presence of perfect fluid configuration. Section **3.2** covers the energy constraints and investigates these bounds for reconstructed $f(\mathcal{G}, T)$ models.

3.1 Dynamics of $f(\mathcal{G},T)$ Gravity

In this section, we construct the field equations as well as discuss the motion of test particles in the presence of perfect fluid. For this fluid configuration, the matter Lagrangian density can be taken as $\mathcal{L}_m=-P$ and the corresponding energy-momentum tensor is given by

$$T_{\alpha\beta}=(\rho+P)v_\alpha v_\beta-Pg_{\alpha\beta}. \tag{3.1.1}$$

In this case, Eq.(1.1.20) becomes

$$\Theta_{\alpha\beta}=-2T_{\alpha\beta}-Pg_{\alpha\beta}. \tag{3.1.2}$$

Equation (1.1.16) can be expressed identical to Einstein field equations as

$$G_{\alpha\beta}=\kappa^2T^{\text{eff}}_{\alpha\beta}=\kappa^2(T_{\alpha\beta}+T^{\mathcal{G}T}_{\alpha\beta}), \tag{3.1.3}$$

where $T^{\mathcal{G}T}_{\alpha\beta}$ represents the $f(\mathcal{G},T)$ contribution. In the background of perfect fluid, the expression for $T^{\mathcal{G}T}_{\alpha\beta}$ is given by

$$\begin{aligned}\kappa^2T^{\mathcal{G}T}_{\alpha\beta} &= \frac{1}{2}g_{\alpha\beta}f(\mathcal{G},T)+(\rho+P)f_T(\mathcal{G},T)v_\alpha v_\beta-(2RR_{\alpha\beta}-4R^\mu_\alpha R_{\mu\beta}\\ &- 4R_{\alpha\mu\beta\nu}R^{\mu\nu}+2R_\alpha^{\ \mu\nu\gamma}R_{\beta\mu\nu\gamma})f_{\mathcal{G}}(\mathcal{G},T)-(2Rg_{\alpha\beta}\nabla^2-2R\nabla_\alpha\nabla_\beta\\ &- 4g_{\alpha\beta}R^{\mu\nu}\nabla_\mu\nabla_\nu-4R_{\alpha\beta}\nabla^2+4R^\mu_\alpha\nabla_\beta\nabla_\mu+4R^\mu_\beta\nabla_\alpha\nabla_\mu\\ &+ 4R_{\alpha\mu\beta\nu}\nabla^\mu\nabla^\nu)f_{\mathcal{G}}(\mathcal{G},T).\end{aligned} \tag{3.1.4}$$

The line element for flat FRW universe model in Cartesian coordinates is

$$ds^2=dt^2-a^2(t)(dx^2+dy^2+dz^2). \tag{3.1.5}$$

The expressions of R and GB invariant for this metric are given by

$$R=-6(H^2+\dot{H}),\quad \mathcal{G}=24H^2(H^2+\dot{H}). \tag{3.1.6}$$

The corresponding fourth order field equations are

$$3H^2=\kappa^2\rho_{\mathrm{eff}},\quad -(2\dot{H}+3H^2)=\kappa^2P_{\mathrm{eff}}, \tag{3.1.7}$$

where

$$\begin{aligned}\rho_{\mathrm{eff}} &= \rho+\frac{1}{\kappa^2}\left[(\rho+P)f_T(\mathcal{G},T)+\frac{1}{2}f(\mathcal{G},T)-12H^2(H^2+\dot{H})f_{\mathcal{G}}(\mathcal{G},T)\right.\\ &+ \left.12H^3\dot{f}_{\mathcal{G}}(\mathcal{G},T)\right], \quad (3.1.8)\\ P_{\mathrm{eff}} &= P-\frac{1}{\kappa^2}\left[\frac{1}{2}f(\mathcal{G},T)-12H^2(H^2+\dot{H})f_{\mathcal{G}}(\mathcal{G},T)+8H(H^2+\dot{H})\right.\\ &\times \left.\dot{f}_{\mathcal{G}}(\mathcal{G},T)+4H^2\ddot{f}_{\mathcal{G}}(\mathcal{G},T)\right]. \quad (3.1.9)\end{aligned}$$

The covariant divergence takes the form

$$\dot{\rho}+3H(\rho+P)=\frac{-1}{\kappa^2+f_T(\mathcal{G},T)}\left[\left(\dot{P}+\frac{1}{2}\dot{T}\right)f_T(\mathcal{G},T)+(\rho+P)\dot{f}_T(\mathcal{G},T)\right]. \tag{3.1.10}$$

We need an additional constraint to obtain the standard conservation equation as

$$\dot{\rho}+3H(\rho+P)=0, \tag{3.1.11}$$

with

$$\left(\dot{P}+\frac{1}{2}\dot{T}\right)f_T(\mathcal{G},T)+(\rho+P)\dot{f}_T(\mathcal{G},T)=0. \tag{3.1.12}$$

Now, we briefly discuss the motion of test particles in $f(\mathcal{G},T)$ gravity. For this purpose, using Eqs.(3.1.1) and (3.1.2) in (1.1.17), the divergence of energy-momentum tensor for perfect fluid gives the following expression

$$\begin{aligned}&\nabla_\beta(\rho+P)v^\alpha v^\beta+(\rho+P)[v^\beta\nabla_\beta v^\alpha+v^\alpha\nabla_\beta v^\beta]-g^{\alpha\beta}\nabla_\beta P\\ &=\frac{-2}{2\kappa^2+3f_T(\mathcal{G},T)}\left[T^{\alpha\beta}\nabla_\beta f_T(\mathcal{G},T)+g^{\alpha\beta}\nabla_\beta(Pf_T(\mathcal{G},T))\right].\end{aligned}$$

The contraction of this equation with projection operator ($h_{\alpha\mu}=g_{\alpha\mu}-v_\alpha v_\mu$) yields

$$g_{\alpha\mu}v^\beta\nabla_\beta v^\alpha=\frac{(2\kappa^2+f_T(\mathcal{G},T))\nabla_\beta P}{(\rho+P)(2\kappa^2+3f_T(\mathcal{G},T))}h^\beta_\mu, \tag{3.1.13}$$

where we have used the relations $h_{\alpha\mu}v^{\alpha}=0,\ v^{\alpha}\nabla_{\beta}v_{\alpha}=0$ and $h_{\alpha\mu}T^{\alpha\beta}=-Ph^{\beta}_{\mu}$. Multiplying Eq.(3.1.13) with $g^{\nu\mu}$ and using the following identity [42]

$$v^{\beta}\nabla_{\beta}v^{\alpha}=\frac{d^2x^{\alpha}}{ds^2}+\Gamma^{\alpha}_{\beta\mu}v^{\beta}v^{\mu},$$

we obtain the equation of motion for massive test particles in this gravity as

$$\frac{d^2x^{\alpha}}{ds^2}+\Gamma^{\alpha}_{\beta\mu}v^{\beta}v^{\mu}=\mathbb{F}^{\alpha}, \tag{3.1.14}$$

where

$$\mathbb{F}^{\alpha}=\frac{(2\kappa^2+f_T(\mathcal{G},T))}{(\rho+P)(2\kappa^2+3f_T(\mathcal{G},T))}(g^{\alpha\beta}-v^{\alpha}v^{\beta})\nabla_{\beta}P, \tag{3.1.15}$$

represents the extra force acting on the test particles and is perpendicular to v^{α} of matter configuration ($\mathbb{F}^{\alpha}v_{\alpha}=0$). For pressureless fluid, this extra force vanishes and hence the geodesic trajectories are followed by dust particles both in GR as well as in $f(\mathcal{G},T)$ gravity. It is worth mentioning here that the equation of motion for perfect fluid in GR is recovered in the absence of curvature-matter non-minimal coupling [56].

3.2 Energy Constraints

In this section, we formulate energy bounds in the context of $f(\mathcal{G},T)$ gravity and express in terms of cosmological parameters. In section **1.5**, we have found that the concept of energy bounds in modified theories of gravity can be extended in a similar way as in GR due to purely geometric nature of Raychaudhuri equations. The energy conditions are formulated for non-spacelike geodesic congruences and dust particles move along geodesic trajectories in this theory while the massive test particles follow non-geodesic lines of motion. Due to this reason, we consider pressureless fluid to discuss the energy conditions. These bounds take the following form

$$\begin{aligned}
&\textbf{NEC:}\quad \rho_{\text{eff}}+P_{\text{eff}}=\rho+\frac{1}{\kappa^2}\Big[\rho f_T(\mathcal{G},T)+4H(H^2-2\dot{H})\dot{f}_{\mathcal{G}}(\mathcal{G},T)\\
&-4H^2\ddot{f}_{\mathcal{G}}(\mathcal{G},T)\Big]\geq 0, \qquad (3.2.1)\\
&\textbf{WEC:}\quad \rho_{\text{eff}}=\rho+\frac{1}{2\kappa^2}\Big[2\rho f_T(\mathcal{G},T)+f(\mathcal{G},T)-24H^2(H^2+\dot{H})\\
&\times f_{\mathcal{G}}(\mathcal{G},T)+24H^3\dot{f}_{\mathcal{G}}(\mathcal{G},T)\Big]\geq 0, \qquad (3.2.2)\\
&\textbf{DEC:}\quad \rho_{\text{eff}}-P_{\text{eff}}=\rho+\frac{1}{\kappa^2}\Big[\rho f_T(\mathcal{G},T)+f(\mathcal{G},T)-24H^2(H^2+\dot{H})\\
&\times f_{\mathcal{G}}(\mathcal{G},T)+4H(5H^2+2\dot{H})\dot{f}_{\mathcal{G}}(\mathcal{G},T)+4H^2\ddot{f}_{\mathcal{G}}(\mathcal{G},T)\Big]\geq 0, \qquad (3.2.3)\\
&\textbf{SEC:}\quad \rho_{\text{eff}}+3P_{\text{eff}}=\rho-\frac{1}{\kappa^2}\Big[f(\mathcal{G},T)-\rho f_T(\mathcal{G},T)-24H^2(H^2+\dot{H})\\
&\times f_{\mathcal{G}}(\mathcal{G},T)+12H(H^2+2\dot{H})\dot{f}_{\mathcal{G}}(\mathcal{G},T)+12H^2\ddot{f}_{\mathcal{G}}(\mathcal{G},T)\Big]\geq 0. \qquad (3.2.4)
\end{aligned}$$

Using Eqs.(1.2.3) and (1.2.4), we can express $H(t)$, R, $\mathcal{G}$ and their derivatives in terms of cosmological parameters as

$$\begin{aligned}
\dot{H} &= -H^2(1+q), \quad \ddot{H}=H^3(j+3q+2),\\
\dddot{H} &= H^4(s-4j-3q^2-12q-6), \qquad (3.2.5)\\
R &= -6H^2(1-q), \quad \dot{R}=-6H^3(j-q-2),\\
\ddot{R} &= -6H^4(s+8q+q^2+6), \qquad (3.2.6)\\
\mathcal{G} &= -24qH^4, \quad \dot{\mathcal{G}}=24H^5(j+3q+2q^2),\\
\ddot{\mathcal{G}} &= 24H^6(s-6j-6qj-12q-15q^2-2q^3). \qquad (3.2.7)
\end{aligned}$$

The energy conditions (3.2.1)-(3.2.4) in the form of above cosmic parameters are

$$\begin{aligned}
&\textbf{NEC:}\quad \rho_{\text{eff}}+P_{\text{eff}}=\rho+\frac{1}{\kappa^2}\Big[\rho f_T+4H^3(3+2q)(f_{\mathcal{G}\mathcal{G}}\dot{\mathcal{G}}+f_{\mathcal{G}T}\dot{T})\\
&-4H^2(f_{\mathcal{G}\mathcal{G}\mathcal{G}}\dot{\mathcal{G}}^2+2f_{\mathcal{G}\mathcal{G}T}\dot{\mathcal{G}}\dot{T}+f_{\mathcal{G}TT}\dot{T}^2+f_{\mathcal{G}\mathcal{G}}\ddot{\mathcal{G}}+f_{\mathcal{G}T}\ddot{T})\Big]\geq 0, \qquad (3.2.8)
\end{aligned}$$

$$\textbf{WEC:} \quad \rho_{\text{eff}} = \rho + \frac{1}{2\kappa^2}\Big[f + 2\rho f_T + 24qH^4 f_{\mathcal{G}} + 24H^3(f_{\mathcal{G}\mathcal{G}}\dot{\mathcal{G}}$$
$$+f_{\mathcal{G}T}\dot{T})\Big] \geq 0, \tag{3.2.9}$$
$$\textbf{DEC:} \quad \rho_{\text{eff}} - P_{\text{eff}} = \rho + \frac{1}{\kappa^2}\Big[f + \rho f_T + 24qH^4 f_{\mathcal{G}} + 4H^3(3-2q)(f_{\mathcal{G}\mathcal{G}}\dot{\mathcal{G}}$$
$$+f_{\mathcal{G}T}\dot{T}) + 4H^2(f_{\mathcal{G}\mathcal{G}\mathcal{G}}\dot{\mathcal{G}}^2 + 2f_{\mathcal{G}\mathcal{G}T}\dot{\mathcal{G}}\dot{T} + f_{\mathcal{G}TT}\dot{T}^2 + f_{\mathcal{G}\mathcal{G}}\ddot{\mathcal{G}} + f_{\mathcal{G}T}\ddot{T})\Big] \geq 0, \tag{3.2.10}$$
$$\textbf{SEC:} \quad \rho_{\text{eff}} + 3P_{\text{eff}} = \rho + \frac{1}{\kappa^2}\big[-f + \rho f_T - 24qH^4 f_{\mathcal{G}} + 12H^3(1+2q)$$
$$\times(f_{\mathcal{G}\mathcal{G}}\dot{\mathcal{G}} + f_{\mathcal{G}T}\dot{T}) - 12H^2(f_{\mathcal{G}\mathcal{G}\mathcal{G}}\dot{\mathcal{G}}^2 + 2f_{\mathcal{G}\mathcal{G}T}\dot{\mathcal{G}}\dot{T} + f_{\mathcal{G}TT}\dot{T}^2 + f_{\mathcal{G}\mathcal{G}}\ddot{\mathcal{G}}$$
$$+f_{\mathcal{G}T}\ddot{T})\Big] \geq 0. \tag{3.2.11}$$

In the following, we discuss energy constraints for de Sitter and power-law universe models.

3.2.1 de Sitter Universe Model

The de Sitter cosmic evolution is an interesting and well-known as it elegantly describes the expanding feature of the universe. Here energy density of matter and radiation are negligible as compared to vacuum energy (energy density for DE dominated era) and thus the universe expands forever at a constant rate. The scale factor of this cosmological model grows exponentially with constant Hubble expansion rate $H(t) = H_0$, defined as

$$a(t) = a_0 e^{H_0 t}. \tag{3.2.12}$$

The values of R and GB invariant are

$$R = -12H_0^2, \quad \mathcal{G} = 24H_0^4. \tag{3.2.13}$$

For dust matter configuration, the trace of energy-momentum tensor and its derivatives have the following expressions

$$T=\rho,\quad \dot{T}=-3H_0T,\quad \ddot{T}=9H_0^2T, \tag{3.2.14}$$

where $\rho=\rho_0e^{-3H_0t}$ is obtained from Eq.(3.1.11). Using Eqs.(3.2.12)-(3.2.14) in (3.1.7), we obtain

$$\kappa^2T+\frac{1}{2}f(\mathcal{G},T)-12H_0^4f_{\mathcal{G}}(\mathcal{G},T)+Tf_T(\mathcal{G},T)-36H_0^4Tf_{\mathcal{G}T}(\mathcal{G},T)-3H_0^2=0,$$

whose solution is given by

$$f(\mathcal{G},T)=c_1c_2\left(e^{c_1\mathcal{G}}T^{\xi_1}+T^{\xi_2}\right)+\xi_3T+\xi_4, \tag{3.2.15}$$

where c_1 and c_2 are integration constants and

$$\xi_1=-\frac{1}{2}\left(\frac{1-24c_1H_0^4}{1-36c_1H_0^4}\right),\quad \xi_2=-\frac{1}{2},\quad \xi_3=-\frac{2}{3}\kappa^2,\quad \xi_4=6H_0^2.$$

The additional constraint (3.1.12) becomes

$$c_1c_2\frac{(1-24c_1H_0^4)(1-30c_1H_0^4)}{(1-36c_1H_0^4)^2}e^{c_1\mathcal{G}}T^{\xi_1}+c_1c_2T^{\xi_2}+\xi_3T=0.$$

This equation splits Eq.(3.2.15) into two $f(\mathcal{G},T)$ functions with some additional constant relations between the coefficients. The model (3.2.15) can be written as a combination of those functions, hence we analyze the energy conditions for this model instead of analyzing them separately. For this model, the energy conditions (3.2.1)-(3.2.4) become

$$\textbf{NEC:}\quad \rho_{\text{eff}}+P_{\text{eff}}=\rho+\frac{1}{\kappa^2}\left[\rho\left\{c_1c_2(\xi_1e^{c_1\mathcal{G}}T^{(\xi_1-1)}+\xi_2T^{(\xi_2-1)})+\xi_3\right\}\right.$$
$$\left.+12c_1^2c_2\xi_1H_0^4(1-3\xi_1)e^{c_1\mathcal{G}}T^{\xi_1}\right]\geq 0,$$

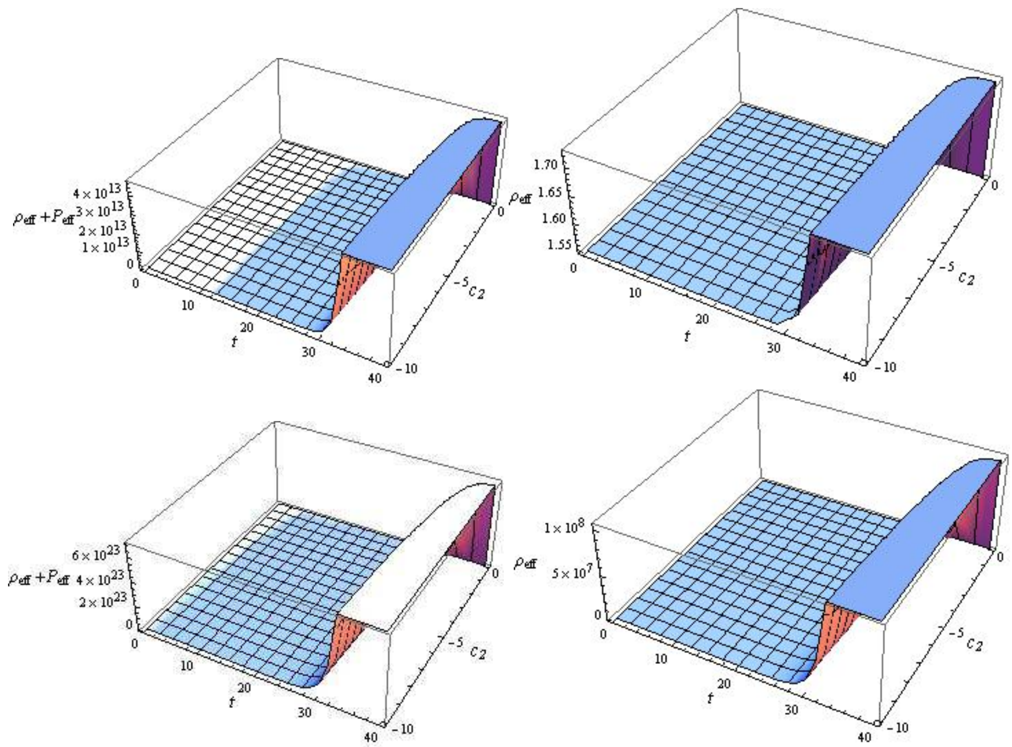


Figure 3.1: Plots of energy conditions for $c_1 = 0.001$ (upper panel) and $c_1 = 4$ (lower panel).

$$\textbf{WEC:}\quad \rho_{\rm eff} = \rho + \frac{1}{2\kappa^2}\left[2\rho\{c_1c_2(e^{c_1\mathcal{G}}\xi_1T^{(\xi_1-1)} + \xi_2T^{(\xi_2-1)}) + \xi_3\}\right.$$
$$\left.+\{c_1c_2(e^{c_1\mathcal{G}}T^{\xi_1} + T^{\xi_2}) + \xi_3T + \xi_4\} - 24c_1^2c_2H_0^4e^{c_1\mathcal{G}}T^{\xi_1}(1+3\xi_1)\right] \geq 0,$$
$$\textbf{DEC:}\quad \rho_{\rm eff} - P_{\rm eff} = \rho + \frac{1}{\kappa^2}[\rho\{c_1c_2(e^{c_1\mathcal{G}}\xi_1T^{(\xi_1-1)} + \xi_2T^{(\xi_2-1)}) + \xi_3\}$$
$$+\{c_1c_2(e^{c_1\mathcal{G}}T^{\xi_1} + T^{\xi_2}) + \xi_3T + \xi_4\} - 12c_1^2c_2H_0^4e^{c_1\mathcal{G}}T^{\xi_1}$$
$$\times\{2 + \xi_1(5 - 3\xi_1)\}] \geq 0,$$
$$\textbf{SEC:}\quad \rho_{\rm eff} + 3P_{\rm eff} = \rho - \frac{1}{\kappa^2}\left[c_1c_2(e^{c_1\mathcal{G}}T^{\xi_1} + T^{\xi_2}) + \xi_3T + \xi_4 - \rho\right.$$
$$\times\{c_1c_2\left(\xi_1e^{c_1\mathcal{G}}T^{(\xi_1-1)} + \xi_2T^{(\xi_2-1)}\right) + \xi_3\} - 12c_1^2c_2e^{c_1\mathcal{G}}H_0^4T^{\xi_1}$$
$$\left.\times\left\{2 + 3\xi_1 - 9\xi_1^2\right\}\right] \geq 0.$$

Figures **3.1** shows the variation of NEC and WEC for the case $c_1 > 0$ and $c_2 < 0$ with

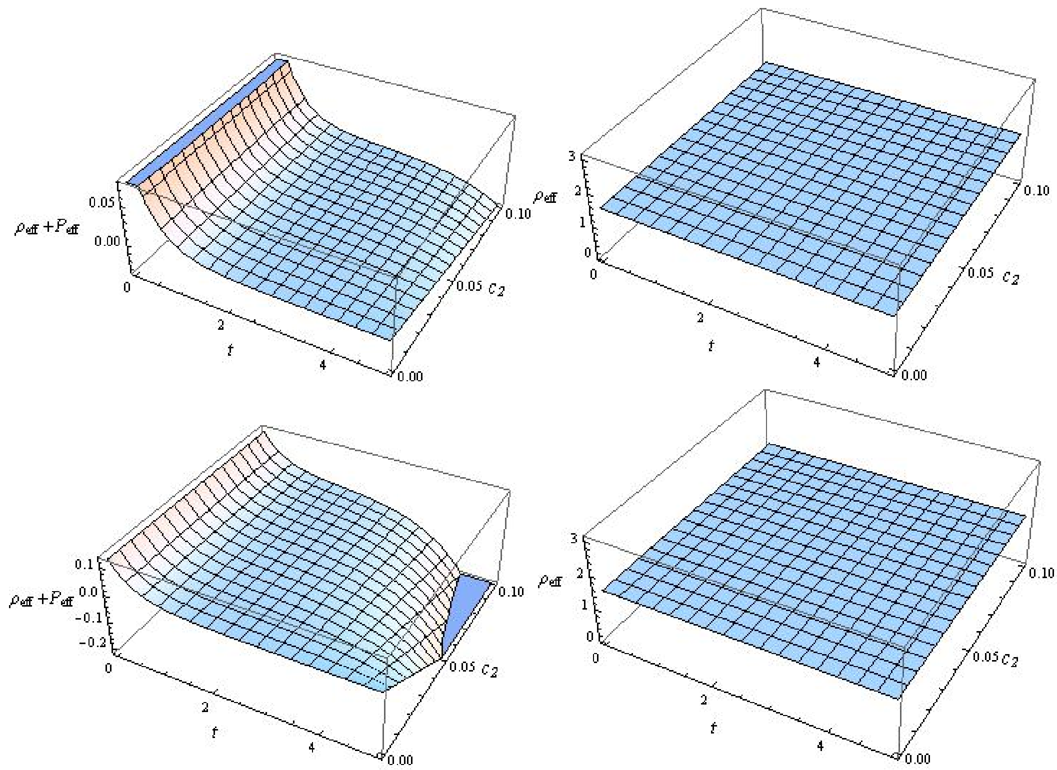


Figure 3.2: Plots of energy conditions for $c_1 = 0.001$ (upper panel) and $c_1 = 0.01$ (lower panel).

$\kappa^2 = 1$. We use the following values of cosmological parameters: $H_0 = 0.718$, $q = -0.64$, $j = 1.02$ and $s = -0.39$ [57]. In these plots, we fix the constant c_1 for two arbitrarily chosen values while $c_2 \in [-10, 0]$. The upper case of Figure **3.1** shows the positively increasing behavior of NEC as well as WEC with respect to time in the considered interval of c_2 while lower panel shows similar behavior for $c_1 = 4$. In this case, both conditions are satisfied for all considered values of c_1 and c_2. The energy conditions for $(c_1, c_2) > 0$ are discussed in Figure **3.2**. The left plot of Figure **3.2** (upper panel) shows that NEC is satisfied for $t < 3$, $t < 2.28$ and $t = 2$ for $c_2 = 0.005$, 0.05 and 0.1, respectively. The lower case of Figure **3.2** (left) shows similar decreasing behavior of time as the value of c_2 increases for $c_1 = 0.01$. It is also

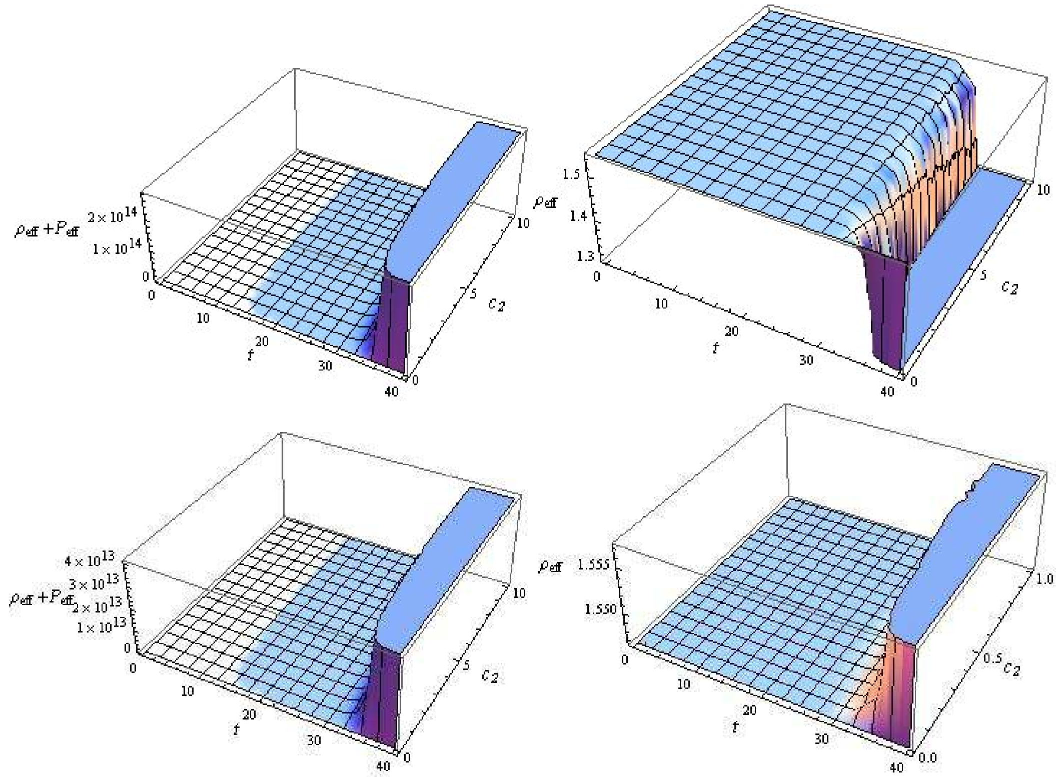


Figure 3.3: Plots of energy conditions for $c_1 = -0.01$ (upper panel) and $c_1 = -0.001$ (lower panel).

observed that as the value of c_1 increases, the time interval for valid NEC decreases while the positivity of $\rho_{\rm eff}$ is shown in the right plots of both panels. For the case $(c_1, c_2) > 0$, both NEC and WEC are satisfied for small values of c_1 and c_2 in a very small time interval.

Figures **3.3** deals with the case $c_1 < 0$ and $c_2 > 0$. For arbitrarily chosen values of c_1, the increasing behavior of NEC with respect to time is observed in the left plot of both panels for all values of c_2. The right plot of Figure **3.3** (upper panel) shows positivity of $\rho_{\rm eff}$ for $t < 34$ while it remains positive throughout the time interval for $c_1 = -0.001$ as shown in the lower case of Figure **3.3** (right). The last possibility, i.e., $c_1 < 0$ and $c_2 < 0$ is examined in Figure **3.4**. The left plot of upper as well

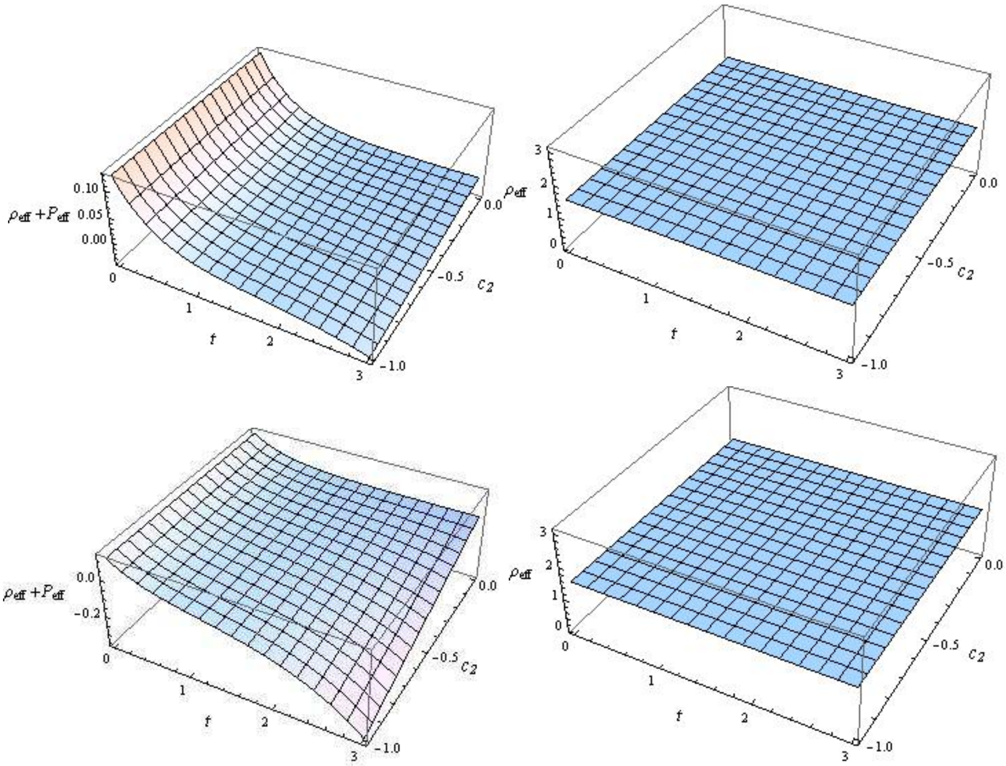


Figure 3.4: Plots of energy conditions for $c_1 = -0.001$ (upper panel) and $c_1 = -0.01$ (lower panel).

as lower panels show the decreasing and increasing behavior of NEC as time and integration constant c_2 increase, respectively. The effective energy density exhibits constant behavior for assumed values of c_1 in the considered interval of c_2.

3.2.2 Power-law Solution

The power-law solution has a significant importance to discuss different phases of cosmic evolution described in subsection **1.2.1**. The Ricci scalar and GB invariant take the form

$$R = \frac{6\lambda}{t^2}(1 - 2\lambda), \quad \mathcal{G} = \frac{24\lambda^3}{t^4}(\lambda - 1). \tag{3.2.16}$$

In this case, the value of T and its time derivatives are

$$T=\rho,\quad \dot{T}=-\frac{3\lambda}{t}T,\quad \ddot{T}=\frac{3\lambda}{t^2}(1+3\lambda)T, \tag{3.2.17}$$

where $\rho=\rho_0t^{-3\lambda}$. Substituting Eqs.(3.2.16) and (3.2.17) in the first field equation (3.1.7), we obtain

$$\kappa^2T+\frac{1}{2}f(\mathcal{G},T)-\frac{1}{2}\mathcal{G}f_{\mathcal{G}}(\mathcal{G},T)+Tf_T(\mathcal{G},T)-\left(\frac{2}{\lambda-1}\right)\mathcal{G}^2f_{\mathcal{G}\mathcal{G}}(\mathcal{G},T)$$
$$-\quad\left(\frac{3\lambda}{2(\lambda-1)}\right)\mathcal{G}Tf_{\mathcal{G}T}(\mathcal{G},T)-3\lambda^2\left(\frac{T}{\rho_0}\right)^{\frac{2}{3\lambda}}=0,$$

whose solution is given by

$$\begin{aligned}f(\mathcal{G},T)&=d_1d_3T^{d_2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)}+d_2d_3T^{d_2}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)}+\mathcal{X}_3T\\&+\ d_1d_2T^{\mathcal{X}_4}+\mathcal{X}_5T^{\mathcal{X}_6},\end{aligned}\tag{3.2.18}$$

where d_i's are constants of integration and

$$\begin{aligned}\mathcal{X}_1&=\frac{1}{2}\left[\lambda^2(1+3d_2(3d_2+2))+2d_2(\lambda-16)+3(2\lambda+3)\right]^{\frac{1}{2}},\\ \mathcal{X}_2&=\frac{1}{2}\left[5-\lambda(1+3d_2)\right],\quad \mathcal{X}_3=-\frac{2}{3}\kappa^2,\quad \mathcal{X}_4=-\frac{1}{2},\\ \mathcal{X}_5&=\left(\frac{18\lambda^3}{2+3\lambda}\right)\rho_0^{-\frac{2}{3\lambda}},\quad \mathcal{X}_6=\frac{2}{3\lambda}.\end{aligned}$$

In this case, Eq.(3.1.12) becomes

$$d_1d_3T^{d_2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)}\left[\frac{d_2}{6\lambda}\{3\lambda(2d_2-1)+2(\mathcal{X}_1+\mathcal{X}_2)\}\right]+d_2d_3T^{d_2}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)}$$
$$\times\quad\left[\frac{d_2}{6\lambda}\{3\lambda(2d_2-1)-2(\mathcal{X}_1-\mathcal{X}_2)\}\right]+\mathcal{X}_3T+d_1d_2\mathcal{X}_4^2T^{\mathcal{X}_4}+\mathcal{X}_5\mathcal{X}_6^2T^{\mathcal{X}_6}=0.$$

Solving Eq.(3.2.18) with the above equation as in the previous section, we obtain two functions whose combination is equivalent to the model (3.2.18).

Inserting the value of $f(\mathcal{G},T)$ from Eq.(3.2.18) in energy constraints (3.2.8)-(3.2.11), we obtain

$$
\begin{aligned}
&\textbf{NEC:}\quad \rho_{\rm eff}+P_{\rm eff}=\rho+\frac{1}{\kappa^2}\Bigg[4H^3(3+2q)([\frac{1}{4}d_1d_3\\
&\times(\mathcal{X}_1+\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1\right]T^{d_2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-2}+\frac{1}{4}d_2d_3(\mathcal{X}_1-\mathcal{X}_2)\\
&\times\left[\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)+1\right]T^{d_2}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-2}]\dot{\mathcal{G}}+[\frac{1}{4}d_1d_2d_3(\mathcal{X}_1+\mathcal{X}_2)T^{d_2-1}\\
&\times\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1}-\frac{1}{4}d_2^2d_3(\mathcal{X}_1-\mathcal{X}_2)T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-1}]\dot{T})\\
&-4H^2([\frac{1}{4}d_1d_3(\mathcal{X}_1+\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1\right]\left[\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-2\right]T^{d_2}\\
&\times\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-3}-\frac{1}{4}d_2d_3(\mathcal{X}_1-\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)+1\right]\left[\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)+2\right]\\
&\times T^{d_2}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-3}]\dot{\mathcal{G}}^2+2[\frac{1}{4}d_1d_2d_3(\mathcal{X}_1+\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1\right]T^{d_2-1}\\
&\times\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-2}+\frac{1}{4}d_2^2d_3(\mathcal{X}_1-\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)+1\right]T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-2}]\\
&\times\dot{\mathcal{G}}\dot{T}+[\frac{1}{4}d_1d_2d_3(d_2-1)(\mathcal{X}_1+\mathcal{X}_2)T^{d_2-2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1}-\frac{1}{4}d_2^2d_3(d_2-1)\\
&\times(\mathcal{X}_1-\mathcal{X}_2)T^{d_2-2}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-1}]\dot{T}^2+[\frac{1}{4}d_1d_3(\mathcal{X}_1+\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1\right]\\
&\times T^{d_2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-2}+\frac{1}{4}d_2d_3(\mathcal{X}_1-\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)+1\right]T^{d_2}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-2}]\\
&\times\ddot{\mathcal{G}}+[\frac{1}{4}d_1d_2d_3(\mathcal{X}_1+\mathcal{X}_2)T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1}-\frac{1}{4}d_2^2d_3(\mathcal{X}_1-\mathcal{X}_2)T^{d_2-1}\\
&\times\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-1}]\ddot{T})+\rho[d_1d_2d_3T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)}+d_2^2d_3T^{d_2-1}\\
&\times\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)}-\mathcal{X}_3+d_1d_2\mathcal{X}_4T^{\mathcal{X}_4-1}+\mathcal{X}_5\mathcal{X}_6T^{\mathcal{X}_6-1}]\Bigg]\geq 0,\\
&\textbf{WEC:}\quad \rho_{\rm eff}=\rho+\frac{1}{2\kappa^2}\Big[d_1d_3T^{d_2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)}+d_2d_3T^{d_2}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)}\\
&-\mathcal{X}_3T+d_1d_2T^{\mathcal{X}_4}+\mathcal{X}_5T^{\mathcal{X}_6}+2\rho[d_1d_2d_3T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)}+d_2^2d_3T^{d_2-1}\\
&\times\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)}-\mathcal{X}_3+d_1d_2\mathcal{X}_4T^{\mathcal{X}_4-1}+\mathcal{X}_5\mathcal{X}_6T^{\mathcal{X}_6-1}]+24qH^4\\
&\times[\frac{1}{4}d_1d_3(\mathcal{X}_1+\mathcal{X}_2)T^{d_2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1}-\frac{1}{4}d_2d_3(\mathcal{X}_1-\mathcal{X}_2)T^{d_2}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-1}]
\end{aligned}
$$

$$+24H^3([\frac{1}{4}d_1d_3(\mathcal{X}_1+\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1\right]T^{d_2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-2}+\frac{1}{4}d_2d_3$$
$$\times(\mathcal{X}_1-\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)+1\right]T^{d_2}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-2}]\dot{\mathcal{G}}+[\frac{1}{4}d_1d_2d_3(\mathcal{X}_1+\mathcal{X}_2)$$
$$\times T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1}-\frac{1}{4}d_2^2d_3(\mathcal{X}_1-\mathcal{X}_2)T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-1}]\dot{T})\Big]\geq 0,$$

DEC: $$\rho_{\text{eff}}-P_{\text{eff}}=\rho+\frac{1}{\kappa^2}\Big[[d_1d_3T^{d_2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)}+d_2d_3T^{d_2}$$
$$\times\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)}-\mathcal{X}_3T+d_1d_2T^{\mathcal{X}_4}+\mathcal{X}_5T^{\mathcal{X}_6}]+\rho[d_1d_2d_3T^{d_2-1}$$
$$\times\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)}+d_2^2d_3T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)}-\mathcal{X}_3+d_1d_2\mathcal{X}_4T^{\mathcal{X}_4-1}+\mathcal{X}_5\mathcal{X}_6T^{\mathcal{X}_6-1}]$$
$$+24qH^4[\frac{1}{4}d_1d_3(\mathcal{X}_1+\mathcal{X}_2)T^{d_2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1}-\frac{1}{4}d_2d_3(\mathcal{X}_1-\mathcal{X}_2)T^{d_2}$$
$$\times\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-1}]+4H^3(3-2q)([\frac{1}{4}d_1d_3(\mathcal{X}_1+\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1\right]$$
$$\times T^{d_2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-2}+\frac{1}{4}d_2d_3(\mathcal{X}_1-\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)+1\right]T^{d_2}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-2}]$$
$$\times\dot{\mathcal{G}}+[\frac{1}{4}d_1d_2d_3(\mathcal{X}_1+\mathcal{X}_2)T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1}-\frac{1}{4}d_2^2d_3(\mathcal{X}_1-\mathcal{X}_2)T^{d_2-1}$$
$$\times\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-1}]\dot{T})+4H^2([\frac{1}{4}d_1d_3(\mathcal{X}_1+\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1\right]$$
$$\times\left[\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-2\right]T^{d_2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-3}-\frac{1}{4}d_2d_3(\mathcal{X}_1-\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)\right.$$
$$\left.+1\right]\left[\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)+2\right]T^{d_2}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-3}]\dot{\mathcal{G}}^2+2[\frac{1}{4}d_1d_2d_3(\mathcal{X}_1+\mathcal{X}_2)$$
$$\times\left[\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1\right]T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-2}+\frac{1}{4}d_2^2d_3(\mathcal{X}_1-\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)\right.$$
$$\left.+1\right]T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-2}]\dot{\mathcal{G}}\dot{T}+[\frac{1}{4}d_1d_2d_3(d_2-1)(\mathcal{X}_1+\mathcal{X}_2)$$
$$\times T^{d_2-2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1}-\frac{1}{4}d_2^2d_3(d_2-1)(\mathcal{X}_1-\mathcal{X}_2)T^{d_2-2}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-1}]\dot{T}^2$$
$$+[\frac{1}{4}d_1d_3(\mathcal{X}_1+\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1\right]T^{d_2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-2}+\frac{1}{4}d_2d_3(\mathcal{X}_1-\mathcal{X}_2)$$
$$\times\left[\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)+1\right]T^{d_2}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-2}]\ddot{\mathcal{G}}+[\frac{1}{4}d_1d_2d_3(\mathcal{X}_1+\mathcal{X}_2)T^{d_2-1}$$
$$\times\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1}-\frac{1}{4}d_2^2d_3(\mathcal{X}_1-\mathcal{X}_2)T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-1}]\ddot{T})\Big]\geq 0,$$

$$
\begin{aligned}
&\textbf{SEC:}\quad \rho_{\text{eff}}+3P_{\text{eff}}=\rho+\frac{1}{\kappa^2}\Big[-[d_1d_3T^{d_2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)}+d_2d_3T^{d_2}\\
&\times\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)}-\mathcal{X}_3T+d_1d_2T^{\mathcal{X}_4}+\mathcal{X}_5T^{\mathcal{X}_6}]+\rho[d_1d_2d_3T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)}\\
&+d_2^2d_3T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)}-\mathcal{X}_3+d_1d_2\mathcal{X}_4T^{\mathcal{X}_4-1}+\mathcal{X}_5\mathcal{X}_6T^{\mathcal{X}_6-1}]-24qH^4\\
&\times[\frac{1}{4}d_1d_3(\mathcal{X}_1+\mathcal{X}_2)T^{d_2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1}-\frac{1}{4}d_2d_3(\mathcal{X}_1-\mathcal{X}_2)T^{d_2}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-1}]\\
&+12H^3(1+2q)([\frac{1}{4}d_1d_3(\mathcal{X}_1+\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1\right]T^{d_2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-2}\\
&+\frac{1}{4}d_2d_3(\mathcal{X}_1-\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)+1\right]T^{d_2}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-2}]\dot{\mathcal{G}}+[\frac{1}{4}d_1d_2d_3\\
&\times(\mathcal{X}_1+\mathcal{X}_2)T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1}-\frac{1}{4}d_2^2d_3(\mathcal{X}_1-\mathcal{X}_2)T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-1}]\dot{T})\\
&-12H^2([\frac{1}{4}d_1d_3(\mathcal{X}_1+\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1\right]\left[\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-2\right]T^{d_2}\\
&\times\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-3}-\frac{1}{4}d_2d_3(\mathcal{X}_1-\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)+1\right]\left[\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)+2\right]\\
&\times T^{d_2}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-3}]\dot{\mathcal{G}}^2+2[\frac{1}{4}d_1d_2d_3(\mathcal{X}_1+\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1\right]T^{d_2-1}\\
&\times\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-2}+\frac{1}{4}d_2^2d_3(\mathcal{X}_1-\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)+1\right]T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-2}]\\
&\times\dot{\mathcal{G}}\dot{T}+[\frac{1}{4}d_1d_2d_3(d_2-1)(\mathcal{X}_1+\mathcal{X}_2)T^{d_2-2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1}-\frac{1}{4}d_2^2d_3(d_2-1)\\
&\times(\mathcal{X}_1-\mathcal{X}_2)T^{d_2-2}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-1}]\dot{T}^2+[\frac{1}{4}d_1d_3(\mathcal{X}_1+\mathcal{X}_2)\Big[\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)\\
&-1\Big]T^{d_2}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-2}+\frac{1}{4}d_2d_3(\mathcal{X}_1-\mathcal{X}_2)\left[\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)+1\right]T^{d_2}\\
&\times\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-2}]\ddot{\mathcal{G}}+[\frac{1}{4}d_1d_2d_3(\mathcal{X}_1+\mathcal{X}_2)T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\mathcal{X}_1+\mathcal{X}_2)-1}-\frac{1}{4}d_2^2d_3\\
&\times(\mathcal{X}_1-\mathcal{X}_2)T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\mathcal{X}_1-\mathcal{X}_2)-1}]\ddot{T})\Big]\geq 0.
\end{aligned}
$$

The NEC and WEC depend on four parameters $t,\ d_1,\ d_2$ and d_3. We plot these conditions against t and d_1 for $\lambda=\frac{2}{3}$ with possible signs of d_2 and d_3. The left plot of Figure **3.5** shows positively increasing behavior of NEC for $-10\leq d_1\leq 0$ with

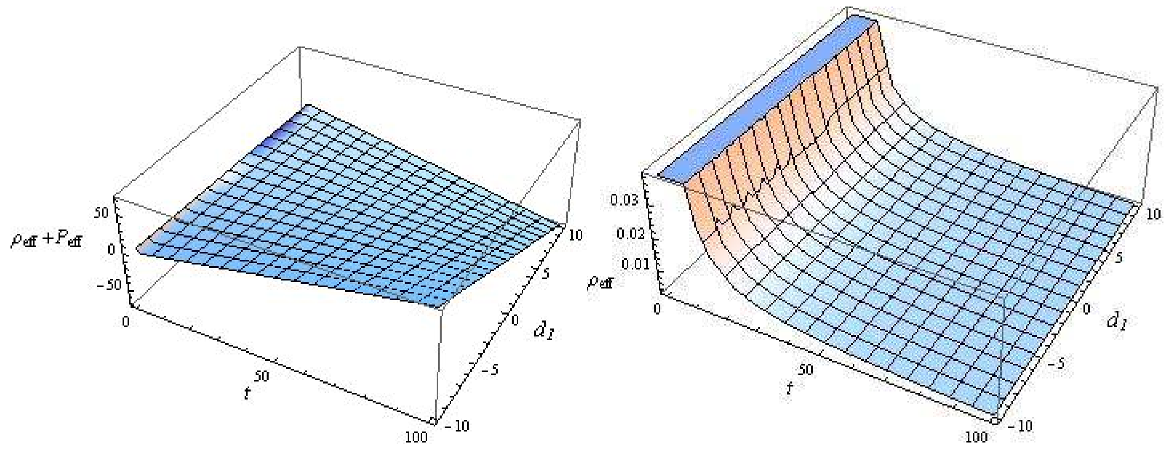


Figure 3.5: Energy conditions for $d_2 = 0.1$ and $d_3 = 1$.

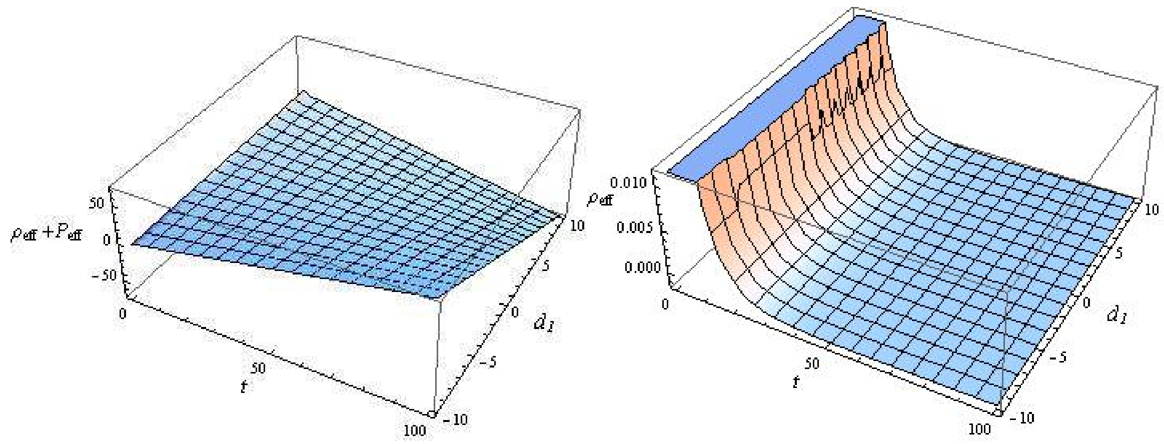


Figure 3.6: Energy conditions for $d_2 = 0.1$ and $d_3 = -0.5$.

respect to time while invalid for $d_1 > 0$. The effective energy density remains positive for all values of (t, d_1) as shown in Figure **3.5** (right). The same behavior of both conditions are obtained for $0 < d_2 \leq 0.51$ with $d_3 > 0$ as well as for $d_2 > 0$ with $d_3 = 0$. The left plot of Figure **3.6** shows similar behavior of NEC for $d_2 > 0$ and $d_3 < 0$ while $\rho_{\rm eff}$ remains positive for $0 < t < 23$. Similarly, for $d_3 = -1$ and -10, WEC is valid for $0 < t < 14$ and $0 < t < 4.5$, respectively with $d_2 = 0.1$. The right plots of Figures **3.7** and **3.8** show the validity of NEC for $d_1 \geq 0$ while does

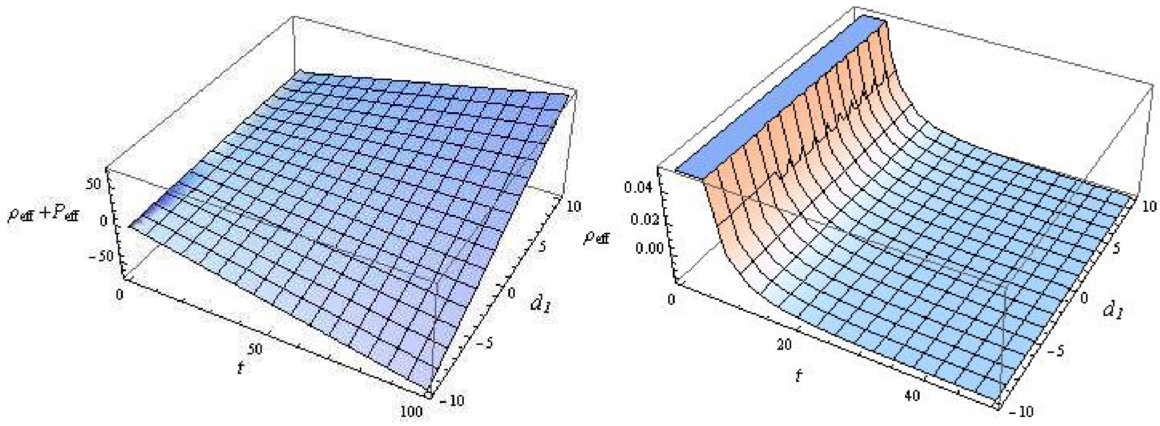


Figure 3.7: Energy conditions for $d_2 = -0.1$ and $d_3 = 0.5$.

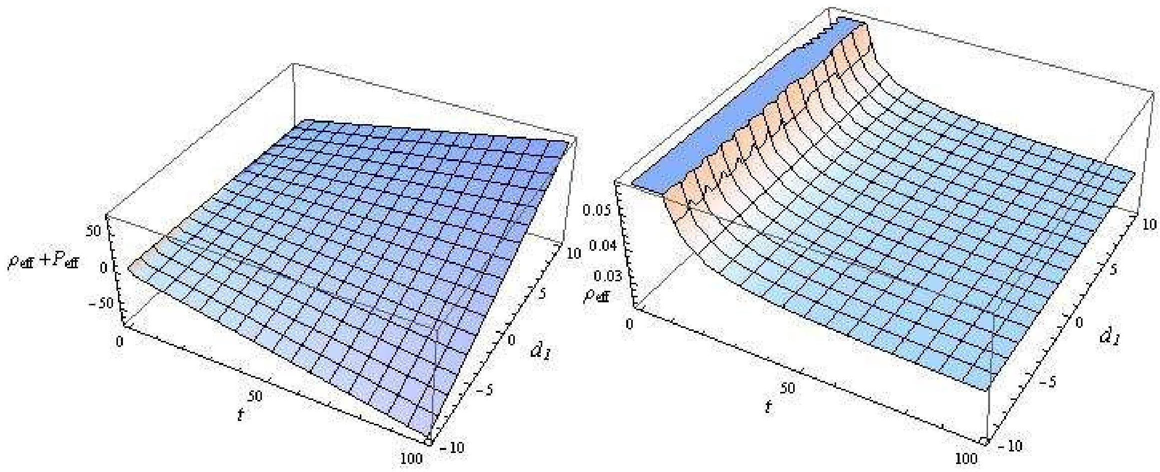


Figure 3.8: Energy conditions for $d_2 = -0.1$ and $d_3 = -1$.

not hold for negative values of d_1. The effective energy density remains positive for time interval $1 \leq t \leq 10$ with $d_3 = 0.5$ as shown in Figure **3.7** (right panel) while for $d_3 = 1$ and 10, the acceptable intervals are $1 \leq t \leq 7$ and $1 \leq t \leq 3$, respectively. This shows that the validity region of WEC decreases as the value of integration constant d_3 increases. The right plot of Figure **3.8** shows the positivity of ρ_{eff} for $(d_2, d_3) < 0$ which confirms the positivity of WEC with $d_1 > 0$.

Chapter 4

Stability Analysis of Einstein Universe

In this chapter, we investigate stability of EU against homogeneous as well as inhomogeneous scalar perturbations in the context of $f(\mathcal{G}, T)$ gravity. We formulate static and perturbed field equations in the presence of perfect fluid to analyze the stability regions. This is done for particular $f(\mathcal{G}, T)$ models corresponding to conserved as well as non-conserved energy-momentum tensor. Results of this chapter have been complied in the form of two papers in which one paper for homogeneous perturbations has been published [58] while the other for inhomogeneous perturbations is submitted for publication [59].

The chapter is organized in three sections. Section **4.1** covers the basic formalism of EU in $f(\mathcal{G}, T)$ gravity. In section **4.2**, we analyze the stability of EU against homogeneous scalar perturbations for zero as well as non-zero covariant divergence of the energy-momentum tensor while section **4.3** deals with inhomogeneous perturbations.

4.1 Einstein Universe in $f(\mathcal{G},T)$ Gravity

This section provides a basic framework to discuss EU in the context of $f(\mathcal{G},T)$ gravity using perfect fluid as matter configuration. We consider positive curvature FRW universe model given in Eq.(1.2.1) to construct the corresponding field equations as follows

$$\begin{aligned}
\frac{3}{a^2}(1+\dot{a}^2) &= \kappa^2\rho+\frac{1}{2}f(\mathcal{G},T)+(\rho+P)f_T(\mathcal{G},T)-12\frac{\ddot{a}}{a^3}(1+\dot{a}^2)\\
&\times f_{\mathcal{G}}(\mathcal{G},T)+12\frac{\dot{a}}{a^3}(1+\dot{a}^2)\dot{f}_{\mathcal{G}}(\mathcal{G},T), \qquad (4.1.1)\\
-(1+\dot{a}^2)-2a\ddot{a} &= \kappa^2a^2P-\frac{1}{2}a^2f(\mathcal{G},T)+12\frac{\ddot{a}}{a}(1+\dot{a}^2)f_{\mathcal{G}}(\mathcal{G},T)\\
&- 8\dot{a}\ddot{a}\dot{f}_{\mathcal{G}}(\mathcal{G},T)-4(1+\dot{a}^2)\ddot{f}_{\mathcal{G}}(\mathcal{G},T), \qquad (4.1.2)
\end{aligned}$$

where

$$\mathcal{G}=\frac{24}{a^3}(1+\dot{a}^2)\ddot{a},\quad T=\rho-3P. \qquad (4.1.3)$$

In terms of conformal time (η), the FRW universe model takes the form [28]

$$ds^2=\tilde{a}^2(\eta)\left[d\eta^2-\frac{dR^2}{1-\mathcal{K}R^2}-R^2(d\theta^2+\sin^2\theta d\phi^2)\right], \qquad (4.1.4)$$

where $\tilde{a}(\eta)$ represents the conformal scale factor satisfying the relation $dt=\tilde{a}(\eta)d\eta$. The field equations are

$$\begin{aligned}
3\left[\left(\frac{\tilde{a}_{,\eta}}{\tilde{a}}\right)^2+\mathcal{K}\right] &= \kappa^2\tilde{a}^2\rho+\frac{1}{2}\tilde{a}^2f(\mathcal{G},T)+\tilde{a}^2(\rho+P)f_T(\mathcal{G},T)\\
&+ \frac{12}{\tilde{a}^3}\left(\frac{\tilde{a}^2_{,\eta}}{\tilde{a}}-\tilde{a}_{,\eta\eta}\right)\left[\left(\frac{\tilde{a}_{,\eta}}{\tilde{a}}\right)^2+\mathcal{K}\right]f_{\mathcal{G}}(\mathcal{G},T)+12\frac{\tilde{a}_{,\eta}}{\tilde{a}}\left[\tilde{a}^2_{,\eta}\right.\\
&- \left.\tilde{a}\tilde{a}_{,\eta\eta}\left(1-\frac{1}{\tilde{a}^4}\right)+\frac{\mathcal{K}}{\tilde{a}^2}\right]\partial_\eta f_{\mathcal{G}}(\mathcal{G},T)-12(\tilde{a}^2_{,\eta}-\tilde{a}\tilde{a}_{,\eta\eta})\\
&\times \left(1-\frac{1}{\tilde{a}^4}\right)\partial_{\eta\eta}f_{\mathcal{G}}(\mathcal{G},T), \qquad (4.1.5)
\end{aligned}$$

$$\begin{aligned}\left(\frac{\tilde{a}_{,\eta}}{\tilde{a}}\right)^2-\frac{2\tilde{a}_{,\eta\eta}}{\tilde{a}}-\mathcal{K} &= \kappa^2\tilde{a}^2P-\frac{1}{2}\tilde{a}^2f(\mathcal{G},T)+\frac{12}{\tilde{a}^3}\left(\tilde{a}_{,\eta\eta}-\frac{\tilde{a}^2_{,\eta}}{\tilde{a}}\right)\left[\left(\frac{\tilde{a}_{,\eta}}{\tilde{a}}\right)^2+\mathcal{K}\right]\\ &\times\ f_{\mathcal{G}}(\mathcal{G},T)+4\tilde{a}_{,\eta}\left[3\tilde{a}^3(\tilde{a}^2_{,\eta}-\tilde{a}\tilde{a}_{,\eta\eta})+\frac{1}{\tilde{a}^3}(\tilde{a}_{,\eta\eta}+\mathcal{K})\right]\\ &\times\ \partial_\eta f_{\mathcal{G}}(\mathcal{G},T)-4\left[3\left(\tilde{a}^4\tilde{a}^2_{,\eta}-\tilde{a}^5\tilde{a}_{,\eta\eta}+\frac{\tilde{a}_{,\eta\eta}}{\tilde{a}^3}\right)-2\left(\frac{\tilde{a}_{,\eta}}{\tilde{a}^2}\right)^2\right.\\ &+\left.\frac{\mathcal{K}}{\tilde{a}^2}\right]\partial_{\eta\eta}f_{\mathcal{G}}(\mathcal{G},T), \qquad (4.1.6)\end{aligned}$$

where $\tilde{a}_{,\eta}$ is the derivative of $\tilde{a}(\eta)$ with respect to η and GB invariant is given by

$$\mathcal{G}=\frac{24}{\tilde{a}^5}\left(\tilde{a}_{,\eta\eta}-\frac{\tilde{a}^2_{,\eta}}{\tilde{a}}\right)\left[\mathcal{K}+\left(\frac{\tilde{a}_{,\eta}}{\tilde{a}}\right)^2\right].$$

The Einstein universe is defined by static FRW universe model as

$$a(t)=\text{constant}=\tilde{a}(\eta)=a_0.$$

At Einstein static state, the expressions for GB invariant and T become

$$\mathcal{G}(a_0)=\mathcal{G}_0=0,\quad T_0=\rho_0-3P_0,$$

where ρ_0 and P_0 denote the unperturbed energy density and pressure, respectively. The field equations for static configuration are

$$3\mathcal{K} = a_0^2\left(\kappa^2\rho_0+\frac{1}{2}f(\mathcal{G}_0,T_0)+(P_0+\rho_0)f_T(\mathcal{G}_0,T_0)\right), \qquad (4.1.7)$$

$$-\mathcal{K} = a_0^2\left(\kappa^2P_0-\frac{1}{2}f(\mathcal{G}_0,T_0)\right). \qquad (4.1.8)$$

It is interesting to mention here that EU is shear, expansion as well as rotation free universe.

4.2 Homogeneous Perturbations

In this section, we explore stability regions of EU against homogeneous scalar perturbations. For this purpose, we consider linear perturbations in the scale factor and energy density as follows

$$a(t)=a_0+a_0\delta a(t),\quad \rho(t)=\rho_0+\rho_0\delta\rho(t), \tag{4.2.1}$$

where $\delta a(t)$ and $\delta\rho(t)$ are the perturbed scale factor and energy density, respectively. Applying Taylor series expansion in two variables upto first order, we have

$$f(\mathcal{G},T)=f(\mathcal{G}_0,T_0)+f_{\mathcal{G}}(\mathcal{G}_0,T_0)\delta\mathcal{G}+f_T(\mathcal{G}_0,T_0)\delta T, \tag{4.2.2}$$

provided that $f(\mathcal{G},T)$ is an analytic function whereas $\delta\mathcal{G}$ and δT have the following expressions

$$\delta\mathcal{G}=\frac{24}{a_0^2}\delta\ddot{a},\quad \delta T=T_0\delta\rho, \tag{4.2.3}$$

where $\delta\ddot{a}=\frac{d^2}{dt^2}(\delta a)$. Using Eqs.(1.2.5), (4.1.7)-(4.2.3) in (4.1.1) and (4.1.2), we obtain the linearized perturbed field equations as follows

$$\begin{aligned}
&6\delta a+24\rho_0(1+\omega)f_{\mathcal{G}T}(\mathcal{G}_0,T_0)\delta\ddot{a}+a_0^2\rho_0[\kappa^2+(1+\omega)f_T(\mathcal{G}_0,T_0)\\
+&\frac{1}{2}(1-3\omega)f_T(\mathcal{G}_0,T_0)+\rho_0(1+\omega)(1-3\omega)f_{TT}(\mathcal{G}_0,T_0)]\delta\rho=0, \qquad (4.2.4)\\
-&\frac{2}{a_0^2}\delta a+2\delta\ddot{a}-\frac{96}{a_0^4}f_{\mathcal{G}\mathcal{G}}(\mathcal{G}_0,T_0)\delta a^{(iv)}+\rho_0[\kappa^2\omega-\frac{1}{2}(1-3\omega)f_T(\mathcal{G}_0,T_0)]\delta\rho\\
-&4\frac{\rho_0}{a_0^2}(1-3\omega)f_{\mathcal{G}T}(\mathcal{G}_0,T_0)\delta\ddot{\rho}=0. \qquad (4.2.5)
\end{aligned}$$

These equations show that perturbations in $a(t)$ are related with density perturbations. In the following subsections, we investigative stability regions for conserved as well as non-conserved energy-momentum tensor.

4.2.1 Case I: $\nabla^{\alpha}T_{\alpha\beta}=0$

In this case, we assume that general conservation law (3.1.11) holds in $f(\mathcal{G},T)$ gravity. The conserved matter contents of the universe satisfy the relation given by

$$\delta\dot{\rho}=-3(1+\omega)\delta\dot{a}. \tag{4.2.6}$$

To eliminate $\delta\rho$ from Eqs.(4.2.4) and (4.2.5), we use the above equation and obtain the fourth-order differential equation in perturbed $a(t)$ as follows

$$\begin{aligned}
&\left[6\kappa^2a_0\omega-3a_0(1-3\omega)f_T(\mathcal{G}_0,T_0)+2a_0\left\{\kappa^2+(1+\omega)f_T(\mathcal{G}_0,T_0)\right.\right.\\
+&\left.\left.\frac{1}{2}(1-3\omega)f_T(\mathcal{G}_0,T_0)+\rho_0(1+\omega)(1-3\omega)f_{TT}(\mathcal{G}_0,T_0)\right\}\right]\delta a\\
+&\left[24a_0\rho_0(1+\omega)\left\{\kappa^2\omega-\frac{1}{2}(1-3\omega)f_T(\mathcal{G}_0,T_0)\right\}f_{\mathcal{G}T}(\mathcal{G}_0,T_0)\right.\\
-&\left\{2+\frac{12\rho_0}{a_0^2}(1+\omega)(1-3\omega)f_{\mathcal{G}T}(\mathcal{G}_0,T_0)\right\}\left\{\kappa^2a_0^3+a_0^3(1+\omega)f_T(\mathcal{G}_0,T_0)\right.\\
+&\left.\left.\frac{1}{2}a_0^3(1-3\omega)f_T(\mathcal{G}_0,T_0)+a_0^3\rho_0(1+\omega)(1-3\omega)f_{TT}(\mathcal{G}_0,T_0)\right\}\right]\delta\ddot{a}\\
+&\frac{96}{a_0^4}\left\{\kappa^2a_0^3+a_0^3(1+\omega)f_T(\mathcal{G}_0,T_0)+\frac{1}{2}a_0^3(1-3\omega)f_T(\mathcal{G}_0,T_0)+a_0^3\rho_0\right.\\
\times&\left.(1+\omega)(1-3\omega)f_{TT}(\mathcal{G}_0,T_0)\right\}f_{\mathcal{G}\mathcal{G}}(\mathcal{G}_0,T_0)\delta a^{(iv)}=0.
\end{aligned} \tag{4.2.7}$$

Adding Eqs.(4.1.7) and (4.1.8), it follows that

$$\frac{2}{a_0^2}=\rho_0(1+\omega)(\kappa^2+f_T(\mathcal{G}_0,T_0)),\quad \mathcal{K}=1. \tag{4.2.8}$$

Using this expression in Eq.(4.2.7), we have

$$\begin{aligned}
&\left[\rho_0(1+\omega)\{\kappa^2+f_T(\mathcal{G}_0,T_0)\}\left\{\kappa^2(1+3\omega)+(1+\omega)f_T(\mathcal{G}_0,T_0)\right.\right.\\
-&\left.\left.(1-3\omega)f_T(\mathcal{G}_0,T_0)+\rho_0(1+\omega)(1-3\omega)f_{TT}(\mathcal{G}_0,T_0)\right\}\right]\delta a\\
+&\left[12\rho_0^2(1+\omega)^2\{\kappa^2+f_T(\mathcal{G}_0,T_0)\}\left\{\kappa^2\omega-\frac{1}{2}(1-3\omega)f_T(\mathcal{G}_0,T_0)\right\}\right.
\end{aligned}$$

$$\begin{aligned}
&\times \quad f_{\mathcal{G}T}(\mathcal{G}_0,T_0)-[2+6\rho_0^2(1+\omega)^2(1-3\omega)\{\kappa^2+f_T(\mathcal{G}_0,T_0)\}f_{\mathcal{G}T}(\mathcal{G}_0,T_0)]\\
&\times \quad \Big\{\kappa^2+(1+\omega)f_T(\mathcal{G}_0,T_0)+\rho_0(1+\omega)(1-3\omega)f_{TT}(\mathcal{G}_0,T_0)+\frac{1}{2}(1-3\omega)\\
&\times \quad f_T(\mathcal{G}_0,T_0)\}]\,\delta\ddot{a}+24\rho_0^2(1+\omega)^2\{\kappa^2+f_T(\mathcal{G}_0,T_0)\}^2\left\{\kappa^2+(1+\omega)\right.\\
&\times \quad \left.f_T(\mathcal{G}_0,T_0)+\rho_0(1+\omega)(1-3\omega)f_{TT}(\mathcal{G}_0,T_0)+\frac{1}{2}(1-3\omega)f_T(\mathcal{G}_0,T_0)\right\}\\
&\times \quad f_{\mathcal{G}\mathcal{G}}(\mathcal{G}_0,T_0)\delta a^{(iv)}=0. \qquad (4.2.9)
\end{aligned}$$

The solution of this differential equation helps to determine the stability regimes in EU. It is a complicated differential equation, therefore we consider the particular form of generic function $f(\mathcal{G},T)$ to find stable/unstable solutions as follows

$$f(\mathcal{G},T)=f_1(\mathcal{G})+f_2(T), \qquad (4.2.10)$$

which shows that the direct non-minimal curvature-matter coupling is absent. This choice of model is considered as the correction term to $f(\mathcal{G})$ gravity. For conserved matter distribution, Eq.(3.1.12) leads to the following second order differential equation for the above model as

$$2(1+\omega)Tf_{2_{TT}}(T)+(1-\omega)f_{2_T}(T)=0,$$

whose solution is the unique representation of $f_2(T)$ for which the energy-momentum tensor is conserved given by

$$f_2(T)=C_1\left(\frac{1+\omega}{1+3\omega}\right)T^{\frac{1}{2}\left(\frac{1+3\omega}{1+\omega}\right)}+C_2, \qquad (4.2.11)$$

where C_1 and C_2 are constants of integration. It is worth mentioning here that $f(\mathcal{G})$ gravity is recovered for this choice of $f(\mathcal{G},T)$ model if $f_2(T)=0$. Using the values from Eqs.(4.2.10) and (4.2.11) in (4.2.9), we have

$$\mathbb{A}_2(\mathbb{A}_1+\mathbb{A}_3)\delta a-2\mathbb{A}_1\delta\ddot{a}+24\mathbb{A}_1\mathbb{A}_2^2f_{1_{\mathcal{G}\mathcal{G}}}(\mathcal{G}_0)\delta a^{(iv)}=0, \qquad (4.2.12)$$

where

$$\begin{aligned}
\mathbb{A}_1 &= \kappa^2+\frac{1}{4}C_1(1+5\omega)[\rho_0(1-3\omega)]^{\frac{\omega-1}{2(\omega+1)}}-\frac{1}{4}C_1\rho_0(1-\omega)(1-3\omega)\\
&\times\ [\rho_0(1-3\omega)]^{\frac{-(3+\omega)}{2(1+\omega)}},\\
\mathbb{A}_2 &= \rho_0(1+\omega)\left[\kappa^2+\frac{1}{2}C_1\{\rho_0(1-3\omega)\}^{\frac{\omega-1}{2(\omega+1)}}\right],\\
\mathbb{A}_3 &= 3\kappa^2\omega-\frac{3}{4}C_1(1-3\omega)\{\rho_0(1-3\omega)\}^{\frac{\omega-1}{2(\omega+1)}}.
\end{aligned}$$

The solution of Eq.(4.2.12) is given by

$$\delta a(t)=\bar{d}_1e^{\Omega_1 t}+\bar{d}_2e^{-\Omega_1 t}+\bar{d}_3e^{\Omega_2 t}+\bar{d}_4e^{-\Omega_2 t},$$

where $\bar{d}_k$'s $(k=1...4)$ are integration constants whereas the parameters Ω_1 and Ω_2 represent frequencies of small perturbations given as

$$\Omega^2_{1,2}=\frac{\mathbb{A}_1\pm\sqrt{\mathbb{A}_1^2-24\mathbb{A}_1\mathbb{A}_2^3(\mathbb{A}_1+\mathbb{A}_3)f_{1_{\mathcal{GG}}}(\mathcal{G}_0)}}{24\mathbb{A}_1\mathbb{A}_2^2f_{1_{\mathcal{GG}}}(\mathcal{G}_0)}, \tag{4.2.13}$$

where $\Omega^2_{1,2}$ represent the frequencies Ω_1^2 and Ω_2^2.

The existence of stable/unstable regions in EU depend on the nature of perturbations. In order to avoid the exponential growth of $\delta a(t)$ or collapse, the frequencies should be purely complex which lead to the existence of stable EU. Thus, the condition of stability is achieved when $\Omega^2_{1,2}<0$. In the limit of GR, Ω_1^2 diverges while Ω_2^2 takes the form

$$\Omega_2^2=\frac{1}{2}\kappa^2\rho_0(1+3\omega)(1+\omega),$$

which provide stable region in the range $-1<\omega<-\frac{1}{3}$ [26]. To discuss graphical analysis of stable EU, we introduce a new parameter $\zeta_1=24f_{1_{\mathcal{GG}}}(\mathcal{G}_0)$ for the sake of simplicity and use $\kappa^2=1,\ \rho_0=0.3$. Figure **4.1** shows the stable regions under homogeneous perturbations of EU for Ω_1^2. It is found that for $C_1=1$ (left plot) the

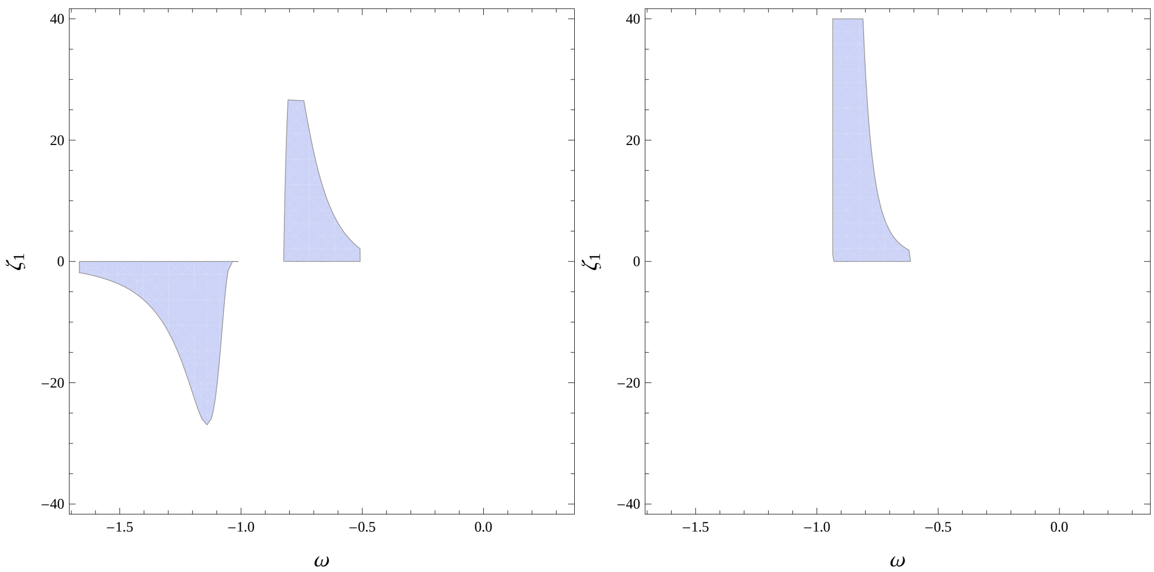


Figure 4.1: Stable regions in (ω, ζ_1) space for Ω_1^2 with $C_1 = 1$ (left) and $C_1 = 5$ (right).

stable EU exists for negative values of ω while no stable region exists for its positive values. The right panel shows the stable region for $C_1 = 5$ and hence the stability regions decrease as the value of integration constant increases while for negative values of C_1, no stable regions are found. The regions of stability for Ω_2^2 are shown in Figure **4.2** for both positive as well as negative values of C_1. The negative values of ζ_1 are obtained for $f_{1\mathcal{G}\mathcal{G}}(\mathcal{G}_0) < 0$ which is in agreement with stability condition of $f(\mathcal{G})$ models [39]. The stability of the whole system depends on all frequencies present in a system such that their common regions reflect the stable solution of EU. Figures **4.1** and **4.2** show that stable EU is completely described by Ω_1^2 for $C_1 = 1$ and $C_1 = 5$ while no stable region exists for $C_1 = -1$.

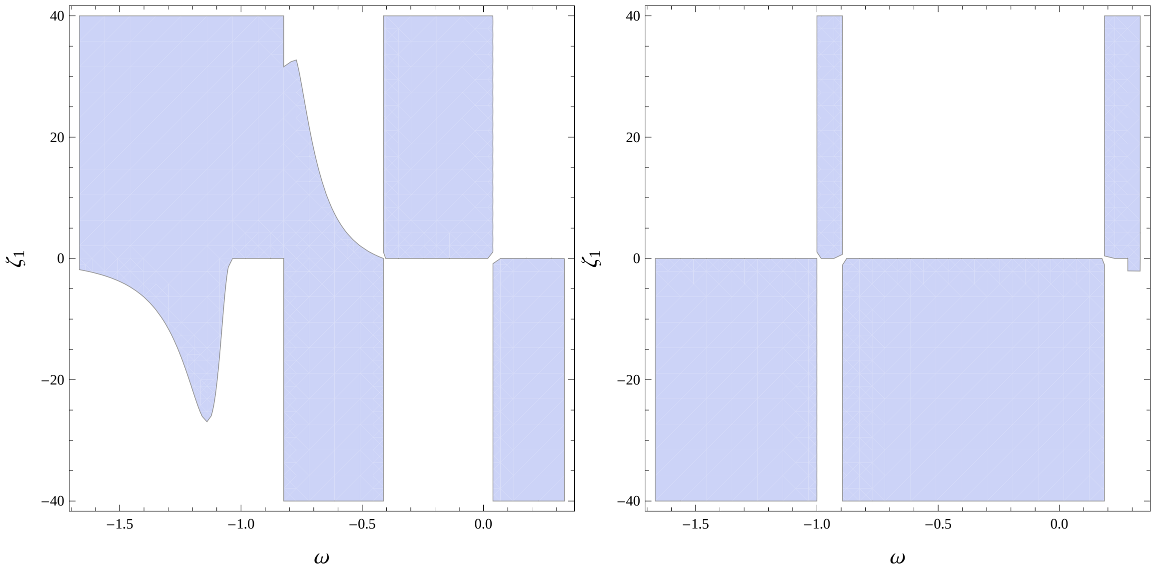


Figure 4.2: Stable regions in (ω, ζ_1) space for Ω_2^2 with $C_1 = 1$ (left) and $C_1 = -1$ (right).

4.2.2 Case II: $\nabla^\alpha T_{\alpha\beta} \neq 0$

Here, we analyze the stability of $f(\mathcal{G}, T)$ model when the energy-momentum tensor is not conserved. We consider generic function $f_1(\mathcal{G})$ and a linear form of $f_2(T)$ in Eq.(4.2.10) as follows

$$f(\mathcal{G}, T) = f_1(\mathcal{G}) + \kappa^2 \varphi T, \tag{4.2.14}$$

where φ is an arbitrary constant. Substituting in Eq.(3.1.10), we obtain

$$\rho = \tilde{\rho}_0 a^{-3\varepsilon}, \quad \varepsilon = \frac{2(1+\varphi)(1+\omega)}{2+\varphi(3-\omega)},$$

where $\tilde{\rho}_0$ is an integration constant. The perturbed field equations (4.2.4) and (4.2.5) take the following form

$$6\delta a + \kappa^2 a_0^2 \rho_0 \left[1 - \frac{\varphi}{2}(\omega - 3)\right] \delta\rho = 0, \tag{4.2.15}$$

$$2\delta\ddot{a} - \frac{2}{a_0^2}\delta a + \kappa^2 \left[\omega - \frac{\varphi}{2}(1 - 3\omega)\right] \rho_0 \delta\rho - \frac{96}{a_0^4} f_{1\mathcal{G}\mathcal{G}}(\mathcal{G}_0)\delta a^{(iv)} = 0. \tag{4.2.16}$$

The first equation shows the relationship between the perturbed energy density and scale factor perturbations. Eliminating $\delta\rho$ from Eqs.(4.2.15) and (4.2.16), we obtain

$$2\delta\ddot{a}-\frac{2}{a_0^2}\left[1+\frac{3(2\omega-\varphi(1-3\omega))}{2-\varphi(\omega-3)}\right]\delta a-\frac{96}{a_0^4}f_{1_{\mathcal{G}\mathcal{G}}}(\mathcal{G}_0)\delta a^{(iv)}=0. \tag{4.2.17}$$

In this case, the addition of static field equations gives

$$\frac{2}{a_0^2}=\kappa^2\rho_0(1+\varphi)(1+\omega). \tag{4.2.18}$$

Inserting this value of $\frac{2}{a_0^2}$ in Eq.(4.2.17), the differential equation in perturbed $a(t)$ becomes

$$\begin{aligned}
&\kappa^2\rho_0[\varphi(1+\varphi)(1-\omega^2)-(1+\varphi)^2(1+\omega)(1+3\omega)]\delta a+[2(1+\varphi)\\
+\;&\varphi(1-\omega)]\delta\ddot{a}-12\kappa^4\rho_0^2[2(1+\varphi)^3(1+\omega)^2+\varphi(1+\varphi)^2(1-\omega)(1+\omega)^2]\\
\times\;&f_{1_{\mathcal{G}\mathcal{G}}}(\mathcal{G}_0)\delta a^{(iv)}=0,
\end{aligned} \tag{4.2.19}$$

whose solution provides the following four frequencies as

$$\Omega_{3,4}^2=\frac{-2(1+\varphi)-\varphi(1-\omega)\pm\sqrt{[2(1+\varphi)+\varphi(1-\omega)]^2-48\kappa^6\rho_0^3\mathbb{A}_4f_{1_{\mathcal{G}\mathcal{G}}}(\mathcal{G}_0)}}{24\kappa^4\rho_0^2[\varphi(1+\varphi)^2(\omega-1)(1+\omega)^2-2(1+\varphi)^3(1+\omega)^2]f_{1_{\mathcal{G}\mathcal{G}}}(\mathcal{G}_0)},$$

where

$$\begin{aligned}
\mathbb{A}_4&=[2(1+\varphi)^3(1+\omega)^2+\varphi(1+\varphi)^2(1-\omega)(1+\omega)^2][(1+\varphi)^2(1+\omega)\\
&\times(1+3\omega)-\varphi(1+\varphi)(1-\omega^2)].
\end{aligned}$$

When $f_1(\mathcal{G}_0)=0=\varphi$, the frequency Ω_3^2 recovers the GR result as obtained in the previous case while Ω_4^2 diverges. We simplify the expression by introducing a new parameter $\zeta_2=-48\kappa^6\rho_0^3f_{1_{\mathcal{G}\mathcal{G}}}(\mathcal{G}_0)$ which remains positive for $f_{1_{\mathcal{G}\mathcal{G}}}(\mathcal{G}_0)<0$. Figure **4.3** shows stable regions against homogeneous perturbations of EU for Ω_3^2. It is found

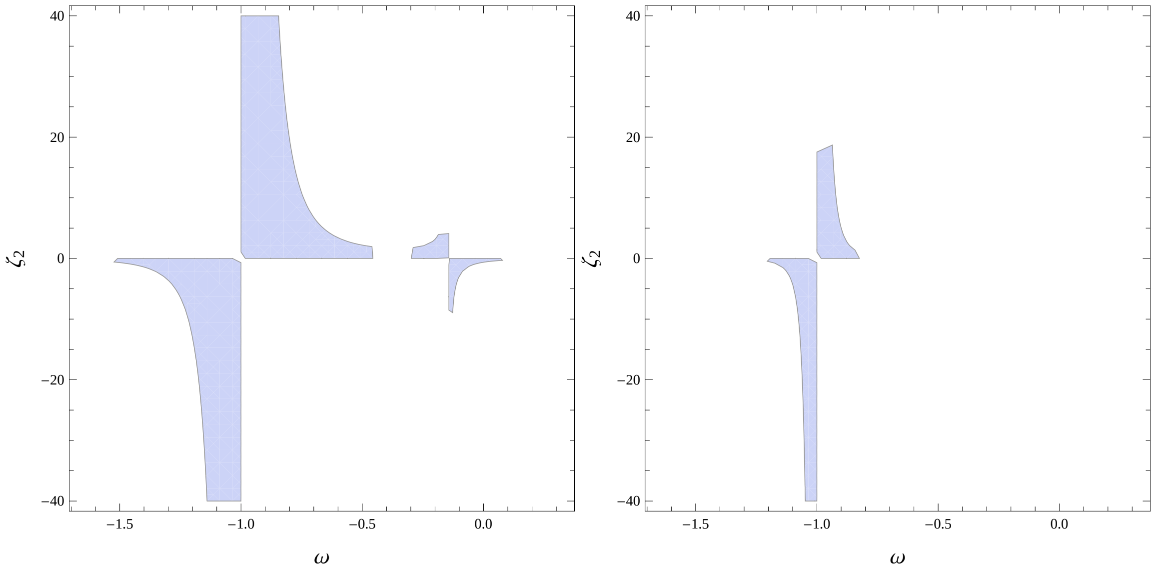


Figure 4.3: Stable regions in (ω, ζ_2) space for Ω_3^2 with $\varphi = 1$ (left) and $\varphi = 5$ (right).

that when $\varphi = 1$ (left panel), the stable EU exists for all values of ω with suitable choice of ζ_2 while less stable regions are obtained when $\varphi = 5$ as shown in the right plot. In the case of non-conserved energy-momentum tensor, the stability regions decrease as the value of model parameter φ increases while we have checked that no stable regions are observed for $\varphi < -0.6$. The regions of stability in EU for Ω_4^2 are shown in Figure **4.4**. In this case, the stable regions of the whole system is obtained by Ω_3^2 for all considered values of φ.

Now, we consider the generalized model given by

$$f(\mathcal{G}, T) = f_1(\mathcal{G}) + \kappa^2 \varphi T^b, \quad b \neq 0. \tag{4.2.20}$$

Following the same procedure, we obtain the following fourth-order differential equation in perturbed $a(t)$ as follows

$$\begin{aligned} & 24\kappa^4 \rho_0^2 (1+\omega)^2 \left[1 + b\varphi\rho_0^{b-1}(1-3\omega)^{b-1}\right]^2 f_{1_{\mathcal{G}\mathcal{G}}}(\mathcal{G}_0)\delta a^{(iv)} - 2\delta\ddot{a} \\ + \quad & \kappa^2 \rho_0 (1+\omega)\left(1 + b\varphi\rho_0^{b-1}(1-3\omega)^{b-1}\right)\left[1 + 3\left(\omega - \frac{b}{2}\varphi(1-3\omega)^b \rho_0^{b-1}\right)\right. \end{aligned}$$

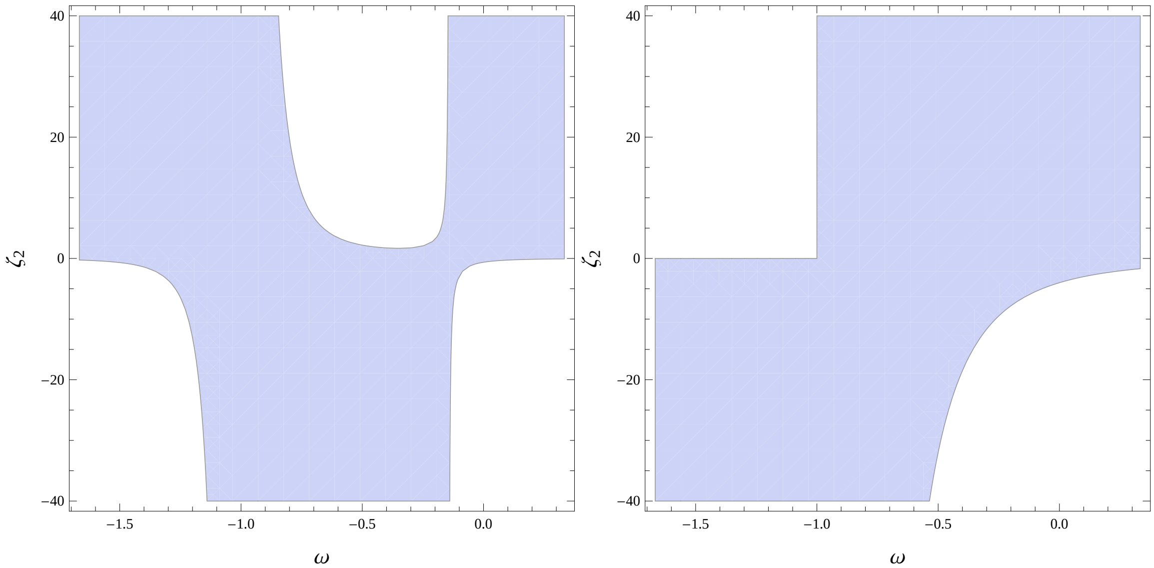


Figure 4.4: Stable regions in (ω, ζ_2) space for Ω_4^2 with $\varphi = 1$ (left) and $\varphi = -0.5$ (right).

$$\begin{aligned} \times & \quad \left(1 + b\varphi(1-3\omega)^{b-1}\rho_0^{b-1}\left[(1+\omega) + \frac{1}{2}(1-3\omega) + (b-1)\right.\right. \\ \times & \quad \left.\left.(1+\omega)\right]\right)^{-1}\Big]\,\delta a = 0. \end{aligned}$$

In this case, the four frequencies are

$$\Omega_{5,6}^2 = \frac{1 \pm \sqrt{1 - 24\kappa^6 \mathbb{A}_5 f_{1\mathcal{G}\mathcal{G}}(\mathcal{G}_0)}}{24\left[\kappa^2\rho_0(1+\omega)\left(1 + b\varphi\rho_0^{b-1}(1-3\omega)^{b-1}\right)\right]^2 f_{1\mathcal{G}\mathcal{G}}(\mathcal{G}_0)},$$

where

$$\begin{aligned} \mathbb{A}_5 &= \rho_0^3(1+\omega)^3\left(1 + b\varphi\rho_0^{b-1}(1-3\omega)^{b-1}\right)^3\left[1 + 3\left(\omega - \frac{b}{2}\varphi(1-3\omega)^b\right.\right. \\ &\times \left.\rho_0^{b-1}\right)\left[1 + b\varphi(1-3\omega)^{b-1}\rho_0^{b-1}\left((1+\omega) + \frac{1}{2}(1-3\omega) + (b-1)\right.\right. \\ &\times \left.\left.\left.(1+\omega)\right)\right]^{-1}\right]. \end{aligned}$$

The graphical analysis of Ω_6^2 is shown in Figures **4.5** and **4.6** where we have used $\zeta_3 = -24\kappa^6 f_{1\mathcal{G}\mathcal{G}}(\mathcal{G}_0)$ and $\varphi = 1$. It is found that stable regions are obtained for all

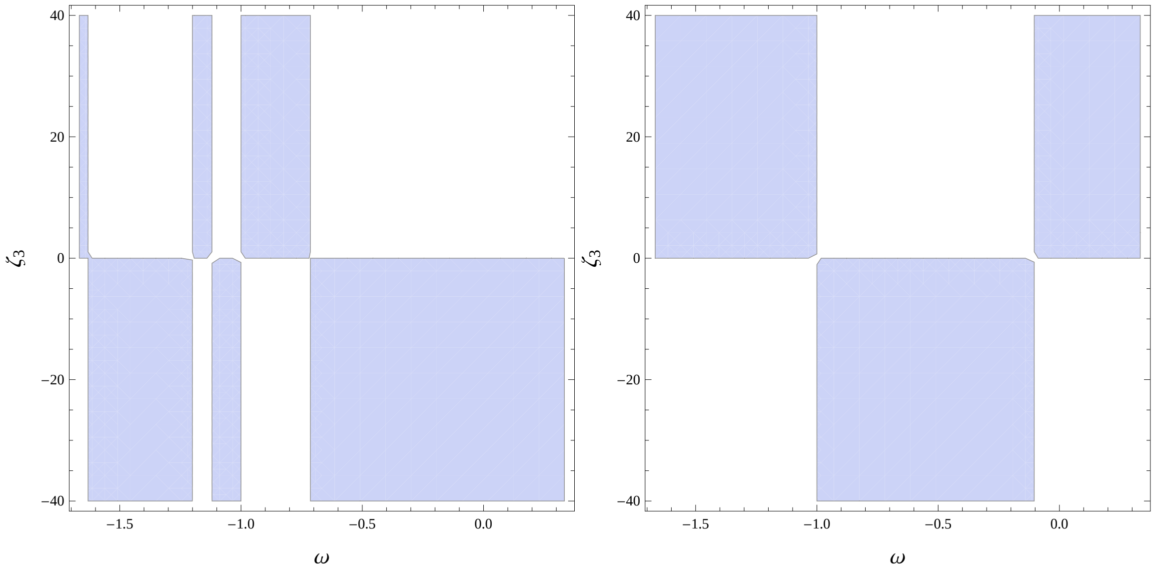


Figure 4.5: Stable regions in (ω, ζ_3) space for Ω_6^2 with $b = -5$ (left) and $b = 0.5$ (right).

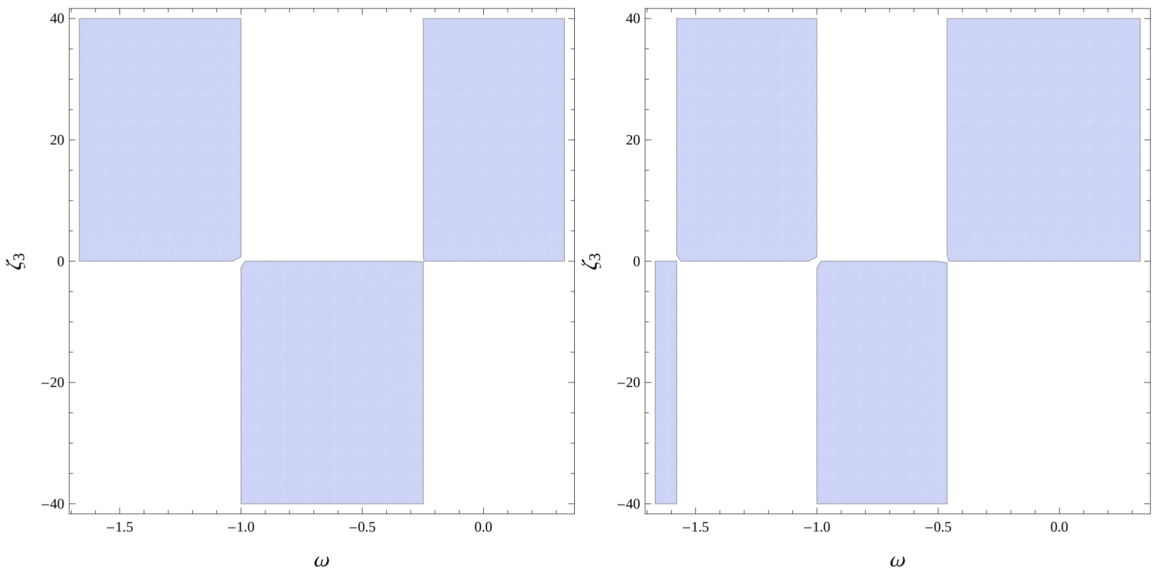


Figure 4.6: Stable regions in (ω, ζ_3) space for Ω_6^2 with $b = 2$ (left) and $b = 5$ (right).

the considered values of b for Ω_6^2 while stable EU does not exist for Ω_5^2. Thus, the EU possesses no stable region in this case. It is interesting to mention here that for $f_1(\mathcal{G}_0) = 0 = \varphi$, the frequency Ω_5^2 diverges while GR is recovered for Ω_6^2 as in the previous case.

4.3 Inhomogeneous Perturbations

In this section, we study the stability regions of EU against inhomogeneous scalar perturbations using Newtonian gauge (also known as longitudinal gauge). In this gauge transformation, the perturbed line element is given by [28]

$$ds^2 = a_0^2(1-2\Phi)d\eta^2 - a_0^2(1+2\Psi)\left[\frac{dR^2}{1-\mathcal{K}R^2} + R^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \qquad (4.3.1)$$

where Φ and Ψ represent the Bardeen potential and perturbation to spatial curvature, respectively. The linear perturbations in energy density and pressure are

$$\rho = \rho_0 + \rho_0\delta\rho, \quad P = P_0 + P_0\delta P.$$

The harmonic decomposition of these scalar perturbations are [60]

$$\begin{aligned}
\delta\rho &= \delta\rho_\psi(\eta)\Upsilon_\psi(x^i), \quad \delta P = \delta P_\psi(\eta)\Upsilon_\psi(x^i),\\
\Phi &= \Phi_\psi(\eta)\Upsilon_\psi(x^i), \quad \Psi = \Psi_\psi(\eta)\Upsilon_\psi(x^i),
\end{aligned}$$

where x^i denotes the spatial coordinates (R,θ,ϕ) while summation over ψ is taken. The harmonic function $\Upsilon_\psi \equiv \Upsilon_\psi(x^i)$ satisfies the following relations corresponding to different geometries of the universe as

$$\Delta\Upsilon_\psi \equiv -\hbar^2\Upsilon_\psi = \begin{cases} -(\psi^2+1)\Upsilon_\psi, & \psi^2 \geq 0, & \mathcal{K} = -1,\\ -\psi^2\Upsilon_\psi, & \psi^2 \geq 0, & \mathcal{K} = 0,\\ -\psi(\psi+2)\Upsilon_\psi, & \psi = 0,1,2,..., & \mathcal{K} = 1, \end{cases}$$

where Δ is the three-dimensional Laplacian operator.

These perturbations generate continuous spectrum for flat as well as closed spatial cosmic geometries while discrete spectrum is observed for open universe model [28]. The first inhomogeneous mode ($\psi = 1$) is known as gauge mode associated with the gauge degree of freedom which depicts the freedom to change v_α of fundamental observers, so the physical modes have $\psi \geq 2$ [22]. The homogeneous scalar perturbations are recovered for $\psi = 0$. Using these perturbations in Eq.(1.1.16), we obtain linearized $\eta\eta$-component for the perturbed metric (4.3.1) as follows

$$\begin{aligned} & 6\mathcal{K}\Psi + 2a_0^2\hbar^2\Psi + a_0^2\rho_0\left[\kappa^2 - \frac{1}{2}(\omega-3)f_T(\mathcal{G}_0,T_0) + \rho_0(1-3\omega)(1+\omega)\right. \\ \times & \left. f_{TT}(\mathcal{G}_0,T_0) - \frac{4\mathcal{K}\hbar^2}{a_0^2}(1-3\omega)f_{\mathcal{G}T}(\mathcal{G}_0,T_0)\right]\delta\rho + \left[a_0^2\rho_0(1+\omega)f_{\mathcal{G}T}(\mathcal{G}_0,T_0)\right. \\ - & \left. 4\mathcal{K}\hbar^2 f_{\mathcal{G}\mathcal{G}}(\mathcal{G}_0,T_0)\right]\delta\mathcal{G} = 0, \end{aligned} \tag{4.3.2}$$

where the linearized perturbed GB invariant is given by

$$\delta\mathcal{G} = \frac{8\mathcal{K}}{a_0^2}\left(\frac{3\Psi_{,\eta\eta}}{a_0^2} + \hbar^2\Phi\right). \tag{4.3.3}$$

The linearized diagonal components of Eq.(1.1.16) give the differential equation of the form

$$\begin{aligned} & 2\left[6\mathcal{K}\Psi - 3\Psi_{,\eta\eta} + a_0^2\hbar^2(2\Psi-\Phi)\right] + a_0^2\rho_0\left[\kappa^2(1-3\omega) + 2(1-3\omega)f_T(\mathcal{G}_0,T_0)\right. \\ + & (1+\omega)f_T(\mathcal{G}_0,T_0) + \rho_0(1-3\omega)(1+\omega)f_{TT}(\mathcal{G}_0,T_0) + \frac{4\mathcal{K}\hbar^2}{a_0^2}(1-3\omega) \\ \times & \left. f_{\mathcal{G}T}(\mathcal{G}_0,T_0)\right]\delta\rho + \left[a_0^2\rho_0(1+\omega)f_{\mathcal{G}T}(\mathcal{G}_0,T_0) - 4\mathcal{K}\hbar^2 f_{\mathcal{G}\mathcal{G}}(\mathcal{G}_0,T_0)\right]\delta\mathcal{G} \\ + & \frac{12\mathcal{K}}{a_0^2}f_{\mathcal{G}\mathcal{G}}(\mathcal{G}_0,T_0)(\delta\mathcal{G})_{,\eta\eta} + \frac{4\mathcal{K}}{a_0^2}\rho_0(1-3\omega)f_{\mathcal{G}T}(\mathcal{G}_0,T_0)(\delta\rho)_{,\eta\eta} = 0. \end{aligned} \tag{4.3.4}$$

The corresponding off-diagonal components in the presence of perfect fluid provide the relation $\Phi(\eta) = \Psi(\eta)$ while it does not hold for anisotropic matter distribution.

Making use of Eq.(4.2.10), the field equations (4.3.2) and (4.3.4) reduce to

$$
\begin{aligned}
& 2(3\mathcal{K}+a_0^2\hbar^2)\Psi+a_0^2\rho_0\left[\kappa^2-\frac{1}{2}(\omega-3)f_{2_T}(T_0)+\rho_0(1-3\omega)(1+\omega)\right.\\
\times\ & \left. f_{2_{TT}}(T_0)\right]\delta\rho-4\mathcal{K}\hbar^2 f_{1_{\mathcal{GG}}}(\mathcal{G}_0)\delta\mathcal{G}=0, \qquad (4.3.5)\\
& 2\left[6\mathcal{K}\Psi-3\Psi_{,\eta\eta}+a_0^2\hbar^2(2\Psi-\Phi)\right]+a_0^2\rho_0\left[\kappa^2(1-3\omega)+\{2(1-3\omega)\right.\\
+\ & \left.(1+\omega)\}f_{2_T}(T_0)+\rho_0(1-3\omega)(1+\omega)f_{2_{TT}}(T_0)\right]\delta\rho-4\mathcal{K}\hbar^2 f_{1_{\mathcal{GG}}}(\mathcal{G}_0)\delta\mathcal{G}\\
+\ & \frac{12\mathcal{K}}{a_0^2}f_{1_{\mathcal{GG}}}(\mathcal{G}_0)(\delta\mathcal{G})_{,\eta\eta}=0. \qquad (4.3.6)
\end{aligned}
$$

Using Eqs.(4.3.3) and (4.3.5) in the elimination of Φ, $\delta\mathcal{G}$ and $\delta\rho$ from Eq.(4.3.6), it follows that

$$
\begin{aligned}
& \frac{288\mathcal{K}^2}{a_0^6}\left[\kappa^2-\frac{1}{2}(\omega-3)f_{2_T}(T_0)+\rho_0(1+\omega)(1-3\omega)f_{2_{TT}}(T_0)\right]f_{1_{\mathcal{GG}}}(\mathcal{G}_0)\Psi^{(iv)}\\
-\ & 6\left[\kappa^2-\frac{1}{2}(\omega-3)f_{2_T}(T_0)+\rho_0(1+\omega)(1-3\omega)f_{2_{TT}}(T_0)-\frac{16\mathcal{K}^2\hbar^2}{a_0^4}\right.\\
\times\ & [\kappa^2(1-3\omega)+2\{2(1-3\omega)+(1+\omega)\}f_{2_T}(T_0)+\rho_0(1+\omega)(1-3\omega)\\
\times\ & \left.f_{2_{TT}}(T_0)]f_{1_{\mathcal{GG}}}(\mathcal{G}_0)\right]\Psi_{,\eta\eta}+2\left[\left(6\mathcal{K}+a_0^2\hbar^2-\frac{16\mathcal{K}^2\hbar^4}{a_0^2}f_{1_{\mathcal{GG}}}(\mathcal{G}_0)\right)\right.\\
\times\ & \left(\kappa^2-\frac{1}{2}(\omega-3)f_{2_T}(T_0)+\rho_0(1+\omega)(1-3\omega)f_{2_{TT}}(T_0)\right)-\left(3\mathcal{K}+a_0^2\hbar^2\right.\\
-\ & \left.\frac{16\mathcal{K}^2\hbar^4}{a_0^2}f_{1_{\mathcal{GG}}}(\mathcal{G}_0)\right)\left(\kappa^2(1-3\omega)+2\{2(1-3\omega)+(1+\omega)\}f_{2_T}(T_0)\right.\\
+\ & \left.\left.\rho_0(1+\omega)(1-3\omega)f_{2_{TT}}(T_0)\right)\right]\Psi=0. \qquad (4.3.7)
\end{aligned}
$$

Using Eqs.(4.1.7), (4.1.8) and (4.2.10), we obtain

$$
\frac{2\mathcal{K}}{a_0^2}=\rho_0(1+\omega)[\kappa^2+f_{2_T}(T_0)].
$$

Substituting this equation in Eq.(4.3.7), the resultant fourth-order perturbed differential equation in Ψ takes the form

$$
18\rho_0^4(1+\omega)^4[\kappa^2+f_{2_T}(T_0)]^4\left[\kappa^2-\frac{1}{2}(\omega-3)f_{2_T}(T_0)+\rho_0(1+\omega)(1-3\omega)\right.
$$

$$
\begin{aligned}
\times &\quad f_{2_{TT}}(T_0)]\, f_{1_{\mathcal{G}\mathcal{G}}}(\mathcal{G}_0)\Psi^{(iv)} - 6\mathcal{K}\rho_0(1+\omega)[\kappa^2 + f_{2_T}(T_0)]\left[\frac{1}{2}\left(\kappa^2 - \frac{1}{2}(\omega-3)f_{2_T}(T_0)\right.\right.\\
+ &\quad \left.\rho_0(1+\omega)(1-3\omega)f_{2_{TT}}(T_0)\right) - 2\rho_0^2\hbar^2(1+\omega)^2[\kappa^2 + f_{2_T}(T_0)]^2[\kappa^2(1-3\omega)\\
+ &\quad \left.2\{2(1-3\omega)+(1+\omega)\}f_{2_T}(T_0) + \rho_0(1+\omega)(1-3\omega)f_{2_{TT}}(T_0)]f_{1_{\mathcal{G}\mathcal{G}}}(\mathcal{G}_0)\right]\Psi_{,\eta\eta}\\
+ &\quad 2\mathcal{K}^2\left[\left\{3\rho_0(1+\omega)[\kappa^2 + f_{2_T}(T_0)] + \hbar^2 - 4\rho_0^2\hbar^4(1+\omega)^2[\kappa^2 + f_{2_T}(T_0)]^2 f_{1_{\mathcal{G}\mathcal{G}}}(\mathcal{G}_0)\right\}\right.\\
\times &\quad \left(\kappa^2 - \frac{1}{2}(\omega-3)f_{2_T}(T_0) + \rho_0(1+\omega)(1-3\omega)f_{2_{TT}}(T_0)\right) - \left\{\frac{3}{2}\rho_0(1+\omega)\right.\\
\times &\quad \left.[\kappa^2 + f_{2_T}(T_0)] + \hbar^2 - 4\rho_0^2\hbar^4(1+\omega)^2[\kappa^2 + f_{2_T}(T_0)]^2 f_{1_{\mathcal{G}\mathcal{G}}}(\mathcal{G}_0)\right\}[\kappa^2(1-3\omega)\\
+ &\quad \left.2\{2(1-3\omega)+(1+\omega)\}f_{2_T}(T_0) + \rho_0(1+\omega)(1-3\omega)f_{2_{TT}}(T_0)]\right]\Psi = 0. \qquad (4.3.8)
\end{aligned}
$$

Now, we explore stability regions for conserved as well as non-conserved energy-momentum tensor.

4.3.1 Case I: $\nabla^\alpha T_{\alpha\beta} = 0$

In this case, the perturbed differential equation (4.3.8) takes the form

$$
\mathbb{A}_6\Psi^{(iv)} + \mathbb{A}_7\Psi_{,\eta\eta} + \mathbb{A}_8\Psi = 0,
$$

where

$$
\begin{aligned}
\mathbb{A}_6 &= 18\rho_0^4(1+\omega)^4\left[\kappa^2 + \frac{1}{2}C_1[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}\right]^4\left[\kappa^2 - \frac{1}{4}C_1(\omega-3)\right.\\
&\times \quad \left.[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)} - \frac{1}{4}C_1\rho_0(1-\omega)(1-3\omega)[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{3+\omega}{1+\omega}\right)}\right]\\
&\times \quad f_{1_{\mathcal{G}\mathcal{G}}}(\mathcal{G}_0),\\
\mathbb{A}_7 &= -6\mathcal{K}\rho_0(1+\omega)\left[\kappa^2 + \frac{1}{2}C_1[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}\right]\\
&\times \quad \left[\frac{1}{2}\left\{\kappa^2 - \frac{1}{4}C_1(\omega-3)[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)} - \frac{1}{4}C_1\rho_0(1-\omega)(1-3\omega)\right.\right.\\
&\times \quad \left.\left.[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{3+\omega}{1+\omega}\right)}\right\} - 2\rho_0^2\hbar^2(1+\omega)^2\left[\kappa^2 + \frac{1}{2}C_1[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}\right]^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times \quad \left[\kappa^2(1-3\omega)+C_1\{2(1-3\omega)+(1+\omega)\}[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}\right.\\
&- \quad \left.\frac{1}{4}C_1\rho_0(1-\omega)(1-3\omega)[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{3+\omega}{1+\omega}\right)}\right] f_{1\mathcal{G}\mathcal{G}}(\mathcal{G}_0)\Bigg],\\
\mathbb{A}_8 \quad &= \quad 2\mathcal{K}^2\left[\left\{3\rho_0(1+\omega)\left[\kappa^2+\frac{1}{2}C_1[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}\right]+\hbar^2-4\rho_0^2\hbar^4(1+\omega)^2\right.\right.\\
&\times \quad \left.\left[\kappa^2+\frac{1}{2}C_1[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}\right]^2 f_{1\mathcal{G}\mathcal{G}}(\mathcal{G}_0)\right\}\left(\kappa^2-\frac{1}{4}C_1(\omega-3)\right.\\
&\times \quad \left.[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}-\frac{1}{4}C_1\rho_0(1-\omega)(1-3\omega)[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{3+\omega}{1+\omega}\right)}\right)\\
&- \quad \left\{\frac{3}{2}\rho_0(1+\omega)\left[\kappa^2+\frac{1}{2}C_1[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}\right]+\hbar^2-4\rho_0^2\hbar^4(1+\omega)^2\right.\\
&\times \quad \left.\left[\kappa^2+\frac{1}{2}C_1[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}\right]^2 f_{1\mathcal{G}\mathcal{G}}(\mathcal{G}_0)\right\}\left(\kappa^2(1-3\omega)+C_1\{2(1-3\omega)\right.\\
&+ \quad (1+\omega)\}[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}-\frac{1}{4}C_1\rho_0(1-\omega)(1-3\omega)\\
&\times \quad \left.\left.[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{3+\omega}{1+\omega}\right)}\right)\right].
\end{aligned}
$$

The solution is given by

$$\Psi(\eta)=\tilde{d}_1e^{-\Omega_7\eta}+\tilde{d}_2e^{\Omega_7\eta}+\tilde{d}_3e^{-\Omega_8\eta}+\tilde{d}_4e^{\Omega_8\eta},$$

where $\tilde{d}_k$'s represent integration constants and frequencies $\Omega_{7,8}$ are

$$\Omega_{7,8}^2=\frac{-\mathbb{A}_7\mp\sqrt{\mathbb{A}_7^2-4\mathbb{A}_6\mathbb{A}_8}}{2\mathbb{A}_6}. \tag{4.3.9}$$

To discuss stability of EU against inhomogeneous perturbations, we introduce the notation $\zeta_4=-2\rho_0^2f_{1\mathcal{G}\mathcal{G}}(\mathcal{G}_0)$. Figures **4.7** and **4.8** show the stable regions for closed and open universe models, respectively with $C_1=-1$. Blue regions indicate the existence of stable EU for Ω_7^2 while the regions occupied by red dashed lines correspond to Ω_8^2, respectively. The stability of the system is described by regions

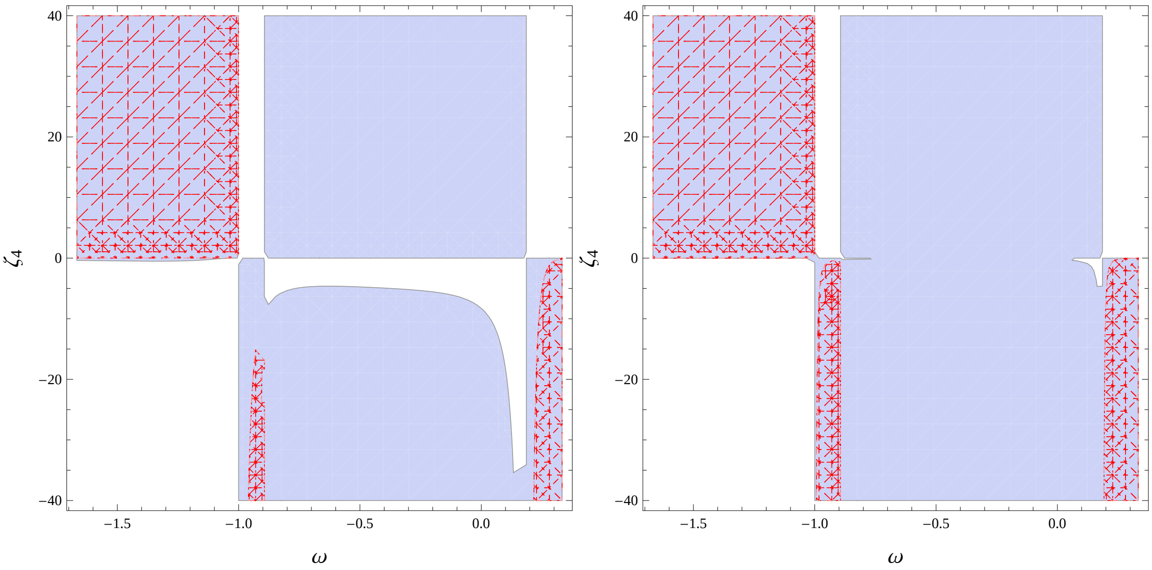


Figure 4.7: Regions of stability in (ω, ζ_4) space for $\mathcal{K} = 1$ with $\psi = 2$ (left) and $\psi = 15$ (right).

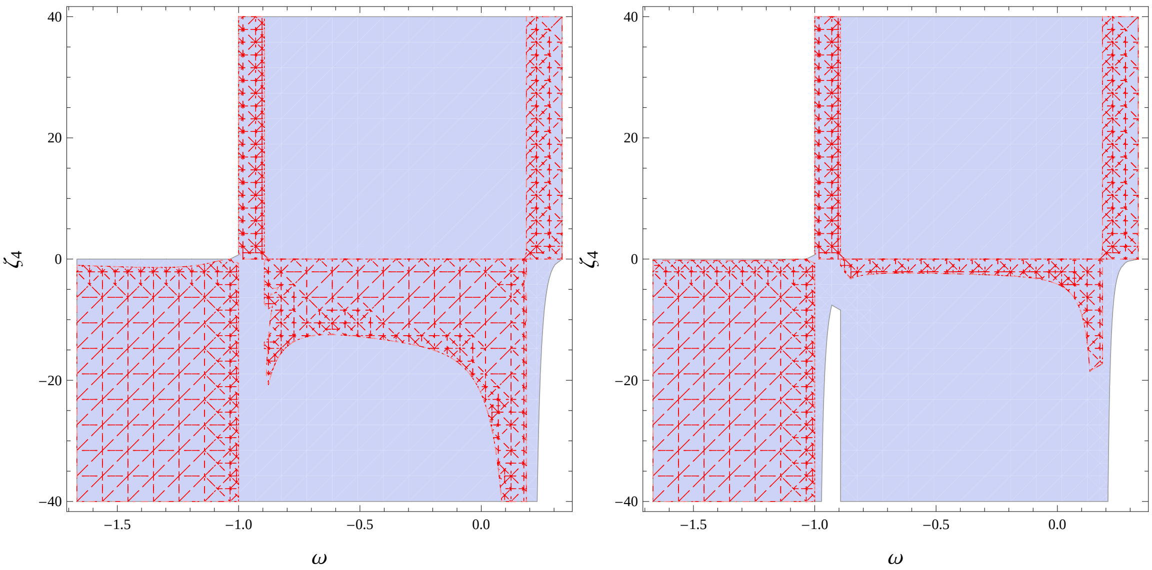


Figure 4.8: Regions of stability in (ω, ζ_4) space for $\mathcal{K} = -1$ with $\psi^2 = 2$ (left) and $\psi^2 = 15$ (right).

shared by all frequencies. Figure **4.7** shows that the stability of EU increases as the value of ψ increases in the background of closed universe whereas it decreases for $\mathcal{K}=-1$ as ψ^2 increases (Figure **4.8**). The $f(\mathcal{G})$ models satisfy the stability condition $f_{1\mathcal{G}\mathcal{G}}(\mathcal{G}_0)<0$ for positive values of parameter ζ_4. For $C_1=0=f_{1\mathcal{G}\mathcal{G}}(\mathcal{G}_0)$, the frequency Ω_7^2 diverges while $\Omega_8^2=\frac{1}{2}\kappa^2\rho_0(1+\omega)(1+3\omega)$, which is in agreement with the results of GR for $-1<\omega<-1/3$ [26].

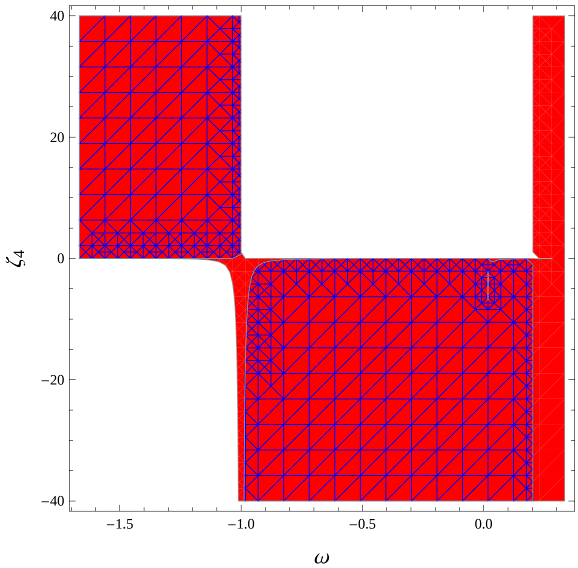
4.3.2 Case II: $\nabla^\alpha T_{\alpha\beta}\neq 0$

Here, we explore the stability of EU when the energy-momentum tensor is not conserved. Using (4.2.20) in Eq.(4.3.8), a fourth-order differential equation in Ψ is obtained whose solution provides four frequencies as follows

$$\Omega_{9,10}^2=\frac{-\mathbb{A}_{10}\mp\sqrt{\mathbb{A}_{10}^2-4\mathbb{A}_9\mathbb{A}_{11}}}{2\mathbb{A}_9},$$

where

$$\begin{aligned}
\mathbb{A}_9 &= 18\rho_0^4(1+\omega)^4[\kappa^2(1+\varphi b[\rho_0(1-3\omega)]^{b-1})]^4\left[\kappa^2\left\{1-\frac{1}{2}\varphi b(\omega-3\right.\right.\\
&- \left.\left.2(1+\omega)(b-1))[\rho_0(1-3\omega)]^{b-1}\right\}\right]f_{1\mathcal{G}\mathcal{G}}(\mathcal{G}_0),\\
\mathbb{A}_{10} &= -6\mathcal{K}\rho_0(1+\omega)[\kappa^2(1+\varphi b[\rho_0(1-3\omega)]^{b-1})]\left[\frac{1}{2}\left(\kappa^2\left\{1-\frac{1}{2}\varphi b(\omega-3\right.\right.\right.\\
&- \left.\left.2(1+\omega)(b-1))[\rho_0(1-3\omega)]^{b-1}\right\}\right)-2\kappa^2\rho_0^2\hbar^2(1+\omega)^2[\kappa^2(1+\varphi b\\
&\times [\rho_0(1-3\omega)]^{b-1})]^2[1-3\omega+\varphi b\{2(3-5\omega)+(1+\omega)(b-1)\}\\
&\times \left.[\rho_0(1-3\omega)]^{b-1}]f_{1\mathcal{G}\mathcal{G}}(\mathcal{G}_0)\right],\\
\mathbb{A}_{11} &= 2\mathcal{K}^2\left[\left\{3\rho_0(1+\omega)[\kappa^2(1+\varphi b[\rho_0(1-3\omega)]^{b-1})]+\hbar^2-4\rho_0^2\hbar^4(1+\omega)^2\right.\right.\\
&\times \left.[\kappa^2(1+\varphi b[\rho_0(1-3\omega)]^{b-1})]^2f_{1\mathcal{G}\mathcal{G}}(\mathcal{G}_0)\right\}\left[\kappa^2\left\{1-\frac{1}{2}\varphi b(\omega-3\right.\right.\\
&- \left.\left.2(1+\omega)(b-1))[\rho_0(1-3\omega)]^{b-1}\right\}\right]-\kappa^2\left\{\frac{3}{2}\rho_0(1+\omega)[\kappa^2(1+\varphi b\right.
\end{aligned}$$



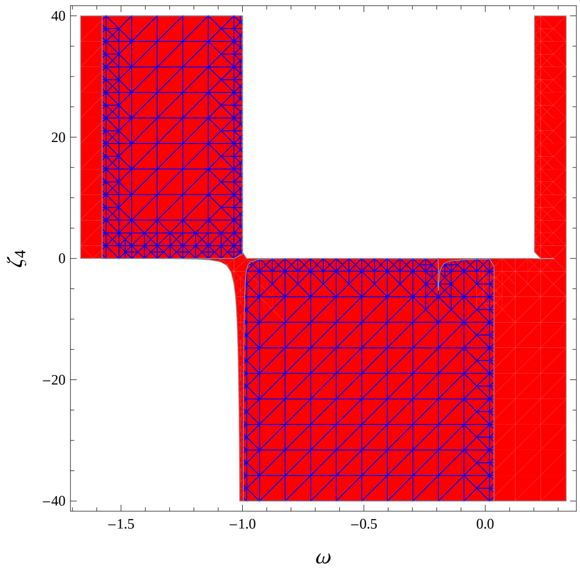


Figure 4.9: Regions of stability in (ω, ζ_4) space for $\mathcal{K} = 1$, $\psi = 2$ with $b = 2$ (left) and $b = 5$ (right).

$$\begin{aligned} &\times \quad [\rho_0(1-3\omega)]^{b-1})] + \hbar^2 - 4\rho_0^2\hbar^4(1+\omega)^2[\kappa^2(1+\varphi b[\rho_0(1-3\omega)]^{b-1})]^2 \\ &\times \quad f_{1_{\mathcal{G}\mathcal{G}}}(\mathcal{G}_0)\} \left[1 - 3\omega + \varphi b\{2(3-5\omega) + (1+\omega)(b-1)\}[\rho_0(1-3\omega)]^{b-1}\right]. \end{aligned}$$

Figures **4.9** and **4.10** show the stability regions of EU against inhomogeneous perturbations for frequencies $\Omega^2_{9,10}$. We have taken $\varphi = 1$ and $\mathcal{K} = 1$ with different values of $f(\mathcal{G}, T)$ model parameter b and inhomogeneous perturbation mode ψ. Red regions represent the existence of stable regimes for Ω^2_9 while the regions occupied by blue lines indicate Ω^2_{10}. It is observed that the region of stability for whole system decreases as the value of b increases while stability region increases with the increase in ψ. The graphical behavior of stable EU with $\mathcal{K} = -1$ is shown in Figures **4.11** and **4.12**. The effects of b and ψ on the stability plots are the same as for the closed universe.

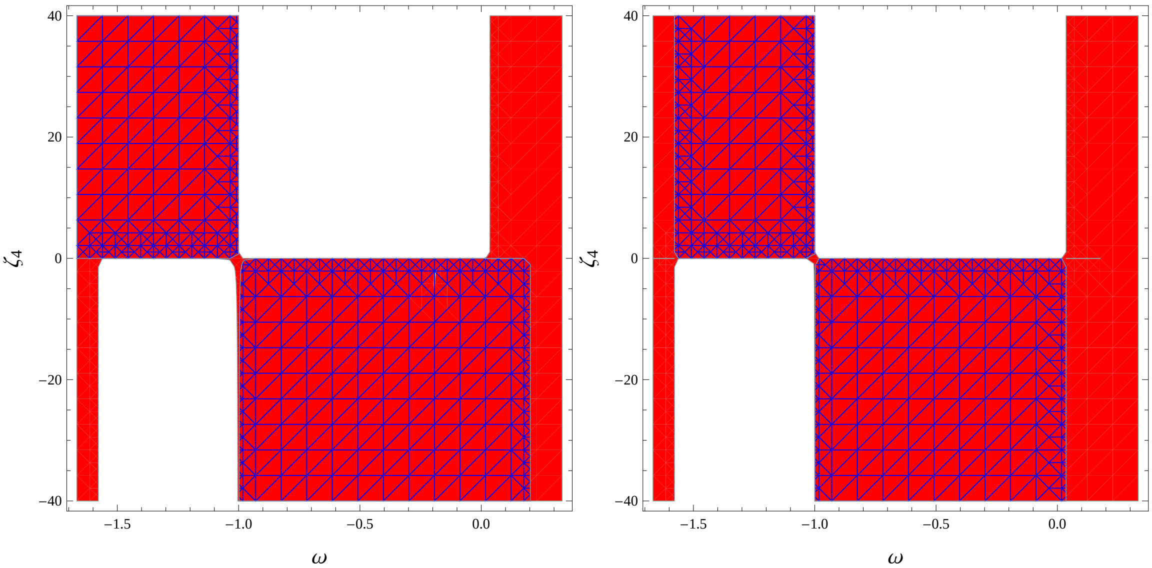


Figure 4.10: Regions of stability in (ω, ζ_4) space for $\mathcal{K} = 1,\ \psi = 15$ with $b = 2$ (left) and $b = 5$ (right).

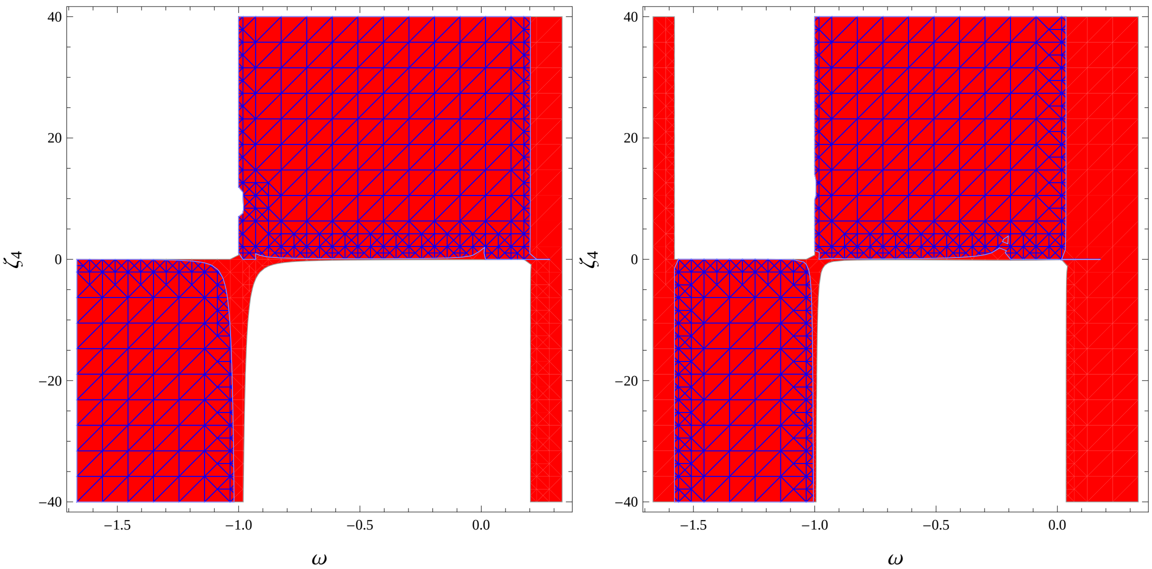


Figure 4.11: Regions of stability in (ω, ζ_4) space for $\mathcal{K} = -1,\ \psi^2 = 2$ with $b = 2$ (left) and $b = 5$ (right).

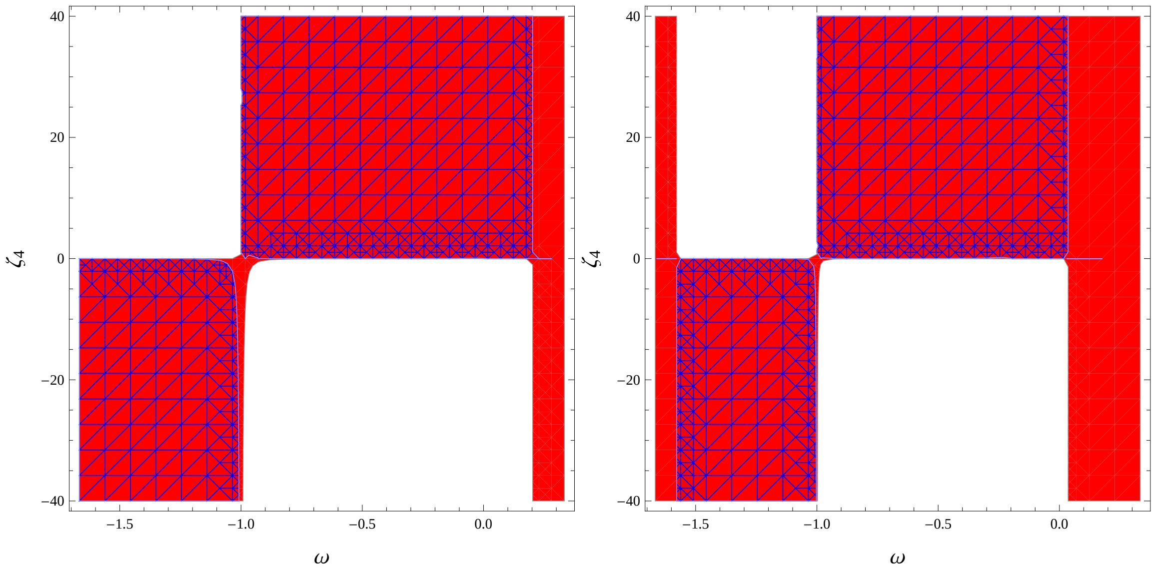


Figure 4.12: Regions of stability in (ω, ζ_4) space for $\mathcal{K} = -1$, $\psi^2 = 15$ with $b = 2$ (left) and $b = 5$ (right).

Chapter 5

Reconstruction and Stability

This chapter explores the cosmic evolution and analyzes the stability of some cosmological models against linear perturbations in $f(\mathcal{G},T)$ gravity. We apply the reconstruction technique to reproduce different evolutionary phases corresponding to de Sitter universe, power-law solutions as well as phantom/non-phantom eras in the context of FRW universe model. The perturbation scheme has been employed upto first order to investigate the stability of reconstructed models describing de Sitter as well as power-law cosmological background. Results of this chapter have been published [61].

The format of this chapter is as follows. Section **5.1** is devoted to reconstruct some well-known cosmological solutions for general as well as particular form of $f(\mathcal{G},T)$ function. In section **5.2**, we analyze the stability of reconstructed models against linear perturbations around FRW universe model.

5.1 Cosmological Reconstruction

In this section, we reconstruct some cosmological backgrounds such as de Sitter universe, power-law solution as well as phantom/non-phantom eras in $f(\mathcal{G},T)$ gravity. The cosmic history can also be discussed through red-shift parameter (z) instead of cosmic time. We rewrite the field as well as conservation equations in terms of new variable $\mathcal{N}$ known as e-folding which is related to z as [29]

$$\mathcal{N}=-\ln(1+z)=\ln(a/a_0).$$

Using the above definition of $\mathcal{N}$, Eqs.(3.1.7) and (3.1.10) become

$$\begin{aligned}
3H^2 &= \kappa^2\rho+\frac{1}{2}f(\mathcal{G},T)+\rho(1+\omega)f_T(\mathcal{G},T)-12H^3(H+H_{,\mathcal{N}})f_{\mathcal{G}}(\mathcal{G},T)\\
&+ 288H^6(HH_{,\mathcal{N}\mathcal{N}}+3H^2_{,\mathcal{N}}+4HH_{,\mathcal{N}})f_{\mathcal{G}\mathcal{G}}(\mathcal{G},T)\\
&+ 12H^4T_{,\mathcal{N}}f_{\mathcal{G}T}(\mathcal{G},T), \qquad (5.1.1)\\
\rho_{,\mathcal{N}}+3(1+\omega)\rho &= \frac{-1}{\kappa^2+f_T(\mathcal{G},T)}\left[\left(\omega\rho_{,\mathcal{N}}+\frac{1}{2}T_{,\mathcal{N}}\right)f_T(\mathcal{G},T)+\rho(1+\omega)\right.\\
&\times \left.\left(\mathcal{G}_{,\mathcal{N}}f_{\mathcal{G}T}(\mathcal{G},T)+T_{,\mathcal{N}}f_{TT}(\mathcal{G},T)\right)\right], \qquad (5.1.2)
\end{aligned}$$

where $H=d\mathcal{N}/dt$ and $d/dt=H(d/d\mathcal{N})$ and $H_{,\mathcal{N}}=dH/d\mathcal{N}$. The simplest choice of $f(\mathcal{G},T)$ model (4.2.10) splits the first field equation (5.1.1) into a set of two ordinary differential equations as

$$288H^6(HH_{,\mathcal{N}\mathcal{N}}+3H^2_{,\mathcal{N}}+4HH_{,\mathcal{N}})f_{1_{\mathcal{G}\mathcal{G}}}(\mathcal{G})-12H^3(H+H_{,\mathcal{N}})f_{1_{\mathcal{G}}}(\mathcal{G})+\frac{1}{2}f_1(\mathcal{G})-3H^2=0,$$

$$\rho(1+\omega)f_{2_T}(T)+\frac{1}{2}f_2(T)+\kappa^2\rho=0.$$

The field equations for perfect fluid matter distribution in $f(\mathcal{G})$ gravity is recovered if $f_2(T)$ vanishes while GR is achieved for $f(\mathcal{G},T)=0$.

5.1.1 de Sitter Universe

This cosmological model explains exponential expansion of the universe with constant Hubble expansion rate defined in Eq.(3.2.12). Equations (1.2.5) and (3.1.11) give energy density of the form

$$\rho=\rho_0 e^{-3(1+\omega)H_0t},\quad \omega\neq-1. \tag{5.1.3}$$

Using Eqs.(3.2.12) and (5.1.3) in the first field equation (3.1.7), we obtain

$$\begin{aligned}&\frac{1}{2}f(\mathcal{G}_0,T)-12H_0^4f_{\mathcal{G}}(\mathcal{G}_0,T)+\left(\frac{1+\omega}{1-3\omega}\right)Tf_T(\mathcal{G}_0,T)-36(1+\omega)H_0^4T\\ \times\ & f_{\mathcal{G}T}(\mathcal{G}_0,T)+\frac{\kappa^2T}{1-3\omega}-3H_0^2=0,\end{aligned} \tag{5.1.4}$$

whose solution is

$$\begin{aligned}f(\mathcal{G},T)&=c_1c_2e^{c_1\mathcal{G}}T^{-\frac{1}{2}\ \frac{(1-24c_1H_0^4)(1-3\omega)}{1+\omega-36c_1H_0^4(1-3\omega)}}+c_1c_2T^{-\frac{1}{2}}\left(\frac{1-3\omega}{1+\omega}\right)\\ &-\frac{2\kappa^2}{3-\omega}T+6H_0^2.\end{aligned} \tag{5.1.5}$$

It is worth mentioning here that this equation reduces to Eq.(3.2.15) for pressureless matter configuration. Since we have used the continuity equation (3.1.11) in Eq.(5.1.4), so we must constrain its solution. Solving the above equation with Eq.(3.1.12), we obtain the following two functions

$$f_3(\mathcal{G},T)=c_1c_2\mathcal{A}_1e^{c_1\mathcal{G}}T^{-\frac{1}{2}\ \frac{(1-24c_1H_0^4)(1-3\omega)}{1+\omega-36c_1H_0^4(1-3\omega)}}+\frac{2\kappa^2\omega}{1-3\omega}T+6H_0^2, \tag{5.1.6}$$

$$f_4(\mathcal{G},T)=c_1c_2\mathcal{A}_2T^{-\frac{1}{2}\left(\frac{1-3\omega}{1+\omega}\right)}+\frac{2\kappa^2}{3-\omega}\mathcal{A}_3T+6H_0^2, \tag{5.1.7}$$

where $\mathcal{A}_i$'s are constants in terms of H_0 and ω provided in Appendix **B**. For the model (4.2.10), we have

$$3H_0^2-\frac{1}{2}f_1(\mathcal{G}_0)+12H_0^4f_{1_{\mathcal{G}}}(\mathcal{G})=0,\quad \kappa^2\rho+\frac{1}{2}f_2(T)+(1+\omega)\rho f_{2_T}(T)=0, \tag{5.1.8}$$

where the first equation corresponds to de Sitter universe in the absence of matter contents in $f(\mathcal{G})$ gravity [3]. Using the constraint (3.1.12), the second differential equation takes the form

$$\kappa^2(1-\omega)T+\frac{1}{2}(1-3\omega)(1-\omega)f_2(T)-2(1+\omega)^2T^2f_{2_{TT}}(T)=0. \qquad (5.1.9)$$

The solution of Eqs.(5.1.8) and (5.1.9) gives

$$\begin{aligned} f(\mathcal{G},T) &= \hat{c}_1e^{\frac{\mathcal{G}}{24H_0^4}}+\hat{c}_2T^{\frac{1}{2}\left(1+\frac{\sqrt{2(1-\omega+2\omega^2)}}{1+\omega}\right)}+\hat{c}_3T^{\frac{1}{2}\left(1-\frac{\sqrt{2(1-\omega+2\omega^2)}}{1+\omega}\right)} \\ &- \frac{2\kappa^2T}{1-3\omega}+6H_0^2, \end{aligned} \qquad (5.1.10)$$

where $\hat{c}_i$'s are integration constants. Equations (5.1.5) and (5.1.10) indicate that de Sitter expansion background can be reproduced in $f(\mathcal{G},T)$ gravity.

5.1.2 Power-law Solutions

In this case, the first field equation (3.1.7) becomes

$$\begin{aligned} &\frac{1}{2}f(\mathcal{G},T)-\frac{1}{2}\mathcal{G}f_{\mathcal{G}}(\mathcal{G},T)+\frac{(1+\omega)T}{1-3\omega}f_T(\mathcal{G},T)-\frac{2}{\lambda-1}\mathcal{G}^2f_{\mathcal{G}\mathcal{G}}(\mathcal{G},T)+\frac{\kappa^2T}{1-3\omega} \\ -&\frac{2\lambda(1+\omega)\mathcal{G}T}{2(\lambda-1)}f_{\mathcal{G}T}(\mathcal{G},T)-3\lambda^2\left(\frac{T}{\rho_0(1-3\omega)}\right)^{\frac{2}{3\lambda(1+\omega)}}=0, \end{aligned} \qquad (5.1.11)$$

where we have used Eqs.(3.1.11) and (3.2.16). The solution of this differential equation is given by

$$f(\mathcal{G},T)=d_1d_3T^{d_2}\mathcal{G}^{\frac{1}{4}(\varrho_1+\varrho_2)}+d_2d_3T^{d_2}\mathcal{G}^{\frac{1}{4}(\varrho_1-\varrho_2)}+d_1d_2T^{\varrho_3}+\varrho_4T+\varrho_5T^{\varrho_6}, \quad (5.1.12)$$

where the constants $\varrho_{\hat{k}}$'s ($\hat{k}=1...6$) are given in Appendix **B**. Inserting Eq.(5.1.12) in (3.1.12), we obtain

$$f_5(\mathcal{G},T) = d_1d_3\mathcal{B}_1T^{d_2}\mathcal{G}^{\frac{1}{4}(\varrho_1+\varrho_2)}+d_1d_2\mathcal{B}_2T^{\varrho_3}+\mathcal{B}_3T+\mathcal{B}_4T^{\varrho_6}, \qquad (5.1.13)$$

$$f_6(\mathcal{G},T) \;=\; d_2d_3\mathcal{R}_1T^{d_2}\mathcal{G}^{\frac{1}{4}(\varrho_1-\varrho_2)}+d_1d_2\mathcal{R}_2T^{\varrho_3}+\mathcal{R}_3T+\mathcal{R}_4T^{\varrho_6}. \quad (5.1.14)$$

where $\mathcal{B}_k$'s and $\mathcal{R}_k$'s are mentioned in Appendix **B**. Now we find the expression of $f(\mathcal{G},T)$ for the choice of model (4.2.10). The differential equation (5.1.11) splits into two ordinary differential equations given by

$$f_1(\mathcal{G})-\mathcal{G}f_{1_{\mathcal{G}}}(\mathcal{G})-\frac{4}{\lambda-1}\mathcal{G}^2f_{1_{\mathcal{G}\mathcal{G}}}(\mathcal{G})=0,$$
$$f_2(T)-\frac{4(1+\omega)^2}{(1-\omega)(1-3\omega)}T^2f_{2_{TT}}(T)+\frac{2\kappa^2T}{1-3\omega}-6\lambda^2\left(\frac{T}{\rho_0(1-3\omega)}\right)^{\frac{2}{3\lambda(1+\omega)}}=0.$$

The solution of these equations provide $f(\mathcal{G},T)$ model as

$$\begin{aligned} f(\mathcal{G},T) &= \tilde{c}_1\mathcal{G}+\tilde{c}_2\mathcal{G}^{\frac{1-\lambda}{4}}+\tilde{c}_3T^{\frac{1}{2}\left(1+\frac{\sqrt{2(1-\omega+2\omega^2)}}{1+\omega}\right)}+\tilde{c}_4T^{\frac{1}{2}\left(1-\frac{\sqrt{2(1-\omega+2\omega^2)}}{1+\omega}\right)}\\ &- \frac{2\kappa^2T}{1-3\omega}+\frac{54\lambda^4(1-\omega)(1-3\omega)}{9\lambda^2(1-\omega)(1-3\omega)-8[2-3\lambda(1+\omega)]}\\ &\times \left(\frac{T}{\rho_0(1-3\omega)}\right)^{\frac{2}{3\lambda(1+\omega)}}, \end{aligned} \quad (5.1.15)$$

where $\tilde{c}_k$'s are constants of integration. Thus, the power-law solutions are reconstructed which may be helpful to discuss the cosmic background in $f(\mathcal{G},T)$ gravity.

5.1.3 Phantom and non-Phantom Cosmic Eras

Here, we reconstruct $f(\mathcal{G},T)$ model which can explain the system including both phantom as well as non-phantom cosmic eras. In GR, the Hubble parameter describing the phantom and non-phantom matter configuration is given by [29]

$$H^2=\frac{\kappa^2}{3}(\rho_pa^{\hat{b}}+\rho_qa^{-\hat{b}}), \quad (5.1.16)$$

where $\hat{b}$, ρ_p and ρ_q represent the model parameter, energy densities of phantom and non-phantom matter distributions, respectively. When $a(t)$ is large, the first term on right hand side dominates which corresponds to the phantom era of the universe with $\omega=-1-\hat{b}/3<-1$. The non-phantom era is observed for $\omega=-1+\hat{b}/3>-1$ when $a(t)$ is small and the second term dominates. We rewrite $H(t)$ in terms of a new function $S(\mathcal{N})$ as $H^2=S(\mathcal{N})$ so that Eq.(5.1.16) becomes

$$S(\mathcal{N})=S_pe^{\hat{b}\mathcal{N}}+S_qe^{-\hat{b}\mathcal{N}}, \tag{5.1.17}$$

where $S_p=\frac{\kappa^2}{3}\rho_pa_0^{\hat{b}}$ and $S_q=\frac{\kappa^2}{3}\rho_qa_0^{-\hat{b}}$. The GB invariant takes the form

$$\mathcal{G}=24S^2(\mathcal{N})+12S(\mathcal{N})S_{,\mathcal{N}}(\mathcal{N}). \tag{5.1.18}$$

Substituting Eq.(5.1.17) in (5.1.18), we obtain a quadratic equation in $e^{2\hat{b}\mathcal{N}}$ whose solution is given by

$$e^{2\hat{b}\mathcal{N}}=\frac{-(48S_pS_q-\mathcal{G})\pm\sqrt{(48S_pS_q-\mathcal{G})^2-576(4-\hat{b}^2)S_p^2S_q^2}}{24(2+\hat{b})S_p^2},\quad \hat{b}\neq 2.$$

For the sake of simplicity, we consider $\hat{b}=2$ so that it reduces to

$$e^{2\hat{b}\mathcal{N}}=\frac{\mathcal{G}-48S_pS_q}{48S_p^2}. \tag{5.1.19}$$

Using Eqs.(5.1.17) and (5.1.19) in (3.1.7), we obtain

$$\begin{aligned}&\frac{1}{2}f(\mathcal{G},T)-\frac{1}{2}\mathcal{G}f_{\mathcal{G}}f(\mathcal{G},T)+\left(\frac{1+\omega}{1-3\omega}\right)Tf_Tf(\mathcal{G},T)+\mathcal{G}^2f_{\mathcal{G}\mathcal{G}}f(\mathcal{G},T)\\ &-\frac{3(1+\omega)\mathcal{G}^2T}{4(\mathcal{G}-48S_pS_q)}f_{\mathcal{G}T}f(\mathcal{G},T)-\frac{1}{4}\sqrt{\frac{3\mathcal{G}^2}{\mathcal{G}-48S_pS_q}}+\frac{\kappa^2T}{1-3\omega}=0,\end{aligned}$$

which is a complicated partial differential equation whose analytical solution cannot be found. To find the reconstructed $f(\mathcal{G},T)$ model, we consider Eq.(4.2.10) which gives the following set of differential equations

$$\frac{1}{2}f_1(\mathcal{G})-\frac{1}{2}\mathcal{G}f_{1_{\mathcal{G}}}(\mathcal{G})+\mathcal{G}^2f_{1_{\mathcal{G}\mathcal{G}}}(\mathcal{G})-\frac{1}{4}\sqrt{\frac{3\mathcal{G}^2}{\mathcal{G}-48S_pS_q}}=0,$$

$$f_2(T) - \frac{4(1+\omega)^2}{(1-\omega)(1-3\omega)}T^2 f_{2_{TT}}(T) + \frac{2\kappa^2 T}{1-3\omega} = 0,$$

where we have used the additional constraint (3.1.12) in the second equation. The solution of these equations leads to

$$\begin{aligned} f(\mathcal{G},T) &= \tilde{d}_1\mathcal{G}^{\frac{1}{2}} + \tilde{d}_2\mathcal{G} + \tilde{d}_3 T^{\frac{1}{2}\left(1+\frac{\sqrt{2(1-\omega+2\omega^2)}}{1+\omega}\right)} + \tilde{d}_4 T^{\frac{1}{2}\left(1-\frac{\sqrt{2(1-\omega+2\omega^2)}}{1+\omega}\right)} \\ &+ \frac{1}{4\sqrt{S_pS_q}}\left[\mathcal{G}\tan^{-1}\left(\frac{1}{12}\sqrt{\frac{3(\mathcal{G}-S_pS_q)}{S_pS_q}}\right) - 2\sqrt{3S_pS_q\mathcal{G}}\right. \\ &\times \left.\ln\left(\mathcal{G} - 24S_pS_q + \sqrt{\mathcal{G}(\mathcal{G}-48S_pS_q)}\right)\right] - \frac{2\kappa^2 T}{1-3\omega}. \end{aligned} \tag{5.1.20}$$

Thus, phantom and non-phantom cosmic expansion history can be discussed in this gravity.

5.2 Stability Analysis

In this section, we analyze stability of some cosmic evolutionary solutions about linear homogeneous perturbations in this modified gravity. We assume a general solution

$$H(t) = H_*(t), \tag{5.2.1}$$

which satisfies the basic field equations for FRW universe model in $f(\mathcal{G},T)$ gravity. In terms of the above solution, the expressions for $\mathcal{G}_*$ and T_* are

$$\mathcal{G}_* = 24H_*^2(H_*^2 + \dot{H}_*) = 24H_*^3(H_* + H_{*,_\mathcal{N}}), \quad T_* = \rho_*(t)(1-3\omega).$$

For any particular $f(\mathcal{G},T)$ model that can regenerate the above solution (5.2.1), the following equation of motion as well as non-zero covariant divergence of the energy-momentum tensor must be satisfied

$$3H_*^2 = \kappa^2\rho_* + (1+\omega)\rho_* f_T^* + \frac{1}{2}f^* - 12H_*^3(H_* + H_{*,_\mathcal{N}})f_\mathcal{G}^* + 288$$

$$\begin{aligned}
&\times \quad H_*^6(H_*H_{*,_{\mathcal{N}\mathcal{N}}}+3H^2_{*,_{\mathcal{N}}}+4H_*H_{*,_{\mathcal{N}}})f^*_{\mathcal{G}\mathcal{G}}+12H_*^4T_{*,_{\mathcal{N}}}f^*_{\mathcal{G}T},\\
\rho_{*,_{\mathcal{N}}}+3(1+\omega)\rho_* &= \frac{-1}{\kappa^2+f^*_T}\left[\frac{1}{2}(T_{*,_{\mathcal{N}}}+2\omega\rho_{*,_{\mathcal{N}}})f^*_T+(1+\omega)\rho_*\left(\mathcal{G}_{*,_{\mathcal{N}}}f^*_{\mathcal{G}T}\right.\right.\\
&+ \left.\left.T_{*,_{\mathcal{N}}}f^*_{TT}\right)\right],
\end{aligned}$$

where superscript $*$ denotes that the function $f(\mathcal{G},T)$ and its corresponding derivatives are calculated at $\mathcal{G}=\mathcal{G}_*$ and $T=T_*$. If the conservation law holds, we obtain energy density in terms of $H_*(t)$ as

$$\rho_*(t)=\rho_0e^{-3(1+\omega)\int H_*(t)dt}.$$

The first order perturbations in Hubble parameter and energy density are defined as

$$H(t)=H_*(t)(1+\delta(t)),\quad \rho(t)=\rho_*(t)(1+\delta_m(t)), \tag{5.2.2}$$

where $\delta(t)$ and $\delta_m(t)$ are the perturbation parameters.

In order to analyze first order perturbations about the solution (5.2.1), we apply the series expansion on the function $f(\mathcal{G},T)$ as

$$f(\mathcal{G},T)=f^*+f^*_{\mathcal{G}}(\mathcal{G}-\mathcal{G}_*)+f^*_T(T-T_*)+\mathcal{O}^2, \tag{5.2.3}$$

where $\mathcal{O}^2$ involves the terms proportional to quadratic or higher powers of $\mathcal{G}$ and T while only the linear terms are considered. Using Eqs.(5.2.2) and (5.2.3) in (3.1.7), we obtain the following perturbed field equation

$$\mathfrak{X}_1\ddot{\delta}+\mathfrak{X}_2\dot{\delta}+\mathfrak{X}_3\delta+\mathfrak{X}_4\dot{\delta}_m+\mathfrak{X}_5\delta_m=0, \tag{5.2.4}$$

where $\mathfrak{X}_h$'s ($h=1...5$) are given in Appendix **B**. Inserting these perturbations in Eq.(3.1.10), the perturbed continuity equation is

$$\mathfrak{U}_1\delta+\mathfrak{U}_2\dot{\delta}+\mathfrak{U}_3\ddot{\delta}+\mathfrak{U}_4\delta_m+\mathfrak{U}_5\dot{\delta}_m=0, \tag{5.2.5}$$

where $\mathfrak{U}_h$'s are provided in Appendix **B**. If the conversation law holds in this modified gravity, Eq.(5.2.5) reduces to

$$\dot{\delta}_m + 3(1+\omega)H_*\delta = 0. \tag{5.2.6}$$

The perturbed equations (5.2.4) and (5.2.5) are helpful to analyze the stability of any specific FRW cosmological evolutionary model in $f(\mathcal{G}, T)$ gravity. For the particular model (4.2.10), these perturbed equations reduce to

$$\hat{\mathfrak{X}}_1\ddot{\delta} + \hat{\mathfrak{X}}_2\dot{\delta} + \hat{\mathfrak{X}}_3\delta + \hat{\mathfrak{X}}_5\delta_m = 0,$$
$$\hat{\mathfrak{U}}_1\delta + \hat{\mathfrak{U}}_4\delta_m + \hat{\mathfrak{U}}_5\dot{\delta}_m = 0,$$

where the coefficients of $(\delta,\ \delta_m)$ and their derivatives are provided in Appendix **B**. In the following subsections, we investigate the stability of de Sitter and power-law solutions.

5.2.1 de Sitter Universe Models

Consider the de Sitter solution $H_*(t) = H_0$, the perturbed equation (5.2.4) takes the form

$$\begin{aligned}
& 288H_0^6 f_{\mathcal{G}\mathcal{G}}^0\ddot{\delta} + \big(864H_0^7 f_{\mathcal{G}\mathcal{G}}^0 + 24\rho_* H_0^3(1+\omega)f_{\mathcal{G}T}^0 - 864\rho_* H_0^7(1-3\omega) \\
\times\ & (1+\omega)f_{\mathcal{G}\mathcal{G}T}^0\big)\,\dot{\delta} + \big(-6H_0^2 - 1152H_0^8 f_{\mathcal{G}\mathcal{G}}^0 + 12\rho_* H_0^2(1+\omega)[8H_0^2 \\
-\ & 9H_0^2(1-3\omega)]f_{\mathcal{G}T}^0 - 3456\rho_* H_0^8(1-3\omega)(1+\omega)f_{\mathcal{G}\mathcal{G}T}^0\big)\,\delta + 12\rho_* H_0^3 \\
\times\ & (1-3\omega)f_{\mathcal{G}T}^0\dot{\delta}_m + \bigg(\kappa^2\rho_* + \frac{1}{2}\rho_*(3-\omega)f_T^0 + \rho_*^2(1-3\omega)(1+\omega)f_{TT}^0 \\
-\ & 12\rho_* H_0^2(1-3\omega)[H_0^2 + 3(1+\omega)H_0^2]f_{\mathcal{G}T}^0 - 36\rho_*^2 H_0^4(1-3\omega)^2(1+\omega) \\
\times\ & f_{\mathcal{G}TT}^0\big)\,\delta_m = 0,
\end{aligned} \tag{5.2.7}$$

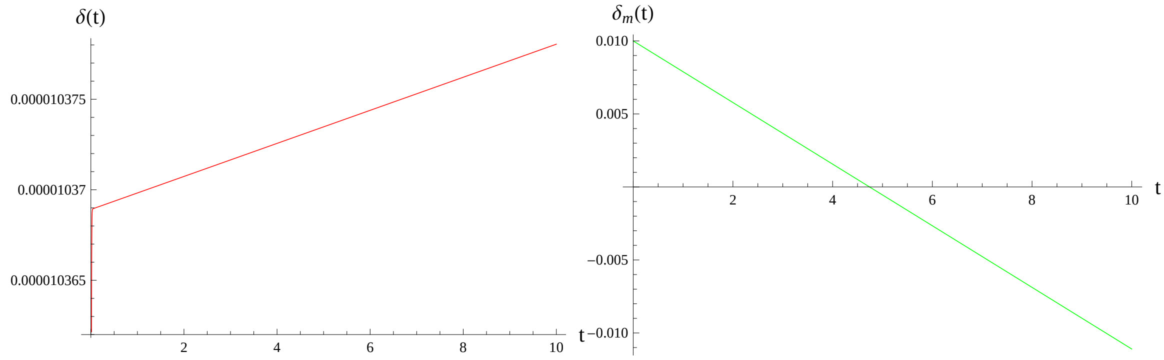


Figure 5.1: Evolution of perturbations $\delta(t)$ and $\delta_m(t)$ for model (5.1.6) with $\omega = 0$.

where the superscript 0 represents that the function $f(\mathcal{G}, T)$ and its corresponding derivatives are evaluated at $\mathcal{G}_0$ and T_0. We consider the conserved perturbed equation for stability analysis since the de Sitter solutions are constructed using the relation (3.1.10) in the previous section. The numerical technique is used to solve Eqs.(5.2.6) and (5.2.7) for the model (5.1.6). The evolution of $\delta(t)$ and $\delta_m(t)$ are shown in Figure **5.1**. We consider $H_0 = 67.8$ and $\kappa^2 = 1$ throughout the stability analysis of de Sitter universe models whereas integration constants are $c_1 = 1 \times 10^{-6}$ and $c_2 = -1 \times 10^{-3}$. Figure **5.1** shows smooth behavior of $\delta(t)$ (left) and $\delta_m(t)$ (right) which do not decay in late times indicating that de Sitter model (5.1.6) is unstable.

The stability analysis of model (5.1.7) with same integration constants is shown in Figure **5.2**. In the left panel, it is observed that small oscillations are produced about $t = 4$ while it decays in late times, thus the model (5.1.7) shows stable behavior against perturbations. For model (4.2.10), Eq.(5.2.7) becomes

$$\begin{aligned} &288H_0^6 f^0_{1_{\mathcal{G}\mathcal{G}}}\ddot{\delta} + 864H_0^7 f^0_{1_{\mathcal{G}\mathcal{G}}}\dot{\delta} - \left(6H_0^2 + 1152H_0^8 f^0_{1_{\mathcal{G}\mathcal{G}}}\right)\delta \\ +\ &\left(\kappa^2\rho_* + \frac{1}{2}\rho_*(3-\omega)f^0_{2_T} + \rho_*^2(1-3\omega)(1+\omega)f^0_{2_{TT}}\right)\delta_m = 0, \qquad (5.2.8) \end{aligned}$$

Figure **5.3** represents the behavior of $\delta(t)$ and $\delta_m(t)$ for model (5.1.10) with inte-

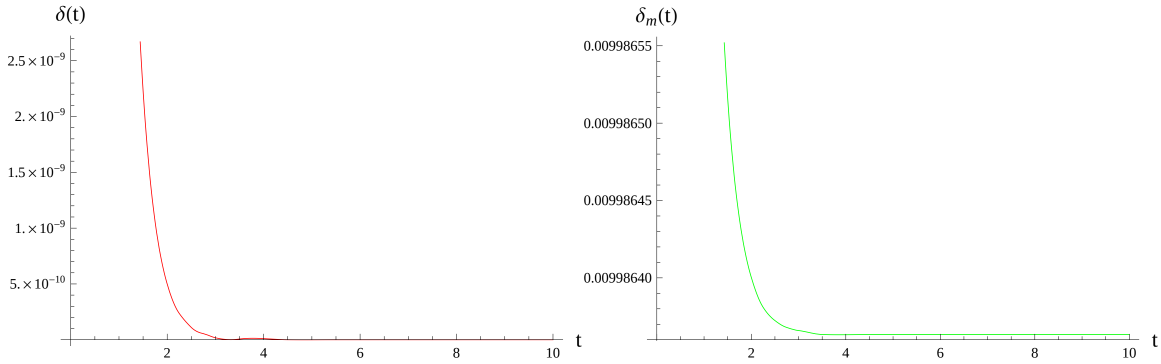


Figure 5.2: Evolution of perturbations $\delta(t)$ and $\delta_m(t)$ for model (5.1.7) with $\omega = 0$.

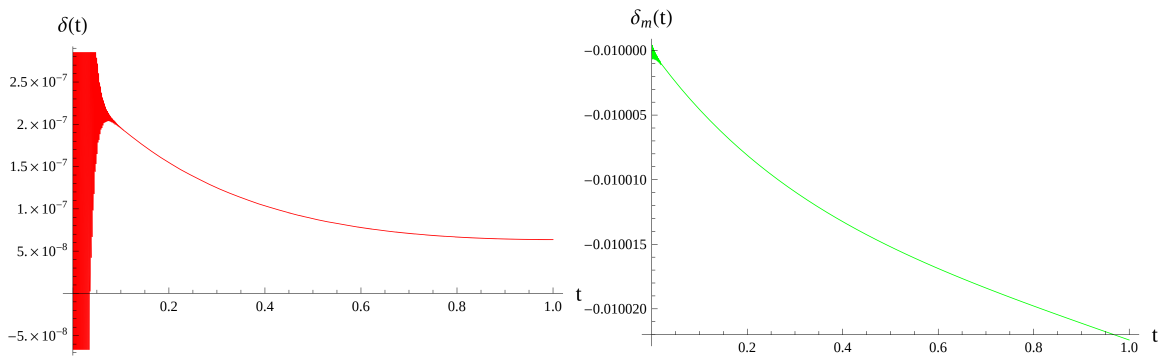


Figure 5.3: Evolution of perturbations $\delta(t)$ and $\delta_m(t)$ for model (5.1.10) with $\omega = 0$.

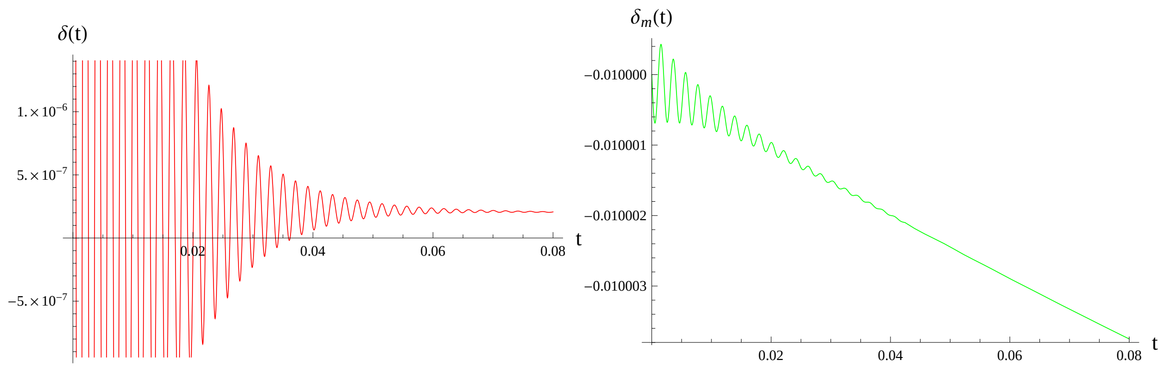


Figure 5.4: Evolution of perturbations $\delta(t)$ and $\delta_m(t)$ for model (5.1.10) with $\omega = 0$.

gration constants $\hat{c}_1 = -10,\ \hat{c}_2 = 0.001$ and $\hat{c}_3 = 1$. It is observed that oscillations in perturbation parameters are produced initially. This oscillating behavior is clearly observed in Figure **5.4** which decays in future for both $\delta(t)$ as well as $\delta_m(t)$ and hence the solution becomes stable.

5.2.2 Stability of Power-law Solutions

Here we investigate the stability of power-law solutions. We first consider the reconstructed power-law solution (5.1.13) and numerically solve Eqs.(5.2.4) and (5.2.6). For this model, we choose integration constants $d_1 = 10,\ d_2 = -0.5$ and $d_3 = -1000$. Figure **5.5** shows the oscillating behavior of perturbed parameters $(\delta(t),\ \delta_m(t))$ for the cosmic accelerated era. The perturbations around the power-law solutions decay in future leading to stable results. The radiation as well as matter dominated eras cannot be discussed for the model (5.1.13) because singular as well as complex terms appear which lead to non-physical case.

Secondly, we consider the model (5.1.14) and analyze its behavior against linear perturbations. Figure **5.6** shows the fluctuating behavior of considered perturbations in the cosmic accelerated phase for $d_1 = 0.001,\ d_2 = -0.61$ and $d_3 = -1000$. It is observed that the oscillating behavior disappears in future while both perturbation parameters will not decay in late times leading to unstable cosmological solutions. The considered model cannot explain the cosmological evolution corresponding to matter and radiation dominated eras like previous model (5.1.13). Lastly, we explore the stability of model (5.1.15) with integration constants $\tilde{c}_1 = -2,\ \tilde{c}_2 = -0.6,\ \tilde{c}_3 = 1000$ and $\tilde{c}_4 = 0.01$. Figure **5.7** represents the evolution of $(\delta,\ \delta_m)$ versus time in which the left panel shows that the oscillations of $\delta(t)$ decay in late times while fluctuations of

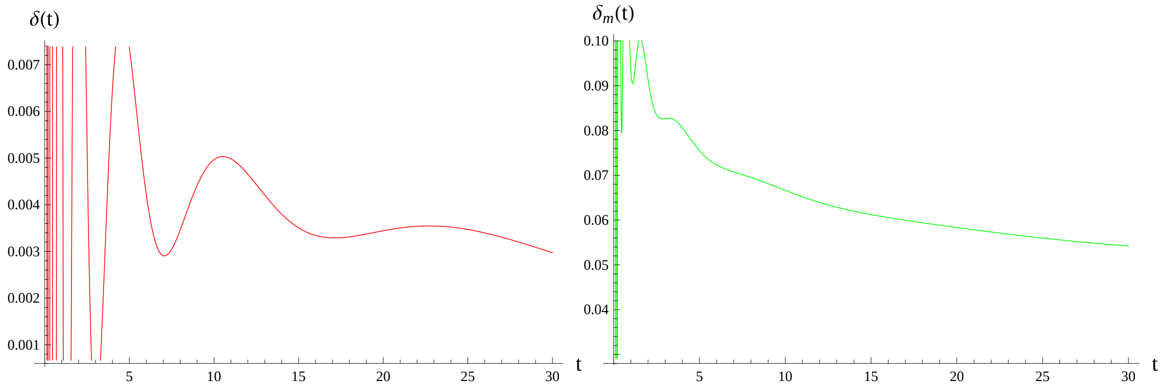


Figure 5.5: Evolution of perturbations $\delta(t)$ and $\delta_m(t)$ for model (5.1.13) with $\omega = -0.5$ and $\lambda = 2$.

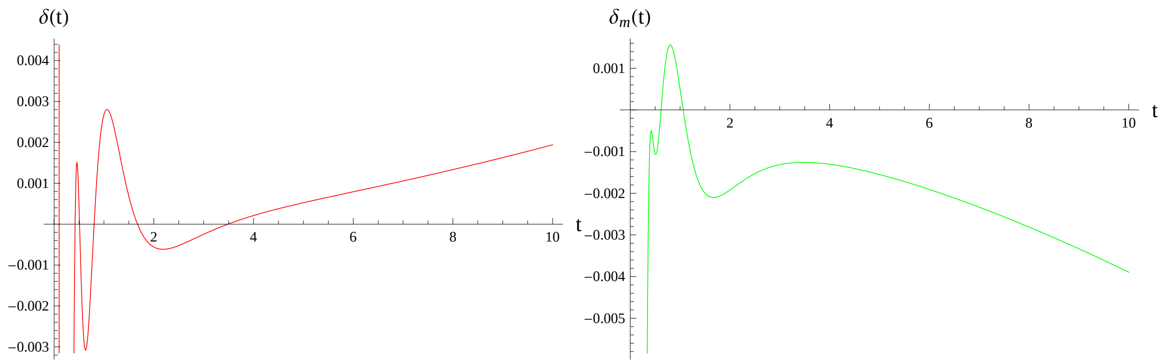


Figure 5.6: Evolution of perturbations $\delta(t)$ and $\delta_m(t)$ for model (5.1.14) with $\omega = -0.5$ and $\lambda = 2$.

$\delta_m(t)$ remain present in future indicating that power-law model (5.1.15) is unstable.

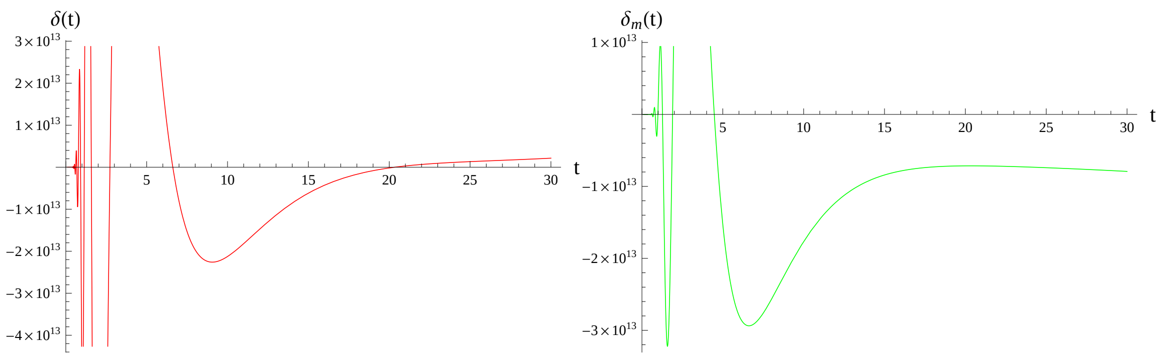


Figure 5.7: Evolution of perturbations $\delta(t)$ and $\delta_m(t)$ for model (5.1.15) with $\omega = -0.5$ and $\lambda = 1.1$.

Chapter 6

Concluding Remarks

In this thesis, we have discussed dynamical instability for spherical star filled with anisotropic fluid evolving under expansion-free condition in $f(\mathcal{G})$ gravity. We have presented a new framework to explore the cosmic evolution as $f(\mathcal{G},T)$ theory of gravity. We have investigated the effects of non-minimal curvature-matter coupling on test particles. The energy constraints have been studied in the presence of pressureless fluid for two reconstructed $f(\mathcal{G},T)$ models. We have analyzed the stability regions for EU against homogeneous as well as inhomogeneous perturbations. Some cosmic evolutionary backgrounds have been reproduced and examined their stability against linear perturbations. In the following, we summarize and briefly discuss the main results of this thesis.

In chapter **TWO**, we have investigated the problem of instability in the background of $f(\mathcal{G})$ gravity for expansion-free spherical anisotropic fluid configuration. We have applied the perturbation approach upto first order to all metric and matter variables. The field as well as dynamical equations are formulated for both static and non-static configurations. We have characterized the terms belonging to N, pN and parameterized pN approximations by expanding Eq.(2.3.5) upto order $\mathcal{C}^{-4}$. The instability of expansion-free fluid has been discussed by Eq.(2.3.9) upto pN approximation. We have found that instability of spherical matter distribution under expansion-free condition depends on metric functions, matter variables and higher order curvature terms but is independent of Γ. The absence of Γ shows that range of instability has not been affected by fluid stiffness. For constant GB invariant, the inequality $r > r_{\Sigma(i)}$ appears in N regime due to $f(\mathcal{G})$ gravity. This range of instability confirms the presence of vacuum cavity within the fluid configuration which indicates compatibility of our results with expansion-free scenario.

In chapter **THREE**, we have investigated energy constraints in the context of FRW universe model filled with dust fluid for the two reconstructed $f(\mathcal{G},T)$ models. The results are summarized as follows.

- For de Sitter reconstructed model, the energy bounds depend on three parameters t, c_1 and c_2. We have plotted NEC and WEC against t and c_2 with four possible signatures of c_1 and c_2 (Figures **3.1-3.4**). It is found that NEC and WEC are satisfied for $c_1 > 0$ and $c_2 < 0$ throughout the time interval while for cases $(c_1, c_2) > 0$ and $(c_1, c_2) < 0$, energy conditions are satisfied for small values of c_i's in a very small time interval. It is observed that NEC shows positively increasing behavior for all negative values of c_1 with $c_2 > 0$ while the validity ranges of WEC have dependence on c_1.

- For power-law reconstructed model, we have explored the behavior of four parameters t, d_1, d_2 and d_3. We have taken $-10 \leq d_1 \leq 10$ and found the valid regions where energy conditions are satisfied for appropriately chosen values of remaining parameters.

Chapter **FOUR** explores the stability of EU against homogeneous as well as inhomogeneous scalar perturbations in the presence of perfect fluid in $f(\mathcal{G},T)$ gravity. We have considered specific $f(\mathcal{G},T)$ form (4.2.10) and constructed the fourth-order perturbed differential equation whose solution provides four frequencies to analyze the stability of EU. The stability analysis is carried out for both zero as well as non-zero divergence of the energy-momentum tensor. The results are summarized as follows.

- For conserved case, the stable EU exists for all considered positive values of integration constant C_1 while no stable region exists for $C_1 = -1$.
- For non-conserved case, the stable regions exist for all considered values of model parameter φ with $b = 1$.

In case of inhomogeneous perturbations, we have obtained the following results

- For conserved case, the stable solutions are found for both closed and open cosmic geometries for all considered values of parameters. The stability of EU increases as the value of inhomogeneous mode ψ increases for closed universe while it decreases for open geometry of the universe as ψ^2 increases.
- For non-conserved case, the region of stability decreases as the values of b increases while stability regime increases with the increase in inhomogeneous mode for closed as well as open geometries of the universe.

In chapter **FIVE**, we have employed the reconstruction scheme to $f(\mathcal{G},T)$ gravity in the background of FRW universe model to reproduce some important cosmological models and have analyzed the stability of these reconstructed models. The results are summarized in Table **1**. In this table, ✓ and $\times$ represent that $f(\mathcal{G},T)$ gravity reproduces and fails to reproduce the corresponding cosmological backgrounds, respectively.

Table 1: Cosmological evolution in $f(\mathcal{G},T)$ gravity.

Cosmological Backgrounds	**General $f(\mathcal{G},T)$ Model**	**Particular $f(\mathcal{G},T)$ Model**
de Sitter Universe	✓	✓
Power-law Solutions	✓	✓
Phantom/non-Phantom Eras	×	✓

In order to analyze the stability of $f(\mathcal{G},T)$ models which reproduce de Sitter and power-law cosmic history, we have applied the perturbations to $H(t)$ and $\rho(t)$ upto first order. The perturbed configuration of field equation as well as conservation law are constructed whose numerical solutions provide stable/unstable results.

- For the de Sitter universe, the evolution of perturbation indicate that models (5.1.7) and (5.1.10) are stable against linear perturbations.
- For the power-law universe, the stability analysis shows that $f(\mathcal{G},T)$ gravity fails to reproduce matter and radiation dominated eras while stable results are obtained for accelerated phase of the universe for model (5.1.13).

There are many open issues that need to be resolved in the newly developed $f(\mathcal{G},T)$ gravity. It would be interesting

- To explore ghost instabilities due to the presence of curvature-matter coupling.
- To construct viable models satisfying the solar system constraints.
- To discuss the dynamics of self-gravitating system with and without expansion-free condition as well as determine the role of electromagnetic field.

- To investigate the stability of EU against vector, tensor as well as anisotropic perturbations.

- To study different cosmic eras corresponding to anisotropic solutions.

- To discuss black hole thermodynamics.

Appendix A

The dark source terms in Eqs.(2.1.10) and (2.1.11) are

$$
\begin{aligned}
\Xi_1 &= \frac{1}{\kappa^2 A}\left[\frac{1}{A^2}\left\{\frac{1}{2}A^2 f(\mathcal{G})+\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left[-\frac{4A}{BC^2}\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right.\right.\right.\right.\\
&- \left.\left.\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\} f_{\mathcal{G}}(\mathcal{G})+\frac{4\dot{B}}{BC^2}\dot{f}_{\mathcal{G}}(\mathcal{G})+\frac{4A^2B'}{B^3C^2}f'_{\mathcal{G}}(\mathcal{G})-\frac{4A^2}{B^2C^2}f''_{\mathcal{G}}(\mathcal{G})\right]\\
&+ \left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)\left(-\frac{8\dot{C}}{BC^2}\dot{f}_{\mathcal{G}}(\mathcal{G})-\frac{8A^2C'}{B^3C^2}f'_{\mathcal{G}}(\mathcal{G})\right)-\frac{4Af_{\mathcal{G}}(\mathcal{G})}{BC^2}\\
&\times \left[\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\right.\\
&\times \left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)\\
&+ \left.\left.\left.\frac{2}{AB}\left(\dot{C'}^2-\ddot{C}C''\right)\right]\right\}_{,0}+\frac{1}{B^2}\left\{\frac{2}{C}\left[\frac{2}{C}\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left(\dot{f}'_{\mathcal{G}}(\mathcal{G})-\frac{A'}{A}\dot{f}_{\mathcal{G}}(\mathcal{G})\right.\right.\right.\\
&- \left.\left.\left.\frac{\dot{B}}{B}f'_{\mathcal{G}}(\mathcal{G})\right)-\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)\left(\frac{4\dot{C}}{A^2C}\dot{f}_{\mathcal{G}}(\mathcal{G})-\frac{4C'}{B^2C}f'_{\mathcal{G}}(\mathcal{G})\right)\right]\right\}_{,1}\\
&- \left(2\frac{\dot{A}}{A^3}-\frac{\dot{B}}{A^2B}-\frac{2\dot{C}}{A^2C}\right)\left\{\frac{1}{2}A^2 f(\mathcal{G})+\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left[-\frac{4A}{BC^2}\left\{\frac{A}{B}\right.\right.\right.\\
&\times \left.\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)-\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\} f_{\mathcal{G}}(\mathcal{G})+\frac{4\dot{B}}{BC^2}\dot{f}_{\mathcal{G}}(\mathcal{G})+\frac{4A^2B'}{B^3C^2}f'_{\mathcal{G}}(\mathcal{G})\\
&- \left.\frac{4A^2}{B^2C^2}f''_{\mathcal{G}}(\mathcal{G})\right]+\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)\left(-\frac{8\dot{C}}{BC^2}\dot{f}_{\mathcal{G}}(\mathcal{G})-\frac{8A^2C'}{B^3C^2}f'_{\mathcal{G}}(\mathcal{G})\right)
\end{aligned}
$$

$$
\begin{aligned}
- & \frac{4Af_{\mathcal{G}}(\mathcal{G})}{BC^2}\left[\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\right. \\
\times & \left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right) \\
+ & \left.\left.\frac{2}{AB}\left(\dot{C}'^2-\ddot{C}C''\right)\right]\right\}+\left(\frac{A'}{AB^2}-\frac{B'}{B^3}+\frac{2C'}{B^2C}\right)\left\{\frac{2}{C}\left[\frac{2}{C}\left(1+\frac{\dot{C}^2}{A^2}\right.\right.\right. \\
- & \left.\frac{C'^2}{B^2}\right)\left(\dot{f}'_{\mathcal{G}}(\mathcal{G})-\frac{A'}{A}\dot{f}_{\mathcal{G}}(\mathcal{G})-\frac{\dot{B}}{B}f'_{\mathcal{G}}(\mathcal{G})\right)-\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)\left(\frac{4\dot{C}}{A^2C}\dot{f}_{\mathcal{G}}(\mathcal{G})\right. \\
- & \left.\left.\left.\frac{4C'}{B^2C}f'_{\mathcal{G}}(\mathcal{G})\right)\right]\right\}+\frac{\dot{B}}{B^3}\left\{-\frac{1}{2}B^2f(\mathcal{G})+\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left[\frac{4B}{AC^2}\left\{\frac{A}{B}\left(\frac{A'B'}{AB}\right.\right.\right.\right. \\
- & \left.\left.\left.\frac{A''}{A}\right)-\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}f_{\mathcal{G}}(\mathcal{G})+\frac{4B^2\dot{A}}{A^3C^2}\dot{f}_{\mathcal{G}}(\mathcal{G})+\frac{4A'}{AC^2}f'_{\mathcal{G}}(\mathcal{G})-\frac{4B^2}{A^2C^2}\ddot{f}_{\mathcal{G}}(\mathcal{G})\right] \\
+ & \left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)\left(\frac{8B^2\dot{C}}{A^3C^2}\dot{f}_{\mathcal{G}}(\mathcal{G})-\frac{8C'}{AC^2}f'_{\mathcal{G}}(\mathcal{G})\right)+\frac{4Bf_{\mathcal{G}}(\mathcal{G})}{AC^2} \\
\times & \left[\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\right. \\
\times & \left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right) \\
+ & \left.\left.\frac{2}{AB}\left(\dot{C}'^2-\ddot{C}C''\right)\right]\right\}+\frac{2\dot{C}}{C^3}\left\{-\frac{1}{2}C^2f(\mathcal{G})+\frac{4f_{\mathcal{G}}(\mathcal{G})}{AB}\right. \\
\times & \left[\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)-\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}\right. \\
+ & \frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right) \\
\times & \left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right) \\
+ & \left.\frac{2}{AB}\left(\dot{C}'^2-\ddot{C}C''\right)\right]+\frac{4C}{A^3B}\left[\dot{B}\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)+\dot{A}\right. \\
\times & \left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)-\frac{2A'}{B}\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)+\dot{C}\left\{\frac{B}{A}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times \quad \left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)-\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\Bigg\}\Bigg]\dot{f}_{\mathcal{G}}(\mathcal{G})+\frac{4C}{AB^3}\Bigg[B'\left(\frac{A'C'}{B^2}\right.\\
&+ \quad \left.\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)+A'\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)-\frac{2\dot{B}}{A}\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}\right.\\
&- \quad \left.\dot{C}'\right)-C'\left\{\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)-\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right\}\Bigg]f'_{\mathcal{G}}(\mathcal{G})+\frac{8C}{A^2B^2}\\
&\times \quad \left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)\dot{f}'_{\mathcal{G}}(\mathcal{G})-\frac{4C}{AB^2}\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)f''_{\mathcal{G}}(\mathcal{G})\\
&- \quad \frac{4C}{A^2B}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)\ddot{f}_{\mathcal{G}}(\mathcal{G})\Bigg\}\Bigg], \qquad \text{(A1)}
\end{aligned}
$$

$$
\begin{aligned}
\Xi_2 \quad &= \quad -\frac{1}{\kappa^2B}\Bigg[\frac{1}{A^2}\Bigg\{\frac{2}{C}\Bigg[\frac{2}{C}\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left(\dot{f}'_{\mathcal{G}}(\mathcal{G})-\frac{A'}{A}\dot{f}_{\mathcal{G}}(\mathcal{G})-\frac{\dot{B}}{B}f'_{\mathcal{G}}(\mathcal{G})\right)\\
&- \quad \left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)\left(\frac{4\dot{C}}{A^2C}\dot{f}_{\mathcal{G}}(\mathcal{G})-\frac{4C'}{B^2C}f'_{\mathcal{G}}(\mathcal{G})\right)\Bigg]\Bigg\}_{,0}+\frac{1}{B^2}\Bigg\{-\frac{1}{2}B^2f(\mathcal{G})\\
&+ \quad \left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\Bigg[\frac{4B}{AC^2}\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)-\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}f_{\mathcal{G}}(\mathcal{G})\\
&+ \quad \frac{4B^2\dot{A}}{A^3C^2}\dot{f}_{\mathcal{G}}(\mathcal{G})+\frac{4A'}{AC^2}f'_{\mathcal{G}}(\mathcal{G})-\frac{4B^2}{A^2C^2}\ddot{f}_{\mathcal{G}}(\mathcal{G})\Bigg]+\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)\\
&\times \quad \left(\frac{8B^2\dot{C}}{A^3C^2}\dot{f}_{\mathcal{G}}(\mathcal{G})-\frac{8C'}{AC^2}f'_{\mathcal{G}}(\mathcal{G})\right)+\frac{4Bf_{\mathcal{G}}(\mathcal{G})}{AC^2}\Bigg[\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\\
&\times \quad \left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)\\
&- \quad \frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)+\frac{2}{AB}\left(\dot{C'}^2-\ddot{C}C''\right)\Bigg]\Bigg\}_{,1}+\frac{A'}{A^3}\Bigg\{\frac{1}{2}A^2f(\mathcal{G})\\
&+ \quad \left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\Bigg[-\frac{4A}{BC^2}\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)-\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}f_{\mathcal{G}}(\mathcal{G})\\
&+ \quad \frac{4\dot{B}}{BC^2}\dot{f}_{\mathcal{G}}(\mathcal{G})+\frac{4A^2B'}{B^3C^2}f'_{\mathcal{G}}(\mathcal{G})-\frac{4A^2}{B^2C^2}f''_{\mathcal{G}}(\mathcal{G})\Bigg]+\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)
\end{aligned}
$$

$$
\begin{aligned}
\times\quad & \left(-\frac{8\dot{C}}{BC^2}\dot{f}_{\mathcal{G}}(\mathcal{G})-\frac{8A^2C'}{B^3C^2}f'_{\mathcal{G}}(\mathcal{G})\right)-\frac{4Af_{\mathcal{G}}(\mathcal{G})}{BC^2}\left[\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\right.\\
\times\quad & \left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)\\
-\quad & \left.\left.\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)+\frac{2}{AB}\left(\dot{C'}^2-\ddot{C}C''\right)\right]\right\}-\left(\frac{\dot{A}}{A^3}-\frac{\dot{B}}{A^2B}-\frac{2\dot{C}}{A^2C}\right)\\
\times\quad & \left\{\frac{2}{C}\left[\frac{2}{C}\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left(\dot{f}'_{\mathcal{G}}(\mathcal{G})-\frac{A'}{A}\dot{f}_{\mathcal{G}}(\mathcal{G})-\frac{\dot{B}}{B}f'_{\mathcal{G}}(\mathcal{G})\right)\right.\right.\\
-\quad & \left.\left.\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)\left(\frac{4\dot{C}}{A^2C}\dot{f}_{\mathcal{G}}(\mathcal{G})-\frac{4C'}{B^2C}f'_{\mathcal{G}}(\mathcal{G})\right)\right]\right\}+\left(\frac{A'}{AB^2}-\frac{2B'}{B^3}\right.\\
+\quad & \left.\frac{2C'}{B^C}\right)\left\{-\frac{1}{2}B^2f(\mathcal{G})+\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left[\frac{4B}{AC^2}\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right.\right.\right.\\
-\quad & \left.\left.\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}f_{\mathcal{G}}(\mathcal{G})+\frac{4B^2\dot{A}}{A^3C^2}\dot{f}_{\mathcal{G}}(\mathcal{G})+\frac{4A'}{AC^2}f'_{\mathcal{G}}(\mathcal{G})-\frac{4B^2}{A^2C^2}\ddot{f}_{\mathcal{G}}(\mathcal{G})\right]\\
+\quad & \left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)\left(\frac{8B^2\dot{C}}{A^3C^2}\dot{f}_{\mathcal{G}}(\mathcal{G})-\frac{8C'}{AC^2}f'_{\mathcal{G}}(\mathcal{G})\right)+\frac{4Bf_{\mathcal{G}}(\mathcal{G})}{AC^2}\left[\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}\right.\right.\\
+\quad & \left.\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\\
\times\quad & \left.\left.\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)+\frac{2}{AB}\left(\dot{C'}^2-\ddot{C}C''\right)\right]\right\}\\
-\quad & \frac{2C'}{C^3}\left\{-\frac{1}{2}C^2f(\mathcal{G})+\frac{4f_{\mathcal{G}}(\mathcal{G})}{AB}\left[\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right.\right.\right.\\
-\quad & \left.\frac{B}{A}\left(\frac{\dot{f}_{\mathcal{G}}(\mathcal{G})\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}+\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}\right.\\
+\quad & \left.\frac{A'C'}{B^2}\right)\left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)\\
+\quad & \left.\frac{2}{AB}\left(\dot{C'}^2-\ddot{C}C''\right)\right]+\frac{4C}{A^3B}\left[\dot{B}\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)+\dot{A}\left(\frac{B'C'}{B^2}\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
+ &\quad \frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\bigg)-\frac{2A'}{B}\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)+\dot{C}\left\{\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right.\\
- &\quad \left.\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right\}\bigg]\dot{f}_{\mathcal{G}}(\mathcal{G})+\frac{4C}{AB^3}\left[B'\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)+A'\left(\frac{B'C'}{B^2}\right.\right.\\
+ &\quad \left.\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)-\frac{2\dot{B}}{A}\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)-C'\left\{\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right.\\
- &\quad \left.\left.\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right\}\right]f'_{\mathcal{G}}(\mathcal{G})+\frac{8C}{A^2B^2}\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)\dot{f}'_{\mathcal{G}}(\mathcal{G})\\
- &\quad \frac{4C}{AB^2}\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)f''_{\mathcal{G}}(\mathcal{G})-\frac{4C}{A^2B}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)\ddot{f}_{\mathcal{G}}(\mathcal{G})\bigg\}\bigg].
\end{aligned}
\tag{A2}
$$

The GB invariant and field equations for non-static first order perturbation scheme are

$$
\begin{aligned}
&\quad \frac{1}{A_0}\left(1-\frac{1}{B_0^2}\right)\left(\frac{\bar{a}'}{B_0}\right)'+\frac{2B_0'}{B_0^4}\left(\frac{\bar{a}}{A_0}\right)'-\frac{\bar{a}}{A_0}\left(1-\frac{1}{B_0^2}\right)\left(\frac{A_0'}{B_0}\right)'\\
- &\quad \frac{A_0'}{A_0B_0}\left(1-\frac{3}{B_0^2}\right)\left(\frac{\bar{b}}{B_0}\right)'-\frac{2\bar{b}}{A_0B_0}\left(1-\frac{2}{B_0^2}\right)\left(\frac{A_0'}{B_0}\right)'-\frac{8\bar{b}A_0'B_0'}{A_0B_0^5}\\
- &\quad \frac{2}{A_0B_0^3}(\bar{c}'A_0')'+\frac{6rA_0'B_0'}{A_0B_0^4}\left(\frac{\bar{c}}{r}\right)'-\frac{2\bar{c}}{rA_0}\left(\frac{A_0'}{B_0}\right)'+\frac{2\bar{c}A_0''}{rA_0B_0^3}+\frac{1}{8}r^2\bar{g}B_0\\
- &\quad \frac{1}{A_0^2}\left(\bar{b}+\frac{\bar{c}B_0'}{B_0^2}-\frac{\bar{b}}{B_0^2}\right)\frac{\ddot{\mathcal{T}}}{\mathcal{T}}=0,
\end{aligned}
\tag{A3}
$$

$$
\begin{aligned}
&\quad \kappa^2\left(2\bar{a}\frac{\rho_0}{A_0}+\frac{\bar{\rho}}{\mathcal{T}}\right)=-\frac{2}{B_0^2}\left[\frac{\bar{a}}{rA_0}\left\{\frac{B_0}{r}\left(\frac{r}{B_0}\right)'-\frac{B_0^2}{r}-\frac{B_0'}{B_0}\right\}-\frac{\bar{b}}{r^2}\left(\frac{r}{B_0}\right)'\right.\\
+ &\quad \left.\frac{\bar{b}B_0'}{rB_0^2}-\frac{1}{r}\left(\frac{\bar{b}}{B_0}\right)'+\left(\frac{\bar{c}}{r}\right)''+\left(\frac{3}{r}-\frac{B_0'}{B_0}\right)\left(\frac{\bar{c}}{r}\right)'-\frac{\bar{c}B_0^2}{r^3}\right]-\frac{1}{2}\chi\mathcal{G}_0^n\left(n\bar{g}\mathcal{G}_0^{-1}\right.\\
+ &\quad \left.2\frac{\bar{a}}{A_0}\right)-4\chi n\mathcal{G}_0^{n-1}\left[\frac{1}{r^2A_0B_0}\left\{\left(\frac{\bar{a}'}{B_0}\right)'+\frac{\bar{a}}{A_0}\left(\frac{A_0'}{B_0}\right)'\right\}\left(1-\frac{1}{B_0^2}\right)\right.\\
+ &\quad \frac{2B_0'}{r^2A_0^2B_0^5}\left(\bar{a}A_0\right)'-\frac{A_0'}{r^2A_0B_0^2}\left(1-\frac{3}{B_0^2}\right)\left(\frac{\bar{b}}{B_0^2}\right)'-\frac{2\bar{b}}{r^2A_0B_0^2}\left(1-\frac{2}{B_0^2}\right)\\
\times &\quad \left(\frac{A_0'}{B_0}\right)'-\frac{8\bar{b}A_0'B_0'}{r^2A_0B_0^6}-2\left\{\frac{(\bar{c}'A_0')'}{r^2A_0B_0^4}-\frac{3A_0'B_0'}{rA_0B_0^5}\left(\frac{\bar{c}}{r}\right)'+\frac{\bar{c}}{r^3A_0B_0}\left(\frac{A_0'}{B_0}\right)'\right.
\end{aligned}
$$

$$
\begin{aligned}
- & \quad \left.\frac{\bar{c}A_0''}{r^3A_0B_0^4}\right\} - (n-1)\frac{\bar{g}}{\mathcal{G}_0}\left\{\frac{A_0'B_0'}{r^2A_0B_0^3}\left(1-\frac{3}{B_0^2}\right) - \frac{A_0''}{r^2A_0B_0^2}\left(1-\frac{1}{B_0^2}\right)\right\}\Bigg] \\
- & \quad 4\chi n(n-1)\frac{1}{r^2B_0^2}\left[\left\{\mathcal{G}_0^{n-2}\mathcal{G}_0'\left[\frac{2B_0'}{B_0}\left(\frac{\bar{a}}{A_0}-\frac{\bar{c}}{r}\right)+\frac{\bar{b}'}{B_0}\right]+\frac{B_0'}{B_0}\left(\bar{g}\mathcal{G}_0^{n-2}\right)'\right\}\right. \\
\times & \quad \left(1-\frac{3}{B_0^2}\right) - 3\frac{\bar{b}B_0'}{B_0^2}\left(1-\frac{5}{B_0^2}\right)\mathcal{G}_0^{n-2}\mathcal{G}_0' + 2B_0\left(\frac{\bar{c}'}{B_0^3}\right)'\mathcal{G}_0^{n-2}\mathcal{G}_0' \\
+ & \quad \left\{2\left(\frac{\bar{c}}{r}-\frac{\bar{a}}{A_0}\right)\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)' - \left(\bar{g}\mathcal{G}_0^{n-2}\right)' - (n-2)\left(\bar{g}\mathcal{G}_0^{n-3}\mathcal{G}_0'\right)'\right\}\left(1-\frac{1}{B_0^2}\right) \\
+ & \quad 2\left\{\frac{\bar{b}}{B_0}\left(1-\frac{2}{B_0^2}\right)+\frac{\bar{c}'}{B_0^2}\right\}\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)'\Bigg], \qquad \text{(A4)} \\
& \quad \left(\frac{\bar{c}}{r}\right)' - \frac{\bar{b}}{rB_0} + \frac{\bar{c}}{r}\left(\frac{1}{r}-\frac{A_0'}{A_0}\right) - 2\chi n(n-1)\frac{1}{r^2}\left(1-\frac{1}{B_0^2}\right)\left[A_0\left(\frac{\bar{g}}{A_0}\right)'\right. \\
+ & \quad \left.(n-2)\bar{g}\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right]\mathcal{G}_0^{n-2} + 4\chi n(n-1)\frac{A_0}{r^2B_0^2}\left(\frac{\bar{c}}{A_0}\right)'\mathcal{G}_0^{n-2}\mathcal{G}_0' + 2\chi n(n-1) \\
\times & \quad \frac{\bar{b}}{r^2B_0}\left(1-\frac{3}{B_0^2}\right)\mathcal{G}_0^{n-2}\mathcal{G}_0' = 0, \qquad \text{(A5)} \\
& \quad \kappa^2\left(\frac{B_0^2\bar{P}_r}{\mathcal{T}} + 2\bar{b}B_0P_{r_0}\right) = \left(1-\frac{1}{B_0^2}\right)\left[-\frac{2}{r^2}\left(\bar{b}B_0 - \frac{\bar{c}B_0^2}{r}\right) - 4\chi n\frac{B_0}{r^2A_0}\right. \\
\times & \quad \left\{\frac{1}{B_0^2}\left(\bar{a}'B_0' - \bar{b}A_0'\frac{B_0'}{B_0} - \bar{a}B_0'\frac{A_0'}{A_0} + \frac{\bar{b}'A_0'}{A_0B_0}\right) - \frac{1}{B_0}\left(\bar{a}'' - \frac{\bar{a}A_0''}{A_0}\right)\right. \\
+ & \quad \left.\frac{1}{B_0}\left(\bar{a} - \frac{\bar{b}A_0}{B_0}\right)\left(\frac{A_0'B_0'}{A_0B_0} - \frac{A_0''}{A_0}\right) + \frac{\bar{b}}{A_0}\left(\frac{\ddot{\mathcal{T}}}{\mathcal{T}}\right)\right\}\mathcal{G}_0^{n-1} - \frac{4\chi n}{r^2A_0} \\
\times & \quad \left(\frac{A_0'B_0'}{B_0^2} - \frac{A_0''}{B_0}\right)\left\{\left(\bar{b} - 2B_0\frac{\bar{c}}{r} - \frac{\bar{a}B_0}{A_0}\right)\mathcal{G}_0^{n-1} + (n-1)\bar{g}B_0\mathcal{G}_0^{n-2}\right\} \\
- & \quad 4\chi n(n-1)\frac{A_0'}{r^2A_0}\left(\bar{g}'\mathcal{G}_0^{n-2} + (n-2)\bar{g}\mathcal{G}_0^{n-3}\mathcal{G}_0'\right) - \frac{4\chi n(n-1)}{r^2A_0}\left(\bar{a}' - 2A_0'\frac{\bar{c}}{r}\right. \\
- & \quad \left.\frac{\bar{a}A_0'}{A_0}\right)\mathcal{G}_0^{n-2}\mathcal{G}_0' + 4\chi n(n-1)\bar{g}\frac{B_0^2}{r^2A_0^2}\left(\frac{\ddot{\mathcal{T}}}{\mathcal{T}}\right)\mathcal{G}_0^{n-2}\Bigg] - \frac{2}{B_0^2}\left(\bar{c}' - \frac{\bar{b}}{B_0}\right) \\
\times & \quad \left[\frac{4\chi n}{r^2A_0}\mathcal{G}_0^{n-1}\left(\frac{A_0'B_0'}{B_0} - A_0'' + (n-1)A_0'\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right) - \frac{B_0^2}{r^2}\right] + \left[\frac{2A_0'}{rA_0}\left(\frac{2\bar{b}}{B_0} - \frac{\bar{a}}{A_0}\right.\right. \\
- & \quad \left.\frac{\bar{c}}{r}\right) + 8\chi n(n-1)\frac{A_0'}{r^2A_0B_0^2}\left\{\bar{g}' + (n-2)\bar{g}\frac{\mathcal{G}_0'}{\mathcal{G}_0} + \left(\bar{c}' - \frac{2\bar{c}}{r} - \frac{\bar{a}}{A_0}\right)\mathcal{G}_0'\right\}\mathcal{G}_0^{n-2} \\
+ & \quad \left(\frac{2B_0^2}{rA_0} + \frac{8\chi n(n-1)}{r^2A_0}\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)\left\{\frac{1}{B_0^2}\left(\bar{a}' + \bar{c}'A_0' - \frac{2\bar{b}A_0'}{B_0}\right) - \frac{\bar{c}}{A_0}\left(\frac{\ddot{\mathcal{T}}}{\mathcal{T}}\right)\right\}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & \frac{1}{2}\chi B_0 \left(n\bar{g}B_0\mathcal{G}_0^{n-1} + 2\bar{b}\mathcal{G}_0^{n}\right) - \Bigg[4\chi n \frac{B_0}{r^2 A_0}\Bigg\{\frac{4A_0'B_0'}{B_0^4}\left(\bar{c}' - \frac{\bar{b}}{B_0}\right) - \frac{2\bar{c}''A_0'}{B_0^3} \\
+ \quad & \frac{2}{B_0^4}\left(\bar{a}'B_0' + \bar{b}'A_0'\right) + \frac{2\bar{c}B_0'}{A_0B_0^2}\left(\frac{\ddot{\mathcal{T}}}{\mathcal{T}}\right)\Bigg\}\mathcal{G}_0^{n-1} - \frac{2A_0'B_0'}{B_0^4}\Bigg\{4\chi n\frac{B_0}{r^2A_0}\Bigg(\frac{\bar{b}}{B_0} - \frac{2\bar{c}}{r} \\
- \quad & \frac{\bar{a}}{A_0} + (n-1)\frac{\bar{g}}{\mathcal{G}_0}\Bigg)\mathcal{G}_0^{n-1}\Bigg\}\Bigg],
\end{aligned}
\tag{A6}
$$

$$
\begin{aligned}
& \kappa^2\left(\frac{2\bar{c}}{r}P_{t_0} + \frac{\bar{P}_t}{\mathcal{T}}\right) = \frac{1}{A_0^2}\left(\frac{\bar{b}}{B_0} + \frac{\bar{c}}{r}\right)\frac{\ddot{\mathcal{T}}}{\mathcal{T}} + \frac{1}{B_0^2}\Bigg[\left(\frac{\bar{a}}{A_0}\right)'' + \left(\frac{\bar{c}}{r}\right)'' \\
+ \quad & \left(\frac{2A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{1}{r}\right)\left(\frac{\bar{a}}{A_0}\right)' - \left(\frac{A_0'}{A_0} + \frac{1}{r}\right)\left(\frac{\bar{b}}{B_0}\right)' + \left(\frac{A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{2}{r}\right)\left(\frac{\bar{c}}{r}\right)' \\
+ \quad & 2\left(\frac{A_0'B_0'}{A_0B_0} + \frac{B_0'}{rB_0} - \frac{A_0'}{rA_0} - \frac{A_0''}{A_0}\right)\left(\frac{\bar{b}}{B_0} - \frac{\bar{c}}{r}\right)\Bigg] + \frac{1}{2}\chi\left(\frac{n\bar{g}}{\mathcal{G}_0} + \frac{2\bar{c}}{r}\right)\mathcal{G}_0^{n} \\
- \quad & 4\chi n\mathcal{G}_0^{n-1}\frac{1}{r^2}\Bigg[-\frac{1}{A_0B_0}\left(\frac{\bar{a}'}{B_0}\right)'\left(1 - \frac{1}{B_0^2}\right) + \frac{\bar{a}}{A_0^2B_0}\left(\frac{A_0'}{B_0}\right)'\left(1 - \frac{1}{B_0^2}\right) \\
- \quad & \frac{2B_0'}{B_0^5}\left(\frac{\bar{a}}{A_0}\right)' + \frac{A_0'}{A_0B_0^2}\left(\frac{\bar{b}}{B_0}\right)'\left(1 - \frac{3}{B_0^2}\right) + \frac{2\bar{b}}{A_0B_0^2}\left(\frac{A_0'}{B_0}\right)'\left(1 - \frac{2}{B_0^2}\right) \\
- \quad & 8\bar{b}\frac{A_0'B_0'}{A_0B_0^6} + \frac{2}{A_0B_0^4}\left\{(\bar{c}'A_0')' - 3\bar{c}'\frac{A_0'B_0'}{B_0}\right\} - (n-1)\frac{\bar{g}}{\mathcal{G}_0}\Bigg\{\frac{1}{A_0B_0}\left(\frac{A_0'}{B_0}\right)' \\
\times \quad & \left(1 - \frac{1}{B_0^2}\right) + \frac{2A_0'B_0'}{A_0B_0^5}\Bigg\} + r(n-1)\Bigg\{\frac{\bar{a}}{A_0^2B_0^3}\left(\frac{A_0'}{B_0}\right)' - \frac{1}{A_0B_0^3}\left(\frac{\bar{a}'}{B_0}\right)' \\
+ \quad & \frac{2B_0'}{B_0^5}\left(\frac{\bar{a}}{A_0}\right)' + \frac{4\bar{b}}{A_0B_0^4}\left(\frac{A_0'}{B_0}\right)' + \frac{3A_0'}{A_0B_0^4}\left(\frac{\bar{b}}{B_0}\right)' - 8\bar{b}\frac{A_0'B_0'}{A_0B_0^6} - \frac{1}{A_0B_0^4}(\bar{c}'A_0')' \\
- \quad & 3\frac{(r\bar{c})'}{r}\frac{A_0'B_0'}{A_0B_0^5} - \frac{\bar{c}A_0''}{rA_0B_0^4}\Bigg\}\frac{\mathcal{G}_0'}{\mathcal{G}_0} - (n-1)\frac{r}{B_0\mathcal{G}_0^{n-1}}\left(\frac{A_0'}{B_0^3}\right)'\left(\bar{g}\mathcal{G}_0^{n-2}\right)' \\
- \quad & (n-1)\frac{r}{B_0^4\mathcal{G}_0^{n-1}}\Bigg\{\frac{A_0'}{A_0}(\bar{g}'\mathcal{G}_0^{n-2})' + \left(\frac{A_0'}{rA_0}(r\bar{c})' + \left(\frac{\bar{a}}{A_0}\right)' - \frac{4\bar{b}A_0'}{A_0B_0}\right)\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)' \\
+ \quad & (n-2)\frac{A_0'}{A_0}\left(\bar{g}\mathcal{G}_0^{n-3}\mathcal{G}_0'\right)'\Bigg\}\Bigg] - \frac{4\chi n}{r^2}\mathcal{G}_0^{n-1}\Bigg[\frac{\bar{b}}{A_0^2B_0}\left(1 - \frac{1}{B_0^2}\right) + \frac{2\bar{c}B_0'}{A_0^2B_0^3} \\
+ \quad & (n-1)\frac{r}{A_0B_0^3}\left(\bar{b} - \frac{\bar{c}B_0'}{A_0}\right)\frac{\mathcal{G}_0'}{\mathcal{G}_0} + (n-1)(n-2)\frac{r\bar{c}}{A_0^2B_0^2}\left(\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right)^2 \\
- \quad & (n-1)\frac{rB_0'}{A_0^2B_0^3}\left(\frac{\bar{g}}{\mathcal{G}_0}\right)\Bigg]\frac{\ddot{\mathcal{T}}}{\mathcal{T}}.
\end{aligned}
\tag{A7}
$$

The expressions for Ξ_{1p} and Ξ_{2p} in Eqs.(2.2.18) and (2.2.19), respectively are

$$\begin{aligned}
\Xi_{1p} &= 4\chi n\mathcal{G}_0^{n-1}\frac{1}{\kappa^2}\left[\frac{\bar{g}}{8}+\frac{1}{r^2B_0^2}\left\{\frac{\bar{a}''}{A_0}-\frac{\bar{a}A_0''}{A_0^2}-\frac{2\bar{c}A_0''}{rA_0}\right\}\left(1-\frac{1}{B_0^2}\right)\right.\\
&+ \frac{1}{r^2B_0^2}\left\{\frac{2\bar{c}A_0'B_0'}{rA_0B_0}-\frac{\bar{b}'A_0'}{A_0B_0}-\left(\frac{\bar{a}}{A_0}\right)'\frac{B_0'}{B_0}\right\}\left(1-\frac{3}{B_0^2}\right)-\frac{2\bar{b}A_0''}{r^2A_0B_0^3}\\
&\times \left.\left(1-\frac{3}{B_0^2}\right)-\frac{3\bar{b}A_0'B_0'}{r^2A_0B_0^4}\left(1-\frac{5}{B_0^2}\right)-\frac{2A_0'}{r^2A_0B_0}\left(\frac{\bar{c}'}{B_0^3}\right)'-\frac{2\bar{c}'B_0''}{r^2A_0B_0^4}\right],
\end{aligned} \tag{A8}$$

$$\begin{aligned}
\Xi_{2p} &= 4\chi n\mathcal{T}\mathcal{G}_0^{n-1}\frac{1}{\kappa^2}\left[\left[\frac{1}{r^2B_0^4}\left(1-\frac{3}{B_0^2}\right)\left\{B_0''+2\frac{A_0'B_0'}{A_0}+\frac{2B_0'}{r}\right\}+\frac{2B_0'^2}{r^2B_0^5}\right.\right.\\
&\times \left.\left(1-\frac{5}{B_0^2}\right)\right]\left(\frac{\bar{a}}{A_0}\right)'+\frac{1}{A_0B_0^3}\left(1-\frac{1}{B_0^2}\right)\left\{\frac{\bar{a}}{A_0}\left(\frac{A_0''}{r^2}\right)'-\frac{1}{r^2A_0}(\bar{a}'A_0')'\right.\\
&+ \left.\left(\frac{\bar{a}''}{r^2}\right)'+\frac{2\bar{a}A_0'A_0''}{r^2A_0^2}\right\}+\frac{B_0'}{r^2A_0B_0^4}\left(3-\frac{7}{B_0^2}\right)\left\{\bar{a}\frac{A_0''}{A_0}-\bar{a}''\right\}+\frac{1}{r^2}\\
&\times \left(\frac{7A_0''}{A_0}-\frac{6A_0'}{rA_0}\right)\left(\frac{\bar{b}}{B_0^2}\right)'-\left\{\frac{2A_0''}{r^2A_0}-\left(\frac{A_0'}{r^2A_0}\right)'\right\}\left(\frac{\bar{b}}{B_0^4}\right)'+\frac{3A_0'B_0'}{r^2A_0}\\
&\times \left(\frac{\bar{b}}{B_0^5}\right)'-\frac{15A_0'B_0'}{r^2A_0}\left(\frac{\bar{b}}{B_0^7}\right)'-\frac{\bar{b}}{r^2B_0^4}\left(3-\frac{5}{B_0^2}\right)\left\{\left(\frac{A_0''}{A_0}\right)'-\frac{2A_0''}{rA_0}\right\}\\
&- \frac{18\bar{b}A_0'}{r^2B_0^7}\left(\frac{B_0'}{A_0}\right)'+\frac{3A_0'}{r^2B_0^6}\left(\frac{\bar{b}'}{A_0}\right)'+\frac{A_0'}{r^2A_0B_0^4}\left(4\bar{b}\frac{B_0''}{B_0}-15\frac{\bar{b}'B_0'}{B_0^3}-\bar{b}''\right)\\
&- \frac{\bar{b}}{8B_0^2}\mathcal{G}_0'-\frac{1}{B_0^2}\left(1-\frac{1}{B_0^2}\right)\left(\frac{\bar{b}}{r^2A_0^2}\right)'\frac{\ddot{\mathcal{T}}}{\mathcal{T}}+\frac{\bar{b}B_0'}{r^2A_0^2B_0^3}\left(1-\frac{3}{B_0^2}\right)\frac{\ddot{\mathcal{T}}}{\mathcal{T}}\\
&+ \left(\frac{\bar{c}}{r}\right)'\left\{\frac{2A_0'A_0''}{rA_0^2B_0^5}+\frac{14B_0'}{A_0B_0^6}\left(\frac{A_0'}{r}\right)'-\frac{6A_0'}{rB_0^6}\left(\frac{B_0'}{A_0}\right)'+4\frac{A_0'B_0'}{A_0}\left(\frac{1}{rB_0^6}\right)'\right.\\
&- \left.\frac{6A_0'B_0'^2}{rA_0B_0^7}+\frac{6A_0''}{r^2A_0B_0^5}\right\}+\frac{2A_0'}{A_0B_0^5}\left(\frac{\bar{c}''}{r^2}\right)'-\frac{2}{A_0B_0^2}\left(\frac{\bar{c}}{r^3}\right)'\left(\frac{A_0'}{B_0}\right)'\\
&- \frac{2\bar{c}''}{r^2B_0^5}\left(\frac{A_0'}{A_0}\right)'-\frac{2}{r^2A_0B_0^5}(\bar{c}'A_0'')'+\frac{2\bar{c}A_0'B_0'}{r^3B_0}\left(\frac{1}{A_0B_0^3}\right)'+\frac{2\bar{c}}{r^3A_0B_0^4}\\
&\times (A_0'B_0')'-\frac{2\bar{c}A_0'''}{r^3A_0B_0^3}\left(1-\frac{1}{B_0^2}\right)+\frac{2\bar{c}A_0'A_0''}{r^3A_0^2B_0^3}+\frac{14\bar{c}''A_0'B_0'}{r^2A_0B_0^6}+\frac{4\bar{c}A_0''B_0'}{r^3A_0B_0^4}\\
&- \left.2\left\{\frac{1}{r^2A_0^2B_0^4}(\bar{c}B_0')'+\frac{\bar{c}B_0'}{B_0}\left(\frac{1}{r^2A_0^2B_0^3}\right)'\right\}\frac{\ddot{\mathcal{T}}}{\mathcal{T}}\right]+4\chi n(n-1)\mathcal{T}\bar{g}\mathcal{G}_0^{n-2}\\
&\times \frac{1}{\kappa^2}\left[\frac{1}{B_0^3}\left(\frac{A_0''}{r^2A_0}\right)'\left(1-\frac{1}{B_0^2}\right)-\frac{A_0'}{B_0^4}\left(\frac{B_0'}{r^2A_0}\right)'\left(1-\frac{3}{B_0^2}\right)+\frac{3A_0'B_0'^2}{r^2A_0B_0^5}\right.
\end{aligned}$$

$$\times \quad \left(1-\frac{5}{B_0^2}\right)-\frac{A_0''B_0'}{r^2A_0B_0^4}\left(3-\frac{7}{B_0^2}\right)+\frac{1}{8B_0}\mathcal{G}_0'\Bigg]+\frac{1}{2\kappa^2B_0}\chi n\mathcal{T}\bar{g}'\mathcal{G}_0^{n-1}. \quad \text{(A9)}$$

The value of expansion-free higher curvature term Ξ_3 in Eq.(2.3.6) is of the form

$$\begin{aligned}
\Xi_3 &= 4\chi n\mathcal{T}\mathcal{G}_0^{n-1}\frac{1}{\kappa^2}\left[\frac{\bar{g}}{8}+\frac{1}{r^2B_0^2}\left\{\frac{\bar{a}''}{A_0}-\frac{\bar{a}A_0''}{A_0^2}\right\}\left(1-\frac{1}{B_0^2}\right)+\left\{\frac{2A_0'}{r^3A_0B_0^2}\right.\right.\\
&\times \left(5\bar{c}\frac{B_0'}{B_0}+\bar{c}'-\frac{\bar{c}}{r}\right)-\frac{1}{r^2B_0^2}\left(\frac{\bar{a}}{A_0}\right)'\frac{B_0'}{B_0}\bigg\}\left(1-\frac{3}{B_0^2}\right)+\frac{2\bar{c}A_0''}{r^3A_0B_0^2}\\
&\times \left(1-\frac{5}{B_0^2}\right)+\frac{6A_0'B_0'}{A_0B_0^5}\left(\frac{\bar{c}}{r^2}\right)'-\frac{2\bar{c}''A_0'}{r^2A_0B_0^4}-\frac{2\bar{c}'B_0''}{r^2A_0B_0^4}+\frac{2\bar{c}}{r^2A_0^2}\\
&\times \left.\left\{\frac{1}{r}\left(1-\frac{1}{B_0^2}\right)+\frac{B_0'}{B_0^3}\right\}\frac{\ddot{\mathcal{T}}}{\mathcal{T}}\right]. \qquad \text{(A10)}
\end{aligned}$$

The expressions for Δ_i's in Eq.(2.3.9) are given as follows

$$\begin{aligned}
\Delta_1 &= \left(\frac{r+2m_0}{r^3}\left[\frac{\kappa^2r^3}{2}(\rho_0+P_{r_0})+\frac{m_0}{r^2}\left\{r^2-2\chi n(n-1)(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right\}\right.\right.\\
&- \left.\left.4\chi n(n-1)r^2m_0\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)\right]\left[1+\frac{2}{r^2}\chi n(n-1)(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right]\right),\\
\Delta_2 &= \left(\frac{r+2m_0}{r}\left[\kappa^2P_{t_0}-\frac{1}{2}\chi\mathcal{G}_0^n-\frac{m_0}{r^3}-\left\{\frac{m_0}{r^5}\left(r^3-8\chi n(r-3m_0)\mathcal{G}_0^{n-1}\right.\right.\right.\right.\\
&+ \left.12\chi n(n-1)r(r-2m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)+\frac{1}{r^3}(r-2m_0)\left(r+4\chi n(n-1)\right.\\
&\times \left.\left.(r-2m_0)\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)'\right)\right\}\left\{\frac{r+2m_0}{r^3}\left[\frac{\kappa^2r^3}{2}(\rho_0+P_{r_0})+\frac{m_0}{r^2}\left\{r^2-2\chi n\right.\right.\right.\\
&\times \left.(n-1)(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right\}-4\chi n(n-1)r^2m_0\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)\Big]\Big[1+\frac{2}{r^2}\\
&\times \left.\left.\chi n(n-1)(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right]\right\}\Big]\left[1-\frac{8}{r^3}\chi nm_0\mathcal{G}_0^{n-1}-\frac{4}{r^2}\chi n(n-1)\right.\\
&\times \left.\left.(r-2m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right]\right),\\
\Delta_3 &= \left[r^4\left(\kappa^2\rho_0'-\frac{6m_0^2}{r^6}(r+2m_0)+\frac{1}{2}\chi n\mathcal{G}_0^{n-1}\mathcal{G}_0'\right)/\left(8m_0(r-2m_0)\right.\right.\\
&\times \left.\chi n\mathcal{G}_0^{n-1}\right)-\left(\frac{2}{r}-(n-1)\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right)\left(\frac{r+2m_0}{r}\left[\kappa^2P_{t_0}-\frac{1}{2}\chi\mathcal{G}_0^n-\frac{m_0}{r^3}\right.\right.\\
&- \left\{\frac{m_0}{r^5}\left(r^3-8\chi n(r-3m_0)\mathcal{G}_0^{n-1}+12\chi n(n-1)r(r-2m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)\right.\\
&+ \left.\frac{1}{r^3}(r-2m_0)\left(r+4\chi n(n-1)(r-2m_0)\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)'\right)\right\}\left\{\frac{r+2m_0}{r^3}\right.
\end{aligned}$$

$$
\begin{aligned}
\times &\quad \left[\frac{\kappa^2 r^3}{2}(\rho_0 + P_{r_0}) + \frac{m_0}{r^2}\left\{r^2 - 2\chi n(n-1)(6m_0 - 2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right\} - 4\chi n\right. \\
\times &\quad \left.(n-1)r^2 m_0 \left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)'\right]\left[1 + \frac{2}{r^2}\chi n(n-1)(6m_0 - 2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right]\Bigg\}\Bigg] \\
\times &\quad \left[1 - \frac{8}{r^3}\chi n m_0 \mathcal{G}_0^{n-1} - \frac{4}{r^2}\chi n(n-1)(r - 2m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right]\Bigg) + 7m_0(r + 2m_0) \\
\times &\quad (n-1)\frac{\mathcal{G}_0'}{r^4\mathcal{G}_0} - 3(n-1)(n-2)\frac{\mathcal{G}_0'\mathcal{G}_0''}{\mathcal{G}_0^2} + (n-1)(n-2)(n-3)\left(\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right)^3 \\
+ &\quad (n-1)(n-2)(4r - 11m_0)(r + 2m_0)\frac{1}{r^3}\left(\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right)^2 - (n-1)\frac{\mathcal{G}_0'''}{\mathcal{G}_0} \\
+ &\quad (n-1)(r + 2m_0)(4r - 7m_0)\frac{\mathcal{G}_0''}{r^3\mathcal{G}_0}\Bigg].
\end{aligned}
$$

Appendix B

The expressions for $\mathcal{A}_i$'s in Eqs.(5.1.6) and (5.1.7) are

$$\begin{aligned}
\mathcal{A}_1 &= 18c_1H_0^4[8c_1H_0^4\{2(2+59\omega^2)-11\omega(5-3\omega^2)\}-(1-11\omega)(1-\omega^2)] \\
&\times [1+\omega-36c_1H_0^4(1-3\omega)]^{-2}, \\
\mathcal{A}_2 &= 18c_1H_0^4[(1-11\omega)(1-\omega^2)-8c_1H_0^4\{2(2+59\omega^2)-11\omega(5-3\omega^2)\}] \\
&\times [(1+\omega)(1-24c_1H_0^4)\{1+\omega-6c_1H_0^4(5-4\omega-33\omega^2)\}]^{-1}, \\
\mathcal{A}_3 &= -[18c_1H_0^4(1-32c_1H_0^4)-3\omega\{1-6c_1H_0^4(3-352c_1H_0^4)\} \\
&- 2\omega^2\{1-9c_1H_0^4(7-1248c_1H_0^4)\}+\omega^3\{1-54c_1H_0^4(7-480c_1H_0^4)\}] \\
&\times [(1-3\omega)(1-24c_1H_0^4)\{1+\omega-6c_1H_0^4(5-4\omega-33\omega^2)\}]^{-1}.
\end{aligned}$$

The values for $\varrho_{\hat{k}}$'s in Eq.(5.1.12) are

$$\begin{aligned}
\varrho_1 &= \frac{1}{2}\left[5-\lambda\{1+3d_2(1+\omega)\}\right], \\
\varrho_2 &= \left[\frac{3}{4}\lambda d_2(1+\omega)\{3d_2\lambda(1+\omega)+2(\lambda-1)-8\}+\frac{1}{4}(\lambda-1)(\lambda+7)+4\right. \\
&+ \left. 8d_2(\lambda-1)\left(\frac{1+\omega}{1-3\omega}\right)\right]^{\frac{1}{2}}, \quad \varrho_3=-\frac{1}{2}\left(\frac{1-3\omega}{1+\omega}\right), \quad \varrho_4=\frac{2\kappa^2}{\omega-3}, \\
\varrho_5 &= \left(\frac{18\lambda^3(1-3\omega)^{\frac{3\lambda(1+\omega)-2}{3\lambda(1+\omega)}}}{3\lambda(1-3\omega)+4}\right)\rho_0^{\frac{-2}{3\lambda(1+\omega)}}, \quad \varrho_6=\frac{2}{3\lambda(1+\omega)}.
\end{aligned}$$

The expressions for $\mathcal{B}_k$'s and Ω_k's in Eqs.(5.1.13) and (5.1.14), respectively are

$$\begin{aligned}
\mathcal{B}_1 &= 1-\frac{\varrho_7}{\varrho_8}, \quad \mathcal{B}_2=1-\frac{\varrho_3^2}{\varrho_8}, \quad \mathcal{B}_3=\varrho_4\left(1-\frac{1}{\varrho_8}\right), \quad \mathcal{B}_4=\varrho_5\left(1-\frac{\varrho_6^2}{\varrho_8}\right), \\
\Omega_1 &= 1-\frac{\varrho_8}{\varrho_7}, \quad \Omega_2=1-\frac{\varrho_3^2}{\varrho_7}, \quad \Omega_3=\varrho_4\left(1-\frac{1}{\varrho_7}\right), \quad \Omega_4=\varrho_5\left(1-\frac{\varrho_6^2}{\varrho_7}\right),
\end{aligned}$$

where

$$\varrho_7 = \frac{d_2}{6\lambda}\left[6d_2\lambda(1+\omega)^2-3\lambda(1+5\omega+2\omega^2)+2(\varrho_1+\varrho_2)\right],$$

$$\varrho_8 = \frac{d_2}{6\lambda}\left[6d_2\lambda(1+\omega)^2 - 3\lambda(1+5\omega+2\omega^2) + 2(\varrho_1-\varrho_2)\right].$$

The values of $\mathfrak{X}_h$'s in Eq.(5.2.4) are given as follows

$$\begin{aligned}
\mathfrak{X}_1 &= 288H_*^6 f^*_{\mathcal{G}\mathcal{G}},\\
\mathfrak{X}_2 &= 288H_*^5(3H_*^2+5\dot{H}_*)f^*_{\mathcal{G}\mathcal{G}} + 6912H_*^7(4H_*^2\dot{H}_* + 2\dot{H}_*^2 + H_*\ddot{H}_*)f^*_{\mathcal{G}\mathcal{G}\mathcal{G}}\\
&+ 24(1+\omega)\rho_* H_*^3 f^*_{\mathcal{G}T} - 864(1+\omega)(1-3\omega)H_*^7\rho_* f^*_{\mathcal{G}\mathcal{G}T},\\
\mathfrak{X}_3 &= -6H_*^2 - 24H_*^2\dot{H}_* f^*_{\mathcal{G}} - 288H_*^4(4H_*^4 - 23H_*^2\dot{H}_* - 11\dot{H}_*^2 - 6H_*\ddot{H}_*)f^*_{\mathcal{G}\mathcal{G}}\\
&+ 6912H_*^6(4H_*^2+\dot{H}_*)(4H_*^2\dot{H}_* + 2\dot{H}_*^2 + H_*\ddot{H}_*)f^*_{\mathcal{G}\mathcal{G}\mathcal{G}} + 12(1+\omega)\rho_* H_*^2\\
&\times [2(4H_*^2+\dot{H}_*) - 9(1-3\omega)H_*^2]f^*_{\mathcal{G}T} - 864(1+\omega)(1-3\omega)\rho_* H_*^6(4H_*^2\\
&+ \dot{H}_*)f^*_{\mathcal{G}\mathcal{G}T},\\
\mathfrak{X}_4 &= 12(1-3\omega)\rho_* H_*^3 f^*_{\mathcal{G}T},\\
\mathfrak{X}_5 &= \kappa^2\rho_* - \frac{1}{2}(\omega-3)\rho_* f^*_T + (1-3\omega)(1+\omega)\rho_*^2 f^*_{TT} - 12(1-3\omega)\rho_* H_*^2\\
&\times [(4+3\omega)H_*^2 + \dot{H}_*]f^*_{\mathcal{G}T} + 288(1-3\omega)\rho_* H_*^4(4H_*^2\dot{H}_* + 2\dot{H}_*^2 + H_*\ddot{H}_*)\\
&\times f^*_{\mathcal{G}\mathcal{G}T} - 36(1+\omega)(1-3\omega)^2\rho_*^2 H_*^4 f^*_{\mathcal{G}TT}.
\end{aligned}$$

The expressions for $\mathfrak{U}_h$'s are

$$\begin{aligned}
\mathfrak{U}_1 &= 3(1+\omega)\rho_* H_*(\kappa^2 + f_T) - 12(1+\omega)\rho_* H_*\Big[3H_*^2(1-\omega)(4H_*^2+\dot{H}_*)\\
&- 2(16H_*^2\dot{H}_* + 6\dot{H}_*^2 + 3H_*\ddot{H}_*)\Big] f^*_{\mathcal{G}T} - 72\rho_*^2 H_*^3(1-3\omega)(1+\omega)^2(4H_*^2\\
&+ \dot{H}_*)f^*_{\mathcal{G}TT} + 576\rho_* H_*^3(1+\omega)(4H_*^2+\dot{H}_*)(4H_*^2\dot{H}_* + 2\dot{H}_*^2\\
&+ 4H_*\ddot{H}_*)f^*_{\mathcal{G}\mathcal{G}T},\\
\mathfrak{U}_2 &= -12\rho_* H_*^2(1+\omega)\Big[3H_*^2(1-\omega) - 4(2H_*^2+3\dot{H}_*)\Big] f^*_{\mathcal{G}T} + 576\rho_* H_*^2\\
&\times (1+\omega)(4H_*^2\dot{H}_* + 2\dot{H}_*^2 + H_*\dot{H}_*)f^*_{\mathcal{G}\mathcal{G}T} - 72\rho_*^2 H_*^4(1-3\omega)\\
&\times (1+\omega^2)f^*_{\mathcal{G}TT},\\
\mathfrak{U}_3 &= 24\rho_* H_*^3(1+\omega)f^*_{\mathcal{G}T},\\
\mathfrak{U}_4 &= -\frac{3}{2}\rho_* H_*(1+\omega)\left[(1-\omega)f^*_T + 2\rho_*^2(1+\omega)(1-3\omega)^2 f^*_{TTT}\right] - \frac{15}{2}\rho_*^2 H_*\\
&\times (1-3\omega)(1+\omega)^2 f^*_{TT} + 24\rho_* H_*(1+\omega)(4H_*^2\dot{H}_* + 2\dot{H}_*^2 + H_*\ddot{H}_*)\\
&\times \left[f^*_{\mathcal{G}T} + \rho_*(1-3\omega)f^*_{TT\mathcal{G}}\right],\\
\mathfrak{U}_5 &= \rho_*\left(\kappa^2 + \frac{1}{2}(3-\omega)f^*_T\right) + (1+\omega)(1-3\omega)\rho_*^2 f^*_{TT}.
\end{aligned}$$

For model (4.2.10), the coefficients of $(\delta,\ \delta_m)$ have the following expressions

$$
\begin{aligned}
\hat{\mathfrak{X}}_1 &= 288H_*^6 f^*_{1_{\mathcal{G}\mathcal{G}}},\\
\hat{\mathfrak{X}}_2 &= 288H_*^5(3H_*^2+5\dot{H}_*)f^*_{1_{\mathcal{G}\mathcal{G}}} + 6912H_*^7(4H_*^2\dot{H}_* + 2\dot{H}_*^2 + H_*\ddot{H}_*)f^*_{1_{\mathcal{G}\mathcal{G}\mathcal{G}}},\\
\hat{\mathfrak{X}}_3 &= -6H_*^2 - 24H_*^2\dot{H}_* f^*_{1_{\mathcal{G}}} - 288H_*^4(4H_*^4 - 23H_*^2\dot{H}_* - 11\dot{H}_*^2 - 6H_*\ddot{H}_*)\\
&\times\ f^*_{1_{\mathcal{G}\mathcal{G}}} + 6912H_*^6(4H_*^2+\dot{H}_*)(4H_*^2\dot{H}_* + 2\dot{H}_*^2 + H_*\ddot{H}_*)f^*_{1_{\mathcal{G}\mathcal{G}\mathcal{G}}},\\
\hat{\mathfrak{X}}_5 &= \kappa^2\rho_* - \frac{1}{2}(\omega-3)\rho_* f^*_{2_T} + (1-3\omega)(1+\omega)\rho_*^2 f^*_{2_{TT}},\\
\hat{\mathfrak{U}}_1 &= 3(1+\omega)\rho_* H_*(\kappa^2 + f_{2_T}),\\
\hat{\mathfrak{U}}_4 &= -\frac{3}{2}\rho_* H_*(1+\omega)\left[(1-\omega)f^*_{2_T} + 2\rho_*^2(1+\omega)(1-3\omega)^2 f^*_{2_{TTT}}\right] - \frac{15}{2}\rho_*^2 H_*\\
&\times\ (1-3\omega)(1+\omega)^2 f^*_{2_{TT}},\\
\hat{\mathfrak{U}}_5 &= \rho_*\left(\kappa^2 + \frac{1}{2}(3-\omega)f^*_{2_T}\right) + (1+\omega)(1-3\omega)\rho_*^2 f^*_{2_{TT}}.
\end{aligned}
$$

Appendix C

List of Publications

The contents of this thesis are based on the following research papers published or submitted in journals of International repute. These papers are also attached herewith.

1. Sharif, M. and **Ikram, A.**: *Instability Analysis of Expansion-free Sphere in $f(\mathcal{G})$ Gravity*,
   Int. J. Mod. Phys. D **26**(2017)1750104.

2. Sharif, M. and **Ikram, A.**: *Energy Conditions in $f(\mathcal{G},T)$ Gravity*,
   Eur. Phys. J. C **76**(2016)640.

3. Sharif, M. and **Ikram, A.**: *Stability Analysis of Einstein Universe in $f(\mathcal{G},T)$ Gravity*,
   Int. J. Mod. Phys. D **26**(2017)1750084.

4. Sharif, M. and **Ikram, A.**: *Inhomogeneous Perturbations and Stability in $f(\mathcal{G},T)$ Gravity*,
   Submitted for Publication.

5. Sharif, M. and **Ikram, A.**: *Stability Analysis of Some Reconstructed Cosmological Models in $f(\mathcal{G},T)$ Gravity*,
   Phys. Dark Universe **17**(2017)1.

Also, the following papers related to this thesis have been published, accepted or submitted for publication.

1. Sharif, M. and **Ikram, A.**: *Warm Inflation in $f(\mathcal{G})$ Theory of Gravity*,
   J. Exp. Theor. Phys. **123**(2016)40.

2. Sharif, M. and **Ikram, A.**: *Inflationary Dynamics $f(\mathcal{G})$ Gravity*, Int. J. Mod. Phys. D **26**(2017)1750030.

3. Sharif, M. and **Ikram, A.**: *Thermodynamics in $f(\mathcal{G},T)$ Gravity*, Adv. High Energy Phys. (to appear, 2018).

4. Sharif, M. and **Ikram, A.**: *Galactic Halo Traversable Wormhole Solutions in $f(\mathcal{G},T)$ Gravity*,
Submitted for Publication.

5. Sharif, M. and **Ikram, A.**: *Existence of Static Wormholes in $f(\mathcal{G},T)$ Gravity*, Int. J. Mod. Phys. D **27**(2018)1750182.

6. Sharif, M. and **Ikram, A.**: *Anisotropic Perturbations and Stability of Static Universe in $f(\mathcal{G},T)$ Gravity*,
Eur. Phys. J. Plus **132**(2017)526.

7. Sharif, M. and **Ikram, A.**: *Cosmic Evolution of Holographic Dark Energy in $f(\mathcal{G},T)$ Gravity*,
Submitted for Publication.

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Instability analysis of expansion-free sphere in $f(\mathcal{G})$ gravity

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The aim of this paper is to study the dynamical instability of expansion-free spherically symmetric anisotropic fluid in the framework of $f(\mathcal{G})$ gravity. We apply perturbation scheme of the first-order to the metric functions as well as matter variables and construct modified field equations for both static and perturbed configurations using power-law $f(\mathcal{G})$ model. To discuss the instability dynamics, we use the contracted Bianchi identities to formulate the dynamical equations in both Newtonian and post-Newtonian regimes. It is found that the range of instability is independent of adiabatic index for expansion-free fluid but depends on anisotropic pressures, energy density and Gauss–Bonnet (GB) terms.

Keywords: Instability; Newtonian and post-Newtonian regimes; $f(\mathcal{G})$ gravity.

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1. Introduction

Dark energy (DE) and dynamical instability of celestial objects are the most fascinating issues of cosmology and astrophysics. The observational data from supernova type Ia, cosmic microwave background radiation, large scale structure, etc. unambiguously proved this expanding behavior of the universe.[1] This cosmic expansion is considered as a result of DE which possesses repulsive nature but its complete characteristics are still not known. The modified theories are taken as the favorable candidates to discuss the nature of DE. Such modified theories are obtained by including the scalar invariants and their corresponding arbitrary functions in the Einstein–Hilbert action.

Gauss–Bonnet (GB) invariant being a linear combination of the Riemann tensor, Ricci tensor and Ricci scalar has attained special attention in cosmology defined as

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\eta}R^{\mu\nu\xi\eta}.$$

This possesses a remarkably interesting feature that it is free from spin-2 ghosts instabilities.[2] It is a four-dimensional topological quantity which has trivial contribution in field equations when added linearly in the Lagrangian. The coupling of $\mathcal{G}$ with scalar field helps to study dynamical effects of $\mathcal{G}$ and naturally appears in low energy string effective actions.[3] Without introducing the scalar field, Nojiri and Odintsov[4] studied the effects of $\mathcal{G}$ in four dimensions by adding $f(\mathcal{G})$ in the Einstein–Hilbert action named as $f(\mathcal{G})$ theory of gravity.

The interesting features of late-time cosmology such as transition from deceleration to acceleration, crossing the phantom divide line and current cosmic acceleration with effective equation-of-state have been discussed for a suitable choice of $f(\mathcal{G})$ model.[4] De Felice and Tsujikawa[5] studied cosmologically viable $f(\mathcal{G})$ gravity models and found that these models are consistent with solar system constraints for a wide range of model parameters as well as responsible for late-time cosmic acceleration. Zhou *et al.*[6] explored the phase space analysis for these models and obtained conditions on their cosmological viability. Sharif and Fatima[7] discussed the role of GB term for the early as well as late-time accelerating phases of the universe in the background of flat FRW universe model.

Gravitational collapse of celestial objects is the captivating issue in general relativity as well as modified theories of gravity. This naturally occurring phenomenon takes place when the hydrostatic equilibrium state of stellar objects is destroyed. During collapse, these heavenly objects pass through several stages which are described by dynamical equations. The existence of static stellar objects are interesting only when they show stable behavior against fluctuations. The issue of dynamical stability/instability in collapsing stars is important in the formation and evolution of celestial objects. In 1964, Chandrasekhar[8] analyzed the dynamical instability for isotropic spherical star through adiabatic index (Γ). This index measures the change in pressure with respect to given variation in density, thus leading to fluid compressibility.

Herrera *et al.*[9] investigated the dynamical instability of spherical star filled with nonadiabatic isotropic fluid and found that the instability range increases at Newtonian (N) approximation. Chan *et al.*[10] studied the instability ranges at N and post-Newtonian (pN) regimes for spherical star filled with matter, anisotropic pressures, dissipation and shear viscosity. Sharif and Bhatti[11] explored the role of Γ in both N and pN regions for nonstatic axially symmetric spacetime with anisotropic fluid distribution.

Expansion-free condition has a significant importance in astrophysics for describing the evolution of collapsing fluid after explosion. The vanishing of expansion scalar forms the cavity inside the fluid which was first observed by Skripin for spherical nondissipative isotropic fluid configuration.[12] Herrera *et al.*[13] discussed that the innermost shell of the collapsing fluid moves away from the center for zero expansion condition and consequently, a vacuum cavity is formed within matter distribution. Herrera *et al.*[14] explored the evolution of cavities using kinematical condition (other than zero expansion) for dissipative anisotropic spherically symmetric matter distribution. Thus, expansion-free state is a sufficient condition for the existence of cavities inside the fluid but the converse does not necessarily hold.

Herrera and his collaborators[15] investigated the dynamical instability under expansion-free condition in N and pN regimes for nondissipative anisotropic matter distribution using spherically symmetric spacetime. Sharif and Azam[16] examined the stability for cylindrical anisotropic fluid distribution and found that the stability only depends on energy density inhomogeneity and pressure anisotropy. It is found that the instability ranges for planar geometry is greater than the spherical under zero expansion condition.[17] Sharif and Bhatti[18] discussed the effects of electromagnetism on the stability for cylindrical as well as planar geometries using the same condition. Abbas and Sarwar[19] investigated the stability of self-gravitating fluids in Einstein GB gravity and analyzed the results graphically. Dynamical instability under expansion-free condition is also widely studied in modified theories of gravity.[20]

In this paper, we study dynamical instability of spherically symmetric anisotropic collapsing fluid under expansion-free condition in the framework of $f(\mathcal{G})$ gravity. The paper has the following format. In Sec. 2, we construct the field equations of $f(\mathcal{G})$ gravity, dynamical equations and junction conditions while Sec. 3 covers the perturbation scheme. Section 4 is devoted to analyze the dynamical equations at N and pN approximations under expansion-free condition and corresponding instability ranges are examined. The results are summarized in the last section.

2. Dynamics of $f(\mathcal{G})$ Gravity

In this section, we construct the field as well as dynamical equations and matching conditions in $f(\mathcal{G})$ gravity for spherically symmetric spacetime. The action for $f(\mathcal{G})$ theory of gravity is defined as[21]

$$\mathcal{S}_{f(\mathcal{G})} = \int d^4x\sqrt{-g}\left(\frac{R+f(\mathcal{G})}{2\kappa} + \mathcal{L}_m\right), \tag{1}$$

where κ, g and $\mathcal{L}_m$ denote the coupling constant, determinant of the metric tensor and Lagrangian density associated with matter distribution, respectively. Variation of $f(\mathcal{G})$ action with respect to $g_{\mu\nu}$ yields the field equations as follows:

$$G_{\mu\nu} = \kappa\mathcal{T}^{(\mathrm{eff})}_{\mu\nu} = \kappa(\mathcal{T}^{(m)}_{\mu\nu} + \mathcal{T}^{\mathcal{G}}_{\mu\nu}), \tag{2}$$

where $G_{\mu\nu}$ and $\mathcal{T}^{(m)}_{\mu\nu}$ represent Einstein tensor and energy–momentum tensor, respectively whereas the GB contribution is given by

$$\begin{aligned}\kappa\mathcal{T}^{\mathcal{G}}_{\mu\nu} = {}& \frac{1}{2}g_{\mu\nu}f(\mathcal{G}) - 2RR_{\mu\nu}F + 4R_{\nu\xi}R^{\xi}_{\mu}F - 2R_{\mu\xi\eta\sigma}R^{\xi\eta\sigma}_{\nu}F - 4R_{\mu\xi\eta\nu}R^{\xi\eta}F\\ &+ 2R\nabla_\mu\nabla_\nu F - 4R^{\xi}_{\nu}\nabla_\mu\nabla_\xi F - 4R^{\xi}_{\mu}\nabla_\nu\nabla_\xi F - 2g_{\mu\nu}R\nabla^2F + 4R_{\mu\nu}\nabla^2F\\ &+ 4g_{\mu\nu}R^{\xi\eta}\nabla_\xi\nabla_\eta F - 4R_{\mu\xi\nu\eta}\nabla^\xi\nabla^\eta F,\end{aligned} \tag{3}$$

where $F = df(\mathcal{G})/d\mathcal{G}$ and $\nabla^2 = \nabla_\mu \nabla^\mu$. The matter distribution for the collapsing fluid is encapsulated by 3D spherical hypersurface $\Sigma_{(e)}$ which separates the 4D line element into interior and exterior regions.

The spherical symmetric spacetime interior to $\Sigma_{(e)}$ in comoving coordinates is of the form

$$ds^2 = -A^2(t,r)dt^2 + B^2(t,r)dr^2 + C^2(t,r)(d\theta^2 + \sin^2\theta d\phi^2). \quad (4)$$

Assume that the metric coefficients are positive functions in which $A(t,r)$ and $B(t,r)$ are dimensionless while $C(t,r)$ has dimension of length. The energy–momentum tensor for nondissipative anisotropic fluid is

$$T_{\mu\nu} = (\rho + P_\perp)V_\mu V_\nu + P_\perp g_{\mu\nu} + (P_r - P_\perp)\chi_\mu\chi_\nu, \quad (5)$$

where $\rho, P_r,\ P_\perp, \chi_\mu$ and V_μ are the energy density, radial pressure, tangential pressure, unit four-vector in radial direction and four velocity of the fluid, respectively. The quantities V_μ and χ_μ satisfy the relations $V_\mu V^\mu = -1$, $\chi_\mu\chi^\mu = 1$, $V_\mu\chi^\mu = 0$ and are defined as

$$V^\mu = A^{-1}\delta^\mu_0, \quad \chi^\mu = B^{-1}\delta^\mu_1. \quad (6)$$

The expansion scalar ($\Theta = V^\mu{}_{;\mu}$) is given by

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + \frac{2\dot{C}}{C}\right), \quad (7)$$

where dot denotes the time derivative. Gauss–Bonnet invariant for Eq. (4) takes the form

$$\begin{aligned}\mathcal{G} = \frac{8}{ABC^2}\Bigg[&\left(1 + \frac{\dot{C}^2}{A^2} - \frac{C'^2}{B^2}\right)\left[\frac{A}{B}\left(\frac{A'B'}{AB} - \frac{A''}{A}\right) - \frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB} - \frac{\ddot{B}}{B}\right)\right]\\
&+ \frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC} + \frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC} - 2\frac{\dot{C}'}{C}\right) + \frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2} + \frac{A'C'}{B^2}\right)\\
&\times\left(C'' - \frac{B'C'}{B}\right) + \frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2} + \frac{\dot{B}\dot{C}}{A^2}\right) + \frac{2}{AB}(\dot{C}'^2 - \ddot{C}C'')\\
&- \frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3} - \frac{\dot{B}C'^2}{B^3}\right)\Bigg],\end{aligned} \quad (8)$$

where prime represents the derivative with respect to r. Using Eqs. (2)–(6), we obtain the following set of field equations:

$$\begin{aligned}\kappa\rho A^2 = \frac{1}{2}A^2 f + &\left(1 + \frac{\dot{C}^2}{A^2} - \frac{C'^2}{B^2}\right)\left[\left(\frac{A}{C}\right)^2 - \frac{4A}{BC^2}\left\{\frac{A}{B}\left(\frac{A'B'}{AB} - \frac{A''}{A}\right)\right.\right.\\
&\left.\left. - \frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB} - \frac{\ddot{B}}{B}\right)\right\}F + \frac{4\dot{B}}{BC^2}\dot{F} + \frac{4A^2B'}{B^3C^2}F' - \frac{4A^2}{B^2C^2}F''\right]\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)\left(\frac{2A^2}{BC}-\frac{8\dot{C}}{BC^2}\dot{F}-\frac{8A^2C'}{B^3C^2}F'\right)-\frac{4AF}{BC^2}\\
&\times\left[\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\right.\\
&\times\left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)\\
&\left.+\frac{2}{AB}({\dot{C}'}^2-\ddot{C}C'')\right],
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
0=\frac{2}{C}&\left[\frac{2}{C}\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left(\dot{F}'-\frac{A'}{A}\dot{F}-\frac{\dot{B}}{B}F'\right)\right.\\
&\left.-\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)\left(1+\frac{4\dot{C}}{A^2C}\dot{F}-\frac{4C'}{B^2C}F'\right)\right],
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
\kappa B^2P_r=&-\frac{1}{2}B^2f+\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left[-\frac{B^2}{C^2}+\frac{4B}{AC^2}\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right.\right.\\
&\left.\left.-\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}F+\frac{4B^2\dot{A}}{A^3C^2}\dot{F}+\frac{4A'}{AC^2}F'-\frac{4B^2}{A^2C^2}\ddot{F}\right]\\
&+\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)\left(\frac{2B^2}{AC}+\frac{8B^2\dot{C}}{A^3C^2}\dot{F}-\frac{8C'}{AC^2}F'\right)+\frac{4BF}{AC^2}\\
&\times\left[\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\right.\\
&\times\left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)\\
&\left.+\frac{2}{AB}({\dot{C}'}^2-\ddot{C}C'')\right],
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\kappa C^2P_\perp=&\frac{C}{A}\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)-\frac{C}{B}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)+\frac{C^2}{AB}\\
&\times\left[\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)-\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right]-\frac{1}{2}C^2f+\frac{4F}{AB}\\
&\times\left[\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)-\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}\right.
\end{aligned}
$$

$$+\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)$$
$$\times\left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)$$
$$\left.+\frac{2}{AB}(\dot{C}'^2-\ddot{C}C'')\right]+\frac{4C}{A^3B}\left[\dot{B}\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)\right.$$
$$+\dot{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)-\frac{2A'}{B}\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)$$
$$\left.+\dot{C}\left\{\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)-\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right\}\right]\dot{F}$$
$$+\frac{4C}{AB^3}\left[B'\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)+A'\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)\right.$$
$$-\frac{2\dot{B}}{A}\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)-C'\left\{\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right.$$
$$\left.\left.-\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right\}\right]F'+\frac{8C}{A^2B^2}\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)\dot{F}'$$
$$-\frac{4C}{AB^2}\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)F''-\frac{4C}{A^2B}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)\ddot{F}. \quad (12)$$

The dynamical equations play a crucial role to discuss the fundamental characteristics of collapsing stars which are formulated through contracted Bianchi identities $G^{\mu\nu}_{;\nu}V_\mu=0$ and $G^{\mu\nu}_{;\nu}\chi_\mu=0$ as follows:

$$\frac{\dot{\rho}}{A}+(\rho+P_r)\frac{\dot{B}}{AB}+2(\rho+P_\perp)\frac{\dot{C}}{AC}-\Delta_1=0, \quad (13)$$
$$\frac{P_r'}{B}+(\rho+P_r)\frac{A'}{AB}+2(P_r-P_\perp)\frac{C'}{BC}+\Delta_2=0, \quad (14)$$

where the expressions for Δ_1 and Δ_2 are given in Appendix A. The mass function for spherical symmetric spacetime is[22]

$$m(t,r)=\frac{C}{2}\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right), \quad (15)$$

which describes the total energy of collapsing star with radius C. The proper time and radial derivatives are defined as

$$D_t=\frac{1}{A}\frac{\partial}{\partial t}, \quad D_C=\frac{1}{C'}\frac{\partial}{\partial r}. \quad (16)$$

The proper time derivative of $C(t,r)$ gives the collapsing fluid velocity, i.e. $U = \frac{\dot{C}}{A}$, $(U<0)$. Taking the proper radial derivative of mass function, we obtain

$$D_C m = \frac{1}{2}\kappa C^2\left[\rho + \frac{\mathcal{T}_{00}^{\mathcal{G}}}{A^2} - \frac{\mathcal{T}_{01}^{\mathcal{G}}}{AB}\left(\frac{U}{E}\right)\right], \tag{17}$$

where $E = \frac{C'}{B} = (1 - \frac{2m}{C} + U^2)^{\frac{1}{2}}$. This radial variation shows the effect of GB terms and energy density on mass between neighboring spherical surfaces which would decrease due to repulsive nature of DE. Integration of Eq. (17) yields

$$m = \frac{1}{2}\kappa\int_0^C \left[C^2\left\{\rho + \frac{\mathcal{T}_{00}^{\mathcal{G}}}{A^2} - \frac{\mathcal{T}_{01}^{\mathcal{G}}}{AB}\left(\frac{U}{E}\right)\right\}\right] dC. \tag{18}$$

We take the following line element outside to $\Sigma_{(e)}$

$$ds_+^2 = -\left(1 - \frac{2M}{r}\right)d\tau^2 - 2drd\tau + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{19}$$

where τ and M represent the retarded time and total mass of the fluid. We consider Darmois junction conditions for the smooth matching of (4) and (19) spacetimes.[23] On the boundary surface $\Sigma_{(e)}$, the continuity of the line elements and extrinsic curvatures yield the following relations

$$M = m(t,r), \quad P_r = -\frac{\mathcal{T}_{01}^{\mathcal{G}}}{AB} - \frac{\mathcal{T}_{11}^{\mathcal{G}}}{B^2}. \tag{20}$$

It is well known that vanishing of expansion scalar forms a vacuum cavity inside the fluid. Such models require an additional boundary surface $\Sigma_{(i)}$ between the cavity and fluid distribution. The matching of Minkowski line element inside the cavity to the collapsing fluid over $\Sigma_{(i)}$ gives

$$m(t,r) = 0, \quad P_r = -\frac{\mathcal{T}_{01}^{\mathcal{G}}}{AB} - \frac{\mathcal{T}_{11}^{\mathcal{G}}}{B^2}.$$

It is worth mentioning here that the cosmic voids (underdense regions and sponge-like structures) are the physical examples of expansion-free models. Voids vary in size from minivoids to macrovoids and occupy 40% of the universe.[24] They are usually considered as spherical vacuum cavities surrounded by the fluid distribution but generally voids are neither spherical nor empty.[25]

3. Perturbation Scheme and $f(\mathcal{G})$ Model

In this section, we take a particular $f(\mathcal{G})$ model and use perturbation approach to discuss the evolution of zero expansion collapsing matter distribution. The considered viable $f(\mathcal{G})$ model is given by[26]

$$f(\mathcal{G}) = \beta\mathcal{G}^n, \tag{21}$$

where $n > 0, \neq 1$ and β is an arbitrary constant. In the perturbation scheme, it is assumed that initially all metric components as well as physical quantities are in static equilibrium, i.e. they possess only radial dependence but as time passes, these quantities also become time dependent. For the first-order perturbation, we consider

$$A(t,r) = A_0(r) + \alpha T(t)a(r), \tag{22}$$

$$B(t,r) = B_0(r) + \alpha T(t)b(r), \tag{23}$$

$$C(t,r) = C_0(r) + \alpha T(t)c(r), \tag{24}$$

$$\rho(t,r) = \rho_0(r) + \alpha\tilde{\rho}(t,r), \tag{25}$$

$$P_r(t,r) = P_{r0}(r) + \alpha\tilde{P}_r(t,r), \tag{26}$$

$$P_\perp(t,r) = P_{\perp 0}(r) + \alpha\tilde{P}_\perp(t,r), \tag{27}$$

$$m(t,r) = m_0(r) + \alpha\tilde{m}(t,r), \tag{28}$$

$$\Theta(t,r) = \alpha\tilde{\Theta}(t,r), \tag{29}$$

where $0 < \alpha \ll 1$ is a perturbation parameter and subscript zero denotes the static part of the corresponding quantities. The expressions for $\mathcal{G}$ and $f(\mathcal{G})$ model are

$$\mathcal{G}(t,r) = \mathcal{G}_0(r) + \alpha T(t)g(r), \tag{30}$$

$$f(\mathcal{G}(t,r)) = \beta\mathcal{G}_0^n + \alpha\beta n T(t)g(r)\mathcal{G}_0^{n-1}, \tag{31}$$

$$F(t,r) = \beta n\mathcal{G}_0^{n-1} + \alpha\beta n(n-1)T(t)g(r)\mathcal{G}_0^{n-2}. \tag{32}$$

We take $C_0(r) = r$ as Schwarzschild coordinate and static configuration of $\mathcal{G}$ and field equations are given by

$$\mathcal{G}_0 = \frac{8}{r^2 A_0 B_0^2}\left[\left(1 - \frac{1}{B_0^2}\right)\left(\frac{A_0' B_0'}{B_0} - A_0''\right) - \frac{2A_0' B_0'}{B_0^3}\right], \tag{33}$$

$$\begin{aligned}\kappa\rho = {} & \frac{1}{r}\left(\frac{1}{r} - \frac{1}{rB_0^2} + \frac{2B_0'}{B_0^3}\right) + \frac{1}{2}\beta\mathcal{G}_0^n + 4\beta n\frac{A_0''}{r^2 A_0 B_0^2}\left(1 - \frac{1}{B_0^2}\right)\mathcal{G}_0^{n-1} \\ & + 4\beta n\frac{A_0 B_0'}{r^2 B_0^3}\left(1 - \frac{3}{B_0^2}\right)\left(\frac{\mathcal{G}_0^{n-1}}{A_0}\right)' - 4\beta n\frac{(n-1)}{r^2 B_0^2}\left(1 - \frac{1}{B_0^2}\right)(\mathcal{G}_0^{n-2}\mathcal{G}_0')',\end{aligned} \tag{34}$$

$$\begin{aligned}\kappa P_{r0} = {} & \frac{1}{r^2 B_0^2}\left(1 - B_0^2 + 2r\frac{A_0'}{A_0}\right) - \frac{1}{2}\beta\mathcal{G}_0^n + \frac{4\beta n}{r^2 A_0 B_0^2}\left[\frac{A_0' B_0'}{B_0}\left(1 - \frac{3}{B_0^2}\right)\right. \\ & \left. - A_0''\left(1 - \frac{1}{B_0^2}\right)\right]\mathcal{G}_0^{n-1} + 4\beta n(n-1)\frac{A_0'}{r^2 A_0 B_0^2}\left(1 - \frac{3}{B_0^2}\right)\mathcal{G}_0^{n-2}\mathcal{G}_0',\end{aligned} \tag{35}$$

$$\begin{aligned}\kappa P_{\perp 0} = {} & \left(\frac{A_0''}{A_0} - \frac{A_0' B_0'}{A_0 B_0}\right)\left(\frac{1}{B_0^2} - \frac{4\beta n}{r^2 B_0^2}\mathcal{G}_0^{n-1}\right) + \frac{1}{rB_0^2}\left(\frac{A_0'}{A_0} - \frac{B_0'}{B_0}\right) - \frac{1}{2}\beta\mathcal{G}_0^n \\ & - 4\beta n\frac{1}{B_0^4}\left(\frac{A_0''}{A_0} - \frac{3A_0' B_0'}{A_0 B_0}\right)\left(\frac{1}{r}\mathcal{G}_0^{n-1}\right)' - 4\beta n(n-1)\frac{A_0'}{rA_0 B_0^4}\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)'.\end{aligned} \tag{36}$$

Using Eqs. (22)–(32), the perturbed configuration of $\mathcal{G}$ and the field equations are given in Appendix B. The static perturbation scheme is identically satisfied for the first dynamical equation (13) whereas for Eq. (14), we obtain the following expression:

$$\begin{aligned}
&P'_{r0}+(\rho_0+P_{r0})\frac{A'_0}{A_0}+\frac{2}{r}(P_{r0}-P_{\perp0})+4\beta n\mathcal{G}_0^{n-1}\frac{1}{\kappa}\left[\frac{1}{8}\mathcal{G}'_0-\frac{2}{r^3B_0^2}\left(1-\frac{1}{B_0^2}\right)\frac{A''_0}{A_0}\right.\\
&\quad+\frac{1}{r^2B_0^2}\left(1-\frac{1}{B_0^2}\right)\frac{A'''_0}{A_0}-\frac{1}{r^2B_0^2}\left(3-\frac{7}{B_0^2}\right)\frac{A''_0B'_0}{A_0B_0}-\frac{1}{r^2B_0^2}\left(1-\frac{1}{B_0^2}\right)\frac{A'_0A''_0}{A_0^2}\\
&\quad-\frac{1}{r^2B_0^2}\left(1-\frac{3}{B_0^2}\right)\frac{A'_0B''_0}{A_0B_0}+\frac{2}{r^3B_0^2}\left(1-\frac{3}{B_0^2}\right)\frac{A'_0B'_0}{A_0B_0}+\frac{1}{r^2B_0^2}\left(1-\frac{3}{B_0^2}\right)\frac{A'^2_0B'_0}{A_0^2B_0}\\
&\quad\left.+\frac{3}{r^2B_0^2}\left(1-\frac{5}{B_0^2}\right)\frac{A'_0B'^2_0}{A_0B_0^2}\right]=0. \qquad (37)
\end{aligned}$$

The perturbed configuration of Eqs. (13) and (14) are

$$\begin{aligned}
&\dot{\bar{\rho}}+\left[(\rho_0+P_{r0})\frac{b}{B_0}+2(\rho+P_{\perp0})\frac{c}{r}+\Delta_{1p}\right]\dot{T}\\
&\quad+4\beta n\mathcal{G}_0^{n-1}\frac{1}{\kappa}\left[\frac{b}{r^2A_0^2B_0}\left(1-\frac{1}{B_0^2}\right)-\frac{2cB'_0}{r^2A_0^2B_0^3}\right]\dddot{T}=0, \qquad (38)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{B_0}\left[\bar{P}'_r+(\bar{\rho}+\bar{P}_r)\frac{A'_0}{A_0}+\frac{2}{r}(\bar{P}_r-\bar{P}_\perp)\right.\\
&\quad\left.+\left\{(\rho_0+P_{r0})\left(\frac{a}{A_0}\right)'+2(P_{r0}-P_{\perp0})\left(\frac{c}{r}\right)'\right\}T\right]\\
&\quad-\frac{b}{B_0^2}\left[P'_{r0}+(\rho_0+P_{r0})\frac{A'_0}{A_0}+\frac{2}{r}(P_{r0}-P_{\perp0})\right]T+\Delta_{2p}=0, \qquad (39)
\end{aligned}$$

where Δ_{1p} and Δ_{2p} are given in Appendix B. Integration of Eq. (38) with respect to time leads to

$$\begin{aligned}
&\bar{\rho}+\left[(\rho_0+P_{r0})\frac{b}{B_0}+2(\rho+P_{\perp0})\frac{c}{r}+\Delta_{1p}\right]T\\
&\quad+4\beta n\mathcal{G}_0^{n-1}\frac{1}{\kappa}\left[\frac{b}{r^2A_0^2B_0}\left(1-\frac{1}{B_0^2}\right)-\frac{2cB'_0}{r^2A_0^2B_0^3}\right]\ddot{T}=0. \qquad (40)
\end{aligned}$$

Using Eqs. (7) and (29), we obtain

$$\tilde{\Theta}=\frac{1}{A_0}\left(\frac{b}{B_0}+\frac{2c}{r}\right)\dot{T}. \qquad (41)$$

The static as well as perturbed configurations of mass function are

$$m_0=\frac{1}{2}r\left(1-\frac{1}{B_0^2}\right),\quad \tilde{m}=-\frac{1}{B_0^2}\left[\frac{c}{2}(1-B_0^2)+r\left(c'-\frac{b}{B_0}\right)\right]T. \qquad (42)$$

The static configuration of second condition in Eq. (20) is obtained by using Eq. (35) while the perturbed one is formulated from Eqs. (B.3) and (B.4). Inserting the corresponding static and nonstatic configurations of the second junction condition in Eq. (B.4), we obtain the following second-order differential equation over $\Sigma_{(e)}$

$$\ddot{T}(t) + u(r)\dot{T}(t) + v(r)T(t) = 0, \tag{43}$$

where

$$\begin{aligned} u(r) &= 2\beta n(n-1)\mathcal{G}_0^{n-2}\frac{A_0}{rcB_0}\left[\frac{b}{B_0}\left(1-\frac{3}{B_0^2}\right)\mathcal{G}_0' \right. \\ &\quad \left. -\left\{(n-2)\frac{g\mathcal{G}_0'}{\mathcal{G}_0} + A_0\left(\frac{g}{A_0}\right)'\right\}\left(1-\frac{1}{B_0^2}\right) + \frac{2A_0}{B_0^2}\left(\frac{c}{A_0}\right)'\mathcal{G}_0'\right], \\ v(r) &= \frac{A_0^2}{c}\left[\frac{1}{r}\left(\frac{b}{B_0}-\frac{c}{r}\right)\left(1-\frac{1}{B_0^2}\right) - \frac{2}{rB_0^2}\left(c'-\frac{b}{B_0}\right)\right. \\ &\quad \left. + \frac{A_0'}{A_0B_0^2}\left(\frac{c}{r}+\frac{a}{A_0}-\frac{2b}{B_0}\right) - \frac{1}{A_0B_0^2}\left(a' - 2b\frac{A_0'}{B_0} + c'A_0'\right)\right]. \end{aligned}$$

Assume that $u(r)$ and $v(r)$ are positive functions to discuss the instability regions. We obtain solution of Eq. (43) as

$$T(t) = -\exp(\Omega_{\Sigma(e)}t), \quad \Omega_{\Sigma(e)} = \frac{-u+\sqrt{u^2-4v}}{2}, \tag{44}$$

which represents that the system initiates collapsing at $t=-\infty$ with $T(-\infty)=0$. At this stage, the system is in static state but it continues collapsing as t increases.

4. Expansion-Free Newtonian and Post-Newtonian Regimes

In this section, we identify terms associated with N, pN as well as parameterized post-Newtonian (ppN) approximations and discuss the dynamical equations using expansion-free condition. For this purpose, the second dynamical equation is converted into centimeter-gram-second (c.g.s.) units from relativistic units. Equation (41) yields the zero expansion condition as

$$\frac{b}{B_0} = -\frac{2c}{r}. \tag{45}$$

For Newtonian approximation, we assume $\rho_0 \gg P_{r0}$ and $\rho_0 \gg P_{\perp 0}$. Adding Eqs. (34) and (35) then using the static configuration of m in Eq. (42), we have

$$\begin{aligned} \frac{A_0'}{A_0} &= \frac{r}{r-2m_0}\left[\frac{\kappa r^3}{2}(\rho_0 + P_{r0}) + \frac{m_0}{r^2}\{r^2 + 2\beta n(n-1)\right. \\ &\quad \left. \times (6m_0 - 2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\} + 4\beta n(n-1)(r-2m_0)\frac{m_0}{r}(\mathcal{G}_0^{n-2}\mathcal{G}_0')'\right] \\ &\quad \times [r^2 + 2\beta n(n-1)(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0']^{-1}. \end{aligned} \tag{46}$$

Equation (36) gives the expression for $\frac{A_0''}{A_0}$ as follows:

$$\frac{A_0''}{A_0}=\frac{r^4}{r-2m_0}\Bigg[\kappa P_{\perp 0}+\frac{1}{2}\beta\mathcal{G}_0^n-\frac{m_0}{r^3}-\Big\{\frac{m_0}{r^5}(r^3+8\beta n(r-3m_0)\mathcal{G}_0^{n-1}$$
$$-12\beta n(n-1)r(r-2m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0')+\frac{1}{r^3}(r-2m_0)$$
$$\times(r-4\beta n(n-1)(r-2m_0)(\mathcal{G}_0^{n-2}\mathcal{G}_0')')\Big\}\frac{A_0'}{A_0}\Bigg]$$
$$\times[r^3-8\beta nm_0\mathcal{G}_0^{n-1}-4\beta n(n-1)r(r-2m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0']^{-1}. \tag{47}$$

Similarly, the value of $\frac{A_0'''}{A_0}$ is obtained by taking the derivative of Eq. (34) with respect to r. Substituting Eqs. (42), (46) and (47) along with value of $\frac{A_0'''}{A_0}$ in Eq. (37), we obtain an expression in relativistic units

$$P_{r0}'=-\Bigg[\frac{1}{r^6}\{r^6(\rho_0+P_{r0})+8\beta nm_0(2r-m_0)\mathcal{G}_0^{n-1}+8\beta n(n-1)rm_0$$
$$\times(r-3m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\}\Bigg\{\frac{r}{r-2m_0}\Bigg[\frac{\kappa r^3}{2}(\rho_0+P_{r0})+\frac{m_0}{r^2}\{r^2+2\beta n$$
$$\times(n-1)(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\}+4\beta n(n-1)(r-2m_0)\frac{m_0}{r}\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)'\Bigg]$$
$$\times[r^2+2\beta n(n-1)(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0']^{-1}\Bigg\}-\frac{8m_0}{r^4}\beta n(n-1)$$
$$\times(r-2m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\Bigg\{\frac{r^4}{r-2m_0}\Bigg[\kappa P_{\perp 0}+\frac{1}{2}\beta\mathcal{G}_0^n-\frac{m_0}{r^3}$$
$$-\Bigg\{\frac{m_0}{r^5}(r^3+8\beta n(r-3m_0)\mathcal{G}_0^{n-1}-12\beta n(n-1)r(r-2m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0')$$
$$+\frac{1}{r^3}(r-2m_0)(r-4\beta n(n-1)(r-2m_0)(\mathcal{G}_0^{n-2}\mathcal{G}_0')')\Bigg\}$$
$$\times\Bigg\{\frac{r}{r-2m_0}\Bigg[\frac{\kappa r^3}{2}(\rho_0+P_{r0})+\frac{m_0}{r^2}\{r^2+2\beta n(n-1)$$
$$\times(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\}+4\beta n(n-1)(r-2m_0)\frac{m_0}{r}\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)'\Bigg]$$
$$\times[r^2+2\beta n(n-1)(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0']^{-1}\Bigg\}\Bigg]$$
$$\times[r^3-8\beta nm_0\mathcal{G}_0^{n-1}-4\beta n(n-1)r(r-2m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0']^{-1}\Bigg\}$$
$$+\frac{2}{r}(P_{r0}-P_{\perp 0})+\kappa\rho_0'+\frac{4m_0(2r-m_0)}{r^4(r-2m_0)}+\frac{8m_0}{r^4}\beta n(n-1)(r-2m_0)$$

$$
\begin{aligned}
&\times \left\{(n-2)\mathcal{G}_0'(\mathcal{G}_0^{n-3}\mathcal{G}_0')' + (\mathcal{G}_0^{n-2}\mathcal{G}_0'')' + (n-2)r^2\mathcal{G}_0^{n-3}\mathcal{G}_0'\left(\frac{\mathcal{G}_0'}{r^2}\right)' - \frac{2}{r}\mathcal{G}_0^{n-2}\mathcal{G}_0''\right\} \\
&- \frac{8m_0}{r^6}\beta n(n-1)(r-3m_0)\left\{(n-2)r\mathcal{G}_0^{n-3}\mathcal{G}_0'^2 - \frac{(2r-m_0)}{(r-2m_0)}\mathcal{G}_0^{n-2}\mathcal{G}_0'\right. \\
&\left. - 2\mathcal{G}_0^{n-2}\mathcal{G}_0'\right\} + \frac{8m_0}{r^5}\beta n(n-1)(2r-7m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'' + \frac{24m_0^2(2r-5m_0)}{r^6(r-2m_0)} \\
&\left.\times \beta n(n-1)\mathcal{G}_0^{n-2}\mathcal{G}_0' + 8\beta n(n-1)(n-2)\frac{m_0}{r^5}(r-4m_0)\mathcal{G}_0^{n-3}\mathcal{G}_0'^2\right]. \qquad (48)
\end{aligned}
$$

In c.g.s. units, this equation becomes

$$
\begin{aligned}
P_{r0}' = &- \left[\frac{1}{r^6}G\left\{r^6(\rho_0 + \mathcal{C}^{-2}P_{r0}) + 8\beta n m_0(2r - \mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-1} + 8\beta n\right.\right. \\
&\times (n-1)rm_0(r-3\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\} \left\{\frac{r}{(r-2\mathcal{C}^{-2}Gm_0)}\right. \\
&\times \left[\frac{\kappa r^3}{2}(\rho_0 + \mathcal{C}^{-2}P_{r0}) + \frac{m_0}{r^2}\left\{r^2 - 4\beta n(n-1)(r-3\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-1}\mathcal{G}_0'\right\}\right. \\
&\left. + 4\beta n(n-1)(r-2\mathcal{C}^{-2}Gm_0)\frac{m_0}{r}(\mathcal{G}_0^{n-2}\mathcal{G}_0')'\right][r^2 - 4\beta n(n-1) \\
&\left.\times (r-3\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-2}\mathcal{G}_0']^{-1}\right\} - \frac{8Gm_0}{r^4}\beta n(n-1)(r-2\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-2}\mathcal{G}_0' \\
&\times \left\{\frac{1}{(r-2\mathcal{C}^{-2}Gm_0)}\left[\kappa P_{\perp 0}r^4\mathcal{C}^{-2} - rm_0 + \frac{1}{2\mathcal{C}^{-2}G}r^4\beta\mathcal{G}_0^n\right.\right. \\
&- \left\{\frac{m_0}{r}(r^3 + 8\beta n(r-3\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-1} - 12\beta n(n-1)(r^2 - 2r\mathcal{C}^{-2}Gm_0)\right. \\
&\left.\times \mathcal{G}_0^{n-2}\mathcal{G}_0') + \frac{r(r-2\mathcal{C}^{-2}Gm_0)}{\mathcal{C}^{-2}G}(r - 4\beta n(n-1)(r-2\mathcal{C}^{-2}Gm_0)(\mathcal{G}_0^{n-1}\mathcal{G}_0')')\right\} \\
&\times \left\{\frac{r\mathcal{C}^{-2}G}{(r-2\mathcal{C}^{-2}Gm_0)}\left[\frac{\kappa r^3}{2}(\rho_0 + \mathcal{C}^{-2}P_{r0}) + \frac{m_0}{r^2}\left\{r^2 - 4\beta n(n-1)\right.\right.\right. \\
&\left.\left.\times (r-3\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-1}\mathcal{G}_0'\right\} + 4\beta n(n-1)(r-2\mathcal{C}^{-2}Gm_0)\frac{m_0}{r}(\mathcal{G}_0^{n-2}\mathcal{G}_0')'\right] \\
&\left.\left.\times \left[r^2 - 4\beta n(n-1)(r-3\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right]^{-1}\right\}\right][r^3 - 8\beta n m_0\mathcal{C}^{-2}G\mathcal{G}_0^{n-1} \\
&\left. - 4r\beta n(n-1)(r-2\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-2}\mathcal{G}_0']^{-1}\right\} + \frac{2}{r}(P_{r0} - P_{\perp 0}) + \frac{\kappa\rho_0'}{\mathcal{C}^{-2}} \\
&+ \frac{4m_0(2r - \mathcal{C}^{-2}Gm_0)}{r\mathcal{C}^{-2}(r-\mathcal{C}^{-2}Gm_0)} + 8\frac{m_0}{r^4\mathcal{C}^{-2}}\beta n(n-1)(r-2\mathcal{C}^{-2}Gm_0)
\end{aligned}
$$

$$\times\left\{(n-2)\mathcal{G}_0'(\mathcal{G}_0^{n-3}\mathcal{G}_0')'+(\mathcal{G}_0^{n-2}\mathcal{G}_0'')'+(n-2)r^2\mathcal{G}_0^{n-3}\mathcal{G}_0'\left(\frac{\mathcal{G}_0'}{r^2}\right)'\right.$$

$$\left.-\frac{2}{r}\mathcal{G}_0^{n-2}\mathcal{G}_0''\right\}-8\beta n(n-1)(r-3\mathcal{C}^{-2}Gm_0)\frac{m_0}{r^6\mathcal{C}^{-2}}\left\{(n-2)r\mathcal{G}_0^{n-3}\mathcal{G}_0'^2\right.$$

$$\left.-\frac{(2r-\mathcal{C}^{-2}Gm_0)}{(r-2\mathcal{C}^{-2}Gm_0)}\mathcal{G}_0^{n-2}\mathcal{G}_0'+2\mathcal{G}_0^{n-2}\mathcal{G}_0'\right\}+\frac{8m_0}{r^5\mathcal{C}^{-2}}\beta n(n-1)$$

$$\times(2r-7\mathcal{C}^{-2}Gm_0)\mathcal{G}_0^{n-2}\mathcal{G}_0''+24\beta n(n-1)m_0^2\frac{G(2r-5\mathcal{C}^{-2}Gm_0)}{r^6(r-2\mathcal{C}^{-2}Gm_0)}$$

$$\left.\times\mathcal{G}_0^{n-2}\mathcal{G}_0'+\frac{8m_0}{r^5\mathcal{C}^{-2}}\beta n(n-1)(n-2)\mathcal{G}_0^{n-3}\mathcal{G}_0'^2(r-4\mathcal{C}^{-2}Gm_0)\right], \tag{49}$$

where $\mathcal{C}$ and G represent the speed of light and gravitational constant, respectively. We expand the above equation upto order $\mathcal{C}^{-4}$ and separate out terms of order $\mathcal{C}^0$ (N approximation), $\mathcal{C}^{-2}$ (pN approximation) and $\mathcal{C}^{-4}$ (ppN approximation). Using Eq. (45) in (40), it follows that

$$\bar{\rho}-\frac{2c}{r}(P_{r0}-P_{\perp 0})T-\Delta_3=0, \tag{50}$$

where Δ_3 is given in Appendix B.

The relationship between $\bar{\rho}$ and $\bar{P}_r$ is given by[9,10]

$$\bar{P}_r=\Gamma\frac{P_{r0}}{\rho_0+P_{r0}}\bar{\rho}. \tag{51}$$

The dynamical instability is well investigated by means of Γ which measures the compressibility of collapsing matter distribution. Since we consider zero expansion condition for which the fluid evolves without being compressed, this means that the instability of the system does not depend on Γ under expansion-free condition. Substituting the value of $\bar{\rho}$ from Eq. (50) in Eq. (51), we have

$$\bar{P}_r=\Gamma\frac{P_{r0}}{\rho_0+P_{r0}}\left[\frac{2c}{r}(P_{r0}-P_{\perp 0})T+\Delta_3\right].$$

In view of Eqs. (46), (50) and (51), it is found that $\bar{P}_r$ as well as $\bar{\rho}\frac{A_0'}{A_0}$ are in ppN regime. Therefore, we discard such terms in order to investigate the instability ranges for N and pN approximations. The metric coefficients for pN approximation in c.g.s. units are[27]

$$A_0=1-\frac{m_0G}{r\mathcal{C}^2}-\frac{1}{6}r^2(n-1)\beta\mathcal{G}_0^n,\quad B_0=1+\frac{m_0G}{r\mathcal{C}^2}. \tag{52}$$

Inserting the value of $\bar{P}_\perp$ from Eq. (B.5) in the second perturbed dynamical equation (39) and applying the expansion-free condition along with assumptions $a=a_0r$,

$c = c_0 r$ and $g = g_0 r$ in the resulting equation, it follows that

$$
\begin{aligned}
&(\rho_0 + P_{r0}) \left(\frac{a_0}{A_0}\right) \left\{1 - r\frac{A_0'}{A_0}\right\} + 2c_0 P_{r0}' + \frac{4c_0}{r}(P_{r0} - P_{\perp 0}) + \frac{4c_0}{r} P_{\perp 0} \\
&\quad - \frac{2}{rB_0^2}\left\{\left(\frac{6c_0}{r} - \frac{a_0}{A_0}\right)\left(r\frac{A_0''}{A_0} + \frac{B_0'}{B_0} + \frac{A_0'}{A_0}\right) + \frac{a_0}{rA_0}\right\} + \beta\mathcal{G}_0^n\left\{\frac{ng_0}{G_0} + \frac{2c_0}{r}\right\} \\
&\quad - 4\beta n\mathcal{G}_0^{n-1}\left[\frac{a_0}{A_0 B_0^2}\left[\frac{1}{r}\left(\frac{A_0'''}{A_0} - \frac{A_0''}{rA_0}\right)\left(1 - \frac{1}{B_0^2}\right) + \frac{1}{r^2 A_0}\left(\frac{B_0''}{B_0} - \frac{2B_0'}{rB_0}\right)\right.\right. \\
&\quad \times \left(1 - \frac{3}{B_0^2}\right)\bigg] - \frac{2c_0}{r^2 B_0^2}\left(\frac{A_0'''}{A_0} - \frac{2A_0''}{rA_0}\right)\left(2 - \frac{5}{B_0^2}\right) - r(n-1)\frac{g_0}{\mathcal{G}_0} \\
&\quad \times \left\{\frac{1}{r^2 B_0^2}\left(1 - \frac{1}{B_0^2}\right)\left(\frac{A_0'''}{A_0} - \frac{2A_0''}{rA_0}\right) + \frac{1}{8}(\mathcal{G}_0' + g_0)\right\} + \frac{2}{r^3}\left[\frac{a_0}{A_0 B_0^2}\left(1 - \frac{3}{B_0^2}\right)\frac{B_0'}{B_0}\right. \\
&\quad + \frac{rA_0''}{A_0 B_0^2}\left[\left(1 - \frac{1}{B_0^2}\right)\left\{\frac{a_0}{A_0} - (n-1)\frac{g_0}{\mathcal{G}_0}\right\} - \frac{4c_0}{r}\left(1 - \frac{3}{2B_0^2}\right)\right] \\
&\quad + r(n-1)\frac{\mathcal{G}_0'}{B_0\mathcal{G}_0}\left[\frac{3a_0 B_0'}{A_0 B_0} + r\left(\frac{a_0}{A_0} - \frac{10c_0}{r}\right)\frac{A_0''}{A_0}\right] - \frac{r(n-1)A_0'' g_0}{B_0^4 \mathcal{G}_0^{n-1}}(r\mathcal{G}_0^{n-2})' \\
&\quad - \frac{r(n-1)}{B_0^4\mathcal{G}_0^{n-1}}\left[\frac{a_0}{A_0}\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)' + \frac{A_0'}{A_0}\left\{r(\mathcal{G}_0^{n-2}\mathcal{G}_0')'\left(\frac{10c_0}{r} - \frac{a_0}{A_0}\right) + g_0(\mathcal{G}_0^{n-2})'\right.\right. \\
&\quad + (n-2)g_0(r\mathcal{G}_0^{n-3}\mathcal{G}_0')'\bigg\}\bigg]\bigg]\bigg] + \left[\frac{2c_0}{rA_0^2} + 4\beta n\mathcal{G}_0^{n-1}\left[-\frac{2c_0}{A_0}\left\{\frac{1}{rA_0 B_0^2}\left(\frac{B_0''}{B_0} - \frac{2B_0'}{rB_0}\right)\right.\right.\right. \\
&\quad - \frac{2}{r^2 A_0}\left(\frac{1}{r} + \frac{A_0'}{A_0}\right)\left(1 - \frac{1}{B_0^2}\right)\bigg\} - \frac{2c_0}{r^3 A_0^2}\left(1 - \frac{1}{B_0^2}\right) + \frac{B_0'}{rA_0^2 B_0^3} \\
&\quad \times \left\{\frac{2c_0}{r} - c_0(n-1)\frac{\mathcal{G}_0'}{\mathcal{G}_0} - (n-1)\frac{g_0}{\mathcal{G}_0}\right\} - \frac{c_0(n-1)}{rA_0 B_0^2}\left\{\frac{2}{r} - \frac{(n-2)\mathcal{G}_0'}{A_0\mathcal{G}_0}\right\}\frac{\mathcal{G}_0'}{\mathcal{G}_0}\bigg]\bigg] \\
&\quad \times \Omega^2(r) = 0.
\end{aligned} \tag{53}
$$

Using Eqs. (42), (46), (47) and (52) with $\mathcal{C} = G = 1$ for pN approximation in the above equation, we obtain

$$
\begin{aligned}
&\frac{a_0}{r}(\rho_0 + P_{r0})\left(r + m_0 + \frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right) + 2c_0 P_{r0}' + \frac{4c_0}{r}(P_{r0} - P_{\perp 0}) \\
&\quad + \frac{4c_0}{r}P_{\perp 0} - \frac{2a_0}{r^4}(r - 2m_0)\left(r + m_0 + \frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right) + \beta n\mathcal{G}_0^{n-1} \\
&\quad \times \left(ng_0 + \frac{2c_0}{r}\mathcal{G}_0\right) + 4\beta n\mathcal{G}_0^{n-1}\left[\frac{r(n-1)g_0}{8\mathcal{G}_0}(\mathcal{G}_0' + g_0) + \frac{2a_0(n-1)}{r^4\mathcal{G}_0^{n-1}}\right. \\
&\quad \times (r - 4m_0)\left(r + m_0 + \frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)(\mathcal{G}_0^{n-1}\mathcal{G}_0')'\bigg]
\end{aligned}
$$

$$+\left[\frac{2c_0}{r^2}\left(r+2m_0+\frac{1}{3}r^3(n-1)\beta\mathcal{G}_0^n\right)+4\beta n\mathcal{G}_0^{n-1}\right.$$
$$\times\left\{\frac{4c_0m_0}{r^5}\left(r+2m_0+\frac{1}{3}r^3(n-1)\beta\mathcal{G}_0^n\right)-\frac{c_0}{r^4}(n-1)\right.$$
$$\times(r-2m_0)\left(r+m_0+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)$$
$$\left.\left.\times\left(2-(n-2)\left(r+m_0+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right)\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right\}\right]\Omega^2(r)$$
$$+\left[-a_0(\rho_0+P_{r0})\left(r+m_0+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)\right.$$
$$-\frac{2}{r^3}(r-2m_0)\left\{6c_0-a_0\left(r+m_0+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)\right\}$$
$$+\frac{8}{r^3}\beta n(n-1)(r-4m_0)\left\{\left[10c_0-a_0\left(r+2m_0+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)\right]\right.$$
$$\left.\times(\mathcal{G}_0^{n-2}\mathcal{G}_0')'+g_0(\mathcal{G}_0^{n-2})'+g_0(n-2)(r\mathcal{G}_0^{n-3}\mathcal{G}_0')'\right\}+\frac{32m_0c_0}{r^4}\beta n\mathcal{G}_0^{n-1}$$
$$\left.\times\left(r+2m_0+\frac{1}{3}r^3(n-1)\beta\mathcal{G}_0^n\right)\Omega^2(r)\right]\Xi_1+\left[-\frac{2}{r^3}(r-2m_0)\right.$$
$$\times\left\{6c_0-a_0\left(r+m_0+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)\right\}+4\beta n\mathcal{G}_0^{n-1}\left[\frac{2a_0m_0}{r^5}\left(1-\frac{2m_0}{r}\right)\right.$$
$$\times\left(r+m_0+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)+\frac{4c_0}{r^5}(r-2m_0)(3r-10m_0)-4(n-1)$$
$$\times(r-2m_0)\frac{m_0g_0}{r^4\mathcal{G}_0}-\frac{2}{r^3}\left[\frac{2m_0}{r}(r-2m_0)\left\{\frac{a_0}{r}\left(r+m_0+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)\right.\right.$$
$$\left.-(n-1)\frac{g_0}{\mathcal{G}_0}\right\}+(n-1)(r-m_0)\left\{a_0\left(r+m_0+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)-10c_0\right\}$$
$$\times\frac{\mathcal{G}_0'}{\mathcal{G}_0}-\frac{2c_0}{r}(r-3m_0)(r-2m_0)+\frac{(n-1)g_0}{r\mathcal{G}_0^{n-1}}(4-4m_0)$$
$$\left.\left.\left.\times\left(r-m_0-\frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)(r\mathcal{G}_0^{n-2})'\right]\right]\right]\Xi_2$$
$$+\left[-4\beta n\mathcal{G}_0^{n-1}\left\{\frac{2a_0m_0}{r^4}(r-m_0)\left(r+m_0+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)\right.\right.$$
$$\left.\left.+\frac{2c_0}{r^4}(r-2m_0)(3r-10m_0)-2(n-1)(r-2m_0)\frac{m_0g_0}{r^3\mathcal{G}_0}\right\}\right]\Xi_3$$

$$
\begin{aligned}
&+\left[-\frac{2}{r^3}(r-2m_0)\left\{6c_0-a_0\left(r+m_0+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)\right\}\right.\\
&-4\beta n\mathcal{G}_0^{n-1}\left[\frac{4a_0}{r^6}(r-3m_0)(r-2m_0)\left(r+2m_0+\frac{1}{3}r^3(n-1)\beta\mathcal{G}_0^n\right)\right.\\
&+\frac{2}{r^3}\left\{-\frac{4a_0}{r^3}(r-2m_0)(r-3m_0)\left(r+m_0+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)+\frac{3a_0}{r}\right.\\
&\left.\left.\times(r-m_0)\left(r+m_0+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right\}\right]+\left[4\beta n\mathcal{G}_0^{n-1}\left\{\frac{1}{r^3}(r-2m_0)\right.\right.\\
&\left.\left.\left.\times\left(r+2m_0+\frac{1}{3}r^3(n-1)\beta\mathcal{G}_0^n\right)\left(\frac{6c_0}{r}-\frac{(n-1)}{\mathcal{G}_0}\left\{c_0\mathcal{G}_0'+g_0\right\}\right)\right\}\right]\Omega^2(r)\right]\\
&\times\left\{-\frac{m_0}{r^3}(r+2m_0)\right\}+\frac{m_0}{r^4}(2r+7m_0)\left[-\frac{8}{r^4}\beta n\mathcal{G}_0^{n-1}(r-2m_0)\right.\\
&\left.\times\left(r+2m_0+\frac{1}{3}r^3(n-1)\beta\mathcal{G}_0^n\right)\left\{rc_0\Omega^2(r)-\frac{a_0}{r}(r-3m_0)\right\}\right]=0, \qquad (54)
\end{aligned}
$$

where $\Xi_i's$ are given in Appendix B. This equation shows that collapsing fluid evolves without being compressed due to the absence of index Γ but instability region completely depends on physical quantities like energy density, anisotropic pressure and some arbitrary constants with considered $f(\mathcal{G})$ model. Thus, the chosen gravity model indicates the compatibility with physical results under the condition of zero expansion. It is observed that some ppN approximation terms also appear in the above equation. In order to investigate the instability ranges for N regime, we drop all terms belonging to the order of pN and ppN in Eq. (54) as follows:

$$
\begin{aligned}
&2c_0|P_{r0}'|+\frac{4c_0}{r}(P_{r0}-P_{\perp 0})+\frac{4c_0}{r}\left(P_{\perp 0}+\frac{3m_0}{r^3}\right)+4\beta n\mathcal{G}_0^{n-1}\\
&\quad\times\left\{\frac{1}{4}\left(ng_0+\frac{2c_0}{r}\mathcal{G}_0\right)+r(n-1)(\mathcal{G}_0'+g_0)\frac{g_0}{8\mathcal{G}_0}\right\}\\
&\quad+\left[\frac{a_0}{r}(\rho_0+P_{r0})-\frac{2a_0}{r^4}(r-3m_0)+4\beta n\mathcal{G}_0^{n-1}\left\{\frac{2a_0(n-1)}{r^4\mathcal{G}_0^{n-1}}(r-4m_0)\right.\right.\\
&\quad\left.\left.\times(\mathcal{G}_0^{n-1}\mathcal{G}_0')'-2(n-1)(r-2m_0)\Omega^2(r)\frac{c_0\mathcal{G}_0'}{r^4\mathcal{G}_0}-\frac{2a_0m_0}{r^5}\left(\frac{4}{r}-\frac{3\mathcal{G}_0'}{\mathcal{G}_0}\right)\right\}\right]\\
&\quad\times\left(r+m_0+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)+24\beta n\mathcal{G}_0^{n-1}\frac{a_0m_0^2}{r^6}\left(r+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)\\
&\quad\times\left(\frac{4}{r}-\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right)+\left[\frac{2c_0^2}{r^2}\Omega^2(r)+4\beta n\mathcal{G}_0^{n-1}\left\{\frac{m_0}{r^5}\left(\frac{8a_0}{r}-3c_0\Omega^2(r)\right)\right.\right.\\
&\quad\left.\left.+\frac{m_0}{\mathcal{G}_0r^4}(n-1)(c_0\mathcal{G}_0'+g_0)\Omega^2(r)\right\}\right]\left(r+2m_0+\frac{1}{3}r^3(n-1)\beta\mathcal{G}_0^n\right)
\end{aligned}
$$

$$
\begin{aligned}
&-8\beta n\mathcal{G}_0^{n-1}\frac{m_0^2}{r^6}\left(7\frac{a_0}{r}+3c_0\Omega^2\right)\left(r+\frac{1}{3}r^3(n-1)\beta\mathcal{G}_0^n\right)\\
&+4\beta n\mathcal{G}_0^{n-1}(n-1)(n-2)(r-2m_0)\left(r+m_0+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)\\
&\times\left(r+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_0^n\right)\frac{c_0}{r^4}\left(\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right)^2\Omega^2(r)=0. \qquad (55)
\end{aligned}
$$

To find out the instability ranges for expansion-free fluids, the positivity of the above equation is required. For this purpose, we assume that all dynamical quantities and constants are positive while $P_{r0}'<0$ shows the decreasing behavior of radial pressure during collapse. Additionally, the following constraints need to be satisfied

$$
P_{r0}>P_{\perp 0},\quad \frac{4}{3}>\frac{r\mathcal{G}_0'}{\mathcal{G}_0}>0,\quad n>2,\quad r>4m_0. \qquad (56)
$$

Under these conditions, the system is unstable at N approximation. If we consider the constant GB curvature ($\mathcal{G}_0(r)=\mathcal{G}_c$) with $g_0=0$ in Eq. (55), we have

$$
\begin{aligned}
&2c_0|P_{r0}'|+\frac{4c_0}{r}(P_{r0}-P_{\perp 0})+\frac{4c_0}{r}\left(P_{\perp 0}+\frac{3m_0}{r^3}\right)+\frac{2c_0}{r}\beta n\mathcal{G}_c^n\\
&+\left[\frac{a_0}{r}(\rho_0+P_{r0})-\frac{2a_0}{r^4}(r-3m_0)-\frac{32a_0m_0}{r^6}\beta n\mathcal{G}_c^{n-1}\right]\\
&\times\left(r+m_0+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_c^n\right)+\frac{96a_0m_0^2}{r^6}\beta n\mathcal{G}_c^{n-1}\left(1+\frac{1}{6}r^2(n-1)\beta\mathcal{G}_c^n\right)\\
&+\left[\frac{2c_0^2}{r^2}\Omega^2(r)+4\beta n\mathcal{G}_c^{n-1}\left(\frac{8a_0m_0}{r^6}-\frac{3c_0m_0}{r^5}\Omega^2(r)\right)\right]\\
&\times\left(r+2m_0+\frac{1}{3}r^3(n-1)\beta\mathcal{G}_c^n\right)-8\beta n\mathcal{G}_c^{n-1}\frac{m_0^2}{r^5}\left(7\frac{a_0}{r}+3c_0\Omega^2\right)\\
&\times\left(1+\frac{1}{3}r^2(n-1)\beta\mathcal{G}_c^n\right)=0. \qquad (57)
\end{aligned}
$$

Using constant GB condition in Eq. (18) and substituting the resulting expression of m_0 in the above equation, we obtain

$$
\begin{aligned}
&2c_0|P_{r0}'|+\frac{4c_0}{r}(P_{r0}-P_{\perp 0})+\frac{4c_0}{r}\left(P_{\perp 0}+\frac{3}{r^3}\left\{\frac{\kappa}{2}\int_{r_{\Sigma(i)}}^{r}r^2\rho_0dr-\frac{1}{12}\beta\mathcal{G}_c^n\right.\right.\\
&\left.\left.\times(r^3-r_{\Sigma(i)}^3)\right\}\right)+2\beta n\mathcal{G}_c^n\frac{c_0}{r}+\left[\frac{a_0}{r}(\rho_0+P_{r0})-\frac{2a_0}{r^4}\left(r-3\left\{\frac{\kappa}{2}\int_{r_{\Sigma(i)}}^{r}r^2\rho_0dr\right.\right.\right.\\
&\left.\left.-\frac{1}{12}\beta\mathcal{G}_c^n(r^3-r_{\Sigma(i)}^3)\right\}\right)-\frac{32a_0}{r^6}\beta n\mathcal{G}_c^{n-1}\left\{\frac{\kappa}{2}\int_{r_{\Sigma(i)}}^{r}r^2\rho_0dr-\frac{1}{12}\beta\mathcal{G}_c^n\right.
\end{aligned}
$$

$$\times(r^3-r^3_{\Sigma(i)})\Big\}\Big]\Bigg(r+\Big\{\frac{\kappa}{2}\int_{r_{\Sigma(i)}}^r r^2\rho_0 dr-\frac{1}{12}\beta\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\Big\}$$

$$+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_c^n\Bigg)+\frac{96a_0}{r^6}\beta n\mathcal{G}_c^{n-1}\left(1+\frac{1}{6}r^2(n-1)\beta\mathcal{G}_c^n\right)$$

$$\times\left\{\frac{\kappa}{2}\int_{r_{\Sigma(i)}}^r r^2\rho_0 dr-\frac{1}{12}\beta\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\right\}^2\left[\frac{2c_0^2}{r^2}\Omega^2(r)+4\beta n\mathcal{G}_c^{n-1}\right.$$

$$\left.+\left(\frac{8a_0}{r^6}-\frac{3c_0}{r^5}\Omega^2(r)\right)\left\{\frac{\kappa}{2}\int_{r_{\Sigma(i)}}^r r^2\rho_0 dr-\frac{1}{12}\beta\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\right\}\right]$$

$$\times\left(r+2\left\{\frac{\kappa}{2}\int_{r_{\Sigma(i)}}^r r^2\rho_0 dr-\frac{1}{12}\beta\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\right\}+\frac{1}{3}r^3(n-1)\beta\mathcal{G}_c^n\right)$$

$$-\frac{8}{r^5}\beta n\mathcal{G}_c^{n-1}\left(7\frac{a_0}{r}+3c_0\Omega^2\right)\left(1+\frac{1}{3}r^2(n-1)\beta\mathcal{G}_c^n\right)$$

$$\times\left\{\frac{\kappa}{2}\int_{r_{\Sigma(i)}}^r r^2\rho_0 dr-\frac{1}{12}\beta\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\right\}^2=0. \tag{58}$$

In order to discuss the effect of energy density, we consider $\rho_0=\zeta r^\lambda$ such that $\zeta>0$ and $\lambda\in(-\infty,+\infty)$. Using this power-law form in Eq. (58), it follows that

$$2c_0|P'_{r0}|+\frac{4c_0}{r}(P_{r0}-P_{\perp 0})+\frac{4c_0}{r}\Bigg(P_{\perp 0}+\frac{3}{r^3}\Bigg\{\frac{\kappa\zeta}{2(\lambda+3)}(r^{\lambda+3}-r^{\lambda+3}_{\Sigma(i)})$$

$$-\frac{1}{12}\beta\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\Bigg\}\Bigg)+2\beta n\mathcal{G}_c^n\frac{c_0}{r}+\Bigg[\frac{a_0}{r}(\rho_0+P_{r0})-\frac{2a_0}{r^4}$$

$$\times\left(r-3\left\{\frac{\kappa\zeta}{2(\lambda+3)}(r^{\lambda+3}-r^{\lambda+3}_{\Sigma(i)})-\frac{1}{12}\beta\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\right\}\right)-\frac{32a_0}{r^6}\beta n\mathcal{G}_c^{n-1}$$

$$\times\left\{\frac{\kappa\zeta}{2(\lambda+3)}(r^{\lambda+3}-r^{\lambda+3}_{\Sigma(i)})-\frac{1}{12}\beta\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\right\}\Bigg]\Bigg(r+\Bigg\{\frac{\kappa\zeta}{2(\lambda+3)}$$

$$\times(r^{\lambda+3}-r^{\lambda+3}_{\Sigma(i)})-\frac{1}{12}\beta\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\Bigg\}+\frac{1}{6}r^3(n-1)\beta\mathcal{G}_c^n\Bigg)+\frac{96a_0}{r^6}\beta n\mathcal{G}_c^{n-1}$$

$$\times\left(1+\frac{1}{6}r^2(n-1)\beta\mathcal{G}_c^n\right)\left\{\frac{\kappa\zeta}{2(\lambda+3)}(r^{\lambda+3}-r^{\lambda+3}_{\Sigma(i)})-\frac{1}{12}\beta\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\right\}^2$$

$$+\Bigg[\frac{2c_0^2}{r^2}\Omega^2(r)+4\beta n\mathcal{G}_c^{n-1}\left(\frac{8a_0}{r^6}-\frac{3c_0}{r^5}\Omega^2(r)\right)\Bigg\{\frac{\kappa\zeta}{2(\lambda+3)}(r^{\lambda+3}-r^{\lambda+3}_{\Sigma(i)})$$

$$-\frac{1}{12}\beta\mathcal{G}_c^n(r^3-r^3_{\Sigma(i)})\Bigg\}\Bigg]\Bigg(r+2\Bigg\{\frac{\kappa\zeta}{2(\lambda+3)}(r^{\lambda+3}-r^{\lambda+3}_{\Sigma(i)})-\frac{1}{12}\beta\mathcal{G}_c^n$$

$$\times (r^3 - r^3_{\Sigma(i)})\Big\} + \frac{1}{3}r^3(n-1)\beta\mathcal{G}_c^n\Big) - \frac{8}{r^5}\beta n\mathcal{G}_c^{n-1}\left(7\frac{a_0}{r} + 3c_0\Omega^2\right)$$
$$\times\left(1+\frac{1}{3}r^2(n-1)\beta\mathcal{G}_c^n\right)\left\{\frac{\kappa\zeta}{2(\lambda+3)}(r^{\lambda+3} - r^{\lambda+3}_{\Sigma(i)}) - \frac{1}{12}\beta\mathcal{G}_c^n(r^3 - r^3_{\Sigma(i)})\right\}^2$$
$$= 0, \tag{59}$$

provided that $\lambda \neq 3$. During collapse, expansion-free matter distribution approaches instability ranges for

$$P_{r0} > P_{\perp 0}, \quad r^{\lambda+3} > r^{\lambda+3}_{\Sigma(i)}, \quad r > r_{\Sigma(i)}, \tag{60}$$

where the last inequality indicates that Minkowskian cavity is present in the center of anisotropic spherical fluid in $f(\mathcal{G})$ theory of gravity under zero expansion condition.

5. Concluding Remarks

In this paper, we have analyzed the problem of instability in the background of $f(\mathcal{G})$ gravity for expansion-free spherical nondissipative anisotropic matter distribution. It is worth mentioning here that voids are the physical examples of expansion-free models which are underdense regions in the large-scale structure of the universe. Modified theories of gravity are considered to be the most promising candidate to disclose some significant facts about the present cosmic large scale structures and its evolution. At the large scales, the stellar objects are interesting only when they are stable against fluctuations. This behavior motivates many researchers to discuss the dynamical instability of celestial objects in modified theories of gravity. Among such theories, $f(\mathcal{G})$ gravity explains remarkable features of quadratic curvature term called GB invariant which originates from string theories.

The presence of generic function $f(\mathcal{G})$ in the action gives rise to the higher order curvature terms in 4D which affects the collapsing phenomenon. In order to discuss dynamical instability, we have chosen the power-law model of $f(\mathcal{G})$ gravity and applied the perturbation approach upto first-order to all metric and matter variables. The corresponding field equations as well as dynamical equations have been evaluated for both static and nonstatic configurations. We have obtained terms belonging to N, pN and ppN approximations by expanding Eq. (49) upto order $\mathcal{C}^{-4}$. The instability of expansion-free fluid has been discussed by Eq. (54) upto pN approximation.

Instability of celestial objects can be well discussed by adiabatic index. Chandrasekhar[8] found the range of instability of isotropic spherical star for $\Gamma < \frac{4}{3}$. We have found that under zero expansion condition, instability of spherical matter distribution depends on matter variables, metric functions and higher order curvature terms but is independent of Γ. The absence of Γ shows that the range of instability has not been affected by fluid stiffness as in general relativity[15] as well as in other modified theories of gravity.[20] For constant curvature, the inequality

$r > r_{\Sigma(i)}$ appears due to $f(\mathcal{G})$ gravity in N regime. As long as this condition holds, the system remains unstable. This range of instability shows the presence of vacuum cavity with the fluid distribution which indicates the compatibility of our results under zero expansion condition.[13] We conclude that the stability of the spherical star filled with anisotropic pressure increases in $f(\mathcal{G})$ gravity as compared to general relativity.

Appendix A.

The dark source terms in Eqs. (13) and (14) are

$$\begin{aligned}
\Delta_1 = &-\frac{1}{\kappa A}\left[-\frac{1}{A^2}\left\{\frac{1}{2}A^2 f + \left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left[-\frac{4A}{BC^2}\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right.\right.\right.\right.\\
&\left.-\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}F + \frac{4\dot{B}}{BC^2}\dot{F} + \frac{4A^2B'}{B^3C^2}F' - \frac{4A^2}{B^2C^2}F''\Bigg]\\
&+\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)\left(-\frac{8\dot{C}}{BC^2}\dot{F}-\frac{8A^2C'}{B^3C^2}F'\right)-\frac{4AF}{BC^2}\\
&\times\left[\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\right.\\
&\times\left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)\\
&\left.\left.+\frac{2}{AB}(\dot{C'}^2-\ddot{C}C'')\right]\right\}_{,0}-\frac{1}{B^2}\left\{\frac{2}{C}\left[\frac{2}{C}\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\right.\right.\\
&\left.\left.\times\left(\dot{F}'-\frac{A'}{A}\dot{F}-\frac{\dot{B}}{B}F'\right)-\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)\left(\frac{4\dot{C}}{A^2C}\dot{F}-\frac{4C'}{B^2C}F'\right)\right]\right\}_{,1}\\
&+\left(2\frac{\dot{A}}{A^3}-\frac{\dot{B}}{A^2B}-\frac{2\dot{C}}{A^2C}\right)\left\{\frac{1}{2}A^2f+\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\right.\\
&\times\left[-\frac{4A}{BC^2}\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)-\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}F+\frac{4\dot{B}}{BC^2}\dot{F}\right.\\
&\left.+\frac{4A^2B'}{B^3C^2}F'-\frac{4A^2}{B^2C^2}F''\right]+\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)\left(-\frac{8\dot{C}}{BC^2}\dot{F}-\frac{8A^2C'}{B^3C^2}F'\right)\\
&-\frac{4AF}{BC^2}\left[\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\right.
\end{aligned}$$

$$\times\left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)$$

$$+\frac{2}{AB}(\dot{C'}^2-\ddot{C}C'')\Bigg]\Bigg\}-\left(\frac{A'}{AB^2}-\frac{B'}{B^3}+\frac{2C'}{B^2C}\right)\left\{\frac{2}{C}\left[\frac{2}{C}\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\right.\right.$$

$$\left.\left.\times\left(\dot{F}'-\frac{A'}{A}\dot{F}-\frac{\dot{B}}{B}F'\right)-\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)\left(\frac{4\dot{C}}{A^2C}\dot{F}-\frac{4C'}{B^2C}F'\right)\right]\right\}$$

$$-\frac{\dot{B}}{B^3}\left\{-\frac{1}{2}B^2f+\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left[\frac{4B}{AC^2}\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right.\right.\right.$$

$$\left.\left.-\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}F+\frac{4B^2\dot{A}}{A^3C^2}\dot{F}+\frac{4A'}{AC^2}F'-\frac{4B^2}{A^2C^2}\ddot{F}\right]$$

$$+\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)\left(\frac{8B^2\dot{C}}{A^3C^2}\dot{F}-\frac{8C'}{AC^2}F'\right)+\frac{4BF}{AC^2}$$

$$\times\left[\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\right.$$

$$\times\left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)$$

$$\left.\left.+\frac{2}{AB}(\dot{C'}^2-\ddot{C}C'')\right]\right\}-\frac{2\dot{C}}{C^3}\left\{-\frac{1}{2}C^2f+\frac{4F}{AB}\left[\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\right.\right.$$

$$\times\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)-\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}+\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)$$

$$\times\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\left(C''-\frac{B'C'}{B}\right)$$

$$\left.+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)+\frac{2}{AB}(\dot{C'}^2-\ddot{C}C'')\right]$$

$$+\frac{4C}{A^3B}\left[\dot{B}\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)+\dot{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)\right.$$

$$\left.-\frac{2A'}{B}\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)+\dot{C}\left\{\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)-\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right\}\right]\dot{F}$$

$$+\frac{4C}{AB^3}\left[B'\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)+A'\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)\right.$$

$$
\begin{aligned}
&-\frac{2\dot{B}}{A}\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)-C'\left\{\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)-\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right\}\Bigg]F' \\
&+\frac{8C}{A^2B^2}\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)\dot{F}'-\frac{4C}{AB^2}\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)F'' \\
&-\frac{4C}{A^2B}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)\ddot{F}\Bigg\}\Bigg],
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
\Delta_2=&\frac{1}{\kappa B}\Bigg[-\frac{1}{A^2}\Bigg\{\frac{2}{C}\Bigg[\frac{2}{C}\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left(\dot{F}'-\frac{A'}{A}\dot{F}-\frac{\dot{B}}{B}F'\right) \\
&-\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)\left(\frac{4\dot{C}}{A^2C}\dot{F}-\frac{4C'}{B^2C}F'\right)\Bigg]\Bigg\}_{,0}-\frac{1}{B^2}\Bigg\{-\frac{1}{2}B^2f \\
&+\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\Bigg[\frac{4B}{AC^2}\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)-\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}F \\
&+\frac{4B^2\dot{A}}{A^3C^2}\dot{F}+\frac{4A'}{AC^2}F'-\frac{4B^2}{A^2C^2}\ddot{F}\Bigg]+\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right) \\
&\times\left(\frac{8B^2\dot{C}}{A^3C^2}\dot{F}-\frac{8C'}{AC^2}F'\right)+\frac{4BF}{AC^2}\Bigg[\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right) \\
&+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right) \\
&-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)+\frac{2}{AB}(\dot{C'}^2-\ddot{C}C'')\Bigg]\Bigg\}_{,1}-\frac{A'}{A^3}\Bigg\{\frac{1}{2}A^2f \\
&+\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\Bigg[-\frac{4A}{BC^2}\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)-\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}F \\
&+\frac{4\dot{B}}{BC^2}\dot{F}+\frac{4A^2B'}{B^3C^2}F'-\frac{4A^2}{B^2C^2}F''\Bigg]+\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right) \\
&\times\left(-\frac{8\dot{C}}{BC^2}\dot{F}-\frac{8A^2C'}{B^3C^2}F'\right)-\frac{4AF}{BC^2}\Bigg[\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right) \\
&+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right) \\
&-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)+\frac{2}{AB}(\dot{C'}^2-\ddot{C}C'')\Bigg]\Bigg\}+\left(\frac{\dot{A}}{A^3}-\frac{\dot{B}}{A^2B}-\frac{2\dot{C}}{A^2C}\right)
\end{aligned}
$$

$$\times\left\{\frac{2}{C}\left[\frac{2}{C}\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left(\dot{F}'-\frac{A'}{A}\dot{F}-\frac{\dot{B}}{B}F'\right)-\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)\right.\right.$$
$$\left.\left.\times\left(\frac{4\dot{C}}{A^2C}\dot{F}-\frac{4C'}{B^2C}F'\right)\right]\right\}-\left(\frac{A'}{AB^2}-\frac{2B'}{B^3}+\frac{2C'}{B^C}\right)\left\{-\frac{1}{2}B^2f\right.$$
$$+\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left[\frac{4B}{AC^2}\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)-\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}F\right.$$
$$\left.+\frac{4B^2\dot{A}}{A^3C^2}\dot{F}+\frac{4A'}{AC^2}F'-\frac{4B^2}{A^2C^2}\ddot{F}\right]+\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)$$
$$\times\left(\frac{8B^2\dot{C}}{A^3C^2}\dot{F}-\frac{8C'}{AC^2}F'\right)+\frac{4BF}{AC^2}\left[\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)\right.$$
$$+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\left(C''-\frac{B'C'}{B}\right)+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)$$
$$\left.\left.-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)+\frac{2}{AB}(\dot{C'}^2-\ddot{C}C'')\right]\right\}+\frac{2C'}{C^3}\left\{-\frac{1}{2}C^2f\right.$$
$$+\frac{4F}{AB}\left[\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\left\{\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)-\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)\right\}\right.$$
$$+\frac{2C^2}{AB}\left(\frac{A'\dot{C}}{AC}+\frac{\dot{B}C'}{BC}\right)\left(\frac{A'\dot{C}}{AC}-\frac{2\dot{C}'}{C}\right)+\frac{2}{B}\left(\frac{\dot{A}\dot{C}}{A^2}+\frac{A'C'}{B^2}\right)\left(C''-\frac{B'C'}{B}\right)$$
$$\left.+\frac{2\ddot{C}}{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}\right)-\frac{2\dot{B}}{A}\left(\frac{\dot{A}\dot{C}^2}{A^3}-\frac{\dot{B}C'^2}{B^3}\right)+\frac{2}{AB}\left(\dot{C'}^2-\ddot{C}C''\right)\right]$$
$$+\frac{4C}{A^3B}\left[\dot{B}\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)+\dot{A}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)\right.$$
$$\left.-\frac{2A'}{B}\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)+\dot{C}\left\{\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)-\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right\}\right]\dot{F}$$
$$+\frac{4C}{AB^3}\left[B'\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)+A'\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)\right.$$
$$\left.-\frac{2\dot{B}}{A}\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)-C'\left\{\frac{B}{A}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}\right)-\frac{A}{B}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\right\}\right]F'$$

$$+\frac{8C}{A^2B^2}\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-\dot{C}'\right)\dot{F}'-\frac{4C}{AB^2}\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\frac{\ddot{C}}{A}\right)F''$$
$$\left.\left.-\frac{4C}{A^2B}\left(\frac{B'C'}{B^2}+\frac{\dot{B}\dot{C}}{A^2}-\frac{C''}{B}\right)\ddot{F}\right\}\right]. \tag{A.2}$$

Appendix B.

Gauss–Bonnet invariant and the field equations for nonstatic first-order perturbation scheme are

$$\frac{1}{A_0}\left(1-\frac{1}{B_0^2}\right)\left(\frac{a'}{B_0}\right)'+\frac{2B_0'}{B_0^4}\left(\frac{a}{A_0}\right)'-\frac{a}{A_0}\left(1-\frac{1}{B_0^2}\right)\left(\frac{A_0'}{B_0}\right)'$$
$$-\frac{A_0'}{A_0B_0}\left(1-\frac{3}{B_0^2}\right)\left(\frac{b}{B_0}\right)'-\frac{2b}{A_0B_0}\left(1-\frac{2}{B_0^2}\right)\left(\frac{A_0'}{B_0}\right)'-\frac{8bA_0'B_0'}{A_0B_0^5}$$
$$-\frac{2}{A_0B_0^3}(c'A_0')'+\frac{6rA_0'B_0'}{A_0B_0^4}\left(\frac{c}{r}\right)'-\frac{2c}{rA_0}\left(\frac{A_0'}{B_0}\right)'+\frac{2cA_0''}{rA_0B_0^3}+\frac{1}{8}r^2gB_0$$
$$-\frac{1}{A_0^2}\left(b+\frac{cB_0'}{B_0^2}-\frac{b}{B_0^2}\right)\frac{\ddot{T}}{T}=0, \tag{B.1}$$

$$\kappa\left(2a\frac{\rho_0}{A_0}+\frac{\bar{\rho}}{T}\right)$$
$$=-\frac{2}{B_0^2}\left[\frac{a}{rA_0}\left\{\frac{B_0}{r}\left(\frac{r}{B_0}\right)'-\frac{B_0^2}{r}-\frac{B_0'}{B_0}\right\}-\frac{b}{r^2}\left(\frac{r}{B_0}\right)'\right.$$
$$\left.+\frac{bB_0'}{rB_0^2}-\frac{1}{r}\left(\frac{b}{B_0}\right)'+\left(\frac{c}{r}\right)''+\left(\frac{3}{r}-\frac{B_0'}{B_0}\right)\left(\frac{c}{r}\right)'-\frac{cB_0^2}{r^3}\right]$$
$$+\frac{1}{2}\beta\mathcal{G}_0^n\left(ng\mathcal{G}_0^{-1}+2\frac{a}{A_0}\right)+4\beta n\mathcal{G}_0^{n-1}\left[\frac{1}{r^2A_0B_0}\left\{\left(\frac{a'}{B_0}\right)'+\frac{a}{A_0}\left(\frac{A_0'}{B_0}\right)'\right\}\right.$$
$$\times\left(1-\frac{1}{B_0^2}\right)+\frac{2B_0'}{r^2A_0^2B_0^5}(aA_0)'-\frac{A_0'}{r^2A_0B_0^2}\left(1-\frac{3}{B_0^2}\right)$$
$$\times\left(\frac{b}{B_0^2}\right)'-\frac{2b}{r^2A_0B_0^2}\left(1-\frac{2}{B_0^2}\right)\left(\frac{A_0'}{B_0}\right)'-\frac{8bA_0'B_0'}{r^2A_0B_0^6}$$
$$-2\left\{\frac{(c'A_0')'}{r^2A_0B_0^4}-\frac{3A_0'B_0'}{rA_0B_0^5}\left(\frac{c}{r}\right)'+\frac{c}{r^3A_0B_0}\left(\frac{A_0'}{B_0}\right)'\frac{cA_0''}{r^3A_0B_0^4}\right\}$$
$$\left.-(n-1)\frac{g}{\mathcal{G}_0}\left\{\frac{A_0'B_0'}{r^2A_0B_0^3}\left(1-\frac{3}{B_0^2}\right)-\frac{A_0''}{r^2A_0B_0^2}\left(1-\frac{1}{B_0^2}\right)\right\}\right]$$

$$+4\beta n(n-1)\frac{1}{r^2B_0^2}\left[\left\{\mathcal{G}_0^{n-2}\mathcal{G}_0'\left[\frac{2B_0'}{B_0}\left(\frac{a}{A_0}-\frac{c}{r}\right)+\frac{b'}{B_0}\right]+\frac{B_0'}{B_0}\left(g\mathcal{G}_0^{n-2}\right)'\right\}\right.$$
$$\times\left(1-\frac{3}{B_0^2}\right)-3\frac{bB_0'}{B_0^2}\left(1-\frac{5}{B_0^2}\right)\mathcal{G}_0^{n-2}\mathcal{G}_0'+2B_0\left(\frac{c'}{B_0^3}\right)'\mathcal{G}_0^{n-2}\mathcal{G}_0'$$
$$+\left\{2\left(\frac{c}{r}-\frac{a}{A_0}\right)\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)'-\left(g\mathcal{G}_0^{n-2}\right)'-(n-2)\left(g\mathcal{G}_0^{n-3}\mathcal{G}_0'\right)'\right\}\left(1-\frac{1}{B_0^2}\right)$$
$$\left.+2\left\{\frac{b}{B_0}\left(1-\frac{2}{B_0^2}\right)+\frac{c'}{B_0^2}\right\}\left(\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)'\right],\tag{B.2}$$

$$\left(\frac{c}{r}\right)'-\frac{b}{rB_0}+\frac{c}{r}\left(\frac{1}{r}-\frac{A_0'}{A_0}\right)+2\beta n(n-1)\frac{1}{r^2}\left(1-\frac{1}{B_0^2}\right)$$
$$\times\left[A_0\left(\frac{g}{A_0}\right)'+(n-2)g\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right]\mathcal{G}_0^{n-2}-4\beta n(n-1)\frac{A_0}{r^2B_0^2}\left(\frac{c}{A_0}\right)'$$
$$\times\mathcal{G}_0^{n-2}\mathcal{G}_0'-2\beta n(n-1)\frac{b}{r^2B_0}\left(1-\frac{3}{B_0^2}\right)\mathcal{G}_0^{n-2}\mathcal{G}_0'=0,\tag{B.3}$$

$$\kappa\left(\frac{B_0^2\bar{P}_r}{T}+2bB_0P_{r0}\right)$$
$$=\left(1-\frac{1}{B_0^2}\right)\left[-\frac{2}{r^2}\left(bB_0-\frac{cB_0^2}{r}\right)+4\beta n\frac{B_0}{r^2A_0}\right.$$
$$\times\left\{\frac{1}{B_0^2}\left(a'B_0'-bA_0'\frac{B_0'}{B_0}-aB_0'\frac{A_0'}{A_0}+\frac{b'A_0'}{A_0B_0}\right)-\frac{1}{B_0}\left(a''-\frac{aA_0''}{A_0}\right)\right.$$
$$\left.+\frac{1}{B_0}\left(a-\frac{bA_0}{B_0}\right)\left(\frac{A_0'B_0'}{A_0B_0}-\frac{A_0''}{A_0}\right)+\frac{b}{A_0}\left(\frac{\ddot{T}}{T}\right)\right\}\mathcal{G}_0^{n-1}+\frac{4\beta n}{r^2A_0}$$
$$\times\left(\frac{A_0'B_0'}{B_0^2}-\frac{A_0''}{B_0}\right)\left\{\left(b-2B_0\frac{c}{r}-\frac{aB_0}{A_0}\right)\mathcal{G}_0^{n-1}+(n-1)gB_0\mathcal{G}_0^{n-2}\right\}$$
$$+4\beta n(n-1)\frac{A_0'}{r^2A_0}\left(g'\mathcal{G}_0^{n-2}+(n-2)g\mathcal{G}_0^{n-3}\mathcal{G}_0'\right)+\frac{4\beta n(n-1)}{r^2A_0}$$
$$\left.\times\left(a'-2A_0'\frac{c}{r}-\frac{aA_0'}{A_0}\right)\mathcal{G}_0^{n-2}\mathcal{G}_0'-4\beta n(n-1)g\frac{B_0^2}{r^2A_0^2}\left(\frac{\ddot{T}}{T}\right)\mathcal{G}_0^{n-2}\right]$$
$$-\frac{2}{B_0^2}\left(c'-\frac{b}{B_0}\right)\left[\frac{4\beta n}{r^2A_0}\mathcal{G}_0^{n-1}\left(\frac{A_0'B_0'}{B_0}-A_0''+(n-1)A_0'\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right)-\frac{B_0^2}{r^2}\right]$$
$$+\left[\frac{2A_0'}{rA_0}\left(\frac{2b}{B_0}-\frac{a}{A_0}-\frac{c}{r}\right)-8\beta n(n-1)\frac{A_0'}{r^2A_0B_0^2}\right.$$
$$\times\left\{g'+(n-2)g\frac{\mathcal{G}_0'}{\mathcal{G}_0}+\left(c'-\frac{2c}{r}-\frac{a}{A_0}\right)\mathcal{G}_0'\right\}\mathcal{G}_0^{n-2}$$

$$+\left(\frac{2B_0^2}{rA_0}-\frac{8\beta n(n-1)}{r^2A_0}\mathcal{G}_0^{n-2}\mathcal{G}_0'\right)\left\{\frac{1}{B_0^2}\left(a'+c'A_0'-\frac{2bA_0'}{B_0}\right)-\frac{c}{A_0}\left(\frac{\ddot{T}}{T}\right)\right\}\Bigg]$$

$$-\frac{1}{2}\beta B_0(ngB_0\mathcal{G}_0^{n-1}+2b\mathcal{G}_0^n)-\Bigg[4\beta n\frac{B_0}{r^2A_0}\left\{\frac{4A_0'B_0'}{B_0^4}\left(c'-\frac{b}{B_0}\right)-\frac{2c''A_0'}{B_0^3}\right.$$

$$\left.+\frac{2}{B_0^4}(a'B_0'+b'A_0')+\frac{2cB_0'}{A_0B_0^2}\left(\frac{\ddot{T}}{T}\right)\right\}\mathcal{G}_0^{n-1}-\frac{2A_0'B_0'}{B_0^4}$$

$$\times\left\{4\beta n\frac{B_0}{r^2A_0}\left(\frac{b}{B_0}-\frac{2c}{r}-\frac{a}{A_0}+(n-1)\frac{g}{\mathcal{G}_0}\right)\mathcal{G}_0^{n-1}\right\}\Bigg], \tag{B.4}$$

$$\kappa\left(\frac{2c}{r}P_{\perp 0}+\frac{\bar{P}_\perp}{T}\right)$$

$$=\frac{1}{A_0^2}\left(\frac{b}{B_0}+\frac{c}{r}\right)\frac{\ddot{T}}{T}+\frac{1}{B_0^2}\Bigg[\left(\frac{a}{A_0}\right)''+\left(\frac{c}{r}\right)''+\left(\frac{2A_0'}{A_0}-\frac{B_0'}{B_0}+\frac{1}{r}\right)\left(\frac{a}{A_0}\right)'$$

$$-\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)\left(\frac{b}{B_0}\right)'+\left(\frac{A_0'}{A_0}-\frac{B_0'}{B_0}+\frac{2}{r}\right)\left(\frac{c}{r}\right)'$$

$$+2\left(\frac{A_0'B_0'}{A_0B_0}+\frac{B_0'}{rB_0}-\frac{A_0'}{rA_0}-\frac{A_0''}{A_0}\right)\left(\frac{b}{B_0}-\frac{c}{r}\right)\Bigg]-\frac{1}{2}\beta\left(\frac{ng}{\mathcal{G}_0}+\frac{2c}{r}\right)\mathcal{G}_0^n$$

$$+4\beta n\mathcal{G}_0^{n-1}\frac{1}{r^2}\Bigg[-\frac{1}{A_0B_0}\left(\frac{a'}{B_0}\right)'\left(1-\frac{1}{B_0^2}\right)+\frac{a}{A_0^2B_0}\left(\frac{A_0'}{B_0}\right)'\left(1-\frac{1}{B_0^2}\right)$$

$$-\frac{2B_0'}{B_0^5}\left(\frac{a}{A_0}\right)'+\frac{A_0'}{A_0B_0^2}\left(\frac{b}{B_0}\right)'\left(1-\frac{3}{B_0^2}\right)+\frac{2b}{A_0B_0^2}\left(\frac{A_0'}{B_0}\right)'\left(1-\frac{2}{B_0^2}\right)$$

$$-8b\frac{A_0'B_0'}{A_0B_0^6}+\frac{2}{A_0B_0^4}\left\{(c'A_0')'-3c'\frac{A_0'B_0'}{B_0}\right\}-(n-1)\frac{g}{\mathcal{G}_0}\left\{\frac{1}{A_0B_0}\left(\frac{A_0'}{B_0}\right)'\right.$$

$$\left.\times\left(1-\frac{1}{B_0^2}\right)+\frac{2A_0'B_0'}{A_0B_0^5}\right\}+r(n-1)\left\{\frac{a}{A_0^2B_0^3}\left(\frac{A_0'}{B_0}\right)'-\frac{1}{A_0B_0^3}\left(\frac{a'}{B_0}\right)'\right.$$

$$+\frac{2B_0'}{B_0^5}\left(\frac{a}{A_0}\right)'+\frac{4b}{A_0B_0^4}\left(\frac{A_0'}{B_0}\right)'+\frac{3A_0'}{A_0B_0^4}\left(\frac{b}{B_0}\right)'-8b\frac{A_0'B_0'}{A_0B_0^6}-\frac{1}{A_0B_0^4}(c'A_0')'$$

$$\left.-3\frac{(rc)'}{r}\frac{A_0'B_0'}{A_0B_0^5}-\frac{cA_0''}{rA_0B_0^4}\right\}\frac{\mathcal{G}_0'}{\mathcal{G}_0}-(n-1)\frac{r}{B_0\mathcal{G}_0^{n-1}}\left(\frac{A_0'}{B_0^3}\right)'(g\mathcal{G}_0^{n-2})'$$

$$-(n-1)\frac{r}{B_0^4\mathcal{G}_0^{n-1}}\left\{\frac{A_0'}{A_0}(g'\mathcal{G}_0^{n-2})'+\left(\frac{A_0'}{rA_0}(rc)'+\left(\frac{a}{A_0}\right)'-\frac{4bA_0'}{A_0B_0}\right)\right.$$

$$\left.\times(\mathcal{G}_0^{n-2}\mathcal{G}_0')'+(n-2)\frac{A_0'}{A_0}(g\mathcal{G}_0^{n-3}\mathcal{G}_0')'\right\}\Bigg]+\frac{4\beta n}{r^2}\mathcal{G}_0^{n-1}\Bigg[\frac{b}{A_0^2B_0}\left(1-\frac{1}{B_0^2}\right)^{n-1}$$

$$+\frac{2cB_0'}{A_0^2B_0^3}+(n-1)\frac{r}{A_0B_0^3}\left(b-\frac{cB_0'}{A_0}\right)\frac{\mathcal{G}_0'}{\mathcal{G}_0}+(n-1)(n-2)\frac{rc}{A_0^2B_0^2}\left(\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right)^2$$
$$-(n-1)\frac{rB_0'}{A_0^2B_0^3}\left(\frac{g}{\mathcal{G}_0}\right)\Bigg]\frac{\ddot{T}}{T}. \qquad \text{(B.5)}$$

The expressions for Δ_{1p} and Δ_{2p} in Eqs. (38) and (39), respectively, are given as follows:

$$\Delta_{1p}=-4\beta n\mathcal{G}_0^{n-1}\frac{1}{\kappa}\left[\frac{g}{8}+\frac{1}{r^2B_0^2}\left\{\frac{a''}{A_0}-\frac{aA_0''}{A_0^2}-\frac{2cA_0''}{rA_0}\right\}\left(1-\frac{1}{B_0^2}\right)\right.$$
$$+\frac{1}{r^2B_0^2}\left\{\frac{2cA_0'B_0'}{rA_0B_0}-\frac{b'A_0'}{A_0B_0}-\left(\frac{a}{A_0}\right)'\frac{B_0'}{B_0}\right\}\left(1-\frac{3}{B_0^2}\right)-\frac{2bA_0''}{r^2A_0B_0^3}$$
$$\left.\times\left(1-\frac{3}{B_0^2}\right)-\frac{3bA_0'B_0'}{r^2A_0B_0^4}\left(1-\frac{5}{B_0^2}\right)-\frac{2A_0'}{r^2A_0B_0}\left(\frac{c'}{B_0^3}\right)'-\frac{2c'B_0''}{r^2A_0B_0^4}\right], \qquad \text{(B.6)}$$

$$\Delta_{2p}=4\beta nT\mathcal{G}_0^{n-1}\frac{1}{\kappa}\left[\left[\frac{1}{r^2B_0^4}\left(1-\frac{3}{B_0^2}\right)\left\{B_0''+2\frac{A_0'B_0'}{A_0}+\frac{2B_0'}{r}\right\}+\frac{2B_0'^2}{r^2B_0^5}\right.\right.$$
$$\times\left.\left(1-\frac{5}{B_0^2}\right)\right]\left(\frac{a}{A_0}\right)'+\frac{1}{A_0B_0^3}\left(1-\frac{1}{B_0^2}\right)\left\{\frac{a}{A_0}\left(\frac{A_0''}{r^2}\right)'-\frac{1}{r^2A_0}(a'A_0')'\right.$$
$$\left.+\left(\frac{a''}{r^2}\right)'+\frac{2aA_0'A_0''}{r^2A_0^2}\right\}+\frac{B_0'}{r^2A_0B_0^4}\left(3-\frac{7}{B_0^2}\right)\left\{a\frac{A_0''}{A_0}-a''\right\}+\frac{1}{r^2}$$
$$\times\left(\frac{7A_0''}{A_0}-\frac{6A_0'}{rA_0}\right)\left(\frac{b}{B_0^2}\right)'-\left\{\frac{2A_0''}{r^2A_0}-\left(\frac{A_0'}{r^2A_0}\right)'\right\}\left(\frac{b}{B_0^4}\right)'+\frac{3A_0'B_0'}{r^2A_0}$$
$$\times\left(\frac{b}{B_0^5}\right)'-\frac{15A_0'B_0'}{r^2A_0}\left(\frac{b}{B_0^7}\right)'-\frac{b}{r^2B_0^4}\left(3-\frac{5}{B_0^2}\right)\left\{\left(\frac{A_0''}{A_0}\right)'-\frac{2A_0''}{rA_0}\right\}$$
$$-\frac{18bA_0'}{r^2B_0^7}\left(\frac{B_0'}{A_0}\right)'+\frac{3A_0'}{r^2B_0^6}\left(\frac{b'}{A_0}\right)'+\frac{A_0'}{r^2A_0B_0^4}\left(4b\frac{B_0''}{B_0}-15\frac{b'B_0'}{B_0^3}-b''\right)$$
$$-\frac{b}{8B_0^2}\mathcal{G}_0'-\frac{1}{B_0^2}\left(1-\frac{1}{B_0^2}\right)\left(\frac{b}{r^2A_0^2}\right)'\frac{\ddot{T}}{T}+\frac{bB_0'}{r^2A_0^2B_0^3}\left(1-\frac{3}{B_0^2}\right)\frac{\ddot{T}}{T}$$
$$+\left(\frac{c}{r}\right)'\left\{\frac{2A_0'A_0''}{rA_0^2B_0^5}+\frac{14B_0'}{A_0B_0^6}\left(\frac{A_0'}{r}\right)'-\frac{6A_0'}{rB_0^6}\left(\frac{B_0'}{A_0}\right)'+4\frac{A_0'B_0'}{A_0}\left(\frac{1}{rB_0^6}\right)'\right.$$
$$\left.-\frac{6A_0'B_0'^2}{rA_0B_0^7}+\frac{6A_0''}{r^2A_0B_0^5}\right\}+\frac{2A_0'}{A_0B_0^5}\left(\frac{c''}{r^2}\right)'-\frac{2}{A_0B_0^2}\left(\frac{c}{r^3}\right)'\left(\frac{A_0'}{B_0}\right)'$$

$$
\begin{aligned}
&-\frac{2c''}{r^2B_0^5}\left(\frac{A_0'}{A_0}\right)'-\frac{2}{r^2A_0B_0^5}(c'A_0'')'+\frac{2cA_0'B_0'}{r^3B_0}\left(\frac{1}{A_0B_0^3}\right)'+\frac{2c}{r^3A_0B_0^4}\\
&\times(A_0'B_0')'-\frac{2cA_0'''}{r^3A_0B_0^3}\left(1-\frac{1}{B_0^2}\right)+\frac{2cA_0'A_0''}{r^3A_0^2B_0^3}+\frac{14c''A_0'B_0'}{r^2A_0B_0^6}+\frac{4cA_0''B_0'}{r^3A_0B_0^4}\\
&-2\left\{\frac{1}{r^2A_0^2B_0^4}(cB_0')'+\frac{cB_0'}{B_0}\left(\frac{1}{r^2A_0^2B_0^3}\right)'\right\}\frac{\ddot{T}}{T}\Bigg]+4\beta n(n-1)Tg\mathcal{G}_0^{n-2}\\
&\times\frac{1}{\kappa}\left[\frac{1}{B_0^3}\left(\frac{A_0''}{r^2A_0}\right)'\left(1-\frac{1}{B_0^2}\right)-\frac{A_0'}{B_0^4}\left(\frac{B_0'}{r^2A_0}\right)'\left(1-\frac{3}{B_0^2}\right)+\frac{3A_0'B_0'^2}{r^2A_0B_0^5}\right.\\
&\left.\times\left(1-\frac{5}{B_0^2}\right)-\frac{A_0''B_0'}{r^2A_0B_0^4}\left(3-\frac{7}{B_0^2}\right)+\frac{1}{8B_0}\mathcal{G}_0'\right]+\frac{1}{2\kappa B_0}\beta nTg'\mathcal{G}_0^{n-1}.
\end{aligned}
\tag{B.7}
$$

The zero expansion condition in Eq. (40) yields the dark source term in the following form:

$$
\begin{aligned}
\Delta_3=&\,4\beta nT\mathcal{G}_0^{n-1}\frac{1}{\kappa}\left[\frac{g}{8}+\frac{1}{r^2B_0^2}\left\{\frac{a''}{A_0}-\frac{aA_0''}{A_0^2}\right\}\left(1-\frac{1}{B_0^2}\right)+\left\{\frac{2A_0'}{r^3A_0B_0^2}\right.\right.\\
&\left.\times\left(5c\frac{B_0'}{B_0}+c'-\frac{c}{r}\right)-\frac{1}{r^2B_0^2}\left(\frac{a}{A_0}\right)'\frac{B_0'}{B_0}\right\}\left(1-\frac{3}{B_0^2}\right)+\frac{2cA_0''}{r^3A_0B_0^2}\\
&\times\left(1-\frac{5}{B_0^2}\right)+\frac{6A_0'B_0'}{A_0B_0^5}\left(\frac{c}{r^2}\right)'-\frac{2c''A_0'}{r^2A_0B_0^4}-\frac{2c'B_0''}{r^2A_0B_0^4}+\frac{2c}{r^2A_0^2}\\
&\left.\times\left\{\frac{1}{r}\left(1-\frac{1}{B_0^2}\right)+\frac{B_0'}{B_0^3}\right\}\frac{\ddot{T}}{T}\right].
\end{aligned}
\tag{B.8}
$$

The expressions for $\Xi_i's$ in Eq. (54) are given as follows:

$$
\begin{aligned}
\Xi_1=&\left(\frac{r+2m_0}{r^3}\left[\frac{\kappa r^3}{2}(\rho_0+P_{r0})+\frac{m_0}{r^2}\left\{r^2+2\beta n(n-1)(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right\}\right.\right.\\
&\left.\left.+4\beta n(n-1)r^2m_0(\mathcal{G}_0^{n-2}\mathcal{G}_0')\right]\left[1-\frac{2}{r^2}\beta n(n-1)(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\right]\right),\\
\Xi_2=&\left(\frac{r+2m_0}{r}\left[\kappa P_{\perp 0}+\frac{1}{2}\beta\mathcal{G}_0^n-\frac{m_0}{r^3}-\left\{\frac{m_0}{r^5}(r^3+8\beta n(r-3m_0)\mathcal{G}_0^{n-1}\right.\right.\right.\\
&-12\beta n(n-1)r(r-2m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0')+\frac{1}{r^3}(r-2m_0)(r-4\beta n(n-1)\\
&\times(r-2m_0)(\mathcal{G}_0^{n-2}\mathcal{G}_0')')\Big\}\left\{\frac{r+2m_0}{r^3}\left[\frac{\kappa r^3}{2}(\rho_0+P_{r0})+\frac{m_0}{r^2}\{r^2+2\beta n\right.\right.
\end{aligned}
$$

$$\times (n-1)(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\} + 4\beta n(n-1)r^2 m_0(\mathcal{G}_0^{n-2}\mathcal{G}_0')\Big]\Big[1-\frac{2}{r^2}$$

$$\times \beta n(n-1)(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\Big]\Big\}\Big]\Big[1+\frac{8}{r^3}\beta n m_0\mathcal{G}_0^{n-1}+\frac{4}{r^2}\beta n(n-1)$$

$$\times (r-2m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\Big]\Big),$$

$$\Xi_3 = \Big[r^4\left(\kappa\rho_0' - \frac{6m_0^2}{r^6}(r+2m_0) - \frac{1}{2}\beta n\mathcal{G}_0^{n-1}\mathcal{G}_0'\right)\Big/(8m_0(r-2m_0)\beta n\mathcal{G}_0^{n-1})$$

$$+\left(\frac{2}{r}-(n-1)\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right)\left(\frac{r+2m_0}{r}\Big[\kappa P_{\perp 0}+\frac{1}{2}\beta\mathcal{G}_0^n - \frac{m_0}{r^3}\right.$$

$$-\Big\{\frac{m_0}{r^5}(r^3+8\beta n(r-3m_0)\mathcal{G}_0^{n-1}-12\beta n(n-1)r(r-2m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0')$$

$$+\frac{1}{r^3}(r-2m_0)(r-4\beta n(n-1)(r-2m_0)(\mathcal{G}_0^{n-2}\mathcal{G}_0')')\Big\}\Big\{\frac{r+2m_0}{r^3}$$

$$\times\Big[\frac{\kappa r^3}{2}(\rho_0+P_{r0})+\frac{m_0}{r^2}\{r^2+2\beta n(n-1)(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\}+4\beta n$$

$$\times (n-1)r^2m_0(\mathcal{G}_0^{n-2}\mathcal{G}_0')'\Big]\Big[1-\frac{2}{r^2}\beta n(n-1)(6m_0-2r)\mathcal{G}_0^{n-2}\mathcal{G}_0'\Big]\Big\}\Big]$$

$$\left.\times\Big[1+\frac{8}{r^3}\beta n m_0\mathcal{G}_0^{n-1}+\frac{4}{r^2}\beta n(n-1)(r-2m_0)\mathcal{G}_0^{n-2}\mathcal{G}_0'\Big]\right) - 7m_0(r+2m_0)$$

$$\times (n-1)\frac{\mathcal{G}_0'}{r^4\mathcal{G}_0}+3(n-1)(n-2)\frac{\mathcal{G}_0'\mathcal{G}_0''}{\mathcal{G}_0^2}-(n-1)(n-2)(n-3)\left(\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right)^3$$

$$-(n-1)(n-2)(4r-11m_0)(r+2m_0)\frac{1}{r^3}\left(\frac{\mathcal{G}_0'}{\mathcal{G}_0}\right)^2-(n-1)\frac{\mathcal{G}_0'''}{\mathcal{G}_0}$$

$$-(n-1)(r+2m_0)(4r-7m_0)\frac{\mathcal{G}_0''}{r^3\mathcal{G}_0}\Big].$$

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Energy conditions in $f(\mathcal{G}, T)$ gravity

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Abstract The aim of this paper is to introduce a new modified gravity theory named $f(\mathcal{G}, T)$ gravity ($\mathcal{G}$ and T are the Gauss–Bonnet invariant and trace of the energy-momentum tensor, respectively) and investigate energy conditions for two reconstructed models in the context of FRW universe. We formulate general field equations, divergence of energy-momentum tensor, equation of motion for test particles as well as corresponding energy conditions. The massive test particles follow non-geodesic lines of geometry due to the presence of an extra force. We express the energy conditions in terms of cosmological parameters like the deceleration, jerk, and snap parameters. The reconstruction technique is applied to this theory using de Sitter and power-law cosmological solutions. We analyze the energy bounds and obtain feasible constraints on the free parameters.

1 Introduction

Current cosmic accelerated expansion has been affirmed from a diverse set of observational data coming from several pieces of astronomical evidence, including supernova type Ia, large scale structure, cosmic microwave background radiation etc. [1–4]. This expanding paradigm is considered as a consequence of mysterious force dubbed dark energy (DE), which possesses a large negative pressure. Modified theories of gravity are considered as the favorite candidates to unveil the enigmatic nature of this energy. These modified theories are usually developed by including scalar invariants and their corresponding generic functions in the Einstein–Hilbert action.

A remarkably interesting gravity theory is the modified Gauss–Bonnet (GB) theory. A linear combination of the form

$$\mathcal{G} = R_{\alpha\beta\xi\eta}R^{\alpha\beta\xi\eta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2,$$

where $R_{\alpha\beta\xi\eta}$, $R_{\alpha\beta}$ and R represent the Riemann tensor, the Ricci tensor, and the Ricci scalar, respectively, is called a Gauss–Bonnet invariant ($\mathcal{G}$). It is a second order Lovelock scalar invariant and thus free from spin-2 ghosts instabilities [5–7]. The Gauss–Bonnet combination is a four-dimensional topological invariant which does not involve the field equations. However, it provides interesting results in the same dimensions when either coupled with a scalar field or when an arbitrary function $f(\mathcal{G})$ is added to the Einstein–Hilbert action [8–10]. The latter approach is introduced by Nojiri and Odintsov; it is known as the $f(\mathcal{G})$ theory of gravity [11]. Like other modified theories, this theory is an alternative to study DE and is consistent with solar system constraints [12]. In this context, there is a possibility to discuss a transition from decelerated to accelerated as well as from non-phantom to phantom phases and also to explain the unification of early and late times accelerated expansion of the universe [13,14].

The fascinating problem of cosmic accelerated expansion has successfully been discussed by taking into account modified theories of gravity with curvature–matter coupling. The motion of test particles is studied in $f(R)$ and $f(\mathcal{G})$ gravity theories non-minimally coupled with the matter Lagrangian density ($\mathcal{L}_m$). Consequently, the extra force experienced by test particles is found to be orthogonal to their four velocities and the motion becomes non-geodesic [15–17]. It is found that, for certain choices of $\mathcal{L}_m$, the presence of the extra force vanishes in a non-minimal $f(R)$ model, while it remains preserved in a non-minimal $f(\mathcal{G})$ model. The geodesic deviation is weaker in $f(\mathcal{G})$ gravity for small curvatures as compared to non-minimal $f(R)$ gravity. Nojiri et al. [18] studied the non-minimally coupling of $f(R)$ and $f(\mathcal{G})$ theories with $\mathcal{L}_m$ and found that such a coupling naturally unifies the inflationary era with current cosmic accelerated expansion.

In order to describe some realistic matter distribution, certain conditions must be imposed on the energy-momentum tensor ($T_{\alpha\beta}$) known as energy conditions. These conditions originate from the Raychaudhuri equations with the requirement that not only gravity is attractive but also the energy

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density is positive. The null (NEC), weak (WEC), dominant (DEC), and strong (SEC) energy conditions are the four fundamental conditions. They play a key role to study the theorems related to singularity and black hole thermodynamics. The null energy condition is important to discuss the second law of black hole thermodynamics while its violation leads to a Big-Rip singularity of the universe [19]. The proof of the positive mass theorem is based on DEC [20], while SEC is useful to study the Hawking–Penrose singularity theorem [21].

The energy conditions have been investigated in different modified theories of gravity like $f(R)$ gravity, Brans–Dicke theory, $f(\mathcal{G})$ gravity, and generalized teleparallel theory [22–25]. Banijamali et al. [26] investigated the energy conditions for non-minimally coupling $f(\mathcal{G})$ theory with $\mathcal{L}_m$ and found that the WEC is satisfied for specific viable $f(\mathcal{G})$ models. Sharif and Waheed [27] explored the energy bounds in the context of generalized second order scalar-tensor gravity with the help of a power-law ansatz for the scalar field. Sharif and Zubair [28] derived these conditions in $f(R, T, R_{\alpha\beta}T^{\alpha\beta})$ theory of gravity for two specific models and also examined the Dolgov–Kowasaki instability for particular models of $f(R, T)$ gravity.

In this paper, we introduced a new modified theory of gravity named $f(\mathcal{G}, T)$ gravity, in which the gravitational Lagrangian is obtained by adding a generic function $f(\mathcal{G}, T)$ in the Einstein–Hilbert action. We study the energy conditions for the reconstructed $f(\mathcal{G}, T)$ models using an isotropic homogeneous universe model. The paper has the following format. In Sect. 2, we formulate the field equations of this gravity and discuss the equation of motion for test particles, while general expressions for the energy conditions as well as formulations in terms of cosmological parameters are discussed in Sect. 3. The reconstruction of models and their energy bounds is analyzed in Sect. 4. In the last section, we summarize our results.

2 Field equations of $f(\mathcal{G}, T)$ gravity

In this section, we formulate the field equations for $f(\mathcal{G}, T)$ gravity. For this purpose, we assume an action of the following form:

$$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}[R + f(\mathcal{G}, T)] + \int d^4x\sqrt{-g}\mathcal{L}_m, \tag{1}$$

where g and κ represent the determinant of the metric tensor $(g_{\alpha\beta})$ and the coupling constant, respectively. The energy-momentum tensor is defined as [29]

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\alpha\beta}}. \tag{2}$$

Assuming that the matter distribution depends on the components of $g_{\alpha\beta}$ but has no dependence on its derivatives, we obtain

$$T_{\alpha\beta} = g_{\alpha\beta}\mathcal{L}_m - 2\frac{\partial\mathcal{L}_m}{\partial g^{\alpha\beta}}. \tag{3}$$

The variation in the action (1) gives

$$\begin{aligned} 0 = \delta\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x[(R + f(\mathcal{G}, T))\delta\sqrt{-g} + \sqrt{-g}(\delta R + f_{\mathcal{G}} \\ \times (\mathcal{G}, T)\delta\mathcal{G} + f_T(\mathcal{G}, T)\delta T)] + \int d^4x\delta(\sqrt{-g}\mathcal{L}_m), \end{aligned} \tag{4}$$

where $f_{\mathcal{G}}(\mathcal{G}, T) = \frac{\partial f(\mathcal{G},T)}{\partial\mathcal{G}}$ and $f_T(\mathcal{G}, T) = \frac{\partial f(\mathcal{G},T)}{\partial T}$. The variations of $\sqrt{-g}$, $R^{\xi}_{\alpha\beta\eta}$, $R_{\alpha\eta}$, and R provide the following expressions:

$$\begin{aligned} \delta\sqrt{-g} &= -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}, \\ \delta R^{\xi}_{\alpha\beta\eta} &= \nabla_\beta(\delta\Gamma^{\xi}_{\eta\alpha}) - \nabla_\eta(\delta\Gamma^{\xi}_{\beta\alpha}), \\ &= (g_{\alpha\lambda}\nabla_{[\eta}\nabla_{\beta]} + g_{\lambda[\beta}\nabla_{\eta]}\nabla_\alpha)\delta g^{\xi\lambda} + \nabla_{[\eta}\nabla^{\xi}\delta g_{\beta]\alpha}, \\ \delta R_{\alpha\eta} &= \delta R^{\xi}_{\alpha\xi\eta}, \quad \delta R = (R_{\alpha\beta} + g_{\alpha\beta}\nabla^2 - \nabla_\alpha\nabla_\beta)\delta g^{\alpha\beta}, \end{aligned} \tag{5}$$

where $\Gamma^{\xi}_{\alpha\beta}$ and ∇_α represent the Christoffel symbol and covariant derivative, respectively. The variations of $\mathcal{G}$ and T yield

$$\begin{aligned} \delta\mathcal{G} &= 2R\delta R - 4\delta(R_{\alpha\beta}R^{\alpha\beta}) + \delta(R_{\alpha\beta\xi\eta}R^{\alpha\beta\xi\eta}), \\ \delta T &= (T_{\alpha\beta} + \Theta_{\alpha\beta})\delta g^{\alpha\beta}, \quad \Theta_{\alpha\beta} = g^{\xi\eta}\frac{\delta T_{\xi\eta}}{\delta g_{\alpha\beta}}. \end{aligned} \tag{6}$$

Using these variational relations in Eq. (4), we obtain the field equations of $f(\mathcal{G}, T)$ gravity after simplification as follows:

$$\begin{aligned} G_{\alpha\beta} &= \kappa^2 T_{\alpha\beta} - (T_{\alpha\beta} + \Theta_{\alpha\beta})f_T(\mathcal{G}, T) + \frac{1}{2}g_{\alpha\beta}f(\mathcal{G}, T) \\ &- (2RR_{\alpha\beta} - 4R^{\xi}_{\alpha}R_{\xi\beta} - 4R_{\alpha\xi\beta\eta}R^{\xi\eta} \\ &+ 2R^{\xi\eta\delta}_{\alpha}R_{\beta\xi\eta\delta})f_{\mathcal{G}}(\mathcal{G}, T) - (2Rg_{\alpha\beta}\nabla^2 \\ &- 2R\nabla_\alpha\nabla_\beta - 4g_{\alpha\beta}R^{\xi\eta}\nabla_\xi\nabla_\eta - 4R_{\alpha\beta}\nabla^2 + 4R^{\xi}_{\alpha}\nabla_\beta\nabla_\xi \\ &+ 4R^{\xi}_{\beta}\nabla_\alpha\nabla_\xi + 4R_{\alpha\xi\beta\eta}\nabla^{\xi}\nabla^{\eta})f_{\mathcal{G}}(\mathcal{G}, T), \end{aligned} \tag{7}$$

where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ and $\nabla^2 = \Box = \nabla_\alpha\nabla^\alpha$ denote the Einstein tensor and the d'Alembert operator, respectively. It is worth mentioning here that, for $f(\mathcal{G}, T) = f(\mathcal{G})$, Eq. (7) reduces to the field equations for $f(\mathcal{G})$ gravity, while $\Lambda(T)$ gravity (Λ is the cosmological constant) is obtained in the absence of the quadratic invariant $\mathcal{G}$ [11,30]. Furthermore, the Einstein field equations are recovered when $f(\mathcal{G}, T) = 0$. The trace of Eq. (7) is given by

$$\begin{aligned} &R + \kappa^2 T - (T + \Theta)f_T(\mathcal{G}, T) + 2f(\mathcal{G}, T) \\ &+ 2\mathcal{G}f_{\mathcal{G}}(\mathcal{G}, T) - 2R\nabla^2 f_{\mathcal{G}}(\mathcal{G}, T) \\ &+ 4R^{\alpha\beta}\nabla_\alpha\nabla_\beta f_{\mathcal{G}}(\mathcal{G}, T) = 0, \end{aligned}$$

where $\Theta = \Theta^{\alpha}_{\alpha}$. In this theory, the covariant divergence of Eq. (7) is non-zero, given by

$$\nabla^{\alpha}T_{\alpha\beta} = \frac{f_T(\mathcal{G},T)}{\kappa^2 - f_T(\mathcal{G},T)}\Big[(T_{\alpha\beta}+\Theta_{\alpha\beta})\nabla^{\alpha}(\ln f_T(\mathcal{G},T)) + \nabla^{\alpha}\Theta_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\nabla^{\alpha}T\Big]. \tag{8}$$

To obtain a useful expression for $\Theta_{\alpha\beta}$, we differentiate Eq. (3) with respect to the metric tensor

$$\frac{\delta T_{\alpha\beta}}{\delta g^{\xi\eta}} = \frac{\delta g_{\alpha\beta}}{\delta g^{\xi\eta}}\mathcal{L}_m + g_{\alpha\beta}\frac{\partial\mathcal{L}_m}{\partial g^{\xi\eta}} - 2\frac{\partial^2\mathcal{L}_m}{\partial g^{\xi\eta}\partial g^{\alpha\beta}}. \tag{9}$$

Using the relations

$$\frac{\delta g_{\alpha\beta}}{\delta g^{\xi\eta}} = -g_{\alpha\mu}g_{\beta\nu}\delta^{\mu\nu}_{\xi\eta}, \quad \delta^{\mu\nu}_{\xi\eta} = \frac{\delta g^{\mu\nu}}{\delta g^{\xi\eta}},$$

where $\delta^{\mu\nu}_{\xi\eta}$ is the generalized Kronecker symbol and putting Eq. (9) into (6), we obtain

$$\Theta_{\alpha\beta} = -2T_{\alpha\beta} + g_{\alpha\beta}\mathcal{L}_m - 2g^{\xi\eta}\frac{\partial^2\mathcal{L}_m}{\partial g^{\alpha\beta}\partial g^{\xi\eta}}. \tag{10}$$

This shows that once the value of $\mathcal{L}_m$ is determined, we can find the expression for the tensor $\Theta_{\alpha\beta}$.

We consider the matter distribution as a perfect fluid given by

$$T_{\alpha\beta} = (\rho + P)V_{\alpha}V_{\beta} - Pg_{\alpha\beta}, \tag{11}$$

where ρ, P and V_{α} are the density, pressure, and four velocity of the fluid, respectively. The four velocity satisfies the relation $V_{\alpha}V^{\alpha} = 1$ and the corresponding Lagrangian density can be taken as $\mathcal{L}_m = -P$ [31]. Thus Eq. (10) yields

$$\Theta_{\alpha\beta} = -2T_{\alpha\beta} - Pg_{\alpha\beta}. \tag{12}$$

Equation (7) can be written in a form identical to the Einstein field equations as

$$G_{\alpha\beta} = \kappa^2 T^{(\mathrm{eff})}_{\alpha\beta} = \kappa^2(T_{\alpha\beta} + T^{\mathcal{G}T}_{\alpha\beta}), \tag{13}$$

where $T^{\mathcal{G}T}_{\alpha\beta}$ is the $f(\mathcal{G},T)$ contribution. For the case of a perfect fluid, the expression for $T^{\mathcal{G}T}_{\alpha\beta}$ is given by

$$\begin{aligned}T^{\mathcal{G}T}_{\alpha\beta} = \frac{1}{\kappa^2}\Big[&(\rho+P)V_{\alpha}V_{\beta}f_T(\mathcal{G},T) + \frac{1}{2}g_{\alpha\beta}f(\mathcal{G},T)\\ &-(2RR_{\alpha\beta} - 4R^{\xi}_{\alpha}R_{\xi\beta} - 4R_{\alpha\xi\beta\eta}R^{\xi\eta} + 2R^{\xi\eta\delta}_{\alpha}R_{\beta\xi\eta\delta})\\ &\times f_{\mathcal{G}}(\mathcal{G},T) - (2Rg_{\alpha\beta}\nabla^2 - 2R\nabla_{\alpha}\nabla_{\beta}\\ &-4g_{\alpha\beta}R^{\xi\eta}\nabla_{\xi}\nabla_{\eta} - 4R_{\alpha\beta}\nabla^2 + 4R^{\xi}_{\alpha}\nabla_{\beta}\nabla_{\xi} + 4R^{\xi}_{\beta}\nabla_{\alpha}\nabla_{\xi}\\ &+4R_{\alpha\xi\beta\eta}\nabla^{\xi}\nabla^{\eta})f_{\mathcal{G}}(\mathcal{G},T)\Big].\end{aligned} \tag{14}$$

The line element for FRW universe model is

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \tag{15}$$

where $a(t)$ represents the scale factor. The corresponding field equations are

$$3H^2 = \kappa^2\rho_{\mathrm{eff}}, \quad -(2\dot{H} + 3H^2) = \kappa^2 P_{\mathrm{eff}}, \tag{16}$$

where

$$\rho_{\mathrm{eff}} = \rho + \frac{1}{\kappa^2}\Big[(\rho+P)f_T(\mathcal{G},T) + \frac{1}{2}f(\mathcal{G},T) - 12H^2 \times (H^2+\dot{H})f_{\mathcal{G}}(\mathcal{G},T) + 12H^3\partial_t f_{\mathcal{G}}(\mathcal{G},T)\Big], \tag{17}$$

$$P_{\mathrm{eff}} = P - \frac{1}{\kappa^2}\Big[\frac{1}{2}f(\mathcal{G},T) - 12H^2(H^2+\dot{H})f_{\mathcal{G}}(\mathcal{G},T) + 8H(H^2+\dot{H})\partial_t f_{\mathcal{G}}(\mathcal{G},T) + 4H^2\partial_{tt}f_{\mathcal{G}}(\mathcal{G},T)\Big], \tag{18}$$

$\mathcal{G} = 24H^2(H^2+\dot{H})$, $H = \dot{a}/a$ is the Hubble parameter and a dot represents the time derivative. The divergence of $T_{\alpha\beta}$ takes the form

$$\dot{\rho} + 3H(\rho+P) = \frac{-1}{\kappa^2 + f_T(\mathcal{G},T)} \times\Big[\Big(\dot{P} + \frac{1}{2}\dot{T}\Big)f_T(\mathcal{G},T) + (\rho+P)\partial_t f_T(\mathcal{G},T)\Big]. \tag{19}$$

To obtain a standard conservation equation,

$$\dot{\rho} + 3H(\rho+P) = 0, \tag{20}$$

we need an additional constraint by taking the right side of Eq. (19) equal to zero:

$$\Big(\dot{P} + \frac{1}{2}\dot{T}\Big)f_T(\mathcal{G},T) + (\rho+P)\partial_t f_T(\mathcal{G},T) = 0. \tag{21}$$

Now, we briefly discuss the motion of test particles in $f(\mathcal{G},T)$ gravity. For this purpose, using Eqs. (11) and (12) in (8), the divergence of the energy-momentum tensor for perfect fluid is given by

$$\begin{aligned}&\nabla_{\beta}(\rho+P)V^{\alpha}V^{\beta} + (\rho+P)[V^{\beta}\nabla_{\beta}V^{\alpha} + V^{\alpha}\nabla_{\beta}V^{\beta}] - g^{\alpha\beta}\nabla_{\beta}P\\ &= \frac{-2}{2\kappa^2 + 3f_T(\mathcal{G},T)}\big[T^{\alpha\beta}\nabla_{\beta}f_T(\mathcal{G},T) + g^{\alpha\beta}\nabla_{\beta}(Pf_T(\mathcal{G},T))\big].\end{aligned}$$

The contraction of the above equation with the projection operator ($h_{\alpha\xi} = g_{\alpha\xi} - V_{\alpha\xi}$) gives the following expression:

$$g_{\alpha\xi}V^{\beta}\nabla_{\beta}V^{\alpha} = \frac{(2\kappa^2 + f_T(\mathcal{G},T))\nabla_{\beta}}{(\rho+P)(2\kappa^2 + 3f_T(\mathcal{G},T))}h^{\beta}_{\xi}, \tag{22}$$

where we have used the relations $V^{\alpha}\nabla_{\beta}V_{\alpha} = 0$, $h_{\alpha\xi}V^{\alpha} = 0$, and $h_{\alpha\xi}T^{\alpha\beta} = -Ph^{\beta}_{\xi}$. Multiplying Eq. (22) with $g^{\mu\xi}$ and using the following identity [31]:

$$V^{\beta}\nabla_{\beta}V^{\alpha} = \frac{d^2x^{\alpha}}{ds^2} + \Gamma^{\alpha}_{\beta\xi}V^{\beta}V^{\xi},$$

we obtain the equation of motion for massive test particles in this model of gravity as

$$\frac{d^2x^\alpha}{ds^2}+\Gamma^\alpha_{\beta\xi}V^\beta V^\xi=\zeta^\alpha, \tag{23}$$

where

$$\zeta^\alpha=\frac{(2\kappa^2+f_T(\mathcal{G},T))}{(\rho+P)(2\kappa^2+3f_T(\mathcal{G},T))}(g^{\alpha\beta}-V^\alpha V^\beta)\nabla_\beta P \tag{24}$$

represents the extra force acting on the test particles and is perpendicular to the four velocity of the fluid ($\zeta^\alpha V_\alpha=0$). For a pressureless fluid, Eq. (24) gives $\zeta^\alpha=0$ and hence the dust particles follow the geodesic trajectories both in general relativity as well as in $f(\mathcal{G},T)$ gravity. The equation of motion for a perfect fluid in general relativity is recovered in the absence of coupling between matter and geometry [32].

3 Energy conditions

The energy conditions are the coordinate invariant which incorporate the common characteristics shared by almost every matter field. The concept of energy conditions came from the Raychaudhuri equations which play a key role in any discussion of the congruence of null and timelike geodesics with the requirement that not only the gravity is attractive but also the energy density is positive. These equations describe the temporal evolution of the expansion scalar (θ) as follows [33]:

$$\frac{d\theta}{d\tau}=-\frac{1}{3}\theta^2+\omega_{\alpha\beta}\omega^{\alpha\beta}-\sigma_{\alpha\beta}\sigma^{\alpha\beta}-R_{\alpha\beta}u^\alpha u^\beta, \tag{25}$$

$$\frac{d\theta}{d\tau}=-\frac{1}{2}\theta^2+\omega_{\alpha\beta}\omega^{\alpha\beta}-\sigma_{\alpha\beta}\sigma^{\alpha\beta}-R_{\alpha\beta}k^\alpha k^\beta, \tag{26}$$

where $\omega_{\alpha\beta}$, $\sigma_{\alpha\beta}$, u^α and k^α represent the rotation, shear tensor, timelike, and null tangent vectors in the congruences, respectively. For non-geodesic congruences, the temporal evolution of θ is affected by the presence of an acceleration term which arises due to a non-gravitational force like pressure gradient as [34,35]

$$\frac{d\theta}{d\tau}=-\frac{1}{3}\theta^2+\omega_{\alpha\beta}\omega^{\alpha\beta}-\sigma_{\alpha\beta}\sigma^{\alpha\beta}+\nabla_\alpha(V^\beta\nabla_\beta V^\alpha)-R_{\alpha\beta}V^\alpha V^\beta. \tag{27}$$

Neglecting the quadratic terms due to rotation-free as well as small distortions described by $\sigma_{\alpha\beta}$, Eqs. (25) and (26) yield

$$\theta=-\tau R_{\alpha\beta}u^\alpha u^\beta,\quad \theta=-\tau R_{\alpha\beta}k^\alpha k^\beta.$$

Using the condition for gravity to be attractive, i.e., $\theta<0$, we obtain $R_{\alpha\beta}u^\alpha u^\beta\geq0$ and $R_{\alpha\beta}k^\alpha k^\beta\geq0$. The equivalent form of these inequalities can be obtained by the inversion of the Einstein field equations as

$$\left(T_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}T\right)u^\alpha u^\beta\geq0,\quad \left(T_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}T\right)k^\alpha k^\beta\geq0.$$

For a perfect fluid matter distribution, these inequalities provide the energy constraints defined by:

- NEC: $\rho+P\geq0$,
- WEC: $\rho+P\geq0,\quad \rho\geq0$,
- SEC: $\rho+P\geq0,\quad \rho+3P\geq0$,
- DEC: $\rho\pm P\geq0,\quad \rho\geq0$.

These conditions show that the violation of the NEC leads to the violation of all other conditions. Due to the purely geometric nature of the Raychaudhuri equations, the concept of energy bounds in modified theories of gravity can be extended with the assumption that the total cosmic matter distribution acts like a perfect fluid. The energy conditions can be formulated by replacing ρ and P with ρ_{eff} and P_{eff}, respectively. The geodesic lines of geometry are followed by dust particles in $f(\mathcal{G},T)$ gravity, therefore we consider a pressureless fluid to discuss the energy conditions. These conditions take the following form:

$$\textbf{NEC}:\ \rho_{\text{eff}}+P_{\text{eff}}=\rho+\frac{1}{\kappa^2}\Big[\rho f_T(\mathcal{G},T)+4H(H^2-2\dot{H})\partial_t f_{\mathcal{G}}(\mathcal{G},T)-4H^2\partial_{tt}f_{\mathcal{G}}(\mathcal{G},T)\Big]\geq0, \tag{28}$$

$$\textbf{WEC}:\ \rho_{\text{eff}}=\rho+\frac{1}{2\kappa^2}\Big[2\rho f_T(\mathcal{G},T)+f(\mathcal{G},T)-24H^2(H^2+\dot{H})f_{\mathcal{G}}(\mathcal{G},T)+24H^3\partial_t f_{\mathcal{G}}(\mathcal{G},T)\Big]\geq0, \tag{29}$$

$$\textbf{SEC}:\ \rho_{\text{eff}}+3P_{\text{eff}}=\rho-\frac{1}{\kappa^2}\times\Big[f(\mathcal{G},T)-\rho f_T(\mathcal{G},T)-24H^2(H^2+\dot{H})\times f_{\mathcal{G}}(\mathcal{G},T)+12H(H^2+2\dot{H})\partial_t f_{\mathcal{G}}(\mathcal{G},T)+12H^2\partial_{tt}f_{\mathcal{G}}(\mathcal{G},T)\Big]\geq0, \tag{30}$$

$$\textbf{DEC}:\ \rho_{\text{eff}}-P_{\text{eff}}=\rho+\frac{1}{\kappa^2}\Big[\rho f_T(\mathcal{G},T)+f(\mathcal{G},T)-24H^2(H^2+\dot{H})\times f_{\mathcal{G}}(\mathcal{G},T)+4H(5H^2+2\dot{H})\partial_t f_{\mathcal{G}}(\mathcal{G},T)+4H^2\partial_{tt}f_{\mathcal{G}}(\mathcal{G},T)\Big]\geq0. \tag{31}$$

The Hubble parameter, the Ricci scalar, the GB invariant, and their derivatives can be written in terms of cosmic parameters as

$$\begin{aligned}&\dot{H}=-H^2(1+q),\quad \ddot{H}=H^3(j+3q+2),\\&\dddot{H}=H^4(s-4j-3q^2-12q-6),\\&R=-6H^2(1-q),\quad \dot{R}=-6H^3(j-q-2),\end{aligned} \tag{32}$$

$$\ddot{R} = -6H^4(s + 8q + q^2 + 6), \tag{33}$$
$$\mathcal{G} = -24qH^4, \quad \dot{\mathcal{G}} = 24H^5(j + 3q + 2q^2),$$
$$\ddot{\mathcal{G}} = 24H^6(s - 6j - 6qj - 12q - 15q^2 - 2q^3), \tag{34}$$

where q, j, and s denote the deceleration, jerk, and snap parameters, respectively, and are defined as [36,37]

$$q = -\frac{1}{H^2}\frac{\ddot{a}}{a}, \quad j = \frac{1}{H^3}\frac{\dddot{a}}{a}, \quad s = \frac{1}{H^4}\frac{\ddddot{a}}{a}. \tag{35}$$

The energy conditions (28)–(31) in the form of the above parameters are

$$\textbf{NEC}: \quad \rho_{\text{eff}} + P_{\text{eff}} = \rho + \frac{1}{\kappa^2}\Big[\rho f_T + 4H^3 \times(3+2q)(f_{\mathcal{G}\mathcal{G}}\dot{\mathcal{G}} + f_{\mathcal{G}T}\dot{T}) - 4H^2(f_{\mathcal{G}\mathcal{G}\mathcal{G}}\dot{\mathcal{G}}^2 + 2f_{\mathcal{G}\mathcal{G}T}\dot{\mathcal{G}}\dot{T} + f_{\mathcal{G}TT}\dot{T}^2 + f_{\mathcal{G}\mathcal{G}}\ddot{\mathcal{G}} + f_{\mathcal{G}T}\ddot{T})\Big] \geq 0, \tag{36}$$

$$\textbf{WEC}: \quad \rho_{\text{eff}} = \rho + \frac{1}{2\kappa^2}\Big[f + 2\rho f_T + 24qH^4 f_{\mathcal{G}} + 24H^3(f_{\mathcal{G}\mathcal{G}}\dot{\mathcal{G}} + f_{\mathcal{G}T}\dot{T})\Big] \geq 0, \tag{37}$$

$$\textbf{SEC}: \quad \rho_{\text{eff}} + 3P_{\text{eff}} = \rho + \frac{1}{\kappa^2}\Big[-f + \rho f_T - 24qH^4 f_{\mathcal{G}} + 12H^3(1+2q) \times(f_{\mathcal{G}\mathcal{G}}\dot{\mathcal{G}} + f_{\mathcal{G}T}\dot{T}) - 12H^2(f_{\mathcal{G}\mathcal{G}\mathcal{G}}\dot{\mathcal{G}}^2 + 2f_{\mathcal{G}\mathcal{G}T}\dot{\mathcal{G}}\dot{T} + f_{\mathcal{G}TT}\dot{T}^2 + f_{\mathcal{G}\mathcal{G}}\ddot{\mathcal{G}} + f_{\mathcal{G}T}\ddot{T})\Big] \geq 0, \tag{38}$$

$$\textbf{DEC}: \quad \rho_{\text{eff}} - P_{\text{eff}} = \rho + \frac{1}{\kappa^2}[f + \rho f_T + 24qH^4 f_{\mathcal{G}} + 4H^3(3-2q)(f_{\mathcal{G}\mathcal{G}}\dot{\mathcal{G}} + f_{\mathcal{G}T}\dot{T}) + 4H^2(f_{\mathcal{G}\mathcal{G}\mathcal{G}}\dot{\mathcal{G}}^2 + 2f_{\mathcal{G}\mathcal{G}T}\dot{\mathcal{G}}\dot{T} + f_{\mathcal{G}TT}\dot{T}^2 + f_{\mathcal{G}\mathcal{G}}\ddot{\mathcal{G}} + f_{\mathcal{G}T}\ddot{T})\Big] \geq 0. \tag{39}$$

4 Reconstruction of $f(\mathcal{G}, T)$ models

In this section, we use the reconstruction technique and discuss the energy conditions for de Sitter and power-law universe models.

4.1 de Sitter universe model

This cosmological model explains the exponential expansion of the universe with constant Hubble expansion rate. The scale factor is defined as [38]

$$a(t) = a_0 e^{H_0 t}, \quad H = H_0, \tag{40}$$

where a_0 is constant at t_0. The values of R and the GB invariant are

$$R = -12H_0^2, \quad \mathcal{G} = 24H_0^4. \tag{41}$$

For pressureless fluid, Eq. (20) gives the energy density of the form

$$\rho = \rho_0 e^{-3H_0 t}. \tag{42}$$

The trace of the energy-momentum tensor and its derivatives have the following expressions:

$$T = \rho, \quad \dot{T} = -3H_0 T, \quad \ddot{T} = 9H_0^2 T. \tag{43}$$

Using Eqs. (40)–(43) in Eq. (16), we obtain a partial differential equation

$$\kappa^2 T + \frac{1}{2}f(\mathcal{G}, T) - 12H_0^4 f_{\mathcal{G}}(\mathcal{G}, T) + T f_T(\mathcal{G}, T) - 36H_0^4 T f_{\mathcal{G}T}(\mathcal{G}, T) - 3H_0^2 = 0, \tag{44}$$

whose solution is given by

$$f(\mathcal{G}, T) = c_1 c_2\big(e^{c_1\mathcal{G}}T^{\gamma_1} + T^{\gamma_2}\big) + \gamma_3 T + \gamma_4, \tag{45}$$

where the c_i are integration constants and

$$\gamma_1 = -\frac{1}{2}\left(\frac{1 - 24c_1H_0^4}{1 - 36c_1H_0^4}\right), \quad \gamma_2 = -\frac{1}{2},$$
$$\gamma_3 = -\frac{2}{3}\kappa^2, \quad \gamma_4 = 6H_0^2.$$

The additional constraint (21) becomes

$$c_1c_2\frac{(1 - 24c_1H_0^4)(1 - 30c_1H_0^4)}{(1 - 36c_1H_0^4)^2}e^{c_1\mathcal{G}}T^{\gamma_1} + c_1c_2T^{\gamma_2} + \gamma_3 T = 0.$$

This equation splits Eq. (45) into two $f(\mathcal{G}, T)$ functions with some additional constant relations between the coefficients. The reconstructed model (45) can be written as a combination of those functions. We analyze the energy conditions for the $f(\mathcal{G}, T)$ model given in Eq. (45) instead of analyzing them separately. Using model (45) in the energy conditions (28)–(31), it follows that

$$\textbf{NEC}: \quad \rho_{\text{eff}} + P_{\text{eff}} = \rho + \frac{1}{\kappa^2}\big[\rho\{c_1c_2(\gamma_1 e^{c_1\mathcal{G}}T^{(\gamma_1-1)} + \gamma_2 T^{(\gamma_2-1)}) + \gamma_3\} + 12c_1^2c_2\gamma_1H_0^4(1 - 3\gamma_1)e^{c_1\mathcal{G}}T^{\gamma_1}\big] \geq 0, \tag{46}$$

$$\textbf{WEC}: \quad \rho_{\text{eff}} = \rho + \frac{1}{2\kappa^2}\big[2\rho\{c_1c_2 \times(e^{c_1\mathcal{G}}\gamma_1T^{(\gamma_1-1)} + \gamma_2T^{(\gamma_2-1)}) + \gamma_3\} + \{c_1c_2(e^{c_1\mathcal{G}}T^{\gamma_1} + T^{\gamma_2}) + \gamma_3T + \gamma_4\} - 24c_1^2c_2H_0^4e^{c_1\mathcal{G}}T^{\gamma_1}(1 + 3\gamma_1)\big] \geq 0, \tag{47}$$

$$\textbf{SEC}: \quad \rho_{\text{eff}} + 3P_{\text{eff}} = \rho - \frac{1}{\kappa^2}\big[c_1c_2(e^{c_1\mathcal{G}}T^{\gamma_1} + T^{\gamma_2}) + \gamma_3T + \gamma_4 - \rho \times\big\{c_1c_2\big(\gamma_1e^{c_1\mathcal{G}}T^{(\gamma_1-1)} + \gamma_2T^{(\gamma_2-1)}\big) + \gamma_3\big\} - 12c_1^2c_2e^{c_1\mathcal{G}}H_0^4T^{\gamma_1} \times\{2 + 3\gamma_1 - 9\gamma_1^2\}\big] \geq 0, \tag{48}$$

DEC: $$\rho_{\rm eff} - P_{\rm eff} = \rho + \frac{1}{\kappa^2}[\rho\{c_1c_2(e^{c_1\mathcal{G}}\gamma_1 T^{(\gamma_1-1)} + \gamma_2 T^{(\gamma_2-1)}) + \gamma_3\} + \{c_1c_2(e^{c_1\mathcal{G}}T^{\gamma_1} + T^{\gamma_2}) + \gamma_3 T + \gamma_4\} - 12c_1^2c_2H_0^4e^{c_1\mathcal{G}}T^{\gamma_1} \times \{2+\gamma_1(5-3\gamma_1)\}] \geq 0. \quad (49)$$

Figures 1 and 2 show the variation of the NEC and WEC for the case $c_1 > 0$ and $c_2 < 0$ with $\kappa = 1$. We use the following values of the cosmological parameters: $H_0 = 0.718$, $q = -0.64$, $j = 1.02$ and $s = -0.39$ [39–41]. In these plots, we fix the constant c_1 for two arbitrarily chosen values, while c_2 varies from $[-10, 0]$. Figure 1 shows the positively increasing behavior of the NEC as well as WEC with respect to time in the considered interval of c_2. Figure 2 shows a similar behavior for $c_1 = 4$. In this case, both conditions are satisfied for all values of c_1 and c_2. The energy conditions for $(c_1, c_2) > 0$ are discussed in Figs. 3 and 4. The left plot of Fig. 3 shows that the NEC is satisfied for $t < 3$, $t < 2.28$ and $t = 2$ for $c_2 = 0.005$, 0.05 and 0.1, respectively. Figure 4 (left) shows a similar decreasing behavior of time as the value of c_2 increases for $c_1 = 0.01$. It is also observed that as the value of c_1 increases, the time interval for a valid NEC decreases, while the positivity of $\rho_{\rm eff}$ is shown in the right panel of both figures. For the case $(c_1, c_2) > 0$, both NEC and WEC are satisfied for small values of c_1 and c_2 in a very small time interval.

Figures 5 and 6 deal with the case $c_1 < 0$ and $c_2 > 0$. For arbitrarily chosen values of c_1, the increasing behavior of the NEC with respect to time is observed in the left panel of both figures for all values of c_2. The right plot of Fig. 5 shows the positivity of $\rho_{\rm eff}$ for $t < 34$, while it remains positive throughout the time interval for $c_1 = -0.001$ as shown in Fig. 6 (right panel). The last possibility, i.e., $c_1 < 0$ and $c_2 < 0$ is examined in Figs. 7 and 8. The left panels of both figures show the decreasing and increasing behavior of the NEC as the time and integration constant c_2 increase, respectively. The effective energy density exhibits a constant behavior for the assumed values of c_1 in the considered interval of c_2.

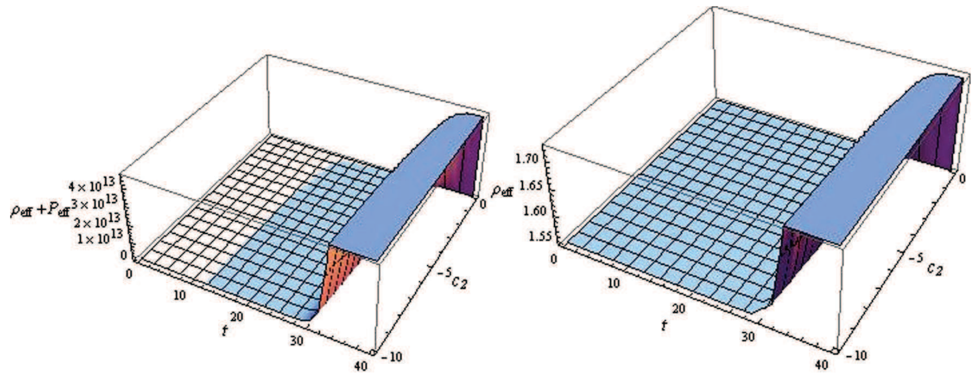

Fig. 1 Energy conditions for $c_1 = 0.001$

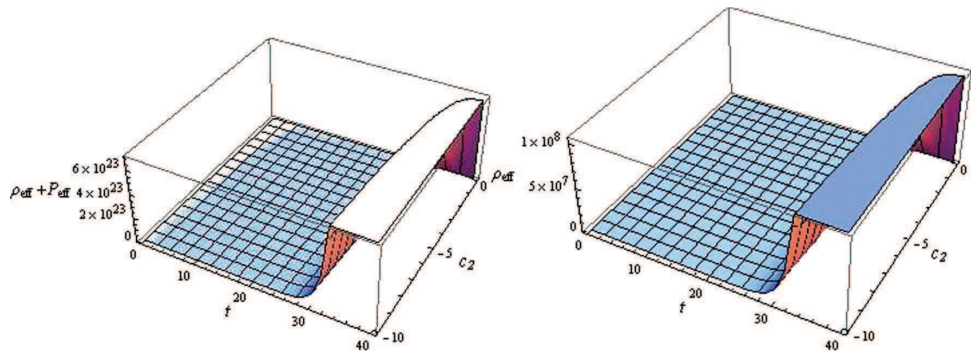

Fig. 2 Energy conditions for $c_1 = 4$

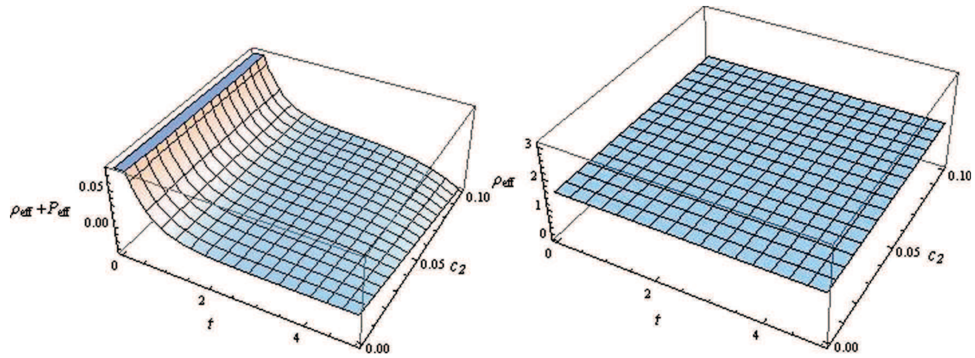

Fig. 3 Energy conditions for $c_1 = 0.001$

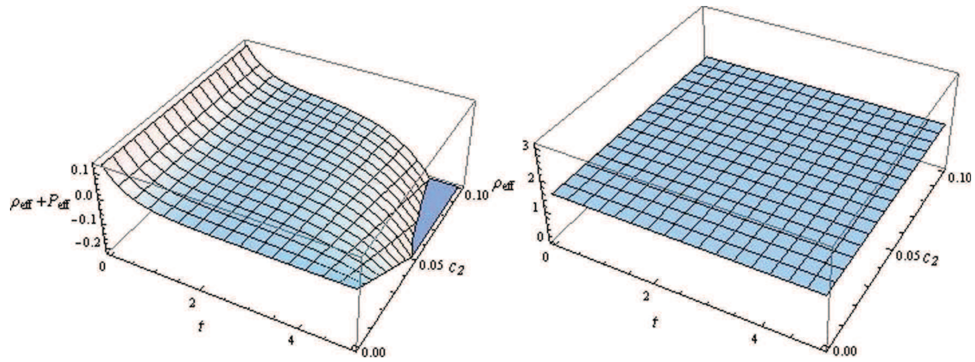

Fig. 4 Energy conditions for $c_1 = 0.01$

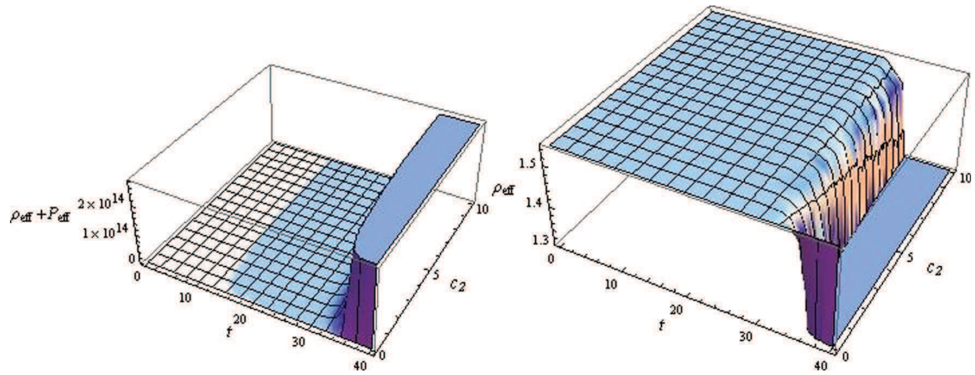

Fig. 5 Energy conditions for $c_1 = -0.01$

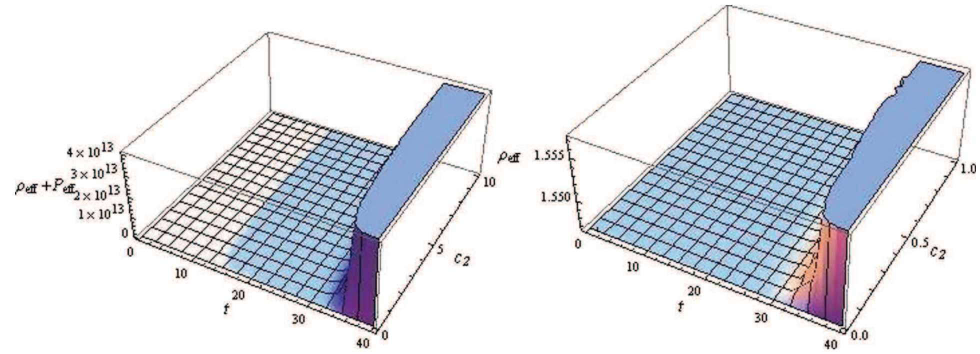

Fig. 6 Energy conditions for $c_1 = -0.001$

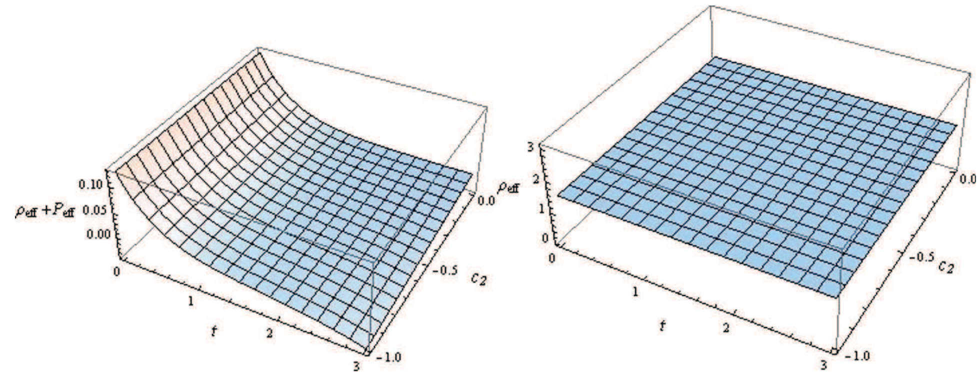

Fig. 7 Energy conditions for $c_1 = -0.001$

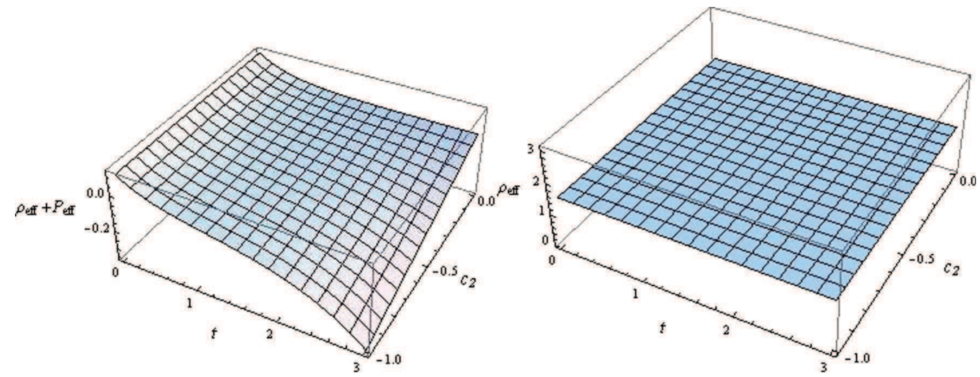

Fig. 8 Energy conditions for $c_1 = -0.01$

4.2 Power-law solution

The power-law solution is of great interest to discuss the cosmic evolution and its scale factor is defined as [38]

$$a(t) = a_0 t^n, \quad H = \frac{n}{t}, \tag{50}$$

where $n > 0$. For $0 < n < 1$, we have a decelerated universe, which leads to a radiation dominated era for $n = \frac{1}{2}$ and a dust dominated era for $n = \frac{2}{3}$, while a cosmic accelerated era is observed for $n > 1$. The Ricci scalar and GB invariant are

$$R = \frac{6n}{t^2}(1 - 2n), \quad \mathcal{G} = \frac{24n^3}{t^4}(n - 1). \tag{51}$$

The energy density for dust fluid is obtained from Eq. (20) as

$$\rho=\rho_0 t^{-3n}. \tag{52}$$

The trace of $T_{\alpha\beta}$ and its time derivatives take the form

$$T=\rho,\quad \dot{T}=-\frac{3n}{t}T,\quad \ddot{T}=\frac{3n}{t^2}(1+3n)T. \tag{53}$$

Inserting Eqs. (50)–(53) in the first field equation (16), we obtain

$$\begin{aligned}&\kappa^2T+\frac{1}{2}f(\mathcal{G},T)-\frac{1}{2}\mathcal{G}f_{\mathcal{G}}(\mathcal{G},T)+Tf_T(\mathcal{G},T)\\&-\left(\frac{2}{n-1}\right)\mathcal{G}^2f_{\mathcal{G}\mathcal{G}}(\mathcal{G},T)\\&-\left(\frac{3n}{2(n-1)}\right)\mathcal{G}Tf_{\mathcal{G}T}(\mathcal{G},T)-3n^2\left(\frac{T}{\rho_0}\right)^{\frac{2}{3n}}=0,\end{aligned} \tag{54}$$

whose solution is given by

$$\begin{aligned}f(\mathcal{G},T)=&\,d_1d_3T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)}+d_2d_3T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)}+\chi_3T\\&+d_1d_2T^{\chi_4}+\chi_5T^{\chi_6},\end{aligned} \tag{55}$$

where d_i are constants of integration and

$$\begin{aligned}\chi_1&=\frac{1}{2}\left[n^2(1+3d_2(3d_2+2))+2d_2(n-16)+3(2n+3)\right]^{\frac{1}{2}},\\ \chi_2&=\frac{1}{2}\left[5-n(1+3d_2)\right],\quad \chi_3=-\frac{2}{3}\kappa^2,\quad \chi_4=-\frac{1}{2},\\ \chi_5&=\left(\frac{18n^3}{2+3n}\right)\rho_0^{-\frac{2}{3n}},\quad \chi_6=\frac{2}{3n}.\end{aligned}$$

In this case, Eq. (21) takes the form

$$\begin{aligned}&d_1d_3T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)}\left[\frac{d_2}{6n}\{3n(2d_2-1)+2(\chi_1+\chi_2)\}\right]\\&+d_2d_3T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)}\\&\times\left[\frac{d_2}{6n}\{3n(2d_2-1)-2(\chi_1-\chi_2)\}\right]\\&+\chi_3T+d_1d_2\chi_4^2T^{\chi_4}+\chi_5\chi_6^2T^{\chi_6}=0.\end{aligned}$$

Solving Eq. (55) with the above equation as in the previous section, we obtain two functions whose combination is equivalent to the reconstructed power-law $f(\mathcal{G},T)$ model.

Inserting the model (55) in the energy conditions (36)–(39), we obtain

$$\begin{aligned}\textbf{NEC}:\ &\rho_{\text{eff}}+P_{\text{eff}}=\rho+\frac{1}{\kappa^2}\Bigg[4H^3(3+2q)\Bigg(\Bigg[\frac{1}{4}d_1d_3\\&\times(\chi_1+\chi_2)\left[\frac{1}{4}(\chi_1+\chi_2)-1\right]\\&\times T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2}+\frac{1}{4}d_2d_3(\chi_1-\chi_2)\\&\times\left[\frac{1}{4}(\chi_1-\chi_2)+1\right]T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}\Bigg]\dot{\mathcal{G}}\\&+\Bigg[\frac{1}{4}d_1d_2d_3(\chi_1+\chi_2)T^{d_2-1}\\&\times\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1}-\frac{1}{4}d_2^2d_3(\chi_1-\chi_2)T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Bigg]\dot{T}\Bigg)\\&-4H^2\Bigg(\Bigg[\frac{1}{4}d_1d_3(\chi_1+\chi_2)\\&\times\left[\frac{1}{4}(\chi_1+\chi_2)-1\right]\left[\frac{1}{4}(\chi_1+\chi_2)-2\right]T^{d_2}\\&\times\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-3}-\frac{1}{4}d_2d_3(\chi_1-\chi_2)\left[\frac{1}{4}(\chi_1-\chi_2)+1\right]\\&\times\left[\frac{1}{4}(\chi_1-\chi_2)+2\right]\\&\times T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-3}\Bigg]\dot{\mathcal{G}}^2+2\Bigg[\frac{1}{4}d_1d_2d_3(\chi_1+\chi_2)\\&\times\left[\frac{1}{4}(\chi_1+\chi_2)-1\right]T^{d_2-1}\\&\times\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2}+\frac{1}{4}d_2^2d_3(\chi_1-\chi_2)\left[\frac{1}{4}(\chi_1-\chi_2)+1\right]\\&\times T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}\Bigg]\\&\times\dot{\mathcal{G}}\dot{T}+\Bigg[\frac{1}{4}d_1d_2d_3(d_2-1)(\chi_1+\chi_2)T^{d_2-2}\\&\times\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1}-\frac{1}{4}d_2^2d_3(d_2-1)\\&\times(\chi_1-\chi_2)T^{d_2-2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Bigg]\dot{T}^2+\Bigg[\frac{1}{4}d_1d_3(\chi_1+\chi_2)\\&\times\left[\frac{1}{4}(\chi_1+\chi_2)-1\right]\\&\times T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2}+\frac{1}{4}d_2d_3(\chi_1-\chi_2)\\&\times\left[\frac{1}{4}(\chi_1-\chi_2)+1\right]T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}\Bigg]\\&\times\ddot{\mathcal{G}}+\Bigg[\frac{1}{4}d_1d_2d_3(\chi_1+\chi_2)T^{d_2-1}\\&\times\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1}-\frac{1}{4}d_2^2d_3(\chi_1-\chi_2)T^{d_2-1}\\&\times\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Bigg]\ddot{T}\Bigg)+\rho\big[d_1d_2d_3T^{d_2-1}\end{aligned}$$

$$\times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)} + d_2^2 d_3 T^{d_2-1} \times \mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)} - \chi_3 + d_1 d_2 \chi_4 T^{\chi_4-1} + \chi_5\chi_6 T^{\chi_6-1}\Big]\Big] \geq 0, \quad (56)$$

WEC: $$\rho_{\text{eff}} = \rho + \frac{1}{2\kappa^2}\Big[d_1 d_3 T^{d_2} \times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)} + d_2 d_3 T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)} - \chi_3 T + d_1 d_2 T^{\chi_4} + \chi_5 T^{\chi_6} + 2\rho[d_1 d_2 d_3 T^{d_2-1} \times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)} + d_2^2 d_3 T^{d_2-1} \times \mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)} - \chi_3 + d_1 d_2 \chi_4 T^{\chi_4-1} + \chi_5\chi_6 T^{\chi_6-1}] + 24qH^4 \times \Big[\frac{1}{4}d_1 d_3(\chi_1+\chi_2) T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4}d_2 d_3(\chi_1-\chi_2) T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Big] + 24H^3\Big(\Big[\frac{1}{4}d_1 d_3(\chi_1+\chi_2) \times \Big[\frac{1}{4}(\chi_1+\chi_2)-1\Big] T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2} + \frac{1}{4}d_2 d_3 \times (\chi_1-\chi_2)\Big[\frac{1}{4}(\chi_1-\chi_2)+1\Big] \times T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}\Big]\dot{\mathcal{G}} + \Big[\frac{1}{4}d_1 d_2 d_3(\chi_1+\chi_2) \times T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4}d_2^2 d_3 \times (\chi_1-\chi_2) T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Big]\dot{T}\Big)\Big] \geq 0, \quad (57)$$

SEC: $$\rho_{\text{eff}} + 3P_{\text{eff}} = \rho + \frac{1}{\kappa^2}\Big[-[d_1 d_3 T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)} + d_2 d_3 T^{d_2} \times \mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)} - \chi_3 T + d_1 d_2 T^{\chi_4} + \chi_5 T^{\chi_6}] + \rho[d_1 d_2 d_3 T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)} + d_2^2 d_3 T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)} - \chi_3 + d_1 d_2 \chi_4 T^{\chi_4-1} + \chi_5\chi_6 T^{\chi_6-1}] - 24qH^4 \times \Big[\frac{1}{4}d_1 d_3(\chi_1+\chi_2) T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4}d_2 d_3(\chi_1-\chi_2) T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Big] + 12H^3(1+2q)\Big(\Big[\frac{1}{4}d_1 d_3(\chi_1+\chi_2) \times \Big[\frac{1}{4}(\chi_1+\chi_2)-1\Big] T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2} + \frac{1}{4}d_2 d_3(\chi_1-\chi_2)\Big[\frac{1}{4}(\chi_1-\chi_2)+1\Big] \times T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}\Big]\dot{\mathcal{G}} + \Big[\frac{1}{4}d_1 d_2 d_3 \times (\chi_1+\chi_2) T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4}d_2^2 d_3(\chi_1-\chi_2) \times T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Big]\dot{T}\Big) - 12H^2\Big(\Big[\frac{1}{4}d_1 d_3(\chi_1+\chi_2)\Big[\frac{1}{4}(\chi_1+\chi_2)-1\Big] \times \Big[\frac{1}{4}(\chi_1+\chi_2)-2\Big] T^{d_2} \times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-3} - \frac{1}{4}d_2 d_3(\chi_1-\chi_2)\Big[\frac{1}{4}(\chi_1-\chi_2)+1\Big] \times \Big[\frac{1}{4}(\chi_1-\chi_2)+2\Big] \times T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-3}\Big]\dot{\mathcal{G}}^2 + 2\Big[\frac{1}{4}d_1 d_2 d_3(\chi_1+\chi_2) \times \Big[\frac{1}{4}(\chi_1+\chi_2)-1\Big] T^{d_2-1} \times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2} + \frac{1}{4}d_2^2 d_3(\chi_1-\chi_2)\Big[\frac{1}{4}(\chi_1-\chi_2)+1\Big] \times T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}\Big] \times \dot{\mathcal{G}}\dot{T} + \Big[\frac{1}{4}d_1 d_2 d_3(d_2-1)(\chi_1+\chi_2) T^{d_2-2} \times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4}d_2^2 d_3(d_2-1) \times (\chi_1-\chi_2) T^{d_2-2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Big]\dot{T}^2 + \Big[\frac{1}{4}d_1 d_3(\chi_1+\chi_2) \times \Big[\frac{1}{4}(\chi_1+\chi_2)-1\Big] T^{d_2} \times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2} + \frac{1}{4}d_2 d_3(\chi_1-\chi_2)\Big[\frac{1}{4}(\chi_1-\chi_2)+1\Big] T^{d_2} \times \mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}\Big]\ddot{\mathcal{G}} + \Big[\frac{1}{4}d_1 d_2 d_3(\chi_1+\chi_2) \times T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4}d_2^2 d_3 \times (\chi_1-\chi_2) T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Big]\ddot{T}\Big)\Big] \geq 0, \quad (58)$$

DEC: $$\rho_{\text{eff}} - P_{\text{eff}} = \rho + \frac{1}{\kappa^2}\Big[[d_1 d_3 T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)} + d_2 d_3 T^{d_2} \times \mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)} - \chi_3 T + d_1 d_2 T^{\chi_4} + \chi_5 T^{\chi_6}] + \rho[d_1 d_2 d_3 T^{d_2-1} \times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)} + d_2^2 d_3 T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)} - \chi_3 + d_1 d_2 \chi_4 T^{\chi_4-1} + \chi_5\chi_6 T^{\chi_6-1}\Big] + 24qH^4\Big[\frac{1}{4}d_1 d_3(\chi_1+\chi_2) T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4}d_2 d_3(\chi_1-\chi_2) T^{d_2}$$

$$\times\, \mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Big] + 4H^3(3-2q)\bigg(\bigg[\frac{1}{4}d_1d_3(\chi_1+\chi_2)$$
$$\times\Big[\frac{1}{4}(\chi_1+\chi_2)-1\Big]$$
$$\times\, T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2} + \frac{1}{4}d_2d_3(\chi_1-\chi_2)\Big[\frac{1}{4}(\chi_1-\chi_2)+1\Big]$$
$$\times\, T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}\bigg]$$
$$\times\, \dot{\mathcal{G}} + \bigg[\frac{1}{4}d_1d_2d_3(\chi_1+\chi_2)T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1}$$
$$-\frac{1}{4}d_2^2d_3(\chi_1-\chi_2)T^{d_2-1}$$
$$\times\, \mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\bigg]\dot{T}\bigg) + 4H^2\bigg(\bigg[\frac{1}{4}d_1d_3(\chi_1+\chi_2)$$
$$\times\Big[\frac{1}{4}(\chi_1+\chi_2)-1\Big]$$
$$\times\Big[\frac{1}{4}(\chi_1+\chi_2)-2\Big]T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-3} - \frac{1}{4}d_2d_3(\chi_1-\chi_2)$$
$$\times\Big[\frac{1}{4}(\chi_1-\chi_2)+1\Big]\Big[\frac{1}{4}(\chi_1-\chi_2)+2\Big]T^{d_2}$$
$$\times\, \mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-3}\bigg]\dot{\mathcal{G}}^2 + 2\bigg[\frac{1}{4}d_1d_2d_3(\chi_1+\chi_2)$$
$$\times\Big[\frac{1}{4}(\chi_1+\chi_2)-1\Big]T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2} + \frac{1}{4}d_2^2d_3(\chi_1-\chi_2)$$
$$\times\Big[\frac{1}{4}(\chi_1-\chi_2)+1\Big]T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}\bigg]$$
$$\times\, \dot{\mathcal{G}}\dot{T} + \bigg[\frac{1}{4}d_1d_2d_3(d_2-1)(\chi_1+\chi_2)$$
$$\times\, T^{d_2-2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4}d_2^2d_3(d_2-1)(\chi_1-\chi_2)$$
$$\times\, T^{d_2-2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\bigg]\dot{T}^2$$
$$+\Big[\frac{1}{4}d_1d_3(\chi_1+\chi_2)\Big[\frac{1}{4}(\chi_1+\chi_2)-1\Big]T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2}$$
$$+\frac{1}{4}d_2d_3(\chi_1-\chi_2)$$

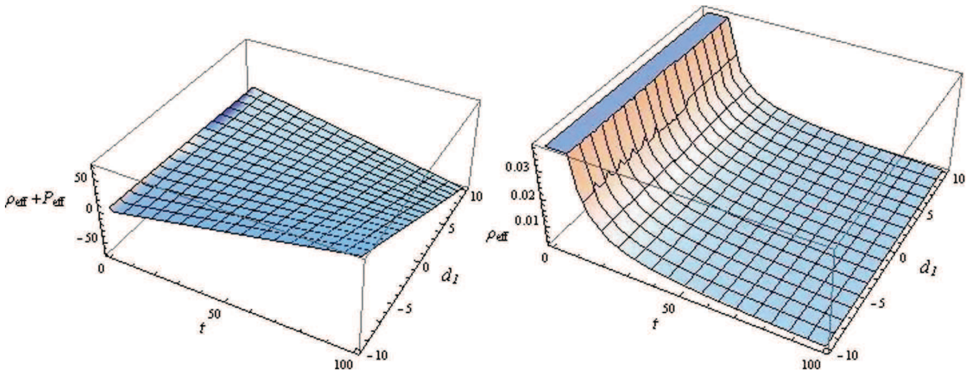

Fig. 9 Energy conditions for $d_2 = 0.1$ and $d_3 = 1$

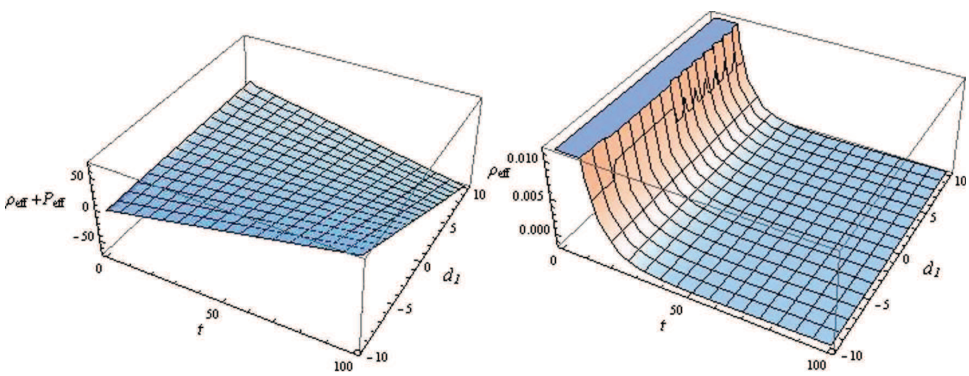

Fig. 10 Energy conditions for $d_2 = 0.1$ and $d_3 = -0.5$

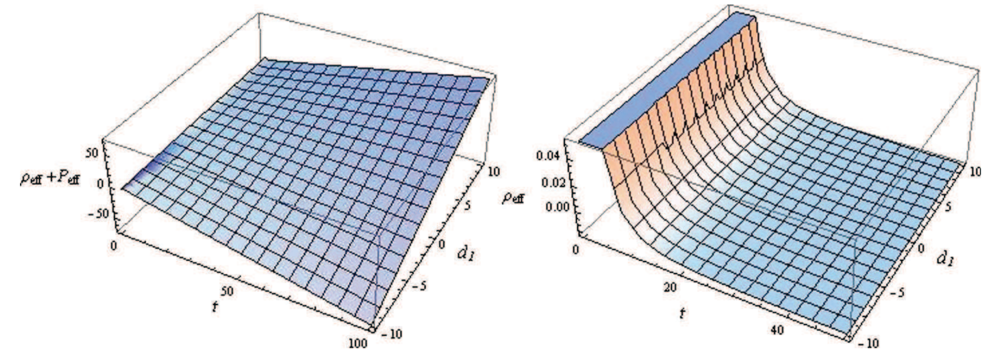

Fig. 11 Energy conditions for $d_2 = -0.1$ and $d_3 = 0.5$

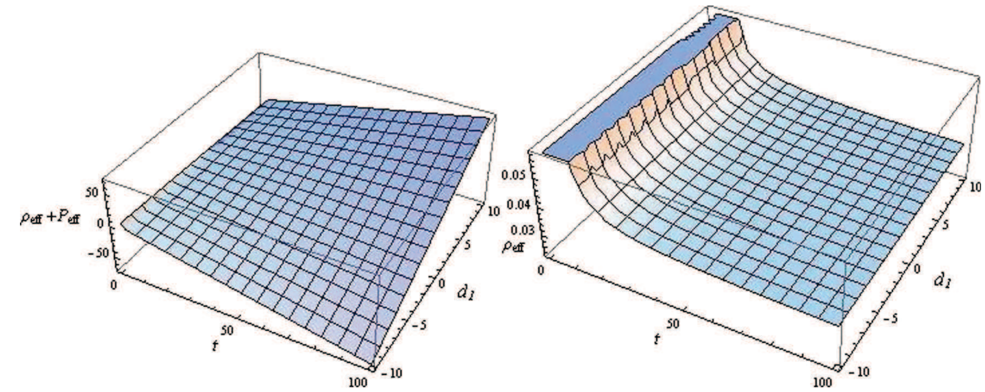

Fig. 12 Energy conditions for $d_2 = -0.1$ and $d_3 = -1$

$$\times \left[\frac{1}{4}(\chi_1 - \chi_2) + 1\right] T^{d_2} \mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}]\ddot{\mathcal{G}}$$
$$+ \left[\frac{1}{4} d_1 d_2 d_3 (\chi_1 + \chi_2) T^{d_2-1}\right.$$
$$\times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4} d_2^2 d_3$$
$$\left.\times (\chi_1 - \chi_2) T^{d_2-1} \mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\right] \ddot{T} \Big) \Big] \geq 0. \tag{59}$$

The NEC and WEC depend on four parameters $t,\ d_1,\ d_2$ and d_3. We plot these conditions against t and d_1 for $n = \frac{2}{3}$ with possible signs of d_2 and d_3. The left plot of Fig. 9 shows a positively increasing behavior of NEC for $-10 \leq d_1 \leq 0$ with respect to time while invalid for $d_1 > 0$. The effective energy density remains positive for all values of (t, d_1) as shown in Fig. 9 (right). The same behavior of both conditions are obtained for $0 < d_2 \leq 0.51$ with $d_3 > 0$ as well as for $d_2 > 0$ with $d_3 = 0$. The left plot of Fig. 10 shows a similar behavior of the NEC for $d_2 > 0$ and $d_3 < 0$, while ρ_{eff} remains positive for $0 < t < 23$. Similarly, for $d_3 = -1$ and -10, WEC is valid for $0 < t < 14$ and $0 < t < 4.5$, respectively, with $d_2 = 0.1$. The right plots of Figs. 11 and 12 show the validity of NEC for $d_1 \geq 0$, while it does not hold for negative values of d_1. The effective energy density remains positive for the time interval $1 \leq t \leq 10$ with $d_3 = 0.5$ as shown in Fig. 11 (right panel), while for $d_3 = 1$ and 10, the acceptable intervals are $1 \leq t \leq 7$ and $1 \leq t \leq 3$, respectively. This shows that the validity region of the WEC decreases as the value of integration constant d_3 increases. The right plot of Fig. 12 shows the positivity of ρ_{eff} for $(d_2, d_3) < 0$, which confirms the positivity of the WEC with $d_1 > 0$.

5 Final remarks

In this paper, we have presented a generalized modified theory of gravity with an arbitrary coupling between geometry and matter. The gravitational Lagrangian is obtained by adding an arbitrary function $f(\mathcal{G}, T)$ in the Einstein–Hilbert action. We have formulated the corresponding field equations using the least action principle and calculated the non-zero covariant divergence of $T_{\alpha\beta}$ consistent with $f(R, T)$ theory [31]. Consequently, the test particles follow non-geodesic trajectories due to the presence of an extra force originating from the non-minimal coupling, while they move along geodesics for a pressureless fluid. We have constructed the energy conditions for an FRW universe model filled with dust fluid in terms of the deceleration, jerk, and snap (q, j, s) cosmological parameters. The reconstruction technique has been applied to $f(\mathcal{G}, T)$ gravity using the well-known de Sitter and power-law universe models. The results are summarized as follows.

- In the de Sitter reconstructed model, the energy bounds have dependence on three parameters t, c_1 and c_2. We have plotted NEC and WEC against t and c_2 with four possible signatures of c_1 and c_2 as shown in Figs. 1, 2, 3, 4, 5, 6, 7, and 8. It is found that NEC and WEC are satisfied for $c_1 > 0$ and $c_2 < 0$ throughout the time interval for cases $(c_1, c_2) > 0$ and $(c_1, c_2) < 0$ that the energy conditions are satisfied for small values of the c_i in a very small time interval. It is observed that the NEC shows a positively increasing behavior for all negative values of c_1 with $c_2 > 0$, while the validity ranges of the WEC show dependence on c_1.
- For a power-law reconstructed model, we have explored the behavior of the four parameters t, d_1, d_2, and d_3 with $n = \frac{2}{3}$. In this case, we have plotted the energy conditions against (t, d_1) and analyzed the possible behavior of remaining constants. In Figs. 9, 10, 11, and 12, we have taken $-10 \leq d_1 \leq 10$ and found the valid regions where the energy conditions are satisfied.

Finally, we conclude that the NEC and WEC are satisfied in both reconstructed $f(\mathcal{G}, T)$ models with a suitable choice of the free parameters.

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Stability analysis of Einstein universe in $f(\mathcal{G}, T)$ gravity

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This paper explores the stability of the Einstein universe against linear homogeneous perturbations in the background of $f(\mathcal{G}, T)$ gravity. We construct static as well as perturbed field equations and investigate stability regions for the specific forms of generic function $f(\mathcal{G}, T)$ corresponding to conserved as well as nonconserved energy-momentum tensor. We use the equation-of-state parameter to parameterize the stability regions. The graphical analysis shows that the suitable choice of parameters lead to stable regions of the Einstein universe.

Keywords: Stability analysis; Einstein universe; $f(\mathcal{G}, T)$ gravity.

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1. Introduction

The current accelerated expansion of the universe is one of the most astonishing discovery in the golden era of cosmology. This has stimulated many researchers to explore the enigmatic nature of dark energy (DE) which is responsible for the phase of cosmic accelerated expansion. Dark energy possesses large negative pressure with repulsive nature but its many salient features are still not known. Modified theories of gravity are considered as the most favorable and optimistic approaches among other proposals to explore the nature of DE. These theories are established by replacing or adding curvature invariants and their corresponding generic functions in the geometric part of general relativity (GR).

The Einstein field equations are derived from the first Lovelock scalar dubbed as the Ricci scalar (R) in the Lagrangian density which corresponds to gravity while a particular form of quadratic curvature invariants yields second Lovelock scalar known as Gauss–Bonnet (GB) invariant. This invariant is a linear combination of the form $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, where $R_{\mu\nu}$ and $R_{\mu\nu\alpha\beta}$ represent the

Ricci and Riemann tensors, respectively. GB invariant is four-dimensional (4D) topological term which has the feature like it is free from spin-2 ghost instabilities.[1] There are two interesting approaches to discuss the dynamics of $\mathcal{G}$ in 4D either by coupling with scalar field or by adding the generic function $f(\mathcal{G})$ in the Einstein–Hilbert action. The first approach naturally appears in the effective low energy action in string theory which effectively discusses the singularity-free cosmological solutions.[2]

Nojiri and Odintsov[3] introduced second approach as an alternative for DE known as $f(\mathcal{G})$ gravity which elegantly studies the fascinating characteristics of late-time cosmology. Cognola *et al.*[4] investigated DE cosmology and found that this theory effectively describes the cosmological structure with a possibility to describe the transition from decelerated to accelerated cosmic phases. De Felice and Tsujikawa[5] constructed some cosmological viable $f(\mathcal{G})$ models and introduced a procedure to avoid numerical instabilities related with a large mass of the oscillating mode. The same authors[6] also found that the solar system constraints are consistent for a wide range of cosmological viable model parameters.

The captivating issue of cosmic accelerated expansion has successfully been discussed by taking into account modified theories of gravity with matter-curvature coupling. Harko *et al.*[7] presented $f(R,T)$ gravity (T is the trace of energy-momentum tensor (EMT)) to study the coupling between geometry and matter. Recently, we introduced another modified theory named as $f(\mathcal{G},T)$ gravity which is a generalization of $f(\mathcal{G})$ gravity.[8] This modification is based on the coupling of quadratic curvature invariant with matter just as $f(R,T)$ gravity. We studied the nonzero covariant divergence of EMT due to matter–curvature coupling and the massive test particles followed nongeodesic trajectories due to the presence of extra force while the dust particles moved along geodesic lines of geometry. In such matter–curvature coupled theories, cosmic expansion can result from geometric as well as matter component.

The stability issue of the Einstein universe (EU) is as old as relativistic cosmology. Einstein tried to find a static solution of his field equations to describe isotropic and homogeneous universe. Since the field equations of GR have no static solution, therefore Einstein introduced the term known as cosmological constant (Λ) to have static solutions. EU is described by static FRW universe model with positive curvature filled with perfect fluid in the presence of Λ. Initially, this model is considered as the most suitable model to discuss static universe but after few years it is found that EU is unstable against small isotropic and homogeneous perturbations.[9] Harrison[10] found that the unstable EU for dust distribution becomes oscillatory in the presence of radiations and also observed that stable EU exists against small inhomogeneous perturbations. Gibbons[11] proved that EU maximizes the entropy against conformal changes if and only if it is stable against speed of sound (c_s) greater than $\frac{1}{\sqrt{5}}$. Barrow *et al.*[12] demonstrated that EU is always neutrally stable in the presence of perfect fluid against small inhomogeneous vector as well as tensor perturbations and also under adiabatic scalar density inhomogeneities until the inequality $5c_s^2 > 1$ holds but unstable otherwise.

EU due to its analytical simplicity and fascinating stability properties has always been of great interest to study in the extensions of GR as well as in quantum gravity models. Emergent universe scenario is based on stable EU to resolve the problem of big-bang singularity which is not successful in GR since EU is unstable against homogeneous perturbations.[13] To find stable static solutions, modified theories have gained much attention to analyze the stability of EU. The stability of EU is studied in braneworld, Einstein–Cartan theory, loop quantum cosmology, nonminimal kinetic coupled gravity, etc.[14] Böhmer *et al.*[15] explored its stability using scalar homogeneous perturbations in $f(R)$ gravity and found that stable EU exists for specific forms of $f(R)$ in contrast to GR. Goswami and his collaborators[16] investigated the existence as well as stability of EU in the background of fourth-order gravity theories. Goheer *et al.*[17] studied the existence of EU for power-law $f(R)$ model and found stable solutions. Böhmer and Lobo[18] discussed the stability of EU in the context of $f(\mathcal{G})$ gravity against scalar homogeneous perturbations and found that stable regions exist for all values of the equation-of-state parameter (ω).

Böhmer[19] studied the stability of EU parameterized by the first and second derivatives of scalar potential for linear homogeneous as well as inhomogeneous perturbations in the context of hybrid metric-Palatini gravity and found a large class of stable solutions. Li *et al.*[20] found stable regions for both open as well as closed universe in modified teleparallel theory against linear homogeneous scalar perturbations. Huang *et al.*[21] obtained stable solutions for EU against homogeneous, inhomogeneous scalar, tensor and anisotropic perturbations in Jordan Brans–Dicke theory. The same authors[22] also found the unstable solutions against homogeneous as well as inhomogeneous scalar perturbations for open universe while stable EU is obtained for a closed universe against homogeneous perturbations in $f(\mathcal{G})$ gravity. Böhmer and his collaborators[23] analyzed stability regions against both homogeneous and inhomogeneous perturbations in scalar-fluid theories and found stable as well as unstable results against inhomogeneous and homogeneous perturbations, respectively. Darabi *et al.*[24] studied the existence and stability of EU in the context of Lyra geometry against scalar, vector as well as tensor perturbations for suitable values of physical parameters. Shabani and Ziaie[25] analyzed the existence as well as stability of EU in $f(R,T)$ gravity and found stable solutions which were unstable in $f(R)$ gravity.

In this paper, we study the stability of EU against scalar homogeneous perturbations in the background of $f(\mathcal{G},T)$ gravity. This analysis is helpful to examine the effects of matter–curvature coupling on the stability of EU. The paper has the following format. In Sec. 2, we construct the field equations of this theory while Sec. 3 is devoted to analyze the stability under linear homogeneous perturbations around EU for conserved as well as nonconserved EMT. The results are summarized in the last section.

2. Dynamics of $f(\mathcal{G},T)$ Gravity

The action for $f(\mathcal{G},T)$ gravity is given by[8]

$$\mathcal{S}=\int d^4x\sqrt{-g}\left[\frac{R+f(\mathcal{G},T)}{2\kappa^2}+\mathcal{L}_m\right], \tag{1}$$

where $T=g_{\mu\nu}T^{\mu\nu}$, κ^2, g and $\mathcal{L}_m$ represent coupling constant, determinant of the metric tensor ($g_{\mu\nu}$) and matter Lagrangian density, respectively. The EMT in terms of $\mathcal{L}_m$ is defined as[26]

$$T_{\mu\nu}=-\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}. \tag{2}$$

If $\mathcal{L}_m$ depends on the components of $g_{\mu\nu}$ but does not depend on its derivatives, then Eq. (2) yields

$$T_{\mu\nu}=g_{\mu\nu}\mathcal{L}_m-2\frac{\partial\mathcal{L}_m}{\partial g^{\mu\nu}}. \tag{3}$$

Varying the action (1) with respect to $g_{\mu\nu}$, we obtain the field equations as follows

$$\begin{aligned}R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\kappa^2T_{\mu\nu}&-(T_{\mu\nu}+\Theta_{\mu\nu})f_T(\mathcal{G},T)+\frac{1}{2}g_{\mu\nu}f(\mathcal{G},T)-(2RR_{\mu\nu}\\&-4R^{\alpha}_{\mu}R_{\alpha\nu}-4R_{\mu\alpha\nu\beta}R^{\alpha\beta}+2R^{\alpha\beta\gamma}_{\mu}R_{\nu\alpha\beta\gamma})f_{\mathcal{G}}(\mathcal{G},T)\\&-(2Rg_{\mu\nu}\Box-4R_{\mu\nu}\Box-2R\nabla_\mu\nabla_\nu+4R^{\alpha}_{\mu}\nabla_\nu\nabla_\alpha+4R^{\alpha}_{\nu}\nabla_\mu\nabla_\alpha\\&-4g_{\mu\nu}R^{\alpha\beta}\nabla_\alpha\nabla_\beta+4R_{\mu\alpha\nu\beta}\nabla^\alpha\nabla^\beta)f_{\mathcal{G}}(\mathcal{G},T),\end{aligned} \tag{4}$$

where $f_{\mathcal{G}}(\mathcal{G},T)=\partial f(\mathcal{G},T)/\partial\mathcal{G}$, $f_T(\mathcal{G},T)=\partial f(\mathcal{G},T)/\partial T$, $\Box=\nabla_\mu\nabla^\mu$ and ∇_μ is a covariant derivative whereas $\Theta_{\mu\nu}$ has the following expression

$$\Theta_{\mu\nu}=g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g_{\mu\nu}}=-2T_{\mu\nu}+g_{\mu\nu}\mathcal{L}_m-2g^{\alpha\beta}\frac{\partial^2\mathcal{L}_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}. \tag{5}$$

The covariant divergence of Eq. (4) is given by

$$\nabla^\mu T_{\mu\nu}=\frac{f_T(\mathcal{G},T)}{\kappa^2-f_T(\mathcal{G},T)}\left[\nabla^\mu\Theta_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\nabla^\mu T+(T_{\mu\nu}+\Theta_{\mu\nu})\nabla^\mu(\ln f_T(\mathcal{G},T))\right]. \tag{6}$$

In this theory, the field equations as well as conservation law depend on the contributions from cosmic matter contents, therefore every suitable selection of $\mathcal{L}_m$ provides the particular scheme of dynamical equations. The line element for positive curvature FRW universe model is[15]

$$ds^2=dt^2-a^2(t)\left(\frac{1}{1-r^2}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)\right), \tag{7}$$

where $a(t)$ is the scale factor. The EMT for perfect fluid is given by

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu - Pg_{\mu\nu}, \tag{8}$$

where ρ, P and u_μ represent the energy density, pressure and four-velocity of the matter distribution, respectively. For perfect fluid as cosmic matter distribution with $\mathcal{L}_m = -P$, Eq. (5) becomes[7]

$$\Theta_{\mu\nu} = -2T_{\mu\nu} - Pg_{\mu\nu}. \tag{9}$$

Using Eqs. (7)–(9) in (4), we obtain the following set of field equations

$$\begin{aligned}\frac{3}{a^2}(1+\dot{a}^2) = \kappa^2\rho + \frac{1}{2}f(\mathcal{G},T) + (\rho+P)f_T(\mathcal{G},T) - 12\frac{\ddot{a}}{a^3}(1+\dot{a}^2) \\ \times f_{\mathcal{G}}(\mathcal{G},T) + 12\frac{\dot{a}}{a^3}(1+\dot{a}^2)\partial_t f_{\mathcal{G}}(\mathcal{G},T),\end{aligned} \tag{10}$$

$$\begin{aligned}-(1+\dot{a}^2) - 2a\ddot{a} = \kappa^2 a^2 P - \frac{1}{2}a^2 f(\mathcal{G},T) + 12\frac{\ddot{a}}{a}(1+\dot{a}^2)f_{\mathcal{G}}(\mathcal{G},T) \\ - 8\dot{a}\ddot{a}\partial_t f_{\mathcal{G}}(\mathcal{G},T) - 4(1+\dot{a}^2)\partial_{tt} f_{\mathcal{G}}(\mathcal{G},T),\end{aligned} \tag{11}$$

where

$$\mathcal{G} = \frac{24}{a^3}(1+\dot{a}^2)\ddot{a}, \quad T = \rho - 3P, \tag{12}$$

and dot represents the time derivative. The conservation equation (6) for perfect fluid yields

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho+P) = \frac{-1}{\kappa^2 + f_T(\mathcal{G},T)}\left[\left(\dot{P} + \frac{1}{2}\dot{T}\right)f_T(\mathcal{G},T) + (\rho+P)\partial_t f_T(\mathcal{G},T)\right]. \tag{13}$$

3. Stability of Einstein Universe

In this section, we analyze the stability of EU against linear homogeneous perturbations in the background of $f(\mathcal{G},T)$ gravity. For EU, $a(t) = a_0 =$ constant and consequently, the field equations (10) and (11) reduce to

$$\frac{3}{a_0^2} = \kappa^2\rho_0 + \frac{1}{2}f(\mathcal{G}_0,T_0) + (\rho_0+P_0)f_T(\mathcal{G}_0,T_0), \tag{14}$$

$$-\frac{1}{a_0^2} = \kappa^2 P_0 - \frac{1}{2}f(\mathcal{G}_0,T_0), \tag{15}$$

where $\mathcal{G}_0 = \mathcal{G}(a_0) = 0$, $T_0 = \rho_0 - 3P_0$, ρ_0 and P_0 are the unperturbed energy density and pressure, respectively. To explore the stability regions, we consider linear form of equation-of-state as $P(t) = \omega\rho(t)$ and define linear perturbations in the scale factor and energy density as follows

$$a(t) = a_0 + a_0\delta a(t), \quad \rho(t) = \rho_0 + \rho_0\delta\rho(t), \tag{16}$$

where $\delta a(t)$ and $\delta\rho(t)$ represent the perturbed scale factor and energy density, respectively. Applying the Taylor series expansion in two variables upto first order with the assumption that $f(\mathcal{G},T)$ is an analytic function, we have

$$f(\mathcal{G},T)=f(\mathcal{G}_0,T_0)+f_{\mathcal{G}}(\mathcal{G}_0,T_0)\delta\mathcal{G}+f_T(\mathcal{G}_0,T_0)\delta T, \tag{17}$$

where $\delta\mathcal{G}$ and δT have the following expressions

$$\delta\mathcal{G}=\frac{24}{a_0^2}\delta\ddot{a},\quad \delta T=T_0\delta\rho, \tag{18}$$

where $\delta\ddot{a}=\frac{d^2}{dt^2}(\delta a)$. Using Eqs. (14)–(18) in (10) and (11), we obtain the linearized perturbed field equations as follows

$$\begin{aligned}&6\delta a+24\rho_0(1+\omega)f_{\mathcal{G}T}(\mathcal{G}_0,T_0)\delta\ddot{a}+a_0^2\rho_0[\kappa^2+(1+\omega)f_T(\mathcal{G}_0,T_0)\\&\quad+\frac{1}{2}(1-3\omega)f_T(\mathcal{G}_0,T_0)+\rho_0(1+\omega)(1-3\omega)f_{TT}(\mathcal{G}_0,T_0)]\delta\rho=0,\end{aligned} \tag{19}$$

$$\begin{aligned}&-\frac{2}{a_0^2}\delta a+2\delta\ddot{a}-\frac{96}{a_0^4}f_{\mathcal{G}\mathcal{G}}(\mathcal{G}_0,T_0)\delta a^{(iv)}+\rho_0\left[\kappa^2\omega-\frac{1}{2}(1-3\omega)f_T(\mathcal{G}_0,T_0)\right]\delta\rho\\&\quad-4\frac{\rho_0}{a_0^2}(1-3\omega)f_{\mathcal{G}T}(\mathcal{G}_0,T_0)\delta\ddot{\rho}=0.\end{aligned} \tag{20}$$

These equations show that the perturbations in $a(t)$ are related with density perturbations. In the following subsections, we discuss the stability modes for conserved as well as nonconserved EMT.

3.1. *Conserved EMT*

In this case, we assume that general conservation law holds in $f(\mathcal{G},T)$ gravity. For this purpose, the right hand side of Eq. (13) must be zero which yields

$$\left(\dot{P}+\frac{1}{2}\dot{T}\right)f_T(\mathcal{G},T)+(\rho+P)\partial_t f_T(\mathcal{G},T)=0. \tag{21}$$

The conserved matter contents of the universe satisfy the relation given by

$$\delta\dot{\rho}=-3(1+\omega)\delta\dot{a}. \tag{22}$$

Using this equation in the elimination of $\delta\rho$ from Eqs. (19) and (20), we obtain the fourth-order perturbation equation in perturbed $a(t)$ as follows

$$\begin{aligned}&\Bigg[6\kappa^2a_0\omega-3a_0(1-3\omega)f_T(\mathcal{G}_0,T_0)+2a_0\Big\{\kappa^2+(1+\omega)f_T(\mathcal{G}_0,T_0)\\&\quad+\frac{1}{2}(1-3\omega)f_T(\mathcal{G}_0,T_0)+\rho_0(1+\omega)(1-3\omega)f_{TT}(\mathcal{G}_0,T_0)\Big\}\Bigg]\delta a\\&\quad+\Bigg[24a_0\rho_0(1+\omega)\left\{\kappa^2\omega-\frac{1}{2}(1-3\omega)f_T(\mathcal{G}_0,T_0)\right\}f_{\mathcal{G}T}(\mathcal{G}_0,T_0)\end{aligned}$$

$$
\begin{aligned}
&- \left\{2 + \frac{12\rho_0}{a_0^2}(1+\omega)(1-3\omega)f_{\mathcal{G}T}(\mathcal{G}_0,T_0)\right\}\left\{\kappa^2 a_0^3 + a_0^3(1+\omega)f_T(\mathcal{G}_0,T_0)\right.\\
&+ \left.\left.\frac{1}{2}a_0^3(1-3\omega)f_T(\mathcal{G}_0,T_0) + a_0^3\rho_0(1+\omega)(1-3\omega)f_{TT}(\mathcal{G}_0,T_0)\right\}\right]\delta\ddot{a}\\
&+ \frac{96}{a_0^4}\left\{\kappa^2 a_0^3 + a_0^3(1+\omega)f_T(\mathcal{G}_0,T_0) + \frac{1}{2}a_0^3(1-3\omega)f_T(\mathcal{G}_0,T_0) + a_0^3\rho_0\right.\\
&\times \left.(1+\omega)(1-3\omega)f_{TT}(\mathcal{G}_0,T_0)\right\} f_{\mathcal{G}\mathcal{G}}(\mathcal{G}_0,T_0)\delta a^{(iv)} = 0. \qquad (23)
\end{aligned}
$$

Adding Eqs. (14) and (15), it follows that

$$\frac{2}{a_0^2} = \rho_0(1+\omega)(\kappa^2 + f_T(\mathcal{G}_0,T_0)). \qquad (24)$$

Using this expression in Eq. (23), the resulting perturbation equation yields

$$
\begin{aligned}
&\left[\rho_0(1+\omega)\{\kappa^2 + f_T(\mathcal{G}_0,T_0)\}\left\{\kappa^2(1+3\omega) + (1+\omega)f_T(\mathcal{G}_0,T_0)\right.\right.\\
&\quad \left.\left.-(1-3\omega)f_T(\mathcal{G}_0,T_0) + \rho_0(1+\omega)(1-3\omega)f_{TT}(\mathcal{G}_0,T_0)\right\}\right]\delta a\\
&\quad + \left[12\rho_0^2(1+\omega)^2\{\kappa^2 + f_T(\mathcal{G}_0,T_0)\}\left\{\kappa^2\omega - \frac{1}{2}(1-3\omega)f_T(\mathcal{G}_0,T_0)\right\}\right.\\
&\quad \times f_{\mathcal{G}T}(\mathcal{G}_0,T_0) - [2 + 6\rho_0^2(1+\omega)^2(1-3\omega)\{\kappa^2 + f_T(\mathcal{G}_0,T_0)\}f_{\mathcal{G}T}(\mathcal{G}_0,T_0)]\\
&\quad \times \left\{\kappa^2 + (1+\omega)f_T(\mathcal{G}_0,T_0) + \rho_0(1+\omega)(1-3\omega)f_{TT}(\mathcal{G}_0,T_0) + \frac{1}{2}(1-3\omega)\right.\\
&\quad \times \left.\left.f_T(\mathcal{G}_0,T_0)\right\}\right]\delta\ddot{a} + 24\rho_0^2(1+\omega)^2\{\kappa^2 + f_T(\mathcal{G}_0,T_0)\}^2\left\{\kappa^2 + (1+\omega)\right.\\
&\quad \times \left.f_T(\mathcal{G}_0,T_0) + \rho_0(1+\omega)(1-3\omega)f_{TT}(\mathcal{G}_0,T_0) + \frac{1}{2}(1-3\omega)f_T(\mathcal{G}_0,T_0)\right\}\\
&\quad \times f_{\mathcal{G}\mathcal{G}}(\mathcal{G}_0,T_0)\delta a^{(iv)} = 0. \qquad (25)
\end{aligned}
$$

The solution of this equation helps to discuss the stability regions in EU. However, it would be difficult to find stable/unstable solutions due to its complicated nature. We, therefore, consider the particular form of $f(\mathcal{G},T)$ as follows

$$f(\mathcal{G},T) = f_1(\mathcal{G}) + f_2(T). \qquad (26)$$

This choice of model does not involve the direct curvature-matter nonminimal coupling but it can be considered as correction to $f(\mathcal{G})$ gravity. In this case, we have assumed that the EMT is conserved, therefore, we first constrain the above model such that the conservation law holds for it. For this purpose, using the considered form in Eq. (21), the resulting second-order differential equation takes the form

$$(1-\omega)f_2'(T) + 2(1+\omega)Tf_2''(T) = 0,$$

where prime represents derivative with respect to x ($x = T$ or $\mathcal{G}$). The solution is given by

$$f_2(T) = \frac{c_1(1+\omega)}{1+3\omega} T^{\frac{1+3\omega}{2(1+\omega)}} + c_2, \tag{27}$$

where c_i's ($i = 1, 2$) are integration constants. This is the unique representation of matter contribution for which conservation law holds with model (26). The modified GB term $f_1(\mathcal{G}_0)$ acts like an effective Λ to the unperturbed field equations. It is worth mentioning here that $f(\mathcal{G})$ gravity is recovered for this choice of $f(\mathcal{G}, T)$ model if $f_2(T) = 0$.[18] Inserting the values from Eqs. (26) and (27) in (25), the differential equation takes the form

$$\Delta_2(\Delta_1 + \Delta_3)\delta a - 2\Delta_1 \delta\ddot{a} + 24\Delta_1\Delta_2^2 f_1''(\mathcal{G}_0)\delta a^{(iv)} = 0, \tag{28}$$

where Δ_j's ($j = 1, 2, 3$) are

$$\begin{aligned}
\Delta_1 &= \kappa^2 + \frac{1}{4}c_1(1+5\omega)[\rho_0(1-3\omega)]^{\frac{\omega-1}{2(\omega+1)}} - \frac{1}{4}c_1\rho_0(1-\omega)(1-3\omega) \\
&\quad \times [\rho_0(1-3\omega)]^{\frac{-(3+\omega)}{2(1+\omega)}}, \\
\Delta_2 &= \rho_0(1+\omega)\left[\kappa^2 + \frac{1}{2}\{\rho_0(1-3\omega)\}^{\frac{\omega-1}{2(\omega+1)}}\right], \\
\Delta_3 &= 3\kappa^2\omega - \frac{3}{4}c_1(1-3\omega)\{\rho_0(1-3\omega)\}^{\frac{\omega-1}{2(\omega+1)}}.
\end{aligned}$$

Equation (28) provides the following solution

$$\delta a(t) = d_1 e^{\Omega_1 t} + d_2 e^{-\Omega_1 t} + d_3 e^{\Omega_2 t} + d_4 e^{-\Omega_2 t},$$

where d_k's ($k = 1, \ldots, 4$) are constants of integration and the parameters Ω_1 and Ω_2 are frequencies of small perturbations given by

$$\Omega_{1,2}^2 = \frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 24\Delta_1\Delta_2^3(\Delta_1+\Delta_3)f_1''(\mathcal{G}_0)}}{24\Delta_1\Delta_2^2 f_1''(\mathcal{G}_0)}. \tag{29}$$

In order to avoid the exponential growth of $\delta a(t)$ or collapse, the frequencies are purely complex which lead to the existence of stable EU. Thus, the condition of stability is achieved when $\Omega_{1,2}^2 < 0$. In the limit of GR, Ω_1^2 diverge while Ω_2^2 are given by

$$\Omega_2^2 = \frac{1}{2}\kappa^2\rho_0(1+3\omega)(1+\omega),$$

which provide stable region in the range $-1 < \omega < -\frac{1}{3}$.[18] For simplicity, we introduce a new parameter $\zeta_1 = 24 f_1''(\mathcal{G}_0)$ as well as use $\kappa^2 = 1$ and $\rho_0 = 0.3$ (present day value of density parameter) to discuss the graphical analysis of stable EU.[27] Figure 1 shows the stable regions under homogeneous perturbations of EU for Ω_1^2. It is found that for $c_1 = 1$ in the left plot, the stable EU exists for negative values of ω while no stable region exists for its positive values. The right

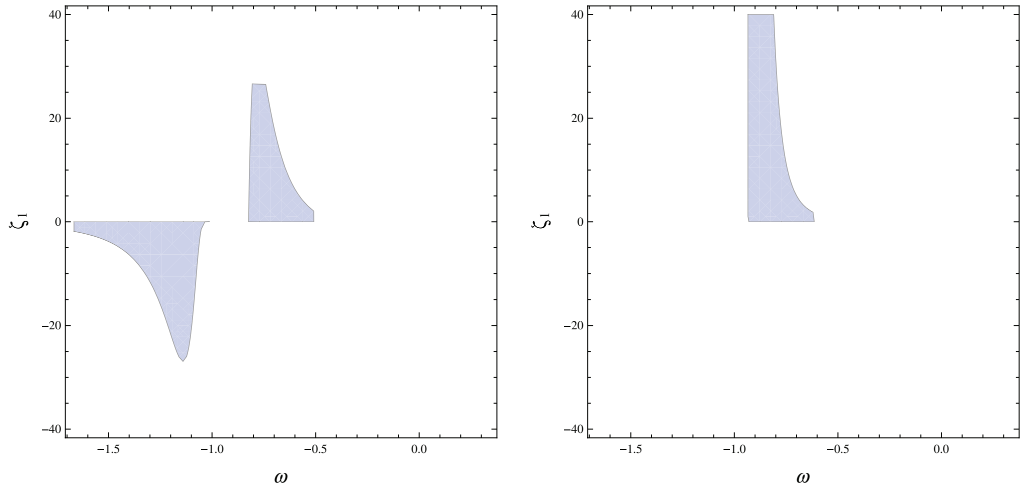


Fig. 1. Stable regions in (ω, ζ_1) space for Ω_1^2 with $c_1 = 1$ (left) and $c_1 = 5$ (right).

panel shows the stable region for $c_1 = 5$ and hence the stability regions decrease as the value of integration constant increases while for negative values of c_1, no stable regions are found. The regions of stability for frequencies Ω_2^2 are shown in Fig. 2 for both positive as well as negative values of c_1. The negative values of ζ_1 are obtained for $f_1''(\mathcal{G}_0) < 0$ which is in agreement with stability condition of $f(\mathcal{G})$ models.[28] Figure 3 shows the stability regions for both Ω_1^2 as well as Ω_2^2 of the whole system.

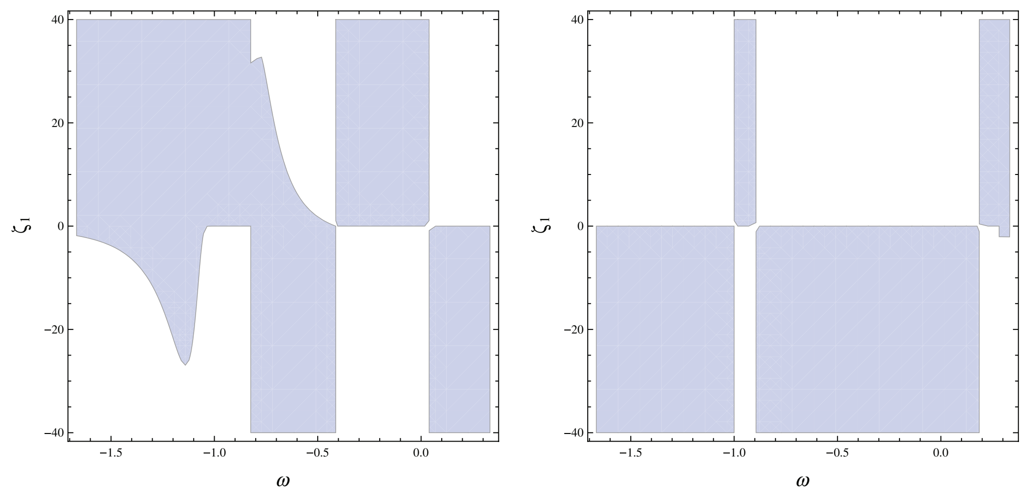


Fig. 2. Stable regions in (ω, ζ_1) space for Ω_2^2 with $c_1 = 1$ (left) and $c_1 = -1$ (right).

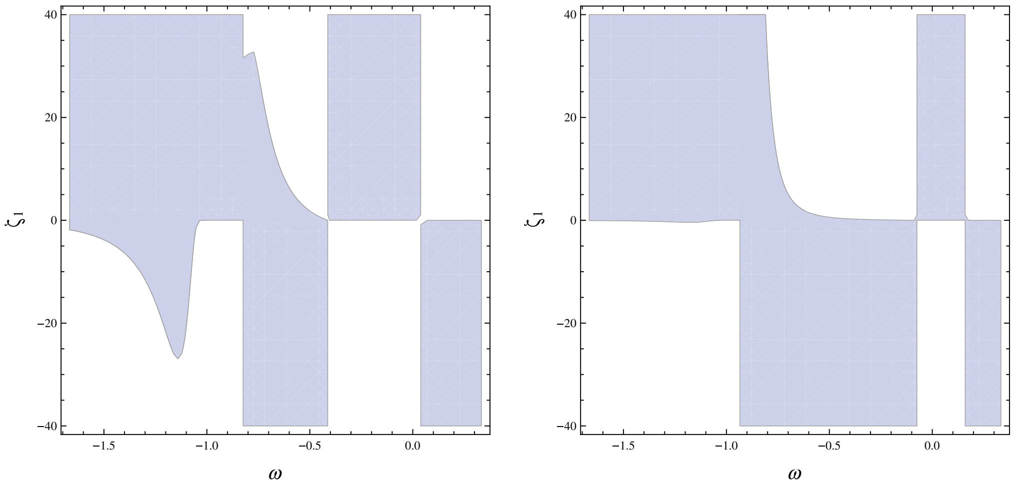


Fig. 3. Stable regions in (ω, ζ_1) space for $\Omega^2_{1,2}$ with $c_1 = 1$ (left) and $c_1 = 5$ (right).

3.2. *Nonconserved EMT*

Here, we analyze the stability of $f(\mathcal{G}, T)$ model when EMT is not conserved. We consider generic function $f_1(\mathcal{G})$ and a linear form of $f_2(T)$ in Eq. (26) as follows

$$f(\mathcal{G}, T) = f_1(\mathcal{G}) + \kappa^2 \chi T, \tag{30}$$

where χ is an arbitrary constant. Substituting in Eq. (13), we obtain

$$\rho = \tilde{\rho}_0 a^{-3\varphi}, \quad \varphi = \frac{2(1+\chi)(1+\omega)}{2+\chi(3-\omega)},$$

where $\tilde{\rho}_0$ is an integration constant. The perturbed field equations (19) and (20) take the following form

$$6\delta a + \kappa^2 a_0^2 \rho_0 \left[1 - \frac{\chi}{2}(\omega - 3)\right] \delta\rho = 0, \tag{31}$$

$$2\delta\ddot{a} - \frac{2}{a_0^2}\delta a + \kappa^2 \left[\omega - \frac{\chi}{2}(1-3\omega)\right] \rho_0 \delta\rho - \frac{96}{a_0^4} f_1''(\mathcal{G}_0)\delta a^{(iv)} = 0. \tag{32}$$

The first field equation shows the relationship between the perturbed energy density and scale factor perturbations. Eliminating $\delta\rho$ from Eqs. (31) and (32), the resulting differential equation in perturbed $a(t)$ is given by

$$2\delta\ddot{a} - \frac{2}{a_0^2}\left[1 + \frac{3(2\omega - \chi(1-3\omega))}{2-\chi(\omega-3)}\right]\delta a - \frac{96}{a_0^4} f_1''(\mathcal{G}_0)\delta a^{(iv)} = 0. \tag{33}$$

In this case, the addition of static field equations yields

$$\frac{2}{a_0^2} = \kappa^2 \rho_0 (1+\chi)(1+\omega). \tag{34}$$

Inserting this value of $\frac{2}{a_0^2}$ in Eq. (33), we obtain

$$\kappa^2\rho_0[\chi(1+\chi)(1-\omega^2)-(1+\chi)^2(1+\omega)(1+3\omega)]\delta a+[2(1+\chi)\\+\chi(1-\omega)]\delta\ddot{a}-12\kappa^4\rho_0^2[2(1+\chi)^3(1+\omega)^2+\chi(1+\chi)^2(1-\omega)(1+\omega)^2]\\\times f_1''(\mathcal{G}_0)\delta a^{(iv)}=0, \tag{35}$$

whose solution provides the following four frequencies as

$$\Upsilon_{1,2}^2=\frac{-2(1+\chi)-\chi(1-\omega)\pm\sqrt{[2(1+\chi)+\chi(1-\omega)]^2-48\kappa^6\rho_0^3\Delta_4 f_1''(\mathcal{G}_0)}}{24\kappa^4\rho_0^2[\chi(1+\chi)^2(\omega-1)(1+\omega)^2-2(1+\chi)^3(1+\omega)^2]f_1''(\mathcal{G}_0)},$$

where

$$\Delta_4=[2(1+\chi)^3(1+\omega)^2+\chi(1+\chi)^2(1-\omega)(1+\omega)^2][(1+\chi)^2(1+\omega)\\\times(1+3\omega)-\chi(1+\chi)(1-\omega^2)].$$

When $f_1(\mathcal{G}_0)=0=\chi$, the frequencies Υ_1^2 recover the GR result as obtained in the previous case while frequencies Υ_2^2 diverge. We simplify the expression by introducing a new parameter $\zeta_2=-48\kappa^6\rho_0^3 f_1''(\mathcal{G}_0)$ which remains positive for $f_1''(\mathcal{G}_0)<0$. Figure 4 shows stable regions against homogeneous perturbations of EU for frequencies Υ_1^2. It is found that when $\chi=1$ (left panel), the stable EU exists for all values of ω with suitable choice of ζ_2 while less stable regions are obtained when $\chi=5$ as shown in the right plot. In the case of nonconserved EMT, the stability regions decrease as the value of model parameter χ increases while no stable regions are observed for $\chi<0$. The regions of stability in EU for frequencies Υ_2^2 are shown in Fig. 5 for considered values of χ while stability regions for whole system is observed in Fig. 6.

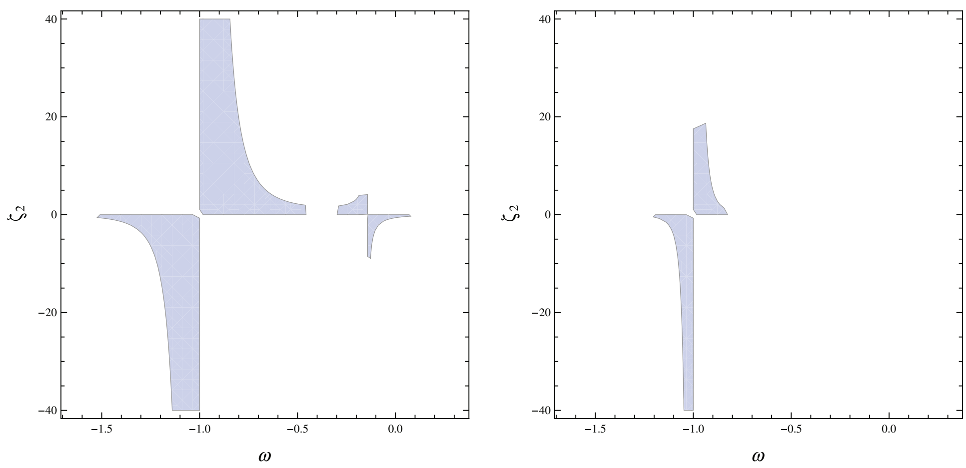


Fig. 4. Stable regions in (ω,ζ_2) space for Υ_1^2 with $\chi=1$ (left) and $\chi=5$ (right).

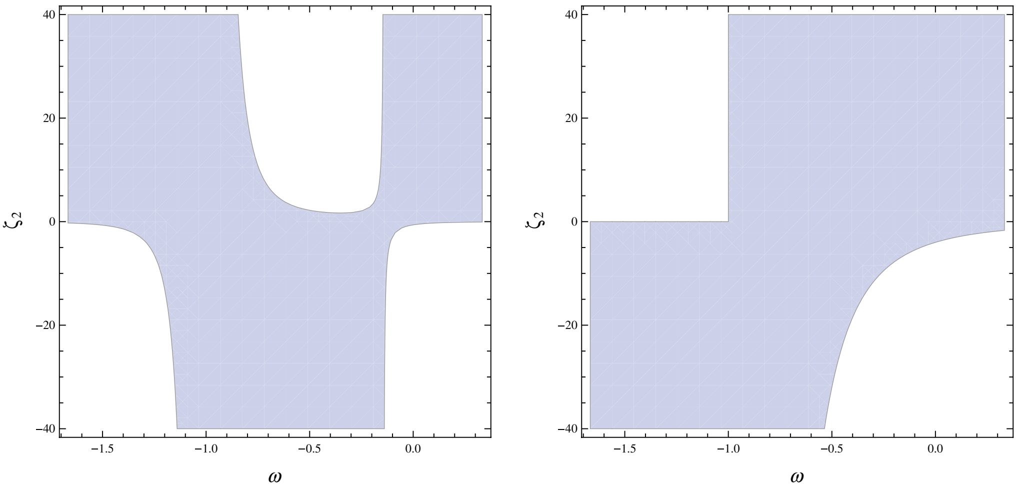


Fig. 5. Stable regions in (ω, ζ_2) space for Υ_2^2 with $\chi = 1$ (left) and $\chi = -0.5$ (right).

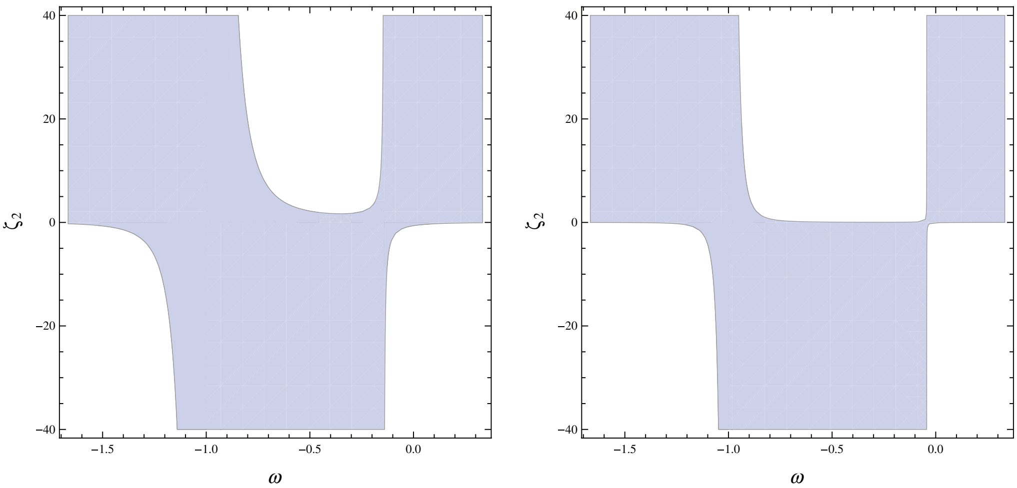


Fig. 6. Stable regions in (ω, ζ_2) space for $\Upsilon_{1,2}^2$ with $\chi = 1$ (left) and $\chi = 5$ (right).

Now, we consider the generalized model given by

$$f(\mathcal{G}, T) = f_1(\mathcal{G}) + \kappa^2 \chi T^n, \quad n \neq 0. \tag{36}$$

Following the same procedure, we obtain the following fourth-order differential equation in perturbed $a(t)$ as follows

$$24\kappa^4 \rho_0^2 (1+\omega)^2 \left[1 + n\chi\rho_0^{n-1}(1-3\omega)^{n-1}\right]^2 f_1''(\mathcal{G}_0)\delta a^{(iv)} - 2\delta\ddot{a}$$
$$+ \kappa^2 \rho_0 (1+\omega)\left(1 + n\chi\rho_0^{n-1}(1-3\omega)^{n-1}\right)\Bigg[1 + 3\left(\omega - \frac{n}{2}\chi(1-3\omega)^n \rho_0^{n-1}\right)$$

$$\times\left(1+n\chi(1-3\omega)^{n-1}\rho_0^{n-1}\left[(1+\omega)+\frac{1}{2}(1-3\omega)+(n-1)\right.\right.$$
$$\left.\left.\left.\times\ (1+\omega)\right]\right)^{-1}\right]\delta a=0,$$

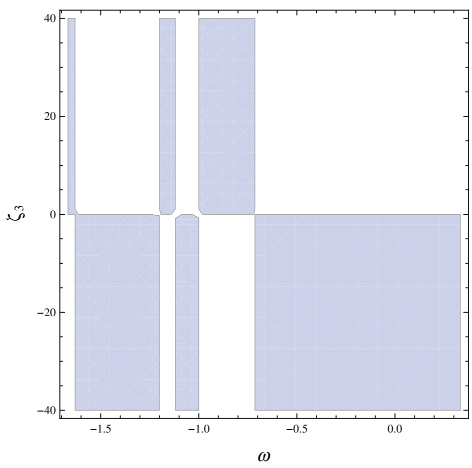
whose solution provides the following four frequencies as

$$\Xi_{1,2}^2=\frac{1\pm\sqrt{1-24\kappa^6\Delta_5 f_1''(\mathcal{G}_0)}}{24\left[\kappa^2\rho_0(1+\omega)\left(1+n\chi\rho_0^{n-1}(1-3\omega)^{n-1}\right)\right]^2 f_1''(\mathcal{G}_0)},$$

where

$$\Delta_5=\rho_0^3(1+\omega)^3\left(1+n\chi\rho_0^{n-1}(1-3\omega)^{n-1}\right)^3\left[1+3\left(\omega-\frac{n}{2}\chi(1-3\omega)^n\right.\right.$$
$$\left.\times\ \rho_0^{n-1}\right)\left[1+n\chi(1-3\omega)^{n-1}\rho_0^{n-1}\left((1+\omega)+\frac{1}{2}(1-3\omega)+(n-1)\right.\right.$$
$$\left.\left.\left.\times\ (1+\omega)\right)\right]^{-1}\right].$$

The graphical analysis of frequencies Ξ_2^2 are shown in Figs. 7 and 8 where we have used $\zeta_3=-24\kappa^6 f_1''(\mathcal{G}_0)$, $\kappa^2=1$, $\rho_0=0.3$ and $\chi=1$. It is found that stable regions are obtained for all the considered values of n while stable EU does not exist for the frequencies Ξ_1^2. In this case, the stability region of whole system is completely described by the frequencies Ξ_2^2. It is interesting to mention here that for $f_1(\mathcal{G}_0)=0=\chi$, the frequencies Ξ_1^2 diverge while GR is recovered for the frequencies Ξ_2^2 as in the previous case.



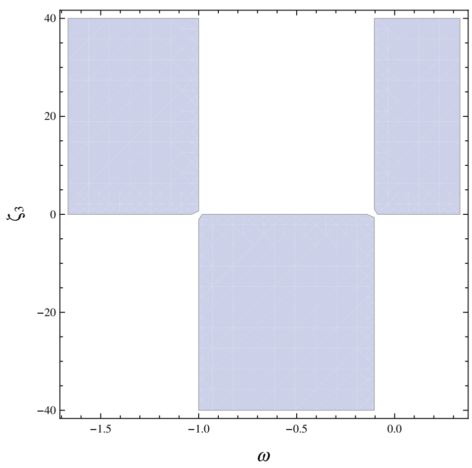


Fig. 7. Stable regions in (ω,ζ_3) space for Ξ_2^2 with $n=-5$ (left) and $n=0.5$ (right).

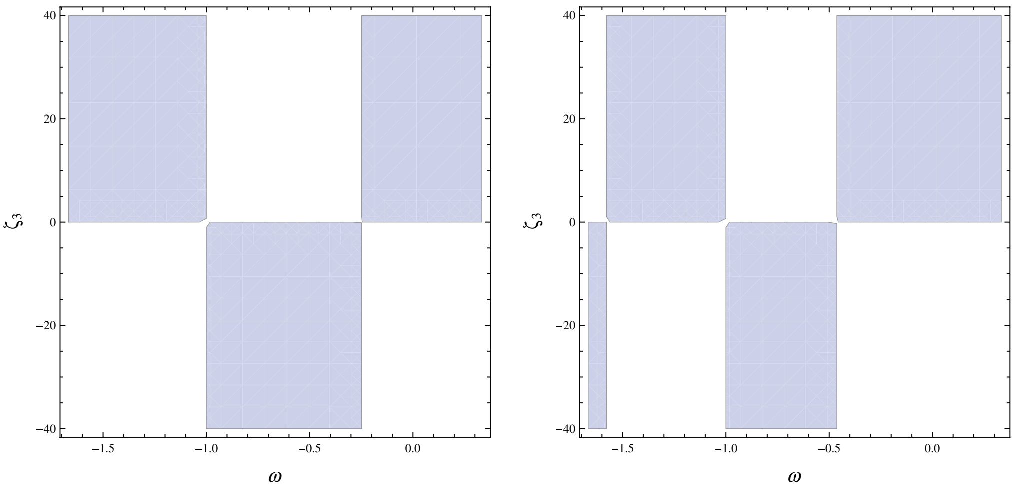


Fig. 8. Stable regions in (ω, ζ_2) space for Ξ_2^2 with $n = 2$ (left) and $n = 5$ (right).

4. Final Remarks

In this paper, we have analyzed the stability issue of EU in the context of $f(\mathcal{G}, T)$ gravity which is the extension of $f(\mathcal{G})$ gravity and is based on the ground of matter–curvature coupling. Due to this coupling, the conservation law does not hold as in $f(R, T)$ gravity.[7] We have considered the isotropic and homogeneous positive curvature FRW line element with perfect fluid as matter content of the universe. The static as well as perturbed field equations are constructed against linear homogeneous perturbations which are parameterized by equation-of-state parameter. We have formulated the fourth-order perturbed differential equation whose solutions are analyzed for the existence and stability of EU for specific form of $f(\mathcal{G}, T) = f_1(\mathcal{G}) + f_2(T)$. For this choice, we have discussed both the models when EMT is conserved as well as not conversed and obtained distinct results as compared to $f(\mathcal{G})$ gravity.

- We have assumed that EMT is conserved in this gravity and obtained a particular form of $f_2(T)$ for which the covariant divergence of EMT becomes zero. We have analyzed the regions of stability around EU and found that stable results are observed for a suitable choice of integration constant c_1.
- Two particular forms of $f_2(T)$ are considered for which the covariant divergence of EMT remains nonzero and the value of energy density in terms of scale factor is evaluated. It is found that stable EU exists in this case for both models if the model parameter χ is chosen appropriately.

We conclude that the stable EU universe exists against scalar homogeneous perturbations in the background of $f(\mathcal{G}, T)$ for all values of the equation-of-state parameter if the model parameters are chosen suitably. EU against vector perturbations (comoving dimensionless vorticity vector) are stable for all equations-of-state on all scales since any initial vector perturbations remain frozen. The mechanism for stability analysis of EU against tensor perturbations (comoving dimensionless traceless shear tensor) suggests that these fluctuations may not break the stability of EU in the background of $f(\mathcal{G},T)$ gravity.[12] It would be interesting to investigate the complete analysis of tensor as well as inhomogeneous perturbations in this gravity which will be helpful to explore the EU. It is worth mentioning here that our results reduce to $f(\mathcal{G})$ gravity in the absence of matter–curvature coupling.[18]

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Inhomogeneous Perturbations and Stability in $f(\mathcal{G}, T)$ Gravity

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Abstract

In this paper, we study stability of the Einstein static universe against inhomogeneous scalar perturbations parameterized by equation of state parameter in $f(\mathcal{G}, T)$ gravity ($\mathcal{G}$ and T represent the Gauss-Bonnet invariant and trace of energy-momentum tensor, respectively). We formulate static as well as perturbed field equations in the presence of perfect fluid and analyze the stability regions. This is accomplished for particular $f(\mathcal{G}, T)$ models corresponding to zero as well as non-zero covariant divergence of the energy-momentum tensor. It is found that stable Einstein universe exists both for spatially closed as well as open universe models for suitable choice of parameters.

Keywords: Stability analysis; Einstein universe; $f(\mathcal{G}, T)$ gravity.
PACS: 04.25.Nx; 04.40.Dg; 04.50.Kd.

1 Introduction

Gauss-Bonnet (GB) invariant being a quadratic curvature invariant has gained special attention in cosmology. It is a linear combination of Ricci scalar (R), Ricci ($R_{\alpha\beta}$) and Riemann ($R_{\alpha\beta\xi\eta}$) tensors. The interesting feature of this four-dimensional topological term is that it is free from spin-2 ghost instabilities [1]. To investigate the dynamics of $\mathcal{G}$ in four dimensions, Nojiri and

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Odintsov [2] presented the modified GB theory (also called $f(\mathcal{G})$ gravity) by adding the generic function $f(\mathcal{G})$ in the Einstein-Hilbert action. This theory acts as an alternative to dark energy (DE) which efficiently describes the late-time cosmological evolution with a possibility to discuss transition between decelerated and accelerated phases of the universe [3]. De Felice and Tsujikawa [4] constructed cosmological viable models and developed an iterative mechanism to avoid numerical instabilities incorporated with large mass of oscillating mode. They also found that these viable $f(\mathcal{G})$ models are consistent with solar system constraints for a suitable choice of model parameters [5].

The significant curvature-matter coupling (CMC) in modified theories of gravity has attained much attention to discuss the fascinating issue of current cosmic accelerated expansion. Harko et al. [6] established such coupling between R and matter referred as $f(R,T)$ gravity. Recently, we have introduced $f(\mathcal{G},T)$ gravity to explore the dynamics of quadratic curvature invariant coupled with matter as the generalization of $f(\mathcal{G})$ gravity [7]. It is found that the covariant divergence of energy-momentum tensor is non-zero due to presence of this coupling. Consequently, an extra force appears due to which the non-geodesic lines of geometry are followed by massive test particles while dust particles move along geodesic trajectories. Some cosmological viable $f(\mathcal{G},T)$ models are constructed via Noether symmetry approach using isotropic and homogeneous universe model [8]. We have reconstructed and analyzed the stability of $f(\mathcal{G},T)$ models corresponding to cosmic evolutionary models such as de Sitter and power-law solutions [9].

In addition to the discovery of current cosmic accelerated expansion, the big-bang singularity is also a captivating issue in modern cosmology. To resolve this singularity problem, several proposals have been presented like emergent universe scenario [10]. In this scenario, the universe stays at an Einstein static state (referred as Einstein universe (EU)) past eternally and then evolves to cosmic inflationary epoch [11]. Einstein universe is described by static isotropic and homogeneous universe model in the presence of prefect fluid. The successful emergent universe scenario requires stable EU against all kinds of isotropic as well as anisotropic perturbations.

The issue of stable EU has stimulated many researchers just after few years of general relativity (GR). Einstein introduced cosmological constant to his famous field equations in order to have static solution since these equations have no such solution. Eddington [12] showed that this initial static model is unstable against small isotropic and spatially homogeneous scalar perturbations in the presence of normal matter. Harrison [13] found that the unstable EU becomes oscillatory and observed the existence of stable EU for dust distribution in the presence of radiations against inhomogeneous perturbations. It is found that the EU is stable around inhomogeneous vector as well as tensor perturbations in the presence of perfect fluid and under adiabatic scalar density inhomogeneities until the inequality $5c_s^2 > 1$ (c_s is the speed of sound) holds [14].

The stability issue of EU has widely been discussed in quantum gravity models as well as in the extensions of GR since EU is unstable against homogeneous perturbations in GR. The existence and stability of EU is investigated in loop quantum cosmology, Einstein-Cartan theory, braneworld, lyra geometry etc [15]. Böhmer et al. [16] observed stable solutions against homogeneous scalar perturbations for particular $f(R)$ models. Goswami and his collaborators [17] studied the existence of stable EU in $f(R)$ gravity and found that EU is neutrally stable against vector as well as tensor perturbations for only one particular choice of $f(R)$ model. Böhmer and Lobo [18] explored stable regions against homogeneous scalar perturbations for all the considered range of equation of state parameter in $f(\mathcal{G})$ gravity.

Böhmer et al. [19] analyzed the stability of EU in the context of hybrid metric-Palatini gravity and found stable results against homogeneous as well as inhomogeneous scalar perturbations. Huang et al. [20] investigated the inhomogeneous scalar as well as tensor perturbations for both perfect fluid and scalar field as matter contents of the universe in Jordan Brans-Dicke theory and found stable solutions. The same authors [21] analyzed the homogeneous as well as inhomogeneous scalar perturbations in $f(\mathcal{G})$ gravity and found stable solutions only for a closed universe against homogeneous perturbations. Stable as well as unstable regions of EU are obtained against inhomogeneous and homogeneous perturbations, respectively in scalar-fluid theories [22]. Shabani and Ziaie [23] discussed the stability of EU against homogeneous scalar perturbations and found stable solutions in $f(R,T)$ gravity. Sharif and Ikram [24] investigated the existence and stability of EU against homogeneous scalar perturbations in $f(\mathcal{G},T)$ gravity for both conserved as well as non-conserved energy-momentum tensor and obtained stable solutions.

In this paper, we analyze the stability of EU against inhomogeneous scalar perturbations for perfect fluid as cosmic matter distribution in $f(\mathcal{G},T)$ gravity. The paper has the following format. In the next section, we formulate the field equations for EU while section **3** is devoted to analyze the stability of EU against inhomogeneous scalar perturbations for zero as well as non-zero covariant divergence of energy-momentum tensor. In section **4**, we summarize the results.

2 Einstein Universe in $f(\mathcal{G},T)$ Gravity

In this section, we review $f(\mathcal{G},T)$ gravity, EU and construct the corresponding field equations for FRW universe model. The $f(\mathcal{G},T)$ gravity is governed by the action [7]

$$\mathcal{A}=\int\left(\frac{R+f(\mathcal{G},T)}{2\kappa}+\mathcal{L}_m\right)\sqrt{-g}d^4x, \tag{1}$$

where $\mathcal{G}=R_{\alpha\beta\xi\eta}R^{\alpha\beta\xi\eta}-4R_{\alpha\beta}R^{\alpha\beta}+R^2,\ \kappa,\ \mathcal{L}_m$ and g represent the coupling constant, Lagrangian density associated with matter configuration and determinant of metric tensor ($g_{\alpha\beta}$), respectively. The variation of action (1) with respect to $g_{\alpha\beta}$ yields the field equations as follows

$$\begin{aligned}
G_{\alpha\beta} &= \kappa T_{\alpha\beta}+\frac{1}{2}g_{\alpha\beta}f(\mathcal{G},T)-2RR_{\alpha\beta}f_{\mathcal{G}}(\mathcal{G},T)+4R^{\xi}_{\alpha}R_{\beta\eta}f_{\mathcal{G}}(\mathcal{G},T)\\
&+ R^{\xi\eta}R_{\alpha\xi\beta\eta}f_{\mathcal{G}}(\mathcal{G},T)-2R_{\alpha}^{\ \gamma\xi\eta}R_{\beta\gamma\xi\eta}f_{\mathcal{G}}(\mathcal{G},T)-(T_{\alpha\beta}+\Theta_{\alpha\beta})f_T(\mathcal{G},T)\\
&- 4R_{\alpha\xi\beta\eta}\nabla^{\xi}\nabla^{\eta}f_{\mathcal{G}}(\mathcal{G},T)+4g_{\alpha\beta}R^{\xi\eta}\nabla_{\xi}\nabla_{\eta}f_{\mathcal{G}}(\mathcal{G},T)-4R^{\xi}_{\alpha}\nabla_{\beta}\nabla_{\xi}f_{\mathcal{G}}(\mathcal{G},T)\\
&- 4R^{\xi}_{\beta}\nabla_{\alpha}\nabla_{\xi}f_{\mathcal{G}}(\mathcal{G},T)+4R_{\alpha\beta}\nabla^2f_{\mathcal{G}}(\mathcal{G},T)+2R\nabla_{\alpha}\nabla_{\beta}f_{\mathcal{G}}(\mathcal{G},T)\\
&- 2g_{\alpha\beta}R\nabla^2f_{\mathcal{G}}(\mathcal{G},T),
\end{aligned} \tag{2}$$

where $\Theta_{\alpha\beta}=g^{\xi\eta}(\delta T_{\xi\eta}/\delta g^{\alpha\beta}),\ \nabla^2=\nabla^{\alpha}\nabla_{\alpha}$ (∇_{α} is the covariant derivative), $f_{\mathcal{G}}(\mathcal{G},T)$ and $f_T(\mathcal{G},T)$ represent derivatives with respect to $\mathcal{G}$ and T, respectively. The CMC gives the non-zero covariant divergence of (2) as

$$\begin{aligned}
\nabla^{\alpha}T_{\alpha\beta} &= \frac{1}{\kappa-f_T(\mathcal{G},T)}\left[(T_{\alpha\beta}+\Theta_{\alpha\beta})\nabla^{\alpha}f_T(\mathcal{G},T)-\left(\frac{1}{2}g_{\alpha\beta}\nabla^{\alpha}T\right.\right.\\
&- \left.\left.\nabla^{\alpha}\Theta_{\alpha\beta}\right)f_T(\mathcal{G},T)\right].
\end{aligned} \tag{3}$$

The above equations show that the dynamics of this modified gravity theory depends on the suitable choice of $\mathcal{L}_m$ and generic function $f(\mathcal{G},T)$. The energy-momentum tensor for perfect fluid is given by

$$T_{\alpha\beta}=(\rho+p)\mu_{\alpha}\mu_{\beta}-pg_{\alpha\beta}, \tag{4}$$

where ρ, p and μ_α denote the energy density, pressure and four-velocity in comoving coordinates, respectively. The tensor $\Theta_{\alpha\beta}$ has the following expression

$$\Theta_{\alpha\beta} = -2T_{\alpha\beta} - pg_{\alpha\beta}, \tag{5}$$

where we have used $\mathcal{L}_m = -p$ with the assumption that $\mathcal{L}_m$ depends only on $g_{\alpha\beta}$ rather than on its derivatives [25].

The line element for homogeneous and isotropic universe is [21]

$$ds^2 = a^2(\nu)\left[d\nu^2 - \frac{d\chi^2}{1-k\chi^2} - \chi^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{6}$$

where $a(\nu)$ and k represent the conformal scale factor depending on the conformal time (ν) and spatial curvature parameter, respectively. The conformal time satisfies the relation $dt = a(\nu)d\nu$ whereas k corresponds to closed ($k = 1$), flat ($k = 0$) and open ($k = -1$) geometries of the universe. The GB invariant and T take the form

$$\mathcal{G} = \frac{24}{a^5}\left(\ddot{a} - \frac{\dot{a}^2}{a}\right)\left[k + \left(\frac{\dot{a}}{a}\right)^2\right], \quad T = \rho - 3p,$$

where dot represents derivative with respect to ν. Using Eqs.(4)-(6) in (2), we obtain the corresponding fourth-order field equations as

$$\begin{aligned} 3\left[\left(\frac{\dot{a}}{a}\right)^2 + k\right] &= \kappa a^2\rho + \frac{1}{2}a^2 f(\mathcal{G},T) + a^2(\rho+p)f_T(\mathcal{G},T) + \frac{12}{a^3}\left(\frac{\dot{a}^2}{a} - \ddot{a}\right) \\ &\times \left[\left(\frac{\dot{a}}{a}\right)^2 + k\right] f_{\mathcal{G}}(\mathcal{G},T) + 12\frac{\dot{a}}{a}\left[\dot{a}^2 - a\ddot{a}\left(1 - \frac{1}{a^4}\right) + \frac{k}{a^2}\right] \\ &\times \dot{f}_{\mathcal{G}}(\mathcal{G},T) - 12(\dot{a}^2 - a\ddot{a})\left(1 - \frac{1}{a^4}\right)\ddot{f}_{\mathcal{G}}(\mathcal{G},T), \end{aligned} \tag{7}$$

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 - \frac{2\ddot{a}}{a} - k &= \kappa a^2 p - \frac{1}{2}a^2 f(\mathcal{G},T) + \frac{12}{a^3}\left(\ddot{a} - \frac{\dot{a}^2}{a}\right)\left[\left(\frac{\dot{a}}{a}\right)^2 + k\right] \\ &\times f_{\mathcal{G}}(\mathcal{G},T) + 4\dot{a}\left[3a^3(\dot{a}^2 - a\ddot{a}) + \frac{1}{a^3}(\ddot{a} + k)\right]\dot{f}_{\mathcal{G}}(\mathcal{G},T) \\ &- 4\left[3\left(a^4\dot{a}^2 - a^5\ddot{a} + \frac{\ddot{a}}{a^3}\right) - 2\left(\frac{\dot{a}}{a^2}\right)^2 + \frac{k}{a^2}\right]\ddot{f}_{\mathcal{G}}(\mathcal{G},T). \end{aligned} \tag{8}$$

The Einstein universe is defined by static FRW universe model as

$$a(\nu) = a_0 = \text{constant}.$$

At Einstein static state, the expressions for GB invariant and T become

$$\mathcal{G}(a_0) = \mathcal{G}_0 = 0, \quad T_0 = \rho_0 - 3p_0,$$

where ρ_0 and p_0 denote the unperturbed energy density and pressure, respectively. The corresponding field equations are

$$3k = a_0^2\left(\kappa\rho_0 + \frac{1}{2}f(\mathcal{G}_0, T_0) + (p_0 + \rho_0)f_T(\mathcal{G}_0, T_0)\right), \tag{9}$$

$$-k = a_0^2\left(\kappa p_0 - \frac{1}{2}f(\mathcal{G}_0, T_0)\right). \tag{10}$$

It is interesting to mention here that EU is shear, expansion as well as rotation free universe.

3 Stability Analysis

In this section, we study the stability regions of EU against inhomogeneous scalar perturbations using Newtonian gauge (also known as longitudinal gauge) in $f(\mathcal{G}, T)$ gravity. In this gauge transformation, the perturbed line element is given by [21]

$$ds^2 = a_0^2(1 - 2\Phi)d\nu^2 - a_0^2(1 + 2\Psi)\left[\frac{d\chi^2}{1 - k\chi^2} + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{11}$$

where Φ and Ψ represent the Bardeen potential and perturbation to spatial curvature, respectively. The linear perturbations in energy density and pressure are

$$\rho = \rho_0 + \rho_0\delta\rho, \quad p = p_0 + p_0\delta p,$$

where $\delta\rho$ and δp are the perturbed energy density and pressure of the matter distribution, respectively. The harmonic decomposition of these scalar perturbations are [26]

$$\begin{aligned}
\delta\rho &= \delta\rho_\lambda(\nu)\Upsilon_\lambda(\vartheta^i), \quad \delta p = \delta p_\lambda(\nu)\Upsilon_\lambda(\vartheta^i),\\
\Phi &= \Phi_\lambda(\nu)\Upsilon_\lambda(\vartheta^i), \quad \Psi = \Psi_\lambda(\nu)\Upsilon_\lambda(\vartheta^i),
\end{aligned}$$

where ϑ^i denotes the spatial coordinates (χ, θ, ϕ) while summation over λ is taken. The harmonic function $\Upsilon_\lambda \equiv \Upsilon_\lambda(\vartheta^i)$ satisfies the following relations corresponding to different geometries of the universe as

$$\Delta\Upsilon_\lambda \equiv -\hbar^2\Upsilon_\lambda = \begin{cases} -(\lambda^2+1)\Upsilon_\lambda, & \lambda^2 \geq 0, \qquad k=-1, \\ -\lambda^2\Upsilon_\lambda, & \lambda^2 \geq 0, \qquad k=0, \\ -\lambda(\lambda+2)\Upsilon_\lambda, & \lambda = 0,1,2,..., \qquad k=+1, \end{cases}$$

where Δ is the three-dimensional Laplacian operator.

These perturbations generate continuous spectrum for flat as well as closed spatial cosmic geometries while discrete spectrum is observed for open universe model [21]. The first inhomogeneous mode ($\lambda = 1$) is known as gauge mode associated with the gauge degree of freedom which depicts the freedom to change μ_α of fundamental observers, so the physical modes have $\lambda \geq 2$ [14]. It is worth mentioning here that the homogeneous scalar perturbations are recovered for $\lambda = 0$. Using these perturbations in Eq.(2), we obtain the linearized $\nu\nu$-component for the perturbed metric (11) as follows

$$\begin{aligned} & 6k\Psi + 2a_0^2\hbar^2\Psi + a_0^2\rho_0\left[\kappa - \frac{1}{2}(\omega-3)f_T(\mathcal{G}_0,T_0) + \rho_0(1-3\omega)(1+\omega)\right. \\ \times\ & \left. f_{TT}(\mathcal{G}_0,T_0) - \frac{4k\hbar^2}{a_0^2}(1-3\omega)f_{\mathcal{G}T}(\mathcal{G}_0,T_0)\right]\delta\rho + \left[a_0^2\rho_0(1+\omega)f_{\mathcal{G}T}(\mathcal{G}_0,T_0)\right. \\ -\ & \left. 4k\hbar^2 f_{\mathcal{G}\mathcal{G}}(\mathcal{G}_0,T_0)\right]\delta\mathcal{G} = 0, \end{aligned} \tag{12}$$

where ω is the equation of state ($p = \omega\rho$) parameter and linearized perturbed GB invariant becomes

$$\delta\mathcal{G} = \frac{8k}{a_0^2}\left(\frac{3\ddot{\Psi}}{a_0^2} + \hbar^2\Phi\right). \tag{13}$$

The linearized diagonal components of Eq.(2) give the differential equation of the form

$$\begin{aligned} & 2\left[6k\Psi - 3\ddot{\Psi} + a_0^2\hbar^2(2\Psi - \Phi)\right] + a_0^2\rho_0\left[\kappa(1-3\omega) + 2(1-3\omega)f_T(\mathcal{G}_0,T_0)\right. \\ +\ & (1+\omega)f_T(\mathcal{G}_0,T_0) + \rho_0(1-3\omega)(1+\omega)f_{TT}(\mathcal{G}_0,T_0) + \frac{4k\hbar^2}{a_0^2}(1-3\omega) \\ \times\ & \left. f_{\mathcal{G}T}(\mathcal{G}_0,T_0)\right]\delta\rho + \left[a_0^2\rho_0(1+\omega)f_{\mathcal{G}T}(\mathcal{G}_0,T_0) - 4k\hbar^2 f_{\mathcal{G}\mathcal{G}}(\mathcal{G}_0,T_0)\right]\delta\mathcal{G} \\ +\ & \frac{12k}{a_0^2}f_{\mathcal{G}\mathcal{G}}(\mathcal{G}_0,T_0)\ddot{\delta\mathcal{G}} + \frac{4k}{a_0^2}\rho_0(1-3\omega)f_{\mathcal{G}T}(\mathcal{G}_0,T_0)\ddot{\delta\rho} = 0. \end{aligned} \tag{14}$$

The corresponding off-diagonal components in the presence of perfect fluid provide the relation $\Phi(\nu)=\Psi(\nu)$ while it does not hold for anisotropic matter distribution.

To discuss the stability of EU, we consider the particular form of generic function $f(\mathcal{G},T)$ as follows [24]

$$f(\mathcal{G},T)=f_1(\mathcal{G})+f_2(T), \tag{15}$$

which shows that the direct non-minimally CMC is absent. This choice of model is considered as the correction term to $f(\mathcal{G})$ gravity and modified GB gravity is recovered for $f_2(T)=0$. Using Eq.(15), the field equations (12) and (14) reduce to

$$\begin{aligned}
& 2(3k+a_0^2\hbar^2)\Psi+a_0^2\rho_0\left[\kappa-\frac{1}{2}(\omega-3)f_2'(T_0)+\rho_0(1-3\omega)(1+\omega)\right.\\
\times\ & \left. f_2''(T_0)\right]\delta\rho-4k\hbar^2f_1''(\mathcal{G}_0)\delta\mathcal{G}=0, \qquad (16)\\
& 2\left[6k\Psi-3\ddot{\Psi}+a_0^2\hbar^2(2\Psi-\Phi)\right]+a_0^2\rho_0\left[\kappa(1-3\omega)+\{2(1-3\omega)\right.\\
+\ & (1+\omega)\}f_2'(T_0)+\rho_0(1-3\omega)(1+\omega)f_2''(T_0)]\,\delta\rho-4k\hbar^2f_1''(\mathcal{G}_0)\delta\mathcal{G}\\
+\ & \frac{12k}{a_0^2}f_1''(\mathcal{G}_0)\delta\ddot{\mathcal{G}}=0, \qquad (17)
\end{aligned}$$

where $f_1'(\mathcal{G})=df_1(\mathcal{G})/d\mathcal{G}$ and $f_2'(T)=df_2(T)/dT$. Using Eqs.(13) and (16) in the elimination of Φ, $\delta\mathcal{G}$ and $\delta\rho$ from Eq.(17), it follows that

$$\begin{aligned}
& \frac{288k^2}{a_0^6}\left[\kappa-\frac{1}{2}(\omega-3)f_2'(T_0)+\rho_0(1+\omega)(1-3\omega)f_2''(T_0)\right]f_1''(\mathcal{G}_0)\Psi^{(iv)}\\
-\ & 6\left[\kappa-\frac{1}{2}(\omega-3)f_2'(T_0)+\rho_0(1+\omega)(1-3\omega)f_2''(T_0)-\frac{16k^2\hbar^2}{a_0^4}\right.\\
\times\ & [\kappa(1-3\omega)+2\{2(1-3\omega)+(1+\omega)\}f_2'(T_0)+\rho_0(1+\omega)(1-3\omega)\\
\times\ & f_2''(T_0)]f_1''(\mathcal{G}_0)]\,\ddot{\Psi}+2\left[\left(6k+a_0^2\hbar^2-\frac{16k^2\hbar^4}{a_0^2}f_1''(\mathcal{G}_0)\right)\right.\\
\times\ & \left(\kappa-\frac{1}{2}(\omega-3)f_2'(T_0)+\rho_0(1+\omega)(1-3\omega)f_2''(T_0)\right)-\Big(3k+a_0^2\hbar^2\\
-\ & \left.\frac{16k^2\hbar^4}{a_0^2}f_1''(\mathcal{G}_0)\right)(\kappa(1-3\omega)+2\{2(1-3\omega)+(1+\omega)\}f_2'(T_0)\\
+\ & \rho_0(1+\omega)(1-3\omega)f_2''(T_0))]\,\Psi=0. \qquad (18)
\end{aligned}$$

Using Eqs.(9), (10) and (15), we obtain

$$\frac{2k}{a_0^2} = \rho_0(1+\omega)[\kappa + f_2'(T_0)].$$

Substituting this equation in Eq.(18), the resultant fourth-order perturbed differential equation in Ψ takes the form

$$\begin{aligned}
& 18\rho_0^4(1+\omega)^4[\kappa + f_2'(T_0)]^4 \left[\kappa - \frac{1}{2}(\omega-3)f_2'(T_0) + \rho_0(1+\omega)(1-3\omega)\right. \\
\times\;& \left. f_2''(T_0)\right] f_1''(\mathcal{G}_0)\Psi^{(iv)} - 6k\rho_0(1+\omega)[\kappa + f_2'(T_0)]\left[\frac{1}{2}\left(\kappa - \frac{1}{2}(\omega-3)f_2'(T_0)\right.\right. \\
+\;& \left.\rho_0(1+\omega)(1-3\omega)f_2''(T_0)\right) - 2\rho_0^2\hbar^2(1+\omega)^2[\kappa + f_2'(T_0)]^2[\kappa(1-3\omega) \\
+\;& \left. 2\{2(1-3\omega) + (1+\omega)\}f_2'(T_0) + \rho_0(1+\omega)(1-3\omega)f_2''(T_0)]f_1''(\mathcal{G}_0)\right]\ddot{\Psi} \\
+\;& 2k^2\left[\left\{3\rho_0(1+\omega)[\kappa + f_2'(T_0)] + \hbar^2 - 4\rho_0^2\hbar^4(1+\omega)^2[\kappa + f_2'(T_0)]^2 f_1''(\mathcal{G}_0)\right\}\right. \\
\times\;& \left(\kappa - \frac{1}{2}(\omega-3)f_2'(T_0) + \rho_0(1+\omega)(1-3\omega)f_2''(T_0)\right) - \left\{\frac{3}{2}\rho_0(1+\omega)\right. \\
\times\;& \left.[\kappa + f_2'(T_0)] + \hbar^2 - 4\rho_0^2\hbar^4(1+\omega)^2[\kappa + f_2'(T_0)]^2 f_1''(\mathcal{G}_0)\right\}[\kappa(1-3\omega) \\
+\;& \left. 2\{2(1-3\omega) + (1+\omega)\}f_2'(T_0) + \rho_0(1+\omega)(1-3\omega)f_2''(T_0)]\right]\Psi = 0. \quad (19)
\end{aligned}$$

In the following subsections, we explore stability regions for zero as well as non-zero covariant divergence of the energy-momentum tensor.

3.1 Case I: $\nabla^\alpha T_{\alpha\beta} = 0$

In CMC theories, generally the conservation law does not hold. The continuity equation (3) for perfect fluid in the background of generic FRW model is

$$\dot{\rho} + 3\rho(1+\omega)\frac{\dot{a}}{a} = \frac{-1}{\kappa + f_T(\mathcal{G},T)}\left[\frac{1}{2}(1-\omega)\dot{\rho}f_T(\mathcal{G},T) + (1+\omega)\rho\dot{f}_T(\mathcal{G},T)\right].$$

For conserved matter distribution, the right hand side of this equation must be zero which leads to the following second order differential equation for the model (15)

$$2(1+\omega)Tf_2''(T) + (1-\omega)f_2'(T) = 0,$$

whose solution is the unique representation of $f_2(T)$ for which the energy-momentum tensor is conserved given by

$$f_2(T) = c_1\left(\frac{1+\omega}{1+3\omega}\right)T^{\frac{1}{2}\left(\frac{1+3\omega}{1+\omega}\right)} + c_2, \quad (20)$$

where c_1 and c_2 are constants of integration. Inserting Eqs.(15) and (20) in (19), the perturbed differential equation takes the form

$$\Omega_1\Psi^{(iv)}+\Omega_2\ddot{\Psi}+\Omega_3\Psi=0,$$

where

$$\begin{aligned}
\Omega_1 &= 18\rho_0^4(1+\omega)^4\left[\kappa+\frac{1}{2}c_1[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}\right]^4\left[\kappa-\frac{1}{4}c_1(\omega-3)\right.\\
&\times\left.[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}-\frac{1}{4}c_1\rho_0(1-\omega)(1-3\omega)[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{3+\omega}{1+\omega}\right)}\right]\\
&\times f_1''(\mathcal{G}_0),\\
\Omega_2 &= -6k\rho_0(1+\omega)\left[\kappa+\frac{1}{2}c_1[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}\right]\\
&\times\left[\frac{1}{2}\left\{\kappa-\frac{1}{4}c_1(\omega-3)[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}-\frac{1}{4}c_1\rho_0(1-\omega)(1-3\omega)\right.\right.\\
&\times\left.[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{3+\omega}{1+\omega}\right)}\right\}-2\rho_0^2\hbar^2(1+\omega)^2\left[\kappa+\frac{1}{2}c_1[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}\right]^2\\
&\times\left[\kappa(1-3\omega)+c_1\{2(1-3\omega)+(1+\omega)\}[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}\right.\\
&-\left.\left.\frac{1}{4}c_1\rho_0(1-\omega)(1-3\omega)[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{3+\omega}{1+\omega}\right)}\right]f_1''(\mathcal{G}_0)\right],\\
\Omega_3 &= 2k^2\left[\left\{3\rho_0(1+\omega)\left[\kappa+\frac{1}{2}c_1[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}\right]+\hbar^2-4\rho_0^2\hbar^4(1+\omega)^2\right.\right.\\
&\times\left.\left[\kappa+\frac{1}{2}c_1[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}\right]^2f_1''(\mathcal{G}_0)\right\}\left(\kappa-\frac{1}{4}c_1(\omega-3)\right.\\
&\times\left.[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}-\frac{1}{4}c_1\rho_0(1-\omega)(1-3\omega)[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{3+\omega}{1+\omega}\right)}\right)\\
&-\left\{\frac{3}{2}\rho_0(1+\omega)\left[\kappa+\frac{1}{2}c_1[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}\right]+\hbar^2-4\rho_0^2\hbar^4(1+\omega)^2\right.\\
&\times\left.\left[\kappa+\frac{1}{2}c_1[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}\right]^2f_1''(\mathcal{G}_0)\right\}(\kappa(1-3\omega)+c_1\{2(1-3\omega)\\
&+(1+\omega)\}[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{1-\omega}{1+\omega}\right)}-\frac{1}{4}c_1\rho_0(1-\omega)(1-3\omega)\\
&\times\left.\left.[\rho_0(1-3\omega)]^{-\frac{1}{2}\left(\frac{3+\omega}{1+\omega}\right)}\right)\right].
\end{aligned}$$

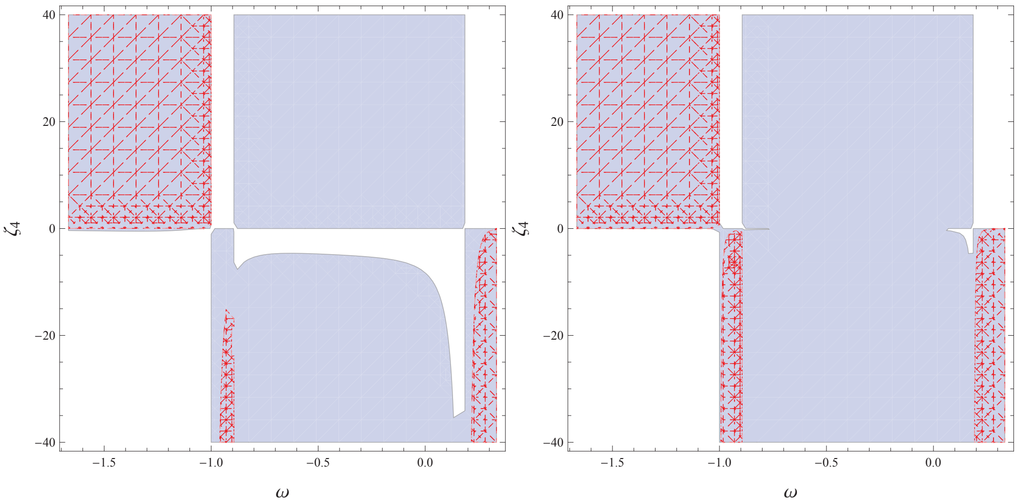

Figure 1: Regions of stability in (ω, τ) space for $k = 1$ with $\lambda = 2$ (left) and $\lambda = 15$ (right).

The solution is given by

$$\Psi(\nu) = d_1 e^{-\zeta_1 \nu} + d_2 e^{\zeta_1 \nu} + d_3 e^{-\zeta_2 \nu} + d_4 e^{\zeta_2 \nu},$$

where d_l's $(l = 1, 2, 3, 4)$ represent integration constants and frequencies $\zeta_{1,2}$ are

$$\zeta_{1,2}^2 = \frac{-\Omega_2 \mp \sqrt{\Omega_2^2 - 4\Omega_1 \Omega_3}}{2\Omega_1}. \tag{21}$$

The existence of stable/unstable regions in EU depend on the growth of perturbations. The exponential growth takes place when the frequencies satisfy the inequality $\zeta_{1,2}^2 > 0$ and give unstable solutions whereas stable EU exists for $\zeta_{1,2}^2 < 0$.

To discuss the stability of EU against inhomogeneous perturbations, we introduce the notation $\tau = -2\rho_0^2 f_1''(\mathcal{G}_0)$. Figures **1** and **2** show the stable regions for closed and open universe models, respectively with $\kappa = 1$, $c_1 = -1$ and $\rho_0 = 0.3$. Blue regions indicate the existence of stable EU for ζ_1^2 while the regions occupied by red dashed lines correspond to ζ_2^2, respectively. The stability of the system is described by regions shared by all frequencies. Figure **1** shows that the stability of EU increases as the value of λ increases in the background of closed universe whereas it decreases for $k = -1$ as λ^2 increases (Figure **2**). The $f(\mathcal{G})$ models satisfy the stability condition

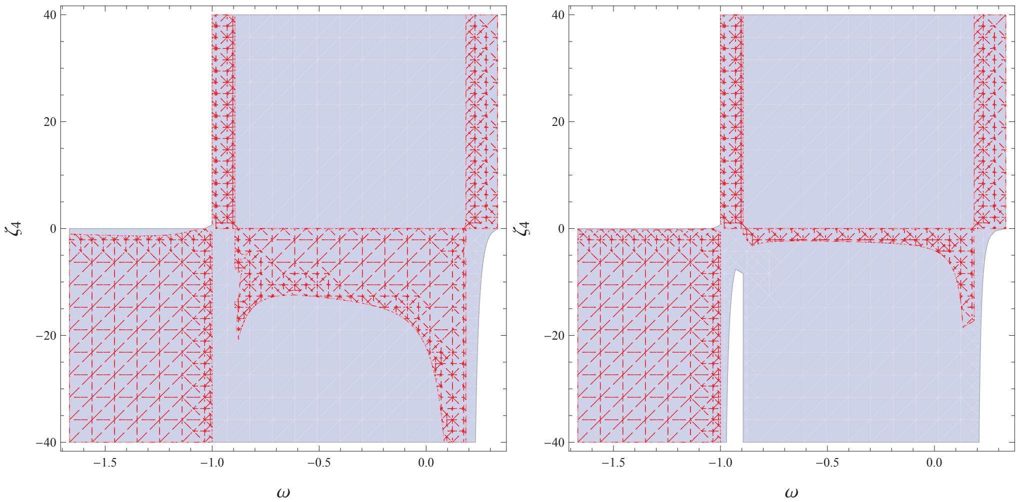

Figure 2: Regions of stability in (ω, τ) space for $k = -1$ with $\lambda^2 = 2$ (left) and $\lambda^2 = 15$ (right).

$f_1''(\mathcal{G}_0) < 0$ for positive values of parameter τ. For $c_1 = 0 = f_1''(\mathcal{G}_0)$, the frequency ζ_1^2 diverges while $\zeta_2^2 = \frac{1}{2}\kappa\rho_0(1+\omega)(1+3\omega)$, which is in agreement with the results of GR for $-1 < \omega < -1/3$ [18].

3.2 Case II: $\nabla^\alpha T_{\alpha\beta} \neq 0$

Here, we explore the stability of EU when the energy-momentum tensor is not conserved. For this purpose, we consider the following form of $f(\mathcal{G}, T)$

$$f(\mathcal{G}, T) = f_1(\mathcal{G}) + \kappa\varphi T^b, \quad b \neq 0, \tag{22}$$

where φ is an arbitrary constant. Substituting in Eq.(19), a fourth-order differential equation in Ψ is obtained whose solution provides four frequencies as follows

$$\zeta_{3,4}^2 = \frac{-\Omega_5 \mp \sqrt{\Omega_5^2 - 4\Omega_4\Omega_6}}{2\Omega_4},$$

where

$$\begin{aligned}\Omega_4 &= 18\rho_0^4(1+\omega)^4[\kappa(1+\varphi b[\rho_0(1-3\omega)]^{b-1})]^4 \left[\kappa\left\{1 - \frac{1}{2}\varphi b(\omega - 3\right.\right. \\ &- \left.\left. 2(1+\omega)(b-1))[\rho_0(1-3\omega)]^{b-1}\right\}\right] f_1''(\mathcal{G}_0),\end{aligned}$$

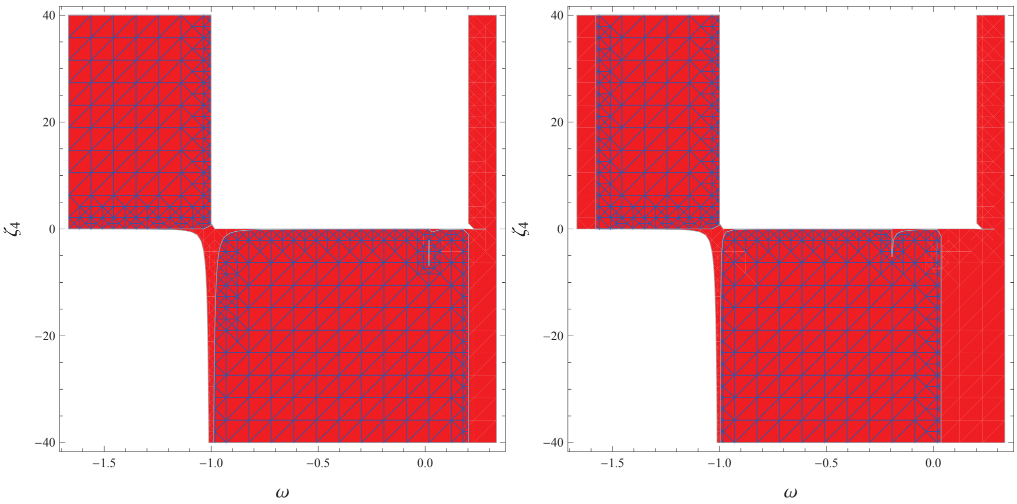

Figure 3: Regions of stability in (ω, τ) space for $k = 1,\ \lambda = 2$ with $b = 2$ (left) and $b = 5$ (right).

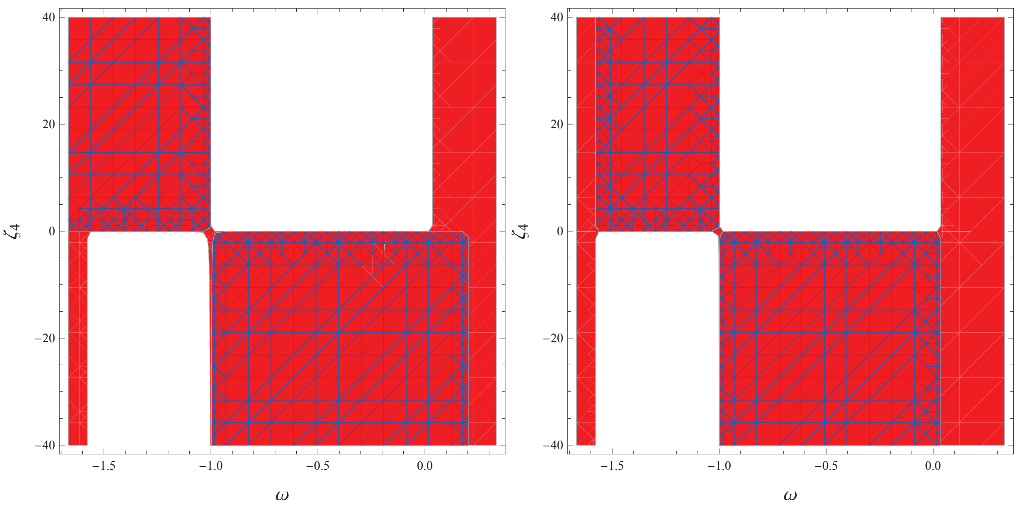

Figure 4: Regions of stability in (ω, τ) space for $k = 1,\ \lambda = 15$ with $b = 2$ (left) and $b = 5$ (right).

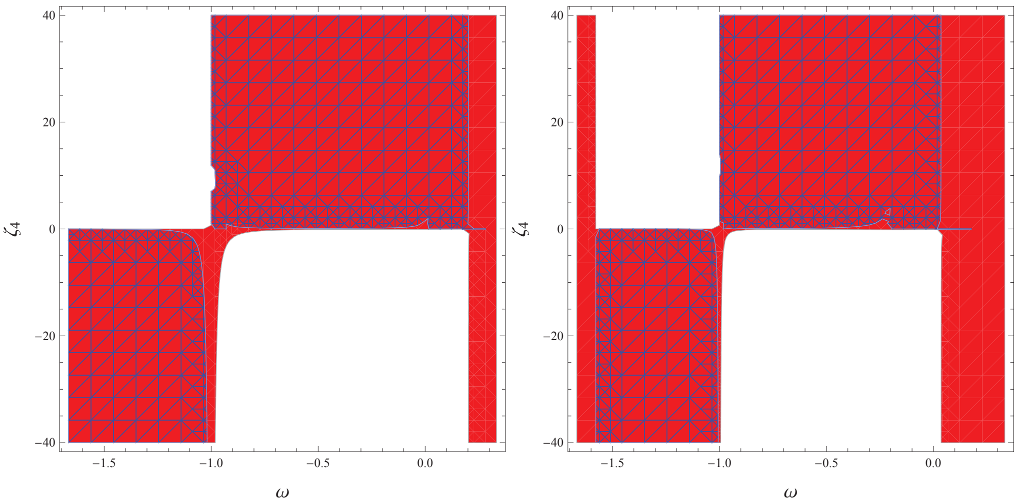

Figure 5: Regions of stability in (ω, τ) space for $k = -1,\ \lambda^2 = 2$ with $b = 2$ (left) and $b = 5$ (right).

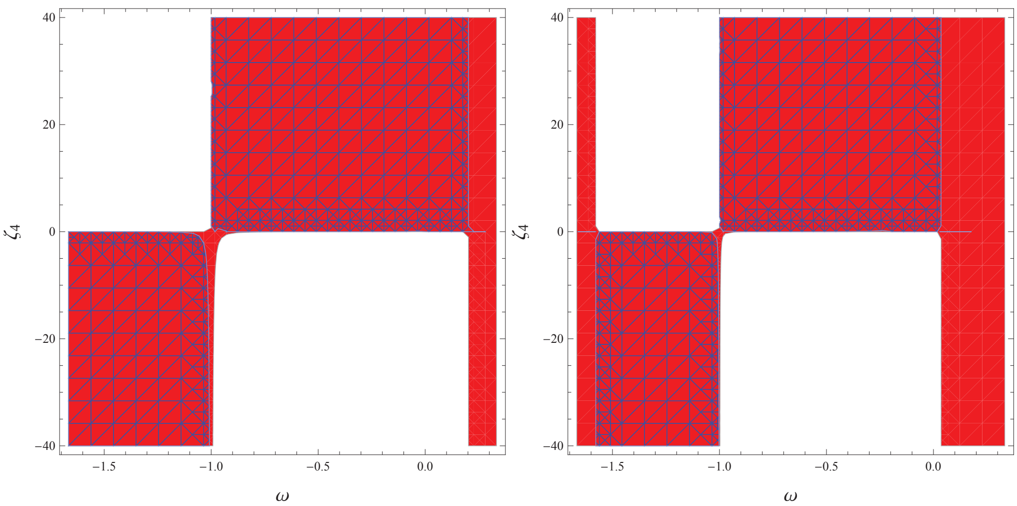

Figure 6: Regions of stability in (ω, τ) space for $k = -1,\ \lambda^2 = 15$ with $b = 2$ (left) and $b = 5$ (right).

$$\begin{aligned}
\Omega_5 &= -6k\rho_0(1+\omega)[\kappa(1+\varphi b[\rho_0(1-3\omega)]^{b-1})]\left[\frac{1}{2}\left(\kappa\left\{1-\frac{1}{2}\varphi b(\omega-3\right.\right.\right.\\
&- \left.\left.2(1+\omega)(b-1))[\rho_0(1-3\omega)]^{b-1}\right\}\right)-2\kappa\rho_0^2\hbar^2(1+\omega)^2[\kappa(1+\varphi b\\
&\times [\rho_0(1-3\omega)]^{b-1})]^2[1-3\omega+\varphi b\{2(3-5\omega)+(1+\omega)(b-1)\}\\
&\times \left.[\rho_0(1-3\omega)]^{b-1}]f_1''(\mathcal{G}_0)\right],\\
\Omega_6 &= 2k^2\left[\left\{3\rho_0(1+\omega)[\kappa(1+\varphi b[\rho_0(1-3\omega)]^{b-1})]+\hbar^2-4\rho_0^2\hbar^4(1+\omega)^2\right.\right.\\
&\times \left.[\kappa(1+\varphi b[\rho_0(1-3\omega)]^{b-1})]^2 f_1''(\mathcal{G}_0)\right\}\left[\kappa\left\{1-\frac{1}{2}\varphi b(\omega-3\right.\right.\\
&- \left.\left.2(1+\omega)(b-1))[\rho_0(1-3\omega)]^{b-1}\right\}\right]-\kappa\left\{\frac{3}{2}\rho_0(1+\omega)[\kappa(1+\varphi b\right.\\
&\times [\rho_0(1-3\omega)]^{b-1})]+\hbar^2-4\rho_0^2\hbar^4(1+\omega)^2[\kappa(1+\varphi b[\rho_0(1-3\omega)]^{b-1})]^2\\
&\times \left.\left.f_1''(\mathcal{G}_0)\right\}[1-3\omega+\varphi b\{2(3-5\omega)+(1+\omega)(b-1)\}[\rho_0(1-3\omega)]^{b-1}]\right].
\end{aligned}$$

Figures **3** and **4** show the stability regions of EU against inhomogeneous perturbations for frequencies $\zeta^2_{3,4}$. We have taken $\kappa=1,\ \rho_0=0.3,\ \varphi=1$ and $k=1$ with different values of $f(\mathcal{G},T)$ model parameter b and inhomogeneous perturbation mode λ. Red regions represent the existence of stable regimes for ζ^2_3 while the regions occupied by blue lines indicate ζ^2_4. It is observed that the region of stability for the whole system decreases as the value of b increases while stability region increases with the increase in λ. The graphical behavior of stable EU with $k=-1$ is shown in Figures **5** and **6**. The effects of b and λ on the stability plots are the same as for the closed universe.

4 Final Remarks

In this paper, we have investigated stability of EU against inhomogeneous scalar perturbations in the presence of perfect fluid in $f(\mathcal{G},T)$ gravity. This modified gravity is based on the CMC due to which the law of conservation does not hold. We have considered the generic FRW spacetime to construct static field equations in EU while perturbed field equations are formulated using perturbed metric in Newtonian gauge. The inhomogeneous scalar perturbations are introduced in energy density, pressure and metric coefficients. We have considered the specific form $f(\mathcal{G},T)=f_1(\mathcal{G})+f_2(T)$ and constructed the fourth-order perturbed differential equation whose solution provides four frequencies to analyze the stability of EU. The two specific models are considered for this particular $f(\mathcal{G},T)$ function for which the energy-momentum tensor is conserved as well as non-conserved. In these cases, we have found that stable EU exists against inhomogeneous perturbations for both closed and open cosmic geometries if the parameters are chosen appropriately.

Emergent universe scenario requires that EU must be stable against all kinds of perturbations. In our previous work [24], we have investigated the EU against homogeneous scalar perturbations and found the existence of stable EU for suitable choice of model parameters. Furthermore, EU against vector perturbations are stable on all scales for all equations of state since initial vector perturbations are frozen. Here, we have found that stable EU also exists against inhomogeneous scalar perturbations for appropriate chosen values of parameters. To discuss the emergent universe successfully in $f(\mathcal{G},T)$ gravity, it would be interesting to investigate EU against tensor as well as anisotropic perturbations. It is worth mentioning here that our results reduce to homogeneous perturbations in $f(\mathcal{G},T)$ gravity for $\lambda=0$ [24] as well as to inhomogeneous perturbations in $f(\mathcal{G})$ gravity for $f_2(T)=0$ [21].

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Stability analysis of some reconstructed cosmological models in $f(\mathcal{G},T)$ gravity

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ABSTRACT

The aim of this paper is to reconstruct and analyze the stability of some cosmological models against linear perturbations in $f(\mathcal{G},T)$ gravity ($\mathcal{G}$ and T represent the Gauss–Bonnet invariant and trace of the energy–momentum tensor, respectively). We formulate the field equations for both general as well as particular cases in the context of isotropic and homogeneous universe model. We reproduce the cosmic evolution corresponding to de Sitter universe, power-law solutions and phantom/non-phantom eras in this theory using reconstruction technique. Finally, we study stability analysis of de Sitter as well as power-law solutions through linear perturbations.

1. Introduction

Modified theories of gravity have attained much attention after the discovery of expanding accelerated universe. The basic ingredient responsible for this tremendous change in cosmic history is some mysterious type force having repulsive nature dubbed as dark energy. The enigmatic nature of this energy has motivated many researchers to unveil its hidden characteristics which are still not known. Modified gravity approach is considered as the promising and optimistic scenario among several other proposals that have been presented to explore the salient features of dark energy. These modified theories are established by adding or replacing curvature invariants and their corresponding generic functions in the Einstein–Hilbert action.

Lovelock theory of gravity is the direct generalization of general relativity (GR) in n-dimensions which coincides with GR in 4-dimensions [1]. The Ricci scalar (R) is known as first Lovelock scalar while Gauss–Bonnet (GB) invariant is the second Lovelock scalar yielding Einstein–Gauss–Bonnet gravity in 5-dimensions [2]. The GB invariant is a linear combination with an interesting feature that it is free from spin-2 ghost instabilities defined as [3]

$$\mathcal{G}=R^2-4R^{\alpha\beta}R_{\alpha\beta}+R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu},$$

where $R_{\alpha\beta}$ and $R_{\alpha\beta\mu\nu}$ are the Ricci and Riemann tensors, respectively. This quadratic curvature invariant is a topological term in 4-dimensions which possesses trivial contribution in the field equations. To discuss the dynamics of GB invariant in 4-dimensions, there are two interesting scenarios either to couple $\mathcal{G}$ with scalar field or to add generic function $f(\mathcal{G})$ in the Einstein–Hilbert action. The first scheme naturally appears in the effective action in string theory which investigates singularity-free cosmological solutions [4]. The second approach known as $f(\mathcal{G})$ gravity is introduced as an alternative for dark energy which successfully discusses the late-time cosmological evolution [5]. This modified theory of gravity is endowed with a quite rich cosmological structure as well as consistent with solar system constraints [6].

The current cosmic accelerated expansion has also been discussed in modified theories of gravity involving the curvature–matter coupling. Harko et al. [7] established $f(R,T)$ gravity to study the curvature–matter coupling. Recently, we introduced the curvature–matter coupling in $f(\mathcal{G})$ gravity named as $f(\mathcal{G},T)$ theory of gravity [8]. This coupling yields non-zero covariant divergence of the energy–momentum tensor and an extra force appears due to which massive test particles follow non-geodesic trajectories while geodesic lines of geometry are followed by the dust particles. Shamir and Ahmad [9] constructed some cosmologically viable models in $f(\mathcal{G},T)$ gravity using Noether symmetry approach. It is mentioned here that cosmic expansion can be obtained from geometric as well as matter components in such coupling.

The reconstruction as well as stability of cosmic evolutionary models in modified theories of gravity are the captivating issues in cosmology. In reconstruction technique, any known cosmic solution is used in the modified field equations to find the corresponding function which reproduces the given evolutionary cosmic history. In stability analysis, the isotropic and homogeneous perturbations are usually considered in which Hubble parameter as well as energy density are perturbed to examine the background stability as time evolves [10]. Nojiri et al. [11] formulated the

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reconstruction scheme to reproduce some cosmological models in $f(R)$ gravity. Elizalde et al. [12] applied the same scenario for ΛCDM cosmology (Λ denotes cosmological constant while CDM stands for cold dark matter) in $f(R,\mathcal{G})$ gravity as well as in modified GB theories of gravity. The stability of power-law solutions are also discussed in modified gravity theories [13].

Sáez-Gómez [14] explored the cosmological solutions in $f(R)$ Hořava–Lifshitz gravity and analyzed their stability against first order perturbations around FRW universe. Myrzakulov and his collaborators [15] discussed the cosmological models and found that $f(\mathcal{G})$ gravity could successfully explain the cosmic evolutionary history. Jamil et al. [16] reconstructed the cosmological models in $f(R,T)$ gravity and found that numerical analysis for Hubble parameter is in good agreement with observational data for redshift parameter <2. The stability of de Sitter, power-law solutions as well as ΛCDM are analyzed in the context of $f(R,\mathcal{G})$ gravity [17]. Salako et al. [18] studied the cosmological reconstruction, stability as well as thermodynamics including first and second laws for ΛCDM model in generalized teleparallel theory of gravity. Sharif and Zubair [19] demonstrated that $f(R,T)$ gravity can reproduce ΛCDM model, phantom or non-phantom eras, de Sitter universe and power-law cosmic history. They also analyzed the stability of reconstructed de Sitter as well as power-law solutions.

In this paper, we reconstruct various cosmological models including de Sitter universe, power-law solutions and phantom/non-phantom eras in $f(\mathcal{G},T)$ theory. We also analyze the stability against linear homogeneous perturbations for de Sitter as well as power-law solutions. The paper has the following format. In Section 2, we formulate the modified field equations while Section 3 is devoted to reconstruct some known cosmological solutions in this gravity. Section 4 analyzes the stability of specific solutions against linear perturbations around FRW universe model. The results are summarized in the last section.

2. $f(\mathcal{G},T)$ gravity

The action for $f(\mathcal{G},T)$ gravity is defined as [8]

$$\mathcal{I}=\int\left(\frac{R+f(\mathcal{G},T)}{2\kappa^2}+\mathcal{L}_m\right)\sqrt{-g}d^4x, \tag{1}$$

where κ^2, g and $\mathcal{L}_m$ represent coupling constant, determinant of the metric tensor ($g_{\alpha\beta}$) and Lagrangian associated with matter distribution, respectively. Varying Eq. (1) with respect to $g_{\alpha\beta}$, we obtain the field equations

$$\begin{aligned}&\kappa^2T_{\alpha\beta}-R_{\alpha\beta}+\frac{1}{2}g_{\alpha\beta}R+\frac{1}{2}g_{\alpha\beta}f(\mathcal{G},T)-(T_{\alpha\beta}+\Theta_{\alpha\beta})f_T(\mathcal{G},T)\\&-[2RR_{\alpha\beta}-4R^{\mu}_{\alpha}R_{\mu\beta}-4R_{\alpha\mu\beta\nu}R^{\mu\nu}+2R^{\mu\nu\xi}_{\alpha}R_{\beta\mu\nu\xi}]f_{\mathcal{G}}(\mathcal{G},T)\\&-[2Rg_{\alpha\beta}\Box-4R_{\alpha\beta}\Box-2R\nabla_\alpha\nabla_\beta+4R^{\mu}_{\beta}\nabla_\alpha\nabla_\mu+4R^{\mu}_{\alpha}\nabla_\beta\nabla_\mu\\&-4g_{\alpha\beta}R^{\mu\nu}\nabla_\mu\nabla_\nu+4R_{\alpha\mu\beta\nu}\nabla^\mu\nabla^\nu]f_{\mathcal{G}}(\mathcal{G},T)=0,\end{aligned} \tag{2}$$

where $f_T(\mathcal{G},T)=\partial f(\mathcal{G},T)/\partial T$, $f_{\mathcal{G}}(\mathcal{G},T)=\partial f(\mathcal{G},T)/\partial\mathcal{G}$, $\Box=\nabla_\alpha\nabla^\alpha$ (∇_α denotes a covariant derivative) and $T_{\alpha\beta}$ is the energy–momentum tensor. The expressions for $T_{\alpha\beta}$ and $\Theta_{\alpha\beta}$ are [20]

$$T_{\alpha\beta}=g_{\alpha\beta}\mathcal{L}_m-2\frac{\partial\mathcal{L}_m}{\partial g^{\alpha\beta}},$$

$$\Theta_{\alpha\beta}=-2T_{\alpha\beta}+g_{\alpha\beta}\mathcal{L}_m-2g^{\mu\nu}\frac{\partial^2\mathcal{L}_m}{\partial g^{\alpha\beta}\partial g^{\mu\nu}},$$

where we have assumed that $\mathcal{L}_m$ depends only on $g_{\alpha\beta}$ rather than its derivatives. The non-zero divergence of $T_{\alpha\beta}$ is given by

$$\begin{aligned}\nabla^\alpha T_{\alpha\beta}=&\frac{1}{\kappa^2-f_T(\mathcal{G},T)}\left[\left(\nabla^\alpha\Theta_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}\nabla^\alpha T\right)f_T(\mathcal{G},T)\right.\\&\left.+(\Theta_{\alpha\beta}+T_{\alpha\beta})\nabla^\alpha f_T(\mathcal{G},T)\right].\end{aligned} \tag{3}$$

The above equations indicate that the complete dynamics of $f(\mathcal{G},T)$ gravity is based on the suitable choice of $\mathcal{L}_m$.

The energy–momentum tensor for perfect fluid is

$$T_{\alpha\beta}=(\rho+p)u_\alpha u_\beta-pg_{\alpha\beta}, \tag{4}$$

where u_α, ρ and p represent the four velocity, energy density and pressure of matter distribution, respectively. In this case, the expression for $\Theta_{\alpha\beta}$ becomes

$$\Theta_{\alpha\beta}=-pg_{\alpha\beta}-2T_{\alpha\beta}, \tag{5}$$

where $\mathcal{L}_m=-p$. The line element for FRW universe model is given by

$$ds^2=dt^2-a^2(t)(dx^2+dy^2+dz^2), \tag{6}$$

where $a(t)$ is the scale factor. Using Eqs. (4)–(6) in (2), we obtain the corresponding field equation as follows

$$\begin{aligned}3H^2=&\kappa^2\rho+\frac{1}{2}f(\mathcal{G},T)+(\rho+p)f_T(\mathcal{G},T)\\&-12H^2(H^2+\dot{H})f_{\mathcal{G}}(\mathcal{G},T)+12H^3\dot{f}_{\mathcal{G}}(\mathcal{G},T),\end{aligned} \tag{7}$$

where $H=\frac{\dot{a}}{a}$, $T=\rho-3p$, $\mathcal{G}=24H^2(\dot{H}+H^2)$ and dot represents derivative with respect to time. The non-zero continuity equation (3) takes the form

$$\begin{aligned}\dot{\rho}+3H(\rho+p)=&\frac{-1}{\kappa^2+f_T(\mathcal{G},T)}\\&\times\left[\left(\dot{p}+\frac{1}{2}\dot{T}\right)f_T(\mathcal{G},T)+(\rho+p)\dot{f}_T(\mathcal{G},T)\right].\end{aligned} \tag{8}$$

The standard conservation law holds if right hand side of this equation vanishes. For equation of state $p=\omega\rho$ (ω is the equation of state parameter), Eq. (8) yields

$$\dot{\rho}=-3H(1+\omega)\rho, \tag{9}$$

with additional constraint

$$\frac{1}{2}\dot{\rho}(1-\omega)f_T+\rho(1+\omega)\left(\dot{\mathcal{G}}f_{\mathcal{G}T}+\dot{T}f_{TT}\right)=0. \tag{10}$$

We rewrite the above equations in terms of new variable $\mathcal{N}$ known as e-folding instead of t which is also related with redshift parameter (z) as [11]

$$\mathcal{N}=-\ln(1+z)=\ln(a/a_0).$$

Using the above definition of $\mathcal{N}$, Eqs. (7) and (8) become

$$\begin{aligned}3H^2=&\kappa^2\rho+\frac{1}{2}f+\rho(1+\omega)f_T-12H^3(H+H')f_{\mathcal{G}}+288H^6\\&\times(HH''+3H'^2+4HH')f_{\mathcal{G}\mathcal{G}}+12H^4T'f_{\mathcal{G}T},\end{aligned} \tag{11}$$

$$\begin{aligned}&\rho'+3(1+\omega)\rho=\frac{-1}{\kappa^2+f_T}\\&\times\left[\left(\omega\rho'+\frac{1}{2}T'\right)f_T+\rho(1+\omega)\left(\mathcal{G}'f_{\mathcal{G}T}+T'f_{TT}\right)\right],\end{aligned}$$

where $H=d\mathcal{N}/dt$, $d/dt=H(d/d\mathcal{N})$ and prime denotes derivative with respect to $\mathcal{N}$. The simplest choice of $f(\mathcal{G},T)$ model is

$$f(\mathcal{G},T)=F(\mathcal{G})+\mathcal{F}(T), \tag{12}$$

which possesses no direct non-minimally coupling between curvature and matter. For this particular model, the field equation (11) splits into a set of two ordinary differential equations as

$$\begin{aligned}&288H^6(HH''+3H'^2+4HH')F_{\mathcal{G}\mathcal{G}}-12H^3(H+H')F_{\mathcal{G}}\\&+\frac{1}{2}F(\mathcal{G})-3H^2=0,\end{aligned}$$

$$\rho(1+\omega)\mathcal{F}_T + \frac{1}{2}\mathcal{F}(T) + \kappa^2\rho = 0,$$

where $F_{\mathcal{G}} = dF(\mathcal{G})/d\mathcal{G}$ and $\mathcal{F}_T = d\mathcal{F}(T)/dT$. The field equations for perfect fluid matter distribution in $f(\mathcal{G})$ gravity is recovered if $\mathcal{F}(T)$ vanishes while GR is achieved for $f(\mathcal{G}, T) = 0$.

3. Cosmological reconstruction

In this section, we reproduce different cosmological scenarios including de Sitter universe, power-law solutions and phantom/non-phantom eras in $f(\mathcal{G}, T)$ gravity.

3.1. de Sitter Universe

The de Sitter cosmic evolution is interesting and well-known as it elegantly describes current expansion of the universe. This solution is considered as the universe in which the energy density of matter and radiation is negligible as compared to vacuum energy (energy density for DE dominated era) and thus the universe expands forever at a constant rate. The scale factor of this evolutionary model grows exponentially with constant Hubble parameter $H(t) = H_0$, defined as [17]

$$a(t) = a_0 e^{H_0 t}, \tag{13}$$

where a_0 is an integration constant. Eq. (9) gives energy density of the form

$$\rho = \rho_0 e^{-3(1+\omega)H_0 t}, \tag{14}$$

where $\omega \neq -1$ and ρ_0 is a constant. Using Eqs. (13) and (14) in (7), we obtain

$$\frac{1}{2}f(\mathcal{G}_0, T) - 12H_0^4 f_{\mathcal{G}}(\mathcal{G}_0, T) + \left(\frac{1+\omega}{1-3\omega}\right)Tf_T(\mathcal{G}_0, T) - 36(1+\omega)H_0^4 Tf_{\mathcal{G}T}(\mathcal{G}_0, T) + \frac{\kappa^2 T}{1-3\omega} - 3H_0^2 = 0, \tag{15}$$

where $\mathcal{G}_0 = 24H_0^4$ is the GB invariant at $H(t) = H_0$. The solution of the above differential equation is

$$f(\mathcal{G}, T) = c_1c_2 e^{c_1\mathcal{G}} T^{-\frac{1}{2}\left(\frac{(1-24c_1H_0^4)(1-3\omega)}{1+\omega-36c_1H_0^4(1-3\omega)}\right)} + c_1c_2 T^{-\frac{1}{2}}\left(\frac{1-3\omega}{1+\omega}\right) - \frac{2\kappa^2}{3-\omega}T + 6H_0^2, \tag{16}$$

where c_i's $(i = 1, 2)$ are integration constants. Since we have used the continuity equation (9) in Eq. (15), so we must constrain its solution. Using the above equation with Eq. (10), we obtain the following functions

$$f_1(\mathcal{G}, T) = c_1c_2\Xi_1 e^{c_1\mathcal{G}} T^{-\frac{1}{2}\left(\frac{(1-24c_1H_0^4)(1-3\omega)}{1+\omega-36c_1H_0^4(1-3\omega)}\right)} + \frac{2\kappa^2\omega}{1-3\omega}T + 6H_0^2, \tag{17}$$

$$f_2(\mathcal{G}, T) = c_1c_2\Xi_2 T^{-\frac{1}{2}}\left(\frac{1-3\omega}{1+\omega}\right) + \frac{2\kappa^2}{3-\omega}\Xi_3 T + 6H_0^2, \tag{18}$$

where Ξ_j's $(j = 1, 2, 3)$ are constants in terms of ω and H_0 given in Appendix. For the model (12), we have

$$3H_0^2 - \frac{1}{2}F + 12H_0^4 F_{\mathcal{G}} = 0, \quad \kappa^2\rho + \frac{1}{2}\mathcal{F} + (1+\omega)\rho\mathcal{F}_T = 0, \tag{19}$$

where the first equation corresponds to de Sitter universe in the absence of matter contents in $f(\mathcal{G})$ gravity [6]. Using the constraint (10), the second equation becomes

$$\kappa^2(1-\omega)T + \frac{1}{2}(1-3\omega)(1-\omega)\mathcal{F} - 2(1+\omega)^2T^2\mathcal{F}_{TT} = 0. \tag{20}$$

The solution of Eqs. (19) and (20) leads to

$$f(\mathcal{G}, T) = \hat{c}_1 e^{\frac{\mathcal{G}}{24H_0^4}} + \hat{c}_2 T^{\frac{1}{2}\left(1+\frac{\sqrt{2(1-\omega+2\omega^2)}}{1+\omega}\right)} + \hat{c}_3 T^{\frac{1}{2}\left(1-\frac{\sqrt{2(1-\omega+2\omega^2)}}{1+\omega}\right)} - \frac{2\kappa^2 T}{1-3\omega} + 6H_0^2, \tag{21}$$

where $\hat{c}_j$'s are constants of integration. Eqs. (16) and (21) indicate that de Sitter expansion can also be described in $f(\mathcal{G}, T)$ gravity.

3.2. Power-law solutions

Power-law solutions have significant importance to discuss different evolutionary phases of the universe in modified theory. These solutions describe the decelerated as well as accelerated cosmic eras which are characterized by the scale factor as [17]

$$a(t) = a_0 t^{\lambda}, \quad H = \frac{\lambda}{t}, \quad \lambda > 0. \tag{22}$$

The cosmic decelerated phase is observed for $0 < \lambda < 1$ including the radiation $(\lambda = \frac{1}{2})$ as well as dust $(\lambda = \frac{2}{3})$ dominated eras while $\lambda > 1$ covers the accelerated phase of the universe. For this scale factor, the GB invariant takes the form

$$\mathcal{G} = 24\frac{\lambda^3}{t^4}(\lambda - 1). \tag{23}$$

Using Eqs. (9), (22) and (23), the field equation becomes

$$\frac{1}{2}f - \frac{1}{2}\mathcal{G}f_{\mathcal{G}} + \frac{(1+\omega)T}{1-3\omega}f_T - \frac{2}{\lambda-1}\mathcal{G}^2 f_{\mathcal{G}\mathcal{G}} - \frac{2\lambda(1+\omega)\mathcal{G}T}{2(\lambda-1)}f_{\mathcal{G}T} - 3\lambda^2\left(\frac{T}{\rho_0(1-3\omega)}\right)^{\frac{2}{3\lambda(1+\omega)}} + \frac{\kappa^2 T}{1-3\omega} = 0, \tag{24}$$

whose solution is given by

$$f(\mathcal{G}, T) = \tilde{c}_1\tilde{c}_3 T^{\tilde{c}_2}\mathcal{G}^{\frac{1}{4}(\gamma_1+\gamma_2)} + \tilde{c}_2\tilde{c}_3 T^{\tilde{c}_2}\mathcal{G}^{\frac{1}{4}(\gamma_1-\gamma_2)} + \tilde{c}_1\tilde{c}_2 T^{\gamma_3} + \gamma_4 T + \gamma_5 T^{\gamma_6}, \tag{25}$$

where $\tilde{c}_j$'s are integration constants and $\gamma_{\hat{j}}$'s $(\hat{j} = 1...6)$ are given in Appendix. Inserting Eq. (25) in (10), we obtain

$$f_1(\mathcal{G}, T) = \tilde{c}_1\tilde{c}_3\Delta_1 T^{\tilde{c}_2}\mathcal{G}^{\frac{1}{4}(\gamma_1+\gamma_2)} + \tilde{c}_1\tilde{c}_2\Delta_2 T^{\gamma_3} + \Delta_3 T + \Delta_4 T^{\gamma_6}, \tag{26}$$

$$f_2(\mathcal{G}, T) = \tilde{c}_2\tilde{c}_3\Omega_1 T^{\tilde{c}_2}\mathcal{G}^{\frac{1}{4}(\gamma_1-\gamma_2)} + \tilde{c}_1\tilde{c}_2\Omega_2 T^{\gamma_3} + \Omega_3 T + \Omega_4 T^{\gamma_6} \tag{27}$$

where Δ_k's and Ω_k's $(k = 1...4)$ are given in Appendix.

Now we find the expression of $f(\mathcal{G}, T)$ for the choice of model (12). The differential equation (24) yields two ordinary differential equations in variables $\mathcal{G}$ and T given by

$$F - \mathcal{G}F_{\mathcal{G}} - \frac{4}{\lambda-1}\mathcal{G}^2 F_{\mathcal{G}\mathcal{G}} = 0,$$

$$\mathcal{F} - \frac{4(1+\omega)^2}{(1-\omega)(1-3\omega)}T^2\mathcal{F}_{TT} + \frac{2\kappa^2 T}{1-3\omega} - 6\lambda^2\left(\frac{T}{\rho_0(1-3\omega)}\right)^{\frac{2}{3\lambda(1+\omega)}} = 0.$$

The solution of these equations provide $f(\mathcal{G}, T)$ model as

$$f(\mathcal{G}, T) = \bar{c}_1\mathcal{G} + \bar{c}_2\mathcal{G}^{\frac{1-\lambda}{4}} + \bar{c}_3 T^{\frac{1}{2}\left(1+\frac{\sqrt{2(1-\omega+2\omega^2)}}{1+\omega}\right)} + \bar{c}_4 T^{\frac{1}{2}\left(1-\frac{\sqrt{2(1-\omega+2\omega^2)}}{1+\omega}\right)} - \frac{2\kappa^2 T}{1-3\omega} + \frac{54\lambda^4(1-\omega)(1-3\omega)}{9\lambda^2(1-\omega)(1-3\omega) - 8[2-3\lambda(1+\omega)]} \times \left(\frac{T}{\rho_0(1-3\omega)}\right)^{\frac{2}{3\lambda(1+\omega)}}, \tag{28}$$

where $\bar{c}_k$'s are integration constants. Thus, the power-law solutions are reconstructed which may be helpful to explore the expansion history of the universe in this modified theory of gravity.

3.3. Phantom and non-phantom matter fluids

Here, we reconstruct $f(\mathcal{G}, T)$ model which can explain the system including both phantom and non-phantom eras. In the Einstein gravity, the Hubble parameter describing the phantom as well as non-phantom matter distribution is given by [11]

$$H^2 = \frac{\kappa^2}{3}(\rho_p a^b + \rho_q a^{-b}), \tag{29}$$

where b, ρ_p and ρ_q represent the model parameter, energy densities of phantom and non-phantom matter fluids, respectively. The violation of all four energy conditions leads to phantom phase and the energy density grows while it decreases in a non-phantom regime. The phantom energy density would become infinite in finite time, causing a huge gravitational repulsion that the universe would lose all structure and end in a big-rip singularity [21]. When the scale factor is large, the first term on right hand side dominates which corresponds to the phantom era of the universe with $\omega = -1 - b/3 < -1$. The non-phantom era in the early universe is observed for $\omega = -1 + b/3 > -1$ when the scale factor is small and the second term dominates. We rewrite $H(t)$ in terms of a new function $S(\mathcal{N})$ as $H^2 = S(\mathcal{N})$ so that Eq. (29) becomes

$$S(\mathcal{N}) = S_p e^{b\mathcal{N}} + S_q e^{-b\mathcal{N}}, \tag{30}$$

where $S_p = \frac{\kappa^2}{3}\rho_p a_0^b$ and $S_q = \frac{\kappa^2}{3}\rho_q a_0^{-b}$. The GB invariant takes the form

$$\mathcal{G} = 24S^2(\mathcal{N}) + 12S(\mathcal{N})S'(\mathcal{N}). \tag{31}$$

Inserting Eq. (30) in (31), we obtain a quadratic equation in $e^{2b\mathcal{N}}$ whose solution is given by

$$e^{2b\mathcal{N}} = \frac{-(48S_pS_q - \mathcal{G}) \pm \sqrt{(48S_pS_q - \mathcal{G})^2 - 576(4 - b^2)S_p^2S_q^2}}{24(2+b)S_p^2}, \quad b \neq 2.$$

For the sake of simplicity, we consider $b = 2$ so that it reduces to

$$e^{2b\mathcal{N}} = \frac{\mathcal{G} - 48S_pS_q}{48S_p^2}. \tag{32}$$

Using Eqs. (30) and (32) in (7), we have

$$\frac{1}{2}f - \frac{1}{2}\mathcal{G}f_{\mathcal{G}} + \left(\frac{1+\omega}{1-3\omega}\right)Tf_T + \mathcal{G}^2 f_{\mathcal{G}\mathcal{G}} - \frac{3(1+\omega)\mathcal{G}^2 T}{4(\mathcal{G} - 48S_pS_q)}f_{\mathcal{G}T} - \frac{1}{4}\sqrt{\frac{3\mathcal{G}^2}{\mathcal{G} - 48S_pS_q}} + \frac{\kappa^2 T}{1-3\omega} = 0,$$

which is a complicated partial differential equation whose analytical solution cannot be found.

To find the reconstructed $f(\mathcal{G}, T)$ model, we consider its particular form (12) which provides the following set of differential equations

$$\frac{1}{2}F - \frac{1}{2}\mathcal{G}F_{\mathcal{G}} + \mathcal{G}^2 F_{\mathcal{G}\mathcal{G}} - \frac{1}{4}\sqrt{\frac{3\mathcal{G}^2}{\mathcal{G} - 48S_pS_q}} = 0,$$

$$\mathcal{F} - \frac{4(1+\omega)^2}{(1-\omega)(1-3\omega)}T^2\mathcal{F}_{TT} + \frac{2\kappa^2 T}{1-3\omega} = 0,$$

where we have used the additional constraint in the second equation. Solving these equations, it follows that

$$\begin{aligned} f(\mathcal{G}, T) &= d_1\mathcal{G}^{\frac{1}{2}} + d_2\mathcal{G} + d_3 T^{\frac{1}{2}\left(1+\frac{\sqrt{2(1-\omega+2\omega^2)}}{1+\omega}\right)} + d_4 T^{\frac{1}{2}\left(1-\frac{\sqrt{2(1-\omega+2\omega^2)}}{1+\omega}\right)} \\ &+ \frac{1}{4\sqrt{S_pS_q}}\left[\mathcal{G}\tan^{-1}\left(\frac{1}{12}\sqrt{\frac{3(\mathcal{G} - S_pS_q)}{S_pS_q}}\right) - 2\sqrt{3S_pS_q\mathcal{G}} \right. \\ &\left. \times \ln\left(\mathcal{G} - 24S_pS_q + \sqrt{\mathcal{G}(\mathcal{G} - 48S_pS_q)}\right)\right] - \frac{2\kappa^2 T}{1-3\omega} \end{aligned} \tag{33}$$

where d_k's are constants of integration. Thus, phantom and non-phantom cosmic history can be discussed in $f(\mathcal{G}, T)$ gravity.

4. Perturbations and stability of cosmological solutions

In this section, we analyze stability of some cosmological evolutionary solutions about linear homogeneous perturbations in this modified gravity. We construct the perturbed field as well as continuity equations using isotropic and homogeneous universe model for both general and particular cases including de Sitter and power-law solutions. We assume a general solution

$$H(t) = H_*(t), \tag{34}$$

which satisfies the basic field equations for FRW universe model in $f(\mathcal{G}, T)$ gravity. In terms of the above solution, the expressions for $\mathcal{G}_*$ and T_* are

$$\mathcal{G}_* = 24H_*^2(H_*^2 + \dot{H}_*) = 24H_*^3(H_* + H_*'), \quad T_* = \rho_*(t)(1-3\omega).$$

For any particular $f(\mathcal{G}, T)$ model that can regenerate the above solution (34), the following equation of motion as well as non-zero divergence of the energy–momentum tensor must be satisfied

$$\begin{aligned} 3H_*^2 &= \kappa^2\rho_* + (1+\omega)\rho_* f_T^* + \frac{1}{2}f^* - 12H_*^3(H_* + H_*')f_{\mathcal{G}}^* + 288 \\ &\times H_*^6(H_*H_*'' + 3H_*'^2 + 4H_*H_*')f_{\mathcal{G}\mathcal{G}}^* + 12H_*^4 T_*' f_{\mathcal{G}T}^*, \end{aligned}$$

$$\begin{aligned} \rho_*' + 3(1+\omega)\rho_* &= \frac{-1}{\kappa^2 + f_T^*}\left[\frac{1}{2}(T_*' + 2\omega\rho_*')f_T^* + (1+\omega)\rho_* \right. \\ &\left. \times \left(\mathcal{G}_*' f_{\mathcal{G}T}^* + T_*' f_{TT}^*\right)\right], \end{aligned}$$

where superscript $*$ denotes that the function and its corresponding derivatives are calculated at $\mathcal{G} = \mathcal{G}_*$ and $T = T_*$. If the conservation law holds, we get energy density in terms of $H_*(t)$ as

$$\rho_*(t) = \rho_0 e^{-3(1+\omega)\int H_*(t)dt}.$$

The first order perturbations in Hubble parameter and energy density are defined as

$$H(t) = H_*(t)(1 + \delta(t)), \quad \rho(t) = \rho_*(t)(1 + \delta_m(t)), \tag{35}$$

where $\delta(t)$ and $\delta_m(t)$ are the perturbation parameters.

In order to analyze first order perturbations about the solution (34), we apply the series expansion on the function $f(\mathcal{G}, T)$ as

$$f(\mathcal{G}, T) = f^* + f_{\mathcal{G}}^*(\mathcal{G} - \mathcal{G}_*) + f_T^*(T - T_*) + \mathcal{O}^2, \tag{36}$$

where $\mathcal{O}^2$ involves the terms proportional to quadratic or higher powers of $\mathcal{G}$ and T while only the linear terms are considered. Using Eqs. (35) and (36) in (7), we obtain the following perturbed field equation

$$\chi_1\ddot{\delta} + \chi_2\dot{\delta} + \chi_3\delta + \chi_4\dot{\delta}_m + \chi_5\delta_m = 0, \tag{37}$$

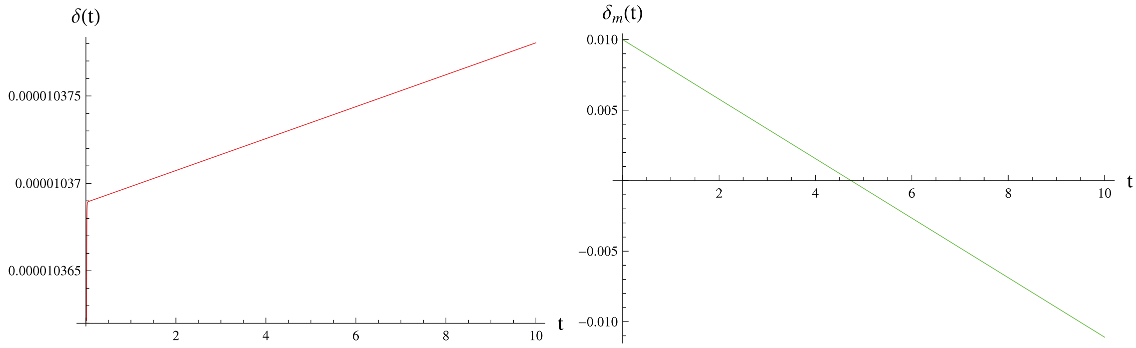

Fig. 1. Evolution of perturbations $\delta(t)$ and $\delta_m(t)$ for model (17) with $\omega = 0$.

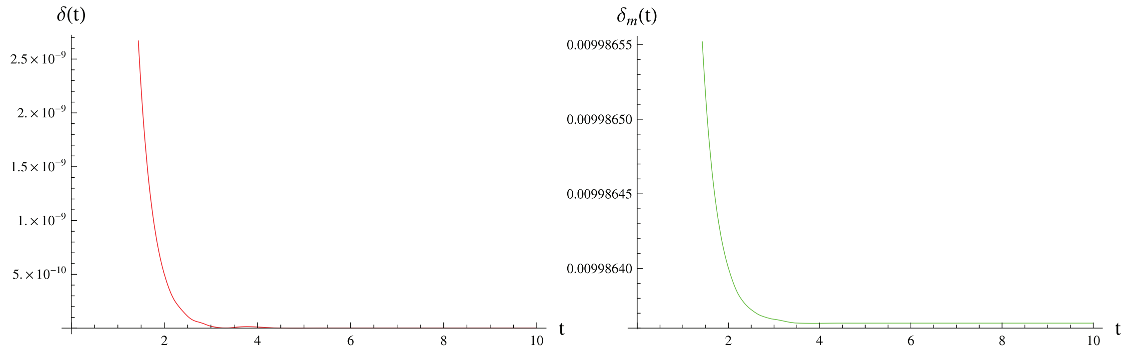

Fig. 2. Evolution of perturbations $\delta(t)$ and $\delta_m(t)$ for model (18) with $\omega = 0$.

where χ_h's $(h = 1\ldots5)$ are given in Appendix. Inserting these perturbations in Eq. (8), the perturbed continuity equation is

$$\Upsilon_1\delta + \Upsilon_2\dot{\delta} + \Upsilon_3\ddot{\delta} + \Upsilon_4\delta_m + \Upsilon_5\dot{\delta}_m = 0, \tag{38}$$

where Υ_h's are provided in Appendix. If the conversation law holds in this modified gravity, Eq. (38) reduces to

$$\dot{\delta}_m + 3(1+\omega)H_*\delta = 0. \tag{39}$$

The perturbed equations (37) and (38) are helpful to analyze the stability of any specific FRW cosmological evolutionary model in $f(\mathcal{G}, T)$ gravity. For the particular model (12), these perturbed equations reduce to

$$\hat{\chi}_1\ddot{\delta} + \hat{\chi}_2\dot{\delta} + \hat{\chi}_3\delta + \hat{\chi}_5\delta_m = 0,$$
$$\hat{\Upsilon}_1\delta + \hat{\Upsilon}_4\delta_m + \hat{\Upsilon}_5\dot{\delta}_m = 0,$$

where the coefficients of $(\delta,\ \delta_m)$ and their derivatives are expressed in Appendix. In the following subsections, we investigate the stability of de Sitter and power-law solutions.

4.1. Stability of de Sitter solutions

Consider the de Sitter solution $H_*(t) = H_0$, the perturbed equation (37) takes the form

$$\begin{aligned}
&288H_0^6 f^0_{\mathcal{G}\mathcal{G}}\ddot{\delta} + \big(864H_0^7 f^0_{\mathcal{G}\mathcal{G}} + 24\rho_* H_0^3(1+\omega) f^0_{\mathcal{G}T}\\
&- 864\rho_* H_0^7(1-3\omega)\\
&\times (1+\omega) f^0_{\mathcal{G}\mathcal{G}T}\big)\dot{\delta} + \big(-6H_0^2 - 1152H_0^8 f^0_{\mathcal{G}\mathcal{G}} + 12\rho_* H_0^2(1+\omega)\\
&\times [8H_0^2 - 9H_0^2(1-3\omega)] f^0_{\mathcal{G}T} - 3456\rho_* H_0^8(1-3\omega)\\
&\times (1+\omega) f^0_{\mathcal{G}\mathcal{G}T}\big)\delta + 12\rho_* H_0^3\\
&\times (1-3\omega) f^0_{\mathcal{G}T}\dot{\delta}_m + \bigg(\kappa^2\rho_* + \frac{1}{2}\rho_*(3-\omega) f^0_T\\
&+ \rho_*^2(1-3\omega)(1+\omega) f^0_{TT}\\
&- 12\rho_* H_0^2(1-3\omega)[H_0^2 + 3(1+\omega)H_0^2] f^0_{\mathcal{G}T}\\
&- 36\rho_*^2 H_0^4(1-3\omega)^2(1+\omega) f^0_{\mathcal{G}TT}\bigg)\delta_m = 0,
\end{aligned} \tag{40}$$

where the superscript 0 represents that the function and its corresponding derivatives are evaluated at $\mathcal{G}_0$ and T_0. We consider the conserved perturbed equation for stability analysis since the de Sitter solutions are constructed using the constraint (10) in the previous section. The numerical technique is used to solve Eqs. (39) and (40) for the model (17). The evolution of $\delta(t)$ and $\delta_m(t)$ are shown in Fig. 1. We consider $H_0 = 67.8$ and $\kappa = 1$ throughout the stability analysis of de Sitter universe models whereas integration constants are $c_1 = 1 \times 10^{-6}$ and $c_2 = -1 \times 10^{-3}$.

Fig. 1 shows smooth behavior of $\delta(t)$ (left) and $\delta_m(t)$ (right) which do not decay in late times indicating that de Sitter model (17) is unstable. The stability analysis of model (18) with same integration constants is shown in Fig. 2. In the left panel, it is observed that small oscillations are produced about $t = 4$ while it decays in late times, thus the model (18) shows stable behavior against perturbations. For model (12), Eq. (40) becomes

$$\begin{aligned}
&288H_0^6 \mathcal{F}^0_{\mathcal{G}\mathcal{G}}\ddot{\delta} + 864H_0^7 \mathcal{F}^0_{\mathcal{G}\mathcal{G}}\dot{\delta} + \big(-6H_0^2 - 1152H_0^8 \mathcal{F}^0_{\mathcal{G}\mathcal{G}}\big)\delta\\
&\quad + \bigg(\kappa^2\rho_* + \frac{1}{2}\rho_*(3-\omega)\mathcal{F}^0_T + \rho_*^2(1-3\omega)(1+\omega)\mathcal{F}^0_{TT}\bigg)\delta_m\\
&= 0.
\end{aligned} \tag{41}$$

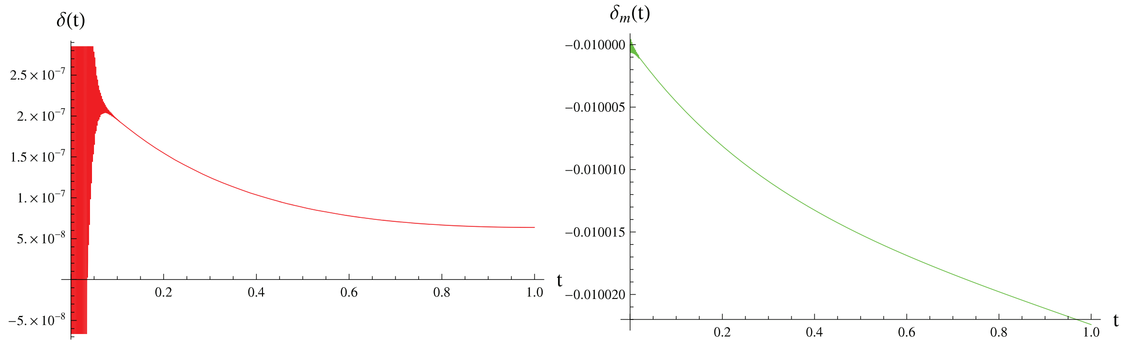

Fig. 3. Evolution of perturbations $\delta(t)$ and $\delta_m(t)$ for model (21) with $\omega = 0$.

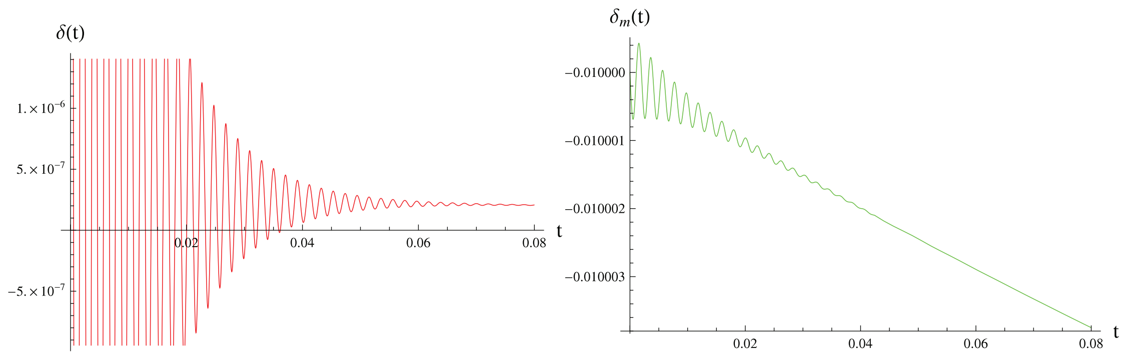

Fig. 4. Evolution of perturbations $\delta(t)$ and $\delta_m(t)$ for model (21) with $\omega = 0$.

Fig. 3 represents the behavior of $\delta(t)$ and $\delta_m(t)$ for model (21) with integration constants $\hat{c}_1 = -10$, $\hat{c}_2 = 0.001$ and $\hat{c}_3 = 1$. It is shown that oscillations in perturbation parameters are produced initially as shown in Fig. 3. This oscillating behavior is clearly observed in Fig. 4 which decays in future for both $\delta(t)$ as well as $\delta_m(t)$ and hence the solution becomes stable.

4.2. Stability of power-law solutions

Here we investigate the stability of power-law solutions. These solutions describe the accelerated as well as decelerated cosmological evolutionary phases in the background of FRW universe. We first consider the reconstructed power-law solution (26) and numerically solve Eqs. (37) and (39). For this model, we choose integration constants $\tilde{c}_1 = 10$, $\tilde{c}_2 = -0.5$ and $\tilde{c}_3 = -1000$. Fig. 5 shows the oscillating behavior of perturbed parameters $(\delta(t),\ \delta_m(t))$ for the cosmic accelerated era with $\omega = -0.5$ and $\lambda = 2$. The perturbations around the power-law solutions decay in future leading to stable results. The radiation ($\lambda = 1/2$ and $\omega = 1/3$) as well as matter ($\lambda = 2/3$ and $\omega = 0$) dominated eras cannot be discussed for the model (26) because singular as well as complex terms appear which lead to non-physical case.

Secondly, we consider the model (27) and analyze its behavior against linear perturbations. Fig. 6 shows the fluctuating behavior of considered perturbations in the cosmic accelerated phase with $\omega = -0.5$ and $\lambda = 2$. Here, we choose $\tilde{c}_1 = 0.001$, $\tilde{c}_2 = -0.61$ and $\tilde{c}_3 = -1000$. It is observed that the oscillating behavior disappears in future while both perturbation parameters will not decay in late times leading to unstable cosmological solutions. The considered model cannot explain the cosmological evolution corresponding to matter and radiation dominated eras like previous model (26).

Lastly, we explore the stability of model (28) with integration constants $\bar{c}_1 = -2$, $\bar{c}_2 = -0.6$, $\bar{c}_3 = 1000$ and $\bar{c}_4 = 0.01$. Fig. 7 represents the evolution of $(\delta,\ \delta_m)$ versus time for $\lambda = 1.1$ with $\omega = -0.5$. The left panel shows that the oscillations of $\delta(t)$ decay in late times while fluctuations of $\delta_m(t)$ remain present in future. Since a complete perturbation against any cosmological solution includes the matter perturbations therefore, the solutions are unstable.

5. Concluding remarks

In this paper, we have employed the reconstruction scheme to $f(\mathcal{G}, T)$ gravity in the background of isotropic and homogeneous universe model to reproduce some important cosmological models. The basic aspect of this modified gravity is the coupling between curvature and matter components which yields non-zero divergence of the energy–momentum tensor. We have imposed additional constraint to obtain the standard conservation equation which has been used to explain the cosmic evolution in this gravity.

The de Sitter and power-law solutions have been reconstructed for general as well as particular cases which are of great interest and have significant importance in cosmology. We have also reconstructed the $f(\mathcal{G}, T)$ model which can explain cosmic history of the phantom as well as non-phantom phases of the universe. Similar reconstruction technique is carried out for ΛCDM model and found that this gravity fails to reproduce it for both general as well as particular $f(\mathcal{G}, T)$ forms. The results are summarized in Table 1. In

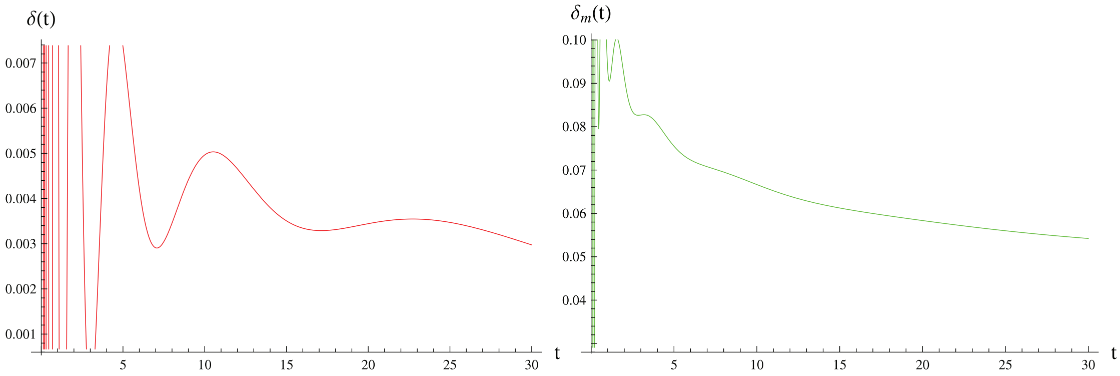

Fig. 5. Evolution of perturbations $\delta(t)$ and $\delta_m(t)$ for model (26) with $\omega = -0.5$ and $\lambda = 2$.

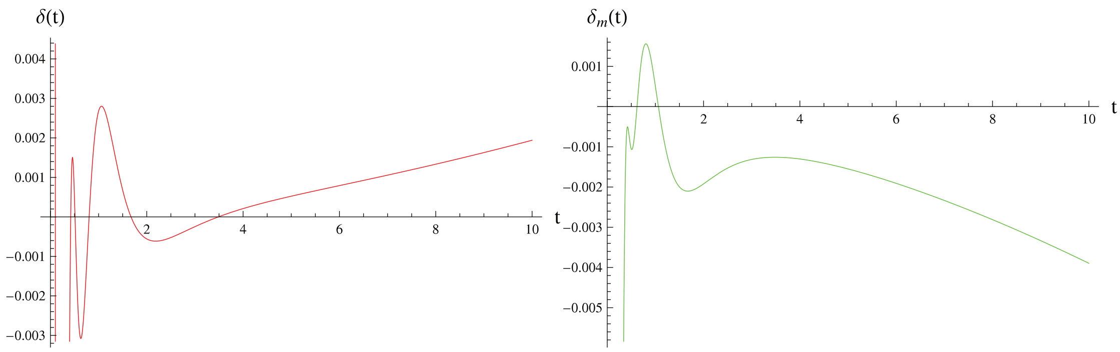

Fig. 6. Evolution of perturbations $\delta(t)$ and $\delta_m(t)$ for model (27) with $\omega = -0.5$ and $\lambda = 2$.

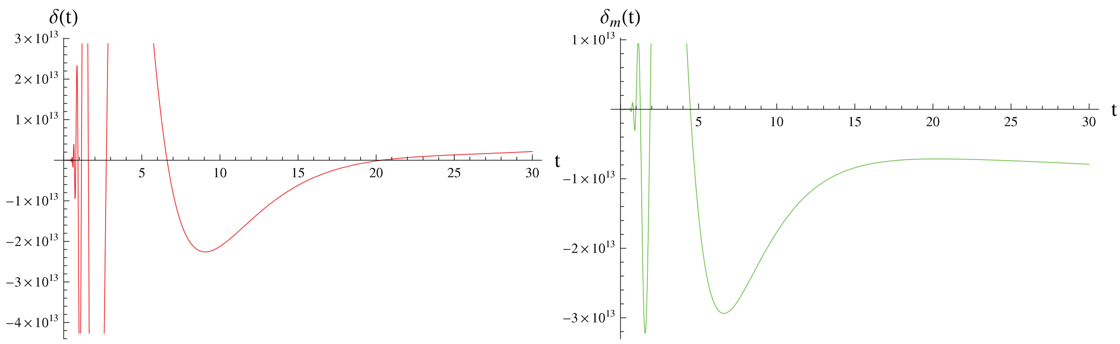

Fig. 7. Evolution of perturbations $\delta(t)$ and $\delta_m(t)$ for model (28) with $\omega = -0.5$ and $\lambda = 1.1$.

Table 1
Cosmological evolution in $f(\mathcal{G}, T)$ gravity.

Cosmological backgrounds	General $f(\mathcal{G}, T)$ model	Particular $f(\mathcal{G}, T)$ model
de Sitter universe	✓	✓
Power-law solutions	✓	✓
Phantom/non-phantom eras	×	✓

this table, ✓ and × represent that $f(\mathcal{G}, T)$ gravity reproduces and fails to reproduce the corresponding cosmological backgrounds, respectively.

On physical grounds, the stability analysis of different forms of generic function leads to classify the modified theories of gravity. We have applied the first order perturbations to Hubble parameter and energy density to analyze the stability of models which reproduce de Sitter and power-law cosmic history. We have perturbed the field equation as well as conservation law whose numerical solutions provide the stable/unstable results.

- For the de Sitter universe, the evolution of perturbation has been plotted against time as shown in Figs. 1–4. These indicate that models (18) and (21) are stable against linear perturbations.
- For the power-law universe, the stability analysis is given in Figs. 5–7. It is found that $f(\mathcal{G},T)$ gravity fails to reproduce matter and radiation dominated eras while stable results are obtained for accelerated phase of the universe for model (26).

We conclude that the cosmological reconstruction and stability analysis might restrict $f(\mathcal{G},T)$ gravity in the background of FRW universe. It would be interesting to discuss ghost instabilities due to the presence of curvature–matter coupling.

Appendix

The expressions for Ξ_i's in Eqs. (17) and (18) are

$$\begin{aligned}\Xi_1 &= 18c_1H_0^4[8c_1H_0^4\{2(2+59\omega^2)-11\omega(5-3\omega^2)\}\\ &\quad-(1-11\omega)(1-\omega^2)]\\ &\quad\times[1+\omega-36c_1H_0^4(1-3\omega)]^{-2},\\ \Xi_2 &= 18c_1H_0^4[(1-11\omega)(1-\omega^2)-8c_1H_0^4\{2(2+59\omega^2)\\ &\quad-11\omega(5-3\omega^2)\}]\\ &\quad\times[(1+\omega)(1-24c_1H_0^4)\{1+\omega\\ &\quad-6c_1H_0^4(5-4\omega-33\omega^2)\}]^{-1},\\ \Xi_3 &= -[18c_1H_0^4(1-32c_1H_0^4)-3\omega\{1-6c_1H_0^4(3-352c_1H_0^4)\}\\ &\quad-2\omega^2\{1-9c_1H_0^4(7-1248c_1H_0^4)\}\\ &\quad+\omega^3\{1-54c_1H_0^4(7-480c_1H_0^4)\}]\\ &\quad\times[(1-3\omega)(1-24c_1H_0^4)\{1+\omega\\ &\quad-6c_1H_0^4(5-4\omega-33\omega^2)\}]^{-1}.\end{aligned}$$

The values for γ_j's in Eq. (25) are

$$\begin{aligned}\gamma_1 &= \frac{1}{2}\left[5-\lambda\{1+3\tilde{c}_2(1+\omega)\}\right],\\ \gamma_2 &= \left[\frac{3}{4}\lambda\tilde{c}_2(1+\omega)\{3\tilde{c}_2\lambda(1+\omega)+2(\lambda-1)-8\}\right.\\ &\quad\left.+\frac{1}{4}(\lambda-1)(\lambda+7)+4+8\tilde{c}_2(\lambda-1)\left(\frac{1+\omega}{1-3\omega}\right)\right]^{\frac{1}{2}},\\ \gamma_3 &= -\frac{1}{2}\left(\frac{1-3\omega}{1+\omega}\right),\quad \gamma_4=\frac{2\kappa^2}{\omega-3},\\ \gamma_5 &= \left(\frac{18\lambda^3(1-3\omega)^{\frac{3\lambda(1+\omega)-2}{3\lambda(1+\omega)}}}{3\lambda(1-3\omega)+4}\right)\rho_0^{\frac{-2}{3\lambda(1+\omega)}},\quad \gamma_6=\frac{2}{3\lambda(1+\omega)}.\end{aligned}$$

The expressions for Δ_k's and Ω_k's in Eqs. (26) and (27) are

$$\begin{aligned}\Delta_1 &= 1-\frac{\gamma_7}{\gamma_8},\quad \Delta_2=1-\frac{\gamma_3^2}{\gamma_8},\quad \Delta_3=\gamma_4\left(1-\frac{1}{\gamma_8}\right),\\ \Delta_4 &= \gamma_5\left(1-\frac{\gamma_6^2}{\gamma_8}\right),\\ \Omega_1 &= 1-\frac{\gamma_8}{\gamma_7},\quad \Omega_2=1-\frac{\gamma_3^2}{\gamma_7},\quad \Omega_3=\gamma_4\left(1-\frac{1}{\gamma_7}\right),\\ \Omega_4 &= \gamma_5\left(1-\frac{\gamma_6^2}{\gamma_7}\right),\end{aligned}$$

where

$$\begin{aligned}\gamma_7 &= \frac{\tilde{c}_2}{6\lambda}\left[6\tilde{c}_2\lambda(1+\omega)^2-3\lambda(1+5\omega+2\omega^2)+2(\gamma_1+\gamma_2)\right],\\ \gamma_8 &= \frac{\tilde{c}_2}{6\lambda}\left[6\tilde{c}_2\lambda(1+\omega)^2-3\lambda(1+5\omega+2\omega^2)+2(\gamma_1-\gamma_2)\right].\end{aligned}$$

The values of χ_h's in Eq. (37) are given as follows

$$\begin{aligned}\chi_1 &= 288H_*^6f_{\mathcal{G}\mathcal{G}}^*,\\ \chi_2 &= 288H_*^5(3H_*^2+5\dot{H}_*)f_{\mathcal{G}\mathcal{G}}^*+6912H_*^7\\ &\quad\times(4H_*^2\dot{H}_*+2\dot{H}_*^2+H_*\ddot{H}_*)f_{\mathcal{G}\mathcal{G}\mathcal{G}}^*\\ &\quad+24(1+\omega)\rho_*H_*^3f_{\mathcal{G}T}^*-864(1+\omega)(1-3\omega)H_*^7\rho_*f_{\mathcal{G}\mathcal{G}T}^*,\\ \chi_3 &= -6H_*^2-24H_*^2\dot{H}_*f_{\mathcal{G}}^*-288H_*^4(4H_*^4-23H_*^2\dot{H}_*\\ &\quad-11\dot{H}_*^2-6H_*\ddot{H}_*)f_{\mathcal{G}\mathcal{G}}^*\\ &\quad+6912H_*^6(4H_*^2+\dot{H}_*)(4H_*^2\dot{H}_*+2\dot{H}_*^2+H_*\ddot{H}_*)f_{\mathcal{G}\mathcal{G}\mathcal{G}}^*\\ &\quad+12(1+\omega)\rho_*H_*^2\\ &\quad\times[2(4H_*^2+\dot{H}_*)-9(1-3\omega)H_*^2]f_{\mathcal{G}T}^*-864(1+\omega)(1-3\omega)\\ &\quad\times\rho_*H_*^6(4H_*^2+\dot{H}_*)f_{\mathcal{G}\mathcal{G}T}^*,\\ \chi_4 &= 12(1-3\omega)\rho_*H_*^3f_{\mathcal{G}T}^*,\\ \chi_5 &= \kappa^2\rho_*-\frac{1}{2}(\omega-3)\rho_*f_T^*+(1-3\omega)(1+\omega)\rho_*^2f_{TT}^*\\ &\quad-12(1-3\omega)\rho_*H_*^2\\ &\quad\times[(4+3\omega)H_*^2+\dot{H}_*]f_{\mathcal{G}T}^*+288(1-3\omega)\\ &\quad\times\rho_*H_*^4(4H_*^2\dot{H}_*+2\dot{H}_*^2+H_*\ddot{H}_*)\\ &\quad\times f_{\mathcal{G}\mathcal{G}T}^*-36(1+\omega)(1-3\omega)^2\rho_*^2H_*^4f_{\mathcal{G}TT}^*.\end{aligned}$$

The expressions for Υ_h's are

$$\begin{aligned}\Upsilon_1 &= 3(1+\omega)\rho_*H_*(\kappa^2+f_T)-12(1+\omega)\rho_*H_*\\ &\quad\times\left[3H_*^2(1-\omega)(4H_*^2+\dot{H}_*)\right.\\ &\quad\left.-2(16H_*^2\dot{H}_*+6\dot{H}_*^2+3H_*\ddot{H}_*)\right]f_{\mathcal{G}T}^*-72\rho_*^2H_*^2\\ &\quad\times(1-3\omega)(1+\omega)^2(4H_*^2+\dot{H}_*)f_{\mathcal{G}TT}^*\\ &\quad+576\rho_*H_*^3(1+\omega)(4H_*^2+\dot{H}_*)(4H_*^2\dot{H}_*+2\dot{H}_*^2+4H_*\ddot{H}_*)f_{\mathcal{G}\mathcal{G}T}^*,\\ \Upsilon_2 &= -12\rho_*H_*^2(1+\omega)\left[3H_*^2(1-\omega)-4(2H_*^2+3\dot{H}_*)\right]f_{\mathcal{G}T}^*\\ &\quad+576\rho_*H_*^2(1+\omega)(4H_*^2\dot{H}_*+2\dot{H}_*^2+H_*\dot{H}_*)f_{\mathcal{G}\mathcal{G}T}^*\\ &\quad-72\rho_*^2H_*^4(1-3\omega)(1+\omega^2)f_{\mathcal{G}TT}^*,\\ \Upsilon_3 &= 24\rho_*H_*^3(1+\omega)f_{\mathcal{G}T}^*,\\ \Upsilon_4 &= -\frac{3}{2}\rho_*H_*(1+\omega)\left[(1-\omega)f_T^*+2\rho_*^2(1+\omega)(1-3\omega)^2f_{TTT}^*\right]\\ &\quad-\frac{15}{2}\rho_*^2H_*(1-3\omega)(1+\omega)^2f_{TT}^*+24\rho_*H_*(1+\omega)\\ &\quad\times(4H_*^2\dot{H}_*+2\dot{H}_*^2+H_*\ddot{H}_*)\left[f_{\mathcal{G}T}^*+\rho_*(1-3\omega)f_{TT\mathcal{G}}^*\right],\\ \Upsilon_5 &= \rho_*\left(\kappa^2+\frac{1}{2}(3-\omega)f_T^*\right)+(1+\omega)(1-3\omega)\rho_*^2f_{TT}^*.\end{aligned}$$

For model (12), the coefficients of $(\delta,\ \delta_m)$ have the following expressions

$$\begin{aligned}\hat{\chi}_1 &= 288H_*^6F_{\mathcal{G}\mathcal{G}}^*,\\ \hat{\chi}_2 &= 288H_*^5(3H_*^2+5\dot{H}_*)F_{\mathcal{G}\mathcal{G}}^*+6912H_*^7(4H_*^2\dot{H}_*\\ &\quad+2\dot{H}_*^2+H_*\ddot{H}_*)F_{\mathcal{G}\mathcal{G}\mathcal{G}}^*,\\ \hat{\chi}_3 &= -6H_*^2-24H_*^2\dot{H}_*F_{\mathcal{G}}^*-288H_*^4(4H_*^4-23H_*^2\dot{H}_*\\ &\quad-11\dot{H}_*^2-6H_*\ddot{H}_*)\\ &\quad\times F_{\mathcal{G}\mathcal{G}}^*+6912H_*^6(4H_*^2+\dot{H}_*)(4H_*^2\dot{H}_*+2\dot{H}_*^2+H_*\ddot{H}_*)F_{\mathcal{G}\mathcal{G}\mathcal{G}}^*,\\ \hat{\chi}_5 &= \kappa^2\rho_*-\frac{1}{2}(\omega-3)\rho_*\mathcal{F}_T^*+(1-3\omega)(1+\omega)\rho_*^2\mathcal{F}_{TT}^*,\\ \hat{\Upsilon}_1 &= 3(1+\omega)\rho_*H_*(\kappa^2+\mathcal{F}_T),\\ \hat{\Upsilon}_4 &= -\frac{3}{2}\rho_*H_*(1+\omega)\left[(1-\omega)\mathcal{F}_T^*+2\rho_*^2(1+\omega)(1-3\omega)^2\mathcal{F}_{TTT}^*\right]\\ &\quad-\frac{15}{2}\rho_*^2H_*(1-3\omega)(1+\omega)^2\mathcal{F}_{TT}^*,\end{aligned}$$

$$\hat{\Upsilon}_5 = \rho_* \left(\kappa^2 + \frac{1}{2}(3-\omega)\mathcal{F}_T^* \right) + (1+\omega)(1-3\omega)\rho_*^2 \mathcal{F}_{TT}^*.$$

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