HIGH-ENERGY PROPERTIES OF THE MANDELSTAM REPRESENTATION

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1. INTRODUCTION

The general theme of this paper is fairly difficult to define, as it is made up of bits and pieces of information which do not appear to have much in common. This illustrates fairly well the present situation in this branch of physics called, until recently, physics of elementary particles. In consequence, this is an attempt to develop a picture of the evolution of the ideas as they developed during the past two years. The order of presentation is therefore more or less historical though not rigidly so. There will be no attempt to give a complete picture of the present situation, as this would be very dull; an up-to-date report will appear in the Proceedings of the 1962 High-Energy Conference held in Geneva and the later issues of the standard reviews.

Having thus set up the plan of this paper as being the historical order, an exception will be made of the comparison between theory and experiment which will be postponed to the end. The general scheme is therefore as follows: There will be first a summary of the situation of the theory before the introduction of Regge poles. Then there will be an explanation of how it came to a deadlock, with more and more paradoxes developing the impossibility of having stable particles of spin greater than one and the impossibility of having cross-sections going to constants at infinite energy.

At that point, Chew had the idea of generalizing certain features of Regge's work on potential theory to the relativistic theory based upon the Mandelstam representation. Then all these paradoxes vanished.

Having thus formulated the basic hypothesis of Chew, which had practically no logical support when it was first proposed, we shall discuss the pros and cons from a mere theoretical standpoint, and finally, we shall examine the predictions and the experimental verifications. By that time, the logical weaknesses of the theory should have been sufficiently exposed and the reader left wondering why the meager experimental results seem to confirm it so well.

2. THE PRE-REGGE DEADLOCK [1]

2.1. Pre-Regge postulates

In the summer of 1960 the last word in elementary particle physics was the Mandelstam representation. It is assumed that it is thoroughly familiar, however, it will be briefly gone over, if only to define the notation. To simplify matters, only spin zero particles and practically always equally massive particles, the mass being taken as unit of mass will be considered.

Furthermore $\hbar = 1$, $c = 1$. The basic assumptions are as follows:

(a) The invariant amplitudes for scattering or production, expressed as functions of the external momenta, possess the same analyticity properties as the formal sum of corresponding Feynman graphs, where all possible intermediate particles are taken into account, regardless of whether they are elementary or composite. This is at least true on one sheet (the "physical" sheet) of the Riemann surface thus defined (Landau).

(b) The discontinuities of the amplitudes across the cuts are given as certain (non-linear) functionals of the amplitudes which generalize the physical unitarity condition (Cutkosky).

(c) In addition, the amplitudes behave at infinite values of the external momentum variables no worse than polynomials, at least in the physical sheet (Mandelstam).

These three assumptions are implied by the Mandelstam representation. Consider a world where there is only one type of particle. The scattering of two particles with momenta p_1 and p_2 into particles with momenta $-p_3$ and $-p_4$ depends only upon two variables

$$
\begin{aligned}
s &= (p_1+p_2)^2, & & \\
t &= (p_1+p_3)^2, & s+t+u &= 4, \\
u &= (p_1+p_4)^2, & p_k^2 &= 1.
\end{aligned}
$$

The scattering amplitude then, as a function of s, t, u, is analytic, (a), except for cuts at s, t or u real, greater than 4. The jump across the cut is then directly given by unitarity, without any Cutkosky generalization, (b), and the amplitude behaves at infinity in the (s, t, u) space at worst like a polynomial, (c).

We shall write the Mandelstam representation

$$
\begin{aligned}
A(s,t,u) = {} & \frac{1}{\pi^2} s^N t^N \iint \frac{\rho(s',t')ds'dt'}{s'^N t'^N (s'-s)(t'-t)} + P_{s,t,u} \\
& + \sum_{p=0}^{M} t^p \frac{s^M}{\pi} \int \frac{\rho_p(s')ds'}{s'^M (s'-s)} + P_{s,t,u} + \sum_{p,q}^{L} t^p s^q \rho_{p,q}. \qquad (1)
\end{aligned}
$$

where L, M, N are some integers and $P_{s,t,u}$ indicates the two terms obtained from the term written just before by circular permutation of (s, t, u).

We shall call $\rho(s,t)$ double spectral function, $\rho_p(s)$ single spectral function of degree p and $\rho_{p,q}$ coefficient of the residual polynomial of degree p+q. (Strictly speaking these "functions" may be distributions). All these quantities are linked together, and to the production amplitudes by an infinite set of non-linear equations which express the unitarity requirement.

If we consider a theory with a more complicated spectrum, i.e. several types of particles, the number of independent amplitudes becomes very large, also the number of spectral functions and even the number of terms may increase in order to include contributions from complex singularities. However, we shall reason only on the very simplified case just mentioned and shall hope that all we say generalizes to more complicated cases. In particular, we shall not consider the question of whether there are complex singularities or not, it being understood that complex singularities are only supposed to bring in more terms. It is often believed that the representation (1) and the representations of an analogous type for the other amplitudes, combined with Cutkosky's rules which insure unitarity, are sufficient basis for a dynamical theory.

2.2. First troubles

Of course, it is very difficult, even with fast computers, to solve these equations expressing axioms a, b and c even without insisting on quantitative predictions. The tendency until 1960 had been to try and do everything feasible by using only functions of one variable: the double spectral functions were neglected altogether (Cini, Fubini), and approximate systems of equations involving single spectral functions of lowest degree were solved (Chew, Mandelstam). By the summer of 1960, all calculations on this scheme had been carried out, at least all of those which did not lead to divergences. Mandelstam* has described how this procedure worked and how one ran into great difficulty as soon as P-wave resonances entered the game. Thus people started to contemplate the double spectral function, thinking that they might help somehow. The ideas that they had at that time were fairly simple and they thought that a fair model of what a double spectral function might look like were, for example, the double spectral functions as they appear in Feynman graphs, quite smooth and without much of a structure.

Something did not seem to fit into this picture very well, however, and that was the occurrence of stable particles with high spins. Indeed, whenever there is a particle with spin j and mass m, some of the amplitudes have a pole of the form $P_j(\cos\theta)/(s-m^2)$ where s is an energy squared variable and $\cos\theta$ is in general proportional to a momentum transfer squared. Such a pole, therefore, fits in the Mandelstam representation (1), under the condition that M be not less than j. This indicates that (1) is valid only with subtractions at least up to a degree equal to the highest spin of the stable intermediate particles. One might then wonder if this could not lead to very large cross-sections at high energy, increasing polynomially with the energy. But, of course, the experiments, even with the most energetic cosmic rays, indicate that it is not so, that the cross-sections behave in a way consistent with a constant within the experimental accuracy.

The model one had at that time for scattering at high energy was that suggested by Pomeranchuk: whenever the particles interact, they have no chance of recombining to scatter elastically because of the competition of the many inelastic channels open at high energy. Therefore, the scattering amplitude becomes purely absorptive and the elastic scattering is simply diffraction scattering. At very high energies, this diffraction can be treated classically, given an absorption coefficient which reflects the distribution of matter in the clouds of virtual particles. One then gets a constant cross-section, a constant diffraction scattering peak (measured in momentum transfer) in the elastic amplitude and a constant elastic cross-section. This picture was more or less in agreement with the experiments which were not

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very precise and which, for some reason, were in general fitted by comparison with classical diffraction scattering by a uniformly gray disc, or sphere, but never with a more sophisticated distribution of the absorption coefficient.

2.3. Gribov's paradox

This picture was shown by Gribov to be inconsistent with the Mandelstam representation. The essence of Gribov's idea was the following: the mathematical expression of Pomeranchuk's model is that the amplitude $A(s,t)$ in the physical region, for large positive s and small negative t has the asymptotic form

$$A(s,t) \approx s f(t) \tag{2}$$

where $f(t)$ determines the shape of the diffraction pattern, and the factor s is there to cancel kinematic coefficients in order to have a constant total cross-section. $f(t)$ in these circumstances has the t-cut of the Mandelstam representation, and we assume that this asymptotic form is valid for t positive, at least up to some value greater than 4. Unitarity in the t channel reads, according to Mandelstam:

$$\rho(s,t) = \frac{1}{\pi\sqrt{t}} \int\int \frac{ds_1 \, ds_2 \, A_s(s_1,t) A_s^*(s_2,t)}{K^{1/2}(t;s,s_1,s_2)} \theta(K), \tag{3}$$

$$K(t;s,s_1,s_2) = (t-4)(s^2 + s_1^2 + s_2^2 - 2 s_1 s_2 - 2 s s_1 - 2 s s_2) - 4 s s_1 s_2. \tag{4}$$

A_s is the absorptive part of A with respect to the s-channel. Substituting the asymptotic form (2), taking into account that $f(t)$ is purely imaginary, $i A_s(s,t) = A(s,t)$:

$$\rho(s,t) = \frac{1}{\pi\sqrt{t}} \int\int \frac{ds_1 \, ds_2 \; s_1 s_2 |f(t)|^2}{K^{1/2}(t;s,s_1,s_2)} \theta(K) \,. \tag{5}$$

Change variables, putting $s_1 = u_1 s^{1/2}$, $s_2 = u_2 s^{1/2}$:

$$\rho(s,t) = \frac{1}{\pi\sqrt{t}} \int\int \frac{s^2 du_1 \, du_2 \; u_1 u_2 |f(t)|^2}{(t-4)[s^2 + 0(s^{3/2}) - 4 s^2 u_1 u_2]^{1/2}} \theta(K). \tag{6}$$

The integration takes place in the hyperbola $u_1 u_2 < (t-4)/4$, asymptotically and diverges logarithmically, except that the term $0(s^{3/2})$ cuts off the integration, thereby introducing a $\ln s$ term which we cannot calculate exactly as it involves non-asymptotic regions. We therefore have the behaviour $\rho(s,t) \approx \text{const.} s \ln s |f(t)|^2$. This is incompatible with $\rho(s,t) = s \cdot \operatorname{Im} f(t)$. We shall see later in a much more transparent fashion the deep reasons for this paradox.

One may generalize this reasoning and one finds that the paradox takes place for any asymptotic behaviour of the form

$$A(s,t) = s^{\alpha}\ln^{\beta}(s) \cdot f(t), \tag{7}$$

if α is real and Re $\beta \geqslant -1$.

Gribov's suggestion was to take $\alpha = 1$, $\beta < -1$, but this is already a little difficult as it implies cross-sections which go to zero at infinity and again, the problem of how to accommodate particles with large spins stays there as the behaviour of the poles corresponds to α real and $\beta = 0$.

2.4. Bound on the asymptotic behaviour in the physical region

Another difficulty arose in this connection, when the author proved that the cross-sections in the framework of the Mandelstam representation cannot increase faster at infinity than $\ln^2 s$. The intuitive basis for the theorem is the following: consider the Pomeranchuk model classically. We may very well suppose that the absorption coefficient changes with energy. However, the distribution of matter in the cloud of virtual particles falls off essentially exponentially, the range of the exponential being given by the mass of the lightest virtual particle. Therefore, all we may expect is an absorption coefficient of the form $g e^{-Kr}$ where g may vary with energy. If the impact parameter b of a collision is such that $g e^{-Kb} \ll 1$, there is practically no effect. If $g e^{-Kb} \gg 1$ there is complete absorption. The cross-section is determined then by the value a of the impact parameter so that $g e^{-Ka} \cong 1$ or $a \cong (1/K)\ln g$, $\sigma_{tot} \cong \pi/K^2(\ln g)^2$. Even if we assume that g grows polynomially with the energy, a increases only logarithmically and the cross-section increases thus like the square of ln s.

A very elegant derivation of this theorem was given by Martin. The only assumption of Martin is that the Legendre polynomial expansion of the amplitude converges for $s > s_0$ up to some positive value t_0 of t and that, at that value of t, the asymptotic behaviour of A(s, t) is polynomial in s. This is automatically guaranteed by the Mandelstam representation [1]. The reasoning of Martin uses the fact that the imaginary part of partial waves is positive and bounded and that the Legendre polynomials $P_\ell(z)$ are positive increasing functions of ℓ for z real, $z > 1$. Let us then write the absorption part of the amplitude:

$$A_s(s,t) = \sum_{\ell}(2\ell+1)\,\mathrm{Im}\, a_\ell(s)\, P_\ell(\cos\theta), \tag{8}$$

$$0 \leqslant \mathrm{Im}\, a_\ell(s) \leqslant \sqrt{s(s-4)}\ , \quad \cos\theta = 1 + 2t/(s-4). \tag{9}$$

If one wants to maximize $A_s(s,0)$, holding $A_s(s,t_0)$ fixed, as $P_\ell[1+2t_0/(s-4)]$ is an increasing function of ℓ whereas $P_\ell(1) = 1$, one has to take as small values of ℓ as possible:

$$\mathrm{Im}\, a_\ell(s) = 0\ ,\quad \ell > \ell_0 \qquad \text{and} \qquad \mathrm{Im}\, a_\ell(s) = \sqrt{s/(s-4)}\ ;\quad \ell < \ell_0.$$

The value of ℓ_0 is determined by

$$A_s(s,t_0) = \sum_{\ell=0}^{\ell_0}(2\ell+1)\sqrt{s/(s-4)}\, P_\ell[1+2t_0/(s-4)] = \sqrt{s/(s-4)}\,(P'_{\ell_0+1} + P'_{\ell_0}).$$

Assume that $A_s(s, t_0) \approx C s^\alpha$; $\mathrm{Re}\,\alpha > 1$. We use the following estimate

$$P_{\ell_0}[1+2t_0/(s-4)] \approx I_0(\ell_0\sqrt{2t_0/(s-4)}) \qquad \text{(modified Bessel function)}$$

and therefore :

$$P'_{\ell_0} = \ell_0\sqrt{(s-4)/8t_0}\, I_1(\ell_0\sqrt{2t_0/(s-4)}).$$

We thus determine ℓ_0 by

$$\ell_0\sqrt{(s-4)/2t_0}\, I_1(\ell_0\sqrt{2t_0/(s-4)}) = C s^\alpha.$$

For $\mathrm{Re}\,\alpha > 1$ the solution of this equation is asymptotically for large s :

$$\ell_0\sqrt{2t_0/(s-4)} \approx (\alpha-1)\ln s \quad \text{or:} \quad \ell_0 \approx (\alpha-1)\, s^{1/2}\ln(s/\sqrt{2t_0})$$

which gives

$$A(s,0) = \sum_0^{\ell_0}(2\ell+1)\sqrt{s/(s-4)} \approx \ell_0^2 \approx (\alpha-1)^2 s\ln^2(s/2t_0)$$

This corresponds to the classical picture given above. One may also estimate the asymptotic behaviour of the amplitude in the physical region, either at fixed momentum transfer, or at fixed angle. The results are the following:

$$|A(s,t)| < M(t)\, s\ln^2 s \ , \quad t < 0, \tag{10a}$$

$$|A(s,(s-4)(\cos\theta-1))| < N(\theta)\, s^{3/4}\ln^{3/2} s \qquad 0 < \theta < \pi. \tag{10b}$$

2.5. Independence of the single spectral functions

An interesting question about the Mandelstam representation was, besides how many subtractions are to be made, whether or not it was possible to change the content of the theory by making more subtractions. This was very interesting particularly in view of the well-known CDD ambiguity which arises when one tries to enforce unitarity on the single spectral functions. It could be that by making more and more subtractions, it becomes possible to introduce more and more particles with higher and higher spins into the theory by introducing CDD poles, just as one may introduce more and more complex terms into a Lagrangian.

However,it is possible to show, using conditions (10), that this is not so. The single spectral functions of degree greater than one and all coefficients of the residual polynomial of degree greater than zero are completely determined by the double spectral function and the conditions (10).

To make this clear, suppose that there are two different amplitudes with the same double spectral function. Their difference satisfies (10), and is expressed by

$$\Delta A(s,t,u) = \sum_{p=0}^{M} t^p \frac{s^M}{\pi}\int\frac{\sigma_p(s')ds'}{(s'-s)(s'^M)} + P_{s,t,u} + \sum_{p,q=0}^{L} t^p s^q \sigma_{p,q}\,. \tag{11}$$

The idea is now to prove that all of these terms must obey conditions (10) individually, in other words that there cannot be any cancellation between different terms. The details of the proof will not be discussed but the principles will be outlined.

Let us take different directions in the (s, t, u) plane, corresponding to different values of the angle θ, and show that an expression like (11) cannot satisfy (10b) for 3M + 2L + 1 different values of cos θ unless

$$\sigma_p(s) < C\, s^{-p+3/4} \ln^{3/2} s,$$

$$\sigma_{p,q} = 0 \text{ except } \sigma_{00}.$$

In that case, if s is fixed negative and t variable, the largest term which contributes to the asymptotic behaviour is

$$t^M \int \frac{\sigma_M(s')ds'}{(s'-s)} .$$

This term violates (10a) if $M > 1$ and therefore must vanish on the negative real s axis and therefore it vanishes everywhere by analytic continuation. Thus we prove that $M = 1$ and $L = 0$ in Eq. (11).

In the same way, *this method allows in principle to compute the single* spectral functions of degree greater than one from the double spectral function. This is in practice very difficult to carry out because of the analytic continuation mentioned above.

Even in principle, it appears very difficult to prove that the partial waves obtained by this method will satisfy unitarity. At any rate, it is sufficiently demonstrated that particles with spin greater than one cannot be elementary, in the sense that one cannot introduce arbitrary CDD poles for the higher waves ($j > 1$) as there is no N/D equation in that case. In that sense, we shall say that all particles with spin greater than one are "dynamical". It is interesting to note that the condition just obtained looks quite similar to the old-fashion "renormalizability" condition. The connection may be deeper in the sense that these two conditions both reflect the fact that, unless very pecular cancellations take place, unitarity is strongly violated at high energies if one introduces a priori high spin particles into the theory. Now the paradox is complete: we have proved that the behaviour of the amplitude in the physical regions is completely different from that in the "spectral" regions: we have an upper bound in the physical region due to unitarity and we have a larger lower bound in the unphysical region as a result of poles of particles with high spin.

How this be reconciled with the analyticity properties ? There must be some kind of oscillation of the amplitude in the spectral region so that the dispersion integrals expressing the amplitude in the physical region do not in fact behave at all like their integrands, but increase more slowly as a result of cancellations inside the integral.

A very natural kind of function with just such a behaviour is, for example:

$$A(s,t) = \beta(t) s^{\alpha(t)} \tag{12}$$

where $\alpha(t)$ is real, less than one for $t < 0$, has a cut for $t > 4$ and is such that $\mathrm{Re}\,\alpha(t)$ stays bounded in the cut plane ; $\beta(t)$ is any function satisfying a dispersion relation in the cut t plane.

Such a function would indeed resolve all paradoxes above. It may be shown (the reader may do this as an exercise) that the Gribov paradox, although holding for any real α, ceases to hold as soon as α is non-real. This is precisely the result of the cancellations introduced by the oscillations described by phase $[s^{\alpha(t)}] = \mathrm{Im}\,\alpha \ln s$.

3. THE INTRODUCTION OF REGGE POLES

Having been compelled to consider amplitudes of the form (12) Chew looked around and found Regge's paper [2] which predicted an asymptotic behaviour of precisely this form in potential scattering with a momentum transfer and t energy variable. This coincidence was very striking because, to reach the form (12) from the relativistic theory, we constantly used the crossing symmetry or equivalently, unitarity in all three channels, which is very specific of the relativistic theory. On the other hand, one may argue that after all it is not so surprising, as the unitarity equation reads much the same for potential scattering and for the elastic regions of the relativistic problem. At any rate, it was very intriguing, and still is, to see whether or not the Regge poles have a logical place in the framework of S-matrix theory. It is very difficult now to expose as nobody yet has produced anything very convincing.

Let me start by describing the way people agree to choose the "best" interpolation.

3.1. Definition of the partial-wave amplitude

In the theory of scattering by superposition of Yukawa potentials (see Regge's lectures*) the amplitude $a(\ell, q^2)$ has the following properties. It is holomorphic for $\mathrm{Re}\,\ell > N$. It decreases exponentially with $\mathrm{Re}\,\ell$ and increases at most like a polynomial with $\mathrm{Im}\,\ell$. Furthermore, it is unitary all along the real axis, even for non-integer points.

If we start from the Mandelstam representation, we have a dispersion relation in $\cos\theta$, which we can write as

$$A(q^2, \cos\theta) = \frac{1}{\pi}\cos^N\theta \int \frac{A_t(q^2, x)dx}{x^N(x-\cos\theta)} + \text{Polynomial.} \tag{13}$$

We have expressed now the amplitude A and its absorptive parts in $\cos\theta$, A_t, in terms of $q^2 = (s-4)/4$ and $\cos\theta = 1 + 2t/(s-4)$.

The integral extends on the real axis, somewhere outside of $[-1, +1]$.

The partial wave computed from this is, for integer ℓ:

$$Q_\ell(q^2) = \frac{1}{2\pi}\int_{-1}^{+1} d\cos\theta\ P_\ell(\cos\theta)\left[\cos^N\theta \int \frac{A_t(q^2, x)dx}{x^N(x-\cos\theta)} + P_{N-1}(\cos\theta)\right]. \tag{14}$$

Let us compute the partial waves for $\ell > N$ only.

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Then we can integrate:

$$\frac{1}{2}\int_{-1}^{+1} d\cos\theta\; P_\ell(\cos\theta)\,\frac{\cos^N\theta}{x^N(x-\cos\theta)} = Q_\ell(x) \tag{15}$$

where $Q_\ell(x)$ is that Legendre function of second kind which is real for $x > 1$ real and we get

$$a_\ell(q^2) = \frac{1}{\pi}\int dx\; A_t(q^2, x) Q_\ell(x). \tag{16}$$

This integral converges, as $Q_\ell(x) \approx 1/(2x)^{\ell+1}$ for large x.

We note at this point that equations (14) and (16) are equivalent only for integer values of ℓ. Furthermore we note that in general $A_t(q^2, x)$ gives contributions to (16) from the side of $x > 1$ and from the side of $x < -1$, thereby introducing terms which behave like $e^{i\pi\ell}$ which do not satisfy our conditions. If however, we introduce the following functions:

$$a_\ell^{\pm}(q^2) = \frac{1}{\pi}\int_{x>1} dx\; Q_\ell(x)\,[A_t(q^2, x) \pm A_t(q^2, -x)], \tag{17}$$

we get an asymptotic behaviour in the half-plane Re $\ell > N$ which is exactly what we want: polynomial at most in Im ℓ, and exponentially decaying with Re ℓ. The physical a_ℓ is equal to either a^+ or a^- according to whether ℓ is even or odd.

Now we may introduce a theorem [3] which is very useful for the following.

3.2. Carlson's Theorem

Let f(z) be regular and of the form $O(e^{\alpha|\mathrm{Im}\, z| + \beta\,\mathrm{Re}\, z})$ in Re $z \geqslant 0$, α and β real, $\alpha < \pi$; let $f(z) = 0$ for $z = 0, 1, 2\ldots$ Then f(z) is identically zero.

Proof

We can write the Cauchy theorem (Re $z > 0$) for the regular function:

$$\frac{e^{\lambda z} f(z)}{\sin\pi z} = -\frac{1}{2\pi i}\int_{0-i\infty}^{0+i\infty} \frac{e^{\lambda z} f(x)\,dx}{\sin\pi x\,(x-z)} \quad \text{for:}\; \begin{array}{l} |\mathrm{Im}\,\lambda| < \pi-\alpha, \\ \mathrm{Re}\,\lambda < -\beta. \end{array}$$

Both sides are analytic in the whole strip $|\mathrm{Im}\,\lambda| < \pi - \alpha$ and the equation holds there. But on the real λ axis, the right hand side is bounded; the left-hand side can only be bounded if $f(z) = 0$.

It is clear that this theorem guarantees the uniqueness of the interpolation $a_\ell^{\pm}$ that we have defined, which satisfies very comfortably the conditions of the theorem, as $Q_\ell(x) \approx 1/(x+\sqrt{x^2-1})^{\ell+1}$ for large ℓ.

This theorem is also useful to prove that, for the regions of energy where the partial wave is unitary (elastic region), the interpolations $a_\ell^{\pm}$ are unitary each on the real ℓ axis. To show this for a_ℓ^{+}, let us write $\ell = 2z + 2N$ and build:

$$a^+_{2z+2N} - (a^+_{2z^*+2N})^* - 2i\sqrt{\frac{s-4}{s}}\, a^+_{2z+2N}(a^+_{2z^*+2N})^* \equiv f(z).$$

$f(z)$ vanishes for every integer value of z, as at that point a^+_ℓ takes on a physical value at an even angular momentum. On the other hand $f(z)$ is regular and satisfies Carlson's asymptotic condition and is therefore identically zero. a^+_ℓ satisfies unitarity in the complex half-plane everywhere in the sense that

$$a^+_\ell - (a^+_{\ell^*})^* = 2i\sqrt{(s-4)/s}\; a^+_\ell (a^+_{\ell^*})^*. \tag{18}$$

The reasoning is the same for a^-_ℓ and leads to the same result.

The reader may show as an excercise that if one is to take only one interpolation, valid for both even and odd partial waves, for example $(a^+ + a^-)/2 + e^{i\pi\ell}(a^+ - a^-)/2$, Carlson's theorem does not apply any more to prove unitarity and in fact the amplitude thus obtained is not unitary in general.

We have so far established a number of properties which are quite interesting in the sense that they remind us strongly of the potential scattering case. Notice also that if one has the Schrödinger equation with an exchange potential, one obtains twice the Regge behaviour: once with the even partial waves and an effective potential which is the sum of the direct and exchange parts, and once with the odd partial waves with the difference. Therefore in that case one also obtains two distinct interpolations a^+_ℓ and a^-_ℓ with the same properties.

3.3. Connection between asymptotic behaviour in cos θ and singularities in the ℓ-plane,

We have not yet reached the interesting part of the ℓ-plane, in the sense that we are still on the right of any Regge pole (if there is any) in the region where Eq. (17) converges.

Indeed, if A displays a behaviour like $(\cos\theta)^\alpha$, the integral (17) only converges for Re $\ell >$ Re α.

If however $A(q^2, \cos\theta) \approx \beta(q^2) P_{\alpha(q^2)}(\cos\theta) + 0[(\cos\theta)^{\alpha'}]$ where Re $\alpha' <$ Re α, then we may analytically continue the integral by writing:

$$\int_1^{+\infty} P_\alpha(x)\, Q_\ell(x)\, dx = 1/(\ell-\alpha)(\ell+\alpha+1) \tag{19}$$

and continuing this term by its exact expression, and the remainder converges further to Re $\ell >$ Re α'. Therefore we may again get Regge poles as a consequence of the behaviour (12).

Incidentally, it might help to see what kind of singularities other asymptotic behaviours may lead to. Consider for example

$$A(q^2, \cos\theta) \approx \cos^\alpha\theta\, \ln^\beta\cos\theta + 0[(\cos\theta)^{\alpha'}].$$

Write

$$\cos^\alpha\theta\, \ln^\beta\cos\theta = \frac{\beta!}{2\pi i}\int_{\alpha'}^{\alpha'} \frac{\cos^\zeta\theta\, d\zeta}{(\zeta-\alpha)^{\beta+1}} + 0[(\cos\theta)^{\alpha'}],$$

the integral being taken around α. We might as well replace $\cos^\ell\theta$ by $P_\ell(\cos\theta)$ and insert into (17) and (19) thus getting the leading singularity

$$[\beta!/(2\alpha+1)][1/(\ell-\alpha)^{\beta+1}].$$

This singularity for β integer negative becomes of logarithmic type. We may remark then that the power in $\cos\theta$ will determine the location of the singularity, whereas the nature of the singularity will depend upon the departures from a simple power behaviour. It is therefore to be expected that any attempt to determine the nature of the singularity by using Eq. (17) is very delicate and it becomes dubious whether it does not at the same time determine the exact location of the singularity.

On the other hand, if one knows by other ways that there are only poles, then the analytic continuation of Eq. (17) is fairly possible: identify the poles by the asymptotic behaviour, and subtract them out. This has been done in practical calculations [4], in particular in potential scattering [5] where one knows that there are only poles.

3.4. Bardacki's method

Recently, BARDACKI [6] has completed some very interesting work which is probably the first step towards a proof of the existence of Regge poles in relativistic S-matrix theory. His basic idea is the following: we assume that the overall number of subtractions for the Mandelstam representation is finite, N. Therefore, for any q^2, $a_\ell^\pm(q^2)$ is regular in the half plane Re $\ell > N$. On the other hand, we have seen that for s real and negative, the maximum power admissible for $\cos\theta$ was one (unitarity in the crossed channel). It is very easy to see that, in fact, this holds also in an infinitesimal neighbourhood of the negative s axis. We therefore have another domain of regularity s negative, Re $\ell > 1$. We may take the holomorphy envelope of these two domains which provides a larger domain of holomorphy for $Q_\ell^\pm(q^2)$.

It turns out that the calculation is fairly trivial. If one makes a conformal mapping to map the s-plane cut from $-\infty$ to 0 and from 4 to $+\infty$ onto a strip:

$$S = 2 + 2\sin z, \qquad -1 < \mathrm{Re}\, z < 1.$$

We can almost use the tube theorem, saying, not rigorously, that we have analyticity in the region:

$$-1 < \mathrm{Re}\, z < 1, \qquad \mathrm{Re}\, \ell > N,$$
$$\mathrm{Re}\, z = -1, \qquad \mathrm{Re}\, \ell > 1.$$

We use the tube theorem, taking the convex hull of the base of the tube:

$$\mathrm{Re}\, \ell > (N+1)/2 + [(N-1)/2]\mathrm{Re}\, z, \qquad \text{or}$$
$$\mathrm{Re}\, \ell > (N+1)/2 + [(N-1)/2]\mathrm{Re} \arcsin (s-2)/2. \qquad (20)$$

This is not quite rigorous because Re z = -1 is not a domain. However, it may be made rigorous. The extension of the domain of holomorphy stops there and one cannot go further. There is, however, a way to extend the analytic properties, but not holomorphy, only meromorphy. This consists in taking exactly into account the two-body unitarity as far as it is valid. It is known that Schwarz's reflection principle allows one to continue through the two-body elastic cut analytically or, alternatively, to write down a function of the partial-wave amplitude which does not have the two-body cut.

To be more specific, consider the partial wave at threshold copying Eq. (17) in the form

$$a_\ell(q^2) = \int_{x_0>1}^{\infty} Q_\ell(x)\, A_t(q^2, x) dx. \tag{21}$$

We should keep in mind that $x = 1 + t/2q^2$. Only large values of x will contribute near threshold. Below threshold, at $q^2 = |q|^2 e^{i\pi}$, $A_t(q^2, x)$ is real, and therefore the phase of $a_\ell(q^2)$ is that of $Q_\ell(x) = Q_\ell[1+t/(2|q|^2 e^{i\pi})]$. It is $\pm \exp i\pi\ell$. Above threshold $q^2 = |q|^2$, $Q_\ell(q^2)$ is unitary, so that $\mathrm{Im}\, 1/[q_\ell(q^2)] = q/\sqrt{1+q^2}$. This gives us the whole set of rules to continue $a_\ell(q^2)$ around $q = 0$ any number of times. The construction of the function

$$R_\ell(q^2) = q^{2\ell}/a_\ell(q^2) + 2i\, q^{2\ell+1}/[1 + \exp(2i\pi\ell)]\sqrt{q^2+1}. \tag{22}$$

so that it turns out to be real for both $q^2 = \pm|q|^2$ is left to the reader. Furthermore, it is bounded at $q^2 = 0$, because

$$a_\ell(q^2) \approx q^{2\ell} \int A_t(q^2, x)\, dt/t^{\ell+1}, (\mathrm{Re}\, \ell > N).$$

It is therefore regular at the origin as a function of q^2. At any rate it is meromorphic wherever $a_\ell(q^2)$ is.

Now if we assume (which is nearly rigorous) that the rules for completion of meromorphy domains are the same as for holomorphy domains, we can play the same game as before except that the initial domain has a cut starting from the first inelastic threshold (somewhere between 4 and 16), say 16. Then we get, for $a_\ell(q^2)$ the meromorphy domain as defined by

$$\mathrm{Re}(\ell) > (N+1)/2 + (N-1)/2\, \mathrm{Re}[\arcsin(s-8)/8]. \tag{23}$$

We see clearly that we are prevented from going further only by our lack of ability: we do not know how to eliminate the further cuts on the real axis. It is conceivable that someone who could master the 4-body unitarity condition could carry on the programme up to the 6-body cut, and so on. At any rate, it is comforting to see a domain of meromorphy which is larger than the domain of holomorphy, as this introduces a kind of proof which depends very little on Eq. (17) as far as the nature of the singularities is concerned. However, it might very well turn out that the 4-body cut introduces other kinds of singularities in the ℓ-plane and that the reason that potential scattering has only poles is precisely the absence of inelastic contributions. This is of course an open question. If however, one makes ad hoc hypotheses on the inelastic contributions, for example, if one assumes [7] analyticity properties of the absorption coefficient $\eta(\ell, q^2)$, then it is possible to carry out the reasoning with threshold of infinity, thus getting meromorphy for Re $\ell > 1$, but it looks a little like assuming what one wants to prove. Another interesting try has been made recently by MANDELSTAM [8], in which he studies a problem where the kinematics are relativistic, the potential energy independent and where there are no inelastic processes. He then succeeds in proving that for a potential weak enough, the Regge-Sommerfeld-Watson formula is applicable down to Re $\ell = 0$, without using the unitarity condition in the crossed channel.

4. DISCUSSION OF CHEW'S HYPOTHESIS

We have seen in the last section how one might think of establishing the existence of Regge poles in S-matrix theory and that a long way still lies in front of us. However, CHEW [9] was bold enough to overcome this lack of logical support and to assume that the only singularities lying in the ℓ-plane are poles and that the partial waves were given even for small ℓ by the analytic continuation of $a_\ell(q^2)$ as defined by [17].

Let us examine how this hypothesis solves and helps to understand the paradoxes encountered in the first section.

Gribov's paradox is now very clear. We have seen that a behaviour of the form $\cos^\alpha\theta \ln^\beta(\cos\theta)$ brings in a singularity in the ℓ-plane at $\ell = \alpha$, of the kind $(\ell - \alpha)^{-(\beta+1)}$. The content of Gribov's paradox is that no such singularity may lie on the real axis, where a_ℓ is bounded by the unitarity condition unless $\beta < -1$, whatever the real value of α is. But, of course, we assume now that the Regge poles move and if $\alpha(q^2)$ is the position of the pole, according to Eq. (22) near the threshold:

$$R_\alpha(q^2) = 2i\, q^{2\alpha+1}/[1 + \exp(2i\pi\alpha)]\sqrt{1+q^2}.$$

The solution of this equation, α, moves out of the real axis just at threshold with an imaginary part [7] of the order of $q^{2\alpha_0+1}$, α_0 being $\alpha(q^2 = 0)$. This is exactly what we need to avoid Gribov's paradox.

Similarly, it is now easier to see through the complexity of the dependence of single spectral functions upon the double spectral functions. For negative real s, the partial waves obtained without subtractions ($\ell > 1$) are indeed the analytic continuation from the region of Re $\ell > N$. Therefore, if this analytic continuation is unique when one analytically continues them to positive values of s, they are still the analytic continuation of a_ℓ as defined by Eq. (17) and therefore are unitary by virtue of Eq. (18). If there are only poles, the analytic continuation is unique and therefore Chew's hypothesis explains the kind of magic which takes place here very well. Furthermore, it may be much easier to continue analytically in ℓ rather than in s, as the continuation path may be shorter, and we have seen that the analytic continuation in ℓ is relatively easy when there are only poles which one can separate out. If this connection is true, we see that the second part of Chew's hypothesis is forced upon us by unitarity in the crossed channels for intermediate partial waves ($2 \leqslant \ell \leqslant N$), and therefore it is very natural to extend it to the S and P-waves.

4.1. Possible range of coupling constants

Let us make a little philosophical digression at this point which may illustrate the possible depth of Chew's hypothesis pretty well. Let us consider that, as in potential theory, the stronger the interactions are, the larger are the values of angular momenta of Regge poles. This is of course without proof of any kind. If, however, one admits this postulate as well as Chew's hypothesis, one is faced with the following situation: the interactions cannot be stronger than they are in the physical world, as this would correspond to amplitudes increasing like S^{α}, $\alpha > 1$ in the physical region, which contradicts the unitarity condition. Chew called this circumstance "saturation of unitarity". It seems that the interactions in nature are "as strong as possible" On the other hand, can they be weaker ? Perhaps, but not vanishingly small, since, according to Chew's hypothesis, if one wants to have one particle, one has to bring at least one Regge pole up to zero. The free-field theory in particular does not satisfy Chew's hypothesis, as its scattering amplitude has no poles and therefore no stable particles. It looks thus as if there was a finite range of interactions possible. If one is very optimistic, one may even hope that there is only one theory possible by this system, but this becomes science-fiction.

4.2. Accumulation of Regge poles

It has been pointed out by Gribov and Pomeranchuk and independently by Wilson that sometimes the Regge poles cluster around some accumulation points. They have used this fact to derive a lower bound on the asymptotic behaviour of cross-sections.

The first case [10] of such an occurrence is essentially kinematic and arises [11] also in potential scattering [5] . We can easily derive it from Eq. (22) in the neighbourhood of the threshold q = 0. The equation of theRegge poles reads

$$R_{\alpha}(q^2) = 2i\, q^{2\alpha+1}/[1+\exp(2i\pi\alpha)]\sqrt{q^2+1}. \tag{24}$$

This equation has an infinite number of solutions near $= -\frac{1}{2}$: this is best seen by taking the logarithm :

$$\ln\, R_{\alpha}(q^2) + \ln\left[\frac{1+\exp(2i\pi\alpha)}{2i}\right] + \tfrac{1}{2}\,\ln[q^2+1] = (2\alpha+1)\ln q + 2\, m\, i\, \pi,$$

$$\ln\, R + \ln[-\pi(\alpha+\tfrac{1}{2})] = (2\alpha+1)\ln q + 2\, m\, i\, \pi + 0(\alpha+\tfrac{1}{2}). \tag{25}$$

We thus have an infinite number of poles labeled by m, going to $-\frac{1}{2}$ roughly like $-1/2 + 2m\, i\, \pi/\ln q$, neglecting a factor of the order $\ln|\ln q|$.

This result leads to a prediction concerning the behaviour of $A(q^2, \cos\theta)$ for $q^2 = 0$, $\cos\theta \to \infty$: $A(0, \cos\theta)$ cannot fall off faster than $(\cos\theta)^{-1/2}$. It is to be expected that such a behaviour will take place at every threshold, at $\ell = -\frac{1}{2}$ for two-body thresholds, possibly at other values of ℓ for many-body thresholds, as it depends upon the phase-space threshold behaviour.

Another point of accumulation of Regge poles [10] is a consequence of a very special feature of relativistic theory, i.e. the existence of a double spectral function at negative energy.

Consider the partial-wave amplitude as defined by Eq.(17). For $q^2 = e^{i\pi}|q|^2$ near zero, $a_\ell^\pm(q^2)$ has a constant phase $\pm\, e^{i\pi\ell}$, that of $Q_\ell(1+t/2q^2)$. For $q^2 < -t_0/4$, a cut appears as a result of the coincidence of the limit of integration $x_0 = 1 + t_0/2q^2$ with -1, which is a branch point for Q_ℓ.

The imaginary part of $b_\ell(q^2) = q^{-2\ell}Q_\ell(q^2)$ above this cut is

$$\operatorname{Im} b_\ell(q^2 + i\epsilon) = -\frac{1}{\pi|q|^2}\int_{-1}^{x_0} \operatorname{Im}[A_t(q^2 + i\epsilon,\ x - i\epsilon)Q_\ell(x - i\epsilon)\, e^{-i\pi\ell}]. \tag{26}$$

This is, in general

$$\operatorname{Im} b_\ell(q^2 + i\epsilon) = -\frac{1}{\pi|q|^{2\ell}}\int_{-1}^{x_0} \operatorname{Im} A_t(q^2 + i\epsilon, x - i\epsilon)\operatorname{Re}[Q_\ell(x - i\epsilon)e^{-i\pi\ell}]\, dx$$

$$- \frac{1}{\pi|q|^{2\ell}}\int_{-1}^{x_0} \operatorname{Re} A_t(q^2 + i\epsilon, x - i\epsilon)\operatorname{Im}[Q_\ell(x - i\epsilon)e^{-i\pi\ell}]\, dx$$

$$\operatorname{Im} b_\ell(q^2) = \frac{1}{\pi|q|^2}\int_{-1}^{x_0} \rho_{t,u}(q^2, x)\, G_\ell(x)\, dx - \frac{1}{\pi|q|^{2\ell}}\int_{-1}^{x_0} \operatorname{Re} A_t(q^2, x)P_\ell(-x)\, dx. \tag{27}$$

where $\rho_{t,u}(q^2, x)$ is the spectral function which lies in $t > 0$, $u > 0$, and where

$$Q_\ell(x) \equiv \operatorname{Re} Q_\ell(x) \quad , \quad -1 < x < 1.$$

The second term is very quiet and, indeed, it is an entire function of ℓ. It is the only one which exists in potential scattering and its nice analytic properties have been used by MANDELSTAM [8] in a recent study where he describes a model of relativistic theory which does not exhibit crossing symmetry. Mandelstam proves there that Chew's hypothesis is verified.

The first term, however, is not regular, but has the poles of $Q_\ell(-x)$ which are at every negative integer ℓ. In particular, the first pole at $\ell = -1$ is very unlikely to vanish, as its residue is

$$\frac{1}{\pi}\int \rho_{t,u}(q^2, x)dx.$$

This can be checked in practice by putting the proper threshold behaviour of $\rho_{t,u}(q^2, x)$ in every particular case of interest. Let us simply assume that the residue is not zero. (In any case, all residues cannot be simultaneously zero, as this would imply $\rho_{t,u} \equiv 0$, because of the completeness of Legendre polynomials which are the residues of $Q_\ell(-x)$.

Consider now the function

$$f(q^2) = \lim_{\ell \to -1}(\ell+1)b_\ell(q^2).$$

It has a non-zero left-hand cut, but, if $a_\ell(q^2)$ is meromorphic down to $\ell = -1$, $a_\ell(q^2)$ is bounded by unitarity on the real positive q^2 axis $f(q^2) = 0$ there. This is a contradiction and proves that $a_\ell(q^2)$ cannot be meromorphic along the real ℓ axis down to $\ell = -1$.

A possible explanation of this phenomenon was furnished by Gribov. As ℓ goes to -1, the importance of the left-hand cut increases. This has the effect in many instances of pulling poles out of the right hand cut ("bound states") in order to counteract the strong left-hand cut. Gribov suggests that more and more of these poles come out as $\ell \rightarrow -1$, until their residue distribution exactly cancels the left hand cut of $f(q^2)$ at the limit $\ell = -1$. This implies an accumulation point of Regge poles around $\ell = -1$, each of which attains -1 only when q^2 is infinite. Notice that in potential scattering with a regular potential, the Regge poles go to negative integers at infinite energies. If Gribov's mechanism is right, the occurrence of the "third" spectral function would only mean that an infinite number of Regge poles reach each negative integer. Obviously this reasoning only applies to the first non-vanishing pole of Eq. (27), but it may be expected that the result holds for all non-vanishing poles.

In all cases, however, we see that it is impossible for the amplitude to fall off faster than $1/s$ as we must have a singularity at least at $\ell = -1$. This should be experimentally checked.

5. CONNECTION WITH THE PHYSICAL WORLD [12]

There are basically two kinds of immediate tests of the whole Regge pole story. The first approach consists in looking in one channel at the Regge poles of the same channel going through physical values of the spin, or nearby, thus producing stable or unstable particles. The second approach consists in studying the asymptotic behaviour of the amplitudes in one physical region, thus getting information on the Regge poles of crossed channels. It is obvious that we cannot get complete information on Regge poles by these methods, but we may get enough to decide whether or not the Regge poles have anything to do with nature.

5.1. First approach: physical ℓ

Consider a well defined channel, that is a well-defined set of quantum numbers, baryon number, charge, parity, strangeness and isotopic spin. In this channel, the S-matrix will be considered expressible in terms of the total angular momentum j and any other set of variables. We assume that, as a function of j, it is meromorphic down to Re $j = 0$ and that this analytic continuation furnishes the right value of the partial waves. Of course, we know already that even for 2-body amplitudes, it is not possible to define one amplitude, but rather two, according to Eq. (17). We thus assume that these two are enough and that every physical S-matrix element is either equal to the value of the interpolation by the S^+ matrix or by the S^- matrix, it being understood that angular momenta differing by two are related to the same interpolation. The sign put in superscript will be called, following Gell-Mann, the signature.

In each channel, then, we may order all stable particles and all resonances according to their signature. Then we could expect these states to belong to the same Regge trajectory, or at least to belong to a finite number (smaller than the number of states) of Regge trajectories. The first attempt in this direction was made by CHEW and FRAUTSCHI [9]. They made a diagram of all then known particles with the squared mass in abscissa and the spin in ordinate. Only one pair of particles could be fitted: the nucleon $P_{1/2}$ and the third nucleon resonance $F_{5/2}$. This corresponds to an average slope $\partial\alpha/\partial s$ of $1/50\,m_\pi^2$. This, quite remarkably, fits with a formula of potential scattering which expresses $\partial\alpha/\partial s$ as $R_{av}^2/4(2\alpha+1)$, where R_{av} is some average radius of the wave function. If we take it to be $1/2\,m_\pi$, we get the result. This, of course, should not be taken too seriously as we are in the relativistic region. However, this figure of $1/50\,m_\pi^2$ should be retained as we shall encounter it many times.

For example, the possibility has been mentioned at the 1962 Geneva Conference of the existence of a resonance at 1920 MeV, B = 1 S = 0, I = 3/2. If the other quantum numbers turn out to be correct, this could correspond to the same Regge pole as the well-known (3,3) resonance. In the same way, the excited hyperon of mass 1815, which appears to have 1 = 0, could be the same pole as the Λ. These two cases would correspond to an average $\partial\alpha/\partial s$ of $1/50\,m_\pi^2$ in the same way.

This is about all the information we can get from this first point of view and is pretty meager. However, the spectroscopy of high-energy resonances is a science in full bloom and the number of pairs associated to the same Regge poles may increase beyond expectation in a few years.

5.2. Second approach: asymptotic properties of cross-sections

The study of the asymptotic properties of the cross-sections at fixed momentum transfer as a function of the energy can also help checking the Regge pole hypothesis. This has to be done in a fairly indirect fashion, as we have seen that it is very difficult to determine from the asymptotic behaviour whether one has to do with poles or with other singularities. However, a number of non-trivial predictions can be made and checked against experiment.

The total cross-sections, being given by the optical theorem as the imaginary part of the amplitude up to some kinematical factor, are a very convenient tool. It should be possible to express them in the form:

$$\sigma_{tot}(s) \approx (1/s)\sum_K \beta_\alpha(0) P_{\alpha_K(0)}(1+s/2), \tag{28}$$

the summation being carried out over all Regge poles having the appropriate quantum numbers, that is the quantum numbers of the particles which could be exchanged in the scattering process. At this point a very tempting assumption can be made, that of factorization [13].

The idea is the following. Consider a matrix M, function of some parameters $\{\lambda\}$. If this matrix is meromorphic in $\{\lambda\}$; the poles are most likely to be simple and their residues to be of rank one in the following sense.

If we consider the inverse matrix $N = M^{-1}$, Det N has a zero at the pole and this zero is most likely to be simple, i.e. we may vary the elements of N by small amounts related by only one condition and still keep a simple zero. If we wanted to keep a double zero, we could only vary the elements of N by small amounts related by 2 conditions and so on. If the zero is simple

N has only one eigenvalue zero and therefore M has only one eigenvalue infinite or, what is the same, the rank of the residue is one.

If this is accepted, then, we find that the S-matrix, as expressed as a function of complex j, is most likely to have residues which are of rank one.

This implies that the factors β in an expression like (28) may be written as follows.

Assume that the reaction under consideration is among particles a and b. Then we have:

$$\beta_\alpha(0) = \beta_K^{(a)}(0) \,.\, \beta_K^{(b)}(0).$$

This has very strong experimental implications, for example, if we assume that the leading term in (28) corresponds to

$$\alpha_1(0) = 1 \qquad \text{("Pomeranchuk pole")} \tag{29}$$

which leads to constant cross-sections at infinity, then

$$\sigma_{tot}(a+a) \;.\; \sigma_{tot}(b+b) = [\sigma_{tot}(a+b)]^2. \tag{30}$$

No such relations has yet been experimentally checked, as they always necessitate targets which are difficult to prepare. However, it is possible that in the future cross-sections like $\sigma_{tot}(\pi+\pi)$ might be measured by some indirect way: extrapolation or the like.

It should also be possible to go farther than that and estimate the next terms of Eq.(28). One gets into trouble here. Take, for example, the case of (p, p) and (p, $\overline{p}$) and (p, $\overline{p}$) scattering. The total cross-sections look as if they were going down slowly towards their limit, the difference decreasing like $S^{-0.5}$. However, σ_{pp} is much nearer to it than σ_{pp}. This is very nice and we hope that it could be the influence of the Regge pole of the ρ resonance. However, this ρ resonance (or the ω resonance), because of its quantum numbers, only contributes to he difference $\sigma_{pp} - p_{pp}$. Therefore one needs another, as yet unknown, Regge pole which has about the same α and β and which has the proper quantum numbers so that it contributes to the sum $\sigma_{p\overline{p}} + \sigma_{pp}$.

5.3. Non-forward scattering

If this last pole exists, one may wonder why it does not correspond to any known particle. This is also true of the dominant ("Pomeranchuk") pole. In fact, the signature of the Pomeranchuk pole is + and therefore it should go through 0 for some negative value of t where a particle should appear. This has been investigated by Gell-Mann and, though not understood in full detail, the situation is pretty well clarified.

The idea, which has been checked by Gell-Mann in a 3-body model, is that for every integer j, there are two different families of Legendre functions which become completely independent. There are those with singularities and those without. The Legendre functions without singularities are connected with the representations of the rotation group. The others may also be connected with the rotation group, but they do not form a basis for a representation because of their singularities.

As a parenthesis, Wigner has shown what the representations of the Poincaré group look like for imaginary mass*. The difference lies in the fact that, for imaginary mass, the relevant surface is not a sphere, but a hyperboloid, and the conditions for the absence of singularities on the hyperboloid are quite different from those for the absence of singularities on a sphere.

For example ℓ, all Legendre functions have singularities on the sphere. When one follows a Regge pole, as a function of ℓ and reaches an integer value of j, one expects the relevant "wave functions", whatever that means precisely, either to keep their singularities on the sphere, or to lose them. In the first case, one will not get any particle or resonance with that spin and this is what happens in the case of the "ghost" of the Pomeranchuk pole at $j = 0$. In the second case, it will furnish an honest particle which can be seen.

Thus, it is getting fairly difficult to trace the Regge poles in their own channel, one may miss them fairly frequently. The behaviour indicated by Eq. (12) $A(s,t) = \beta(t)A^{\alpha(t)}$ which leads to a differential elastic cross-section of the form

$$d\sigma^{el}(s,t)/dt \approx |\beta(t)|^2 s^{2[\alpha(t)-1]} \tag{31}$$

has been experimentally checked, or at least, that it is not incompatible with experiment.

5.4. Classical picture of high-energy scattering - the puzzle of heavy nuclei

Equation (31) can be interpreted classically, as at very-high energies the wave length of the particles is much smaller than any of the dimensions involved in (31). It is therefore tempting to do so. We may rewrite (31), putting $2\,\partial\alpha/\partial t = a$.

$$d\sigma^{el}/dt \approx |\beta(t)|^2 \exp[-\alpha|t|\ln s]. \tag{32}$$

The pattern is that of a shrinking diffraction peak. This corresponds to an increasing size of the target. However, the total cross-section being constant, we end up with a target which blows up like a puff of smoke, as the energy increases, becoming bigger and thinner.

This is a very striking feature of this whole analysis. One may start wondering what happens when the target is a heavy nucleus. It is known that the scattering of a high-energy proton by a heavy nucleus is essentially proportional to the area of the nucleus, therefore going like $A^{2/3}$, and is essentially constant up to cosmic ray energies.

But what if all the nucleons inside the nucleus start blowing up, thus becoming more and more transparent? Gell-Mann and Udgaonkar have pro-

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* These proceedings.

posed such a model and they show that at very high energies, the cross-section should be proportional to A, rather than $A^{2/3}$, as there is no screening effect any more. The transition of one set of cross-sections to the other should take place very slowly, as the increase in size of the nuclei, and finally, we end up with a cross-section which tends towards its limit as $1/\ln s$, which leads to a cut in the ℓ-plane.

Another possibility is interesting to investigate. Let us use the factorization hypothesis in equation (32). We get for the amplitude

$$A(s,t) = is\beta^{(a)}(t)\beta^{(\ell)}(t)\exp\left(-\frac{a}{2}|t|\ln s\right). \tag{33}$$

We now consider that this is pure diffraction scattering, which occurs with a very weak absorption over a large surface. We can therefore trace back the absorption density $\rho(\vec{b})$ as a function of the impact parameter $\vec{b}$: $\rho(\vec{b})$ is the two dimensional Fourrier transform of A(s, t) as expressed in terms of the two-dimensional transverse momentum transfer.

The product (31) is transformed into a convolution by this Fourrier transformation:

$$\rho(\vec{b}) = \rho^{(a)}(\vec{b}) * \rho^{(b)}(\vec{b}) * (2\pi/a\ln s)\exp(-b^2/2a\ln s). \tag{34}$$

Now it seems that this way of writing $\rho(\vec{b})$ is fairly natural and represents a part involving the target and only the target, a part involving the incident particle and only it and a part involving the Pomeramchuk pole and only it. All these parts could be replaced by another of a similar nature and it would only describe another physical phenomenon.

The classical interpretation of (34) is obvious: $\rho^{(a)}(\vec{b})$ and $\rho^{(b)}(\vec{b})$ represents the net probability of emitting or absorbing a Pomeranchuk pole at a place $\vec{b}$, integrated along the line of flight, of particles a and b repectively.

The expression $[\pi/a\ln s]\exp[-b^2/2a\ln s]$ is the probability, again integrated along the line of flight, for a Pomeranchuk pole emitted at the origin, to be absorbed at a distance b from the origin.

We may also think that in fact, all these probabilities should be sperically symmetrically distributed; it is an easy matter then to compute the 3-dimensional distributions out of the integrated ones (Abel's problem).

In this case, however, the puzzle of the heavy nuclei disappears, as only the Pomeranchuk pole blows up and thins out. The screening effect still takes place inside $\rho_{(a)}$ and $\rho_{(b)}$ and the cross-section goes like $A^{2/3}$, even asymptotically.

In conclusion, one should bear in mind the amount of guesses and conjectures which have been used in this whole study. This is a very unscientific situation, in which the bases are so far away from the prediction of experiments that there is no such thing as a decisive experiment to test this or that basic postulate. It is therefore pretty frail and it would be in many ways a miracle if all this is still true in 10 years from now.

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