Applications of the gauge/gravity duality

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Abstract

The gauge/gravity duality maps quantities computed in classical gravity theories to observables in certain gauge theories in the limits of strong coupling and large number of colors. Though no gauge theories with established gravity duals are known to occur in nature, simple models informed by the gauge/gravity duality could still provide intuition about physical phenomena.

This dissertation focuses on two such models. The first model consists of five-dimensional gravity coupled to a single real scalar. Black hole solutions to this model describe a four-dimensional field theory at finite temperature. In Chapter 2, a family of black holes is constructed that closely mimics the equation of state of quantum chromodynamics.

The second model consists of a $U(1)$ gauge field minimally coupled to a complex scalar ψ in four-dimensional gravity. Black hole solutions to this model describe a three-dimensional field theory at finite temperature and finite chemical potential. Above some critical temperature T_c, $\psi = \psi^* = 0$. Below T_c, however, $|\psi|$ is nonzero along the thermodynamically stable branch of solutions, and the $U(1)$ symmetry is spontaneously broken. Since superconductivity is typically understood as a broken gauge symmetry, the model may capture some features of high-temperature superconductors. Chapter 3 probes the model in the zero-temperature limit, where the black hole disappears. The solutions represent flows between infrared and ultraviolet field theories. In some solutions, the speed of light is lower in an infrared conformal theory than in the ultraviolet theory. Other solutions have an infrared Lifshitz symmetry, which features an anisotropy between the space and time coordinates.

Chapter 1 is essentially a user's guide to the gauge/gravity duality. It provides enough information for non-experts to understand the remainder of the dissertation.

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To my parents, my sister, and my '98 Camry.

Relation to previously published work

Chapters 2 and 3 are lightly edited versions of references [46] and [48], respectively. Both articles were published previously in *Physical Review D*; the appropriate notices of copyright are included in the bibliography.

Contents

Chapter 1

Background

1.1 Motivation

Theoretical physicists confront two major challenges in their research:

1. Theories that are known to describe natural phenomena, but that are difficult to solve. An example is quantum chromodynamics (QCD), the theory of the strong force. Ordinary perturbation theory only captures the physics at short distances, where the coupling constant α_s is small. Lattice methods, on the other hand, are computationally intensive and largely ineffective in predicting dynamical quantities like the shear viscosity.

2. Natural phenomena that have eluded satisfactory theoretical description. An example is high-temperature superconductivity. In this context, "high-temperature" means "above ~ 30 K," where the canonical microscopic picture provided by BCS theory breaks down. While the presence of flux quantization and the Josephson effect points towards the formation of Cooper pairs, the dominant pairing mechanism is poorly understood.

This dissertation discusses models rooted in the gauge/gravity duality that may help shed light on the examples cited above. The gauge/gravity duality relates quantum field theories to supergravity theories via string theory, the mathematical machinery that pertains to one-dimensional extended objects, or strings. Some essentials of the gauge/gravity duality are introduced in this chapter. Chapter 2 describes a model whose parameters can be adjusted to approximately reproduce the equation of state of QCD from lattice data. Chapter 3 focuses on the zero-temperature behavior of a model that in some sense realizes superconductivity. A summary of the original research in this dissertation is contained in a concluding chapter.

1.2 Elements of the gauge/gravity duality

The Maldacena conjecture [78] provides us with the quintessential example of the gauge/gravity duality. In its strongest form, it asserts that type IIB superstring theory on $AdS_5 \times S^5$ is equivalent, or dual, to $\mathcal{N} = 4$ $SU(N_c)$ super-Yang-Mills theory (SYM) in four spacetime dimensions. Call the Yang-Mills coupling g_{YM}. When both the t'Hooft coupling $\lambda \equiv g_{YM}^2 N_c$ and N_c are taken to infinity, the dual description of SYM collapses to classical type IIB supergravity (SUGRA) on $AdS_5 \times S^5$. More globally, the gauge/gravity duality is a reservoir of intuition about field theories at strong coupling: quantities computed in a field theory at large λ and large N_c are mapped to quantities computed in a gravity dual. Some generic features of this mapping are reviewed below.

1.2.1 Conformal field theories and anti-de Sitter space

The relativistic conformal group in d spacetime dimensions is generated by translations P_μ, boosts and rotations $M_{\mu\nu}$, dilatations D, and special conformal transformations K_μ with nonvanishing commutators[1]

$$\begin{aligned}[D, K_\mu] = -iK_\mu \qquad [D, P_\mu] = iP_\mu \qquad [K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu}) \\ [K_\rho, M_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \qquad [P_\rho, M_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\ [M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho})\,,\end{aligned} \tag{1.1}$$

where η is the Minkowski metric. A conformal field theory (CFT) is a quantum field theory whose correlation functions are invariant under the action of this group. Conformal symmetry is nearly synonymous with scale invariance, a self-similarity property contained in the dilatation operator; though a conformal theory is always scale-invariant, a scale-invariant theory need not be conformal. Scale invariance is a powerful constraint: ordinarily, a quantum field theory behaves differently at different energy scales according to its renormalization group equations. For example, at one loop, the beta function of pure $SU(N_c)$ Yang-Mills theory is

$$\beta(g_{YM}(\mu)) \equiv \mu\frac{dg_{YM}}{d\mu} = -\frac{11}{3}\frac{g_{YM}^3 N_c}{16\pi^2}\,, \tag{1.2}$$

and we say the coupling constant g_{YM} runs with the energy scale μ. For $\mathcal{N} = 4$ super-Yang-Mills theory, a CFT, g_{YM} does not run with energy scale; the beta function vanishes identically.

In the gravity dual of a given field theory, an extra coordinate plays the role of the energy scale. This "holographic" coordinate is r in the geometry

$$ds^2 = \frac{r^2}{L^2}(-dt^2 + dx_i^2) + \frac{L^2}{r^2}dr^2\,. \tag{1.3}$$

Above, L is the curvature radius, and i runs from 1 to $d-1$. The line element (1.3) describes a Poincaré patch of anti-de Sitter space (AdS) in $d+1$ dimensions.

[1] In this work, Greek indices run across all spacetime dimensions, and Latin indices run across only spatial dimensions.

The extreme ultraviolet corresponds to the conformal boundary at $r = \infty$, while the extreme infrared lies at $r = 0$. AdS is the maximally symmetric vacuum solution to Einstein's equations with a negative cosmological constant. Many properties of AdS run counter to intuition about Minkowski space. For instance, though the boundary is an infinite distance from any point (x_μ, r) at finite r, a photon starting at (x_μ, r) can reach the boundary in finite coordinate time.

The isometry group of AdS_{d+1} is $SO(d, 2)$, which is isomorphic to the relativistic conformal group. In fact, the gauge/gravity duality is often called the AdS/CFT correspondence. Most of the best-understood examples of the duality involve spaces that are at least asymptotically AdS in the ultraviolet and field theories whose conformality is broken, if at all, by one or more relevant scalar operators. In more detail, the duality is between a field theory, typically in three or four dimensions, and a supergravity theory in ten or eleven dimensions. The spacetime dimensions of the field theory are mapped to spacetime dimensions in supergravity. One of the remaining dimensions of the supergravity theory corresponds to the r coordinate described in the previous paragraph, and the others go to form a compact space that encodes part of the physics of the dual field theory. In particular, global symmetries of the dual field theory are represented as isometries of the compact space. In the case of $AdS_5 \times S_5$, the five-sphere S^5 has an $SO(6)$ symmetry. This symmetry is identified with the $SU(4)$ R-symmetry of $\mathcal{N} = 4$ super-Yang-Mills theory.

Let X_n denote some n-dimensional compact manifold. In general, spaces that asymptote to $AdS_{d+1} \times X_n$ in the ultraviolet are related to d-dimensional field theories, where $d + n + 1$ is either ten or eleven. The original research in this dissertation never refers to the compact manifolds. Instead, it treats effective four- or five-dimensional actions that can be imagined to obtain by integrating out the compact dimensions of the full ten- or eleven-dimensional theories. Although they are qualitatively similar to genuine supergravity constructions, the effective actions considered here are always *ad hoc*. The geometries that arise from these effective actions are noncompact and usually approach AdS in the ultraviolet. Field theory quantities are read off from the boundary asymptotics of fields in the noncompact space. As an illustrative example, in the next subsection, we compute the two-point correlation function of a generic scalar operator in a generic field theory dual to pure AdS.

Incidentally, there is growing interest in tentative extensions of the gauge/gravity duality that do not involve AdS. They are constructed by choosing the noncompact manifold to realize a symmetry besides relativistic conformal symmetry. For instance, some strongly correlated electron systems have quantum critical points that exhibit the dynamical scaling

$$t \to \lambda^z t \qquad x_i \to \lambda x_i \,, \tag{1.4}$$

where z is called the scaling exponent. When $z = 1$, (1.4) reduces to the scaling found in AdS, but for other values of z, a different geometry is required [68]:

$$ds^2 = -\frac{r^{2z}}{L^{2z}}dt^2 + \frac{r^2}{L^2}dx_i^2 + \frac{L^2}{r^2}dr^2 \,. \tag{1.5}$$

The isometry group of the metric above defines Lifshitz symmetry. Alternatively, Lifshitz symmetry is defined by a hamiltonian H, translations P_i, rotations M_{ij}, and dilatations D with nonvanishing commutators [1]

$$\begin{aligned} [D, H] = izH \qquad [D, P_i] = iP_i \qquad [P_i, M_{jk}] = i(\delta_{ik}P_j - \delta_{ij}P_k) \\ [M_{ij}, M_{kl}] = i(\delta_{ik}M_{jl} + \delta_{jl}M_{ik} - \delta_{jk}M_{il} - \delta_{il}M_{jk}) \,. \end{aligned} \tag{1.6}$$

Unlike their conformal analogs (1.1), these relations involve neither boosts nor special conformal transformations. Moreover, in the Lifshitz case, dilatations treat time and space differently. A gravity model with an emergent Lifshitz symmetry in the infrared is discussed in Chapter 3.

1.2.2 Correlation functions

The fundamental objects of a CFT are local operators. Physically meaningful quantities are constructed from correlation functions of these operators. To avoid ambiguities in defining correlators, only Euclidean field theories are considered in this subsection. A gravity dual to a Euclidean CFT involves the Euclidean variant of AdS, which is obtained from (1.3) by a simple Wick rotation of the t coordinate:

$$ds^2 = \frac{r^2}{L^2}(dt^2 + dx_i^2) + \frac{L^2}{r^2}dr^2 \,. \tag{1.7}$$

To avoid notational clutter, x is used to denote all the spacetime dimensions of a given field theory. The development here is based on [35, 87, 80, 71].

Consider a d-dimensional CFT with a gravity dual. According to the AdS/CFT dictionary, there is a one-to-one correspondence of "bulk" fields $\phi_i(x, r)$ in some asymptotically AdS space and single-trace, gauge-invariant field theory operators $\mathcal{O}_i(x)$. For example, the CFT stress-energy tensor $T_{\mu\nu}(x)$ is mapped to the metric $g_{\mu\nu}(x, r)$ in gravity. Here, r is again the holographic coordinate. Let $r = \infty$ denote the conformal boundary, and use ϕ_b as a shorthand for the boundary data $\phi_i(x, r \to \infty)$. Now imagine turning on a source $\phi_s(x)$ for only the operator $O_1(x)$. The AdS/CFT dictionary further identifies the string theory partition function

$$Z_{string}(\phi_b) \equiv \int_{\mathcal{M},\phi_b} \mathcal{D}\phi_i \, e^{-S_{string}} \tag{1.8}$$

with the field theory expression

$$Z_{CFT}(\phi_s) \equiv e^{-W_{CFT}(\phi_s)} \equiv \left\langle \exp\left\{ -\int d^d x \, \phi_s(x)\mathcal{O}_1(x) \right\} \right\rangle_{CFT} \,. \tag{1.9}$$

Above, Z_{string} is the generating functional of on-shell string amplitudes on a ten-dimensional manifold $\mathcal{M}$ with an AdS factor in the extreme ultraviolet. The path integral is understood to be taken over all the bulk fields, not just $\phi_1(x,r)$. ϕ_b serves as a set of boundary conditions for the bulk fields; in particular, $\phi_1(x, r \to \infty)$ is linear in $\phi_s(x)$. On the field theory side, Z_{CFT} is the generating functional of correlation functions of O_1, and W_{CFT} is the associated generating functional of connected correlation functions. For instance, the two-point correlator of $\mathcal{O}_1$ is given by

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_1(x_2)\rangle = -\frac{\delta}{\delta\phi_s(x_1)}\frac{\delta}{\delta\phi_s(x_2)}W_{CFT}|_{\phi_s=0} \,. \tag{1.10}$$

So ϕ_s is merely a calculational expedient that is eventually set equal to zero.

When the number of colors N_c in the gauge theory is taken to infinity, string loops can be ignored, and Z_{string} is dominated by its saddle point. When the t'Hooft coupling λ in the gauge theory is also taken to infinity, the dual string theory reduces to some version of supergravity. Equation (1.8) then becomes

$$Z_{string}(\phi_b) = e^{-S_{SUGRA}(\phi_b)}\,, \tag{1.11}$$

where S_{SUGRA} is the on-shell action of the supergravity theory. In practice, "on-shell" means that the action is evaluated on a solution to the classical equations of motion subject to the boundary conditions specified by ϕ_b. Comparing (1.11) with (1.9), we see that

$$W_{CFT} = S_{SUGRA}\,, \tag{1.12}$$

or in words, the generating functional of connected correlation functions in a CFT is the on-shell action of its gravity dual. The quantitative power of the gauge/gravity duality stems directly from this statement: many computations that appear intractable in a strongly coupled field theory are straightforward in its gravity dual.

Now consider, for concreteness, a four-dimensional CFT dual to type IIB SUGRA on $AdS_5 \times X_5$, where X_5 is some five-dimensional compact manifold. Compactify the SUGRA theory over the X^5, and assume that the operator $\mathcal{O}(x)$ in the field theory is dual to a massive scalar $\phi(x,r)$ in the bulk. Our goal is to find the two-point correlator of $\mathcal{O}$. Our starting point is the effective five-dimensional action

$$\begin{aligned} S_{short} &= \frac{1}{2\kappa_5^2}\int d^4x\,dr\,\sqrt{g}\left(\frac{1}{2}(\partial\phi)^2+\frac{1}{2}m^2\phi^2\right) \\ &= \frac{1}{2\kappa_5^2}\int d^4x\,dr\,\sqrt{g}\left(\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)+(\partial_r\phi)(\partial^r\phi)+\frac{1}{2}m^2\phi^2\right)\,. \end{aligned} \tag{1.13}$$

Here, κ_5 is the five-dimensional gravitational constant, m is the mass of the scalar, and g is the determinant of the metric for AdS_5. Once evaluated on shell, S_{short} can be substituted for S_{SUGRA} in (1.12). Since only the source that corresponds to ϕ is turned on, terms involving other bulk fields are not necessary in S_{short}. Moreover, adding higher-order terms in ϕ to S_{short} would simply yield contributions that vanish once ϕ_s is set equal to zero in (1.10).

The equation of motion for ϕ that follows from S_{simple} is the Klein-Gordon equation

$$(\Box - m^2)\phi(x,r) = 0\,, \tag{1.14}$$

where $\Box$ is defined on AdS_5. With AdS_5 as in (1.7), this equation can be rewritten as

$$\left[\frac{L^2}{r^2}\sum_\mu \frac{\partial^2}{\partial x_\mu^2}+\frac{1}{L^2r^3}\frac{\partial}{\partial r}r^5\frac{\partial}{\partial r}-m^2\right]\phi(x,r)=0\,. \tag{1.15}$$

The relevant solution to (1.15) is the bulk-to-boundary propagator

$$\phi(x,r)=K(r,x;x')\equiv\frac{\Gamma(\Delta)}{\pi^2\Gamma(\Delta-2)}\frac{(L^2/r)^\Delta}{[(L^2/r)^2+|x-x'|^2]^\Delta}\,. \tag{1.16}$$

Above, x' is a set of spacetime coordinates introduced for later convenience, and Δ is related to the mass of the scalar through

$$\Delta(\Delta-4)=m^2L^2\,. \tag{1.17}$$

Equation (1.17) has two solutions for Δ. In this subsection, we take

$$\Delta = 2 + \sqrt{4 + m^2 L^2} \tag{1.18}$$

and assume $\Delta > 2$. Reference [72] discusses some subtleties that arise when other values of Δ are considered.

At $r = 0$, K vanishes. As $r \to \infty$, K behaves as

$$K(r, x; x') \approx \delta^{(4)}(x - x') \left(\frac{r}{L^2}\right)^{\Delta - 4} + \frac{\Gamma(\Delta)}{\pi^2 \Gamma(\Delta - 2)} \frac{1}{|x - x'|^{2\Delta}} \left(\frac{r}{L^2}\right)^{-\Delta} . \tag{1.19}$$

More generally, in any space that approaches the AdS_5 metric (1.7) in the extreme ultraviolet, the scalar would take the form

$$\phi(x, r) \approx A(x) \left(\frac{r}{L^2}\right)^{\Delta - 4} + B(x) \left(\frac{r}{L^2}\right)^{-\Delta} \tag{1.20}$$

near the conformal boundary. Up to dimensionless[2] factors that depend on how $\mathcal{O}$ is normalized, the coefficient $A(x)$ of $(r/L^2)^{\Delta-4}$ is simply the source $\phi_s(x)$, while the coefficient $B(x)$ of $(r/L^2)^{-\Delta}$ is the vacuum expectation value of $\mathcal{O}$. To proceed in the present case of pure AdS_5, set $A(x) = \phi_s(x)$ by integrating K over the x' coordinates:

$$\phi(x, r) = \int d^4 x' K_\Delta(r, x; x') \phi_s(x') . \tag{1.21}$$

This construction exploits the delta function in (1.19).

Equation (1.21) can now be inserted in S_{short}, but it is convenient to first massage the action into a more tractable form. Integrate by parts in (1.13) to obtain

$$S_{short} = \frac{1}{2\kappa_5^2} \int d^4 x \, dr \, \left[-\sqrt{g}\phi(\Box - m^2)\phi + \partial_\mu(\sqrt{g}\phi\partial^\mu\phi) + \partial_r(\sqrt{g}\phi\partial^r\phi) \right] . \tag{1.22}$$

[2]Throughout this work, a quantity described as "dimensionless" has zero mass dimension.

Since S_{short} will be evaluated on shell, the term involving the Klein-Gordon operator $\Box - m^2$ will vanish by virtue of the equation of motion (1.14). Moreover, the $\partial_\mu(\phi\partial^\mu\phi)$ term does not contribute if $\phi(x_\mu \to \pm\infty, r) \to 0$. Plugging (1.7) and (1.21) into the remaining term gives

$$\begin{aligned} S_{on-shell} &= \frac{1}{2\kappa_5^2}\int d^4x \int_0^\infty dr\partial_r(\sqrt{g}\phi g^{rr}\partial_r\phi) \\ &= \frac{1}{2\kappa_5^2}\int d^4x_1' \int d^4x_2' \Big[\phi_s(x_1')\phi_s(x_2') \\ &\times \int d^4x \frac{r^5}{L^5} K(r,x;x_1')\partial_r K(r,x;x_2')\Big]_{r\to\infty} . \end{aligned} \tag{1.23}$$

Using the asymptotic expression for K (1.19), $S_{on-shell}$ can be simplified to

$$\begin{aligned} S_{on-shell} &= \frac{L^3}{2\kappa_5^2}\int d^4x_1' \int d^4x_2' \Big[\phi_s(x_1')\phi_s(x_2') \\ &\times \bigg((\Delta-4)\left(\frac{r}{L^2}\right)^{2\Delta-4}\delta^{(4)}(x_1'-x_2') - \frac{4\Gamma(\Delta)}{\pi^2\Gamma(\Delta-2)}\frac{1}{|x_1'-x_2'|^{2\Delta}} \\ &- \frac{\Delta\Gamma(\Delta)^2}{\pi^4\Gamma(\Delta-2)^2}\left(\frac{r}{L^2}\right)^{4-2\Delta}\int d^4x \frac{1}{|x-x_1'|^{2\Delta}|x-x_2'|^{2\Delta}}\bigg)\Big]_{r\to\infty} . \end{aligned} \tag{1.24}$$

In the limit $r \to \infty$, the term proportional to $(r/L^2)^{4-2\Delta}$ vanishes, and of course the term independent of r remains finite. However, the term proportional to $(r/L^2)^{2\Delta-4}$ is a contact term that blows up at the boundary. It must be removed by a procedure called holographic renormalization. The strategy is as follows:

1. Move the boundary from $r = \infty$ to $r = r_0$, where $r_0 \gg L$. As noted in the previous subsection, the holographic coordinate r represents the energy scale of the dual field theory. r_0 serves an an ultraviolet cutoff.

2. Add a boundary counterterm to the bulk action that cancels the contact term. The counterterm will not affect the equation of motion.

3. Take the limit $r_0 \to \infty$.

The appropriate counterterm is

$$S_{counter} = -\frac{\Delta - 4}{2L\kappa_5^2} \int d^4x \sqrt{g_0} \phi(x, r_0)^2 \,, \tag{1.25}$$

where g_0 is the metric on the renormalization surface at $r = r_0$. The full renormalized action is

$$S_{ren} = \frac{1}{2\kappa_5^2} \int d^4x dr \sqrt{g} \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2} m^2 \phi^2 \right] - \frac{\Delta - 4}{2L\kappa_5^2} \int d^4x \sqrt{g_0} \phi(x, r_0)^2 \,. \tag{1.26}$$

Evaluated on shell, this action yields only the finite term

$$S_{ren,on-shell} = -\frac{(\Delta - 2)\Gamma(\Delta)}{\pi^2 \Gamma(\Delta - 2)} \frac{L^3}{\kappa_5^2} \int d^4x_1' \int d^4x_2' \frac{\phi_s(x_1')\phi_s(x_2')}{|x_1' - x_2'|^{2\Delta}} \,. \tag{1.27}$$

The two-point function is finally found by taking $W_{CFT} = S_{ren,on-shell}$ in (1.10):

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle = \frac{(2\Delta - 4)\Gamma(\Delta)}{\pi^2 \Gamma(\Delta - 2)} \frac{L^3}{\kappa_5^2} \frac{1}{|x_1 - x_2|^{2\Delta}} \,. \tag{1.28}$$

L^3/κ_5^2 is related to the number of colors N_c in the dual field theory and is dimensionless. Equation (1.28) therefore implies that Δ is the mass dimension of $\mathcal{O}$. Though only the case $\Delta > 2$ was treated above, the only physical constraint on Δ is the unitary bound, which requires that $\Delta \geq 1$. Note from (1.17) that when $1 \leq \Delta < 4$, m^2 is negative. In fact, the so-called Breitenlohner-Freedman bound [12, 13, 81] stipulates that

$$m^2 L^2 \geq -4 \,. \tag{1.29}$$

This bound is clearly never violated for real Δ.

In a generic field theory with a gravity dual, an n-point correlator is found by taking n functional derivatives of the on-shell gravity action. This action could involve many bulk fields, which need not be "probe" fields like the scalar considered here;

they could help shape the background geometry. The next subsection discusses an example involving a single bulk scalar.

1.2.3 Deviations from conformality

Consider again turning on a source $\phi_s(x)$ for some gauge-invariant operator $\mathcal{O}(x)$ of a four-dimensional CFT with a gravity dual. In the computation of the two-point correlator above, this source was ultimately set equal to zero. It is possible, however, to keep the source finite so it participates in a genuine deformation of the CFT lagrangian $\mathcal{L}_{CFT}(x)$:

$$\mathcal{L}_{deformed}(x) = \mathcal{L}_{CFT}(x) + \phi_s(x)^{4-\Delta}\mathcal{O}(x)\,. \tag{1.30}$$

$\mathcal{L}_{deformed}$ is the lagrangian of a field theory that is in general nonconformal: the source $\phi_s(x)$ usually introduces a new scale and sets up a renormalization group flow. For $\Delta > 4$, the deformation is irrelevant, and its effects are strongest in the ultraviolet. For $\Delta < 4$, the deformation is relevant, and its effects are strongest in the infrared. For $\Delta = 4$, the deformation is marginal. Marginal deformations fall into three classes: exactly marginal deformations, which preserve the conformality of the field theory, and either relevant or irrelevant marginal deformations, which give rise to behavior qualitatively similar to their relevant or irrelevant counterparts. For the deformation term to be renormalizable, $\Delta \leq 4$.

If $\mathcal{O}$ is dual to the bulk scalar $\phi(x,r)$, the effective five-dimensional dual gravity theory could take the form

$$S_{scalar} = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right]\,, \tag{1.31}$$

where the metric is

$$ds^2 = e^{2A(r)}(-dt^2 + dx_i^2) + e^{2B(r)}dr^2 \tag{1.32}$$

for some functions A and B of the holographic coordinate r. Equation (1.32) should asymptote to the real-time variant of AdS_5 as $\phi \to 0$. This is ensured if at small ϕ,

$$V(\phi) = -\frac{12}{L^2} + m^2\phi^2 + \mathcal{O}(\phi^3)\,. \tag{1.33}$$

As mentioned previously, the scalar behaves as (1.20) at small ϕ. The dimension Δ of the operator $\mathcal{O}$ is still determined from (1.17).

Chapter 2 explores the theory (1.31) at finite temperature. Accommodating finite temperature and finite chemical potential in a gravity dual to a field theory is the subject of the next subsection.

1.2.4 Thermodynamics

At zero temperature, a CFT has no scale. Raising the temperature T from zero breaks conformal invariance softly: boundary conditions involving the new energy scale T are imposed on the CFT action.[3] Since T is the sole energy scale in the theory, the behavior of other thermodynamic quantities — for example, the energy E and the entropy S — is constrained. Now consider a CFT in d dimensions; the entropy is extensive, proportional to the $(d-1)$-dimensional volume V of the system, but has no units. To counterbalance the mass dimension $1-d$ of the volume scaling, S must also be proportional to T^{d-1}. Only infinite-volume systems are treated in this dissertation, so it is useful to instead work with the entropy density s:

$$s = \frac{S}{V} \sim T^{d-1}\,. \tag{1.34}$$

[3]When conformal invariance is broken only by thermodynamic scales, the field theory is still called a CFT.

The energy E is also extensive, but has mass dimension 1. By similar logic, $E \sim VT^d$, and the energy density ϵ is

$$\epsilon = \frac{E}{V} \sim T^d \,. \tag{1.35}$$

It is possible to derive a conformal equation of state that relates ϵ, T, and s. Since the dimensionless quantity $\epsilon/s^{d/(d-1)}$ does not depend on temperature, write

$$\epsilon = cs^{d/(d-1)} \tag{1.36}$$

for some constant c. From the first law of thermodynamics,

$$d\epsilon = Tds \ , \text{ or } \frac{d\epsilon}{ds} = T \,. \tag{1.37}$$

Differentiate (1.36) with respect to s and eliminate c as

$$\frac{d\epsilon}{ds} = \frac{d}{d-1} cs^{1/(d-1)} = \frac{d}{d-1}\frac{\epsilon}{s} \,. \tag{1.38}$$

Equations (1.37) and (1.38) together imply that

$$\epsilon = \frac{d-1}{d} Ts \,. \tag{1.39}$$

Now introduce a second energy scale to the CFT: assume the theory has a conserved current J^μ associated with some global $U(1)$ symmetry, and turn on the corresponding chemical potential μ. For $\mathcal{N} = 4$ SYM, this symmetry could be a $U(1)$ subgroup of the $SU(4)$ R-symmetry. The chemical potential μ sources the time component J^t of the operator J^μ. So the CFT lagrangian $\mathcal{L}_{CFT,\mu}$ at finite chemical potential can be considered a deformation of the CFT lagrangian $\mathcal{L}_{CFT}$ at zero chemical potential as

$$\mathcal{L}_{CFT,\mu}(x) = \mathcal{L}_{CFT}(x) - \mu J^t(x) \,. \tag{1.40}$$

The energy density and entropy density no longer have well-defined temperature scalings; these thermodynamic quantities can in general depend nontrivially on both temperature and chemical potential. However, it is still possible to derive a conformal equation of state analogous to (1.39). Write the first law of thermodynamics as

$$d\epsilon = T ds + \mu d\rho \,. \tag{1.41}$$

Here, the charge density ρ is the expectation value of J^t, and ϵ is treated as a function of both s and ρ. Equation (1.41) implies that

$$\frac{\partial \epsilon}{\partial s} = T \qquad \frac{\partial \epsilon}{\partial \rho} = \mu \,. \tag{1.42}$$

The dimensionless quantity $\epsilon / s^{d/(d-1)}$ can depend on s and ρ only through the dimensionless quotient s/ρ, so

$$\epsilon = s^{d/(d-1)} k(s/\rho) \tag{1.43}$$

for some function k of s/ρ. Take partial derivatives of (1.43) with respect to s and ρ to obtain

$$\begin{aligned} \frac{\partial \epsilon}{\partial s} &= \frac{d}{d-1} s^{1/(d-1)} k(s/\rho) + \frac{s^{d/(d-1)}}{\rho} k'(s/\rho) \\ \frac{\partial \epsilon}{\partial \rho} &= -\frac{s^{(2d-1)/(d-1)}}{\rho^2} k'(s/\rho) \,. \end{aligned} \tag{1.44}$$

Eliminate $\partial\epsilon/\partial s$, $\partial\epsilon/\partial\rho$, k, and k' between the five equations in (1.42), (1.43), and (1.44) to arrive at

$$\epsilon = \frac{d-1}{d} (Ts + \mu\rho) \,. \tag{1.45}$$

The relation (1.39) is recovered when μ is set equal to zero.

At finite temperature, the gravity dual to a field theory has a black hole. The Hawking temperature and Bekenstein-Hawking entropy of the black hole correspond to the temperature and entropy of the field theory. To write general temperature and entropy formulas, consider the dual geometry

$$ds^2 = e^{2A(r)}(-h(r)dt^2 + dx_i^2) + \frac{dr^2}{h(r)} \tag{1.46}$$

for some functions A and h of r. Let i run from 1 to $d-1$, and assume that (1.46) approaches AdS_{d+1} in the extreme ultraviolet. The boundary occurs at $r = \infty$. Choose ultraviolet AdS to take the form (1.46) with

$$A(r) = \frac{r}{L} \qquad h(r) = 1\,. \tag{1.47}$$

This way of writing AdS is obtained from (1.3) by the reparametrization $r \to Le^{r/L}$. If a black hole is present in (1.46), its event horizon occurs at the largest value r_H of r where h has a zero. The Hawking temperature T of the black hole obtains from a simple recipe [30, 60]. Wick-rotate to Euclidean time by taking $t \to it$. The metric is then

$$ds^2 = e^{2A(r)}(h(r)dt^2 + dx_i^2) + \frac{dr^2}{h(r)}\,. \tag{1.48}$$

Expanding it to first-order in powers of $(r - r_H)$ gives

$$ds^2 = e^{2A(r_H)}(h'(r_H)(r - r_H)dt^2 + dx_i^2) + \frac{dr^2}{h'(r_H)(r - r_H)}\,. \tag{1.49}$$

The path of an observer who travels through this geometry without stepping in any of the x_i directions has the line element

$$ds^2 = e^{2A(r_H)}h'(r_H)(r - r_H)dt^2 + \frac{dr^2}{h'(r_H)(r - r_H)}\,. \tag{1.50}$$

Define $z^2 = 4(r - r_H)/h'(r_H)$ and $\theta = e^{A(r_H)} h'(r_H) t/2$ to rewrite (1.50) as

$$ds^2 = z^2 d\theta^2 + dz^2 \,. \tag{1.51}$$

The metric (1.51) has a conical singularity at the horizon $z = 0$. Thermodynamic equilibrium is associated with a smooth geometry and requires the removal of this singularity. Impose the period 2π on θ. Then the original coordinate t has period $4\pi/e^{A(r_H)} h'(r_H)$, the singularity is gone, and both (1.51) and (1.50) describe $\mathbf{R}^2$. The Hawking temperature of the black hole is identified with the inverse period of t:

$$T = \frac{1}{4\pi} e^{A(r_H)} h'(r_H) \,. \tag{1.52}$$

The Bekenstein-Hawking entropy of the black hole is proportional to the area of the event horizon, which spans the $d-1$ spatial dimensions. This area diverges, but the entropy density is given by

$$s = \frac{2\pi}{\kappa_{d+1}^2} e^{(d-1)A(r_H)} \,, \tag{1.53}$$

where κ_{d+1} is the gravitational constant in $d+1$ dimensions.

The energy density ϵ of a field theory is simply the tt component of the expectation value of the stress-energy tensor $T^{\mu\nu}$. According to the AdS/CFT dictionary, in a gravity dual, this expectation value is found by evaluating [6]

$$\langle T^{\mu\nu} \rangle = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{grav}}{\delta \gamma_{\mu\nu}} \,. \tag{1.54}$$

Above, $\gamma_{\mu\nu}$ is the metric at boundary AdS, and S_{grav} is the effective d-dimensional gravitational action evaluated on shell. Any ultraviolet divergences can be canceled by adding boundary counterterms to S_{grav}, as illustrated in the two-point correlator computation from subsection 1.2.2. For a CFT dual to the geometry (1.46), the energy density appears in the asymptotic behavior of h near the boundary as

$$h \approx 1 - \frac{2L\kappa_{d+1}^2 \epsilon}{d-1} e^{-dr/L} \,. \tag{1.55}$$

The simplest gravity dual to a d-dimensional CFT is described by the effective action

$$S = \int dr\, dt\, d^{d-1}x \left[R + \frac{d(d-1)}{L^2} \right] , \tag{1.56}$$

where R is the Ricci scalar. When $d = 4$, this CFT could be $\mathcal{N} = 4$ SYM. An extremum of the action (1.56) is the AdS-Schwarzschild solution, given by the metric (1.46) with

$$A = \frac{r}{L} \qquad h = 1 - e^{-d(r-r_H)/L} \,. \tag{1.57}$$

AdS-Schwarzschild describes a class of CFTs at finite temperature. Plugging (1.57) into the formulas for temperature and entropy derived above gives

$$T = \frac{d}{4\pi L} e^{r_H/L} \qquad s = \frac{2\pi}{\kappa_{d+1}^2} e^{(d-1)r_H/L} \,. \tag{1.58}$$

The energy density is read off the ultraviolet behavior of h according to (1.55):

$$\epsilon = \frac{d-1}{2L\kappa_{d+1}^2} e^{dr_H/L} \,. \tag{1.59}$$

As expected for a finite-temperature CFT, $\epsilon = (d-1)sT/d$, and $s \sim T^{d-1}$. The action (1.31), on the other hand, can describe a nonconformal field theory. At finite temperature with the metric as in (1.46), s and T are still obtained from (1.52) and (1.53). However, h does not in general behave as (1.55) near the boundary, and $\epsilon = (d-1)sT/d$ no longer holds.

A chemical potential is accommodated in the bulk theory like the source of a scalar operator. For instance, if a CFT described by (1.40) admits a gravity dual, the conserved field theory current J^μ associated with μ corresponds to a $U(1)$ gauge field A^μ in the bulk. The $U(1)$ gauge symmetry in gravity could be a subgroup of a larger gauge symmetry. Assume the gauge field takes the form

$$A_\mu dx^\mu = \Phi(r) dt \,. \tag{1.60}$$

In subsection 1.2.2, the source $\phi_0(x)$ and the vacuum expectation value of the operator $\mathcal{O}$ were read off the boundary behavior of $\phi(x,r)$. Here, μ sources J^0, and ρ is the vacuum expectation value of J^0. Both μ and ρ participate in the boundary behavior of Φ as

$$\Phi(r) \approx 2L\mu - \frac{\kappa_{d+1}^2 \rho}{d-2} e^{-(d-2)r/L} \,. \tag{1.61}$$

The simplest bulk action that supports finite chemical potential is

$$S = \int dr\, dt\, d^{d-1}x \left[R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{d(d-1)}{L^2} \right] ; \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{1.62}$$

An extremum of the action (1.62) is the Reissner-Nordstrom AdS (RNAdS) solution, which is given by (1.46) and (1.60) with

$$\begin{aligned} A(r) = \frac{r}{L} \qquad & h(r) = 1 - \frac{2L\kappa_{d+1}^2 \epsilon}{d-1} e^{-dr/L} + \frac{\kappa_{d+1}^4 \rho^2}{2(d-1)(d-2)} e^{-(2d-2)r/L} \\ & \Phi(r) = 2L\mu - \frac{\kappa_{d+1}^2 \rho}{d-2} e^{-(d-2)r/L} \,. \end{aligned} \tag{1.63}$$

RNAdS describes a class of CFTs at finite temperature and finite chemical potential. In (1.63), the thermodynamic quantities ϵ, ρ, and μ are written into the solution for convenience. A regular event horizon of a charged black hole occurs at some $r = r_H$, where both h and Φ have simple zeros. The equations $h(r_H) = 0$ and $\Phi(r_H) = 0$ together with the formulas (1.52) and (1.53) can be solved simultaneously to confirm that $\epsilon = (d-1)(Ts + \mu\rho)/d$, as expected for a CFT.

Chapter 3 discusses field theories dual to the candidate bulk action

$$S = \frac{1}{2\kappa_{d+1}^2} \int dr\, dt\, d^{d-1}x \sqrt{-g} \left[R - \frac{1}{4} F_{\mu\nu}^2 - |(\partial_\mu - iqA_\mu)\psi|^2 - V(|\psi|) \right] \qquad (1.64)$$

with $d = 3$. Above, ψ is a complex scalar coupled to the gauge field A^μ with strength q. The bulk scalar ψ is dual to some charged scalar operator $\mathcal{O}_\psi$ in the field theory. When the source of $\mathcal{O}_\psi$ is turned off, the relation (1.45) still holds. This relation connects ϵ, μ, and ρ, quantities read off from boundary asymptotics, with T and s, quantities determined at the horizon. The way the bulk theory enforces (1.45) is through a Noether charge. As explained in detail in Chapter 3, equating the Noether charge evaluated at the horizon with the Noether charge evaluated at the boundary recovers (1.45).

Non-experts should now have enough background on the gauge/gravity duality to understand Chapters 2 and 3. The next section concludes the introduction by outlining these chapters in the broad context of gauge/gravity modeling.

1.3 Gauge/gravity modeling

No gauge theories with established gravity duals are known to be realized in nature. However, a "bottom-up" strategy has evolved for modeling a real-world system using the gauge/gravity duality:

1. Identify a physical regime of the system that may at least qualitatively reflect the limits in which the duality holds. Strongly coupled phenomena are generally good candidates for dual gravitational descriptions.

2. Write down an effective gravitational action that incorporates some features of the system in this physical regime. The action could have a solution with a

particular symmetry, could yield a result that matches experimental data, or could include matter that exhibits some characteristic behavior. For instance, at high temperatures, the model (1.64) admits only solutions for which $\psi = \psi^* = 0$. Below some critical temperature, however, $|\psi|$ becomes nonzero and breaks the $U(1)$ symmetry. This behavior is reminiscent of the Landau-Ginzburg picture of superconductivity, and indeed, (1.64) is the canonical example of a holographic superconductor.

3. Use the machinery of the gauge/gravity duality to compute observables associated with the effective action. Quantities of interest could include correlation functions, transport coefficients, and phase transition temperatures.

A drawback of the bottom-up strategy is that the existence and consistency of a field theory dual to an *ad hoc* gravity action are assumed, not established. Moreover, little is known about the content of this hypothetical field theory. By contrast, a "top-down" strategy starts with a string theoretic construction and derives a class of gravity duals constrained to incorporate some features of a physical system. This method furnishes precise information about the content of the dual field theories, but is more involved and less flexible than the bottom-up strategy. Only bottom-up models are considered in this dissertation.

The gauge/gravity duality has traditionally been used to model QCD physics, a program called AdS/QCD. In recent years, the duality has also been used to model condensed matter systems, a program called AdS/CMT. In Chapter 2, which pertains to AdS/QCD, some mathematical tools are developed to study the single-scalar model (1.31) at finite temperature. This model is arguably the simplest that can describe a nonconformal field theory at large N_c and strong coupling. QCD is an example of a nonconformal theory without a known holographic description. Above some crossover temperature $T_c \sim 10^{12}K$, QCD is in a deconfined phase called the quark-gluon plasma (QGP). At arbitrarily high T, asymptotic freedom sets in, and QCD is studied best with ordinary perturbative methods. However, for $T_c \lesssim T \lesssim 3T_c$, QCD is strongly coupled, and (1.31) may provide some insight into the physics of the QGP. In this spirit, a two-parameter ansatz for the scalar potential $V(\phi)$ in the action (1.31) is adjusted so the squared speed of sound $c_s^2 = \partial(\log T)/\partial(\log s)$ as a function of T mimics the squared speed of sound of QCD from lattice data. The fit captures the sharp rise in the number of degrees of freedom close to T_c. Interestingly, [49, 50] find that the ratio of bulk viscosity to entropy density has a peak near T_c in backgrounds associated with similar fits.

In Chapter 3, which pertains to AdS/CMT, zero-temperature solutions to the equations of motion from the action (1.64) are studied. This action is an Abelian Higgs model in AdS: it includes gravity, a $U(1)$ gauge field, and a complex scalar ψ with charge q and potential $V(|\psi|)$. The local $U(1)$ symmetry on the gravity side is dual to a global $U(1)$ symmetry on the field theory side. As noted previously, when T is greater than some critical temperature T_c, the model admits only solutions with $\psi = \psi^* = 0$. Below T_c, however, $|\psi|$ is nonzero along the thermodynamically favored branch of solutions, and the $U(1)$ symmetry is spontaneously broken. The spontaneous breaking of a $U(1)$ symmetry is characteristic of a superconducting phase transition. Though conventional superconductors are well-described by BCS theory, high-temperature superconductors are strongly correlated electron systems that may admit at least a qualitatively correct holographic description. In fact, it is suspected that the phase diagram of some high-temperature superconductors includes a quantum critical point described by a strongly coupled CFT. Quantum critical points influence low-temperature physics, so it is important to understand the zero-temperature solutions to the equations of motion from the action (1.64). In Chapter 3, it is shown that the infrared symmetries of these solutions depend radically on $V(|\psi|)$ and q. Some solutions interpolate between two copies of AdS, and others between a reduced infrared Lorentz symmetry and ultraviolet AdS. An exotic class of solutions has an infrared geometry with Lifshitz symmetry, which was described at the end of subsection 1.2.1.

The models studied here are straightforward to write down, but give rise to a diverse array of possible phenomena. This dissertation by no means provides an exhaustive treatment of gauge/gravity modeling. Some excellent reviews of AdS/QCD are [40, 43, 53, 24], and of AdS/CMT are [55, 63, 80].

Chapter 2

Mimicking the QCD equation of state with a dual black hole

2.1 Introduction

In the supergravity approximation, the near-extremal D3-brane has equation of state $s \propto T^3$, with a constant of proportionality that is 3/4 of the free-field value for the dual $\mathcal{N} = 4$ super-Yang-Mills theory [34]. The speed of sound is $c_s = 1/\sqrt{3}$, as required by conformal invariance. On the other hand, the speed of sound of a thermal state in QCD has an interesting and phenomenologically important dependence on temperature, with a minimum near the crossover temperature T_c. Lattice studies of the equation of state are too numerous to cite comprehensively, but they include [11] (for pure glue), [70] (a review article), and [3, 18] (studies with $2 + 1$ flavors).

We would like to find a five-dimensional gravitational theory that has black hole solutions whose speed of sound as a function of temperature mimics that of QCD. We will not try to include chemical potentials or to account for chiral symmetry breaking. We will not try to include asymptotic freedom, but instead will limit our computation to $T \lesssim 4T_c$ and assume conformal behavior in the extreme UV. We will not even try to give an account of confinement, except insofar as the steep rise in the number of degrees of freedom near the crossover temperature T_c is recovered in our setup, corresponding to a minimum of c_s near T_c. We will not try to embed our construction in string theory, but instead adjust parameters in a five-dimensional gravitational action to recover approximately the dependence $c_s(T)$ found from the lattice. That action is

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right] , \tag{2.1}$$

up to total derivative terms that affect the evaluation of the free energy, but not the entropy or temperature. We will not include higher derivative corrections, which would arise from α' and loop corrections if the theory (2.1) were embedded explicitly in string theory.

The ansatz we will study is

$$ds^2 = e^{2A}(-hdt^2 + d\vec{x}^2) + e^{2B} \frac{dr^2}{h} , \tag{2.2}$$

where A, B, and h are functions of r, and ϕ is also some function of r. This ansatz is dictated by the symmetries: we want translation invariance in the $\mathbf{R}^{3,1}$ directions parametrized by $(t, \vec{x})$, and we want $SO(3)$ symmetry in the $\vec{x}$ directions but not $SO(3,1)$ boost invariance — because boost invariance is broken by finite temperature. Assuming conformal behavior in the extreme UV means that we assume the geometry (2.2) is asymptotically anti-de Sitter. A regular horizon arises when h has a simple zero. Let's say the first such zero (that is, the one closest to the conformal boundary) is at $r = r_H$. It is assumed that A and B are finite and regular at $r = r_H$. Standard manipulations lead to the following formulas for entropy density and temperature:

$$s = \frac{2\pi}{\kappa_5^2} e^{3A(r_H)} \qquad T = \frac{e^{A(r_H)-B(r_H)}|h'(r_H)|}{4\pi} , \tag{2.3}$$

and once these quantities are known, the speed of sound can be read off from

$$c_s^2 = \frac{d \log T}{d \log s} \,. \tag{2.4}$$

The formula for the entropy density in (2.3) comes from the Bekenstein-Hawking result $S = A/4G_N$, where A is the area of the horizon (really a volume in our case) and $G_N = \kappa_5^2/8\pi$. The formula for the temperature comes from Hawking's result $T = \kappa/2\pi$ where κ is the surface gravity at the horizon.

By adjusting $V(\phi)$ one might expect to be able to recover any pre-specified $c_s(T)$, at least within certain limits — perhaps including that $s(T)/T^3$ should be monotonic or some similar criterion (in this connection see [4]). The main aim of this chapter is to characterize how $V(\phi)$ translates into $c_s(T)$ and vice versa. In section 2.2 we begin with the simplest possible case: $c_s(T)$ constant. It translates into $V(\phi) = V_0 e^{\gamma\phi}$ for some $V_0 < 0$ and γ related to c_s. In section 2.3 we tackle the general case, exploiting a weak form of integrability of the equations resulting from plugging (2.2) into (2.1). In section 2.5 we exhibit several examples. These include a particular $V(\phi)$ whose corresponding $c_s(T)$ curve closely mimics that of QCD.[1] We close with a discussion in section 2.6.

Some topics in this chapter are summarized in [49].

[1] Two earlier studies [69, 2] of thermodynamic properties of putative holographic duals to QCD obtain $s \sim T^3 e^{-T_0^2/T^2}$ for some constant T_0 in a region above the deconfinement transition temperature. But the line elements considered in these studies are simply assumed, rather than derived starting from a fully specified classical action, as our solutions are.

2.2 Chamblin-Reall solutions and an adiabatic generalization of them

In a d-dimensional conformal field theory (meaning a CFT in $d-1$ spatial dimensions plus one time dimension), the entropy density must obey

$$s \propto T^{d-1} \,, \tag{2.5}$$

simply because this expression is dimensionally correct and there is no scale other than the temperature that would permit a more complicated dependence. So the speed of sound is $c_s = 1/\sqrt{d-1}$. If $d > 4$, then we could obtain a nonconformal theory in four dimensions by compactifying our CFT$_d$ on a $(d-4)$-dimensional torus. (A similar idea has been considered in [7, 15].) Doing so should not change the speed of sound: a planar sound wave in the resulting 4-dimensional theory would correspond to a planar sound wave in the original theory whose propagation is in the direction of the uncompactified directions.

The AdS_{d+1}-Schwarzschild solution is an extremum of the action

$$S = \frac{1}{2\kappa_{d+1}^2} \int d^{d+1}x \sqrt{-\hat{g}} \left[\hat{R} + \frac{d(d-1)}{L^2} \right] , \tag{2.6}$$

and it takes the form

$$d\hat{s}^2 = \frac{L^2}{z^2} \left(-h dt^2 + d\widehat{\vec{x}}^2 + \frac{dz^2}{h} \right) , \tag{2.7}$$

where

$$h = 1 - \frac{z^d}{z_H^d} \,. \tag{2.8}$$

We use hats to distinguish $d+1$-dimensional quantities from 4-dimensional ones. It is easy to see that $T \propto 1/z_H$ and $s \propto 1/z_H^{d-1}$, so that $s \propto T^{d-1}$ as the conformal field theory requires. Suppose we now perform the dimensional reduction described in the previous paragraph on the solution (2.7). In slightly more generality than we need, the Kaluza-Klein ansatz is

$$d\hat{s}^2 = \exp\left\{\sqrt{\frac{2}{3}\frac{d-4}{d-1}}\,\phi\right\} ds^2 + \exp\left\{-\sqrt{\frac{6}{(d-1)(d-4)}}\,\phi\right\} ds^2_{d-4}\,, \tag{2.9}$$

where ds^2 is a five-dimensional metric and ds^2_{d-4} is the flat metric on a torus $\mathbf{T}^{d-4}$, whose shape we will assume to be square with side length ℓ, so that $\mathrm{Vol}\,\mathbf{T}^{d-4} = \ell^{d-4}$. All components of the metric, and also ϕ, are assumed to depend only on the five-dimensional coordinates. It is assumed that ℓ is a constant; variation of the size of the torus is taken care of by the exponential prefactor multiplying ds^2_{d-4} in (2.9). The particular coefficients in the exponentials were chosen presciently to obtain a simple five-dimensional action. Comparing the general form (2.9) with the specific solution (2.7), one finds

$$ds^2 = \left(\frac{L}{z}\right)^{\frac{2}{3}(d-1)} \left(-h dt^2 + d\vec{x}^2 + \frac{dz^2}{h}\right) \qquad e^{\phi} = \left(\frac{z}{L}\right)^{\sqrt{\frac{2}{3}(d-1)(d-4)}}\,, \tag{2.10}$$

where $h = 1 - z^d/z_H^d$ as in (2.8). The line element (2.10) was obtained by the authors of [17], but not via Kaluza-Klein reduction; instead, they considered black hole solutions to the equations of motion from an action like (2.1) with potentials of the form

$$V(\phi) = V_0 e^{\gamma\phi}\,, \tag{2.11}$$

with $V_0 < 0$. To see that the solutions have to come out the same in either approach, let's carry through the Kaluza-Klein reduction at the level of the action by plugging (2.9) into (2.6). After performing the trivial integral over T^{d-4}, one obtains

$$S = \frac{\ell^{d-4}}{2\kappa^2_{d+1}} \int d^5x\,\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right]\,, \tag{2.12}$$

where $V(\phi)$ has the form (2.11) with the identifications

$$V_0 = -\frac{d(d-1)}{L^2} \qquad \gamma = \sqrt{\frac{2}{3}\frac{d-4}{d-1}}\,. \tag{2.13}$$

Evidently, the length scale ℓ enters the action only as a prefactor, which can be absorbed into a definition of the five-dimensional gravitational constant: $\kappa_5^2 = \kappa_{d+1}^2/\ell^{d-4}$.

By comparing the expression for γ in (2.13) with the result $c_s = 1/\sqrt{d-1}$ for the speed of sound, we find

$$c_s^2 = \frac{1}{3} - \frac{\gamma^2}{2}\,. \tag{2.14}$$

This result can be derived more directly by showing that $s \propto T^{6/(2-3\gamma^2)}$ for Chamblin-Reall solutions: explicitly,

$$\begin{aligned} s &= \frac{1}{2\kappa_5^2}\left(\frac{L}{z_H}\right)^{d-1} = \frac{1}{2\kappa_5^2}\exp\left\{-\frac{\phi_H}{\gamma}\right\} \\ T &= \frac{d}{4\pi z_H} = \frac{1}{4\pi L}\frac{8-3\gamma^2}{2-3\gamma^2}\exp\left\{\left(\frac{\gamma}{2}-\frac{1}{3\gamma}\right)\phi_H\right\}\,, \end{aligned} \tag{2.15}$$

where ϕ_H is the value of ϕ at the horizon. The dimensional reduction we have described is well-defined only for integer $d > 4$, but for the purposes of the computations presented here, it can be any real number greater than 4.

Suppose we rewrite the result (2.15) as

$$\begin{aligned} \log s &= -\frac{\phi_H}{\gamma} + (\text{constant in } \phi_H) \\ \log T &= \left(\frac{\gamma}{2}-\frac{1}{3\gamma}\right)\phi_H + (\text{constant in } \phi_H)\,. \end{aligned} \tag{2.16}$$

Given (2.16) and the formula $\gamma = V'(\phi)/V(\phi)$, a natural next step would be to guess the following dependence of s and T on ϕ_H when γ is a slowly varying function of ϕ

rather than a constant:

$$\begin{aligned}\log s &= -\int_{\phi_0}^{\phi_H} d\phi\, \frac{V(\phi)}{V'(\phi)} + (\text{slowly varying in } \phi_H) \\ \log T &= \int_{\phi_0}^{\phi_H} d\phi \left(\frac{1}{2}\frac{V'(\phi)}{V(\phi)} - \frac{1}{3}\frac{V(\phi)}{V'(\phi)} \right) + (\text{slowly varying in } \phi_H)\,.\end{aligned} \tag{2.17}$$

The lower limit ϕ_0 in the integrals is an arbitrary cutoff. If we assume that $V(\phi)$ has a maximum at $\phi = 0$ and an expansion of the form (2.37), then $V(\phi)/V'(\phi) \approx -12/(m^2 L^2 \phi)$ near $\phi = 0$. So the integrals in (2.17) diverge if they are continued all the way to $\phi = 0$, and the cutoff ϕ_0 must be chosen to have the same sign as ϕ_H to avoid this divergence.

A consequence of the estimates (2.17) is a simple formula for the speed of sound:

$$c_s^2 = \frac{d\log T/d\phi_H}{d\log s/d\phi_H} \approx \frac{1}{3} - \frac{1}{2}\frac{V'(\phi_H)^2}{V(\phi_H)^2}\,. \tag{2.18}$$

Another consequence is

$$\begin{aligned}\log \frac{s}{T^3} &= -\frac{3}{2}\int_{\phi_0}^{\phi_H} d\phi\, \frac{V'(\phi)}{V(\phi)} + (\text{slowly varying in } \phi_H) \\ &= -\frac{3}{2}\log\frac{V(\phi_H)}{V(\phi_0)} + (\text{slowly varying in } \phi_H)\,.\end{aligned} \tag{2.19}$$

A simpler way of expressing (2.19) is

$$\frac{s}{T^3} \propto |V(\phi_H)|^{-3/2}\,, \tag{2.20}$$

up to corrections from slowly varying terms. This is interesting because s/T^3 is one way of defining the effective number of degrees of freedom available to a system, and we see from (2.20) that it is closely related to the potential evaluated at the horizon.

The results (2.18) and (2.19) are a first attempt at solving the problem of translating an arbitrary $V(\phi)$ to an equation of state, or an arbitrary equation of state into

$V(\phi)$. Here's how the latter process would work. Suppose one specifies the equation of state as $s = s(T)$. Ignoring corrections to (2.19), one has

$$f \equiv -\frac{2}{3} \log \frac{s}{T^3} = \log \frac{V}{V_0} \,, \tag{2.21}$$

where V_0 is some constant. Let's regard f as the independent variable. Because $V = V_0 e^f$, all we need is to find $\phi = \phi(f)$, and we will have a parametric representation of $V(\phi)$. One may rewrite (2.18) as

$$c_s^2 = \frac{1}{3} - \frac{1}{2(d\phi/df)^2} \,, \tag{2.22}$$

where corrections have again been ignored. Knowing $s(T)$ with good precision means one can express c_s^2 as a function of f. Then (2.22) can readily be integrated to give

$$\phi(f) = \int \frac{df}{\sqrt{2\left(\frac{1}{3} - c_s(f)^2\right)}} \,. \tag{2.23}$$

The integral is left in indefinite form because adding a constant to ϕ is obviously allowed.

We stress that the result of plugging (2.23) into the form $V = V_0 e^f$ will result in a $V(\phi)$ that only approximately reproduces the desired $s(T)$. If the speed of sound varies rapidly with T, the approximation may be poor. In section 2.4 we will show how to improve this approximation without resorting to differential equations that cannot be explicitly solved in terms of indefinite integrals.

2.3 A nonlinear master equation

There is a residual gauge freedom in the ansatz (2.2), namely reparametrization of the radial direction. A convenient gauge choice, which should be at least piecewise valid in any geometry where the scalar is nonvanishing, is to set $r = \phi$. Then the line element becomes

$$ds^2 = e^{2A}(-hdt^2 + d\vec{x}^2) + e^{2B}\frac{d\phi^2}{h}\,, \tag{2.24}$$

and the equations of motion following from the action (2.1) take the form

$$A'' - A'B' + \frac{1}{6} = 0 \tag{2.25a}$$

$$h'' + (4A' - B')h' = 0 \tag{2.25b}$$

$$6A'h' + h(24A'^2 - 1) + 2e^{2B}V = 0 \tag{2.25c}$$

$$4A' - B' + \frac{h'}{h} - \frac{e^{2B}}{h}V' = 0\,, \tag{2.25d}$$

where primes denote $d/d\phi$. The first two of these equations come from the tt and x^1x^1 Einstein equations; the third comes from the $\phi\phi$ Einstein equation; and the last comes from the scalar equation of motion. There is typically some redundancy in equations obtained from classical gravity, with or without matter. In the case of (2.25), the redundancy is that the ϕ derivative of the third equation follows algebraically from the four equations listed.

The ansatz (2.24) has one peculiar feature: e^{2B} must have dimensions of length squared. This is because ϕ is dimensionless.

The equations of motion (2.25) enjoy a weak form of integrability, in the following sense: if a smooth "generating function" $G(\phi)$ is specified, then it is possible to find a black hole solution where $A'(\phi) = G(\phi)$ in terms of indefinite integrals of simple functions of $G(\phi)$ and $G'(\phi)$. But $V(\phi)$ itself is expressed in terms of such integrals, and one cannot easily find all the possible $G(\phi)$ that lead to a specified $V(\phi)$. In other words, there can be simple analytic solutions to (2.25) for special $V(\phi)$ at a special value of the temperature, but as far as we know, there is no nontrivial $V(\phi)$ (i.e., none besides the exponential form) for which analytic solutions exist over a continuous range of temperatures.

To understand this claim of integrability, let us consider $A'(\phi) = G(\phi)$ to be fixed as a function of ϕ and work out $A(\phi)$, $B(\phi)$, $h(\phi)$, and $V(\phi)$. The first of these is trivial:

$$A(\phi) = A_0 + \int_{\phi_0}^{\phi} d\tilde{\phi}\, G(\tilde{\phi})\,. \tag{2.26}$$

Computing $B(\phi)$ is immediate once one solves (2.25a) for B':

$$B(\phi) = B_0 + \int_{\phi_0}^{\phi} d\tilde{\phi}\, \frac{G'(\tilde{\phi}) + 1/6}{G(\tilde{\phi})}\,. \tag{2.27}$$

Next one observes that (2.25b) is straightforwardly solved once one knows $A(\phi)$ and $B(\phi)$:

$$h(\phi) = h_0 + h_1 \int_{\phi_0}^{\phi} d\tilde{\phi}\, e^{-4A(\tilde{\phi})+B(\tilde{\phi})}\,. \tag{2.28}$$

Now (2.25c) can be solved for $V(\phi)$ in terms of known quantities:

$$V(\phi) = h(\phi)\frac{e^{-2B(\phi)}}{2}\left(1 - 24G(\phi)^2 - 6G(\phi)\frac{h'(\phi)}{h(\phi)}\right)\,. \tag{2.29}$$

The constraint equation (2.25d) doesn't yield any new information.

If one chooses

$$G(\phi) = -\frac{1}{3\gamma}\,, \tag{2.30}$$

then by working through (2.26)-(2.29) one recovers the Chamblin-Reall solution in the form

$$\begin{aligned}
A(\phi) &= A_0 - \frac{\phi - \phi_0}{3\gamma} \\
B(\phi) &= B_0 - \frac{\gamma}{2}(\phi - \phi_0) \\
h(\phi) &= h_0 + \tilde{h}_1 \exp\left\{\frac{8 - 3\gamma^2}{6\gamma}(\phi - \phi_0)\right\} \\
V(\phi) &= V_0 e^{\gamma\phi}
\end{aligned} \tag{2.31}$$

where

$$\tilde{h}_1 = \frac{6e^{-4A_0+B_0}\gamma}{8-3\gamma^2} h_1 \qquad V_0 = -\frac{8-3\gamma^2}{6\gamma} e^{-2B_0-\gamma\phi_0} h_0 \,. \tag{2.32}$$

By choosing

$$\phi_0 = \frac{1}{\gamma}\left(\log h_0 - 2B_0\right) \,, \tag{2.33}$$

one obtains $V(\phi)$ in a form that doesn't depend on any integration constants at all. This situation is very special and corresponds to the fact that for $V(\phi) \propto -e^{\gamma\phi}$ one can find a whole family of analytic solutions parametrized by ϕ_H.

By differentiating combinations of (2.26)-(2.29), one can derive the following "non-linear master equation:"

$$\frac{G'}{G+V/3V'} = \frac{d}{d\phi} \log\left(\frac{G'}{G} + \frac{1}{6G} - 4G - \frac{G'}{G+V/3V'}\right) \,. \tag{2.34}$$

Describing (2.34) as the master equation is appropriate because if one starts knowing $V(\phi)$ and manages to solve (2.34), then to obtain a black hole solution one only needs to perform the "trivial" integrations (2.26)-(2.28). A numerically efficient strategy for obtaining an equation of state given $V(\phi)$ centers around solving (2.34) numerically. In more detail, the procedure is:

1. Choose the value ϕ_H of the scalar at the horizon.

2. Find a series solution of the nonlinear master equation around $\phi = \phi_H$.

3. Seed a numerical integrator like Mathematica's `NDSolve` close to $\phi = \phi_H$ using the series solution.

4. Integrate the nonlinear master equation up to a value of ϕ close to a maximum, corresponding to the asymptotically anti-de Sitter part of the geometry.

5. Extract s and T from integrals of simple functions of $G(\phi)$.

Most of these steps require further explanation, which will occupy the remainder of this section.

At the horizon, h has a simple zero, and the other quantities appearing in (2.25c) and (2.25d) are finite. Evaluating these two equations at the horizon gives

$$V(\phi_H) = -3e^{-2B(\phi_H)}G(\phi_H)h'(\phi_H) \qquad V'(\phi_H) = e^{-2B(\phi_H)}h'(\phi_H)\,, \tag{2.35}$$

which implies that $G + V/3V'$ vanishes at the horizon. Starting from this condition, one may develop a power series solution around the horizon:

$$G(\phi) = -\frac{1}{3}\frac{V(\phi_H)}{V'(\phi_H)} + \frac{1}{6}\left(\frac{V(\phi_H)V''(\phi_H)}{V'(\phi_H)^2} - 1\right)(\phi - \phi_H) + O\left[(\phi - \phi_H)^2\right]\,. \tag{2.36}$$

This series solution can be developed to any desired order without encountering arbitrary integration constants.

To understand the asymptotic behavior far from the horizon, let's specialize to the case where $V(\phi)$ has a maximum at $\phi = 0$:

$$V(\phi) = -\frac{12}{L^2} + \frac{1}{2}m^2\phi^2 + O(\phi^3)\,, \tag{2.37}$$

where $m^2 < 0$ in order for $\phi = 0$ to be a maximum. The gauge theory operator $\mathcal{O}_\phi$ dual to ϕ has dimension Δ, where

$$\Delta(\Delta - 4) = m^2L^2\,. \tag{2.38}$$

We will be primarily interested in the case where Δ is close to 4. It helps our intuition at this point to pass to a more standard gauge: instead of setting $r = \phi$, we can set $B = 0$ to obtain

$$ds^2 = e^{2A}(-hdt^2 + d\vec{x}^2) + \frac{dr^2}{h}\,. \tag{2.39}$$

We note that A and h appearing in (2.39) are precisely the same as when we use the $r = \phi$ gauge, only expressed as functions of r rather than ϕ. Large r corresponds to the region far from the horizon, and the leading asymptotic behavior of solutions there is

$$A \approx \frac{r}{L} \qquad h \approx 1 \qquad \phi \approx (\Lambda L)^{4-\Delta} e^{(\Delta-4)A} \,. \tag{2.40}$$

Each approximate equality in (2.40) means that the ratio of the two expressions on each side approaches 1 as r becomes large. The behavior of ϕ indicates a relevant deformation of the conformal field theory:

$$\mathcal{L} = \mathcal{L}_{\rm CFT} + \Lambda^{4-\Delta} \mathcal{O}_\phi \,. \tag{2.41}$$

As we vary temperature to compute the equation of state, we should of course keep Λ fixed. A simple way to do this is to set $\Lambda L = 1$. Then the last equation in (2.40) becomes

$$A(\phi) = \frac{\log\phi}{\Delta-4} + o(1) \qquad \text{for small } \phi, \tag{2.42}$$

where $o(1)$ means a quantity that is parametrically smaller than 1 in the limit under consideration — so in the limit $\phi \to 0$ in the case of (2.42).

In order to compute the entropy density using (2.3), we need to know $A(\phi_H)$. This can be extracted by comparing (2.42) to (2.26) with ϕ_0 set equal to ϕ_H and A_0 set equal to $A_H = A(\phi_H)$:

$$A(\phi) = A_H + \int_{\phi_H}^{\phi} d\tilde\phi \, G(\tilde\phi) = \frac{\log\phi}{\Delta-4} + o(1) \,. \tag{2.43}$$

Solving for A_H and then taking $\phi \to 0$, one finds

$$A_H = \frac{\log\phi_H}{\Delta-4} + \int_0^{\phi_H} d\phi \left[G(\phi) - \frac{1}{(\Delta-4)\phi} \right] \,. \tag{2.44}$$

The integral converges because the explicit $1/\phi$ term cancels against the leading behavior of $G(\phi)$ for small ϕ. Plugging (2.44) into the expression for entropy density from (2.3), we find at last

$$s = \frac{2\pi}{\kappa_5^2} \phi_H^{3/(\Delta-4)} \exp\left\{ 3 \int_0^{\phi_H} d\phi \left[G(\phi) - \frac{1}{(\Delta-4)\phi} \right] \right\} . \tag{2.45}$$

A similar formula for the temperature may be derived starting with the observation that

$$\frac{dr}{d\phi} = -e^B , \tag{2.46}$$

where B is the function controlling the $g_{\phi\phi}$ metric component in $r = \phi$ gauge. One obtains (2.46) by comparing the (2.24) to (2.39). The sign is based on assuming that ϕ increases from 0 to positive values as r decreases from $+\infty$ to finite values. The first equation in (2.40) implies $dA/dr \to 1/L$ as $r \to \infty$. Combining this with (2.46) gives

$$G = \frac{dA}{d\phi} = \frac{dr}{d\phi} \frac{dA}{dr} \approx -e^B \frac{1}{L} , \tag{2.47}$$

where the approximate equality means that the ratio of the last expression to the previous ones approaches 1 as ϕ goes to 0. In summary,

$$1 \approx -LG(\phi) e^{-B(\phi)} , \tag{2.48}$$

using the same sense of approximate equalities. (Recall that e^{-B} has dimensions of inverse length, while $G(\phi)$ is dimensionless, so (2.48) is dimensionally correct.) We may cast the expression for temperature from (2.3) in terms of $G(\phi)$ by multiplying by 1 in the form indicated in (2.48):

$$
\begin{aligned}
T &= \frac{e^{A_H - B(\phi_H)}|h'(\phi_H)|}{4\pi} \approx \frac{e^{A_H - B(\phi_H)}h'(\phi_H)}{4\pi} LG(\phi)e^{-B(\phi)} \\
&= \frac{Le^{-2B(\phi_H)}G(\phi_H)h'(\phi_H)}{4\pi} \exp\left\{ A_H + B(\phi_H) - B(\phi) - \log \frac{G(\phi_H)}{G(\phi)} \right\} \\
&= -\frac{LV(\phi_H)}{12\pi} \exp\left\{ A_H + \int_\phi^{\phi_H} \frac{d\tilde{\phi}}{6G(\tilde{\phi})} \right\} .
\end{aligned}
\tag{2.49}
$$

In the second step we assumed that $h'(\phi_H) < 0$, which is the expected sign when ϕ vanishes on the boundary and is positive at the horizon. In the last step we used the first equation from (2.35) to simplify the prefactor and also

$$
B(\phi_H) - B(\phi) = \log \frac{G(\phi_H)}{G(\phi)} + \int_\phi^{\phi_H} \frac{d\tilde{\phi}}{6G(\tilde{\phi})} , \tag{2.50}
$$

which is a consequence of (2.27). The integral in the last expression of (2.51) converges even when $\phi \to 0$. We use (2.44) to eliminate A_H from (2.51) and obtain at last

$$
T = \frac{\phi_H^{1/(\Delta-4)}}{\pi L} \frac{V(\phi_H)}{V(0)} \exp\left\{ \int_0^{\phi_H} d\phi \left[G(\phi) - \frac{1}{(\Delta-4)\phi} + \frac{1}{6G(\phi)} \right] \right\} . \tag{2.51}
$$

The measure of the number of degrees of freedom that is easiest for us to access is

$$
\frac{s}{T^3} = 2\pi^4 \frac{L^3}{\kappa_5^2} \frac{V(0)^3}{V(\phi_H)^3} \exp\left\{ -3 \int_0^{\phi_H} \frac{d\phi}{6G(\phi)} \right\} , \tag{2.52}
$$

where to obtain the right hand side we simply combined (2.45) and (2.51). When ϕ_H is small, entropy and temperature become large because of the factors $\phi_H^{3/(\Delta-4)}$ and $\phi_H^{1/(\Delta-4)}$ in (2.45) and (2.51). In this limit, the integrals in the exponent become negligible, and (2.52) becomes

$$
\frac{s}{T^3} \approx 2\pi^4 \frac{L^3}{\kappa_5^2} . \tag{2.53}
$$

So we recover conformal behavior in the ultraviolet, as expected.

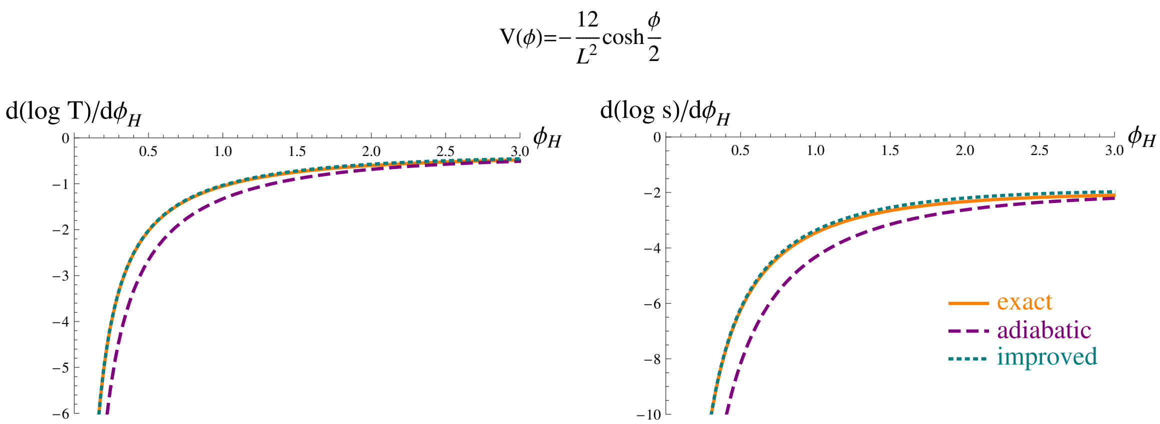

Figure 2.1: A comparison of the exact $\frac{d(\log s)}{d\phi_H}$ and $\frac{d(\log T)}{d\phi_H}$ for $V(\phi) = -\frac{12}{L^2}\cosh\frac{\phi}{2}$ with the adiabatic approximation, (2.17), and the improved approximation scheme, (2.57) with the choice (2.58).

2.4 An approximate determination of the equation of state

The adiabatic formulas (2.17) work well when ϕ_H is in a region where $V(\phi)$ is nearly exponential, but they do not work well for small ϕ_H, where $V(\phi)$ is close to attaining a maximum. This is shown in figure 2.1 for $V(\phi) = -(12/L^2)\cosh(\phi/2)$. On the other hand, it's easy to extract asymptotic formulas valid in the $\phi_H \to 0$ limit from (2.45) and (2.51): using the expansion (2.37), one finds

$$T = \frac{1}{\pi L}\phi_H^{1/(\Delta-4)} \qquad s = \frac{2\pi}{\kappa_5^2}\phi_H^{3/(\Delta-4)} \,. \tag{2.54}$$

We wish to find formulas that interpolate smoothly between (2.17) and (2.54) and involve at most indefinite integrals, not solutions to a difficult, nonlinear, second-order differential equation such as (2.34). We start by noting that the formulas (2.54) together with (2.37) imply

$$\begin{aligned} \frac{d\log T}{d\phi_H} &\approx \frac{\Delta}{4}\left(\frac{1}{2}\frac{V'(\phi_H)}{V(\phi_H)} - \frac{1}{3}\frac{V(\phi_H)}{V'(\phi_H)}\right) \\ \frac{d\log s}{d\phi_H} &\approx \frac{\Delta}{4}\left(-\frac{V(\phi_H)}{V'(\phi_H)}\right) \qquad \text{for small } \phi_H, \end{aligned} \tag{2.55}$$

where approximate equality means that the ratio of the two sides tends to 1 as $\phi_H \to 0$. On the other hand, provided $V(\phi)$ tends to an exponential form $V_0 e^{\gamma\phi}$ for large ϕ, the adiabatic approximation becomes good if ϕ_H is large. So for such potentials, (2.17) can be rephrased as

$$\begin{aligned} \frac{d\log T}{d\phi_H} &\approx \frac{1}{2}\frac{V'(\phi_H)}{V(\phi_H)} - \frac{1}{3}\frac{V(\phi_H)}{V'(\phi_H)} \\ \frac{d\log s}{d\phi_H} &\approx -\frac{V(\phi_H)}{V'(\phi_H)} \qquad \text{for large } \phi_H. \end{aligned} \tag{2.56}$$

Comparing (2.55) and (2.56), we are led to introduce "fudge factors" $\rho_s(\phi_H)$ and $\rho_T(\phi_H)$ such that

$$\begin{aligned} \frac{d\log T}{d\phi_H} &= \rho_T(\phi_H)\left(\frac{1}{2}\frac{V'(\phi_H)}{V(\phi_H)} - \frac{1}{3}\frac{V(\phi_H)}{V'(\phi_H)}\right) \\ \frac{d\log s}{d\phi_H} &= \rho_s(\phi_H)\left(-\frac{V(\phi_H)}{V'(\phi_H)}\right). \end{aligned} \tag{2.57}$$

We can rephrase (2.55) and (2.56) as the statements that both $\rho_T(\phi_H)$ and $\rho_s(\phi_H)$ interpolate between $\Delta/4$ at small ϕ_H and 1 at large ϕ_H. Our improved estimate of the equation of state consists simply of guessing an interpolating function with these properties. The guess is

$$\rho_T(\phi_H) \approx \rho_s(\phi_H) \approx \rho(\phi_H) \equiv 1 + \frac{V(0)}{V(\phi_H)}\left(\frac{\Delta}{4} - 1\right). \tag{2.58}$$

In (2.58) approximate equality means that ρ_T, ρ_s, and ρ are supposed to be nearly equal for all ϕ_H. But away from the small ϕ_H and large ϕ_H limits, (2.58) is not a controlled approximation, in the sense that there isn't a parameter that we can tune to make it better. It is nevertheless useful for quickly determining the qualitative features of an equation of state given $V(\phi)$, as illustrated in figure 2.1. There might be better choices of $\rho_T(\phi_H)$ and $\rho_s(\phi_H)$. Systematic high-temperature expansions of $c_s^2(T)$ are discussed in [19, 65, 20].

Starting from (2.57), we have immediately

$$c_s^2 = \frac{d\log T/d\phi_H}{d\log s/d\phi_H} = \frac{\rho_T(\phi_H)}{\rho_s(\phi_H)} \left(\frac{1}{3} - \frac{1}{2} \frac{V'(\phi_H)^2}{V(\phi_H)^2} \right) . \tag{2.59}$$

Thus, assuming $\rho_T(\phi_H) \approx \rho_s(\phi_H)$ is the same as assuming that the speed of sound, as a function of ϕ_H, is well-approximated by the adiabatic formula, (2.18).

2.5 Examples

In this section we will present results for c_s^2 or s/T^3 versus T based on numerical integration of the nonlinear master equation (2.34), for several different choices of the scalar potential $V(\phi)$.

The simplest analytical form that interpolates between a maximum at $\phi = 0$ and exponential behavior for large ϕ is

$$V(\phi) = V_{\cosh}(\phi) \equiv -\frac{12}{L^2} \cosh \gamma\phi \, . \tag{2.60}$$

The adiabatic treatment discussed in section 2.2 leads us to expect that the speed of sound will be $c_s = \sqrt{\frac{1}{3} - \frac{\gamma^2}{2}}$ for large ϕ_H. So in order to have stable black holes in this regime, we must have $\gamma \leq \sqrt{2/3}$. This bound can be regarded as an application of the correlated stability conjecture (CSC) [44, 45]. But there is a tighter bound on

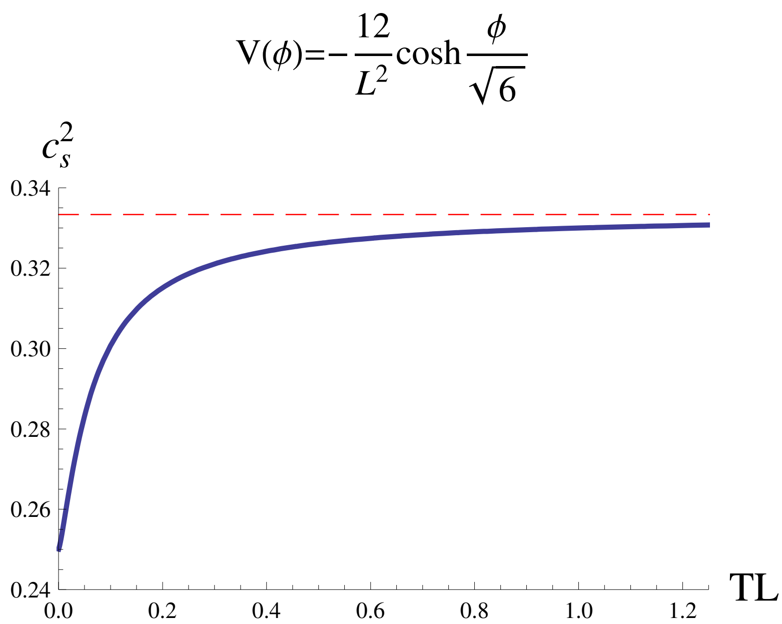

Figure 2.2: The speed of sound for $V(\phi) = -\frac{12}{L^2}\cosh\frac{\phi}{\sqrt{6}}$.

γ coming from the behavior near $\phi = 0$:

$$V_{\cosh}(\phi) = -\frac{12}{L^2} - \frac{6\gamma^2}{L^2}\phi^2 + O(\phi^4)\,, \tag{2.61}$$

so $m^2 = -12\gamma^2/L^2$. In order to satisfy the Breitenlohner-Freedman bound, $m^2L^2 \geq -4$ [12, 13, 81], we must have $\gamma \leq 1/\sqrt{3}$. This means that the minimum speed of sound we can arrange at large ϕ_H using the pure cosh potential (2.60) is $c_s = 1/\sqrt{6} \approx 0.41$. The behavior of c_s^2 as a function of T is shown in figure 2.2 for $\gamma = 1/\sqrt{6}$.

If one uses the potential (2.60), then c_s in the infrared is tied to the dimension Δ of the operator that breaks conformal invariance in the ultraviolet. Let us consider a minimal generalization that loosens this artificial constraint:

$$V(\phi) = V(\gamma, b; \phi) \equiv -\frac{12}{L^2}\cosh\gamma\phi + b\phi^2\,. \tag{2.62}$$

A parameter equivalent to γ, as before, is the speed of sound in the infrared, $c_s^2 = \frac{1}{3} - \frac{\gamma^2}{2}$. With γ fixed, a parameter equivalent to b is the dimension Δ of the operator

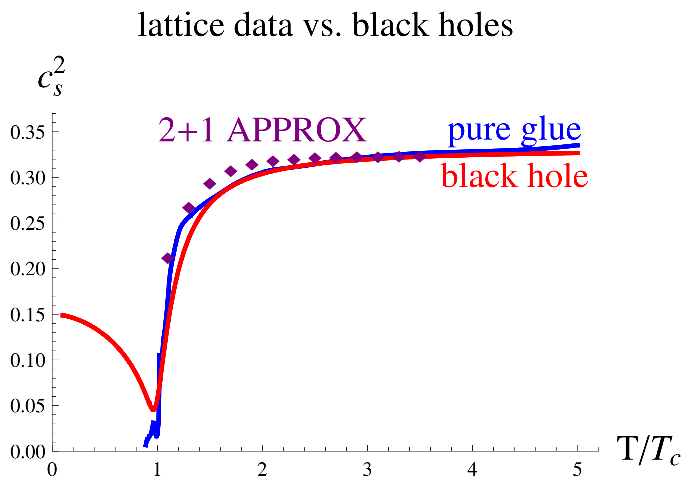

Figure 2.3: The equation of state of a black hole (red) compared to the lattice equation of state for pure glue (blue) and $2+1$-flavor QCD. The pure glue curve is based on [11] and private communications from F. Karsch. The $2+1$-flavor QCD points are based on [18].

dual to ϕ near the UV fixed point:

$$b = \frac{6\gamma^2}{L^2} + \frac{\Delta(\Delta - 4)}{2L^2} \,. \tag{2.63}$$

Note that taking Δ close to 4 amounts to making $V(\phi)$ nearly quartic near its maximum. As reported in more detail in [49], the choice

$$\gamma = 0.606 \qquad b = \frac{2.06}{L^2} \,, \tag{2.64}$$

corresponding to $c_s^2 = 0.15$ in the infrared and $\Delta = 3.93$, leads to an equation of state that bears a striking resemblance to the one expected for QCD: see figure 2.3. It may seem surprising that the most distinctive feature of the equation of state of QCD, namely a smooth but rapid crossover, emerges from a potential that is nearly featureless. To gain some intuition about why this happened, consider again the adiabatic approximation (2.18) to the speed of sound. When ϕ_H is close to where the nearly quartic behavior of $V(\phi)$ rolls over into nearly exponential behavior, this approximate formula predicts that c_s^2 dips to a fairly low value, only to rise back up

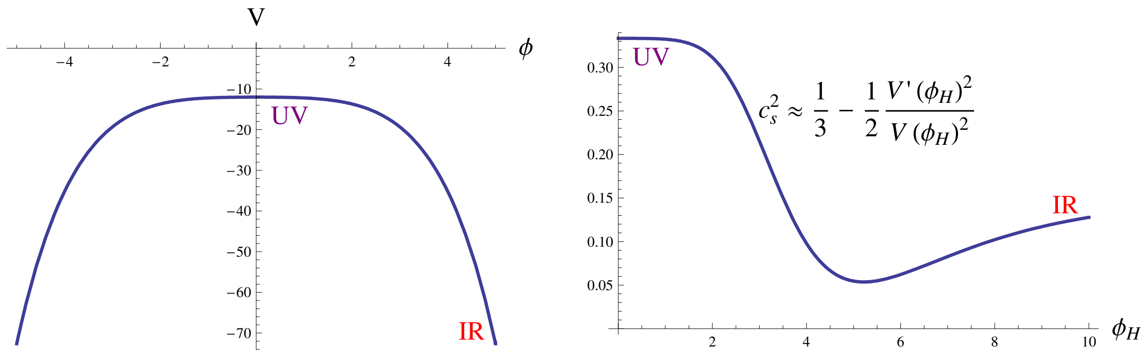

Figure 2.4: Left: The potential (2.62) with the parameter choices (2.63) that give an equation of state resembling QCD's. Right: Although $V(\phi)$ is relatively featureless, the adiabatic formula $c_s^2 \approx \frac{1}{3} - \frac{1}{2}\frac{V'(\phi_H)^2}{V(\phi_H)^2}$ suggests that the equation of state resulting from it will indeed exhibit a low minimum for the speed of sound.

again for larger ϕ_H towards its infrared limit, 0.15. See figure 2.4.

Other behaviors emerge from the potential (2.62) for other choices of b and γ. For example, if $\gamma > \sqrt{2/3}$, the adiabatic approximation suggests that there is a minimum temperature T_{min} for black hole solutions. A particular case is illustrated in figure 2.5. Solutions with very low entropy have high temperature and negative specific heat, and they are always thermodynamically disfavored compared to a branch of high-entropy solutions. Presumably there is a first-order transition to geometries with no horizon, similar to the Hawking-Page transition [61]. This transition probably happens at a temperature above T_{min}. It is worth noting that the specific heat diverges at T_{min} because T reaches a minimum as a function of ϕ_H while S is varying smoothly.

It is also possible to have a first-order transition between high-entropy and low-entropy black holes. An example where this happens is illustrated in figure 2.6. For a finite range of ϕ_H, the speed of sound is imaginary, indicating a Gregory-Laflamme instability. This touches once again on the correlated stability conjecture (CSC), so let us pause to review it. It was proposed in [44, 45] and further argued in [84] that, in the absence of conserved charges related to gauge symmetries, existence of a Gregory-Laflamme instability [31, 32] is equivalent to positivity of the specific

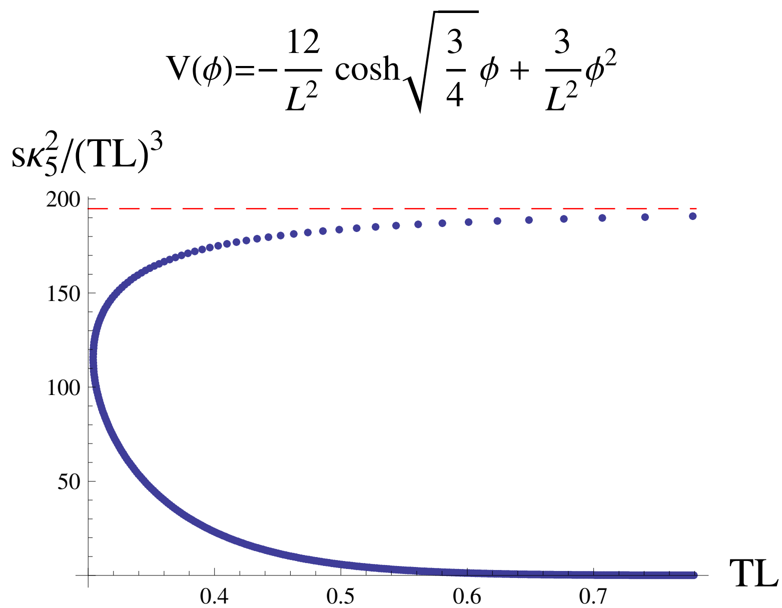

Figure 2.5: The equation of state for $V(\phi) = -\frac{12}{L^2}\cosh\sqrt{\frac{3}{4}}\phi + \frac{3}{L^2}\phi^2$.

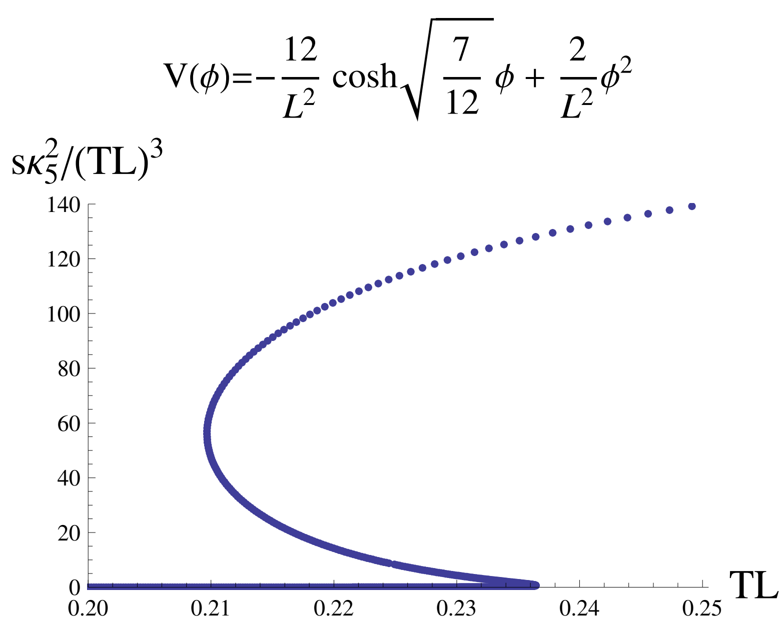

Figure 2.6: The equation of state for $V(\phi) = -\frac{12}{L^2}\cosh\sqrt{\frac{7}{12}}\phi + \frac{2}{L^2}\phi^2$.

heat, $C = T\partial S/\partial T$. According to a more general version of the CSC, dynamical stability of a horizon is equivalent to positivity of an appropriate Hessian matrix of susceptibilities, one of which is the specific heat [44, 45]. As pointed out in [14], $C > 0$ is equivalent to $c_s^2 > 0$, which makes the CSC seem inevitable, at least in the absence of conserved charges and in the presence of some kind of holographic dual. The argument of [14] can probably be extended to cover the general case by considering the dispersion relations for all the hydrodynamical modes, including those arising from the dual conserved currents. However, the CSC remains a conjecture, and there appears to be room for violations: see for example [27, 79, 10].

The CSC relates only to the existence of a linearized instability around a static or stationary horizon. Considerable work has been devoted to the question of what the endpoint of the evolution of the Gregory-Laflamme instability might be: see for example [66, 37, 54, 86, 73, 74]. When there are thermodynamically stable horizons both with larger and smaller entropy, it seems to us likely that the endpoint of the evolution is a mixed phase with uniformly small curvatures outside the horizon, which remains unbroken. A mixed phase is a configuration where high-entropy and low-entropy regions with the same temperature are separated by domain walls. Typical solutions may not be static, but may instead evolve slowly according toward larger domains according to an effective theory with domain walls whose width is eventually negligible compared to the size of the domains. Mixed phases were previously suggested in connection with the Gregory-Laflamme instability in [22].

Finally, it is possible to arrange second-order behavior by tuning the potential $V(\phi)$ so that c_s^2 goes to 0 at some value of ϕ_H but never becomes negative: see figure 2.7. There is a corresponding critical temperature, and near it the equation of state typically takes the form

$$s \approx s_0 + s_{1/3} t^{1/3} \qquad \text{where} \qquad t = \frac{T - T_c}{T_c} \,. \tag{2.65}$$

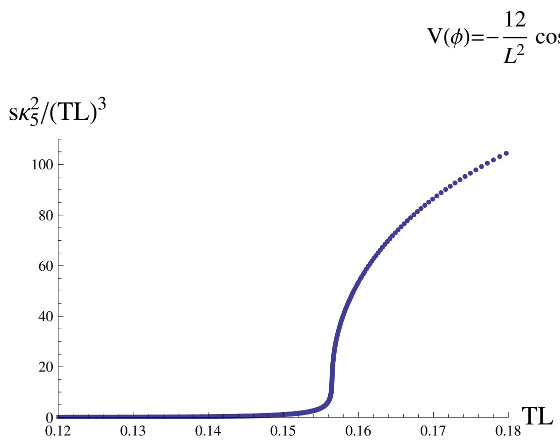

Figure 2.7: The equation of state and the speed of sound for $V(\phi) = -\frac{12}{L^2}\cosh\frac{\phi}{\sqrt{2}} + \frac{1.942}{L^2}\phi^2$. The point where $c_s^2 = 0$ is a second-order phase transition. If one considers instead $V(\phi) = -\frac{12}{L^2}\cosh\frac{\phi}{\sqrt{2}} + b\phi^2$ for $b > 1.942$, the transition becomes first-order, while if $b < 1.942$, it is a crossover.

The specific heat diverges as $C \sim t^{-2/3}$, and consequently the speed of sound behaves as $|t|^{-1/3}$.

2.6 Discussion

Since the inception of the anti-de Sitter / conformal field theory correspondence [78, 35, 87], it has been hoped that it would help solve quantum chromodynamics (QCD). This hope was articulated most clearly in the early literature in [88]. Subsequently, a large and somewhat heterogeneous literature has grown up around the idea of "AdS/QCD." Points of entry into this literature include [21, 82, 85, 25].

The first thermodynamic question one might ask of a putative dual to QCD is whether the equation of state is right. We have shown that the equation of state can be built into the construction by choosing an appropriate potential $V(\phi)$ for a scalar field that describes the breaking of conformal invariance. Indeed, within certain limitations, any equation of state $s = s(T)$ can be translated into a choice of $V(\phi)$, and vice versa. The limitations include that we use the supergravity approximation. This immediately points to a weakness of our approach: the shear viscosity will always satisfy $\eta/s = 1/4\pi$, regardless of temperature [83, 62, 16, 75]. Low shear viscosity is in conflict with expectations for the low-temperature phase of QCD, where the mean free path becomes large. Another reason to be suspicious of any attempt to describe the low-temperature phase using a black hole horizon is that at large N, entropy of a horizon scales as N^2, whereas the number of degrees of freedom in the confined phase of an $SU(N)$ gauge theory scales as N^0. A black hole description may be approximately valid above T_c, and its validity may fail only gradually as one passes through the crossover. But sufficiently far below the transition, the paradigm of weakly interacting hadrons should take over, and that is not part of our construction. One might imagine improvements on our construction, where, for example, higher curvature corrections significantly increase η/s, especially around or below the transition temperature. Eventually — perhaps when curvatures near the horizon become sufficiently large compared to the string scale — there could be a crossover to a gas of strings in a curved spacetime.

Our methods for constructing black holes are more general than the particular problem of mimicking the equation of state of QCD. Smooth crossovers, second-order transitions, first-order transitions, and perhaps even mixed phases may all be accommodated within the framework we have proposed. Our nonlinear master equation approach is special to the case of a single scalar, and it takes advantage of a weak form of integrability of the underlying equations. However, it is straightforward in principle to work with multiple scalars as well as with gauge fields: in this connection see for example [38, 39]. It seems likely that black holes in suitably designed theories exhibit a remarkable diversity of phase transitions.

Chapter 3

Holographic superconductors at zero temperature

3.1 Introduction

The Abelian Higgs model in AdS_4 is specified by the action

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R - \frac{1}{4} F_{\mu\nu}^2 - |(\partial_\mu - iqA_\mu)\psi|^2 - V(\psi, \psi^*) \right] , \qquad (3.1)$$

where V is assumed to depend on ψ and ψ^* only through the product $\psi\psi^*$. The theory (3.1) was introduced in [39] with the aim of describing superconducting black holes, following earlier work [38] on black hole phase transitions and [64, 59, 56], among others, on the possible relation between AdS_4 vacua and quantum critical behavior. In [57], the Abelian Higgs model was treated in a probe approximation, where the matter fields do not backreact appreciably on the metric. This approximation is justified in the limit of large q, and it is a useful starting point for studies of the conductivity and the behavior near the phase transition. However, one must go beyond the probe approximation to discover what the energetically preferred zero-temperature states are. In [52], it was suggested that for the W-shaped quartic potential

$$V(\psi,\psi^*) = -\frac{6}{L^2} + m^2\psi\psi^* + \frac{u}{2}(\psi\psi^*)^2\,, \tag{3.2}$$

with $m^2 < 0$ and $u > 0$, the zero-temperature limit of superconducting black holes is a domain wall that interpolates between the AdS_4 vacuum with $\psi = 0$ and the symmetry-breaking AdS_4 vacuum with

$$|\psi| = \psi_{\rm IR} \equiv \sqrt{\frac{-m^2}{u}}\,. \tag{3.3}$$

An example of such a domain wall was exhibited in [52], but it was left open whether such a domain wall exists for all values of q, and whether it really is a zero-temperature limit of superconducting black holes. The next studies beyond the probe approximation were [58, 47]. These studies focused on the case $m^2 < 0$, $u = 0$, and they provided numerical evidence that in the $T \to 0$ limit, all the charge is expelled from the black hole — at least when q is not too small and/or m^2L^2 is sufficiently negative. Moreover, according to [47], an $SO(2,1)$ boost symmetry appears in the infrared of the $T = 0$ solution, similar to the conformal symmetry of the infrared side of the domain walls studied in [52]. Subsequently, it was shown in [51] that AdS_5-to-AdS_5 and AdS_4-to-AdS_4 domain walls exist in string theory and M-theory, based on theories similar to (3.1) but with curved target spaces for the scalars. The AdS_5-to-AdS_5 case was based on [42] and was reported on earlier in [36], and the AdS_4-to-AdS_4 case was previously suggested in [29], based in part on [28].

In a parallel line of development, it was demonstrated in [68] that a massive gauge field coupled to gravity leads to geometries with anisotropic, Lifshitz-like scaling: $t \to \lambda^z t$ while $\vec{x} \to \lambda\vec{x}$ for some critical exponent z. It was shown in [5] that solutions to type IIB supergravity exist with anisotropic scaling between different spatial dimensions; moreover, such scaling solutions could be obtained as the infrared limit of flows from a suitably deformed AdS_5 vacuum. Some no-go arguments were given in [76] against the existence of Lifshitz solutions in type IIA supergravity and M-theory.

The problem of finding superconducting black hole solutions to the classical equations of motion following from (3.1) can be posed as follows. Consider the ansatz

$$\begin{aligned} ds^2 &= e^{2A(r)}\left[-h(r)dt^2 + d\vec{x}^2\right] + e^{2B(r)}\frac{dr^2}{h(r)} \\ A_\mu dx^\mu &= \Phi(r)dt \qquad \psi = \psi(r) \qquad B = 0\,, \end{aligned} \tag{3.4}$$

where $\psi(r)$ is everywhere real, and $B = 0$ is a gauge choice. The equations of motion and zero-energy constraint are

$$A'' = -\frac{1}{2}\psi'^2 - \frac{q^2}{2h^2e^{2A}}\Phi^2\psi^2 \tag{3.5}$$

$$h'' + 3A'h' = e^{-2A}\Phi'^2 + \frac{2q^2}{he^{2A}}\Phi^2\psi^2 \tag{3.6}$$

$$\Phi'' + A'\Phi' = \frac{2q^2}{h}\Phi\psi^2 \tag{3.7}$$

$$\psi'' + \left(3A' + \frac{h'}{h}\right)\psi' = \frac{1}{h}\frac{\partial V}{\partial\psi^*} - \frac{q^2}{h^2e^{2A}}\Phi^2\psi \tag{3.8}$$

$$h^2\psi'^2 + e^{-2A}q^2\Phi^2\psi^2 - \frac{1}{2}he^{-2A}\Phi'^2 - 2hh'A' - 6h^2A'^2 - hV(\psi,\psi^*) = 0\,. \tag{3.9}$$

It is straightforward to show that

$$Q = e^A(e^{2A}h' - \Phi\Phi') \tag{3.10}$$

is a constant if the equations of motion (3.5)-(3.9) are satisfied. It is the Noether charge associated with the scaling symmetry

$$\begin{gathered} A \to A - \log c \qquad h \to c^6 h \qquad \Phi \to c^2 \Phi \qquad B \to B + 3\log c \\ t \to t/c^2 \qquad \vec{x} \to c\vec{x} \end{gathered} \tag{3.11}$$

of the action (3.1) when evaluated with A_μ and the metric as in (3.4) before B has been fixed. If there is a black hole horizon at $r = r_H$, then the temperature and entropy density are

$$T = \frac{e^{A(r_H)} h'(r_H)}{4\pi} \qquad\qquad s = \frac{2\pi}{\kappa^2} e^{2A(r_H)}\,, \tag{3.12}$$

and one sees that

$$Q = 2\kappa^2 T s\,. \tag{3.13}$$

The $\Phi\Phi'$ term drops out of the relation (3.13) because $\Phi(r_H)$ has to be zero in order for Φdt to be well-defined at the horizon as a one-form, and $\Phi'(r_H)$ has to be finite so as to avoid generating divergent stress-energy. Thus $Q = 0$ is a form of extremality condition: it implies that either there is no horizon at all, or that if there is one, it has $Ts = 0$.

The behavior of the fields near the conformal boundary of AdS_4 is

$$\begin{aligned} A &= \sqrt{H_0}\frac{r}{L} + a_0 + \ldots \\ h &= H_0 + H_3 e^{-3A} + \ldots \\ \Phi &= p_0 + p_1 e^{-A} + \ldots \\ \psi &= \Psi_a e^{(\Delta_\psi - 3)A} + \Psi_b e^{-\Delta_\psi A} + \ldots\,, \end{aligned} \tag{3.14}$$

where $\ldots$ stands for terms that are subleading at large r relative to the ones shown, and

$$\Delta_\psi(\Delta_\psi - 3) = m^2 L^2\,. \tag{3.15}$$

(We will restrict attention to the larger root of this equation even in the window where both roots correspond to valid operator dimensions.) The chemical potential μ, the charge density ρ, and the energy density ϵ of the dual gauge theory are obtained from asymptotics near the boundary as

$$\mu = \frac{p_0}{2L\sqrt{H_0}} \qquad \rho = -\frac{p_1}{\kappa^2\sqrt{H_0}} \qquad \epsilon = -\frac{H_3}{\kappa^2 L H_0}\,. \tag{3.16}$$

Consider fixing p_0 at some definite value and setting $a_0 = 0$, $H_0 = 1$, and $\Psi_a = 0$. This corresponds to studying the dual gauge theory at finite chemical potential but not deforming its lagrangian with the operator $\mathcal{O}_\psi$ dual to ψ. Alternatively, one may leave p_0 free and instead fix p_1: this corresponds to considering the dual gauge theory at fixed charge density. Evaluating the conserved charge (3.13) close to the boundary gives

$$\epsilon = \frac{2}{3}(Ts + \mu\rho)\,. \tag{3.17}$$

This relationship also follows from the tracelessness of the field theory stress-energy tensor, which implies

$$\epsilon - 2p = 0\,. \tag{3.18}$$

Above, p is the pressure. For a large, homogeneous system at finite temperature and chemical potential, the pressure is just $-g$, where the Gibbs free energy density g is defined as

$$g = \epsilon - Ts - \mu\rho\,. \tag{3.19}$$

Equations (3.18) and (3.19) together imply (3.17). The conserved charge (3.13) thus enforces a thermodynamic relationship that holds for the dual conformal theory by connecting bulk thermodynamic variables that appear in horizon and boundary asymptotics.

Solving the equations (3.5)-(3.9) with the boundary conditions described in the previous paragraph, and demanding no singularities in the bulk outside regular black hole horizons, one might find only the AdS_4 Reissner-Nordstrom black hole solution (hereafter RNAdS), where $\psi = 0$ identically; or one may find superconducting solutions, where $\psi \neq 0$ spontaneously breaks the Abelian gauge symmetry. Typically there are several one-parameter families of solutions, each one parametrized by the energy density. The question at issue is what happens when we make this energy density as small as possible. In other words, what is the ground state of the system at finite chemical potential, or at finite charge density? If superconducting black holes are stable and thermodynamically favored over RNAdS, then this question is the same as asking what the zero-temperature limit of superconducting black holes is.

A reasonable guess is that when $m^2 < 0$ and $u > 0$, the zero-temperature limit is always a domain wall like the one in [52], with emergent conformal symmetry in the infrared. By considering expansions around the infrared AdS_4 geometry, we will show in section 3.2 that this cannot be right when q is too small. We propose in section 3.3 that what happens instead, below a certain threshold for q, is that the infrared geometry exhibits Lifshitz-like scaling.[1] This transition to Lifshitz behavior can be understood from a field theory perspective in terms of a non-conserved current operator becoming relevant when q is below its threshold value. We also find Lifshitz behavior when $m^2 > 0$. We exhibit explicit, numerically generated examples of AdS_4-to-AdS_4 and AdS_4-to-Lifshitz domain walls. All our analysis is based on simple four-dimensional gravity theories, not drawn from explicit string theory or M-theory constructions.

[1] This behavior is also mentioned in [67].

3.2 Emergent conformal symmetry

Let's assume that V takes the simple quartic form (3.2). The existence of an AdS_4 vacuum with $\psi = \psi_{\rm IR}$ and $\Phi = 0$ is wholly insensitive to the gauge field dynamics. It is likely that one can flow to this vacuum from the $\psi = 0$ AdS_4 vacuum with the gauge field set uniformly to 0. Such a holographic renormalization group flow, however, would have to be triggered by a relevant deformation of the lagrangian of the conformal field theory dual to the $\psi = 0$ vacuum. We are interested in eliminating such a deformation in favor of a finite density of the charge dual to the gauge field. To inquire whether conformal symmetry can emerge in the infrared in this context, we must ask whether one can perturb the $\psi = \psi_{\rm IR}$ AdS_4 vacuum in such a way that it can match onto a domain wall solution with nonzero gauge field. Let's express the AdS_4 vacuum as

$$ds^2 = e^{2r/L_{\rm IR}}(-dt^2 + d\vec{x}^2) + dr^2\,, \tag{3.20}$$

where

$$L_{\rm IR} = \sqrt{\frac{-6}{V(\psi_{\rm IR}, \psi_{\rm IR})}}\,. \tag{3.21}$$

Then the perturbations of interest are ones that vanish in the $r \to -\infty$ limit (the deep infrared) and are either finite or divergent in the $r \to +\infty$ limit, which is eventually replaced by the domain wall. In field theory terms, we wish to study irrelevant perturbations by operators dual to the fields A_0 and ψ.

As a first step, consider the linearized equations of motion for the scalar and the gauge field, assuming that $A_0 = \delta\Phi$ is the only nonvanishing component of A_μ, that the scalar $\psi = \psi_{\rm IR} + \delta\psi$ is everywhere real, and that both $\delta\Phi$ and $\delta\psi$ depend only on r:

$$\begin{aligned} \left[\partial_r^2 + \frac{1}{L_{\rm IR}}\partial_r - m_\Phi^2\right]\delta\Phi = 0 \\ \left[\partial_r^2 + \frac{3}{L_{\rm IR}}\partial_r - m_{\rm IR}^2\right]\delta\psi = 0\,, \end{aligned} \tag{3.22}$$

where

$$m_\Phi^2 = 2q^2\psi_{\rm IR}^2$$
$$m_{\rm IR}^2 = 2\frac{\partial^2 V}{\partial\psi\partial\psi^*}(\psi_{\rm IR},\psi_{\rm IR})\,. \tag{3.23}$$

Let us further define

$$\Delta_\Phi = \frac{3}{2} + \sqrt{\frac{1}{4} + m_\Phi^2 L_{\rm IR}^2}$$
$$\Delta_{\rm IR} = \frac{3}{2} + \sqrt{\frac{9}{4} + m_{\rm IR}^2 L_{\rm IR}^2}\,, \tag{3.24}$$

where the positive sign on the square root is understood in both cases. The operators $J_\mu^{\rm IR}$ and $\mathcal{O}_{\rm IR}$ dual to A_μ and $\delta\psi$ have dimensions Δ_Φ and $\Delta_{\rm IR}$, respectively. The solutions to (3.22) that vanish in the limit $r \to -\infty$ are

$$\delta\Phi \equiv \Phi_1 = a_\Phi e^{(\Delta_\Phi - 2)r/L_{\rm IR}}$$
$$\delta\psi \equiv \psi_1 = a_\psi e^{(\Delta_{\rm IR}-3)r/L_{\rm IR}}\,, \tag{3.25}$$

where a_ψ and a_Φ are undetermined coefficients.

The second formula in (3.24) shows that $\Delta_{\rm IR} \geq 3$ provided $m_{\rm IR}^2 \geq 0$, which has to be true given that $\psi_{\rm IR}$ is a minimum of the potential. In other words, the operator dual to ψ at the infrared fixed point is an irrelevant perturbation, which makes sense because it participates in a flow toward conformality in the infrared. The first formula in (3.24) shows that $\Delta_\Phi \geq 2$ provided $m_\Phi^2 > 0$, which has to be true given the expression for m_Φ^2 in (3.23).[2] If $\Delta_\Phi > 3$, then the operator J_0 dual to Φ is also an irrelevant perturbation, so again one has a sensible field theory interpretation that J_0 participates in a flow toward Lorentz-invariant conformality in the infrared. On the other hand, if $2 < \Delta_\Phi < 3$, then there seems to be a puzzle: in gravity we have the solution Φ_1 exhibited in (3.25), which vanishes in the limit $r \to -\infty$; but in field theory, J_0 is a *relevant* operator, which should distort the field theory further and further away from Lorentz invariance as one proceeds toward the infrared.

[2] Also, there is a unitarity bound $\Delta_\Phi \geq 2$ for gauge-invariant, primary operators [77] (see also [33]), suggesting that even in a more general setup, one cannot have $m_\Phi^2 < 0$.

The resolution of this puzzle is that gravity solutions describing charged matter in the ultraviolet conformal field theory cannot flow to the symmetry-breaking infrared fixed point if $2 < \Delta_\Phi < 3$. As far as we can tell, this is the only obstacle to the existence of such flows. This line of thought is what led to the Criticality Pairing Conjecture of [51].

To demonstrate the claim that flowing to a conformal fixed point is impossible (or at least fine-tuned) if $2 < \Delta_\Phi < 3$, we need to develop some machinery describing perturbations of the infrared conformal point. Although the presentation of the next couple of paragraphs is a bit lengthy, the final punch line can be stated in advance: for $2 < \Delta_\Phi < 3$, there is strong backreaction on the metric such that the blackening function $-g_{tt}/g_{xx}$, doesn't approach a constant in the infrared. Instead, as we will describe in section 3.3, one finds Lifshitz-like scaling in the infrared.

Consider the expansions

$$\begin{aligned} A &= \frac{r}{L_{\rm IR}} + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3 + \dots \\ h &= 1 + \lambda h_1 + \lambda^2 h_2 + \lambda^3 h_3 + \dots \\ \Phi &= \lambda \Phi_1 + \lambda^2 \Phi_2 + \lambda^3 \Phi_3 + \dots \\ \psi &= \psi_{\rm IR} + \lambda \psi_1 + \lambda^2 \psi_2 + \lambda^3 \psi_3 + \dots\,, \end{aligned} \tag{3.26}$$

where λ is a formal expansion parameter that we eventually want to set to unity. What we are really expanding in is the smallness of all corrections to AdS_4 in the limit $r \to -\infty$. Plugging the expansions (3.26) into the equations of motion (3.5)-

(3.8), one obtains at nth order in λ the conditions

$$\begin{aligned} \partial_r^2 A_n &= \mathcal{S}_n^A \\ \left[\partial_r^2 + \frac{3}{L_{\rm IR}}\partial_r\right] h_n &= \mathcal{S}_n^h \\ \left[\partial_r^2 + \frac{1}{L_{\rm IR}}\partial_r - m_\Phi^2\right] \Phi_n &= \mathcal{S}_n^\Phi \\ \left[\partial_r^2 + \frac{3}{L_{\rm IR}}\partial_r - m_{\rm IR}^2\right] \psi_n &= \mathcal{S}_n^\psi \,, \end{aligned} \tag{3.27}$$

where $\mathcal{S}_n^X$, for $X = A$, h, Φ, or ψ is a polynomial in the coefficient functions A_k, h_k, Φ_k, and ψ_k, and their derivatives, for $k < n$. $\mathcal{S}_1^X = 0$ for $X = A$, h, Φ, and ψ. We choose Φ_1 and ψ_1 as in (3.25), and we set $A_1 = h_1 = 0$. For $n > 1$, the equations (3.27) can be solved iteratively using a method of Green's functions:

$$\begin{aligned} A_n(r) &= \int_{-\infty}^r d\tilde{r} \int_{-\infty}^{\tilde{r}} dr_* \, \mathcal{S}_n^A(r_*) = \int_{-\infty}^r dr_* \, (r - r_*)\mathcal{S}_n^A(r_*) \\ h_n(r) &= \int_{-\infty}^r dr_* \, L_{\rm IR} \frac{1 - e^{3(r_* - r)/L_{\rm IR}}}{3} \mathcal{S}_n^h(r_*) \\ \Phi_n(r) &= \int_{-\infty}^r dr_* \, L_{\rm IR} \frac{e^{(\Delta_\Phi - 2)(r - r_*)/L_{\rm IR}} - e^{-(\Delta_\Phi - 1)(r - r_*)/L_{\rm IR}}}{2\Delta_\Phi - 3} \mathcal{S}_n^\Phi(r_*) \\ \psi_n(r) &= \int_{-\infty}^r dr_* \, L_{\rm IR} \frac{e^{(\Delta_{\rm IR} - 3)(r - r_*)/L_{\rm IR}} - e^{-\Delta_{\rm IR}(r - r_*)/L_{\rm IR}}}{2\Delta_{\rm IR} - 3} \mathcal{S}_n^\psi(r_*) \,. \end{aligned} \tag{3.28}$$

One can check that the solution (3.28) satisfies the zero-energy constraint. Heuristically, this is because the zero-energy constraint is trivially satisfied for the AdS_4 vacuum, and the perturbations (3.28) are constructed so as to approach this limit as rapidly as possible as $r \to -\infty$.

Equation (3.28) represents only one particular set of solutions to the equations (3.27). All others can be obtained by adding solutions to the homogeneous equations. For the purposes of studying irrelevant perturbations to the infrared AdS_4 vacuum, no such additions should be made. To see this, first note that three of those solutions — $h_n = e^{-3r/L_{\rm IR}}$, $\Phi_n = e^{-(\Delta_\Phi - 1)r/L_{\rm IR}}$, and $\psi_n = e^{-\Delta_{\rm IR} r/L_{\rm IR}}$ — are disallowed because they diverge exponentially as $r \to -\infty$. The two solutions $A_n = r$ and $h_n = 1$ can be added, but the zero-energy constraint imposes a relation between their coefficients, and when this constraint is satisfied, the effect of the addition is simply to change the normalization of r. The solution $A_n = 1$ need not be added because it can be offset by rescaling t and $\vec{x}$. This leaves only the solutions $\Phi_n = e^{(\Delta_\Phi - 2)r/L_{\rm IR}}$ and $\psi_n = e^{(\Delta_{\rm IR}-3)r/L_{\rm IR}}$. They are present for $n = 1$ and need not be included at higher orders: doing so would merely adjust the values of a_Φ and a_ψ.

The parameters L, m^2, u, and q that enter into the lagrangian (3.1) with the quartic potential (3.2) can be traded for $L_{\rm IR}$, $\psi_{\rm IR}$, Δ_Φ, and $\Delta_{\rm IR}$. These four parameters, together with a_Φ and a_ψ, completely determine all the X_n, where as usual, X denotes A, h, Φ, or ψ. For generic values of the parameters, the solutions take the form

$$X_n = \sum_\alpha c^X_{n,\alpha} e^{-\gamma^X_{n,\alpha} r/L_{\rm IR}}\,, \tag{3.29}$$

where α runs over some finite set, the $c^X_{n,\alpha}$'s are rational functions of the parameters (independent of r), and

$$\gamma^X_{n,\alpha} = p^X_{n,\alpha}\Delta_\Phi + s^X_{n,\alpha}\Delta_{\rm IR} + r^X_{n,\alpha}\,. \tag{3.30}$$

The coefficients $p^X_{n,\alpha}$ and $s^X_{n,\alpha}$ are nonnegative integers (not both zero for a given value of X, n, and α), and $r^X_{n,\alpha}$ are negative integers.

Clearly, the expansions (3.26) are valid only when all the $\gamma^X_{n,\alpha}$ are positive. The positivity constraints at level $n = 1$ are $\Delta_\Phi > 2$ and $\Delta_{\rm IR} > 3$, which are trivial in the sense that they follow from the definitions (3.24). At the quadratic level, $n = 2$, one finds a tighter constraint from the γ coefficients for A, h, and ψ: $\Delta_\Phi > 3$. It is straightforward to check that the following two versions of the $n = 2$ constraint are equivalent:

$$\Delta_\Phi > 3 \qquad \Longleftrightarrow \qquad qL_{\rm IR}\psi_{\rm IR} > 1\,. \tag{3.31}$$

If this constraint is violated, then there cannot be a domain wall interpolating between the $\psi = 0$ and $\psi = \psi_{\rm IR}$ AdS_4 vacua. No further tightening of constraints occurs at the next two orders, and we conjecture that there is no further tightening at any higher order, at least when the scalar potential is smooth. Assuming this conjecture is correct, there is still a possibility of convergence problems in the infrared expansion, even though no individual term is badly behaved. But numerical investigations suggest that a charged domain wall solution does exist, starting from the undeformed ultraviolet conformal theory, when the constraint (3.31) is satisfied.

To recapitulate: the condition (3.31), in field theory terms, is the statement that the operator dual to Φ is irrelevant. This is precisely the condition one expects in order for a flow to conformal invariance in the infrared to exist. The series expansion machinery introduced in (3.26)-(3.28) confirms this expectation on the gravity side. So we conclude that charged domain wall solutions with conformal invariance in the infrared probably exist when $\Delta_\Phi > 3$.

3.3 Lifshitz-like scaling

The previous section, building upon results of [52, 51], provides a candidate ground state of the Abelian Higgs model in AdS_4, provided there exists an extremum of the potential with $\psi \neq 0$, and provided the charge is not too small. The candidate ground state is a domain wall interpolating between symmetry-preserving AdS_4 on the ultraviolet side and symmetry-breaking AdS_4 on the infrared side. Its explicit form is given in (3.4), with h interpolating between two different constants in the ultraviolet and infrared, and with Φ vanishing in the infrared limit. Slightly nonextremal generalizations of these domain walls would be approximately described as domain walls between the ultraviolet AdS_4 geometry and AdS_4-Schwarzschild in the infrared.

In this section, we propose another candidate ground state. It is like the one just described in that it is a domain wall with symmetry-preserving AdS_4 in the ultraviolet. But its infrared limit is a Lifshitz geometry similar to the ones constructed in [68]. In subsection 3.3.1 we briefly review this construction and indicate how it can be formally embedded in a limit of the Abelian Higgs model. In subsection 3.3.2 we demonstrate that, besides AdS_4, AdS_4-Schwarzschild, and AdS_4-Reissner-Nordstrom, the Lifshitz geometry is the only solution to the equations of motion (3.5)-(3.9) that can have constant ψ. In subsection 3.3.3, we analyze the perturbations of Lifshitz backgrounds at linear order. In subsection 3.3.4, we discuss Lifshitz solutions based on the U-shaped quadratic potential: (3.2) with $m^2 > 0$ and $u = 0$. In subsection 3.3.5 we discuss solutions with Lifshitz-like scaling based on the W-shaped quartic potential: (3.2) with $m^2 < 0$ and $u > 0$.

As in the case of emergent conformal symmetry discussed in the previous section, what we are doing here is constructing a geometry that *may* be the infrared side of a domain-wall ground state of the Abelian Higgs model in AdS_4. To show that such domain walls really exist, the only approach we know of is numerics. We give some examples in sections 3.3.4 and 3.3.5.

Altogether, our results on AdS_4-to-Lifshitz domain walls bear some resemblance to the work of [5]. The main differences are that we do not attempt to embed our solutions into string theory, and that the coordinate in our solutions that scales anisotropically is not spatial but instead timelike. (In [5], a configuration was considered in which the scaling is anisotropic in the timelike direction. However, this configuration involved a slightly unusual feature, namely a continuous density of extended fundamental strings. So it is not wholly described in terms of supergravity, as our solutions are.) Our domain walls are quite different from the one exhibited in [68], in that we have conformal invariance in the ultraviolet and Lifshitz behavior in the infrared, not the other way around.

3.3.1 Embedding Lifshitz solutions in the Abelian Higgs model

In [68], it was explained that four-dimensional gravity with a negative cosmological constant coupled to a two-form field strength $F_{(2)}$ and a three-form field strength $F_{(3)} = dB_{(2)}$, with a $B_{(2)} \wedge F_{(2)}$ interaction, admits solutions with Lifshitz-like symmetry. The metric is

$$ds^2 = -\left(\frac{r}{L_0}\right)^{2z} dt^2 + \frac{r^2}{L_0^2} d\vec{x}^2 + L_0^2 \frac{dr^2}{r^2}\,, \tag{3.32}$$

and the Lifshitz-like scaling symmetry is

$$t \to \lambda^z t \qquad \vec{x} \to \lambda \vec{x} \qquad r \to \frac{r}{\lambda}\,. \tag{3.33}$$

The dynamical exponent z is determined in terms of L_0 and the coupling multiplying the $B_{(2)} \wedge F_{(2)}$ term. Note that in (3.32), we have persisted in letting r be a dimensionful variable. To recover the form of the ansatz discussed in [68], one can use the dimensionless variables t/L_0, $\vec{x}/L_0$, and r/L_0 in place of t, $\vec{x}$, and r.

The $B_{(2)} \wedge F_{(2)}$ theory considered in [68] is a limit of the Abelian Higgs model in which the modulus of ψ is frozen at $\psi_{\rm IR}$. Our main aim in this section is to explain how solutions of the form (3.32) arise in the Abelian Higgs model before any special limit is taken. However, let us briefly detour to explain how to map the frozen modulus limit of the Abelian Higgs model into the $B_{(2)} \wedge F_{(2)}$ theory. First, to define this limit, we consider a potential $V(\psi, \psi^*)$ that depends only on the modulus of ψ and has a very sharp minimum at some finite value ψ_0 of $|\psi|$. Restricting

$$\psi = \psi_0 e^{i\theta}\,, \tag{3.34}$$

one finds from (3.1) the action

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\mathcal{L} \tag{3.35}$$

where

$$\mathcal{L} = R - \frac{1}{4}F_{\mu\nu}^2 - \psi_0^2(\partial_\mu\theta - qA_\mu)^2 - V_0 - \frac{q\sqrt{2}\psi_0}{4\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}\left(F_{\rho\sigma} - 2\partial_\rho A_\sigma\right) , \tag{3.36}$$

and $V_0 = V(\psi_0, \psi_0)$. In the last term of (3.36) we have introduced a Lagrange multiplier field $B_{\mu\nu}$ that enforces $F_{(2)} = dA_{(1)}$ as a constraint. When integrated against $\sqrt{-g}$, this term is topological in the sense that it does not involve the metric. So it doesn't affect the Einstein equations. The momentum conjugate to θ is

$$\Pi^\mu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)} = -2\psi_0^2(\partial^\mu\theta - qA^\mu) , \tag{3.37}$$

and it is conserved because θ enters into $\mathcal{L}$ only through its first derivatives. A convenient way to express this conservation is

$$\Pi^\mu = -\frac{\sqrt{2}\psi_0}{3!\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}F_{\nu\rho\sigma} , \tag{3.38}$$

where $F_{(3)}$ is a closed three-form. To obtain the equations of motion for the other degrees of freedom, one may use the Routhian construction:

$$\begin{aligned}\mathcal{R} \equiv \mathcal{L} - \Pi^\mu\partial_\mu\theta &= R - \frac{1}{4}F_{\mu\nu}^2 + \frac{\Pi_\mu^2}{4\psi_0^2} - q\Pi^\mu A_\mu - V_0 - \frac{q\sqrt{2}\psi_0}{4\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}\left(F_{\rho\sigma} - 2\partial_\rho A_\sigma\right) \\ &= R - \frac{1}{4}F_{\mu\nu}^2 - \frac{1}{12}F_{\mu\nu\rho}^2 - V_0 - \frac{q\sqrt{2}\psi_0}{4\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}F_{\rho\sigma} \\ &\quad + \frac{q\sqrt{2}\psi_0}{3!\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}A_\mu(F_{\nu\rho\sigma} - 3\partial_\nu B_{\rho\sigma}) + (\text{total derivative}) .\end{aligned} \tag{3.39}$$

A_μ enters the final expression for the Routhian only as a lagrange multiplier enforcing the constraint $F_{(3)} = dB_{(2)}$. We may omit the lagrange multiplier term if we elevate the constraint to a definition of $F_{(3)}$; then all equations of motion follow from the second line of (3.39), and we have indeed recovered $B_{(2)} \wedge F_{(2)}$ theory. The second line of (3.39) precisely matches (2.3) of [68], provided we set $\kappa = 1$, $e = 1$, $2\Lambda = V_0$, and $c = q\sqrt{2}\psi_0$.

3.3.2 The uniqueness of Lifshitz solutions

Having established that the $B_{(2)} \wedge F_{(2)}$ theory is equivalent to the frozen-modulus limit of the Abelian Higgs model, a natural follow-up question is whether the Lifshitz solutions (3.32) persist away from this limit. We claim that they do, and that besides AdS_4, AdS_4-Schwarzschild, and AdS_4-Reissner-Nordstrom they are the only other solutions with translation invariance in time, spatial translation and rotation symmetry in the directions x^1 and x^2, and constant value for the scalar field ψ. Without loss of generality, we can assume that this constant value is real and positive. We will make this restriction from now on.

To establish our claim, we start with a general ansatz consistent with the symmetries mentioned:

$$\begin{aligned} ds^2 &= -g(r)^2 dt^2 + \frac{r^2}{L_0^2} d\vec{x}^2 + e^{2B(r)} \frac{L_0^2}{r^2} dr^2 \\ \Phi &= \Phi(r) \qquad\qquad \psi = \psi_0 = (\text{constant})\,. \end{aligned} \tag{3.40}$$

The metric ansatz in (3.40) is equivalent to the one in (3.4) after appropriate redefinitions of coordinates and fields. Plugging the constant value ψ_0 into the scalar equation of motion results in the condition that

$$V_{\text{eff}}(\psi, \psi^*) = V(\psi, \psi^*) - \frac{q^2 \Phi(r)^2 \psi \psi^*}{g(r)^2} \tag{3.41}$$

is extremized at $\psi = \psi_0$. There are two ways in which this can happen, for all r:

1. It could be that both terms of (3.41) are separately extremized.

2. It could be that neither term of (3.41) is separately extremized, but instead that their first derivatives cancel at $\psi = \psi_0$.

Let us refer to solutions with constant ψ as solutions of the first or second kind, according to which of the two possibilities just described is realized. Because of the $U(1)$ symmetry, we can assume that ψ_0 is real and nonnegative.

For solutions of the first kind, we must have $\frac{\partial V}{\partial \psi} = \frac{\partial V}{\partial \psi^*} = 0$ at $\psi = \psi^* = \psi_0$, and also that either Φ or $q\psi_0$ vanishes. If Φ vanishes, then the solution (3.40) can only be AdS_4 or AdS_4-Schwarzschild. If $q\psi_0$ vanishes, then the possibilities are AdS_4 and AdS_4-Schwarzschild if Φ is constant, and AdS_4-Reissner-Nordstrom if it isn't. Thus our claim comes down to demonstrating that solutions of the second type must exhibit Lifshitz-like scaling.

The condition that V_{eff} is extremized for all values of r through non-trivial competition between the two terms is quite restrictive because it implies that

$$g(r) = \frac{1}{\sqrt{2 - 2/\eta}} \Phi(r) \tag{3.42}$$

for some constant η. Plugging this equation into the Einstein equations, one can solve algebraically for $r\Phi''(r)$, $rB'(r)$, and $B(r)$ in terms of $\Phi(r)$, $r\Phi'(r)$, $V(\psi_0, \psi_0)$, $q\psi_0$, and η, with no additional dependence on r. ($B''(r)$ doesn't enter to the Einstein equations because it is essentially a gauge degree of freedom.) The resulting expressions are unenlightening, so we will not exhibit them explicitly. Eliminating $r\Phi''(r)$, $rB'(r)$ and $B(r)$ from the Maxwell equation for Φ, one obtains the relation

$$\frac{V(\psi_0, \psi_0)}{q^2 \psi_0^2} = \frac{4\eta + 8z\eta + z^2(-3 + 2\eta + \eta^2)}{\eta(-z^2 + \eta + 2z\eta)} , \tag{3.43}$$

where

$$z = \frac{r\Phi'(r)}{\Phi(r)} . \tag{3.44}$$

(3.43) shows that z is a constant, so Φ has a power-law dependence, $\Phi \propto r^z$. Plugging this dependence back into the Einstein equations leads to the constraint

$$(\eta - z)(z\eta + 2\eta - z) = 0\,. \tag{3.45}$$

Assuming that the second factor vanishes leads to difficulties: the Maxwell equation for Φ then demands that $p_0 z(1+z) = 0$. But if $z = 0$ or -1 then g_{rr} formally vanishes, while if $p_0 = 0$ then g_{tt} formally vanishes. So we may assume that $\eta = z$. Then (3.43) becomes

$$\frac{V(\psi_0, \psi_0)}{q^2 \psi_0^2} = -\frac{4 + z + z^2}{z}\,, \tag{3.46}$$

and the only additional constraints from the equations of motion are

$$\frac{\partial V}{\partial \psi}(\psi_0, \psi_0) = \frac{\partial V}{\partial \psi^*}(\psi_0, \psi_0) = 2\frac{z-1}{z} q^2 \psi_0 \tag{3.47}$$

$$e^{2B} q^2 \psi_0^2 L_0^2 = z\,. \tag{3.48}$$

The latter implies that B is constant. If one starts with a definite function $V(\psi, \psi^*)$ and a definite value of the charge q, then (3.46) and (3.47) generically admit at most a discrete set of solutions for z and ψ_0 (given that we assume that ψ_0 is real and nonnegative). Then (3.48) can be regarded as determining the product $e^{2B} L_0^2$. By themselves e^{2B} and L_0^2 are not meaningful: a rescaling

$$r \to \lambda_1 r \qquad L_0 \to \lambda_1 L_0 \qquad e^{2B} \to \frac{1}{\lambda_1^2} e^{2B} \tag{3.49}$$

preserves the form of the ansatz (3.40) and the product $e^{2B} L_0^2$. We can use this scaling symmetry to set $B = 0$. A second scaling symmetry,

$$r \to \lambda_2 r \qquad \vec{x} \to \frac{1}{\lambda_2} \vec{x}\,, \tag{3.50}$$

also preserves the form of the ansatz. We know that $g(r) \propto r^z$, and use of the λ_2 symmetry allows us to dictate the constant of proportionality:

$$g = \left(\frac{r}{L_0}\right)^z \qquad \Phi = \sqrt{2 - \frac{2}{z}} \left(\frac{r}{L_0}\right)^z , \tag{3.51}$$

where in the second equation we have used (3.42). Plugging (3.51) into (3.40), we recover the original ansatz (3.32) with Lifshitz-like scaling symmetry. This completes our demonstration that geometries with Lifshitz-like scaling are the only possible solutions to the classical equations of motion following from (3.1), other than AdS_4, AdS_4-Schwarzschild, and AdS_4-Reissner-Nordstrom, in which ψ can be constant.

The Lifshitz solution we have described is a straightforward lift of the solution of [68] to the Abelian Higgs model. The relations (3.46) and (3.48) correspond precisely to relations obtained in the frozen-modulus limit, namely (2.11) (first line) and (2.7) of [68]. In order for Φ and g both to be real, we must have $z \geq 1$ or else $z < 0$. The latter possibility is ruled out by the relation (3.48). Having concluded that $z \geq 1$,[3] we see from (3.47) that $\frac{\partial V}{\partial \psi}(\psi_0, \psi_0) \geq 0$. For the double-well quartic potential (3.2), this is only true when $\psi_0 > \psi_{\rm IR}$, where $\psi_{\rm IR}$ is the positive real minimum of $V(\psi, \psi^*)$, as in (3.3). Finally, noting that $4 + z + z^2 > 0$ when $z \geq 1$, we see from (3.46) that $V(\psi_0, \psi_0) < 0$. For the potential (3.2), this implies $\psi_0 < \psi_*$ where ψ_* is the unique positive root of the equation $V(\psi, \psi^*) = 0$.

Although we did not use the Noether charge Q defined in (3.13) in our demonstration of uniqueness of Lifshitz backgrounds, it is straightforward, starting with the expression

$$Q = \frac{1}{g e^B} \left(\frac{r}{L_0}\right)^3 \left(2gg' - \frac{2g^2}{r} - \Phi\Phi'\right) , \tag{3.52}$$

[3]The inequality $z \geq 1$ was also obtained in [68] for the $B_{(2)} \wedge F_{(2)}$ theory. It can almost be obtained from a null-energy argument, as follows. By calculation, $L_0^2(-R_t^t + R_x^x) = (z+2)(z-1)$. According to the Einstein equations, $-R_t^t + R_x^x = \kappa^2(-T_t^t + T_x^x) = \kappa^2 T_{\mu\nu}\xi^\mu\xi^\nu$ where $\xi^\mu = \left(\frac{1}{r^z}, \frac{1}{r}, 0, 0\right)$ is a null vector. The null energy condition says that $T_{\mu\nu}\xi^\mu\xi^\nu \geq 0$ for any null vector ξ^μ. So $(z+2)(z-1) \geq 0$, implying either $z \leq -2$ or $z \geq 1$. Nothing in this null-energy argument rules out $z \leq -2$, but (3.48) of course does.

to check that Q does vanish.

3.3.3 Perturbing a Lifshitz background

In the previous subsection, we demonstrated that Lifshitz solutions to the equations of motion following from (3.1) exist precisely when we can simultaneously solve (3.46) and (3.47). What we want to find out next is when such solutions can be matched onto an asymptotically AdS_4 geometry in order to describe a ground state of the asymptotically conformal holographic Abelian Higgs model. The answer turns out to be a bit subtle for approximately the same reason that we encountered with emergent conformal symmetry: there may or may not be irrelevant perturbations to the Lifshitz background of a sort that allow it to participate in an AdS_4-to-Lifshitz domain wall.

To study perturbations, let's consider the ansatz (3.40) again, but with ψ allowed to be a function of r, and all functions expressed as perturbations of the Lifshitz solution (3.32):

$$\begin{aligned} g &= \left(\frac{r}{L_0}\right)^z + \lambda g_1 + \dots \\ B &= \lambda B_1 + \dots \\ \Phi &= \sqrt{2 - \frac{2}{z}} \left(\frac{r}{L_0}\right)^z + \lambda \Phi_1 + \dots \\ \psi &= \psi_0 + \lambda \psi_1 + \dots \,, \end{aligned} \tag{3.53}$$

where λ is a formal expansion parameters, and g_1, B_1, Φ_1, and ψ_1 are functions only of r. In the most general ansatz consistent with preservation of the translation symmetries in the t and $\vec{x}$ directions and the rotation symmetry between x^1 and x^2, we would have to include also perturbations δg_{xx} and δg_{tr} to the metric and δA_r to the gauge field. Excluding these additional perturbations amounts to partially gauge-fixing.

The five functions $(g_1, B_1, \Phi_1, \psi_1, \psi_1^*)$ are subject to five second-order differential equations plus three first-order constraints, obtained by linearizing the equations of motion in λ. So there are seven linearly independent solutions. Two of the seven perturbations are trivial:

1. $\psi_1 = -\psi_1^* = i$, corresponding to changing the background value of the scalar from ψ_0 to $e^{i\theta_0}\psi_0$, where θ_0 is some constant phase. (Recall we assume that ψ_0 is real and positive.)

2. $g_1 = (r/L_0)^z$ and $\Phi = \sqrt{2-2/z}(r/L_0)^z$, corresponding to rescaling t by a constant.

Both these two pure gauge modes, and the other five perturbations, can be put into the general form

$$g_1 = c_g r^{\beta_g} \qquad B_1 = c_B r^{\beta_B} \qquad \Phi_1 = c_\Phi r^{\beta_\Phi} \qquad \delta\psi = c_\psi r^{\beta_\psi} \qquad \delta\psi^* = c_{\psi^*} r^{\beta_{\psi^*}} , \tag{3.54}$$

and one always finds the following relations among the exponents:

$$\beta_g = \beta_B + z = \beta_\psi + z = \beta_{\psi^*} + z = \beta_\Phi . \tag{3.55}$$

In order for a perturbation to be "irrelevant," all the $\mathcal{O}(\lambda)$ corrections should become small compared to the leading order solution, except for B_1, which should become small compared to 1. This happens precisely if $\mathrm{Re}\,\beta_\psi > 0$. If instead $\mathrm{Re}\,\beta_\psi < 0$, then the perturbation is "relevant" in the sense of becoming larger as one passes toward the infrared.

The remaining five perturbations fall into two classes (c.f. the analysis of [8]):

3. There is one perturbation with $\beta_\psi = -2 - z$, which we will term the "universal

mode.” One can show that

$$
\begin{aligned}
c_{tt} &= -L_0^{-z}\sqrt{\frac{z(z-1)}{2}}\frac{2(z^2+9z+2)+\psi_0^2 m_0^2 L_0^2(z-2)}{6z(z-1)+\psi_0^2 m_0^2 L_0^2(z^2+2)}c_\Phi \\
c_{rr} &= -L_0^{2+z}\sqrt{\frac{z(z-1)}{2}}(z+2)\frac{2(z+3)+\psi_0^2 m_0^2 L_0^2}{6z(z-1)+\psi_0^2 m_0^2 L_0^2(z^2+2)}c_\Phi \\
c_\psi &= c_{\psi^*} = L_0^z\sqrt{\frac{z(z-1)}{2}}\frac{2(z+1)(z+2)\psi_0}{6z(z-1)+\psi_0^2 m_0^2 L_0^2(z^2+2)}c_\Phi\,,
\end{aligned}
\tag{3.56}
$$

where we have defined

$$
m_0^2 \equiv \frac{\partial^2 V}{\partial\psi\partial\psi^*}(\psi_0,\psi_0)+\frac{\partial^2 V}{\partial\psi^{*2}}(\psi_0,\psi_0)\,. \tag{3.57}
$$

We describe this mode as universal because it is present even in the frozen modulus limit where $m_0^2 \to \infty$. In that limit, one can see from (3.56) that if c_{tt}, c_{rr}, and c_Φ are held finite, then $c_\psi \to 0$, indicating that the scalar stays pinned at its background value.

Although the universal mode is a solution of the linearized equations of motion, it is not a solution of the linearization of the extremality condition $Q = 0$. Its interpretation seems to follow fairly clearly: this mode is related to making Lifshitz backgrounds nonextremal. This is confirmed in the $B_{(2)} \wedge F_{(2)}$ theory by the calculations of [9]: see in particular the ultraviolet asymptotics of the non-extremal backgrounds constructed there.

4. There are four perturbations that we will term “non-universal” because their characteristics depend on details of the potential. Each one is based on one of the following values of β_ψ:

$$
\beta_\psi = \beta_\psi(s_1,s_2) \equiv -\frac{z+2}{2}+\frac{s_1}{\psi_0}\sqrt{D_1+s_2\sqrt{D_2}}\,, \tag{3.58}
$$

where

$$
\begin{aligned}
D_1 &= -z + 1 + \left(\frac{5z^2}{4} - 2z + 3 + \frac{m_0^2 L_0^2}{2}\right)\psi_0^2 \\
D_2 &= \left[\left(z^2 - 3z + 2 - \frac{m_0^2 L_0^2}{2}\right)\psi_0^2 + z - 1\right]^2 - 8\left(z^2 - 1\right)\psi_0^2
\end{aligned}
\tag{3.59}
$$

and s_1 and s_2 are independently chosen to be either $+1$ or -1. The values (3.58) come out of insisting that when $c_\psi = c_{\psi^*}$ and β_ψ is neither $-2-z$ nor 0, the determinant of the matrix that constrains c_{tt}, c_{rr}, c_Φ, and c_ψ must vanish. The closed-form expressions for these coefficients are long and unenlightening. It is easily seen that the real parts of both $\beta_\psi(-1,1)$ and $\beta_\psi(-1,-1)$ are always negative. Therefore, these exponents are always associated with relevant perturbations, and they never participate in a flow toward an infrared Lifshitz fixed point. On the other hand, the $\beta_\psi(1,\pm 1)$ may be associated with relevant or irrelevant perturbations, as we will see in the next section.

3.3.4 The positive-mass quadratic potential

Inspecting (3.47) and recalling that we have to have $z \geq 1$, we see that V has to slope upward in the direction of increasing magnitude of ψ in order for there to be a Lifshitz fixed point. The simplest nontrivial potential for ψ that satisfies this upward slope condition is the positive-mass quadratic potential:

$$
V(\psi, \psi^*) = -\frac{6}{L^2} + m^2 \psi \psi^* \tag{3.60}
$$

with $m^2 > 0$. Since the limiting case $z = 1$ corresponds to AdS_4, we restrict ourselves to $z > 1$. Solving (3.46), (3.47), and (3.48) simultaneously with $B = 0$, one can readily show that

$$q^2 = \frac{zm^2}{2(z-1)} \qquad \psi_0 = \frac{2\sqrt{3}}{mL}\sqrt{\frac{z-1}{(z+1)(z+2)}} \qquad L_0 = L\sqrt{\frac{(z+1)(z+2)}{6}}\,. \quad (3.61)$$

So every ordered pair $(z > 1, m^2L^2 > 0)$ corresponds to a unique Lifshitz solution, and the ordered pairs $(z > 1, m^2L^2 > 0)$ span the space of Lifshitz solutions admitted by positive-mass quadratic potentials. But every choice of $(z > 1, m^2L^2 > 0)$ doesn't necessarily permit a "superconducting" flow from a conformal fixed point in the ultraviolet to a Lifshitz fixed point in the infrared. Ultimately, it appears to require numerical work to determine precisely when such a flow exists. However, two complementary lines of thought provide important partial insight into when such flows should exist:

- The boundary geometry of extremal RNAdS in four dimensions is AdS_4, but the near-horizon geometry is $AdS_2 \times \mathbf{R}^2$. Though the complex scalar ψ satisfies the Breitenlohner-Freedman (BF) bound [12, 13] $m^2L^2 > -9/4$ at the boundary, it may not satisfy the analogous bound in AdS_2 near the horizon. If it doesn't, there is an instability, which suggests that the complex scalar ψ assumes a nontrivial profile and spontaneously breaks the Abelian gauge symmetry. Similar arguments can be found in earlier works, including [38, 58, 47, 23, 26]. The derivation below closely follows the development in [47]. Using the metric convention (3.4), the RNAdS solution is

$$\begin{aligned} &A = \frac{r}{L} \qquad h = 1 - \epsilon L\kappa^2 e^{-3r/L} + \frac{\rho^2\kappa^4}{4}e^{-4r/L} \\ &\Phi = \rho\kappa^2(e^{-r_H/L} - e^{-r/L}) \qquad \psi = 0\,, \end{aligned} \quad (3.62)$$

where the horizon $r = r_H$ occurs where $h = 0$. We are free to set $r_H = 0$. At extremality, h has a double zero at $r = 0$, and

$$\rho = \frac{2\sqrt{3}}{\kappa^2} \qquad \epsilon = \frac{4}{\kappa^2 L} \, . \tag{3.63}$$

It is now straightforward to show that near $r = 0$, the extremal metric takes the form

$$ds^2 = \underbrace{-\frac{r^2}{(L/\sqrt{6})^2}dt^2 + \frac{(L/\sqrt{6})^2}{r^2}dr^2}_{AdS_2} + \underbrace{dx_1^2 + dx_2^2}_{\mathbf{R}^2} \, . \tag{3.64}$$

The curvature radius L_{AdS_2} of near-horizon AdS_2 is thus

$$L_{AdS_2} = L/\sqrt{6} \, . \tag{3.65}$$

The BF bound in near-horizon AdS_2 is violated when

$$m^2_{AdS_2} L^2_{AdS_2} < -\frac{1}{4} \, . \tag{3.66}$$

Above, $m^2_{AdS_2}$ is the limit $r \to 0$ of the effective mass squared m^2_{eff} of ψ, which was defined in [39] as

$$m^2_{\text{eff}} = m^2 + g^{tt} q^2 \Phi^2 \, . \tag{3.67}$$

Plugging the RNAdS metric (3.62) into this equation and taking the limit $r \to 0$, we find that

$$m^2_{AdS_2} = m^2 - 2q^2 . \tag{3.68}$$

So the BF bound in near-horizon AdS_2 suggests that extremal RNAdS is unstable when

$$m^2 L^2 - 2q^2 L^2 < -\frac{3}{2} \, . \tag{3.69}$$

(It is interesting to note that the first relation in (3.61) implies that Lifshitz solutions for the positive-mass potential exist only when $m^2L^2 - 2q^2L^2 < 0$.)

Equation (3.69) translates to an inequality relating m^2L^2 and z when we plug in the expression for q^2 from (3.61):

$$m^2L^2 > \frac{3}{2}(z-1)\,. \tag{3.70}$$

When this inequality is obeyed, we have an *a priori* reason to expect there are symmetry-breaking solutions with nonzero ψ: the instability of extremal RNAdS. When this inequality is not obeyed, there is no known instability in extremal RNAdS, and the existence of symmetry-breaking solutions is less likely. Indeed, we have been unable to numerically construct symmetry-breaking solutions that violate (3.70).

- There are two non-universal perturbations of a given Lifshitz fixed point that can be irrelevant. They are associated with the powers $\beta_\psi(1,\pm 1)$. At least one perturbation must be irrelevant in order for the Lifshitz point to participate in a flow from an ultraviolet conformal field theory to infrared Lifshitz behavior, simply because the approach to Lifshitz behavior is described by some irrelevant perturbation. Usually we are interested in flows to infrared Lifshitz behavior that arise spontaneously from a conformal fixed point: that is, we prescribe that there is no explicit symmetry breaking in the ultraviolet. In order to impose such a constraint, a generic expectation is that one must have not one but two irrelevant perturbations in the infrared, so that one parameter (besides an overall energy scale) can be tuned in the infrared to accommodate the constraint in the ultraviolet. This generic expectation might fail at a codimension one locus in the space of allowed (z, m^2L^2).

  An interesting possibility is that $\beta_\psi(1,1)$ and $\beta_\psi(1,-1)$ could be complex. If they are, then they are complex conjugates of one another, and in order to have an AdS_4-to-Lifshitz flow, their real parts must be positive. Keeping in mind

$m_0 = m$ for a quadratic potential, we can plug the expressions for ψ_0 and L_0 from (3.61) into (3.58) to obtain

$$\beta_\psi(1,\pm 1) = \beta_\psi^{\text{quad}}(1,\pm 1) \equiv -\frac{z+2}{2} + \sqrt{d_1 \pm \sqrt{d_2}}\,, \tag{3.71}$$

where

$$\begin{aligned} d_1 &= \frac{5}{4}z^2 - 2z + 3 \\ d_2 &= 12(z-1)^2(z-2)^2 - 8m^2L^2(z+1)^2(z+2)\,. \end{aligned} \tag{3.72}$$

The quantity d_1 is always positive and greater than $\sqrt{d_2}$ for positive d_2. It follows that the $\beta_\psi^{\text{quad}}(1,\pm 1)$ are only complex when d_2 is negative, and that there is a transition from real $\beta_\psi^{\text{quad}}(1,\pm 1)$ to complex $\beta_\psi^{\text{quad}}(1,\pm 1)$ where d_2 vanishes. More specifically, the $\beta_\psi^{\text{quad}}(1,\pm 1)$ are only real when

$$m^2L^2 < \frac{3}{2}\frac{(z-2)^2(z-1)^2}{(z+1)^2(z+2)}\,. \tag{3.73}$$

One can also easily show that $\operatorname{Re}\beta_\psi^{\text{quad}}(1,\pm 1) > 0$ precisely when

$$m^2L^2 > \frac{3}{2}\frac{(2-z)(z-1)(7z+2)}{(z+1)^2(z+2)}\,. \tag{3.74}$$

If the $\beta_\psi^{\text{quad}}(1,\pm 1)$ are real, then to determine where in parameter space there are irrelevant deformations of Lifshitz solutions, we should ask when $\beta_\psi^{\text{quad}}(1,\pm 1)$ vanishes. It is easily checked that

- $\beta_\psi^{\text{quad}}(1,1)$ only vanishes when $z \leq 2$ and $m^2L^2 \to 0$.
- $\beta_\psi^{\text{quad}}(1,-1)$ only vanishes when $z \geq 2$ and $m^2L^2 \to 0$.

So a critical point occurs at $(z, m^2L^2) = (2,0)$, where both $\beta_\psi^{\text{quad}}(1,1)$ and $\beta_\psi^{\text{quad}}(1,-1)$ are zero. This critical point coincides with a minimum of the RHS of the inequality (3.73), and it is also where the RHS of the inequality (3.74)

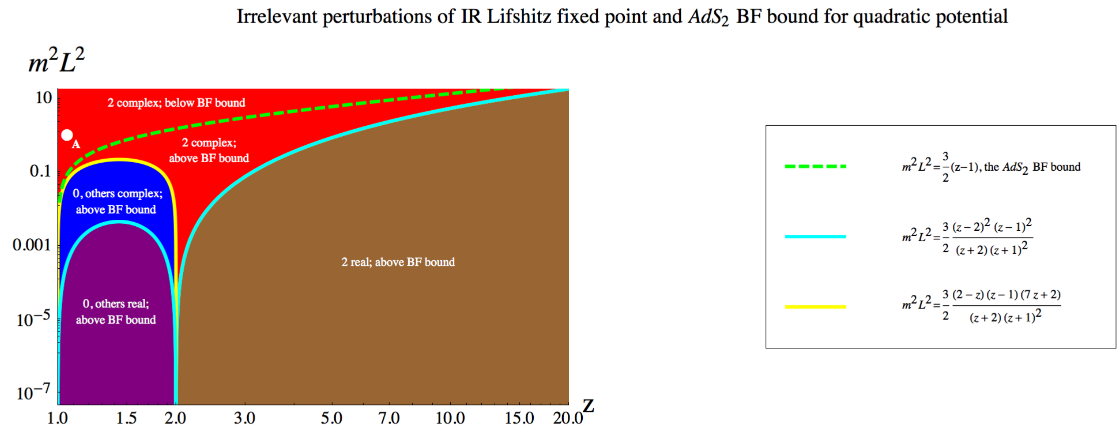

Figure 3.1: The number of irrelevant perturbations to the Lifshitz solution for a quadratic potential, as a function of $z > 1$ and $m^2L^2 > 0$. The two powers $\beta_\psi(1,1)$ and $\beta_\psi(1,-1)$ that characterize infrared perturbations away from this solution fall into one of the four categories described in the text. The four categories meet at the point $(z, m^2L^2) = (2, 0)$. Point A corresponds to an example flow discussed in the text and displayed in figure 3.2.

crosses the z-axis in z-m^2L^2 space.

Figure 3.1 ties together all the features of the above discussion by plotting the AdS_2 BF bound presented in (3.70) and dividing z-m^2L^2 space into four categories:

1. (purple) Both $\beta_\psi^{\text{quad}}(1,1)$ and $\beta_\psi^{\text{quad}}(1,-1)$ are real and negative. They correspond to relevant perturbations.

2. (brown) Both $\beta_\psi^{\text{quad}}(1,1)$ and $\beta_\psi^{\text{quad}}(1,-1)$ are real and positive. They correspond to irrelevant perturbations.

3. (red) $\beta_\psi^{\text{quad}}(1,1) = \beta_\psi^{\text{quad}}(1,-1)^*$ is complex with a positive real part. The $\beta_\psi^{\text{quad}}(1,\pm1)$ correspond to irrelevant perturbations. A flow to a conformal fixed point in the UV would exhibit damped oscillations in the IR.

4. (blue) $\beta_\psi^{\text{quad}}(1,1) = \beta_\psi^{\text{quad}}(1,-1)^*$ is complex with a negative real part. The $\beta_\psi^{\text{quad}}(1,\pm1)$ correspond to relevant perturbations.

Evidently, only complex $\beta_\psi(1,\pm1)$ associated with irrelevant perturbations obey the inequality (3.70). Therefore, the positive-mass quadratic potential probably admits AdS_4-to-Lifshitz flows only in cases where the approach to the Lifshitz point is oscillatory.

As we have already remarked, the considerations going into figure 3.1 provide only partial insight into when AdS_4-to-Lifshitz flows exist, and the only way we know of definitely establishing existence is to construct such flows numerically. In previous works, two different numerical strategies have been pursued:

1. One can construct the zero-temperature solution directly, as in [52], provided one has analytic control over the infrared asymptotics. This "direct" approach to constructing candidate ground states of the holographic Abelian Higgs model has the advantage of speed and simplicity.

2. One can find the hottest AdS_4-Reissner-Nordstrom solution with a static solution to the linearized equation for ψ and then follow the branch of solutions with $\psi \neq 0$ down toward extremality, as in [47]. Although more laborious than the direct approach, this "cooling" approach has the advantage that one knows how the symmetry-breaking ground state connects to the phase with unbroken symmetry.

We pursued the cooling approach to generate a very cold black hole solution to the theory with $qL = 3$ and $m^2L^2 = 1$, which corresponds to point A in figure 3.1. In figure 3.2 we compare the numerically obtained $\psi(r)$ and $g(r)$ with fits to the expected zero-temperature behavior. The blue curves represent the low-temperature solution for $qL = 3$ and $m^2L^2 = 1$, which corresponds to point A in figure 3.1. In the corresponding zero-temperature solution for ψ, the irrelevant perturbations of the Lifshitz fixed point are characterized by the powers $\beta_\psi(1,1) = \beta_\psi(1,-1)^* \approx 0.204 + 0.848i$. The dotted red line in the plot of ψ is a fit of the zero-temperature ansatz

$$\psi(r) = \psi_0 + c_\psi r^{\beta_\psi(1,1)} + c_\psi^* r^{\beta_\psi(1,-1)} \tag{3.75}$$

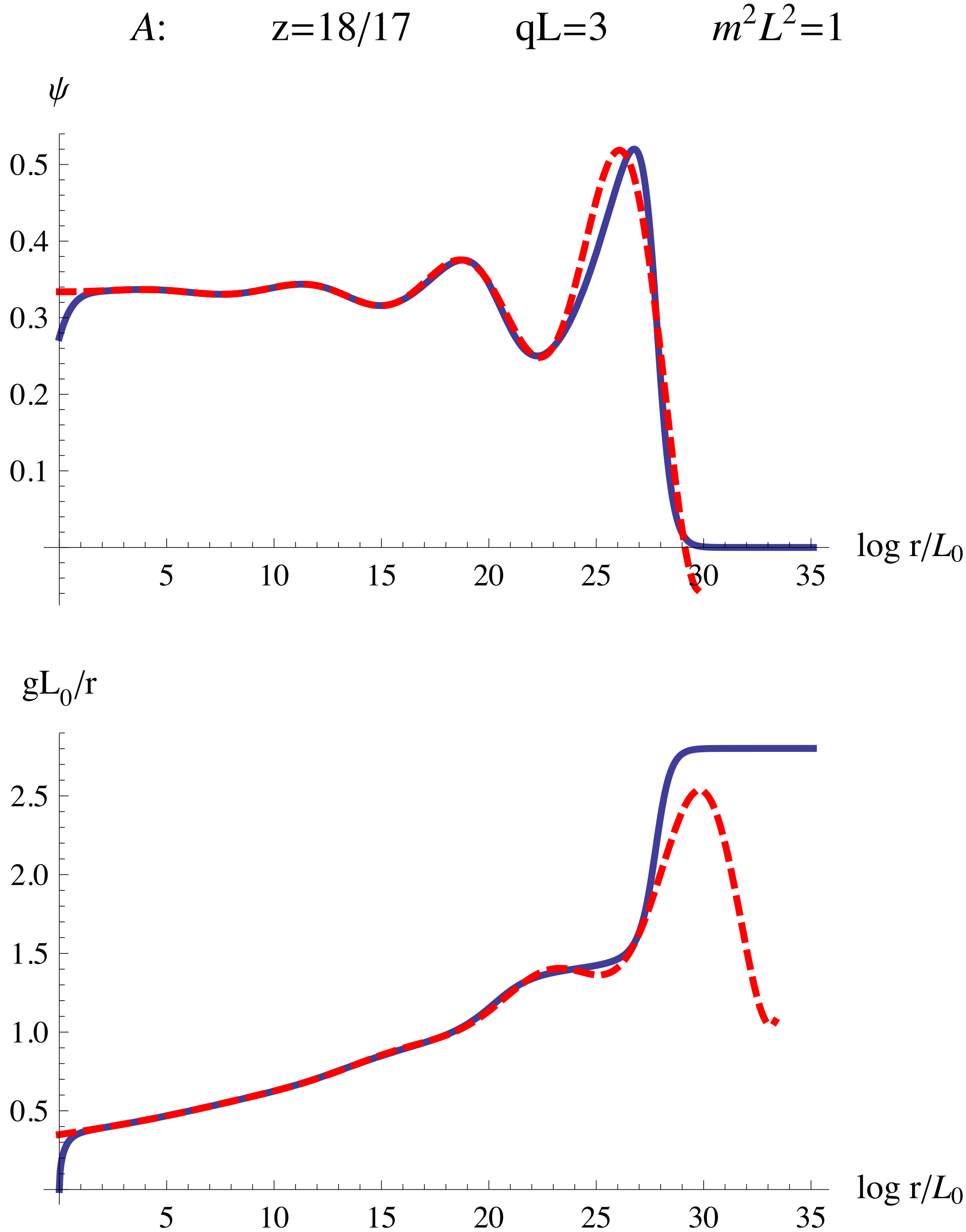

Figure 3.2: The blue curves are ψ and gL_0/r for a very cold superconducting black hole based on the positive-mass quadratic potential with $qL = 3$ and $m^2L^2 = 1$. The temperature of this black hole is $T/\mu \approx 2.356 \times 10^{-14}$, twelve orders of magnitude lower than the highest temperature at which the Abelian gauge symmetry is broken by ψ, $T_c/\mu \approx 0.0864$. The dotted red curves represent near-horizon fits to zero-temperature ansatzes that describe perturbations away from an infrared Lifshitz fixed points.

to the behavior of the low-temperature solution close, but not too close, to the horizon. (In practice, this meant for $\log r/L_0$ approximately between 1 and 8.) Above, $\psi_0 = \sqrt{51/455}$ and $z = 18/17$, as can be determined from (3.61). The fit parameters are the real and imaginary parts of c_ψ. The dotted red line in the plot of gr/L_0 is based on the infrared asymptotics with the same value of c_ψ. An overall scale factor in g can be adjusted as a consequence of a symmetry of the equations of motion:

$$g \to cg \qquad \Phi \to c\Phi \,. \tag{3.76}$$

We fix this scale factor using a fit to the low-temperature solution. The agreement between low-temperature numerics and the analytic zero-temperature asymptotics is evidently excellent, except extremely close to the horizon (i.e., for $0 < \log r/L_0 \lesssim 1.5$), where finite-temperature effects become important, and far from the horizon (i.e., for $\log r/L_0 \gtrsim 25$), where the roll-over from Lifshitz behavior to the ultraviolet conformal behavior occurs.

3.3.5 The W-shaped quartic potential

We argued in section 3.2 that domain wall solutions with conformal invariance in both the ultraviolet and the infrared probably exist provided $qL_{\rm IR}\psi_{\rm IR} > 1$, which is equivalent to $\Delta_\Phi > 3$. Suppose we hold the potential fixed (so that in particular $L_{\rm IR}$ and $\psi_{\rm IR}$ are fixed) and lower q below the value permitted by this inequality. What happens to the domain wall solutions? We expect that they still exist, but have Lifshitz-like symmetry in the infrared instead of emergent conformal symmetry. A heuristic reason to think this is the right idea is that when $\Delta_\Phi = 3$, the second-order corrections to the solution include a constant shift of ψ away from $\psi_{\rm IR}$. So it seems sensible that the system would find a different solution with constant ψ. As we saw in subsection 3.3.2, Lifshitz scaling is the only possibility. It can be further checked that the second-order shift of ψ away from $\psi_{\rm IR}$ is positive when $\Delta_\Phi = 3$, which makes sense since Lifshitz solutions exist only in the region where the potential slopes upward.

The rest of this section is structured as follows. First we give an analysis, for the W-shaped quartic potential, of when Lifshitz solutions exist. The results are summarized in figure 3.3. Next we present a (mostly) analytical study of whether there are irrelevant perturbations to the Lifshitz solutions. The outcome of this study is shown in figure 3.4. Finally, in figures 3.5-3.7 we provide one explicit example of an AdS_4-toAdS_4 flow and two examples of AdS_4-to-Lifshitz flows.

Observe that with the help of (3.21), (3.46) and (3.47) can be brought into the form

$$-\frac{\psi_0}{V(\psi_0,\psi_0)}\frac{\partial V}{\partial\psi^*}(\psi_0,\psi_0) = 2\frac{z-1}{z^2+z+4} \tag{3.77}$$

$$q^2\psi_{\rm IR}^2 L_{\rm IR}^2 = \frac{6z}{4+z+z^2}\frac{V(\psi_0,\psi_0)}{V(\psi_{\rm IR},\psi_{\rm IR})}\frac{\psi_{\rm IR}^2}{\psi_0^2}\,. \tag{3.78}$$

Specializing to the quartic potential (3.2) and defining

$$y \equiv \frac{\psi_0}{\psi_{\rm IR}} \qquad\qquad \tilde{u} = \frac{6u}{m^4L^2}\,, \tag{3.79}$$

we find that (3.77) and (3.78) can be rewritten as

$$\frac{4y^2(y^2-1)}{2\tilde{u}+2y^2-y^4} = 4\frac{z-1}{4+z+z^2} \tag{3.80}$$

$$q^2\psi_{\rm IR}^2 L_{\rm IR}^2 = \frac{2\tilde{u}+2y^2-y^4}{y^2(1+2\tilde{u})}\frac{6z}{4+z+z^2}\,. \tag{3.81}$$

If we also define $y_* = \psi_*/\psi_{\rm IR}$, then it is straightforward to check that for $\tilde{u} > 0$ and $z > 1$, there is a unique solution y to (3.80) with $1 < y < y_*$. This is the allowed range of y because it corresponds to values of ψ between the minimum of $V(\psi,\psi^*)$ at $\psi_{\rm IR}$ and its zero at ψ_*. Plugging this solution y of (3.80) into (3.81), one obtains a unique value for $q^2\psi_{\rm IR}^2 L_{\rm IR}^2$, and hence a definite prediction for Δ_Φ, based on (3.23) and (3.24),

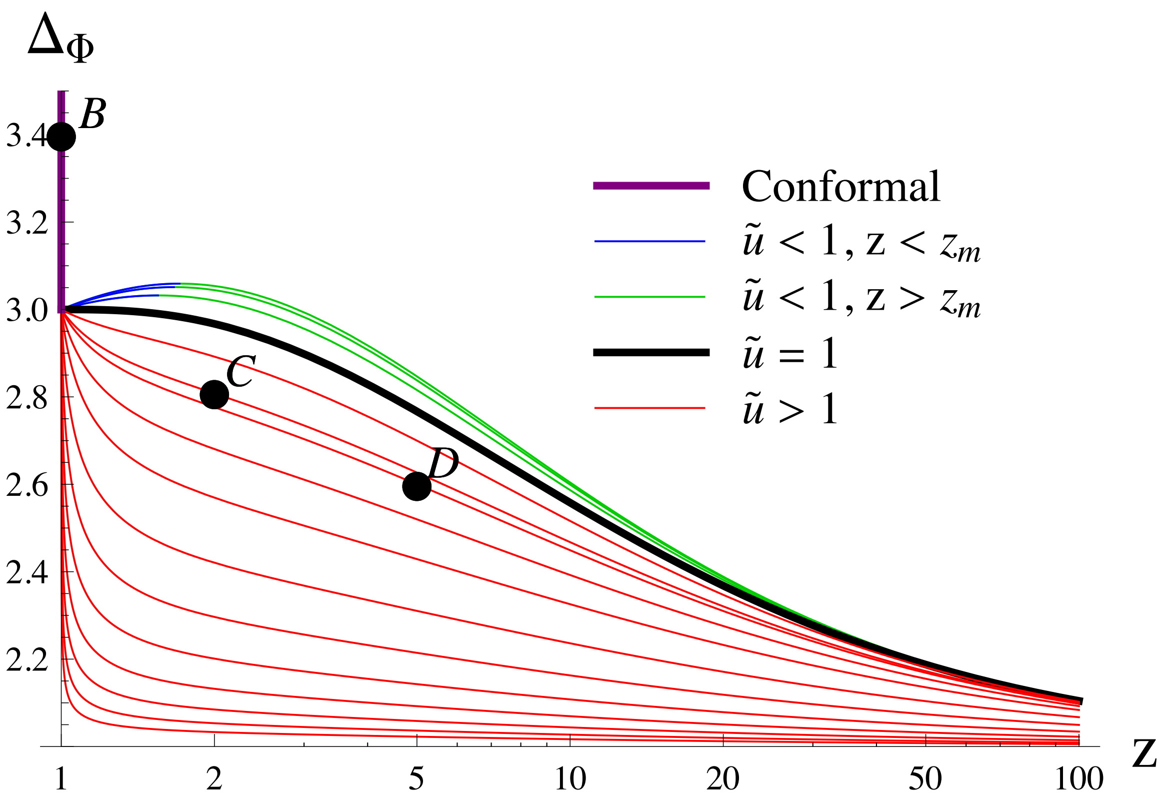

Figure 3.3: The behavior of Δ_Φ as a function of z for the quartic potential (3.2). The vertical purple line reminds us that for $\Delta_\Phi > 3$, domain walls with emergent conformal symmetry are allowed. The various curves show how Δ_Φ behaves as a function of z in backgrounds with Lifshitz scaling. Each curve corresponds to a definite value of the rescaled quartic coupling $\tilde{u}$. For $\tilde{u} < 1$, z_m is the value of z where Δ_Φ is maximized.

at the symmetry-breaking conformal fixed point. Although this conformal fixed point doesn't participate in the infrared dynamics, it is clearly "nearby" in theory space, and Δ_Φ proves to be a useful quantity in tracking the various possible behaviors of the Lifshitz geometry. In any case, for fixed $\tilde{u}$, Δ_Φ can be regarded as a well-defined function of $z > 1$. We plot its behavior in figure 3.3. Recall from section 3.2 that $\Delta_\Phi > 2$ on fairly general grounds.

Figure 3.3 depicts curves of constant $\tilde{u}$ in z-Δ_Φ space. There is a distinction between two regimes:

- Weak quartic coupling, $\tilde{u} < 1$. In this regime, Δ_Φ first increases with z, then decreases, with a maximum at $z = z_m$. When the quartic coupling is weak in this sense, it is possible for a domain wall with emergent conformal symmetry to exist at the same value of q as two different solutions with Lifshitz-like scaling.

- Strong quartic coupling, $\tilde{u} > 1$. This regime is simpler because at every value of Δ_Φ, our analysis leads to only one candidate ground state: an AdS_4-to-AdS_4 domain wall if $\Delta_\Phi > 3$, and an AdS_4-to-Lifshitz domain wall if $\Delta_\Phi < 3$.

If we specify m^2 and L, then each point (z, Δ_Φ) in figure 3.3 that corresponds to a Lifshitz solution can be classified further according to the behaviors of the powers $\beta_\psi(1, \pm 1)$. (Note that with m^2 and L fixed, varying z and Δ_Φ is equivalent to varying q and u.) The analysis of the $\beta_\psi(1, \pm 1)$ proceeds similarly to the analogous analysis of z-m^2L^2 space in the quadratic case. Since it is tedious and only analytical up to a point, we do not present the details here. The $\beta_\psi(1, \pm 1)$ fall into one of five categories, where we have indicated in each case the color of the corresponding region in figure 3.3:

1. (green) $\beta_\psi(1, 1)$ and $\beta_\psi(1, -1)$ are real and positive. They are associated with irrelevant perturbations, and an AdS_4-to-Lifshitz domain wall is probably possible.

2. (gray) $\beta_\psi(1, 1)$ and $\beta_\psi(1, -1)$ are complex with $\beta_\psi(1, 1) = \beta_\psi(1, -1)^*$, and $\mathrm{Re}\,\beta_\psi > 0$. The two associated perturbations are irrelevant, and a flow to a conformal fixed point in the ultraviolet is likely possible. Such a flow would exhibit damped oscillations in the infrared.

3. (blue) $\beta_\psi(1, 1)$ and $\beta_\psi(1, -1)$ are real, but one is negative and the other is positive. The negative power is associated with a relevant perturbation and the positive power is associated with an irrelevant perturbation. In general, a flow to a conformal fixed point in the ultraviolet is not possible when one forbids explicitly symmetry-breaking deformations of the ultraviolet theory.

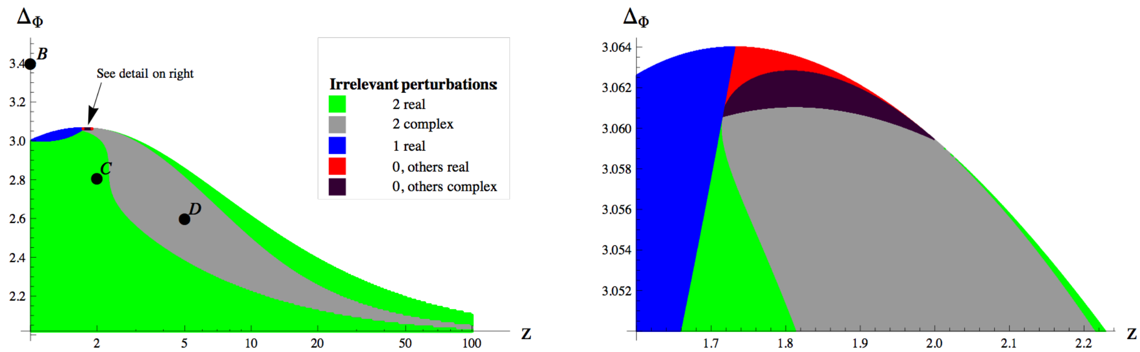

Figure 3.4: When there is a Lifshitz solution at a given point (z, Δ_Φ), the two powers $\beta_\psi(1,1)$ and $\beta_\psi(1,-1)$ that characterize perturbations away from this solution in the infrared fall into one of the five categories described in the text and summarized briefly in the legend. In the plot above, we have taken $m^2 = -2$ and $L = 1$. The detail on the right shows that the five categories meet at the point $(z, \Delta_\Phi) \approx (1.715, 3.061)$.

4. (red) $\beta_\psi(1,1)$ and $\beta_\psi(1,-1)$ are real and negative. They are associated with relevant perturbations, and a flow to a conformal fixed point in the ultraviolet is not possible.

5. (purple) $\beta_\psi(1,1)$ and $\beta_\psi(1,-1)$ are complex with $\beta_\psi(1,1) = \beta_\psi(1,-1)^*$, and $\mathrm{Re}\,\beta_\psi < 0$. The two associated perturbations are relevant, and a flow to a conformal fixed point in the ultraviolet is not possible.

To produce figure 3.4, we chose $m^2 = -2$ and $L = 1$. In the white space, there are no Lifshitz solutions. The curve that separates the white space from the colored regions represents the limit $\tilde{u} \to 0$. At the critical point $(z, \Delta_\Phi) \approx (1.715, 3.061)$ in the weak-coupling regime, both $\beta_\psi(1,1)$ and $\beta_\psi(1,-1)$ vanish, and the five colored regions meet. Note from (3.69) that $m^2L^2 = -2$ always violates the AdS_2 BF bound encoded in the inequality (3.70); if m^2L^2 had been -1, for instance, symmetry-breaking solutions with nonzero ψ probably would not occur in a region of z-Δ_Φ space, approximately where the AdS_2 BF bound is satisfied.

At the points B, C, and D, which are displayed in figure 3.3 as well as figure 3.4, we have numerically obtained flows to conformal fixed points in the ultraviolet for

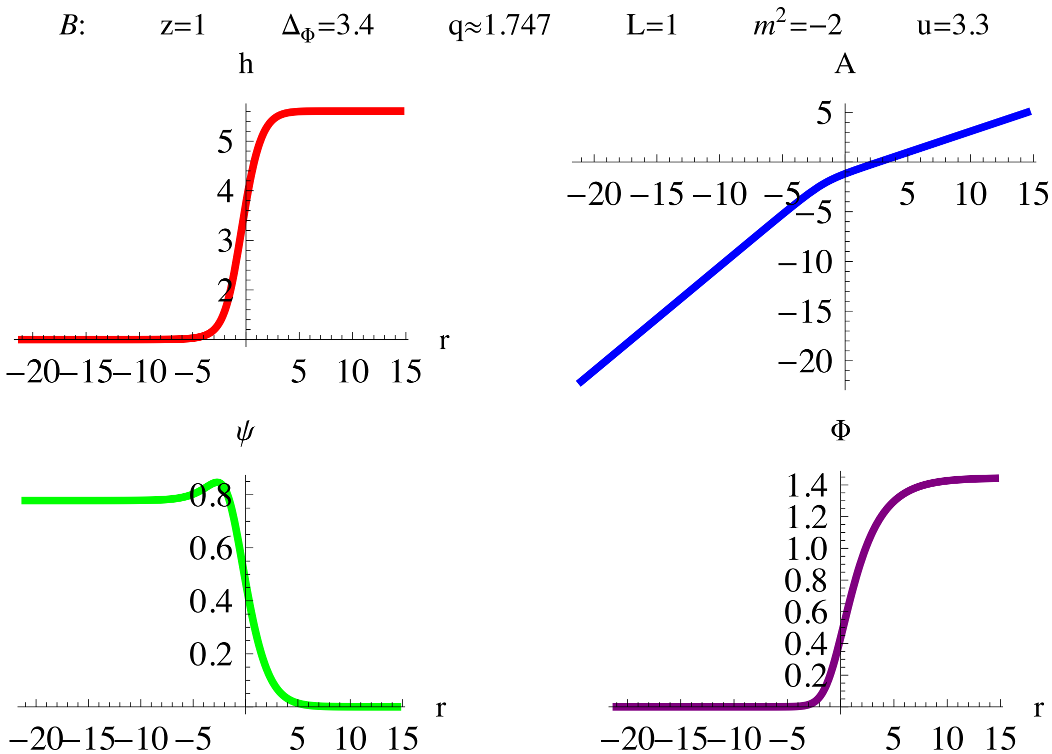

Figure 3.5: A flow between two conformal fixed points.

$m^2 = -2$, $L = 1$. In figure 3.5, we exhibit the solution that corresponds to point B using the metric convention (3.4). This solution interpolates between two copies of AdS_4 and is qualitatively similar to the solution discussed in [52]. The UV-to-IR speed-of-light ratio is approximately 2.368. In figures 3.6 and 3.7, we exhibit the solutions that correspond to points C and D, respectively, using the metric convention in (3.40). Each interpolates between AdS_4 in the ultraviolet and a Lifshitz geometry in the infrared. Though the solutions at points C and D look similar, they represent qualitatively different behavior in the deep infrared, where $\log r/L_0 \to -\infty$. At point D, $\beta_\psi(1,1) = \beta_\psi(1,-1)^* \approx 3.143 + 0.910i$: there is a nonzero imaginary part. However, the real part is over three times larger than the imaginary part, and though the solution exhibits oscillations in the infrared, they are so damped that they cannot be seen in the figure. At point C, $\beta_\psi(1,1) = 2.802$ and $\beta_\psi(1,-1) = 1.245$: both powers are real and positive, so there are no oscillations in the infrared.

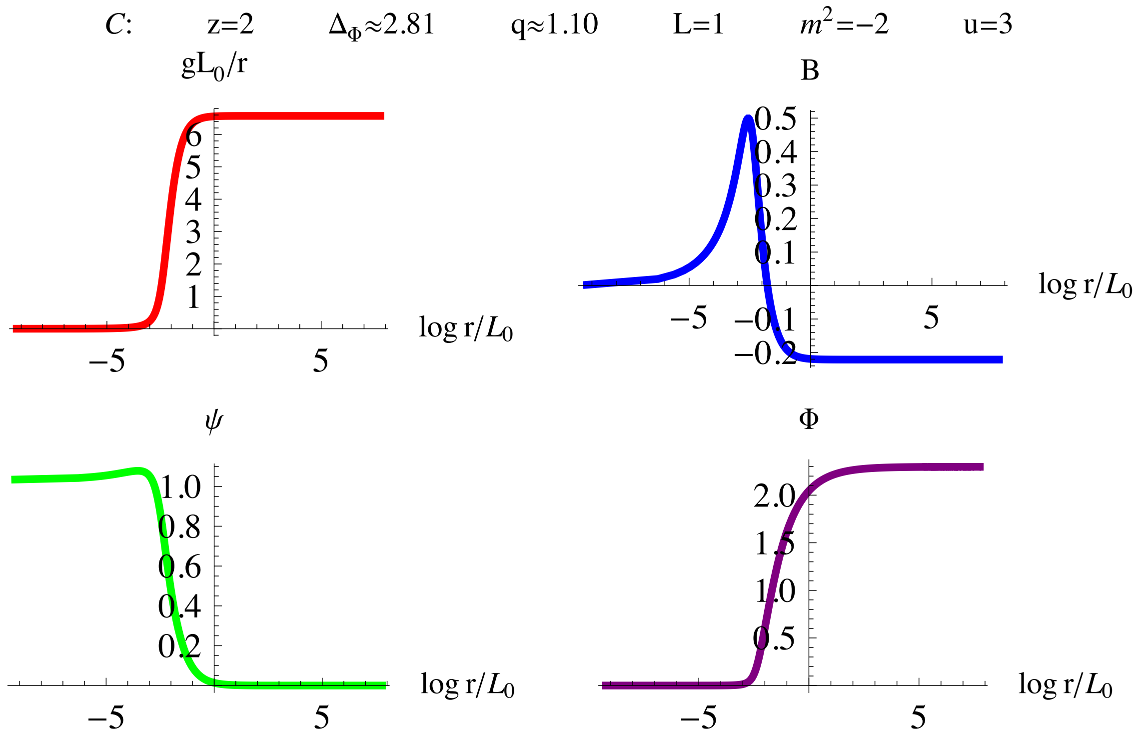

Figure 3.6: A flow between a Lifshitz fixed point in the infrared and a conformal fixed point in the ultraviolet for real $\beta_\psi(1,1)$ and $\beta_\psi(1,-1)$. There are no oscillations in the infrared.

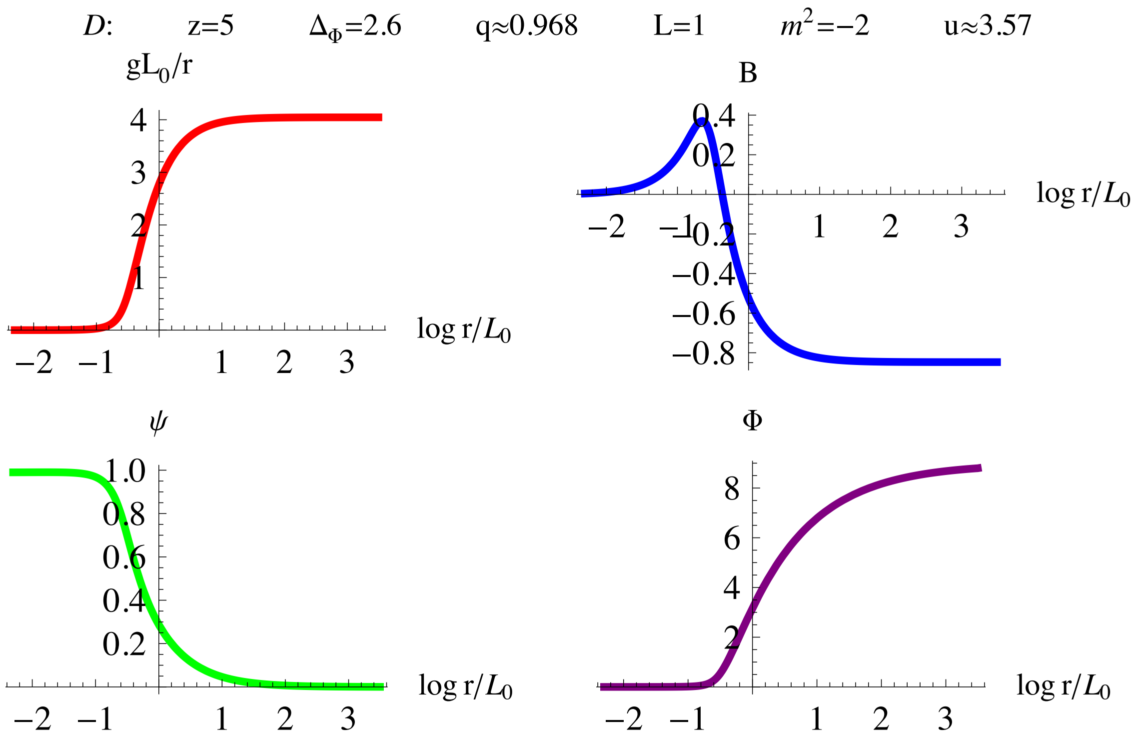

Figure 3.7: A flow between a Lifshitz fixed point in the infrared and a conformal fixed point in the ultraviolet for complex $\beta_\psi(1,1)$ and $\beta_\psi(1,-1)$. The damped oscillations in the deep infrared are imperceptible.

3.4 Discussion

A prominent feature of quantum field theory is that when an operator that usually is irrelevant or marginal becomes relevant, interesting new dynamics arise: for example, BCS superconductivity and confinement can be understood in these terms. Here we have a novel example, where the operator in question is the time component, J_0, of a vector operator, J_μ, which is conserved in the ultraviolet, but which becomes non-conserved due to condensation of a scalar operator at sufficiently large chemical potential for J_0. In the infrared, if J_0 acquires an anomalous dimension large enough to make it irrelevant, relativistic conformal symmetry can be recovered. If J_0 is relevant, then, at least for a broad class of examples typified by the examples in figures 3.2, 3.5, 3.6, and 3.7, the result is Lifshitz-like scaling in the infrared.

An unexpected feature of flows from AdS_4 in the ultraviolet to Lifshitz solutions in the infrared is that the approach to Lifshitz behavior can be oscillatory: in fact, for the positive-mass quadratic potential, this appears to be the only possibility. In field theory, the oscillations presumably represent oscillatory or cyclic approach to the Lifshitz fixed point behavior. In regions of the bulk geometry where the oscillations are strong, the blackening function, $-g_{tt}/g_{xx}$, can be almost constant over a significant range of values of g_{xx}. An example of this can be seen in figure 3.2. Examples with more pronounced shelves can be constructed. A shelf (nearly constant $-g_{tt}/g_{xx}$) indicates the approximate recovery of an $SO(2,1)$ symmetry over a finite range of energy scales. Presumably, Green's functions of the dual gauge theory would reflect such an approximate symmetry: in particular, the spectral measure of two-point functions would have its weight concentrated in a momentum-space light-cone with a speed of light determined by $\sqrt{-g_{tt}/g_{xx}}$, over a range of energies corresponding to the extent of the shelf. An example of this was seen in [41] for the case of true emergent conformal symmetry in the infrared. On the gravity side, one can understand the presence of shelves in $-g_{tt}/g_{xx}$ heuristically as competition between oscillatory behavior and the constraint that $-g_{tt}/g_{xx}$ is a monotonically increasing function of r. This latter constraint follows from (3.6).

A comprehensive study of flows from AdS_4 to Lifshitz behavior for the W-shaped quartic potential is clearly an involved task. There are three dimensionless parameters: m^2L^2, qL, and uL^2.[4] The solutions found in figures 3.5-3.7 are representative, but to work out the full story, one should investigate to what extent the AdS_2 BF bound condition is an accurate guideline to when symmetry breaking solutions exist, and also whether AdS_4-to-Lifshitz solutions win out thermodynamically over AdS_4-to-AdS_4 solutions when they both exist.

We leave open two important questions about stability. First, are the extremal backgrounds we construct stable against linearized perturbations? The oscillatory perturbations of Lifshitz solutions have some similarities with scalars that violate of the BF bound in anti-de Sitter space, but it is not clear to us whether they indicate true instabilities of the domain wall solutions. Second, what is the energetically preferred extremal background at finite chemical potential? Sometimes — for instance, at large q — it is fairly clear that the AdS_4-to-AdS_4 solutions are indeed the preferred ground state. But when Δ_Φ is only slightly larger than 3 and the quartic coupling is small, there can be competition between AdS_4-to-AdS_4 and AdS_4-to-Lifshitz domain walls. It is numerically challenging to ascertain which type of domain wall wins out. We hope these and related issues are resolved in future work.

[4] In principle, κ/L is another dimensionless parameter. But κ doesn't enter into the equations of motion following from (3.1), so its value doesn't affect classical solutions.

Chapter 4

Conclusion

This dissertation has explored models rooted in the gauge/gravity duality that may provide intuition about QCD and condensed matter physics. Chapter 2 studied the action

$$S = \int dr\, dt\, d^3x \left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right] \tag{4.1}$$

as a gravity dual to a nonconformal field theory at finite temperature. Analytical and numerical tools were developed to simplify the treatment of the model. The scalar potential V was then arranged so a family of translationally invariant black holes matched the QCD equation of state from lattice data. Backgrounds with this choice of V can be used to compute transport coefficients and other physical observables. Such predictions can subsequently be compared to results from lattice-based approaches — or to experimental data from the Relativistic Heavy Ion Collider, which probes the strongly coupled quark-gluon plasma.

Chapter 3 studied zero-temperature solutions to the equations of motion from the action

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R - \frac{1}{4} F_{\mu\nu}^2 - |(\partial_\mu - iqA_\mu)\psi|^2 - V(|\psi|) \right] . \tag{4.2}$$

This Abelian Higgs model in AdS was treated as the gravity dual to a CFT at finite temperature and chemical potential. It could give insight into high-temperature superconductivity. Above some critical temperature T_c, the model admits only solutions with $\psi = \psi^* = 0$. Below T_c, however, ψ condenses, which breaks a global $U(1)$ symmetry on the field theory side. For Mexican hat potentials, zero-temperature solutions were shown to fall into two classes: flows between an infrared CFT and an ultraviolet CFT, and flows between an infrared theory with Lifshitz symmetry and an ultraviolet CFT. For quadratic potentials that open upward, all solutions found had Lifshitz symmetry in the infrared, but exhibited oscillations that could point toward instabilities.

Since its inception over forty years ago, string theory has often seemed tantalizingly close to providing quantitative predictions about physical phenomena. The emergence of the gauge/gravity duality in 1997 suggested new lines of inquiry about how string theory could make contact with experiment. Gauge theories with established gravity duals have not yet been found in nature, but do share many features with theories known to describe our world. As new experimental technologies are developed and our understanding of the gauge/gravity duality matures, string theorists may finally find precise, testable predictions to offer experimentalists. In the meantime, developing bottom-up models informed by the gauge/gravity duality could prove useful for qualitative insight into poorly understood physical systems.

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