LARGE MOMENTUM BEHAVIOUR OF FEYNMAN AMPLITUDES

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Abstract: The complete asymptotic expansion of Feynman amplitudes for large values of the scale parameter is derived for Euclidean and Minkowski metrics.

I. Formulation of the Problem

Consider an arbitrary Feynman graph G^o occurring in a Lagrangian field theory which describes the self-interaction of one sort of particles with mass $m \neq 0$. G^o consists of a collection $\mathcal{V}$ of vertices and a collection $\mathcal{L}$ of L internal lines (no tadpoles, no external lines!). Let N stand for the number of independent loops of G^o and let $\mathfrak{U}$ denote the collection of all U external vertices of G^o. The four-momenta $(p_u)_{u \in \mathfrak{U}} = \underline{p}$ enter the graph G^o.

Consider the renormalized Feynman amplitude associated with G^o: $\tilde{\mathcal{T}}(p;m)$. Replace $\underline{p}$ by $\Lambda\underline{p}$, $\Lambda \in \mathbb{R}_+$. Show that the following asymptotic expansion for large values of the scale parameter Λ holds in the sense of distributions

$$\tilde{\mathcal{T}}(\Lambda\underline{p};m) = \sum_{\mu=n}^{M} \sum_{\varkappa=0}^{m_\mu - 1} [\Lambda^2]^{-\mu-2} [\ln \Lambda^2]^{\varkappa} \tilde{\mathcal{T}}_{\mu,\varkappa}(\underline{p};m) + \tilde{\mathcal{R}}_M$$

where M is an arbitrary integer, larger than or equal to n, and set up rules how to calculate the numbers n, m_μ, the coefficient distributions $\tilde{\mathcal{T}}_{\mu,\varkappa}$ and the remainder distribution $\mathcal{R}_M$.

Without loss of generality, G^o may be assumed to be irreducible (I) i.e. one-particle-irreducible and one-vertex-irreducible. Otherwise, $\tilde{\mathcal{T}}(\underline{p};m)$ would factorize into a product of renormalized Feynman amplitudes corresponding to the maximal irreducible components of G^o and propagators.

The numbers n and m_n were determined by S. Weinberg [1] and J. P. Fink [2] respectively for the case of convergent graphs and Euclidean metrics. Under the same restrictions D. A. Slavnov [3] proved the validity of the general form of the asymptotic expansion as written down by us. Finally, M. C. Bergère and Y-M. P. Lam [4] determined the coefficients of the polynomial in $(\ln \Lambda^2)$ accompanying the leading inverse power of Λ^2, again assuming Euclidean metrics. We have formulated the problem in the distribution theoretical frame as opposed to the pointwise discussion of the above mentioned authors. For Minkowski metrics this turns out to be both adequate and helpful (ensuring uniform convergence at a later stage). The difference between distribution theoretical and pointwise discussion may be illustrated by the simple mathematical example: $\frac{\varepsilon}{\varepsilon^2 + p^2}$, $\varepsilon > 0$, p a real scalar variable

	distribution theoretical	pointwise
$\frac{\varepsilon}{\varepsilon^2+(\Lambda p)^2} =$	$\Lambda^{-1}\pi\delta(p) + O(\Lambda^{-2})$	$\begin{cases} \Lambda^{-2}\cdot\frac{\varepsilon}{p^2} + O(\Lambda^{-4}), & p \neq 0 \\ \varepsilon^{-1} = \text{const}, & p = 0 \end{cases}$

II. Resolution of the Ultraviolet Singularities and Renormalization

As to the renormalization scheme, we decide in favour for Speer's analytical renormalization [5] since the theory of analytical functions will be employed anyway: to every line $l \in \mathcal{L}$ we associate a complex variable λ_l: $\underline{\lambda} = (\lambda_l)_{l\in\mathcal{L}}$ (and a Feynman parameter α_l: $\underline{\alpha} = (\alpha_l)_{l\in\mathcal{L}}$) such that the renormalized Feynman amplitude $\tilde{\mathcal{C}}(\underline{p};m)$ is obtained by evaluating the analytical continuation of the analytically regularized Feynman amplitude $\tilde{\mathcal{C}}_{\underline{\lambda}}(\underline{p};m)$, $\underline{\lambda} \in \Omega_2$, $\Omega_2 = \{ \underline{\lambda} \,/\, \mathcal{R}e\lambda_l > 2$ for every $l \in \mathcal{L} \}$ at the point $\underline{\lambda} = \underline{1}$:

$$\tilde{\mathcal{C}}(\underline{p};m) = \mathcal{W}_L\, \tilde{\mathcal{C}}_{\underline{\lambda}}(\underline{p};m)$$

with the help of Speer's generalized evaluator $\mathcal{W} = \{\mathcal{W}_L \,/\, L = 1, 2, \ldots\}$.
In Ω_2 $\tilde{\mathcal{C}}_{\underline{\lambda}}$ is holomorphic. To achieve the analytical continuation from Ω_2 to $\underline{1}$, $\tilde{\mathcal{C}}_{\underline{\lambda}}$ is represented as a Feynman parameter integral which for the case of scalar particles in the absence of derivative coupling - we shall restrict our discussion to this case for the sake of simplicity - reads as follows

$$\tilde{\mathcal{C}}_{\underline{\lambda}}(\underline{p};m) = \lim_{\varepsilon\downarrow 0} \delta\Big(\sum_{u\in\mathcal{V}} p_u\Big)\Gamma(\nu)\Big[\prod_{l\in\mathcal{L}}\int_0^\infty d\alpha_l\, \alpha_l^{\lambda_l-1}\Big]\, d(\underline{\alpha})^{-2}\cdot$$
$$\cdot \exp\Big\{-i\Big[A_{\underline{\alpha}}(\underline{p},\underline{p}) + m^2\sum_{l\in\mathcal{L}}\alpha_l\Big]\Big\}\,;\ \underline{\lambda}\in\Omega_2,\ \nu = \sum_{l\in\mathcal{L}}(\lambda_l - 1) + L - 2N,$$

$A_{\underline{\alpha}}(\underline{p},\underline{p})$ being a homogeneous quadratic form in $\underline{p}$, $d(\underline{\alpha})$ denoting the Feynman determinant. Both $A_{\underline{\alpha}}(\underline{p},\underline{p})$ and $d(\underline{\alpha})$ depend polynomially on $\underline{\alpha}$.

The vanishing of $d(\underline{\alpha})$ when some or all of the α-parameters are zero gives rise to a loss of integrability in the course of the analytical continuation in $\underline{\lambda}$, the ultraviolet divergences. In order to get the complete information about the analytical structure of $\tilde{\mathcal{C}}_{\underline{\lambda}}$ outside of Ω_2, the intersection of the zero-surfaces of $d(\underline{\alpha})$ at $\underline{\alpha} = 0$ has to be resolved. In differential topology this problem is known under the name "resolution of a singularity". The outcome of the analyzis there - suitably adapted to the present situation - is the existence of a covering of the α-space $\mathbb{R}_+^L$ by a minimal number of sectors $\mathcal{D}$ such that the union of all sectors makes up the entire α-space, the intersections of any two different sectors have Lebesgue measure zero and such that for every sector $\mathcal{D}$ there exists a parametrization $\underline{\alpha} : \mathcal{D} = \underline{\alpha}(\Delta)$

$$\underline{\alpha} = \underline{\alpha}((t),(\beta)) \quad \text{real analytical for} \quad ((t),(\beta)) \in \Delta$$

with

$$d(\underline{\alpha}) = \prod_1^N t_i^{N_i} \cdot \hat{d}((t),(\beta))$$

where

$$\hat{d}(t),(\beta)) \geq 1 \quad \text{for} \quad ((t),(\beta)) \in \Delta$$

and where the powers N_i are non-negative integers.

This resolution of the ultraviolet singularities has been accomplished by Speer with the help of the concept of a labeled singularity family. To explain this concept, along with the graph G^o we consider its subgraphs G consisting of a collection $\mathcal{V}(G)$ of vertices and a collection $\mathcal{L}(G)$ of L(G) internal lines. Let N(G) stand for the number of independent loops of G.

A singularity family $\mathcal{E}$ is a maximal collection of non-trivial irreducible non-overlapping subgraphs G of G^o such that if $G', G \in \mathcal{E}$, $G' \subsetneq G$ there exists at least one line $l \in \mathcal{L}(G), \notin \mathcal{L}(G')$.

A labeled singularity family $(\mathcal{E},\sigma)$ arises from $\mathcal{E}$ by distinguishing for every $G \in \mathcal{E}$ one line $l = \sigma(G) \in \mathcal{L}(G), \notin \mathcal{L}(G')$ for any $G' \in \mathcal{E}$, $G' \subsetneq G$, the line whose α-parameter is largest. Thus σ is a map from $\mathcal{E}$ into $\mathcal{L}$.

A labeled singularity family leads to a sector $\mathcal{D} = \mathcal{D}(\mathcal{E},\sigma)$ in α-space

$$\mathcal{D} = \{ \alpha \,/\, \alpha_l \geq 0 \ \text{ for every } l \in \mathcal{L}, \ \ \alpha_l \leq \alpha_{\sigma(G)} \ \text{ for every } l \in \mathcal{L}(G),\ G \in \mathcal{E} \ \}$$

characterized by a partial ordering of the α-parameters. This partial ordering is made explicit by the parametrization

$$\alpha_l = \begin{cases} \prod\limits_{G \subset G' \in \mathcal{E}} t_{G'} & \text{if } \ l = \sigma(G) \ \text{ for some } \ G \in \mathcal{E} \\ \beta_l \cdot \prod\limits_{G(l) \subset G' \in \mathcal{E}} t_{G'} & \text{if } \ l \in \mathcal{L} \setminus \sigma(\mathcal{E}) \end{cases},$$

$$0 \leq t_{G^o} < \infty, \quad ((t_G)_{G \in \mathcal{E} \setminus \{G^o\}}, (\beta_l)_{l \in \mathcal{L} \setminus \sigma(\mathcal{E})}) = (\underline{t},\underline{\beta}) \in I^{L-1} \quad \text{where } I = [0\,,\,1]$$

and

$$G(l) = \text{minimal element of the set } \{G \,/\, G \in \mathcal{E}\ ,\ l \in \mathcal{L}(G)\}\ .$$

On the sector $\mathcal{D} = \mathcal{D}(\mathcal{E},\sigma)$

$$d(\underline{\alpha}) = \prod_{G\in\mathcal{E}} t_G^{N(G)} \cdot \hat{d}(\underline{t},\underline{\beta}) \qquad ; \quad \hat{d}(\underline{t},\underline{\beta}) \geqslant 1 \quad \text{for} \quad (\underline{t},\underline{\beta}) \in I^{L-1}.$$

The analytically renormalized Feynman amplitude then takes the following form

$$\tilde{\mathcal{C}}_{\underline{\lambda}}(\underline{p};m) = \sum_{\mathcal{D}} \tilde{\mathcal{C}}_{\underline{\lambda}}^{\mathcal{D}}(\underline{p};m)$$

with

$$\tilde{\mathcal{C}}_{\underline{\lambda}}^{\mathcal{D}}(\underline{p};m) = \delta\Big(\sum_{u\in\mathfrak{V}} p_u\Big)\Big[\prod_{G\in\mathcal{E}} \Gamma(\nu(G))\Big]\Big[\prod_{\ell\in\mathcal{L}\setminus\sigma(\mathcal{E})} \int_0^1 d\beta_\ell\, \beta_\ell^{\lambda_\ell - 1}\Big] \cdot$$

$$\cdot\Big[\prod_{G\in\mathcal{E}\setminus\{G^0\}} \Gamma(\nu(G))^{-1} \int_0^1 dt_G\, t_G^{\nu(G)-1}\Big]\Big[D_{\underline{t},\underline{\beta}}(\underline{p},\underline{p}) + m^2 \sum_{\ell\in\mathcal{L}} \frac{\alpha_\ell}{\alpha_{\sigma(G^0)}} - i0\Big]^{-\nu} \hat{d}(\underline{t},\underline{\beta})^{-2}$$

and $\nu(G) = \sum_{l\in\mathcal{L}(G)} (\lambda_l - 1) + L(G) - 2N(G) \quad , \quad \nu = \nu(G^0)$.

The Γ-functions have been distributed in such a way that the λ-singularities are entirely contained in the first factor. This is so because the massive Feynman denominator $[\ldots\ldots]^{-\nu}$ is an entire function of its negative power ν and an infinite differentiable function in the t- and β-variables.

III. Simultaneous Resolution of the Ultraviolet and Infrared Singularities

If we replace $\underline{p}$ by $\Lambda\underline{p}$ and let Λ tend to plus infinity, we are dealing essentially with an inhomogeneous quadratic form with an ever decreasing inhomogeneous term raised to the power $-\nu$:

$$\Big[D_{\underline{t},\underline{\beta}}(\Lambda\underline{p},\Lambda\underline{p}) + m^2\sum_{\ell\in\mathcal{L}} \frac{\alpha_\ell}{\alpha_{\sigma(G^0)}} - i0\Big]^{-\nu} = \big[\Lambda^2\big]^{-\nu} \cdot \Big[D_{\underline{t},\underline{\beta}}(\underline{p},\underline{p}) + \frac{m^2}{\Lambda^2}\sum_{\ell\in\mathcal{L}} \frac{\alpha_\ell}{\alpha_{\sigma(G^0)}} - i0\Big]^{-\nu}.$$

In the limit of an infinitely large Λ the inhomogeneous term disappears altogether and the situation changes drastically: the massless Feynman denominator is no longer an entire function of ν nor is it an infinite differentiable function of the t-variables. Instead, it has simple poles as a function of ν and branching singularities in the t-variables arising whenever and wherever the quadratic form $D_{\underline{t},\underline{\beta}}(\underline{p},\underline{p}) / \sum_{u\in\mathfrak{V}} p_u = 0$ in the 4(U-1) p-variables degenerates.

To illustrate the latter point we consider the following parameter dependent distribution of the real four-vector variables $p_i \quad i = 1,2,3 : [-p_1^2 - \Theta\cdot p_2^2 - \Theta\cdot\theta\cdot p_3^2 - i0]^{-\nu}$, $\Theta, \theta \in I$, $\nu \in \mathbb{C}$, $\mathcal{R}e\,\nu < 6$, p^2 either the Euclidean or the Minkowski length of p, as a mathematical example for a massless Feynman denominator. Its singularities as a distribution valued function of Θ and θ are exhibited by the following representation

$$[-p_1^2 - \Theta\cdot p_2^2 - \Theta\cdot\theta\cdot p_3^2 - i0]^{-\nu} = F_\nu(\underline{p};\Theta,\theta) + \Theta^{2-\nu} \cdot G_\nu(\underline{p};\Theta,\theta) + \Theta^{2-\nu}\cdot\theta^{4-\nu} \cdot H_\nu(\underline{p},\Theta,\theta)$$

where F_ν, G_ν and H_ν as distribution valued functions of Θ and θ are infinite differentiable in the unit square I^2 and where the sum on the right hand side depends holomorphically on ν for $\mathcal{R}e\,\nu < 6$.

Thus, in order to keep track of the formation of these singularities we have

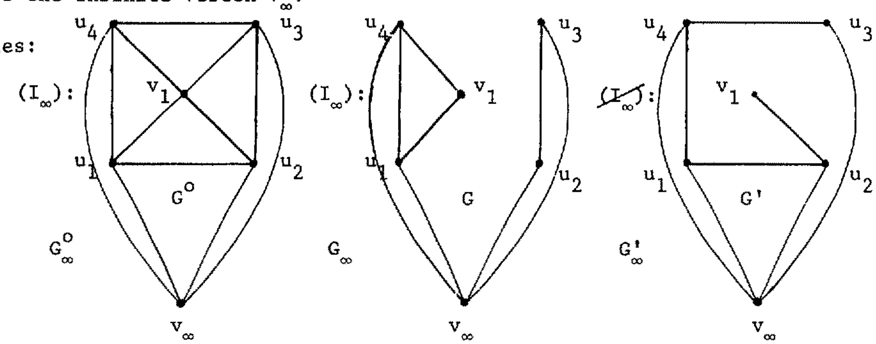
i) to resolve the intersection of the zero-surfaces of the Feynman determinant $d(\underline{\alpha})$ <u>and</u> the Jacobian of the transformation $\underline{p} \to (D_{\underline{t},\underline{\beta}}(\underline{p},\underline{p})$, coordinates on the surfaces $D_{\underline{t},\underline{\beta}}(\underline{p},\underline{p})$ = constant)

ii) extract explicitly the branching singularities in the t-variables from the Feynman denominator.

The first task is accomplished by the introduction of a coarser concept of irreducibility: the so-called irreducibility in view of infinity (I_∞). In order to define the property (I_∞) we embed the graph G^o into a bigger graph G^o_∞ by adding to $\mathcal{V}$ one more vertex v_∞, the infinite vertex, and joining every external vertex $u \in \mathcal{U}$ to v_∞ by one separate line l_u. Similarly, we embed the subgraphs G of G^o into subgraphs G_∞ of G^o_∞.

Definition: A subgraph G of G^o possesses the property (I_∞) if G is irreducible (I) or if G_∞ is one-particle irreducible and one-vertex-irreducible with the possible exception of the infinite vertex v_∞.

Examples:



With the help of the new irreducibility concept (I_∞) in complete analogy to Speer, we define new singularity families $\mathcal{E}_\infty$, labeled singularity families $(\mathcal{E}_\infty, \sigma_\infty)$, sectors $\mathcal{D}_\infty = \mathcal{D}_\infty(\mathcal{E}_\infty, \sigma_\infty)$ and appropriate parametrizations. However, now we have to distinguish two subfamilies $\mathcal{F}_\infty$ and $\mathcal{H}_\infty$ which are disjoint and which make up $\mathcal{E}_\infty$:

$G \in \mathcal{F}_\infty$ iff the removal of the line $\sigma_\infty(G)$ from $\mathcal{L}(G)$ opens a loop, $G \in \mathcal{H}_\infty$ iff the removal of the line $\sigma_\infty(G)$ from $\mathcal{L}(G)$ dissolves a connection between external vertices.

The number of elements in $\mathcal{F}_\infty$ is equal to N, the number of elements in $\mathcal{H}_\infty$ equal to U - 1 .

The second task is accomplished in two steps:

First, the quadratic form $D_{\underline{t},\underline{\beta}}(\underline{p},\underline{p})$ is diagonalized

$$D_{\underline{t},\underline{\beta}}(\underline{p},\underline{p}) = E_{\underline{t},\underline{\beta}}\left(\underline{q}(\underline{t},\underline{\beta}),\underline{q}(\underline{t},\underline{\beta})\right)$$

$$E_{\underline{t},\underline{\beta}}(q,q) = -t_{G^\circ}^{-1}\cdot\sum_{H\in\mathcal{H}_\infty}\left(\prod_{H\subset G\in\mathcal{E}_\infty} t_G\right)\cdot\hat{e}_H(\underline{t},\underline{\beta})\, q_H^2$$

$$\hat{e}_H(\underline{t},\underline{\beta}) \geq 1 \quad \text{for all } H\in\mathcal{H}_\infty \text{ and } ((\underline{t}),(\underline{\beta}))\in I^{L-1}$$

with the help of an invertible linear transformation $T_{\underline{t},\underline{\beta}}$ which as a function of $\underline{t}$ and $\underline{\beta}$ is infinite differentiable [6], $(\underline{q}(\underline{t},\underline{\beta}),\sum_{u\in\mathcal{U}} p_u) = ((q_H(\underline{t},\underline{\beta}))_{H\in\mathcal{H}_\infty},\sum_{u\in\mathcal{U}} p_u) = T_{\underline{t},\underline{\beta}}\,\underline{p}$.

Second, the additive occurrence of $\frac{m^2}{\Lambda^2}\cdot\sum_{\ell\in\mathcal{L}}\frac{\alpha_1}{\alpha_{\sigma(G^\circ)}}$ and the (U-1) terms of $E_{\underline{t},\underline{\beta}}$ in the Feynman denominator is converted into a multiplicative appearance in the Feynman parameter integral by means of an (U-1)-fold Mellin transformation.

The points i) and ii) being settled in this way, we arrive at the following representation for $\tilde{\mathcal{T}}_{\underline{\lambda}}(\Lambda\underline{p};m)$:

$$\tilde{\mathcal{T}}_{\underline{\lambda}}(\Lambda\underline{p};m) = \sum_{\mathcal{D}_\infty}{}' \tilde{\mathcal{T}}_{\underline{\lambda}}^{\mathcal{D}_\infty}(\Lambda\underline{p};m)$$

$$\tilde{\mathcal{T}}_{\underline{\lambda}}^{\mathcal{D}_\infty}(\Lambda\underline{p};m) = \delta\Big(\sum_{u\in\mathcal{U}} p_u\Big)\left[\prod_{H\in\mathcal{H}_\infty}\frac{1}{2\pi i}\int_{-\gamma_H-i\infty}^{-\gamma_H+i\infty} ds_H\,\Gamma(-s_H)\,\Gamma(2+s_H)\right]\cdot$$

$$\left[\Lambda^2\right]^{\sum_{H\in\mathcal{H}_\infty}s_H}\left[\prod_{G\in\mathcal{E}_\infty}\Gamma\Big(\nu(G)+\sum_{G\subset H\in\mathcal{H}_\infty}s_H\Big)\right]\cdot h_{\underline{\lambda},\underline{s}}(\underline{p};m)$$

where $h_{\underline{\lambda},\underline{s}}(\underline{p};m)$ is an entire function of $\underline{\lambda}$ and $\underline{s} = (s_H)_{H\in\mathcal{H}_\infty}$, $0<\gamma_H<1$, $H\in\mathcal{H}_\infty$. Note the presence of simple poles contained in the factors $\Gamma(2+s_H)$. The residues of these infrared poles bring about δ-functions: $\delta(\sum' p_u)$, where $\sum'$ stands for a partial sum of external momenta, or derivatives of them.

The integrations converge uniformly in $\underline{\lambda}$ in the topology of $\mathcal{S}'$ for Euclidean as well as for Minkowski metrics. It is at this point that the distribution theoretical discussion pays off.

As $\underline{\lambda}$ tends to $\underline{1}$, poles in s_H move in from the left of the contours. Together with the fixed poles on the right they may or may not pinch the contours. In the latter case we pick up extra contributions. In view of the uniform convergence, the generalized evaluator may be applied under the integration signs. Thus we obtain the following form for the renormalized amplitude

$$\tilde{\mathcal{T}}(\Lambda\underline{p};m) = \frac{1}{2\pi i}\int_{\varrho-i\infty}^{\varrho+i\infty} dz\left[\Lambda^2\right]^{z-2}\tilde{\mathcal{T}}_{\mathcal{W}}^{z}(\underline{p};m),\quad L-2N<\varrho<L-2N+1$$

where $\tilde{\mathcal{T}}_{\mathcal{W}}^{z}(\underline{p};m)$ is a meromorphic function of z. The localization and order of its poles can be read off directly from the graph G^{o} [7]. Its asymptotic behaviour in z is entirely known.

The complete asymptotic expansion is now obtained by shifting the z-contour to the left and picking up contributions from the poles according to their order.

References

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Discussion

A. Ukawa (question): Can you treat exceptional momenta cases by your method?

K. Pohlmeyer (answer): Yes, you can. If you introduce the restriction $\sum' p_u = 0$, $\sum'$ denoting a partial sum of external momenta, right from the beginning you have to resolve the intersection of the zero-surfaces of the Feynman determinant $d(\underline{\alpha})$ and the Jacobian of the transformation $\underline{p}_{/\sum' p_u = 0} \to (D_{\underline{t},\underline{\beta}}(\underline{p},\underline{p})_{/\sum' p_u = 0}$, coordinates on the surfaces: $D_{\underline{t},\underline{\beta}}(\underline{p},\underline{p})_{/\sum' p_u = 0} =$ constant). This can be done by introducing one more infinite vertex with the help of which an appropriate irreducibility concept - finer than (I_∞) but still coarser than (I) - is defined [8].