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IDEALS IN ALGEBRAS OF UNBOUNDED OPERATORS. II

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Идеалы в алгебрах неограниченных операторов.II

В предыдущей статье были введены два класса идеалов в алгебрах неограниченных операторов. В настоящей статье рассматриваются алгебраические и топологические свойства одного из этих классов. Кроме того, с помощью дуальности исследована связь между этими классами.

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Ideals in Algebras of Unbounded Operators. II

In an earlier paper two classes of ideals in algebras of unbounded operators were introduced. This paper deals with some algebraical and topological properties of one class. Moreover it is shown how these two classes of ideals are connected by duality.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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This paper is part II of the investigations begun in /4/. There two classes of ideals in algebras of unbounded operators were defined: $\mathcal{S}_{\Phi}(\mathfrak{D})$ and $M(\mathcal{S}_{\Phi}(\mathfrak{D}), \mathcal{S}_{\Psi}(\mathfrak{D}))$, where Φ is a symmetric norming function /1/. In /4/ algebraical and topological properties of $\mathcal{S}_{\Phi}(\mathfrak{D})$ were investigated. Now we indicate some properties of $M(\mathcal{S}_{\Phi}(\mathfrak{D}), \mathcal{S}_{\Psi}(\mathfrak{D}))$ and show how these ideals are connected with $\mathcal{S}_{\Phi}(\mathfrak{D})$ by duality properties. All results are taken from /5/. In section 1 we collect some definitions (cf./1/,/2/) and results from /4/. Section 2 contains properties of M(.,.) while section 3 deals with the duality mentioned above.

1. PRELIMINARIES AND BASIC DEFINITIONS

For a dense linear manifold $\mathfrak{D}$ in a separable Hilbert space $\mathfrak{H}$ we denote by $\mathcal{L}^+(\mathfrak{D})$ the $*$-algebra of all operators A for which $A\mathfrak{D} \subset \mathfrak{D}$ and $A^*\mathfrak{D} \subset \mathfrak{D}$. The involution is given by $A \longrightarrow A^+ = A^* \restriction \mathfrak{D}$. $\mathcal{L}^+(\mathfrak{D})$ defines a natural topology t on the domain $\mathfrak{D}$ given by the directed system of seminorms $\phi \longrightarrow \| A\phi \|$ for all $A \in \mathcal{L}^+(\mathfrak{D})$. An Op*-algebra $\mathcal{A}(\mathfrak{D})$ is a $*$-subalgebra of $\mathcal{L}^+(\mathfrak{D})$ with unit I. By $t_{\mathcal{A}}$ we denote the topology induced by $\mathcal{A}(\mathfrak{D})$ on $\mathfrak{D}$. $\mathcal{A}(\mathfrak{D})$ is said to be selfadjoint if $\mathfrak{D} = \bigcap_{A \in \mathcal{A}} \mathfrak{D}(A^*)$.

In this paper we consider only selfadjoint $\mathcal{L}^+(\mathfrak{D})$!
$\mathfrak{F}(\mathfrak{D})$ denotes the set of all finite dimensional operators of $\mathcal{L}^+(\mathfrak{D})$ which is the minimal two-sided $*$-ideal of $\mathcal{L}^+(\mathfrak{D})$.
For a completely continuous operator $T \in \mathfrak{B}(\mathfrak{H})$ (the set of all bounded linear operators on $\mathfrak{H}$) $(s_n(T))$ stands for the sequence of s-numbers $s_1(T) \geq s_2(T) \geq \ldots$ (each number repeated according to its multiplicity), $\Phi(.)$ is a symmetric norming function and

$\mathfrak{S}_\Phi(\mathcal{H})$, or simple $\mathfrak{S}_\Phi$, denotes the corresponding symmetrically normed ideal with norm $\|\cdot\|_\Phi$ given by $\|T\|_\Phi = \Phi(s_1(T), s_2(T), \dots)$.
For details the reader may consult /1/. In /4/ the following classes of ideals were introduced:

$$\mathfrak{S}_\Phi(\mathfrak{D}) = \{T \in \mathcal{L}^+(\mathfrak{D}) : ATB \in \mathfrak{S}_\Phi(\mathcal{H}) \text{ for all } A,B \in \mathcal{L}^+(\mathfrak{D})\}$$

and

$$M(\mathfrak{S}_\Phi(\mathfrak{D}), \mathfrak{S}_\Psi(\mathfrak{D})) = \{A \in \mathcal{L}^+(\mathfrak{D}) : AT, A^+T \in \mathfrak{S}_\Psi(\mathfrak{D}) \text{ for all } T \in \mathfrak{S}_\Phi(\mathfrak{D})\}.$$

($\mathfrak{B}(\mathfrak{D})$ and $M(\mathfrak{B}(\mathfrak{D}), \mathfrak{S}_\Psi(\mathfrak{D}))$ are defined analogously.)
It was a useful result that for selfadjoint $\mathcal{L}^+(\mathfrak{D})$ the ideals $\mathfrak{S}_\Phi(\mathfrak{D})$ can be characterized equivalently by

$$\begin{aligned}\mathfrak{S}_\Phi(\mathfrak{D}) &= \{T \in \mathcal{L}^+(\mathfrak{D}) : AT, AT^* \in \mathfrak{S}_\Phi(\mathcal{H}) \text{ for all } A \in \mathcal{L}^+(\mathfrak{D})\} \\ &= \{T \in \mathcal{L}^+(\mathfrak{D}) : TA, T^*A \in \mathfrak{S}_\Phi(\mathcal{H}) \text{ for all } A \in \mathcal{L}^+(\mathfrak{D})\} \\ &= \{T \in \mathcal{L}^+(\mathfrak{D}) : AT, TA \in \mathfrak{S}_\Phi(\mathcal{H}) \text{ for all } A \in \mathcal{L}^+(\mathfrak{D})\}.\end{aligned}$$

A simple equivalent characterization of $M(.,.)$ is given by

$$M(\mathfrak{S}_\Phi(\mathfrak{D}), \mathfrak{S}_\Psi(\mathfrak{D})) = \{A \in \mathcal{L}^+(\mathfrak{D}) : XAT, TAX \in \mathfrak{S}_\Psi(\mathfrak{D}) \text{ for all } X \in \mathcal{L}^+(\mathfrak{D}), \text{ for all } T \in \mathfrak{S}_\Phi(\mathfrak{D})\}.$$

2. PROPERTIES OF M(.,.)

We collect some simple properties of $M(\mathfrak{S}_\Phi(\mathfrak{D}), \mathfrak{S}_\Psi(\mathfrak{D}))$ (in short $M(\Phi,\Psi)$ or $M(\infty,\Psi)$ for $\mathfrak{S}_\Phi = \mathfrak{S}_\infty$).

Lemma 1

Let $\mathfrak{D} = \mathcal{H}$ and let $\|\ \|_\Phi$ be not equivalent to the operator-norm $\|\ \|$. Then $M(\infty,\Phi) = \mathfrak{S}_\Phi(\mathcal{H}) = M(\mathfrak{B}(\mathfrak{D}),\Phi)$.

Proof

$\mathfrak{D} = \mathcal{H}$ implies $\mathcal{L}^+(\mathfrak{D}) = \mathfrak{B}(\mathcal{H}) = \mathfrak{B}(\mathfrak{D})$. Let $A \in \mathfrak{S}_\Phi(\mathcal{H})$, $C \in \mathfrak{B}(\mathfrak{D})$, $X \in \mathcal{L}^+(\mathfrak{D})$. Then the estimation $\|XAC\|_\Phi + \|CAX\|_\Phi \le 2\|X\|\|C\|\|A\|_\Phi$ means $A \in M(\mathfrak{B}(\mathfrak{D}),\Phi) \subseteq M(\infty,\Phi)$, that is $\mathfrak{S}_\Phi(\mathcal{H}) \subseteq M(\mathfrak{B}(\mathcal{H}),\Phi)$, $\mathfrak{S}_\Phi(\mathcal{H}) \subseteq M(\infty,\Phi)$.
(Here the equivalent characterization of M(.,.) was used). Now we show $\mathfrak{S}_\Phi(\mathcal{H}) \supseteq M(\mathfrak{S}_\infty(\mathcal{H}),\Phi)$ which completes the proof. To see this let $A \notin \mathfrak{S}_\Phi(\mathcal{H})$, i.e. $\|A\|_\Phi = \infty$. Without loss of generality let $A = A^* \ge 0$. First consider completely continuous A. The s-numbers are identical with the eigenvalues of A and because $\Phi^{(n)}(s_j(A)) \equiv \Phi(s_1(A),\dots,s_n(A),0,0\dots)$ is non-decreasing, $\Phi^{(n)}(s_j(A)) \longrightarrow \infty$ for $n \longrightarrow \infty$. Select a sequence (n_k) with $\Phi^{(n_k)}(s_j(A)) > k$ and choose an operator C as follows: the eigenvectors of C coincide with that of A, the eigenvalues (c_k) fulfill
$c_k = 1$ for $1 \le k \le n_1$, $c_k = 1/j^{1/2}$ for $n_{j-1} < k \le n_j$, $j = 2,3,\dots$
Obviously $C \in \mathfrak{S}_\infty(\mathcal{H})$ but $\Phi^{(n_k)}(AC) = \Phi(s_1(AC),\dots,s_{n_k}(AC),0\dots) = \Phi(s_1(A)s_1(C),\dots,s_{n_k}(A)s_{n_k}(C),0\dots) \ge k/k^{1/2} \longrightarrow \infty$, i.e. $AC \notin \mathfrak{S}_\Phi(\mathcal{H})$ and hence $A \notin M(\infty,\Phi)$ (the last estimation holds because of Lemma 3.1 chap.III of /1/).
If A is not completely continuous, then it is trivial to see that there is a completely continuous A such that $AC \notin \mathfrak{S}_\Phi(\mathcal{H})$.

Q.E.D.

Similarly as in $\mathfrak{S}_\Phi(\mathfrak{D})$ (cf./4/) one can introduce a lot of locally convex topologies in $M(.,.)$. In this section we mention three of them. To get results on duality other topologies are useful. They will be given explicitely in the propositions where they are needed.
Let us denote by $\sigma^l_{\Phi,\Psi}$, $\sigma^r_{\Phi,\Psi}$ and $\sigma_{\Phi,\Psi}$ the following topologies on $M(\Phi,\Psi)$ given by the systems of seminorms:

$\sigma^l_{\Phi,\Psi}$: $A \longrightarrow \|XAT\|_\Psi$

$\sigma^r_{\Phi,\Psi}$: $A \longrightarrow \|TAX\|_\Psi$

$\sigma_{\Phi,\Psi}$: $A \longrightarrow \max\{\|XAT\|_\Psi, \|TAX\|_\Psi\}$

where $X \in \mathcal{L}^+(\mathfrak{D})$ and $T \in \mathfrak{S}_\Phi(\mathfrak{D})$ are arbitrary.
Clearly $\sigma^l_{\Phi,\Psi} \prec \sigma_{\Phi,\Psi}$, $\sigma^r_{\Phi,\Psi} \prec \sigma_{\Phi,\Psi}$.
The corresponding topologies on $M(\mathfrak{B}(\mathfrak{D}),\Phi)$ are defined obviously.
The following lemmas are valid for these ideals, too.

Lemma 2

The systems of seminorms defining the topologies $\sigma^l_{\Phi,\Psi}$, $\sigma^r_{\Phi,\Psi}$ and $\sigma_{\Phi,\Psi}$ are directed.

Proof

We give the proof for the topology $\sigma^l_{\Phi,\Psi}$, the other cases are handled analogously. Let $A \longrightarrow \|X_1AT_1\|_\Psi$, $A \longrightarrow \|X_2AT_2\|_\Psi$, $X_{1,2} \in \mathcal{L}^+(\mathfrak{D})$, $T_{1,2} \in \mathfrak{S}_\Phi(\mathfrak{D})$ two seminorms. Because the system of seminorms defining the topology t is directed (cf. section 1), there is an $X = X^+ \in \mathcal{L}^+(\mathfrak{D})$ with $\|X_{1,2}\phi\| \le \|X\phi\|$ for all $\phi \in \mathfrak{D}$. Therefore $\|X_{1,2}AT_{1,2}\phi\| \le \|XAT_{1,2}\phi\|$ for all $\phi \in \mathfrak{D}$. But this implies $\|X_{1,2}AT\|_\Psi \le \|XAT_{1,2}\|_\Psi = \|T^+_{1,2}A^+X\|_\Psi$. Moreover with the operator T given by $T^2 = T_1T_1^+ + T_2T_2^+$ from $\|T^+_{1,2}\phi\| \le$

$= \| T\phi \|$ the estimation $\| X_{1,2} A T_{1,2} \|_{\Psi} \le \| T^{+}_{1,2} A^{+} X \|_{\Psi} \le \| T A^{+} X \|_{\Psi} =$ $= \| XAT \|_{\Psi}$ follows.
The assertion of the Lemma follows if we prove $T \in \mathfrak{S}_{\Phi}(\mathfrak{D})$.
T is completely continuous and $\| T\phi \| \le \| T_1^{+}\phi \| + \| T_2^{+}\phi \|$ implies $\| TA\phi \| \le \| T_1^{+}A\phi \| + \| T_2^{+}A\phi \|$ for all $A \in \mathcal{L}^{+}(\mathfrak{D})$ and $\phi \in \mathfrak{D}$.
Because $T_1^{+}A,\ T_2^{+}A \in \mathfrak{S}_{\Phi}(\mathcal{H})$, the operator TA lies in $\mathfrak{S}_{\Phi}(\mathcal{H})$.
Hence $T \in \mathfrak{S}_{\Phi}(\mathfrak{D})$ by the equivalent characterization of $\mathfrak{S}_{\Phi}(\mathfrak{D})$.

Q.E.D.

Lemma 3

$M(\Phi,\Psi)[\sigma_{\Phi,\Psi}]$ are complete locally convex spaces.

The proof uses the completeness of $\mathfrak{S}_{\Psi}(\mathfrak{D})$ in an appropriate topology (cf. /4/ Lemma 17). The following Lemmas are also simple. They generalize the corresponding situation for $\mathfrak{S}_{\Phi}(\mathcal{H})$ (/1/). Because the proofs are similarly as for the ideals $\mathfrak{S}_{\Phi}(\mathfrak{D})$ (see /4/) they are omitted.

Lemma 4

i) For arbitrary $A \in M(\Phi,\Psi)$, $X \in \mathcal{L}^{+}(\mathfrak{D})$, $T \in \mathfrak{S}_{\Phi}(\mathfrak{D})$ it is

$$\min \| X(A-F)T \|_{\Psi} = \Psi(s_{n+1}(XAT), s_{n+2}(XAT), \dots)$$

$$\min \| T(A-F)X \|_{\Psi} = \Psi(s_{n+1}(TAX), s_{n+2}(TAX), \dots)$$

The minimum is taken over all $F \in \mathfrak{F}(\mathfrak{D})$, $\dim F \le n$.

ii) If Ψ is mono-norming then $\mathfrak{F}(\mathfrak{D})$ is dense in $M(\Phi,\Psi)$ with respect to $\sigma^{r}_{\Phi,\Psi}$ and $\sigma^{l}_{\Phi,\Psi}$.

Let us denote by $M^{0}(\Phi,\Psi)$ the $\sigma_{\Phi,\Psi}$ -closure of $\mathfrak{F}(\mathfrak{D})$ in $M(\Phi,\Psi)$

Lemma 5

i) $M^{0}(\Phi,\Psi)$ is a two-sided $*$-ideal in $\mathcal{L}^{+}(\mathfrak{D})$.

ii) If $\mathfrak{D}[t]$ is separabel, then also $M^{0}(\Phi,\Psi)$ with respect to $\sigma_{\Phi,\Psi}$ and hence also with respect to $\sigma^{r}_{\Phi,\Psi}$ and $\sigma^{l}_{\Phi,\Psi}$.

Lemma 6

Let $\mathfrak{S}_{\Psi}(\mathcal{H}) \neq \mathfrak{S}_{\infty}(\mathcal{H})$.

i) If $(A_m) \subset M(\Phi,\Psi)$ converges weakly on $\mathfrak{D}$ to $A \in \mathcal{L}^{+}(\mathfrak{D})$ and $\sup_m \max\{\| XA_mT \|_{\Psi}, \| TA_mX \|_{\Psi}\} < \infty$ for all $X \in \mathcal{L}^{+}(\mathfrak{D})$, $T \in \mathfrak{S}_{\Phi}(\mathfrak{D})$, then $A \in M(\Phi,\Psi)$.

ii) Suppose that there is a sequence $(P_n) \subset \mathcal{L}^{+}(\mathfrak{D})$, P_n finite dimensional projections and $\| X(P_n - I)\phi \| \longrightarrow 0$ for all $X \in \mathcal{L}^{+}(\mathfrak{D})$ and $\phi \in \mathfrak{D}$ (i.e. $P_n \longrightarrow I$ t-strongly). If for $A \in \mathcal{L}^{+}(\mathfrak{D})$ the sequence (P_nAP_n) is $\sigma_{\Phi,\Psi}$-bounded, then $A \in M(\Phi,\Psi)$.

3. DUALITY

In this section we give some examples to show how the ideals $\mathfrak{S}_{\Phi}(\mathfrak{D})$ are related to some of the ideals $M(\cdot,\cdot)$ by duality. Here topologies are useful which are obtained from $\sigma_{\cdot,\cdot}$ if we fix $X = I \in \mathcal{L}^{+}(\mathfrak{D})$. Furthermore we use topologies $t_{\Phi,\Psi}$ in $\mathfrak{D}$ induced by $M(\Phi,\Psi)$ in the same way as t, i.e. $t_{\Phi,\Psi}$ is given by the system of seminorms $\phi \longrightarrow \| A\phi \|$ for all $A \in M(\Phi,\Psi)$. This system is directed. Remark that $R \in \mathcal{L}(\mathfrak{D}[t_{\Phi,\Psi}], \mathcal{H})$ means $\| R\phi \| \le \| A\phi \|$ for some $A \in M(\Phi,\Psi)$ and all $\phi \in \mathfrak{D}$. Therefore $RT \in \mathfrak{S}_{\Psi}(\mathfrak{D})$ for all $T \in \mathfrak{S}_{\Phi}(\mathfrak{D})$.

Theorem 7

Let $\mathfrak{S}_{\Phi}(\mathfrak{D})$ be equipped with a topology given by the system of seminorms $T \longrightarrow \| AT \|_{1}$ for all $A \in M(\Phi,1)$ $(= M(\mathfrak{S}_{\Phi}(\mathfrak{D}), \mathfrak{S}_{1}(\mathfrak{D})))$.
Then any linear functional $\omega \in \mathfrak{S}_{\Phi}(\mathfrak{D})'$ is given by

$$(+) \qquad \omega(T) = \operatorname{Tr} RT$$

where $R \in \mathcal{L}(\mathfrak{D}[t_{\Phi,1}], \mathcal{H})$.

Proof

$R \in \mathcal{L}(\mathfrak{D}[t_{\Phi,1}], \mathcal{H})$ means RT nuclear for all $T \in \mathfrak{S}_{\Phi}(\mathfrak{D})$, $\| RT\phi \| \le \| AT\phi \|$ for all $\phi \in \mathfrak{D}$ and appropriate $A \in M(\Phi,1)$. Hence $|\omega(T)| = |\operatorname{Tr} RT| \le \| RT \|_1 \le \| AT \|_1$. Thus any functional ω given by (+) is continuous in the topology defined above. On the other hand, let ω be a continuous linear functional on $\mathfrak{S}_{\Phi}(\mathfrak{D})$, i.e. $|\omega(T)| \le \| AT \|_1$.
For $\phi, \psi \in \mathfrak{D}$ the operator $\langle \phi, . \rangle \psi$ is in $\mathfrak{S}_{\Phi}(\mathfrak{D})$ and $|\omega(\langle \phi, . \rangle \psi)| \le \| \langle \phi, . \rangle A\psi \|_1 = \| \phi \| \cdot \| A\psi \|$. This means $\omega(\langle \phi, . \rangle \psi)$ is a continuous (anti-)linear form in ϕ (on $\mathfrak{D}$) for any fixed ψ. By the Riesz-Theorem $\omega(\langle \phi, . \rangle \psi) = \langle \phi, \chi \rangle$ for some $\chi \in \mathcal{H}$. Define R by $R\psi = \chi$, then

$\omega(\langle \phi, . \rangle \psi) = \langle \phi, R\psi \rangle = \operatorname{Tr} R(\langle \phi, . \rangle \psi)$ and

$|\omega(\langle \phi, . \rangle \psi)| = |\langle \phi, R\psi \rangle| \le \| \phi \| \| A\psi \|$. This implies $\| R\psi \| \le \| A\psi \|$ for all $\psi \in \mathfrak{D}$. Therefore $R \in \mathcal{L}(\mathfrak{D}[t_{\Phi,1}], \mathcal{H})$

By linearity

(++) $\omega(F) = \mathrm{Tr}\, RF$ for all $F \in \mathfrak{F}(\mathcal{D})$.

In (++) there stand continuous linear functionals which can be extended by continuity to $\mathfrak{S}_\Phi(\mathcal{D})$. Here the fact is used that the symmetric norming function corresponding to $\|\ \|_1$ is mono-norming and therefore $\mathfrak{F}(\mathcal{D})$ is dense in $\mathfrak{S}_\Phi(\mathcal{D})$. Thus

$$\omega(T) = \mathrm{Tr}\, RT \quad \text{for all } T \in \mathfrak{S}_\Phi(\mathcal{D}).$$

Q.E.D.

Theorem 8

On $M(\Phi,1)$ consider two topologies given by the directed systems of seminorms

$M(\Phi,1) \ni A \longrightarrow \|AT\|_1$ for all $T \in \mathfrak{S}_\Phi(\mathcal{D})$

and

$M(\Phi,1) \ni A \longrightarrow \|TA\|_1$ for all $T \in \mathfrak{S}_\Phi(\mathcal{D})$.

The linear functionals ω on $M(\Phi,1)$ which are continuous with respect to both of these topologies are given by

$$\omega(A) = \mathrm{Tr}\, AT \quad \text{for } T \in \mathfrak{S}_\Phi(\mathcal{D}).$$

Proof

For $T \in \mathfrak{S}_\Phi(\mathcal{D})$ form $\omega(A) = \mathrm{Tr}\, AT$, then $|\omega(A)| = |\mathrm{Tr}\, AT| \leq \|AT\|_1$. Moreover, because $M(\Phi,1)$ and $\mathfrak{S}_\Phi(\mathcal{D})$ are $*$-ideals AT nuclear implies TA nuclear and $\mathrm{Tr}\, AT = \mathrm{Tr}\, TA$, i.e. $|\omega(A)| = |\mathrm{Tr}\, TA| \leq \|TA\|_1$.

Conversely, let ω be continuous in the sense described in the theorem. Because the systems of seminorms are directed, we may assume

(3) $|\omega(A)| \leq \|AT\|_1$

(4) $|\omega(A)| \leq \|TA\|_1$

for all $A \in M(\Phi,1)$ and appropriate $T = T^* \geq 0$, $T \in \mathfrak{S}_\Phi(\mathcal{D})$. For $\langle\varphi,.\rangle\psi \in \mathfrak{F}(\mathcal{D})$ (3) gives $|\omega(\langle\varphi,.\rangle\psi)| \leq \|T\varphi\|\cdot\|\psi\|$ i.e. $\omega(\langle\varphi,.\rangle\psi)$ is for any fixed $\psi \in \mathcal{D}$ a $\|\ \|$-continuous antilinear functional on $\mathcal{D}$, hence there is a $\rho \in \mathcal{H}$ with $\omega(\langle\varphi,.\rangle\psi) = \langle\varphi,\rho\rangle$. Put $S\psi = \rho$ then

(5) $\omega(\langle\varphi,.\rangle\psi) = \langle\varphi,S\psi\rangle$, $|\langle\varphi,S\psi\rangle| \leq \|T\varphi\|\,\|\psi\|$ for all $\varphi,\psi \in \mathcal{D}$.

Analogously from (4) we get

(6) $\omega(\langle\varphi,.\rangle\psi) = \langle R\varphi,\psi\rangle$, $|\langle R\varphi,\psi\rangle| \leq \|\varphi\|\,\|T\psi\|$ for all $\varphi,\psi \in \mathcal{D}$.

Hence on $\mathcal{D}$ $R^* = S$ and $S^* = R$. (5) implies for $\varphi \in \mathcal{D}$, $Z \in \mathcal{L}^+(\mathcal{D})$ $|\langle Z\varphi,S\psi\rangle| \leq \|TZ\varphi\|\|S\psi\| \leq K\|\varphi\|\|\psi\|$, i.e. $S \in \mathcal{D}(Z^*)$ for all $Z \in \mathcal{L}^+(\mathcal{D})$. Because $\mathcal{L}^+(\mathcal{D})$ was assumed to be selfadjoint we have $S\mathcal{D} \subset \mathcal{D}$. In the same way (6) gives $R\mathcal{D} \subset \mathcal{D}$, i.e. $S \in \mathcal{L}^+(\mathcal{D})$.

Putting (5) and (6) together one obtains

$|\omega(\langle\varphi,.\rangle\psi)| = |\langle\varphi,S\psi\rangle| \leq \|\varphi\|\,\|T\psi\|$

$|\omega(\langle\varphi,.\rangle\psi)| = |\langle S^*\varphi,\psi\rangle| \leq \|T\varphi\|\,\|\psi\|$.

Thus $\|S\psi\| \leq \|T\psi\|$, $\|S^*\varphi\| \leq \|T\varphi\|$ and consequently $S \in \mathfrak{S}_\Phi(\mathcal{D})$. The relation (5) gives

(7) $\omega(F) = \mathrm{Tr}\, FS$ for all $F \in \mathfrak{F}(\mathcal{D})$.

Both the functionals in (7) are continuous and $\mathfrak{F}(\mathcal{D})$ is dense in $M(\Phi,1)$ with respect to the two topologies. Thus from (7) the desired result follows:

$$\omega(A) = \mathrm{Tr}\, AS \quad \text{for all } A \in M(\Phi,1).$$

Q.E.D.

These two theorems are also valid for $\mathfrak{S}_1(\mathcal{D})$ and $M(1,1) = \mathcal{L}^+(\mathcal{D})$. Because $\mathfrak{S}_1(\mathcal{D})$ is a very important ideal we state the result as a corollary.

Corollary 9

i) Let $\mathfrak{S}_1(\mathcal{D})$ be equipped with the topology given by the system of seminorms $T \longrightarrow \|AT\|_1$ for all $A \in \mathcal{L}^+(\mathcal{D})$. Then the continuous linear functionals ω on $\mathfrak{S}_1(\mathcal{D})$ are given by

$\omega(T) = \mathrm{Tr}\, RT$, $R \in \mathcal{L}(\mathcal{D}[t], \mathcal{H})$.

ii) On $\mathcal{L}^+(\mathcal{D})$ consider the topologies given by the systems of seminorms

$A \longrightarrow \|AT\|_1$ for all $T \in \mathfrak{S}_1(\mathcal{D})$ and $A \longrightarrow \|TA\|_1$ for all $T \in \mathfrak{S}_1(\mathcal{D})$.

The linear functionals ω on $\mathcal{L}^+(\mathcal{D})$ which are continuous with respect to both of these topologies are given by

$$\omega(A) = \mathrm{Tr}\, AT, \quad T \in \mathfrak{S}_1(\mathcal{D}).$$

Remark that i) was already proved in /3/, the proof of ii) is the same as in Theorem 8.

The situation and results described in Theorems 7,8 and Corollary 9 can be extended to more general objects. We give an example which indicates one of such a possibility. For this we need the following definition.

Definition

Let $\mathcal{A}(\mathcal{D})$ be a selfadjoint Op^*-algebra. Define $\mathcal{S}_1(\mathcal{A})$ by
$\mathcal{S}_1(\mathcal{A}) = \{ T \in \mathcal{L}^+(\mathcal{D}) :$ ATB nuclear for all $A,B \in \mathcal{A} \}$.

This set is identical with $\{ T \in \mathcal{L}^+(\mathcal{D}) : AT, AT^*$ nuclear $\forall A \in \mathcal{A} \}$ and with $\{ T \in \mathcal{L}^+(\mathcal{D}) : TA, T^* A$ nuclear for all $A \in \mathcal{A}(\mathcal{D}) \}$. Clearly, $\mathcal{S}_1(\mathcal{A})$ is a $*$-algebra and moreover $\mathcal{S}_1(\mathcal{A})$ can be regarded as an $\mathcal{A}$-modul. On $\mathcal{S}_1(\mathcal{A})$ we define the topology $\tau_1(\mathcal{A})$ given by the directed system of seminorms $T \longrightarrow \| AT \|_1$, $A \in \mathcal{A}(\mathcal{D})$. Then the following result is valid.

Proposition 10

The $\tau_1(\mathcal{A})$-continuous linear functionals ω on $\mathcal{S}_1(\mathcal{A})$ are given by

$$\omega(T) = \mathrm{Tr}\, RT \quad , \quad R \in \mathcal{L}(\mathcal{D}[t_{\mathcal{A}}] , \mathcal{H}).$$

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