CRITICAL BEHAVIOUR IN FINITE TEMPERATURE QCD

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ABSTRACT:

We survey strong interaction thermodynamics as obtained from lattice quantum chromodynamics. At low temperature, quarks and gluons are confined to colour singlets, forming hadronic matter. At the deconfinement transition, colour screening decouples the constituents; subsequently, chiral symmetry restoration renders the quarks massless. At sufficiently high temperature, we obtain a plasma of non-interacting quarks and gluons.

I. INTRODUCTION

Quarks and gluons appear confined as long as we consider hadrons in the physical vacuum: any constituent can then travel a spatial distance of not more than 10^{-13}cm before it reaches confinement constraints.

In sufficiently dense matter, we expect this situation to change. Colourless hadronic matter should then undergo a phase transition to colour conducting quark matter. With sufficient overlap, the constituents can no longer be associated to a given hadron and can move over macroscopic distances without ever leaving an overall confinement environment.

The idea of a phase transition from hadronic to quark matter is as old[1)] as the quark structure of hadrons. Since then, a great variety of phenomenological approaches to the two-phase nature of strongly interacting matter have been pursued[2)]. They all have in common two phases as input. The advent of quantum chromodynamics (QCD) gave rise to the hope that both the two-phase character and the transition might be obtained from one basic theory[3)]. Exciting recent developments in lattice QCD at finite temperature seem to indicate that such a hope is justified[4-14)]; these developments will be the subject of my survey.

Quantum chromodynamics specifies the basic interaction of quarks and gluons; from this we are to obtain the description of strongly interacting matter in its different states. Not surprisingly, the first attempts concentrated on limiting behaviour.

Asymptotic freedom makes interactions at very short distances (or high momenta) arbitrarily weak, so that a perturbation expansion in powers of the effective coupling may be expected to converge in the limit of high temperature or density[15)]. Sufficiently hot and/or dense matter should therefore become a gas of non-interacting quarks and gluons.

In the confinement region, at lower temperatures and densities, strongly interacting matter should exhibit hadronic behaviour. One has here considered on the lattice an expansion in terms of the inverse coupling (strong coupling expansion[3)]) or used in the continuum semiclassical solutions to the field equations (instanton gas[16)]). The resulting description indeed provided many aspects of hadron phenomenology, as given e.g. by dual string or bag models.

Strong and weak coupling approaches have evidently specified regions of applicability and thus basically give one-phase descriptions. Nevertheless, they already provide hints for a phase transition near the boundary of their regions of validity. In the perturbative treatment of the quark-gluon gas, the pressure (in first order) becomes negative at some temperature value, and this has been interpreted as the onset of confinement[15]. In the strong coupling expansion, there are indications for a phase transition due to Debye screening of colour charges[3]. The suppression of large scale instantons leads to similar conclusions[16]. It is clear, however, that these limiting approaches cannot give us the unified "whole-range" description we would like to obtain from a basic theory.

The Monte Carlo evaluation of finite temperature QCD on the lattice now provides us with such a unified picture. The evaluation method itself was devised for and first applied to the study of the confinement problem[17]. Its application to finite temperature statistical mechanics is, however, quite straight-forward - perhaps it is even more natural here, where the real physical temperature plays the role of the Euclidean time in the confinement problem. In either case, the lattice acts as scaffolding during the evaluation: both discreteness and finiteness are to be removed at the end, to give us continuum theory results.

II. TWO-PHASE PHENOMENOLOGY

In order to introduce some of the questions and concepts of critical strong interaction physics, we shall in this section briefly consider simple phenomenological models for the two "limiting" phases.

For the statistical mechanics of hadronic matter, we consider an ideal gas of ground state hadrons and all their resonance excitations. The partition function of such a resonance gas is

$$\ln Z_H(\beta, V) = \frac{V}{(2\pi)^3} \int_0^\infty dm\ \tau(m) \int d^3k\ e^{-\beta\sqrt{k^2+m^2}} \quad , \tag{2.1}$$

for a system with zero chemical potential and, for simplicity, with Boltzmann statistics; $\beta^{-1} = T$ is the temperature, V the spatial volume. From hadron dynamics (dual string[18], bag[19] models), the resonance spectrum $\tau(m)$ is known to have the form

$$\begin{aligned} &\tau(m) = d\ \delta(m-m_0) + c\ \theta(m-2m_0)\ m^{-a}\ e^{bm} \quad , \\ &a,\ b,\ c,\ d = \text{const.} \quad , \end{aligned} \tag{2.2}$$

as first proposed by Hagedorn[20]. The first term in eq. (2.2) corresponds to a d-fold degenerate ground state; for $c = 0$, we would thus simply obtain an ideal gas of ground state hadrons. While a depends on the details of the model used[2], b is related quite generally to the string tension σ ($b^2 = 3\sigma/4\pi$ in four dimensions) or equivalently to the Regge slope.

From eq. (2.1) and (2.2) we obtain for the energy density of the resonance gas

$$\varepsilon_H(\beta) \equiv \lim_{V\to\infty} \frac{-1}{V} \left(\frac{\partial \ln Z_H}{\partial \beta}\right)_V \tag{2.3}$$

$$\simeq \varepsilon_0(\beta) + \frac{c}{(2\pi\beta)^{3/2}} \int_{2m_0}^{\infty} dm\, m^{-a+5/2}\, e^{-m(\beta-b)} \tag{2.4}$$

with $\varepsilon_0(\beta)$ denoting the energy density of an ideal gas of ground state hadrons m_0 . The corresponding specific heat becomes

$$c_H(\beta) = c_0(\beta) + \beta \left\{ \frac{3}{2}(\varepsilon_H - \varepsilon_0) + \frac{c}{(2\pi\beta)^{3/2}} \int_{2m_0}^{\infty} dm\, m^{-a+7/2}\, e^{-m(\beta-b)} \right\} ; \tag{2.5}$$

again the first term describes the ideal ground state gas.

It is well-known[22] and from eqs. (2.3/2.4) also immediately evident that the spectral form (2.2) leads to critical behaviour. Depending on the exact value of a , from some derivative of the partition function on we will have divergent expressions at and/or above the critical temperature $T_c = b^{-1}$.

To illustrate what happens, we choose $a = 4$, $c = d = 1$. The specific heat of the resonance gas then diverges at $T_c = b^{-1}$, while the energy density remains finite there. Both are not defined for $T > T_c$. We have thus reached the end of hadron physics when $T = T_c$; without further information, we cannot say what lies beyond T_c .

The statistical mechanics of an ideal gas of massless quarks and gluons is obtained from the partition function

$$\ln Z_P(\beta, V) = \frac{V\beta}{(2\pi)^3} \int d^3k \left[g_B \ln \left\{\frac{1}{1-e^{-\beta|k|}}\right\} + g_F \ln \left\{\frac{1}{1+e^{-\beta|k|}}\right\}\right] . \tag{2.6}$$

Here g_B and g_F denote the bosonic and fermionic degrees of freedom, respectively; for colour SU(3) and two quark flavours, we have (with spins and antifermions)

$$g_B = 8 \times 2 = 16 \quad ,$$

$$g_F = 3 \times 2 \times 2 \times 2 = 24 \quad .$$

This leads to the Stefan-Boltzmann form of the energy density

$$\varepsilon_{SB}^{P}/T^4 = [(8\pi^2/15) + (7\pi^2/10)] \simeq 12.2 \tag{2.7}$$

with the first term corresponding to the gluon and the second to the quark component of the gas.

In fig. 1, we compare the energy density of hadronic matter, eq. (2.4), with that of an ideal gas of quarks and gluons, eq. (2.7). We expect that with increasing temperature, the constituent degrees of freedom, "frozen" in the hadronic state, will "thaw" to make ε attain its plasma value. We hope that QCD will give us a unified description of this development, including the phase transition(s) separating the two limiting states. In the next sections we shall find these expectations at least in part fulfilled.

III. YANG-MILLS THERMODYNAMICS ON THE LATTICE

We shall follow the historical development of QCD thermodynamics and treat first the case of pure Yang-Mills theory. This restriction to finite temperature gluon matter allows us to introduce both formalism and evaluation method for a simpler system already exhibiting many of the essential features of the full theory; also at present the precision of the evaluation is definitely higher in this simplified case. Since calculations based on colour SU(2)[3-6,12] and those using SU(3)[8,10] do lead to essentially the same results, it is moreover possible to reduce computer times by considering the smaller colour group. - The extension to full QCD with fermions will be presented in section IV.

The partition function for a quantum system described in terms of fields $A(x)$ by a Hamiltonian $H(A)$ is defined as

$$Z = \mathrm{Tr}\, e^{-\beta H} \quad , \tag{3.1}$$

where $T = \beta^{-1}$ is again the physical temperature. The conventional lattice formalism is obtained from this in three steps: (1) reformulation of Z as path integral; (2) introduction of the lattice; (3) change of "variables" from gauge field to gauge group. Let us look at this procedure in a little more detail.

The Lagrange density of gluon QCD is given by

$$\mathcal{L} = -\frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu}_{a} \tag{3.2}$$

with

$$F^{a}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^a_{bc} A^b_\mu A^c_\nu \quad . \tag{3.3}$$

Here the f^{abc} are the structure functions of the relevant underlying gauge group, whose generators λ_i satisfy $[\lambda_a, \lambda_b] = i f^c_{ab} \lambda_c$; for SU(2), the colour indices a, b, c each run from one to two, for SU(3) from one to three. If we set the structure functions equal to zero, we recover the photon gas structure: it is the non-Abelian nature which gives us the gluon-gluon interaction. The partition function Z can now be written[23] in the form of a path integral

$$Z(\beta, V) = \int [d\, A] \exp \{ \int_0^\beta d\tau \int_V d^3x\, \mathcal{L} [A(x, \tau)] \} \tag{3.4}$$

using the Euclidean Lagrange density, $\mathcal{L}$, with $it = \tau$, and with periodicity in τ , $A(x, 0) = A(x, \beta)$. The three-dimensional integral of the Hamiltonian form ($H \sim \int d^3x \, \mathcal{H}(x)$) thus becomes an asymmetric four-dimensional one, with the "special" dimension measuring the temperature.

In the next step, we replace the Euclidean $x - \tau$ continuum by a finite lattice[24], with N_σ sites and spacing a_σ in the spatial part, N_β sites and spacing a_β in the temperature direction. To assure the required periodicity in τ , we chose a lattice closed on itself: $1 \hat{=} 1 + N_\beta$. For economy in the later calculations, we work with lattices which are symmetric and also periodic in the space part, although neither property would be necessary. - The integrals in the exponent of eq. (3.4) now become sums, and we have $V = (N_\sigma a_\sigma)^3$, $\beta = N_\beta a_\beta$. The thermodynamic limit requires $N_\sigma \to \infty$ at fixed a_σ ; the continuum limit is obtained by a_σ , $a_\beta \to 0$ with fixed $N_\beta a_\beta$, which forces also $N_\beta \to \infty$. The success of the approach rests on the (lucky) facts that already rather small lattices ($N_\sigma \sim 5 - 10$, $N_\beta \sim 3 - 5$) seem to be asymptotic, and such that scale changes (changes in lattice spacings) can be connected to changes in the coupling strength g by the renormalization group relation, indicating continuum behaviour.

In the last step[24], we replace the gauge field "variable" $A_\mu((x_i + x_j)/2)$ associated to the link between two adjacent sites i and j by the gauge group element

$$U_{ij} = \exp \{ -i(x_i - x_j)^\mu A_\mu(\frac{x_i + x_j}{2}) \} \quad , \tag{3.5}$$

where $A_\mu(x) = \lambda_a A^a_\mu(x)$. With this transformation, the partition function becomes

$$Z(\beta, V) = \int \prod_{\{links\}} dU_{ij} \exp\{-S(U)\} \quad , \tag{3.6}$$

where the SU(N) lattice action is given by

$$\begin{aligned} S(U) = \frac{2N}{g^2} \{ & \frac{a_\beta}{a_\sigma} \sum_{\{P_\sigma\}} [1 - \frac{1}{N} \mathrm{Re\,Tr}\, U_{ij} U_{jk} U_{kl} U_{li}] \\ & + \frac{a_\sigma}{a_\beta} \sum_{\{P_\beta\}} [1 - \frac{1}{N} \mathrm{Re\,Tr}\, U_{ij} U_{jk} U_{kl} U_{li}] \} \quad . \end{aligned} \tag{3.7}$$

Here the sum $\{P_\sigma\}$ runs over all purely spacelike lattice plaquettes $(ijkl)$, while $\{P_\beta\}$ runs over all those with two spacelike and two "temperature-like" links. - If we insert eq. (3.5) in eq. (3.6/3.7) and expand for small lattice spacings $(|x_i - x_j| \to 0)$, then we recover in leading order the starting form (3.4).

From eq. (3.6/3.7), the energy density

$$\varepsilon \equiv (-1/V) \, (\partial \ln Z/\partial\beta)_V = -(N_\sigma^3 N_\beta a_\sigma^3)^{-1} \, (\partial \ln Z/\partial a_\beta)_{a_\sigma} \tag{3.8}$$

is found to be[6,25])

$$\varepsilon \simeq 2N(N_\sigma^3 N_\beta a_\sigma^3 a_\beta g^2)^{-1} \{ < \frac{a_\beta}{a_\sigma} \sum_{\{P_\sigma\}} [1 - \frac{1}{N} \mathrm{Re\,Tr\,UUUU}] >$$

$$- < \frac{a_\sigma}{a_\beta} \sum_{\{P_\beta\}} [1 - \frac{1}{N} \mathrm{Re\,Tr\,UUUU}] > \} \qquad (3.9)$$

with $<>$ denoting the usual thermodynamic average

$$< X > \equiv \{ \int \Pi \, dU \, e^{-S(U)} \, X(U) \} / \{ \int \Pi \, dU \, e^{-S(U)} \} \quad . \qquad (3.10)$$

Eq. (3.9) is our starting point for the Monte Carlo evaluation of gluon thermodynamics.

The evaluation is now carried out as follows[26]). The computer simulates an $N_\sigma^3 \times N_\beta$ lattice; for convenience we choose $a_\sigma = a_\beta = a$. Starting from a given ordered (all $U = 1$, "cold start") or disordered (all U random, "hot start") initial configuration, successively each link is assigned a new element U' , chosen randomly with the weight $\exp\{-S(U)\}$. One traverse of this procedure through the entire lattice is called one iteration. In general, it is found that five hundred or so iterations provide reasonable first indications about the behaviour of the energy density (3.9), but for some precision one should have more. The results to be shown here are obtained for colour SU(2), with typically around three thousand iterations, after which we observe quite stable behaviour; we have moreover reproduced our results also with the finite subgroup approximation to SU(2)[27]). Our work was done with $N_\sigma = 7, 9, 10$ for $N_\beta = 2, 3, 4, 5$; apart from expected finite lattice size effects[9]) there was no striking N_σ dependence of ε , suggesting that in general the thermodynamic limit is reached. To give at least some intuitive grounds for this, note that a $10^3 \times 3$ lattice has about 12,000 link degrees of freedom.

As result of the Monte Carlo evaluation, we obtain for a lattice of given size (N_σ, N_β) the energy density ε as function of g . In the continuum limit, g and the lattice spacing a are for colour SU(N) related through

$$a \, \Lambda_L = (11Ng^2/48\pi^2)^{-51/121} \exp\{-24\pi^2/11Ng^2\} \quad ; \qquad (3.11)$$

this relation is found by requiring a dimensional parameter Λ_L to remain constant under scale changes accompanied by corresponding changes in coupling strength. Hence once we are in the region of validity of the continuum limit, eq. (3.11) gives us the connection between g and a . Since $(N_\beta a)^{-1}$ is the temperature in units of Λ_L , we then have the desired continuum form of $\varepsilon(\beta)$.

In fig. 2, we show the resulting energy density ε as function of the temperature T . We first note that at high temperatures $(T/\Lambda_L \gtrsim 100)$, the results of the Monte Carlo evaluation agree quite well with the anticipated Stefan-Boltzmann form

$$\varepsilon/T^4 = \pi^2/5 \quad . \tag{3.12}$$

Let us now go to lower T. At about $T = 50\,\Lambda_L$, ε drops sharply. The derivative of ε gives us the specific heat, shown in fig. 3. At $T \simeq 43\,\Lambda_L$, it has a singularity-like peak, which signals the transition from bound to free gluons. With Λ_L taken in physical units[28], this gives us $T_c \simeq 200$ MeV. How do we know that it is the deconfinement transition which occurs here? There are two separate pieces of evidence. We shall see shortly that below T_c the SU(2) Yang-Mills system follows the behaviour of hadronic matter[6], as given in section II. Alternatively, one can study the behaviour of a static $q\bar{q}$ pair immersed in a gluon system of temperature T[4,5,8]; the free energy F of an isolated quark then serves to define the thermal Wilson loop $\langle L \rangle = \exp\{-\beta F\}$ as order parameter. It is found that $\langle L \rangle$ is essentially zero below and non-zero above T_c. Since $\langle L \rangle = 0$ corresponds to an infinite free energy of an isolated colour source, we have confinement below T_c.

Coming now, as promised, to the temperature region just below T_c, we show in fig. 4 the difference between energy density and pressure,

$$\Delta \equiv (\varepsilon - 3P)/T^4 \tag{3.13}$$

as taken from the Monte Carlo evaluation, compared to the corresponding hadronic gas form Δ_H from section II; both are given as functions of $x = (T_c/T) - 1$. This comparison, if it leads to agreement on functional behaviour, also allows us to determine the mass m_G of the glueball, as lowest gluonium state. We see from fig. 4 that $m_G \simeq 4.5\, T_c \simeq 190\,\Lambda_L$ provides quite good agreement with the Monte Carlo data. Moreover, this value of m_G (with physical parameters about 850 - 1000 MeV) is in reasonable accord with other lattice QCD determinations[34].

Finally let us have a look at how the Yang-Mills system behaves just above deconfinement[12]. While we expect perturbative behaviour at very high temperature, it seems likely that the form just above T_c is still non-perturbative. If we parametrize the contributions of the physical vacuum bubbles still present in the plasma close to T_c in terms of a bag description, we have in the case of colour SU(2) for the pressure

$$P = \frac{\pi^2}{15} T^4 \left[1 - \frac{5\alpha_s}{2\pi}\right] - B \tag{3.14}$$

and

$$\varepsilon = 3P + \frac{11}{18}\alpha_s^2 T^4 + 4B \tag{3.15}$$

for the energy density. Here B denotes the bag pressure and $\alpha_s = 3\pi/(11 \ln 4T/\Lambda)$

is the running coupling constant, with Λ as scale parameter. In fig. 5 we see that $B^{1/4} \simeq 190$ MeV and $\Lambda \simeq 100$ MeV yield a very good description of the functional form of $\Delta \equiv (\varepsilon - 3P)/T^4$. This implies a basically non-perturbative approach towards asymptotic freedom at least up to $T \sim 2T_c$. The role of higher order perturbation corrections is presently still unclear [29].

All lattice results presented here were obtained with the Wilson form (3.7) of the action, which provides the correct continuum limit. There are, however, other lattice actions which also do this, and we may therefore ask if deconfinement, both qualitatively and quantitatively, is independent of the choice of action. It was recently shown that this is indeed the case[30].

In closing this section, we note that also the extension to the SU(3) system has now been carried out[8,10]; it requires greater computational efforts, because there are eight group parameters instead of three. The behaviour observed is, however, in good agreement with that of the SU(2) system. In particular, we note that at high temperature the energy density now approaches[10]

$$\varepsilon/T^4 \simeq 8\pi^2/15 \quad , \tag{3.16}$$

instead of eq. (3.12) in the SU(2) case. Both times we thus find the number of degrees of freedom of a system of massless, non-interacting gluons for the corresponding colour group. The deconfinement transition in the SU(3) case occurs at $T_c/\Lambda_L \simeq 75 - 83$, which with the string tension relation[28] gives $T_c \simeq 150\text{-}170$ MeV, also in accord with the SU(2) value.

In conclusion: we have seen that Monte Carlo techniques applied to lattice QCD allow us to evaluate gluon thermodynamics over the whole temperature range. The resulting behaviour shows the expected two-phase nature: at low temperatures, we have a hadronic resonance gas of gluonium states; heating brings us to a deconfinement transition and beyond that to an ideal gluon gas.

IV. QCD THERMODYNAMICS WITH QUARKS

In this section we want to extend the considerations of the previous chapter to include quarks and antiquarks. We shall see that this brings in a basically new feature - the question of chiral symmetry restoration at high temperature. The lattice formulation encounters as a result the problem of species doubling[24,31], and in addition the Monte Carlo evaluation becomes considerably more complex. Nevertheless, first results both on the full QCD energy density[13] and on chiral symmetry restoration[13,14] have now appeared; we shall here consider the former, returning to chiral symmetry in the following section.

The QCD Lagrangian density for massless quarks of one flavour only can be written

$$\mathcal{L}(\psi, A) = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \bar{\psi}_\alpha (i \not\partial - g \not{A}_a \lambda^a)_{\alpha\beta} \psi_{\alpha\beta} \tag{4.1}$$

Here, μ, ν denote tensor, α, β spinor and a, b, c colour indices. The finite temperature Euclidean action becomes

$$S_\beta(\psi, A) = -\int_V d^3x \int_0^\beta d\tau \, \mathcal{L}(\psi, A) \tag{4.2}$$

with periodic (antiperiodic) boundary conditions in the temperature integration of the boson (fermion) fields. The full action thus is a sum

$$S_\beta(\psi, A) = S^G_\beta(A) + S^F_\beta(\psi, A) \tag{4.3}$$

of the pure Yang-Mills part S^G and the quark-gluon part S^F. The Yang-Mills system was treated in the previous section; we shall concentrate here on S^F.

Fermion theories on the lattice generally lead to species doubling[31], unless one is willing to accept chiral symmetry breaking[24]. We shall here use Wilson's form[24], in which chiral symmetry is recovered only in the continuum limit. We consider again an asymmetric lattice, with N_σ spatial and N_β temporal sites.

On this lattice, the action S^F of the quark-gluon sector is written[13,25]

$$\begin{aligned} S^F = \sum_n \Big\{ & \bar{\psi}_n \psi_n - \frac{k_F(g^2)}{8} \frac{4a_\sigma}{3a_\beta + a_\sigma} [\bar{\psi}_n (1-\gamma_0) U_{n,n+\hat{0}} \psi_{n+\hat{0}} \\ & + \bar{\psi}_n (1+\gamma_0) U^+_{n-\hat{0},n} \psi_{n-\hat{0}}] \\ & - \frac{k_F(g^2)}{8} \frac{4a_\beta}{3a_\beta + a_\sigma} \sum_{\mu=1}^{3} [\bar{\psi}_n (1-\gamma_\mu) U_{n,n+\hat{\mu}} \psi_{n+\hat{\mu}} \\ & + \bar{\psi}_n (1+\gamma_\mu) U^+_{n-\hat{\mu},n} \psi_{n-\hat{0}}] \Big\} \end{aligned} \tag{4.4}$$

where we have suppressed all but symbolic lattice indices. The second term in eq. (4.4) refers to that part of the lattice summation in which the gauge group elements U_{nm} are associated with timelike lattice links n, m ; in the third, the links are spacelike. The fermionic coupling $K(g^2) = k_F^2(g^2)/8$ is the usual "hopping" parameter[24,32].

In terms of S^F and S^G, the Euclidean form of the QCD partition function on the lattice is now given by

$$Z = \int \prod_{\text{links}} dU \prod_{\text{sites}} d\psi \, d\bar{\psi} \, e^{-S^G(U) - S^F(U, \psi, \bar{\psi})} \tag{4.5}$$

with the dU integration to be carried out for all links, the $d\psi\, d\bar{\psi}$ integrations for all sites of the lattice. Since the fermion action S^F has the form

$$S^F = \bar{\psi}(1 - KM)\,\psi \quad , \tag{4.6}$$

$$M_\mu = (1-\gamma_\mu)\, U_{nm}\, \delta_{n,m-\hat{\mu}} + (1+\gamma_\mu)\, U^+_{mn}\, \delta_{n,m+\hat{\mu}} \quad , \tag{4.7}$$

the integration over the anti-commuting spinor fields can be carried out[33] to give an effective boson form

$$Z = \int \prod_{\text{links}} dU\, e^{-S^G(U)}\, \det(1-KM) \quad . \tag{4.8}$$

The Euclidean energy density ε is obtained from Z ; it also becomes the sum $\varepsilon = \varepsilon^G + \varepsilon^F$ of the pure gluon part (eq. (3.9)) and the quark-gluon part[25])

$$\varepsilon^F \equiv -\xi^2 (N_\sigma^3 N_\beta a_\sigma^4 Z)^{-1} \int \prod_{\text{links}} dU\, e^{-S^G(U)}\, \det Q\, *$$

$$* \left\{ \frac{3K(g^2)}{4}\, \mathrm{Tr}(M_0\, Q^{-1}) - \frac{K\,(g^2)}{4} \sum_{\mu=1}^{3} \mathrm{Tr}(M_\mu\, Q^{-1}) \right\} \tag{4.9}$$

with $Q \equiv 1 - KM(U)$.

The computational problem beyond what is encountered in the pure Yang-Mills case lies in the evaluation of $\det Q$ and of Q^{-1} . We shall here use the expansion of these quantities in powers of the fermionic coupling K ("hopping parameter expansion"[32]), and retain in both cases only the leading term. By calculating an ideal gas of massless fermions in the same approximation, we shall then get some idea of how valid this procedure may be.

For $\det Q$ the leading term is

$$\det Q = \det(1-KM) \simeq 1 \tag{4.10}$$

("quenched approximation"[34]), while in the expansion

$$Q^{-1} = [1 - KM]^{-1} = \sum_{\ell=0}^{\infty} K^\ell [M(U)]^\ell \quad , \tag{4.11}$$

because of gauge invariance, the first contribution to $\mathrm{Tr}(Q^{-1}M)$ arises for the shortest non-vanishing closed loop obtained from $M(U) \sim U$. For $N_\beta = 2$ and 3 , this is a thermal loop, i.e., one closed in the temperature direction; hence in that case, the first term is $\ell = N_\beta - 1$. For $N_\beta \geq 4$, these loops are not the only ones; but the non-thermal loops lead to negligibly small contributions, so that we obtain on an isotropic lattice

$$\varepsilon^F a^4 \simeq \frac{3}{4} [K(g^2)]^{N_\beta} 2^{N_\beta + 2} \langle L \rangle \tag{4.12}$$

with $\langle L \rangle$ for the expectation value of the thermal Wilson loop, and a for the lattice spacing.

To test the convergence of the hopping parameter expansion, we compare[13] in table 1 the resulting energy density of an ideal gas of massless quarks with the exact form [9] for such a system, both calculated on lattices of the same size. For low N_β values, the approximation given by just the leading term is found to be quite reasonable, with less than 10% errors for $N_\beta = 2$ and 3 . This leads us to expect that also for QCD we can obtain an indicative estimate by retaining that term only. This expectation is supported by preliminary results for SU(2) fermions[35]: the energy density obtained by including all terms up to $\ell = 20$ in eq. (4.11) differs only by 10% from the leading term.

We now return to eq. (4.12) for the quark-gluon energy density of SU(N) QCD. The fermion coupling $K(g^2)$ for massless quarks has been evaluated numerically both at large[36] and at small[37] g^2 . The thermal Wilson loop $\langle L \rangle$ can be calculated by the usual finite temperature Monte Carlo techniques.

With the connection between g^2 , the lattice spacing a and the lattice scale Λ_L , as given by the renormalization group relation[38] (3.11) we can then from eq. (4.12) obtain ε^F as function of the temperature $T = \beta^{-1} = (N_\beta a)^{-1}$.

Comparing the leading term of the hopping parameter expansion for ε^F with that of an ideal gas of massless fermions, ε^F_{SB} , we have from eq. (4.12)

$$\varepsilon^F / \varepsilon^F_{SB} = [8K(g^2)]^{N_\beta} \langle L \rangle / N \tag{4.13}$$

with $K = 1/8$, $\langle L \rangle = N$ for the ideal gas analog of the SU(N) case.

For the SU(3) case, which is obviously the physically most interesting one, we display in table 2, for $N_\beta = 3$ and 4 , the values of $\langle L \rangle$ from ref.10), together with the coupling $K(g^2)$, which is taken from the u, d form of ref. 36), and the resulting energy density ratio $\varepsilon^F/\varepsilon^F_{SB}$. We note that the energy density very quickly approaches its asymptotic value - and not because K and $\langle L \rangle$ separately do so, but rather because these quantities, for each N_β , together provide an almost asymptotic energy density. In fig. 6 we display the temperature behaviour of the combined $N_\beta = 3$ and 4 results. We note a sharp drop around $T \sim 80\ \Lambda_L$ ($\sim 0.4\ \sigma^{1/2}$) , which presumably corresponds to the onset of confinement.

In fig. 7 we show the overall energy density ε/T^4, obtained by combining our above results for ε^F with the pure Yang-Mills results of section III. We conclude that full quantum chromodynamics with fermions indeed appears to lead to the deconfinement behaviour observed in the study of Yang-Mills systems alone. In particular, we note that at temperatures $T \gtrsim 2T_c$ essentially all constituent degrees of freedom have been "thawed".

V. DECONFINEMENT AND CHIRAL SYMMETRY RESTORATION

Quantum chromodynamics, for massless quarks a priori free of dimensional scales, contains the intrinsic potential for the spontaneous generation of two scales: one for the confinement force coupling quarks to form hadrons, and one for the chiral force binding the collective excitations to Goldstone bosons[39]. These two lead in thermodynamics to two possible phase transitions, characterized by two critical temperatures, T_c and T_{ch}. Above T_c, the density is high enough to render confinement unimportant: hadrons dissolve into quarks and gluons. Above T_{ch}, chiral symmetry is restored, so that quarks must be massless. For T below both T_c and T_{ch}, we have a gas of massive hadrons; for T above both T_c and T_{ch}, we have a plasma of massless quarks and gluons. Conceptually simplest would be $T_c = T_{ch}$; the possibility $T_c > T_{ch}$ appears rather unlikely[40]. On the other hand, $T_c < T_{ch}$ would correspond to a regime of unbound massive "constituent" quarks[40], as they appear in the additive quark model for hadron-hadron and hadron-lepton interactions[41]. The question of deconfinement vs. chiral symmetry restoration thus confronts us with one of the most intriguing aspects of quark-gluon thermodynamics.

The fermionic action of Wilson[24] used in the last section avoids species doubling at the cost of chiral invariance. Even an ideal gas of massless quarks in this formulation is not chirally invariant[42], since the expectation value $\langle\bar{\psi}\psi\rangle$ is always different from zero. It has therefore been suggested[42] to use the difference between this "Stefan-Boltzmann" value and the corresponding QCD value for Wilson fermions as the physically meaningful order parameter: it would vanish when the behaviour of a non-interacting system of massless fermions is reached.

In fig. 8 we show this order parameter as calculated for colour SU(3), in leading power of the hopping parameter expansion[13]. It is non-zero up to $T_{ch} \simeq 100\ \Lambda_L$, and vanishes for higher temperatures. This suggests chiral symmetry restoration slightly above deconfinement, with

$$T_{ch} / T_c \simeq 1.3 \quad . \tag{5.1}$$

It remains open at present to what extent this will be modified by the inclusion of virtual quark loops, or if there are any significant finite lattice effects.

Using for the SU(2) case a chirally invariant action with the resulting species doubling, it was found in ref. 14) that chiral symmetry restoration occurs at

$$T_{ch} = (0.55 \pm 0.07)\sqrt{\sigma} \quad ; \tag{5.2}$$

here also virtual quark loops are neglected. Since in this determination only $\langle\bar{\Psi}\Psi\rangle$ is studied, it does not provide any information about T_c. To obtain T_{ch}/T_c, one therefore has to rely on some other T_c determination. With the rather low value of ref. 5), it is found that[14)]

$$T_{ch}/T_c = 1.6 \pm 0.2 \quad . \tag{5.3}$$

Using the largest T_c obtained[6)], we have instead

$$T_{ch}/T_c = 1.0 \pm 0.1 \tag{5.4}$$

so that the question of whether or not $T_c = T_{ch}$ appears to remain open.

VI. PHASE TRANSITION PARAMETERS

In the lattice evaluation of QCD thermodynamics, we have calculated all physical quantities in terms of the dimensional lattice scale Λ_L. To convert Λ_L into physical units, we just have to measure one of these physical observables. String tension considerations give for Yang-Mills systems

$$\Lambda_L = \left\{ \begin{array}{l} (1.1 \pm 0.2) \times 10^{2}\ \sqrt{\sigma} = (4.4 \pm 0.8)\ \text{MeV}^{43)} \\ (1.3 \pm 0.2) \times 10^{-2}\ \sqrt{\sigma} = (5.2 \pm 0.8)\ \text{MeV}^{44)} \end{array} \right\} \tag{6.1}$$

in case of colour SU(2) and

$$\Lambda_L = (5.0 \pm 1.5) \times 10^{-3}\ \sqrt{\sigma} = (2.0 \pm 0.6)\ \text{MeV}^{45)} \tag{6.2}$$

for colour SU(3). The deconfinement temperature is found to be

$$T_c = (38^{\,4)} - 43^{\,6)})\ \Lambda_L \tag{6.3}$$

for SU(2) and

$$T_c = (75^{\,8)} - 83^{10)})\ \Lambda_L \tag{6.4}$$

for SU(3). Taking the average of eq. (6.1), we have

$$T_c = \left\{ \begin{array}{ll} [(170-210) \pm 30]\ \text{MeV} & \text{SU(2)} \\ [(150-170) \pm 50]\ \text{MeV} & \text{SU(3)} \end{array} \right\} \tag{6.5}$$

and thus little or no dependence of T_c on the colour group. This Yang-Mills value of the deconfinement temperature remains uneffected by the introduction of quarks in the scheme of section IV. The temperature for chiral symmetry restoration is accordingly given by relation (5.1).

From eq. (6.5) and the form of fig. 7, we can now estimate the energy density values at the two transition points. For the SU(3) Yang-Mills case, we obtain

$$\varepsilon(T_c) \simeq 200 - 300 \text{ MeV/fm}^3 \quad , \tag{6.6}$$

where we have assumed that the turn-over in ε occurs at about half the Stefan-Boltzmann value. This range, corresponding roughly to hadronic energy density, seems physically quite reasonable. It is not known at present if and how much it would be increased by the introduction of quarks; a shift proportional to that of the Stefan-Boltzmann limit would double the value of eq. (6.6). This suggests twice standard nuclear density (n_o = 150 MeV/fm^3) as lower and four times nuclear density as upper bound for the deconfinement transition. Present estimates for the energy density expected in ultrarelativistic heavy ion collisions[46] thus put deconfinement within reach.

Chiral symmetry restoration, even if it occurs at only slightly higher temperatures, seems to be considerably more difficult to attain. Just a small increase beyond T_c brings us to the top of the Stefan-Boltzmann "shelf", where the energy density is above 2 GeV/fm^3 .

VII. CONCLUSIONS

Our basic conclusion is certainly that the lattice formulation of quantum chromodynamics appears to be an extremely fruitful approach to the thermodynamics of strongly interacting matter. It is so far the only way to describe within one theory the whole temperature range from hadronic matter to the quark-gluon plasma. It leads to deconfinement and provides first hints on chiral symmetry restoration.

We are still at the beginning. It is not really clear if $T_c \neq T_{ch}$, finite size scaling near the phase transitions has not been studied at all for $T \neq 0$, and the lattice thermodynamics of systems with non-zero baryon number has not been touched. Nevertheless, there seems to emerge today from QCD something already suggested by percolation methods[47], instanton considerations[48] and mean field calculations[49]: a three state picture of strongly interacting matter.

ACKNOWLEDGEMENT

It is a pleasure to thank J. Engels, P. Hasenfratz, F. Karsch, L. McLerran, I. Montvay and B. Petersson for stimulating discussions.

FIGURE CAPTIONS

Figure 1 : Energy density of hadronic matter (H) and of an ideal quark-gluon plasma (P).

Figure 2 : Energy density of the SU(2) Yang-Mills system, compared to the ideal gas value ε_{SB} , as function of temperature T . The curve is a fit.

Figure 3 : The specific heat of the SU(2) Yang-Mills system as function of temperature T , obtained by differentiating the fit of fig. 2.

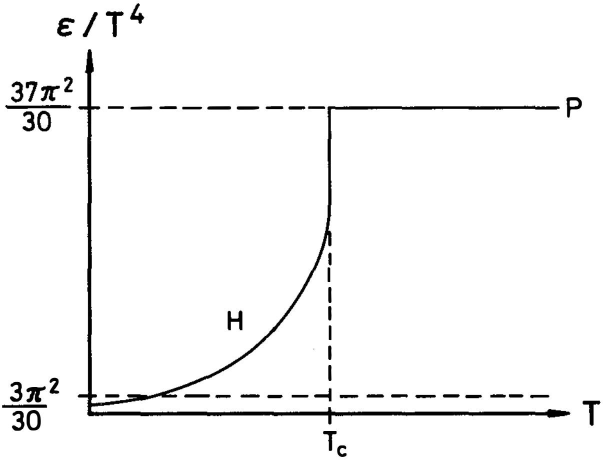
Figure 4 : Interaction measure $\Delta \equiv (\varepsilon - 3P)/T^4$ as function of $X = T_c/T - 1$, compared to the resonance gas prediction with glueball mass $m_G = 4$, 4.5 and 5 T_c .

Figure 5 : Interaction measure $\Delta \equiv (\varepsilon - 3P)/T^4$ as function of temperature, compared to the leading order perturbative form with bag correction $B^{1/4} = 180$ (a), 190 (b) and 200 (c) MeV .

Figure 6 : Fermionic energy density of the SU(3) system, compared to the ideal gas value ε^F_{SB} , as function of the temperature. Circles correspond to $N_\beta = 3$, triangles to $N_\beta = 4$.

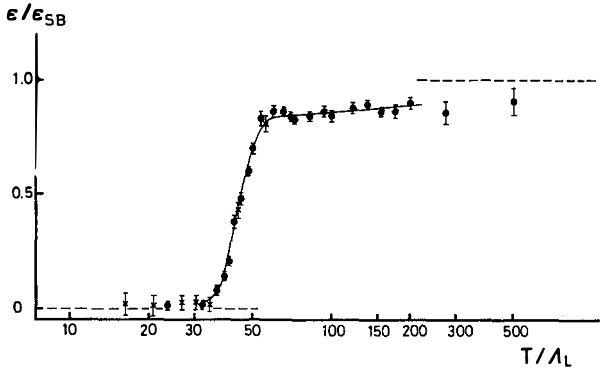
Figure 7 : Comparison of the energy density of full QCD with that of the SU(3) Yang-Mills theory, as obtained from a fit to fig. 6 and from ref. 10).

Figure 8 : The chiral order parameter $[(\langle\bar{\psi}\psi\rangle_{SB} - \langle\bar{\psi}\psi\rangle) / \langle\bar{\psi}\psi\rangle_{SB}]$, where $\langle\bar{\psi}\psi\rangle_{SB}$ measures the chiral symmetry breaking of an ideal gas of massless Wilson fermions on a finite lattice.



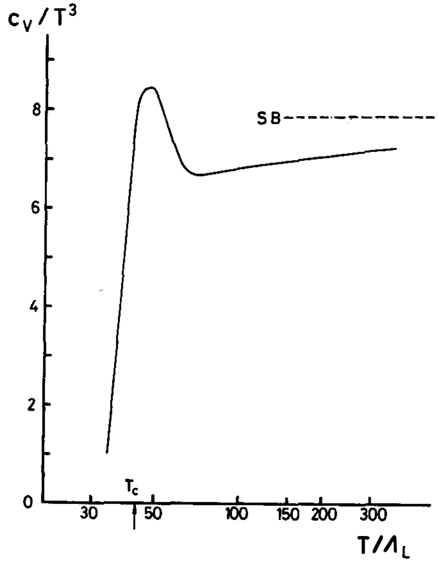
ε / T⁴
37π²/30
P
H
3π²/30
T
Tc

Fig. 1



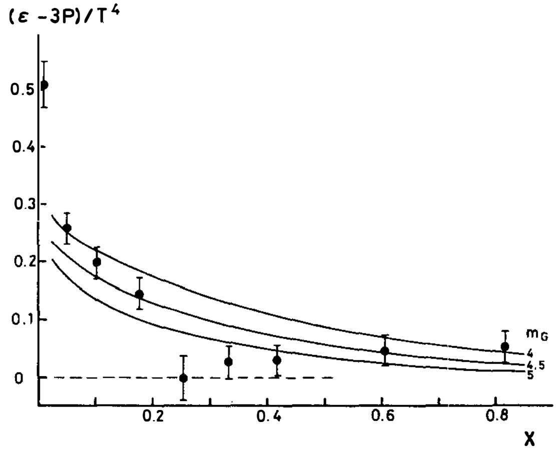
ε/εSB
1.0
0.5
0
10
20
30
50
100
150
200
300
500
T/ΛL

Fig. 2



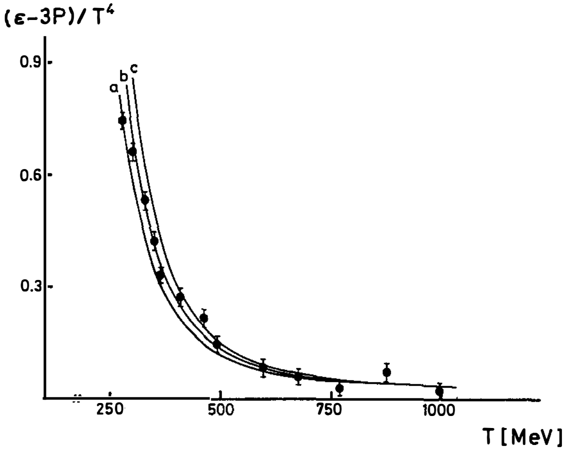
c_V/T^3
8
6
4
2
0
SB
T_c
30
50
100
150
200
300
T/Λ_L

Fig. 3



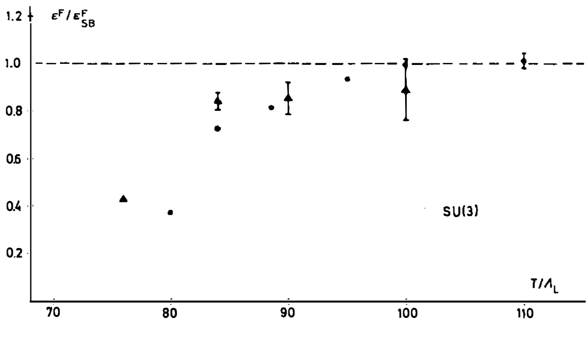
$(\varepsilon - 3P)/T^4$
0.5
0.4
0.3
0.2
0.1
0
m_G
4
4.5
5
0.2
0.4
0.6
0.8
X

Fig. 4



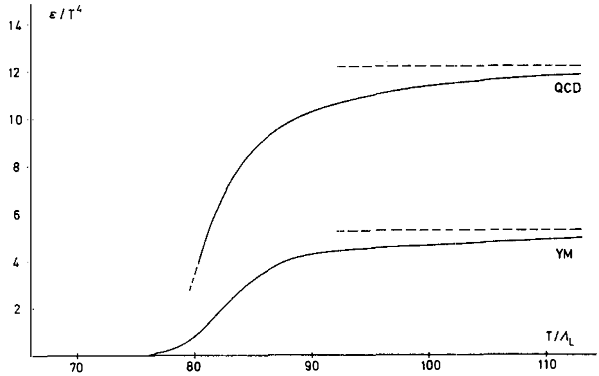
(ε-3P)/T⁴
0.9
0.6
0.3
a
b
c
250
500
750
1000
T [MeV]

Fig. 5



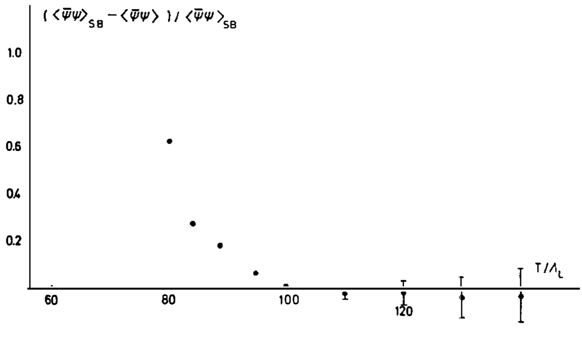
1.2
ε^F / ε^F_SB
1.0
0.8
0.6
0.4
0.2
SU(3)
T/Λ_L
70
80
90
100
110

Fig. 6



ε / T^4
14
12
10
8
6
4
2
QCD
YM
T/Λ_L
70
80
90
100
110

Fig. 7



(⟨ψ̄ψ⟩_SB − ⟨ψ̄ψ⟩) / ⟨ψ̄ψ⟩_SB
1.0
0.8
0.6
0.4
0.2
T/Λ_L
60
80
100
120

Fig. 8

N_β	R
2	1.086
3	0.944
4	0.764
5	0.557

Table 1 : Ratio R of the leading term of the hopping parameter expansion for the energy density of an ideal gas of massless fermions to the exact energy density on an infinite spatial lattice, at several N_β values.

N_β	T/Λ_L	8K	$\langle L \rangle$	$\varepsilon^F/\varepsilon^F_{SB}$
3	80	1.536	0.31	0.374
	84	1.512	0.63	0.726
	89	1.496	0.73	0.815
	95	1.472	0.88	0.935
	100	1.456	0.96	0.988
	110	1.416 - 1.448	1.04	1.01 ± 0.03
	120	1.384 - 1.440	1.08	1.02 ± 0.07
	130	1.360 - 1.440	1.13	1.04 ± 0.10
	140	1.328 - 1.432	1.17	1.03 ± 0.12
4	76	1.456	0.29	0.434
	84	1.416 - 1.448	0.60	0.84 ± 0.04
	90	1.384 - 1.440	0.64	0.85 ± 0.07
	100	1.328 - 1.432	0.73	0.89 ± 0.13

Table 2 : Hopping parameter K , thermal Wilson loop $\langle L \rangle$ and ratio $\varepsilon^F/\varepsilon^F_{SB}$ for SU(3).

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