A STUDY OF COMPACTIFICATION IN SUPERSTRING THEORIES

A THESIS

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BY

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Certificate

This is to certify that the thesis entitled '**A Study of Compactification in Superstring Theories**' submitted by Ms **Sampada Shrivastava** to Dayalbagh Educational Institute (Deemed University), Dayalbagh, Agra, for the award of the degree of **Doctor of Philosophy**, is a record of bona fide research work carried out by her under my supervision and guidance. This work has not been submitted for the award of any other degree.

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Declaration

The work presented in the thesis entitled '**A Study of Compactification in Superstring Theories**' is an original piece of research work carried out by me under the supervision and guidance of **Dr Gunjan Agrawal** at the Department of Mathematics, Faculty of Science, Dayalbagh Educational Institute (Deemed University), Dayalbagh, Agra. This work has not been submitted, either in part or full, for the award of any other degree.

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Preface

The present thesis entitled "A Study of Compactification in Superstring Theories" is an outcome of research work carried out by me at the Department of Mathematics of Dayalbagh Educational Institute (Deemed University), since September 2006, under the supervision and guidance of Dr Gunjan Agrawal, Reader, Department of Mathematics, Faculty of Science, Dayalbagh Educational Institute (Deemed University), Dayalbagh, Agra.

The focus of the present study is the n-dimensional Minkowski space $\mathbb{M}$, an essential component of superstring compactification. The study undertaken revolves around various non-Euclidean topologies on $\mathbb{M}$, namely t, s, f, time and space topologies and deals with its topological properties and compact sets.

The thesis consists of eight chapters together with an Introduction in the beginning and a Bibliography at the end. Chapter 1 comprises notation, some known definitions and results that are used in the subsequent chapters. In Chapter 2, a review of the relevant literature has been presented. Chapter 3 is devoted to the comparison of the non-Euclidean topologies under study. In Chapter 4, topological properties of $\mathbb{M}$ with each of the non-Euclidean topologies are studied. In Chapter 5, the notion of Zeno sequence in $\mathbb{M}$ with each of the non-Euclidean topologies under study, is introduced and rigorously studied. This leads to the characterization of those subsets of $\mathbb{M}$

on which the subspace topologies induced from the Euclidean and a non-Euclidean topology are same. Analogue of Heine-Borel theorem in $\mathbb{M}$ with each of the non-Euclidean topologies under study is obtained in Chapter 6, using the study of Zeno sequences made in Chapter 5. In fact, the technique used here has emerged as a tool to study the compact sets in $\mathbb{M}$ with a non-Euclidean topology. In Chapter 7, several applications of the study carried out in Chapters 5 and 6 can be found. The main highlight is the study of simple connectedness of $\mathbb{M}$ with the non-Euclidean topologies. Finally, Chapter 8, concludes the thesis.

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Acknowledgement

This thesis owes its existence to the help, support, and inspiration of many people. First and foremost, I convey my deep sense of gratitude to my supervisor Dr **Gunjan Agrawal**.

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My appreciation goes to Dr **Amit Kulshrestha** for making me realize the beauty of Mathematics. I sincerely thank him for listening to my problems, mathematical and otherwise, for hours on phone. Word are less to thank my sister Mrs **Nidhi Srivastava**, my brother in law Mr **Amit Srivastava** and my friend Mr **Ankur Prakash** for giving me the much required emotional strength and for solving my computer related problems.

I owe a great deal to my mother Mrs **Abha Shrivastava**, for standing by me in the true sense right from my childhood. Its her hard work alone that has shaped me into what I am today. Last but not the least, I take this opportunity to express my heartfelt gratitude to my father (late) Mr **B. K. Shrivastava** for his enormous love, blessings and guidance.

At last, I acknowledge all those people who have helped me at any stage of my life in any way whatsoever.

December 29, 2009 Sampada Shrivastava

Contents

Introduction

Presenting briefly the background of the subject and motivation for the work undertaken, this part of the thesis provides the organization of the thesis.

0.1 Background

The presence of four forces, namely Electromagnetic, Weak, Strong and Gravitational, is the essence of all the physical phenomena in nature. To find a unified theory encompassing all these forces and the particles which are subjected to these forces is one of the fundamental quest of physicists. The Standard Model describes the behaviour of the particles and their interactions through the strong, weak and electromagnetic forces, but not gravitational, within a single theoretical frame work. String theory provides a promising candidate for a physical model that could accomplish the hunt of unification of all the four forces.

In string theory, there is only one fundamental object namely string which is a 1-dimensional object. Different particles are essentially strings with different modes of vibrations and all the interactions are explained by the splitting and joining of these strings. String theory unifies not only the forces of the standard model in an elegant way, it incorporates gravity in a way which is apparently consistent with quantum mechanics.

Superstring theory is a version of string theory that incorporates the notion of supersymmetry. As of now there are five consistent different superstring theories namely- Type I SO(32), Type IIA, Type IIB, SO(32) Heterotic and $E_8 \times E_8$ Heterotic, each formulated in 10-dimensional spacetime. To obtain a candidate theory explaining our four dimensional universe, it is required to find a solution of one of these theories whose low energy physics is well described by the four dimensional effective field theory, containing the well established Standard Model of particle physics coupled to Einstein's general relativity.

The standard paradigm for finding such solutions is compactification which refers to the compactification in Physics and not the compactification in Mathematics. Indeed, it is the mechanism of obtaining four dimensional physics at low energies out of a D-dimensional theory ($D > 4$) by postulating that the D-dimensional manifold is the product space $\mathbb{M} \times K$, where $\mathbb{M}$ is the 4-dimensional Minkowski space and K is a $(D-4)$-dimensional compact manifold known as compactification manifold or internal manifold. Informally speaking, compactification in physics means curling up of extra dimensions into a compact manifold of size of the order of 10^{-33} cm which cannot be detected with the probes available to us. For a detailed study on compactification in superstring theories, we refer to [7, 13, 20].

While studying compactification in superstring theories, it has been observed that the effective theory in Minkowski space depends on the geometrical and topological properties of the internal manifold. In Kaluja-Klein compactification models, detailed properties of elementary particles are determined by the structure of core compact spaces. The study of the fundamental group of some internal manifolds, has led to a rank five low energy gauge group [1, 7, 8].

0.2 Motivation

The 4-dimensional Minkowski space, an essential component of compactification in superstring theories, is the mathematical framework to formulate Einstein's special theory of relativity and is regarded as the spacetime construct of special relativity. Motivated by this physical relevance of Minkowski space together with the significant impact of topological structure of internal manifolds on the study of compactification in superstring theories, the present thesis focuses on a topological study of the n-dimensional Minkowski space.

To carry out a topological study on a structure, the first step is to look for a suitable topology. In case of the n-dimensional Minkowski space $\mathbb{M}$, the most natural topology is the Euclidean topology. However, motivation for defining non-Euclidean topologies on $\mathbb{M}$ comes from the facts that (i) the Euclidean topology does not take into account the causal structure of Minkowski space and (ii) the homeomorphism group of Minkowski space with Euclidean topology is too large to be of any physical significance. Some of the non-Euclidean topologies of interest on Minkowski space include the names of t, s, f, time space and fine topologies which were originally defined by E. C. Zeeman [22] for the 4-dimensional case. The present thesis revolves around a detailed study of these non-Euclidean topologies.

0.3 Relevance in the Context of Superstring Compactification

In recent years topology has firmly established itself as an important part of the physicist's mathematical arsenal. The significance of topology has increased further with the development of string theory. Ever since the "first superstring revolution" and the compactification of the heterotic string on Calabi-Yau manifolds, interaction of Physics with Mathematics has been one of the primary forces driving progress in superstring theory. It is remarkable to note that, the cohomology groups of a internal manifold probe important fundamental information about its geometrical structure and play a central role in the physical analysis concerning string theory. Also, the effective theory observed in the Minkowski's space depends on the geometrical and topological properties of the internal manifold [7, 13, 20]. Because of this power of topology, study of the n-dimensional Minkowski space with various non-Euclidean topologies, the subject matter of the present thesis, forms a part of the study of string compactification.

0.4 Objectives

In the present work, a comprehensive study of the n-dimensional Minkowski space with the non-Euclidean t, s, f, time and space topologies incorporating the causal structure of spacetime, is undertaken. The study is aimed at the following:

(i) Comparison of Euclidean, t, s, f, time, space and fine topologies.

(ii) Investigation of various topological properties, namely separation axioms, countability axioms, separability, path connectedness, compactness, local compactness, metrizability, Lindelöfness etc.

(iii) Characterization of compact sets.

(iv) Study of the fundamental group.

0.5 Techniques

In this section, techniques of the proofs of the main results in the thesis along with their cruxes have been described. The symbol $\mathbb{M}$ stands for the n-dimensional Minkowski space.

Topological Properties: It is well known that the Euclidean n-space is path connected because for any two distinct points in it, the straight line joining them is a path. However, when $\mathbb{M}$ is considered with the non-Euclidean topologies undertaken in the present study, which are finer than the Euclidean topology, this straight line path need to be checked for continuity. It has been found that in $\mathbb{M}$ with t topology, it is continuous if the difference

$x - y$ of the two given points x and $y \in \mathbb{M}$ is a timelike vector. Otherwise, a third point $z \in \mathbb{M}$ is chosen so that $x - z$ and $z - y$ are timelike vectors. Similar technique is used to prove the path connectedness of $\mathbb{M}$ with the other non-Euclidean topologies.

To obtain the separability of $\mathbb{M}$ with non-Euclidean topologies, it is proved that $\mathbb{K}^n$, where $\mathbb{K}$ is the set of rationals, is dense in the corresponding space by showing that each of its nonempty open set intersects with $\mathbb{K}^n$. However, it is worthwhile mentioning here that for $\mathbb{M}$ with time and space topologies, it is proved in several steps and the technique used is completely different from the one used in the Euclidean n-space. For the other non-Euclidean topologies the result is proved using the fact that $\mathbb{K}^n$ is dense in $\mathbb{M}$ with the Euclidean topology.

The other topological properties are proved or disproved by using respective definitions or method of contradiction.

Compact Sets: The celebrated Heine-Borel theorem that characterizes the compact sets of the Euclidean n-space gives no clue about the compact sets of $\mathbb{M}$ with the non-Euclidean topologies under study except that they will be closed and bounded in the Euclidean n-space. To characterize the compact sets in these spaces a new line of thought, namely the notion of Zeno sequence in Minkowski space with each of the non-Euclidean topologies, is introduced and developed. The characterization is then obtained in terms of this new notion of Zeno sequences.

Subsets Having Same Subspace Topologies: For a nonempty subset

A of $\mathbb{M}$, the subspace topologies induced on A from the non-Euclidean topologies are in general finer than the Euclidean topology because the non-Euclidean topologies under present study are all finer than the Euclidean topology. The reverse containment is proved to be true for certain sets by using the notion of Zeno sequence and the method of contradiction.

Simple Connectedness: It is well-known that the Euclidean 2-space is simply connected, for it is path connected and any two of its loops are path homotopic by straight line path homtopy. However, the map defined by straight line path homotopy may not be continuous, if the topology under consideration is a non-Euclidean topology. In the present work, two loops have been constructed in the 2-dimensional Minkowski space with each of the non-Euclidean topologies under study and it has been found using the study of Zeno sequences and compact sets, that indeed no path homotopy exists between them, thus proving the spaces to be non-simply connected. Further, the n-dimensional Minkowski space with the t and time topologies is proved to be non-simply connected by exploiting the 2-dimensional case and using the theory of retracts.

0.6 Chapter-wise Description

The thesis begins with an 'Introduction' followed by Chapters 1 and 2 entitled 'Notation and Preliminaries' and 'Survey of the Relevant Literature', respectively, providing foundation for the thesis.

Chapter 3, entitled 'Comparison of Topologies', explores the relationship between the non-Euclidean t, s, f, time, space and fine topologies on the n-dimensional Minkowski space $\mathbb{M}$ by meticulously studying the mathematics of some of its fundamental subsets like cones, hyperplanes, straight lines etc. It is proved that the time topology is strictly finer than the t and fine topologies, the space topology is strictly finer than the s and fine topologies, the fine topology is strictly finer than the f topology, the t topology is strictly finer than the f topology and the s topology is strictly finer than the f topology while the t, s and fine topologies are non-comparable and the time and space topologies are also non-comparable.

Topological properties of $\mathbb{M}$ with each of the t, s, f, time and space topologies have been dealt with in Chapter 4 entitled 'Topological Properties', which have been further studied in a later chapter after characterizing compact sets in these spaces in Chapter 6. It is proved that each of these spaces is a path connected, separable, non-regular, non-locally compact, non-Lindelöf, non-second countable space with the exception of first countability: $\mathbb{M}$ with t or s or f topology is proved to be first countable while with time or space is not.

In Chapter 5, entitled 'Zeno Sequence', the concept of Zeno sequence is introduced in $\mathbb{M}$ with each of the non-Euclidean topologies under study, which was originally defined by Zeeman [22] in 4-dimensional Minkowski space with fine topology to study the homeomorphism group of it. Relevant examples have been explored and it is proved that a Zeno sequence in $\mathbb{M}$ with t topology admits a subsequence whose image is closed in it but not in Euclidean n-space. Those subsets of $\mathbb{M}$, that have the same subspace topologies as induced from the Euclidean and t topologies have been characterized. Further, necessary condition for a set to be open in $\mathbb{M}$ with t topology is obtained, besides many other results. Analogous results are obtained for s and f topologies using similar techniques with suitable changes as used in the corresponding results for t topology. Further, this study has been carried out for time and space topologies as well. However the techniques employed this time differ a lot from that of the ones in t topology. This study leads to important contributions in the succeeding chapters.

An analogue of the well known Heine-Borel theorem in $\mathbb{M}$ with each of the non-Euclidean topologies has been obtained in Chapter 6 which is entitled 'Compact Sets' by proving that a subset of $\mathbb{M}$ is a compact subspace of $\mathbb{M}$ with t topology or s topology or f topology or time topology or space topology if and only if it is a compact subspace of Euclidean n-space and does not contain completed image of any Zeno Sequence in the corresponding space. There by, it is shown that I^n, the unit n-cube, which is compact in Euclidean n-space is not compact in $\mathbb{M}$ with any of the non-Euclidean topologies under study.

The study carried out in Chapters 5 and 6 is used in Chapter 7 entitled 'Applications' to study the continuity of maps and simple connectedness of $\mathbb{M}$ with the non-Euclidean topologies under study. It is proved that the 2-dimensional Minkowski space with each of the t, s, f, time and space topologies has a non-trivial fundamental group and is not simply connected. Further, the case $n = 2$, has been exploited to prove the non-simple connectedness of $\mathbb{M}$ with t and time topologies, for $n > 2$.

Finally, Chapter 8 entitled 'Conclusion' concludes the thesis. Thereafter a list of research papers and books referred to during the course of the present work is provided in the Bibliography.

Note: Detailed proofs of the results obtained have been provided. However, if the technique is same to prove a particular result for different topologies, then only suitable changes have been mentioned and detailed proofs are skipped to avoid complexity of the presentation, although an independent visualization is required for each topology.

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Chapter 1

Notation and Preliminaries

In this chapter, some basic notation, known definitions and results which are used in the succeeding chapters are provided.

1.1 Basic Notation

Throughout, $\mathbb{R}$, $\mathbb{K}$ and $\mathbb{N}$ denote the set of reals, the set of rationals and the set of naturals respectively. For $n \in \mathbb{N}$, $\mathbb{R}^n$ denotes the n-dimensional real vector space. For x, y in $\mathbb{R}^n$, $d(x, y)$ denotes the Euclidean distance between x and y and for $\epsilon > 0$, $N_\epsilon^E(x)$ denotes the Euclidean neighbourhood of radius ϵ centered at x given by the set $\{y \in \mathbb{R}^n : d(x, y) < \epsilon\}$. For x, y in $\mathbb{R}^n$, the map $\gamma : [0, 1] \longrightarrow \mathbb{R}^n$ defined by $\gamma(t) = (1 - t)x + ty, t \in [0, 1]$, is denoted by γ_{xy}. The image of it, which is the line segment joining x and y, is denoted by $[x, y]$. For $A \subseteq X$, the complement of A in X is denoted by $X - A$.

1.2 Minkowski Space

For $n \in \mathbb{N}$ and $n > 1$, $\mathbb{R}^n$ with the bilinear form $g : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, satisfying the following properties:

(i) the bilinear form is symmetric, i.e., for all $x, y \in \mathbb{R}^n, g(x,y) = g(y,x)$

(ii) the bilinear form is non-degenerate, i.e, if for all $y \in \mathbb{R}^n, g(x,y) = 0$, then $x = 0$, and

(iii)the bilinear form is of index one, i.e., there exists a basis $\{e_0, e_1, \ldots, e_{n-1}\}$ for $\mathbb{R}^n$ with

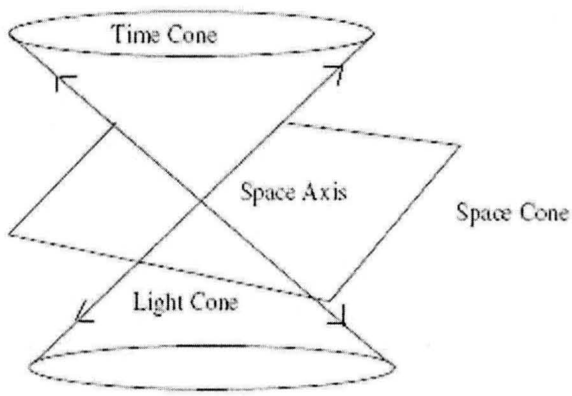
$$g(e_i, e_j) \equiv \eta_{ij} = \begin{cases} 1 & \text{if } i = j = 0 \\ -1 & \text{if } i = j = 1, \ldots, n-1 \\ 0 & \text{if } i \neq j \end{cases}$$

is called the *n-dimensional Minkowski space*, henceforth denoted by $\mathbb{M}$. The bilinear form g is called the *Lorentz inner product* and the matrix $(\eta_{ij})_{n \times n}$ is known as the *Minkowski metric.*

Elements of $\mathbb{M}$ are called *events.* For an event, $x \equiv \sum_{i=0}^{n-1} x^i e_i$, the coordinate x^0 is called the *time component* and the coordinates $x^1, \ldots, x^{n-1}$ are called the *spatial components* of x relative to the basis $\{e_0, e_1, \ldots, e_{n-1}\}$. In terms of components, the Lorentz inner product $g(x,y)$ of two events $x \equiv \sum_{i=0}^{n-1} x^i e_i$ and $y \equiv \sum_{i=0}^{n-1} y^i e_i$ is given by $x^0 y^0 - \sum_{i=1}^{n-1} x^i y^i$. The Lorentz inner product induces an indefinite characteristic quadratic form Q on $\mathbb{M}$ defined by $Q(x) = g(x,x)$. The group of all linear operators T on $\mathbb{M}$ which leave the quadratic form Q invariant, i.e., $Q(x) = Q(T(x))$, for all $x \in \mathbb{M}$, is called the *Lorentz group.* For more details, we refer to [10, 15].

An event $x \in \mathbb{M}$ is called *timelike, lightlike* (also called *null*) or *space-*

like, according as $Q(x)$ is positive, zero or negative. The sets $C^T(x) = \{y \in \mathbb{M} : Q(y-x) > 0\} \cup \{x\}, C^L(x) = \{y \in \mathbb{M} : Q(y-x) = 0\}, C^S(x) = \{y \in \mathbb{M} : Q(y-x) < 0\} \cup \{x\}$ are likewise respectively called the *time cone*, *light cone* (also called *null cone*) and *space cone* at x. A straight line is called *timelike straight line* or *light ray* or *spacelike straight line* according as it is parallel to a timelike or lightlike or spacelike vector. A hyperplane of $\mathbb{M}$ is called *spacelike* if each of its nonzero element is a spacelike vector. *Space axes* refer to the spacelike hyperplanes together with their translates. For further details, we refer to [15, 19, 22].



3-Dimensional Minkowski Space

1.3 Topologies on Minkowski Space

In this section, we define those non-Euclidean topologies on the n-dimensional Minkowski space $\mathbb{M}$ that are undertaken in the present work: these topologies were originally defined by Zeeman [22] on the 4-dimensional Minkowski space and continue to be studied by various researchers till date.

Definition 1.3.1. The *Euclidean topology* on the n-dimensional Minkowski space $\mathbb{M}$ is defined to be the topology generated by the basis $\{N_\epsilon^E(x) : \epsilon > 0, x \in \mathbb{M}\}$. Henceforth, $\mathbb{M}$ with Euclidean topology will be denoted by $\mathbb{M}^E$. For $A \subseteq \mathbb{M}$, the subspace A of $\mathbb{M}^E$ will be denoted by A^E.

Definition 1.3.2. The t *topology* on the n-dimensional Minkowski space $\mathbb{M}$ is defined to be the topology generated by the basis $\{N_\epsilon^E(x) \cap C^T(x) : \epsilon > 0, x \in \mathbb{M}\}$. Now onwards, $\mathbb{M}$ with t topology will be denoted by $\mathbb{M}^t$. Further for $x \in \mathbb{M}$, the set $N_\epsilon^E(x) \cap C^T(x)$ will be denoted by $N_\epsilon^t(x)$ and will be called the t neighbourhood of radius ϵ centered at x. For $A \subseteq \mathbb{M}$, the subspace A of $\mathbb{M}^t$ will be denoted by A^t.

Definition 1.3.3. The s *topology* on the n-dimensional Minkowski space $\mathbb{M}$ is defined to be the topology generated by the basis $\{N_\epsilon^E(x) \cap C^S(x) : \epsilon > 0, x \in \mathbb{M}\}$. Henceforward, $\mathbb{M}$ with s topology will be denoted by $\mathbb{M}^s$. For $x \in \mathbb{M}$, the set $N_\epsilon^E(x) \cap C^S(x)$ will be denoted by $N_\epsilon^s(x)$ and will be called the s neighbourhood of radius ϵ centered at x. For $A \subseteq \mathbb{M}$, the subspace A of $\mathbb{M}^s$ will be denoted by A^s.

Definition 1.3.4. The f *topology* on the n-dimensional Minkowski space $\mathbb{M}$ is defined to be the topology generated by the basis $\{N_\epsilon^E(x) \cap (C^S(x) \cup C^T(x)) : \epsilon > 0, x \in \mathbb{M}\}$. From now on, $\mathbb{M}$ with f topology will be denoted by $\mathbb{M}^f$. For $x \in \mathbb{M}$, the set $N_\epsilon^E(x) \cap (C^S(x) \cup C^T(x))$ will be denoted by

$N^f_\epsilon(x)$ and will be called the f neighbourhood of radius ϵ centered at x. For $A \subseteq \mathbb{M}$, the subspace A of $\mathbb{M}^f$ will be denoted by A^f.

Definition 1.3.5. The *time topology* on the n-dimensional Minkowski space $\mathbb{M}$ is defined to be the finest topology on $\mathbb{M}$ that induces 1-dimensional Euclidean topology on every timelike straight line. Henceforth, $\mathbb{M}$ with time topology will be denoted by $\mathbb{M}^T$. For $A \subseteq \mathbb{M}$, the subspace A of $\mathbb{M}^T$ will be denoted by A^T.

Definition 1.3.6. The *space topology* on the n-dimensional Minkowski space $\mathbb{M}$ is defined to be the finest topology on $\mathbb{M}$ that induces $(n-1)$-dimensional Euclidean topology on every space axis. Now onwards, $\mathbb{M}$ with space topology will be denoted by $\mathbb{M}^S$. For $A \subseteq \mathbb{M}$, the subspace A of $\mathbb{M}^S$ will be denoted by A^S.

Definition 1.3.7. The *fine topology* on the n-dimensional Minkowski space $\mathbb{M}$ is defined to be the finest topology on $\mathbb{M}$ that induces 1-dimensional Euclidean topology on every timelike straight line and $(n-1)$-dimensional Euclidean topology on every space axis. From now on, $\mathbb{M}$ with fine topology will be denoted by $\mathbb{M}^F$. For $A \subseteq \mathbb{M}$, the subspace A of $\mathbb{M}^F$ will be denoted by A^F.

Remark 1.3.8. Since, for $x \in \mathbb{M}$, $N^t_\epsilon(x)$, $N^s_\epsilon(x)$, $N^f_\epsilon(x)$ are contained in $N^E_\epsilon(x)$, t, s and f topologies on $\mathbb{M}$ are finer than the Euclidean topology on $\mathbb{M}$. Further, the time and space topologies are finer than the Euclidean topology on $\mathbb{M}$, for the Euclidean topology induces 1-dimensional Euclidean topology on every timelike straight line and $(n-1)$-dimensional Euclidean topology on every space axis.

1.4 Topological Properties

Throughout this section, X denotes a topological space.

Separation Axioms: A space X is called a T_1 space if finite points sets are closed in it. X is said to be *Hausdorff* if for each pair $x, y \in X$ of distinct points of X, there exists disjoint open sets containing x and y respectively. X is called *regular* if for each pair consisting of a point x and a closed set A not containing x, there exists disjoint open sets containing x and A, respectively. X is said to be completely regular if for each point and each closed set A not containing x, there is a continuous function $f : X \longrightarrow [0,1]$ such that $f(x) = 1$ and $f(A) = \{0\}$. X is said to be *normal* if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B respectively. A T_1 normal space is completely regular, T_1 completely regular space is regular, T_1 regular space is Hausdorff and Hausdorff space is T_1.

Countability Axioms and Separability: A space X is said to have a countable basis at $x \in X$ if there is a countable collection $\mathcal{B}$ of open sets containing x such that each open set containing x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each of its points is said to be the *first countable.* A space X is called *second countable* if X has a countable basis for its topology. A space having a countable dense subset is said to be *separable.* It may be noted that a second countable space is first countable and separable.

Connectedness: A space X is said to be *connected* if there does not exist disjoint nonempty open subsets of X whose union is X. A space X is called *path connected* if for every pair of points x, y, there exists a continuous map $f : [0,1] \longrightarrow X$ such that $f(0) = x$ and $f(1) = y$. A path connected space is a connected space.

Compactness: A collection $\mathcal{A}$ of subsets of X is said to be *locally finite* in X if every point of X has an open neighbourhood that intersects only finitely many elements of $\mathcal{A}$. A collection $\mathcal{B}$ of subsets of X is said to be a *refinement* of a collection $\mathcal{A}$ of subsets of X if for each element B of $\mathcal{B}$, there is an element A of $\mathcal{A}$ containing B. An open covering of a space X is a collection of its open sets whose union is X. A space X is said to be *compact* if every open covering of X contains a finite subcollection that also covers X. A space X is said to be *Lindelöf* if every open covering contains a countable subcollection that also covers X. A space X is said to be locally compact at $x \in X$ if there is some compact subspace of X that contains a neighbourhood of x. If X is locally compact at each of its points, X is called *locally compact.* A space X is said to be *paracompact* if every open covering $\mathcal{A}$ of X has a locally finite open refinement $\mathcal{B}$ that covers X. A compact Hausdorff or locally compact Hausdorff or paracompact Hausdorff space is regular.

Metrizability: A space X is said to be *metrizable* if there exists a metric on the set X that induces the topology of X.

Locally Euclidean: A space X is said to be *locally m-Euclidean* if for each $x \in X$ there is a neighbourhood of x that is homeomorphic to an open subset of $\mathbb{R}^m$.

Simple Connectedness: Two paths $f, g : [0,1] \longrightarrow X$ are said to be *path homotopic* if they have the same initial point x and the same final point y, and if there is a continuous map $F : [0,1] \times [0,1] \longrightarrow X$ such that

$$F(s,0) = f(s) \text{ and } F(s,1) = g(s),$$

$$F(0,t) = x \text{ and } F(1,t) = y,$$

for each $s \in [0,1]$ and each $t \in [0,1]$. If f is a path in X from x to y and if g is a path in X from y to z, then the product $f * g$ is defined to be the path given by the join of f and g. The set of path homotopy classes of loops based at $x \in X$, with the operation $*$ is called the *fundamental group* of X relative to the base point x. A space is said to be *simply connected* if it is path connected and has trivial fundamental group.

For more topological details, we refer to [3, 11, 14].

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Chapter 2

Survey of the Relevant Literature

The study of topological and differential geometric structures on spacetime models has emerged as a front line area of research during the past few decades. The present thesis is directed towards a detailed topological study of a spacetime model, namely Minkowski space. In this chapter, a brief account of the development of this topic is presented, as found in the literature.

2.1 Review on Topological Study of Minkowski Space

Non-Euclidean topologies on 4-dimensional Minkowski space were first introduced by Zeeman [22] in 1967. These topologies include fine and some other non-Euclidean topologies which were later named as space topology [16], time topology [17], t topology [17], s topology [17] etc. Studying the homeomorphism group of 4-dimensional Minkowski space with fine topology, Zeeman in his paper [22] mentioned that it is Hausdorff, connected, locally connected space that is not normal, not locally compact and not first countable. His results were interesting both topologically and physically; topologically, because Minkowski space with fine topology had a homeomorphism group that one could calculate explicitly and the calculation was quite nontrivial; physically, because that homeomorphism group was the group generated by the Lorentz group, translations and dilatations which was exactly the one physicists would want it to be.

Continuing the study of non-Euclidean topologies, Nanda in his papers [16, 17] mentioned that the 4-dimensional Minkowski space, with the space topology is Hausdorff but neither normal nor locally compact nor second countable and that $\mathbb{M}$ with each of the t topology and s topology is a non-normal, non-compact Hausdorff space besides proving that the 4-dimensional Minkowski space with space, t and s topologies have their homeomorphism group generated by the Lorentz group, translations and dilatations.

Further, Nanda and Panda [18] introduced the notion of a non-Euclidean topology namely order topology and obtained that it is a non-compact, non-Hausdorff, locally connected, connected, path connected, simply connected space.

Quite recently in 2007, Dossena [2] proved that the n-dimensional Minkowski space, $n > 1$, with the fine topology is separable, Hausdorff, non-normal, non-locally compact, non-Lindelöff and non-first countable. He further obtained that 2-dimensional Minkowski space with fine topology is path connected but not simply connected and characterized its compact sets.

Research has also been carried out on the n-dimensional Minkowski space from differential geometric view-point, e.g. Formiga *et al.* [4] established the Serret-Frenet equations in the 4-dimensional Minkowski spacetime and consequently provided a simple proof of the fundamental theorem of curves in Minkowski space.

2.2 Review on Other Spacetime Manifolds

The seminal work of Zeeman [22], in due course of time, inspired several generalizations to curved spacetimes of general relativity with topologies based on the idea of causal structure. Göbel [6] defined and investigated the general relativistic analogue of the Zeeman's fine topology; Hawking [9] *et al.* proposed path topology, which is defined in terms of continuous timelike curves for strongly causal spacetimes and determines the causal, differential and conformal structure of spacetime. Further, the path topology on the 4-dimensional Minkowski space is same as that of the t topology on it [15]. Fullwood [5] suggested a physically elegant topology on spacetime defined solely in terms of causal structure. In 2006, Kim [12], showed that the path topology of Hawking *et al.* can be extended to the causal completion of a globally hyperbolic Lorentzian manifold.

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Chapter 3

Comparison of Topologies

In this chapter, relationship between the non-Euclidean t, s, f, time, space and fine topologies on the n-dimensional Minkowski space has been explored by meticulously studying the mathematics of some of its fundamental subsets like cones, hyperplanes, straight lines etc. with respect to each of these topologies. It is proved that the time topology is strictly finer than the t and fine topologies, the space topology is strictly finer than the s and fine topologies, the fine topology is strictly finer than the f topology and the t and s topologies are strictly finer than the f topology while the t, s and fine topologies are non-comparable and time and space topologies are also non-comparable.

3.1 Open Sets of Minkowski Space with Time and Space Topologies

In this section, equivalent conditions for a set to be open in $\mathbb{M}$ with each of the time and space topologies are obtained.

Proposition 3.1.1. *Let $\mathbb{M}^T$ be the n-dimensional Minkowski space $\mathbb{M}$ with time topology and $G \subseteq \mathbb{M}$. Then G is open in $\mathbb{M}^T$ if and only if $G \cap \tau$ is open in τ^E, for every timelike straight line τ. Consequently, a subset F of $\mathbb{M}$ is closed in $\mathbb{M}^T$ if and only if $F \cap \tau$ is closed in τ^E, for every timelike straight line τ.*

Proof. The "only if" part follows trivially. To prove the "if" part, it is sufficient to notice that if $G \cap \tau$ is open in τ^E, for every timelike straight line τ and G is not open in $\mathbb{M}^T$, then the topology generated by G and the open sets of $\mathbb{M}^T$ is strictly finer than the time topology and induces Euclidean topology on every timelike straight line. This is a contradiction in view of the fact that the time topology is the finest topology that induces Euclidean topology on every timelike straight line. □

Proposition 3.1.2. *Let $\mathbb{M}^S$ be the n-dimensional Minkowski space $\mathbb{M}$ with space topology and $G \subseteq \mathbb{M}$. Then G is open in $\mathbb{M}^S$ if and only if $G \cap \sigma$ is open in σ^E, for every space axis σ. Consequently, a subset F of $\mathbb{M}$ is closed in $\mathbb{M}^S$ if and only if $F \cap \sigma$ is closed in σ^E, for every space axis σ.*

Proof. It can be proved in the same way as that of Proposition 3.1.1, by considering space axes in place of timelike straight lines. □

3.2 Subsets of Minkowski Space

In this section, a detailed study of the important subsets, namely cones, neighbourhoods, straight lines, hyperplanes etc., of the n-dimensional Minkowski space has been carried out. This analysis has been used rigorously in the succeeding section and forthcoming chapters.

In the following proposition, it is proved that the time and space cones based at a point are not open in the Euclidean n-space.

Proposition 3.2.1. *Let $\mathbb{M}^E$ be the n-dimensional Minkowski space $\mathbb{M}$ with Euclidean topology and $x \in \mathbb{M}$. Then $C^T(x)$, $C^S(x)$ and $C^T(x) \cup C^S(x)$ are not open in $\mathbb{M}^E$.*

Proof. For $\epsilon > 0$ and $x = \Sigma_{i=0}^{n-1} x^i e_i$, consider $y \in \mathbb{M}$ such that $y^1 = x^1 + \frac{\epsilon}{2}$ and $y^i = x^i$, for $i \neq 1, 0 \leq i \leq n-1$. Then $y \in N_\epsilon^E(x)$ and $y \notin C^T(x)$. Hence $N_\epsilon^E(x) \nsubseteq C^T(x)$, for any $\epsilon > 0$. This proves that x is not an interior point of $C^T(x)$ in $\mathbb{M}^E$ and therefore $C^T(x)$ is not open in $\mathbb{M}^E$. That $C^S(x)$ and $C^T(x) \cup C^S(x)$ are not open in $\mathbb{M}^E$ can be proved analogously by showing that x is not an interior point of $C^S(x)$and $C^T(x) \cup C^S(x)$ respectively in $\mathbb{M}^E$. □

In the following proposition, it is proved that the time and space cones punctured at their base points are open in the Euclidean n-space while the light cone is closed.

Proposition 3.2.2. *Let $\mathbb{M}^E$ be the n-dimensional Minkowski space $\mathbb{M}$ with Euclidean topology and $x \in \mathbb{M}$. Then $C^T(x) - \{x\}$, $C^S(x) - \{x\}$ and $(C^T(x) \cup C^S(x)) - \{x\}$ are open in $\mathbb{M}^E$ while $C^L(x)$ is closed in $\mathbb{M}^E$.*

Proof. Let the map $f : \mathbb{M}^E \longrightarrow \mathbb{R}$ be defined by $f(u) = (u^0 - x^0)^2 - \sum_{i=1}^{n-1}(u^i - x^i)^2$, for $u \in \mathbb{M}$. Then f being the product and sum of real valued continuous functions is continuous. The result now follows by noting that $f^{-1}(0,\infty) = C^T(x) - \{x\}, f^{-1}(-\infty, 0) = C^S(x) - \{x\}$, $f^{-1}(\mathbb{R} - \{0\}) = (C^T(x) \cup C^S(x)) - \{x\}$ and $f^{-1}\{0\} = C^L(x)$. □

The following three propositions discuss the openness of time and space cones in the n-dimensional Minkowski space with each of the t, s and f topologies.

Proposition 3.2.3. *Let $\mathbb{M}^t$ be the n-dimensional Minkowski space $\mathbb{M}$ with t topology and $x \in \mathbb{M}$. Then $C^T(x)$ is open in $\mathbb{M}^t$ while $C^S(x)$ is not.*

Proof. Let $y \in C^T(x) - \{x\}$. Then by Proposition 3.2.2, y is an interior point of $C^T(x) - \{x\}$ in $\mathbb{M}^E$ and hence in $\mathbb{M}^t$. Further, for $\epsilon > 0$, $N^t_\epsilon(x) \subseteq C^T(x)$. This implies that each point of $C^T(x)$ is an interior point of it in $\mathbb{M}^t$. Hence $C^T(x)$ is open in $\mathbb{M}^t$. That $C^S(x)$ is not open in $\mathbb{M}^t$ follows by noting that $N^t_\epsilon(x)$, for any $\epsilon > 0$, being not contained in $C^S(x)$, x is not an interior point of $C^S(x)$ in $\mathbb{M}^t$. □

Proposition 3.2.4. *Let $\mathbb{M}^s$ be the n-dimensional Minkowski space $\mathbb{M}$ with s topology and $x \in \mathbb{M}$. Then $C^S(x)$ is open in $\mathbb{M}^s$ while $C^T(x)$ is not.*

Proof. It can be proved in the same way as that of Proposition 3.2.3 by considering s neighbourhood in place of t neighbourhood in the proof. □

Proposition 3.2.5. *Let $\mathbb{M}^f$ be the n-dimensional Minkowski space $\mathbb{M}$ with f topology and $x \in \mathbb{M}$. Then $C^T(x) \cup C^S(x)$ is open in $\mathbb{M}^f$ while $C^T(x)$ and $C^S(x)$ are not.*

Proof. It can be proved in the same way as that of Proposition 3.2.3 by considering f neighbourhood in place of t neighbourhood in the proof. $\square$

In the following two propositions, it is proved that the t, s and f neighbourhoods are not open in $\mathbb{M}^E$ while these neighbourhoods punctured at their base points are open in $\mathbb{M}^E$.

Proposition 3.2.6. *Let $\mathbb{M}^E$ be the n-dimensional Minkowski space $\mathbb{M}$ with Euclidean topology and $x \in \mathbb{M}$. Then none of $N_\epsilon^t(x)$, $N_\epsilon^s(x)$ and $N_\epsilon^f(x)$ is open in $\mathbb{M}^E$.*

Proof. The result follows by noting that x is not an interior point of any of $N_\epsilon^t(x)$, $N_\epsilon^s(x)$ and $N_\epsilon^f(x)$ in $\mathbb{M}^E$ which can be proved in the same way as it has been proved in Proposition 3.2.1 that x is not an interior point of $C^T(x)$. $\square$

Proposition 3.2.7. *Let $\mathbb{M}^E$ be the n-dimensional Minkowski space $\mathbb{M}$ with Euclidean topology and $x \in \mathbb{M}$. Then for $\epsilon > 0$, $N_\epsilon^t(x) - \{x\}$, $N_\epsilon^s(x) - \{x\}$ and $N_\epsilon^f(x) - \{x\}$ are open in $\mathbb{M}^E$.*

Proof. Since $N_\epsilon^t(x) - \{x\} = N_\epsilon^E(x) \cap (C^T(x) - \{x\})$ and $N_\epsilon^s(x) - \{x\} = N_\epsilon^E(x) \cap (C^S(x) - \{x\})$, $N_\epsilon^f(x) - \{x\} = N_\epsilon^E(x) \cap [(C^T(x) \cup C^S(x)) - \{x\}]$, the result follows from Proposition 3.2.2. $\square$

It is mentioned without proof in [17] that the t topology on the 4-dimensional Minkowski space induces Euclidean topology on a timelike straight line and discrete topology on a light ray and a space axis. In the following two propositions, these results are proved for the n-dimensional case.

Proposition 3.2.8. *The t topology on the n-dimensional Minkowski space $\mathbb{M}$ induces 1-dimensional Euclidean topology on a timelike straight line.*

Proof. Let τ be a timelike straight line. Since, for $x \in \mathbb{M}$, $N_\epsilon^t(x) \subseteq N_\epsilon^E(x)$, it follows that the topology on τ^E is coarser than the topology on τ^t. Hence it is sufficient to prove that $N_\epsilon^t(x) \cap \tau$ is open in τ^E. Since

$$N_\epsilon^t(x) \cap \tau = \begin{cases} N_\epsilon^E(x) \cap \tau & \text{if } x \in \tau \\ (N_\epsilon^t(x) - \{x\}) \cap \tau & \text{if } x \notin \tau, \end{cases}$$

the assertion follows from Proposition 3.2.7. □

Proposition 3.2.9. *The t topology on the n-dimensional Minkowski space $\mathbb{M}$ induces discrete topology on a light ray and on a space axis.*

Proof. Let λ be a light ray or a space axis and $p \in \lambda$. Then for $\epsilon > 0$, $N_\epsilon^t(p) \cap \lambda = \{p\}$. This proves the result. □

It is mentioned without proof in [17] that the s topology on the 4-dimensional Minkowski space induces Euclidean topology on a space axis and discrete topology on a light ray and a timelike straight line. In the following two propositions, these results are proved for the n-dimensional case.

Proposition 3.2.10. *The s topology on the n-dimensional Minkowski space $\mathbb{M}$ induces 3-dimensional Euclidean topology on a space axis.*

Proof. In view of Proposition 3.2.7, the result can be proved in the same way as that of Proposition 3.2.8, by considering s neighbourhood in place of t neighbourhood. □

Proposition 3.2.11. *The s topology on the n-dimensional Minkowski space $\mathbb{M}$ induces discrete topology on a timelike straight line and on a light ray.*

Proof. It can be proved in the same way as that of Proposition 3.2.9, by considering s neighbourhood in place of t neighbourhood. $\square$

The following two propositions discuss the subspace topologies induced on timelike straight lines, light rays and spacelike hyperplanes from f topology.

Proposition 3.2.12. *Let $\mathbb{M}$ be the n-dimensional Minkowski space. Then the f topology on $\mathbb{M}$ induces Euclidean topology on a timelike straight line and on a space axis.*

Proof. In view of Proposition 3.2.7, the result can be proved in the same way as that of Proposition 3.2.8, by considering f neighbourhood in place of t neighbourhood. $\square$

Proposition 3.2.13. *The f topology on the n-dimensional Minkowski space $\mathbb{M}$ induces discrete topology on a light ray.*

Proof. It follows in the same way as that of Proposition 3.2.9, by considering f neighbourhood in place of t neighbourhood. $\square$

3.3 Relationship between Topologies

In this section, a comparison of various topologies on $\mathbb{M}$, undertaken in the present work, has been carried out.

Proposition 3.3.1. *Let $\mathbb{M}$ be the n-dimensional Minkowski space. Then the t topology on $\mathbb{M}$ is strictly finer than the Euclidean topology on $\mathbb{M}$.*

Proof. For $x \in \mathbb{M}$, $N_\epsilon^t(x) \subseteq N_\epsilon^E(x)$, hence the t topology on $\mathbb{M}$ is finer than the Euclidean topology. Further, from Propositions 3.2.1 and 3.2.3, it follows that the time cone based at a point is open in $\mathbb{M}$ with t topology but not in $\mathbb{M}$ with Euclidean topology. This proves the result. □

Proposition 3.3.2. *Let $\mathbb{M}$ be the n-dimensional Minkowski space. Then the s topology on $\mathbb{M}$ is strictly finer than the Euclidean topology on $\mathbb{M}$.*

Proof. It follows from Propositions 3.2.1 and 3.2.4 that the space cone based at a point is open in $\mathbb{M}$ with s topology but not in $\mathbb{M}$ with Euclidean topology. Since for $x \in \mathbb{M}$, $N_\epsilon^s(x) \subseteq N_\epsilon^E(x)$, s topology is finer than the Euclidean topology. Hence the result. □

Proposition 3.3.3. *Let $\mathbb{M}$ be the n-dimensional Minkowski space. Then the f topology on $\mathbb{M}$ is strictly finer than the Euclidean topology on $\mathbb{M}$.*

Proof. Let $x \in \mathbb{M}$. Then $N_\epsilon^f(x) \subseteq N_\epsilon^E(x)$. Hence f topology is finer than the Euclidean topology. From Propositions 3.2.1 and 3.2.5, the union of time and space cones based at a point is open in $\mathbb{M}$ with f topology but not in $\mathbb{M}$ with Euclidean topology. This proves the result. □

Proposition 3.3.4. *Let $\mathbb{M}$ be the n-dimensional Minkowski space. Then the t topology on $\mathbb{M}$ is strictly finer than the f topology on $\mathbb{M}$.*

Proof. For $x \in \mathbb{M}$, since $N_\epsilon^t(x) \subseteq N_\epsilon^f(x)$, it follows that the t topology on $\mathbb{M}$ is finer than the f topology. Now from Propositions 3.2.3 and 3.2.5, $C^T(x)$ is open in $\mathbb{M}$ with t topology but not in $\mathbb{M}$ with f topology. This proves the result. □

Proposition 3.3.5. *Let $\mathbb{M}$ be the n-dimensional Minkowski space. Then the s topology on $\mathbb{M}$ is strictly finer than the f topology on $\mathbb{M}$.*

Proof. From Propositions 3.2.4 and 3.2.5, space cone based at a point is open in $\mathbb{M}$ with s topology but not in $\mathbb{M}$ with f topology. The result now follows by noting that for $x \in \mathbb{M}$, $N_\epsilon^s(x) \subseteq N_\epsilon^f(x)$. $\square$

Proposition 3.3.6. *Let $\mathbb{M}$ be the n-dimensional Minkowski space. Then the t and s topologies on $\mathbb{M}$ are non-comparable.*

Proof. From Propositions 3.2.3 and 3.2.4, the time cone based at a point is open in $\mathbb{M}$ with t topology but not in $\mathbb{M}$ with s topology and the space cone based at a point is open in $\mathbb{M}$ with s topology but not in $\mathbb{M}$ with t topology. This proves the result. $\square$

It is proved in [17] that the time topology on the 4-dimensional Minkowski space is strictly finer than the t topology. In the following proposition, this results is proved for the n-dimensional case.

Proposition 3.3.7. *The time topology on the n-dimensional Minkowski space $\mathbb{M}$ is strictly finer than the t topology on $\mathbb{M}$. Consequently, time topology is strictly finer than the Euclidean topology.*

Proof. From Proposition 3.2.8, the t topology induces Euclidean topology on every timelike straight line. Thus time topology is finer than the t topology on $\mathbb{M}$. To see that it is infact strictly finer, consider $z \in \mathbb{M}$ and a sequence of distinct timelike straight lines $\{\tau_k : k \in \mathbb{N}\}$ passing through z. For $k \in \mathbb{N}$, choose $z_k \in \tau_k$ such that $0 < d(z_k, z) < 1/k$. Then $(z_k)_{k\in\mathbb{N}}$ converges to z in $\mathbb{M}^E$. We assert that $Z \cap \tau$, where $Z = \{z_k : k \in \mathbb{N}\}$ and τ is a timelike straight line, is finite. To prove the assertion, let $Z \cap \tau$ be infinite, for some timelike straight line τ . Then the sequence of infinite terms in $Z \cap \tau$ converges in τ^E, since τ^E is complete. Therefore $z \in \tau$. By construction, for any timelike straight line τ passing through z, $Z \cap \tau$ is atmost a singleton. This contradiction proves that Z meets any timelike straight line in finitely many points. Hence from Proposition 3.1.1, Z is closed in $\mathbb{M}^T$. Further, since z is a limit point of Z in $\mathbb{M}^t$ and $z \notin Z$, Z is not closed in $\mathbb{M}^t$. This completes the proof. □

Corollary 3.3.8. *The time topology on the n-dimensional Minkowski space $\mathbb{M}$ induces discrete topology on a light ray and on a space axis.*

Proof. Follows from Propositions 3.2.9 and 3.3.7. □

It is proved in [16] that the space topology on the 4-dimensional Minkowski space is strictly finer than the s topology. In the following proposition, this results is proved for the n-dimensional case.

Proposition 3.3.9. *The space topology on the n-dimensional Minkowski space $\mathbb{M}$ is strictly finer than the s topology. Consequently, space topology is strictly finer than the Euclidean topology.*

Proof. It follows from Proposition 3.2.10 that the s topology induces Euclidean topology on every space axis. Hence space topology is finer than the s topology on $\mathbb{M}$. Consider now $z \in \mathbb{M}$ and a sequence of distinct spacelike straight lines $\{\sigma_k : k \in \mathbb{N}\}$ passing through z such that any space axis contains only finitely many σ_k's. For $k \in \mathbb{N}$, choose $z_k \in \sigma_k$ such that $0 < d(z_k, z) < 1/k$. Then $Z \equiv \{z_k : k \in \mathbb{N}\}$ can be proved to be closed in $\mathbb{M}^S$

and not in $\mathbb{M}^s$ from the same argument as used in Proposition 3.3.7. This completes the proof. □

Corollary 3.3.10. *The space topology on the n-dimensional Minkowski space $\mathbb{M}$ induces discrete topology on a light ray and on a timelike straight line.*

Proof. Follows from Propositions 3.2.11 and 3.3.9. □

The following is a characterization of open sets in $\mathbb{M}$ with fine topology obtained in [2].

Lemma 3.3.11. *[2]. Let $\mathbb{M}^F$ be the n-dimensional Minkowski space with fine topology and $G \subseteq \mathbb{M}$. Then G is open in $\mathbb{M}^F$ if and only if $G \cap \Omega$ is open in Ω^E, for every Ω, where Ω denotes a timelike straight line or space axis.*

Proposition 3.3.12. *The fine topology on the n-dimensional Minkowski space $\mathbb{M}$ is strictly finer than the f topology. Consequently, fine topology is strictly finer than the Euclidean topology.*

Proof. From Proposition 3.2.12, f topology induces Euclidean topology on every timelike straight line and space axis. Since fine topology is the finest such topology, fine topology is finer than the f topology on $\mathbb{M}$. To prove that it is strictly finer, consider $z \in \mathbb{M}$ and a sequence of distinct timelike straight lines $\{\tau_k : k \in \mathbb{N}\}$ passing through z. For $k \in \mathbb{N}$, choose $z_k \in \tau_k$ such that $0 < d(z_k, z) < 1/k$. Then in view of Lemma 3.3.11, $Z \equiv \{z_k : k \in \mathbb{N}\}$ can be proved to be closed in $\mathbb{M}^F$ and not in $\mathbb{M}^f$ in the same way as shown in Proposition 3.3.7. Hence the result. □

Proposition 3.3.13. *The time topology on the n-dimensional Minkowski space $\mathbb{M}$ is strictly finer than the fine topology.*

Proof. It follows from the definitions of time and fine topologies that the time topology on $\mathbb{M}$ is finer than the fine topology on it. Now for $x \in \mathbb{M}$, $C^T(x)$ is open in $\mathbb{M}^t$ and hence in $\mathbb{M}^T$. The result now follows, in view of Lemma 3.3.11, by noting that $C^T(x)$ is not open in $\mathbb{M}^F$, for $C^T(x) \cap H = \{x\}$ is not open in H^E, where H is a space axis passing through $x \in \mathbb{M}$. $\square$

Proposition 3.3.14. *The space topology on the n-dimensional Minkowski space $\mathbb{M}$ is strictly finer than the fine topology.*

Proof. From the definitions of space and fine topologies, it follows that the space topology on $\mathbb{M}$ is finer than the fine topology on it. The remaining part can now be proved in a way similar to that of Proposition 3.3.13, by considering space cone and timelike straight line in place of time cone and space axis respectively, in its proof. $\square$

Proposition 3.3.15. *The space and time topologies on the n-dimensional Minkowski space $\mathbb{M}$ are non-comparable.*

Proof. Let $x \in \mathbb{M}$. Then by respective definitions, $C^T(x)$ is open in $\mathbb{M}^T$ but not in $\mathbb{M}^S$ while $C^S(x)$ is open in $\mathbb{M}^S$ but not in $\mathbb{M}^T$. This completes the proof. $\square$

Proposition 3.3.16. *The t and fine topologies on the n-dimensional Minkowski space $\mathbb{M}$ are non-comparable.*

Proof. Let $z \in \mathbb{M}$ and $Z \equiv \{z_k : k \in \mathbb{N}\}$ be as in the proof of Proposition 3.3.12. Then $\mathbb{M} - Z$ is open in $\mathbb{M}^F$. That it is not open in $\mathbb{M}^t$ follows by noting that z is not an interior point of $\mathbb{M} - Z$ in $\mathbb{M}^t$. Further, for $x \in \mathbb{M}$, $C^T(x) \cap H = \{x\}$, where H is a space axis passing through x. Since $\{x\}$ is not open in H^E, in view of Lemma 3.3.11, $C^T(x)$ is not open in $\mathbb{M}^F$. Since $C^T(x)$ is open in $\mathbb{M}^t$, the result follows. □

Proposition 3.3.17. *The s and fine topologies on the n-dimensional Minkowski space* $\mathbb{M}$ *are non-comparable.*

Proof. The space cone based at a point is open in $\mathbb{M}^s$ but not in $\mathbb{M}^F$, by Proposition 3.2.4 and Lemma 3.3.11. For the remaining part of the proof, consider the nonclosed set Z in $\mathbb{M}^s$ constructed in the proof of Proposition 3.3.9. In view of Lemma 3.3.11, Z is closed in $\mathbb{M}^F$. This completes the proof. □

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Chapter 4

Topological Properties

In this chapter, topological properties, namely, Hausdorffness, regularity, complete regularity, normality, path connectedness, separability, regularity, metrizability, compactness, local compactness, paracompactness, Lindelöfness, first countability and second countability, of the n-dimensional Minkowski space with each of the non-Euclidean t, f, s, time and space topologies have been dealt with. It is found that except for first countability, all the topological properties are commonly shared or commonly not shared by the spaces under consideration.

4.1 Separation Axioms

In this section, separation axioms of the n-dimensional Minkowski space with non-Euclidean t, f, s, time and space topologies are explored. It is proved that each of $\mathbb{M}^t$, $\mathbb{M}^f$, $\mathbb{M}^s$, $\mathbb{M}^T$ and $\mathbb{M}^S$ is Hausdorff, non-regular, non-completely regular and non-normal.

Since t, f, s, time and space topologies are all finer than the Euclidean topology on $\mathbb{M}$ and $\mathbb{M}^E$ is Hausdorff, we have the following proposition.

Proposition 4.1.1. *The n-dimensional Minkowski space $\mathbb{M}$ with each of the t, f, s, time and space topologies is Hausdorff and hence T_1.*

It is mentioned in [15] that the 4-dimensional Minkowski space with t topology is not regular. We generalize below this result for the n-dimensional case.

Proposition 4.1.2. *Let $\mathbb{M}^t$ be the n-dimensional Minkowski space $\mathbb{M}$ with t topology. Then $\mathbb{M}^t$ is not regular.*

Proof. Let λ be a light ray passing through $0 \in \mathbb{M}$, where 0 denotes the zero vector of $\mathbb{M}$. Then by Proposition 3.2.9, λ is a discrete subspace of $\mathbb{M}^t$. Since t topology is finer than the Euclidean topology, λ is closed in $\mathbb{M}^t$. Hence $\lambda - \{0\}$ is closed in $\mathbb{M}^t$. We claim that $\lambda - \{0\}$ and 0 cannot be separated by disjoint open sets in $\mathbb{M}^t$. For, if G and H are open sets in $\mathbb{M}^t$ containing 0 and $\lambda - \{0\}$ respectively, then for some $\epsilon > 0$, $N_\epsilon^t(0) \subseteq G$. Further, for $x \in (\lambda - \{0\}) \cap N_\epsilon^E(0)$, there exists $\delta > 0, N_\delta^t(x) \subseteq H$. Notice that $N_\epsilon^t(0) \cap N_\delta^t(x) \neq \phi$ and hence $G \cap H \neq \phi$. This completes the proof. □

In the following three propositions, it is proved that the n-dimensional Minkowski space with each of the s, f, time and space topologies is not regular.

Proposition 4.1.3. *The n-dimensional Minkowski space $\mathbb{M}$ with each of the s and f topologies is not regular.*

Proof. The result for the s topology, can be proved in the same way as Proposition 4.1.2, in view of Proposition 3.2.11, and for the f topology, in view of Proposition 3.2.13. □

Proposition 4.1.4. *Let $\mathbb{M}^T$ be the n-dimensional Minkowski space $\mathbb{M}$ with time topology. Then $\mathbb{M}^T$ is not regular.*

Proof. Consider the time cone $C^T(0)$, where 0 denotes the zero vector of $\mathbb{M}$. By Proposition 3.2.3, $C^T(0)$ is open in $\mathbb{M}^t$. Since time topology is finer than the t topology on $\mathbb{M}$, $C^T(0)$ is open in $\mathbb{M}^T$. If V is a neighborhood of 0 in $\mathbb{M}^T$ such that $V \subseteq C^T(0)$, then $\mathrm{Cl}(V) \nsubseteq C^T(0)$, for $\mathrm{Cl}(V)$ contains some points of $C^L(0)$, where $\mathrm{Cl}(V)$ denotes the closure of V in $\mathbb{M}^T$. Hence there does not exist a neighborhood of 0 contained in $C^T(0)$ whose closure is also contained in $C^T(0)$. Since $\mathbb{M}^T$ is Hausdorff, the result follows. □

It is proved in [2], that the n-dimensional Minkowski space with fine topology is not normal. In the following proposition, it is shown that $\mathbb{M}^F$ is not even regular.

Proposition 4.1.5. *The n-dimensional Minkowski space $\mathbb{M}$ with each of the space and fine topologies is not regular.*

Proof. The result for the space topology and for the fine topology can be proved in the same way as Proposition 4.1.4 by considering respectively $C^S(0)$ and $C^T(0) \cup C^S(0)$ in place of $C^T(0)$, in the proof of the proposition. □

Corollary 4.1.6. *The n-dimensional Minkowski space $\mathbb{M}$ with each of the non-Euclidean t, s, f, time, space and fine topologies is not completely regular.*

Proof. Since a T_1 completely regular space is regular, the result follows. □

Corollary 4.1.7. *The n-dimensional Minkowski space $\mathbb{M}$ with each of the t, f, s, time and space topologies is not normal.*

Proof. Since a T_1 normal space is regular, the result follows. □

4.2 Countability Axioms

In this section, countability axioms of the n-dimensional Minkowski space with respect to each of the non-Euclidean topologies undertaken in the present study is discussed. It is proved that $\mathbb{M}^t$, $\mathbb{M}^f$, $\mathbb{M}^s$, $\mathbb{M}^T$ and $\mathbb{M}^S$ are not second countable while $\mathbb{M}^t$, $\mathbb{M}^f$, $\mathbb{M}^s$ are first countable, $\mathbb{M}^T$ and $\mathbb{M}^S$ are not.

It is mentioned in [17] that the 4-dimensional Minkowski space with t topology is first countable. In the following proposition this result is proved for the n-dimensional case.

Proposition 4.2.1. *Let $\mathbb{M}^t$ be the n-dimensional Minkowski space $\mathbb{M}$ with t topology. Then $\mathbb{M}^t$ is first countable.*

Proof. Let $x \in \mathbb{M}$. Then the collection $\eta(x) = \{N_\epsilon^t(x) : \epsilon > 0, \epsilon \in \mathbb{K}\}$ is a countable local base at x for the t topology on $\mathbb{M}$. Hence $\mathbb{M}^t$ is first countable. □

Proposition 4.2.2. *The n-dimensional Minkowski space $\mathbb{M}$ with each of the s and f topologies is first countable.*

Proof. Given $x \in \mathbb{M}$, the collection $\eta(x) = \{N_\epsilon^s(x) : \epsilon > 0, \epsilon \in \mathbb{K}\}$ forms a countable local base at x for the s topology and the collection $\mu(x) = \{N_\epsilon^f(x) : \epsilon > 0, \epsilon \in \mathbb{K}\}$ forms a countable local base at x for the f topology. Hence the result. □

Proposition 4.2.3. *Let $\mathbb{M}^T$ be the n-dimensional Minkowski space $\mathbb{M}$ with time topology. Then $\mathbb{M}^T$ is not first countable.*

Proof. Suppose to the contrary that $\mathbb{M}^T$ is first countable. For $z \in \mathbb{M}$, let $\{G_k : k \in \mathbb{N}\}$ be a countable local base at z in $\mathbb{M}^T$. For $k \in \mathbb{N}$, set $U_1 = G_1$ and $U_k = U_{k-1} \cap G_k$, for $k \geq 2, k \in \mathbb{N}$. Then $\{U_k : k \in \mathbb{N}\}$ is a local base at z such that $U_{k+1} \subset U_k, k \geq 1$, $k \in \mathbb{N}$. Consider now distinct timelike straight lines $\tau_k, k \in \mathbb{N}$, passing through z and choose $z_k \in \tau_k \cap U_k, z_k \neq z$, for $k \in \mathbb{N}$. Then $(z_k)_{k\in\mathbb{N}}$ converges to z in $\mathbb{M}^E$. We assert that any timelike straight line τ intersects $Z \equiv \{z_k : k \in \mathbb{N}\}$ in finitely many points so that $\tau \cap Z$ is closed in τ^E. If τ intersects Z in infinitely many points then τ must not pass through z because otherwise $\tau \cap Z$ would be either empty or singleton. Then the sequence of these infinite terms, τ^E being complete, converges in τ^E. Further, it converges to z. Hence $z \in \tau$. In other words, τ passes through z, a contradiction. By Proposition 3.1.1, Z is closed in $\mathbb{M}^T$ and hence $\mathbb{M} - Z$ is open. Since no U_k is contained in $\mathbb{M} - Z$ and $z \in \mathbb{M} - Z$, $\{U_k : k \in \mathbb{N}\}$ is not a local base at z, a contradiction. Hence the result. □

Proposition 4.2.4. *Let $\mathbb{M}^S$ be the n-dimensional Minkowski space $\mathbb{M}$ with space topology. Then $\mathbb{M}^S$ is not first countable.*

Proof. The result can be proved in the same way as the previous proposition by choosing distinct spacelike straight lines σ_k, $k \in \mathbb{N}$ passing through $z \in \mathbb{M}$ such that any space axis passing through z contains only finitely many σ_k's, in place of distinct timelike straight lines, in its proof. □

It is stated in [17], that $\mathbb{M}^t$, for $n = 4$, is second countable. This result is disproved in the following proposition.

Proposition 4.2.5. *Let $\mathbb{M}$ be the n-dimensional Minkowski space. Then $\mathbb{M}^t$ is not second countable.*

Proof. To the contrary, let $\mathbb{M}^t$ be second countable. Then since second countability is hereditary property, a light ray is second countable. From Proposition 3.2.9, the induced topology on a light ray is discrete. Then, being uncountable, light ray is not second countable, a contradiction. □

Proposition 4.2.6. *The n-dimensional Minkowski space $\mathbb{M}$ with each of the s and f topologies is not second countable.*

Proof. It can be proved in the same way as that of Proposition 4.2.5, in view of Propositions 3.2.11 and 3.2.13. □

Proposition 4.2.7. *The n-dimensional Minkowski space $\mathbb{M}$ with each of the time and space topologies is not second countable.*

Proof. Since a second countable space is first countable, the result follows in view of Propositions 4.2.3 and 4.2.4.

Alternate Proof. It can be proved in the same way as that of Proposition 4.2.5, in view of Corollaries 3.3.8 and 3.3.10.

□

4.3 Separability

In this section, we prove that the n-dimensional Minkowski space $\mathbb{M}$ with each of the non-Euclidean topologies undertaken in the present study is a separable space which leads to an alternate proof for the non-normality of each these spaces.

Proposition 4.3.1. *Let $\mathbb{M}^t$ be the n-dimensional Minkowski space $\mathbb{M}$ with t topology. Then $\mathbb{M}^t$ is separable.*

Proof. Let $x \in \mathbb{M}$ and $\epsilon > 0$. Then $N_\epsilon^t(x) - \{x\}$ is nonempty and by Proposition 3.2.7, it is open in $\mathbb{M}^E$. Since $\mathbb{K}^n$ is dense in $\mathbb{M}^E$, $(N_\epsilon^t(x) - \{x\}) \cap \mathbb{K}^n \neq \phi$. This proves that $N_\epsilon^t(x) \cap \mathbb{K}^n \neq \phi$, that is, $\mathbb{K}^n$ is dense in $\mathbb{M}^t$. The fact that K^n is countable, completes the proof. □

Proposition 4.3.2. *The n-dimensional Minkowski space $\mathbb{M}$ with each of the s and f topologies is separable.*

Proof. The result can be proved in the same way as the previous proposition by choosing punctured s neighbourhood and f neighbourhood of $x \in \mathbb{M}$ in place of $N_\epsilon^t(x) - \{x\}$, the punctured t neighbourhood, in the proof. □

Proposition 4.3.3. *Let $\mathbb{M}^T$ be the n-dimensional Minkowski space $\mathbb{M}$ with time topology. Then $\mathbb{M}^T$ is separable.*

Proof. In view of the fact that $\mathbb{K}^n$ is countable, it is sufficient to prove that $\mathbb{K}^n$ is dense in $\mathbb{M}^T$. For this, let G be a nonempty open set in $\mathbb{M}^T$, $B \equiv \{e_0, e_1, \ldots, e_{n-1}\}$ be an ordered orthonormal basis for $\mathbb{M}$ and let $x^0, x^1, \ldots, x^{n-1}$ be the components of $x \in \mathbb{M}$ with respect to the basis B. Then to obtain a point in $G \cap \mathbb{K}^n$, we first obtain a point in G with rational spatial coordinates by the following process.

Choose $z \in G$ and consider a timelike straight line τ_1 passing through z but not parallel to e_0. Since $G \cap \tau_1$ is open in τ_1^E, there exists a $w \in G \cap \tau_1$ such that $w^1 \in \mathbb{K}$. To continue the process, notice that, having obtained a point $y \in G$ with first j rational spatial coordinates, where $1 \leq j < n-1$, choose a timelike straight line τ_{j+1} passing through y, contained in $H + y$, but not parallel to e_0, where H is the vector subspace of $\mathbb{M}$ spanned by $\{e_0, e_{j+1}, \ldots, e_{n-1}\}$. Since $G \cap \tau_{j+1}$ is open in τ_{j+1}^E, there exists $u \in G \cap \tau_{j+1}$ such that $u^{j+1} \in \mathbb{K}$. Since $u^i = y^i$, for $i = 1, \ldots, j$, hence u has its first $j+1$ spatial coordinates as rationals. This process yields a point, say v, in G with rational spatial coordinates.

Finally, choose a timelike straight line τ_0 passing through v and parallel to e_0. Now, $G \cap \tau_0$ being open in τ_0^E, there exists a $p \in G \cap \tau^E$ such that $p^0 \in \mathbb{K}$. Since $p^i = v^i$ and $v^i \in K$, for $i = 1, \ldots, n-1$, hence $p \in \mathbb{K}^n$ so that $G \cap \mathbb{K}^n \neq \phi$. This completes the proof. □

Proposition 4.3.4. *Let $\mathbb{M}^S$ be the n-dimensional Minkowski space $\mathbb{M}$ with space topology. Then $\mathbb{M}^S$ is separable.*

Proof. Let $B \equiv \{e_0, e_1, \ldots, e_{n-1}\}$ be an ordered orthonormal basis for $\mathbb{M}$ and let $x^0, x^1, \ldots, x^{n-1}$ be the components of $x \in \mathbb{M}$ with respect to the basis B.

We begin by showing that $\mathbb{K}^n$ is dense in $\mathbb{M}^S$. For this, let G be a nonempty open set in $\mathbb{M}^S$ and $x \in G$. If $x^0 \notin \mathbb{K}$, then there exists $z \in G \cap \sigma$ such that $z^0 \in \mathbb{K}$, where σ denotes the spacelike straight line passing through x and not parallel to $e_i, 1 \leq i \leq n-1$. Consider now the space axis $H = \{z + \Sigma_{i=1}^{n-1} t^i e_i : t^i \in \mathbb{R}\}$. Then since G is open in $\mathbb{M}^S$ and $G \cap H$ is open in H^E, there exists $\epsilon^i \in \mathbb{R}$ such that $\{z + \Sigma_{i=1}^{n-1} t^i e_i : t^i \in (-\epsilon^i, \epsilon^i)\} \subseteq G$. By choosing rationals $r^i \in (z^i - \epsilon^i, z^i + \epsilon^i)$, the point $p = (z^0, r^1, \ldots, r^{n-1}) \in G \cap \mathbb{K}^n$. Hence $\mathbb{K}^n$ is dense in $\mathbb{M}^s$. This proves the result in view of the fact that $\mathbb{K}^n$ is countable. $\square$

The following remark gives an alternate proof for the separability of $\mathbb{M}$ with t or s or f topology which is in fact more complicated than the proofs previously obtained.

Remark 4.3.5. *Alternate proof for the separability of* $\mathbb{M}$ *with each of the* t, s *and* f *topologies:*

From Propositions 4.3.3 and 4.3.4, it follows that $\mathbb{M}^T$ and $\mathbb{M}^S$ are separable. Further in [2], it is proved that the $\mathbb{M}^F$ is separable. In view of the facts that the t, s and f topologies are respectively coarser than the time, space and fine topologies and separability is preserved under a coarser topology, it follows that $\mathbb{M}^t$, $\mathbb{M}^s$ and $\mathbb{M}^f$ are separable.

Let $|A|$ denotes the cardinality of a subset A of $\mathbb{M}$. The following corollary provides an upper bound for $|C(\mathbb{M}^t, \mathbb{R})|$, the cardinality of the set of all continuous real valued functions on $\mathbb{M}^t$, which generates an alternate proof of the non-normality of $\mathbb{M}$ with t topology.

Corollary 4.3.6. *Let* $\mathbb{M}^t$ *be the* n*-dimensional Minkowski space* $\mathbb{M}$ *with* t *topology. Then the cardinality of the set* $C(\mathbb{M}^t, \mathbb{R})$ *of all continuous real-valued functions on* $\mathbb{M}^t$ *is at most* c.

Proof. From Proposition 4.3.1, there exists a countable dense subset of $\mathbb{M}^t$, say D. Then $|C(D, \mathbb{R})|$ is at most equal to $(|\mathbb{R}|)^{|D|} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} = c$. Since two continuous maps are equal if they agree on a dense subset, hence $|C(\mathbb{M}^t, \mathbb{R})|$ is at most $|C(D, \mathbb{R})|$. This completes the proof. $\square$

By the same technique as used in Corollary 4.3.6, we have the following result.

Corollary 4.3.7. *The cardinality of the set of all continuous real-valued functions on the n-dimensional Minkowski space $\mathbb{M}$ with the each of the f, s, time and space topologies is atmost c.*

Following remark provides an alternate proof for the non-normality of the n-dimensional Minkowski space with the non-Euclidean topologies under study.

Remark 4.3.8. *Alternate proof for the non-normality of the n-dimensional Minkowski space $\mathbb{M}$ with each of the non-Euclidean t, s, f, time and space topologies:*

We first provide the alternate proof for $\mathbb{M}^t$. To the contrary, let $\mathbb{M}^t$ be normal. Let λ be a light ray and A be a nonempty subset of λ. Since λ is a closed discrete subspace of $\mathbb{M}^t$, hence A and $\lambda - A$ are closed in $\mathbb{M}^t$. By Urysohn lemma, there exists a continuous map $f : \mathbb{M}^t \longrightarrow R$ such that $f(A) = \{0\}$ and $f(\lambda - A) = \{1\}$. This implies that there would be at least as many real-valued continuous functions on $\mathbb{M}^t$ as many there are subsets of λ. Hence $|C(\mathbb{M}^t, R)|$ is at least $(2^{2^{\aleph_0}}) = 2^c$, a contradiction to Corollary 4.3.6.

The alternate proofs for the non-normality of $\mathbb{M}$ with the other non-Euclidean topologies can be proved on the same lines as above in view of Corollary 4.3.7.

4.4 Path Connectedness

In this section, we prove that the n-dimensional Minkowski space $\mathbb{M}$ with each of the non-Euclidean topologies undertaken in the present study is a path connected space.

Recall that, for $x, y \in \mathbb{M}$, the map $\gamma_{xy} : [0,1] \longrightarrow \mathbb{M}^E$ is always continuous. However the continuity of this map is not assured when $\mathbb{M}$ is assigned the finer t or s or f or time or space topology. We investigate below the conditions under which this map becomes continuous, when $\mathbb{M}$ is considered with a non-Euclidean topology under study.

Lemma 4.4.1. *Let $\mathbb{M}^t$ be the n-dimensional Minkowski space $\mathbb{M}$ with t topology and $x, y \in \mathbb{M}$, $x \neq y$. Then the map $\gamma_{xy} : [0,1] \longrightarrow \mathbb{M}^t$ defined by $\gamma_{xy}(t) = (1-t)x + ty$ is continuous if and only if $y - x$ is a timelike vector.*

Proof. Let the map $\gamma_{xy} : [0,1] \longrightarrow \mathbb{M}^t$ be continuous. Then its image $[x,y]^t$ is compact. Since t topology induces discrete topology a light ray and spacelike straight line and Euclidean topology on a timelike straight line, it follows that $y - x$ is a timelike vector. Conversely, if $y - x$ is a timelike vector, then from Proposition 3.2.8, $[x,y]^t = [x,y]^E$. Since the map $\gamma_{xy} : [0,1] \longrightarrow \mathbb{M}^E$ is continuous, $\gamma_{xy} : [0,1] \longrightarrow [x,y]^E$ is continuous. This implies that $\gamma_{xy} : [0,1] \longrightarrow [x,y]^t$ and hence $\gamma_{xy} : [0,1] \longrightarrow \mathbb{M}^t$ is continuous. This completes the proof. □

Lemma 4.4.2. *Let $\mathbb{M}^s$ be the n-dimensional Minkowski space $\mathbb{M}$ with s topology and $x, y \in \mathbb{M}$, $x \neq y$. Then the map $\gamma_{xy} : [0,1] \longrightarrow \mathbb{M}^s$ defined by $\gamma_{xy}(t) = (1-t)x + ty$ is continuous if and only if $y - x$ is a spacelike vector.*

Proof. It can be proved in a same way as that of Lemma 4.4.1, in view of Propositions 3.2.10 and 3.2.11. □

Lemma 4.4.3. *Let $\mathbb{M}^f$ be the n-dimensional Minkowski space $\mathbb{M}$ with f topology and $x, y \in \mathbb{M}$, $x \neq y$. Then the map $\gamma_{xy} : [0,1] \longrightarrow \mathbb{M}^f$ defined by $\gamma_{xy}(t) = (1-t)x + ty$ is continuous if and only if $y - x$ is a timelike or spacelike vector.*

Proof. In view of Propositions 3.2.12 and 3.2.13, the result can be analogously proved as that of Lemma 4.4.1. $\square$

Lemma 4.4.4. *Let $\mathbb{M}^T$ be the n-dimensional Minkowski space $\mathbb{M}$ with time topology and $x, y \in \mathbb{M}$, $x \neq y$. Then the map $\gamma_{xy} : [0,1] \longrightarrow \mathbb{M}^T$ defined by $\gamma_{xy}(t) = (1-t)x + ty$ is continuous if and only if $y - x$ is a timelike vector.*

Proof. Let the map $\gamma_{xy} : [0,1] \longrightarrow \mathbb{M}^T$ be continuous. Then its image $[x,y]^T$ is compact. Hence in view of the definition of time topology, $y - x$ is a timelike vector. Conversely, if $y - x$ is a timelike vector, then by the definition of time topology, $[x,y]^T = [x,y]^E$. Now since the map $\gamma_{xy} : [0,1] \longrightarrow \mathbb{M}^E$, $\gamma_{xy} : [0,1] \longrightarrow [x,y]^E$ is continuous. This implies that $\gamma_{xy} : [0,1] \longrightarrow [x,y]^T$ and hence $\gamma_{xy} : [0,1] \longrightarrow \mathbb{M}^T$ is continuous. Hence the result. $\square$

Lemma 4.4.5. *Let $\mathbb{M}^S$ be the n-dimensional Minkowski space $\mathbb{M}$ with space topology and $x, y \in \mathbb{M}$, $x \neq y$. Then the map $\gamma_{xy} : [0,1] \longrightarrow \mathbb{M}^S$ defined by $\gamma_{xy}(t) = (1-t)x + ty$ is continuous if and only if $y - x$ is a spacelike vector.*

Proof. In view of the definition of space topology, the result can be proved in the same way as that of Lemma 4.4.4. $\square$

Lemma 4.4.6. *Let $\mathbb{M}^F$ be the n-dimensional Minkowski space $\mathbb{M}$ with fine topology and $x, y \in \mathbb{M}$, $x \neq y$. Then the map $\gamma_{xy} : [0,1] \longrightarrow \mathbb{M}^F$ defined by $\gamma_{xy}(t) = (1-t)x + ty$ is continuous if and only if $y - x$ is a timelike or spacelike vector.*

Proof. The result can be proved similarly as that of Lemma 4.4.4, in view of the definition of fine topology. □

Proposition 4.4.7. *Let $\mathbb{M}^t$ be the n-dimensional Minkowski space $\mathbb{M}$ with t topology. Then $\mathbb{M}^t$ is path connected.*

Proof. Let $x, y \in \mathbb{M}$. If $y-x$ is a timelike vector, then from Lemma 4.4.1, γ_{xy} is a path in $\mathbb{M}^t$ from x to y. If $y - x$ is a lightlike vector or spacelike vector, choose $z \in C^T(x) \cap C^T(y)$. Then $z - x$ and $y - z$ are timelike vectors and hence by Lemma 4.4.1, the maps $\gamma_{xz} : [0,1] \longrightarrow \mathbb{M}^t$ and $\gamma_{zy} : [0,1] \longrightarrow \mathbb{M}^t$ are continuous. This proves that the map $\gamma : [0,1] \longrightarrow \mathbb{M}^t$ defined by

$$\gamma(t) = \begin{cases} x + 2t(z-x); & t \in [0, \frac{1}{2}] \\ z + (2t-1)(y-z); & t \in [\frac{1}{2}, 1] \end{cases}$$

which infact is the join of γ_{xz} and γ_{zy}, is a path from x to y in $\mathbb{M}^t$. □

Proposition 4.4.8. *The n-dimensional Minkowski space $\mathbb{M}$ with each of the s and f topologies is path connected.*

Proof. In view of Lemma 4.4.2, the result for s topology and in view of Lemma 4.4.3, the result for f topology can be proved as that of Proposition 4.4.7, by making suitable changes. □

Proposition 4.4.9. *Let $\mathbb{M}^T$ be the n-dimensional Minkowski space $\mathbb{M}$ with time topology. Then $\mathbb{M}^T$ is path connected.*

Proof. Let $x, y \in \mathbb{M}$. If $y - x$ is a timelike vector, then from Lemma 4.4.4, γ_{xy} is a path in $\mathbb{M}^T$ from x to y. If $y - x$ is a lightlike vector or a spacelike vector, choose $z \in C^T(x) \cap C^T(y)$. Then $z - x$ and $y - z$ are timelike vectors and from Lemma 4.4.4, γ_{xz} and γ_{zy} are paths from x to z and z to y respectively in $\mathbb{M}^T$. Further, the join of γ_{xz} and γ_{zy}, is a path from x to y in $\mathbb{M}^T$, as required. □

It is proved in [2], that 2-dimensional Minkowski space with fine topology is path connected. This result is proved below for n-dimensional Minkowski space.

Proposition 4.4.10. *The n-dimensional Minkowski space $\mathbb{M}$ with each of the space and fine topologies is path connected.*

Proof. In view of Lemma 4.4.5, the result for space topology and in view of Lemma 4.4.6, the result for fine topology can be proved as that of Proposition 4.4.9, by making suitable changes. □

Remark 4.4.11. *Alternate proof for the path connectedness of n-dimensional Minkowski space with each of the t, s and f topologies:*

Since t topology is coarser than the time topology, s topology is coarser than the space topology and f topology is coarser than the fine topology and $\mathbb{M}^T$, $\mathbb{M}^S$ and $\mathbb{M}^F$ are path connected [cf. Propositions 4.4.9 and 4.4.10], the result follows.

Corollary 4.4.12. *The n-dimensional Minkowski space $\mathbb{M}$ with each of the t, s, f, time, space and fine topologies is connected.*

Proof. Since a path connected space is connected, the result follows. $\square$

4.5 A Few More Topological Properties

In this section, some more topological properties, namely compactness, local compactness, metrizability, Lindelöfness etc., of the n-dimensional Minkowski space with each of the non-Euclidean t, s, f, time and space topologies are explored.

Proposition 4.5.1. *The n-dimensional Minkowski space $\mathbb{M}$ with each of the t, s, f, time and space topologies is non-compact.*

Proof. The result follows in view of the fact that $\mathbb{M}$ with Euclidean topology is not compact and the t, s, f, time and space topologies on $\mathbb{M}$ are finer than the Euclidean topology on $\mathbb{M}$. $\square$

Proposition 4.5.2. *The n-dimensional Minkowski space $\mathbb{M}$ with each of the t, s, f, time and space topologies is not metrizable.*

Proof. Since a metrizable space is regular [14], the result for the t topology, s topology, f topology, time topology and space topology follows in view of Proposition 4.1.2, Proposition 4.1.3, Proposition 4.1.3, Proposition 4.1.4 and Proposition 4.1.5 respectively. $\square$

Proposition 4.5.3. *The n-dimensional Minkowski space $\mathbb{M}$ with each of the t, s, f, time and space topologies is not locally compact.*

Proof. Since a locally compact Hausdorff is regular [14], the result for the t topology, s topology, f topology, time topology and space topology follows in view of Proposition 4.1.2, Proposition 4.1.3, Proposition 4.1.3, Proposition 4.1.4 and Proposition 4.1.5 respectively. $\square$

Proposition 4.5.4. *The n-dimensional Minkowski space $\mathbb{M}$ with each of the t, s, f, time and space topologies is not paracompact.*

Proof. Since a paracompact Hausdorff is regular [14], the result for the t topology, s topology, f topology, time topology and space topology follows in view of Proposition 4.1.2, Proposition 4.1.3, Proposition 4.1.3, Proposition 4.1.4 and Proposition 4.1.5 respectively. $\square$

Corollary 4.5.5. *The n-dimensional Minkowski space $\mathbb{M}$ with each of the t, s, f, time and space topologies is not locally m-Euclidean.*

Proof. Since a locally Euclidean space is locally compact [14], the result follows in view of Proposition 4.5.3. $\square$

Proposition 4.5.6. *The n-dimensional Minkowski space $\mathbb{M}$ with each of the t, s, f, time and space topologies is not Lindelöf.*

Proof. Suppose to the contrary that $\mathbb{M}^t$, $\mathbb{M}$ with the t topology, is Lindelöf. Let λ be a light ray. Since λ is a closed subspace of $\mathbb{M}^E$, by Proposition 3.3.1, λ is a closed subspace of $\mathbb{M}^t$. Further, by Proposition 3.2.9, λ is discrete. Hence λ is Lindelöf, for Lindelöfness is closed hereditary. This is a contradiction since an infinite discrete space is not Lindelöf. The proof for the other topologies follow in a similar way. $\square$

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Chapter 5

Zeno Sequence

The concept of Zeno sequence in the 4-dimensional Minkowski space with fine topology was originally defined by Zeeman [22] to study its homeomorphism group. In this chapter, we introduce the notion of Zeno sequence in the n-dimensional Minkowski space with each of the non-Euclidean t, s, f, time and space topologies and obtain a necessary condition for a set to be open in the respective space. Further, subsets of $\mathbb{M}$, that have the same subspace topologies as induced from the Euclidean topology and a non-Euclidean topology have been characterized. This study leads to important contributions in the succeeding chapters.

5.1 Definition and Examples

In this section, introducing the notion of Zeno sequences in the n-dimensional Minkowski space with each of the non-Euclidean topologies undertaken, some examples of Zeno sequences are provided.

Definition 5.1.1. Recall that a sequence in a topological space X is said to converge to $x \in X$, if every open set in X containing x, contains all but finitely many terms of the sequence.

Let $z \in \mathbb{M}$ and let $(z_k)_{k\in\mathbb{N}}$ be a sequence of distinct terms in $\mathbb{M}$ such that $z_k \neq z$, for every $n \in \mathbb{N}$. Then

(i) $(z_k)_{k\in\mathbb{N}}$ is called a *Zeno sequence in* $\mathbb{M}^t$ *converging to* $z \in \mathbb{M}$, if $(z_k)_{k\in\mathbb{N}}$ converges to z in $\mathbb{M}^E$ but not in $\mathbb{M}^t$.

(ii) $(z_k)_{k\in\mathbb{N}}$ is called a *Zeno sequence in* $\mathbb{M}^s$ *converging to* $z \in \mathbb{M}$, if $(z_k)_{k\in\mathbb{N}}$ converges to z in $\mathbb{M}^E$ but not in $\mathbb{M}^s$.

(iii) $(z_k)_{k\in\mathbb{N}}$ is called a *Zeno sequence in* $\mathbb{M}^f$ *converging to* $z \in \mathbb{M}$, if $(z_k)_{k\in\mathbb{N}}$ converges to z in $\mathbb{M}^E$ but not in $\mathbb{M}^f$.

(iv) $(z_k)_{k\in\mathbb{N}}$ is called a *Zeno sequence in* $\mathbb{M}^T$ *converging to* $z \in \mathbb{M}$, if $(z_k)_{k\in\mathbb{N}}$ converges to z in $\mathbb{M}^E$ but not in $\mathbb{M}^T$.

(v) $(z_k)_{k\in\mathbb{N}}$ is called a *Zeno sequence in* $\mathbb{M}^S$ *converging to* $z \in \mathbb{M}$, if $(z_k)_{k\in\mathbb{N}}$ converges to z in $\mathbb{M}^E$ but not in $\mathbb{M}^S$.

The *image* of a Zeno sequence $(z_k)_{k\in\mathbb{N}}$ will mean the set $Z = \{z_k | k \in \mathbb{N}\}$ and the *completed image* of a Zeno sequence $(z_k)_{k\in\mathbb{N}}$ will mean the set $Z \cup \{z\}$.

Remark 5.1.2. From the finer/coarser relationship of the Euclidean and non-Euclidean topologies discussed in Chapter 1, we have the following:

(i) $(z_k)_{k\in\mathbb{N}}$ is not a Zeno sequence in $\mathbb{M}^t$ converging to $z \Longrightarrow (z_k)_{k\in\mathbb{N}}$ is not a Zeno sequence in $\mathbb{M}^f$ converging to z.

(ii) $(z_k)_{k\in\mathbb{N}}$ is not a Zeno sequence in $\mathbb{M}^s$ converging to $z \Longrightarrow (z_k)_{k\in\mathbb{N}}$ is not a Zeno sequence in $\mathbb{M}^f$ converging to z.

(iii) $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^f$ converging to $z \Longrightarrow (z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^t$ converging to z.

(iv) $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^f$ converging to $z \Longrightarrow (z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^s$ converging to z.

(v) $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^t$ converging to $z \Longrightarrow (z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^T$ converging to z.

(vi) $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^s$ converging to $z \Longrightarrow (z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^S$ converging to z.

Recall that the timelike straight lines, space axes and light rays passing through a point are respectively contained in time cone, space cone and light cone based at that point. Based on this thought, some examples of Zeno sequences are constructed below.

Examples 5.1.3. **(Light Sequence)** Let $z \in \mathbb{M}$ and $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence of light rays passing through z. For $k \in \mathbb{N}$, choose $z_k \in \lambda_k$ such that $0 < d(z_k, z) < 1/k$ and $z_i \neq z_j, i, j \geq 1, i \neq j$. Then $(z_k)_{k\in\mathbb{N}}$ converges to z in $\mathbb{M}^E$, for any $\epsilon > 0$, by Archimedian property of real numbers, there exists $m \in \mathbb{N}$, such that $1/m < \epsilon$ and therefore $N_\epsilon^E(z)$ and hence any open set containing z contains all but finitely many terms of $(z_k)_{k\in\mathbb{N}}$. Further, we have the following:

(i) The sequence $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^t$ converging to z, for given $\epsilon > 0$, the open set $N^t_\epsilon(z)$ does not contain any term of $(z_k)_{k\in\mathbb{N}}$ and hence does not converge to z in $\mathbb{M}^t$.

(ii) The sequence $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^s$ converging to z, for $(z_k)_{k\in\mathbb{N}}$ does not converge to z in $\mathbb{M}^s$ as the open set $N^s_\epsilon(z)$ does not contain any term of $(z_k)_{k\in\mathbb{N}}$.

(iii) The sequence $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^f$ converging to z, since for $\epsilon > 0$, the open set $N^f_\epsilon(z)$ does not contain any term of $(z_k)_{k\in\mathbb{N}}$ and hence does not converge to z in $\mathbb{M}^f$.

(iv) The sequence $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^T$ converging to z, for $(z_k)_{k\in\mathbb{N}}$ does not converge to z in $\mathbb{M}^t$ and time topology being finer than the t topology, it does not converge to z in $\mathbb{M}^T$.

(v) The sequence $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^S$ converging to z, for $(z_k)_{k\in\mathbb{N}}$ does not converge to z in $\mathbb{M}^s$ and space topology being finer than the s topology, it does not converge to z in $\mathbb{M}^S$.

Examples 5.1.4. **(Time Sequence)** Let $z \in \mathbb{M}$ and $(\tau_k)_{k\in\mathbb{N}}$ be a sequence of distinct timelike straight lines passing through z. For $k \in \mathbb{N}$, choose $z_k \in \tau_k$ such that $0 < d(z_k, z) < 1/k$. Then $z_i \neq z_j, i, j \geq 1, i \neq j$ and as shown in Example 5.1.3, $(z_k)_{k\in\mathbb{N}}$ converges to z in $\mathbb{M}^E$. Further, we have the following:

(i) The sequence $(z_k)_{k\in\mathbb{N}}$ is a not Zeno sequence in $\mathbb{M}^t$, for any $\epsilon > 0$, the open set $N^t_\epsilon(z)$ and hence any open set containing z in $\mathbb{M}^t$ contains all but finitely many term of $(z_k)_{k\in\mathbb{N}}$ and therefore converges in $\mathbb{M}^t$ as well.

(ii) The sequence $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^s$ converging to z, for $(z_k)_{k\in\mathbb{N}}$ does not converge to z in $\mathbb{M}^s$ as the open set $N^s_\epsilon(z)$ does not contain any term of $(z_k)_{k\in\mathbb{N}}$.

(iii) The sequence $(z_k)_{k\in\mathbb{N}}$ is not a Zeno sequence in $\mathbb{M}^f$ converging to z, for $(z_k)_{k\in\mathbb{N}}$ converges to z in $\mathbb{M}^t$ and f topology being coarser than the t topology, it converges to z in $\mathbb{M}^f$.

(iv) The sequence $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^T$ converging to z, for $(z_k)_{k\in\mathbb{N}}$ does not converge to z in $\mathbb{M}^T$. To prove that $(z_k)_{k\in\mathbb{N}}$ does not converge to z in $\mathbb{M}^T$, let τ be a timelike straight line. Then $Z \cap \tau$, where $Z = \{z_n : n \in \mathbb{N}\}$, is finite, for otherwise $z \notin \tau$ and then limit of $(z_n)_{n\in\mathbb{N}}$ being z in $\mathbb{M}^E$ and τ^E being a complete metric space, $z \in \tau$, a contradiction. By Proposition 3.1.1, Z is closed in $\mathbb{M}^T$ and thereby $\mathbb{M} - Z$ is an open set containing z in $\mathbb{M}^T$. This proves that $(z_k)_{k\in\mathbb{N}}$ does not converge to z in $\mathbb{M}^T$, for $\mathbb{M} - Z$ does not contain any term of $(z_k)_{k\in\mathbb{N}}$.

(v) The sequence $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^S$ converging to z, for $(z_k)_{k\in\mathbb{N}}$ does not converge to z in $\mathbb{M}^s$ and space topology being finer than the s topology, it does not converge to z in $\mathbb{M}^S$.

Examples 5.1.5. (Space Sequence) Let $z \in \mathbb{M}$ and $(\sigma_k)_{k\in\mathbb{N}}$ be a sequence of distinct spacelike straight lines passing through z such that any space axis passing through z contains only finitely many spacelike straight lines σ_k's. For $k \in \mathbb{N}$, choose $z_k \in \sigma_k$ such that $0 < d(z_k, z) < 1/k$. Then $z_i \neq z_j, i, j \geq 1, i \neq j$ and as shown in Example 5.1.3, $(z_k)_{k\in\mathbb{N}}$ converges to z in $\mathbb{M}^E$. Further, we have the following:

(i) The sequence $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^t$ converging to z, for given $\epsilon > 0$, the open set $N_\epsilon^t(z)$ does not contain any term of $(z_k)_{k\in\mathbb{N}}$ and hence doesn't converge to z in $\mathbb{M}^t$.

(ii) The sequence $(z_k)_{k\in\mathbb{N}}$ is not a Zeno sequence in $\mathbb{M}^s$ converging to z, for $(z_k)_{k\in\mathbb{N}}$ converges to z in $\mathbb{M}^s$ as the open set $N_\epsilon^s(z)$ and hence any open set containing z in $\mathbb{M}^s$ contains all but finitely many terms of $(z_k)_{k\in\mathbb{N}}$.

(iii) The sequence $(z_k)_{k\in\mathbb{N}}$ is not a Zeno sequence in $\mathbb{M}^f$ converging to z, for $(z_k)_{k\in\mathbb{N}}$ converges to z in $\mathbb{M}^s$ and f topology being coarser than s topology, it converges to z in $\mathbb{M}^f$.

(iv) The sequence $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^T$ converging to z, for $(z_k)_{k\in\mathbb{N}}$ does not converge to z in $\mathbb{M}^t$ and time topology being finer than the t topology, it does not converge to z in $\mathbb{M}^T$.

(v) The sequence $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^S$ converging to z, for $(z_k)_{k\in\mathbb{N}}$ doesn't converge to z in $\mathbb{M}^S$. To see that $(z_k)_{k\in\mathbb{N}}$ does not converge to z in $\mathbb{M}^S$, let H be a space axis. Then $Z\cap H$, where $Z = \{z_n : n \in \mathbb{N}\}$, is finite, for otherwise $z \notin H$ and then limit of $(z_n)_{n\in\mathbb{N}}$ being z in $\mathbb{M}^E$ and H^E being a complete metric space, $z \in H$, a contradiction. By Proposition 3.1.2, Z is closed in $\mathbb{M}^S$. Hence $\mathbb{M} - Z$ is an open set in $\mathbb{M}^S$ containing z. This proves that $(z_k)_{k\in\mathbb{N}}$ does not converge to z in $\mathbb{M}^S$, for $\mathbb{M} - Z$ does not contain any term of $(z_k)_{k\in\mathbb{N}}$.

5.2 Some Results on Zeno Sequences

Recall that the non-Euclidean t, s, f, time and space topologies are all finer than the Euclidean topology. Hence these non-Euclidean topologies induce finer subspace topologies on $A \subseteq \mathbb{M}$ than the Euclidean topology. In this section, we characterize those subsets of $\mathbb{M}$ for which the induced non-Euclidean topology is same as that of the induced Euclidean topology. Further, necessary conditions for a set to be open in $\mathbb{M}$ with each of these non-Euclidean topologies are obtained, besides many other results.

Proposition 5.2.1. *Let $\mathbb{M}^t$ be the n-dimensional Minkowski space $\mathbb{M}$ with t topology and $(z_k)_{k\in\mathbb{N}}$ be a Zeno sequence in $\mathbb{M}^t$ converging to $z \in \mathbb{M}$. Then there exists a subsequence of $(z_k)_{k\in\mathbb{N}}$ whose image is closed in $\mathbb{M}^t$ but not in $\mathbb{M}^E$, $\mathbb{M}$ with Euclidean topology.*

Proof. Since $(z_k)_{k\in\mathbb{N}}$ does not converge to z in $\mathbb{M}^t$, there exists an open set U in $\mathbb{M}^t$ containing z that leaves outside infinitely many terms of the sequence. These terms form a subsequence $(z_{k_i})_{i\in\mathbb{N}}$ of $(z_k)_{k\in\mathbb{N}}$ such that $z_{k_i} \notin U$, for $i \in \mathbb{N}$. We assert that the image $A \equiv \{z_{k_i} : i \in \mathbb{N}\}$ of $(z_{k_i})_{i\in\mathbb{N}}$ is closed in $\mathbb{M}^t$ but not in $\mathbb{M}^E$. The subsequence $(z_{k_i})_{i\in\mathbb{N}}$ converges to z in $\mathbb{M}^E$, for $(z_k)_{k\in\mathbb{N}}$ converges to z in $\mathbb{M}^E$. Since $z \notin A$, A is not closed in $\mathbb{M}^E$. Further, no point of $\mathbb{M}$ other than z is a limit point of A in $\mathbb{M}^E$ and hence in $\mathbb{M}^t$. That z is not a limit point of A in $\mathbb{M}^t$ follows by noting that U does not intersect with A. This proves that A is closed in $\mathbb{M}^t$. Hence the result. □

Proposition 5.2.2. *Let $\mathbb{M}^*$ be the n-dimensional Minkowski space $\mathbb{M}$ with s or f or time or space topology. Then a Zeno sequence in $\mathbb{M}^*$ admits a* *subsequence whose image is closed in $\mathbb{M}^*$ but not in $\mathbb{M}^E$, $\mathbb{M}$ with Euclidean topology.*

Proof. Similar to that of Proposition 5.2.1 □

As obtained in Proposition 3.2.8, the subspace topology on a timelike straight line induced from the t topology on $\mathbb{M}$ is Euclidean. The following proposition determines precisely the class of subsets C of $\mathbb{M}$ for which this is true.

Proposition 5.2.3. *Let $\mathbb{M}^t$ be the n-dimensional Minkowski space $\mathbb{M}$ with t topology and C be a nonempty subset of $\mathbb{M}$. Then the topologies on the subspaces C^t and C^E are same if and only if C does not contain completed image of any Zeno sequence in $\mathbb{M}^t$.*

Proof. We first prove the "if" part. In view of the fact that t topology on $\mathbb{M}$ is finer than the Euclidean topology, it is sufficient to prove that for every $z \in C$ and every open set G_z^t in $\mathbb{M}^t$ containing z, there exists an open set G_z^E containing z of $\mathbb{M}^E$ such that $C \cap G_z^E \subseteq C \cap G_z^t$. Suppose to the contrary that for some $z \in C$ and an open set G_z^t in $\mathbb{M}^t$ containing z, there is no open set G_z^E in $\mathbb{M}^E$ such that $C \cap G_z^E \subset C \cap G_z^t$. Thus, in particular, for no $k \in \mathbb{N}$, $C \cap N_{1/k}^E(z) \subseteq C \cap G_z^t$. It now follows that there exists an strictly increasing sequence $(m_k)_{k\in\mathbb{N}}$ of natural numbers so that for each $k \in \mathbb{N}$, we can choose $z_k \in C \cap N_{1/m_k}^E(z)$ such that $z_k \notin C \cap G_z^t$ and $z_k \neq z_i, 1 \leq i < k$. Since $\{N_\epsilon^E(x) : \epsilon > 0, x \in \mathbb{M}\}$ is a basis for $\mathbb{M}^E$ and for $\epsilon > 0$, $N_\epsilon^E(z)$ contains all but finitely many terms of the sequence $(z_k)_{k\in\mathbb{N}}$, hence $(z_k)_{k\in\mathbb{N}}$ converges to z in $\mathbb{M}^E$. That it does not converge to z in $\mathbb{M}^t$, follows by noting that $G_z^t \cap \{z_k : k \in \mathbb{N}\} = \phi$. Hence $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^t$ converging to z with its completed image contained in C, a contradiction.

To prove the converse, assume to the contrary that C contains the completed image $\{z_k : k \in \mathbb{N}\} \cup \{z\} \equiv Z \cup \{z\}$ of a Zeno sequence $(z_k)_{k\in\mathbb{N}}$ in $\mathbb{M}^t$ converging to $z \in C$. Since $(z_k)_{k\in\mathbb{N}}$ does not converge to z in $\mathbb{M}^t$, there is an open set U containing z in $\mathbb{M}^t$ which does not meet some subsequence of $(z_k)_{k\in\mathbb{N}}$. On the other hand, since each open set containing z in $\mathbb{M}^E$ meets such a subsequence, it follows that $U \cap (Z \cup \{z\}) \neq B \cap (Z \cup \{z\})$, for any open set B of z in $\mathbb{M}^E$. Hence t and Euclidean topologies on $\mathbb{M}$ induce different topologies on $Z \cup \{z\}$ and hence also on C. □

The following proposition characterizes those subsets C of $\mathbb{M}$ for which the subspace topology on C induced by a non-Euclidean topology is equal to subspace topology induced by the Euclidean topology.

Proposition 5.2.4. *Let $\mathbb{M}^*$ be the n-dimensional Minkowski space $\mathbb{M}$ with s or f or time or space topology and C be a nonempty subset of $\mathbb{M}$. Then the topologies on the subspaces C^* and C^E are same if and only if C does not contain completed image of any Zeno sequence in $\mathbb{M}^*$.*

Proof. Same as that of Proposition 5.2.3 □

In the following Lemma, necessary conditions for a set to be open in $\mathbb{M}^E$ are obtained.

Lemma 5.2.5. *Let $\mathbb{M}^E$ be the n-dimensional Minkowski space $\mathbb{M}$ with Euclidean topology, G be a nonempty open set in $\mathbb{M}^E$ and $z \in G$. Then the following statements hold:*

(i) G contains completed image of a Zeno sequence in $\mathbb{M}^t$ converging to z.

(ii) G contains completed image of a Zeno sequence in $\mathbb{M}^s$ converging to z.

(iii) G contains completed image of a Zeno sequence in $\mathbb{M}^f$ converging to z.

Proof. (i) Since $\{N_\epsilon^E(z) : \epsilon > 0, z \in \mathbb{M}\}$ is a basis for the Euclidean topology on $\mathbb{M}$, hence for some $\epsilon > 0$, $N_\epsilon^E(z) \subseteq G$. For $k \in \mathbb{N}$, choose $z_k \in N_\epsilon^E(z) \cap C^L(z)$ such that $0 < d(z_k, z) < \epsilon/k$ and $z_k \neq z_i, 1 \leq i < k$. Then $(z_k)_{k\in\mathbb{N}}$ converges to z in $\mathbb{M}^E$. Since $N_\epsilon^t(z)$ contains no term of $(z_k)_{k\in\mathbb{N}}$, it follows that $(z_k)_{k\in\mathbb{N}}$ does not converge to z in $\mathbb{M}^t$. Thus $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^t$ converging to z. This proves the result.

(ii) Let $(z_k)_{k\in\mathbb{N}}$ be as constructed in the proof of Lemma 5.2.5, Part (i). Then $(z_k)_{k\in\mathbb{N}}$ converges to z in $\mathbb{M}^E$. That $(z_k)_{k\in\mathbb{N}}$ does not converge to z in $\mathbb{M}^s$ follows by noting that $N_\epsilon^s(z)$ contains no term of $(z_k)_{k\in\mathbb{N}}$. Hence $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^s$ converging to z. This completes the proof.

(iii) Let $(z_k)_{k\in\mathbb{N}}$ be as constructed in the proof of Lemma 5.2.5, Part (i). Then $(z_k)_{k\in\mathbb{N}}$ converges to z in $\mathbb{M}^E$. Since $N_\epsilon^f(z)$ contains no term of $(z_k)_{k\in\mathbb{N}}$, hence $(z_k)_{k\in\mathbb{N}}$ does not converge to z in $\mathbb{M}^f$. Then $(z_k)_{k\in\mathbb{N}}$ is a Zeno sequence in $\mathbb{M}^f$ converging to z. Hence the result. □

The following proposition, provides a necessary condition for a set to be open in $\mathbb{M}$ with a non-Euclidean t, s or f topology. The time and space topologies counterparts are dealt with separately in Proposition 5.2.8 and Proposition 5.2.9, because of the difference in statements and proofs.

Proposition 5.2.6. *Let $\mathbb{M}$ be the n-dimensional Minkowski space $\mathbb{M}$ and G be a nonempty subset of $\mathbb{M}$. Then the following statements hold:*

(i) If G is open in $\mathbb{M}^t$ then G contains completed image of a Zeno sequence in $\mathbb{M}^t$.

(ii) If G is open in $\mathbb{M}^s$ then G contains completed image of a Zeno sequence in $\mathbb{M}^s$.

(iii) If G is open in $\mathbb{M}^f$ then G contains completed image of a Zeno sequence in $\mathbb{M}^f$.

Proof. (i) Let $z \in G$. Then since G is open in $\mathbb{M}^t$, for some $\epsilon > 0$, $N^t_\epsilon(z) \subseteq G$ and hence $N^t_\epsilon(z) - \{z\} \subseteq G$. From Proposition 3.2.7, $N^t_\epsilon(z) - \{z\}$ is open in $\mathbb{M}^E$. Now in view of Lemma 5.2.5, Part (i), $N^t_\epsilon(z) - \{z\}$ and hence G contains completed image of some Zeno sequence in $\mathbb{M}^t$. This completes the proof.

(ii) Let $z \in G$. Then for some $\epsilon > 0$, $N^s_\epsilon(z) \subseteq G$. In view of Proposition 3.2.7 and Lemma 5.2.5, Part (ii), $N^s_\epsilon(z) - \{z\}$ and therefore G contains completed image of a Zeno sequence in $\mathbb{M}^s$. Hence the result.

(iii) Let $z \in G$. Then for some $\epsilon > 0$, $N^f_\epsilon(z) \subseteq G$. From Proposition 3.2.7 and Lemma 5.2.5, Part (iii), $N^f_\epsilon(z) - \{z\}$ and hence G contains completed image of a Zeno sequence in $\mathbb{M}^f$. This proves the result. □

Remark 5.2.7. In Lemma 5.2.5, the Zeno sequence obtained in an open set G in $\mathbb{M}^E$ converges to the chosen point in G whereas in Proposition 5.2.6, the Zeno sequence obtained in an open set in $\mathbb{M}^t$ or $\mathbb{M}^s$ or $\mathbb{M}^f$ may not converge to the chosen point. However, it is proved in the following two propositions that the Zeno sequence obtained in an open set G in $\mathbb{M}^T$ or $\mathbb{M}^S$ converges to the chosen point in G.

Proposition 5.2.8. *Let $\mathbb{M}^T$ be the n-dimensional Minkowski space $\mathbb{M}$ with time topology, G be a nonempty open set in $\mathbb{M}^T$ and $z \in G$. Then G contains completed image of a Zeno sequence in $\mathbb{M}^T$ converging to z.*

Proof. Let $\{\tau_k\}_{k\in\mathbb{N}}$ be a sequence of distinct timelike straight lines passing through z. Since $\tau_k \cap G$ is an open interval in τ_k^E containing z, hence for $k \in \mathbb{N}$, we can choose $z_k \in \tau_k \cap G$ such that $0 < d(z_k, z) < 1/k$ and $z_k \neq z_i, 1 \leq i < k$. Then $(z_k)_{k\in\mathbb{N}}$ can be shown to be a Zeno sequence in $\mathbb{M}^T$ converging to z as it is shown in Examples 5.1.4 (iv). This completes the proof. □

Proposition 5.2.9. *Let $\mathbb{M}^S$ be the n-dimensional Minkowski space $\mathbb{M}$ with space topology, G be a nonempty open set in $\mathbb{M}^S$ and $z \in G$. Then G contains completed image of a Zeno sequence in $\mathbb{M}^S$ converging to z.*

Proof. Let $\{\sigma_k\}_{k\in\mathbb{N}}$ be a sequence of distinct spacelike straight lines passing through z such that any space axis passing through z contains only finitely many $\sigma_k' s$. As $\sigma_k \cap G$ is an open interval in σ_k^E containing z, hence for $k \in \mathbb{N}$, we can choose $z_k \in \sigma_k \cap G$ such that $0 < d(z_k, z) < 1/k$ and $z_k \neq z_i, 1 \leq i < k$. Then $(z_k)_{k\in\mathbb{N}}$ can be shown to be a Zeno sequence in $\mathbb{M}^S$ converging to z as it is proved in Examples 5.1.5 (v). This completes the proof.

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□

Chapter 6

Compact Sets

The celebrated Heine-Borel theorem characterizes the compact sets in the Euclidean n-space. It states that a set in the Euclidean n-space is compact if and only if it is closed and bounded in it. In this chapter, necessary and sufficient conditions for a set to be compact in the n-dimensional Minkowski space, with each of the non-Euclidean topologies undertaken in the present study, are obtained.

6.1 Necessary Conditions

In this section, necessary conditions for a set to be compact in Minkowski space with each of the t, s, f, time and space topologies are obtained.

Proposition 6.1.1. *Let $\mathbb{M}^t$ be the n-dimensional Minkowski space $\mathbb{M}$ with t topology and C be a nonempty subset of $\mathbb{M}$ such that C^t is compact. Then C^E is compact and hence is closed and bounded in $\mathbb{M}^E$, $\mathbb{M}$ with Euclidean topology.*

Proof. Since the t topology on $\mathbb{M}$ is finer than the Euclidean topology and compactness is preserved under a coarser topology, C^E is compact. The remaining part of the result follows from the Heine-Borel theorem. $\square$

Proposition 6.1.2. *Let $\mathbb{M}^*$ be the n-dimensional Minkowski space $\mathbb{M}$ with s or f or time or space topology and C be a nonempty subset of $\mathbb{M}$ such that C^* is compact. Then C^E is compact and hence is closed and bounded in $\mathbb{M}^E$, $\mathbb{M}$ with Euclidean topology.*

Proof. Since each of the s, f, time and space topologies on $\mathbb{M}$ is finer than the Euclidean topology and compactness is preserved under a coarser topology, C^E is compact. The remaining part of the result follows from the Heine-Borel theorem. $\square$

Proposition 6.1.3. *Let $\mathbb{M}^t$ be the n-dimensional Minkowski space $\mathbb{M}$ with t topology and C be a nonempty subset of $\mathbb{M}$ such that C^t is compact. Then C does not contain image of a Zeno sequence in $\mathbb{M}^t$.*

Proof. To the contrary, let C contain a Zeno sequence $(z_k)_{k\in\mathbb{N}}$ in $\mathbb{M}^t$ converging to $z \in \mathbb{M}$. Then from Proposition 5.2.1, there exists a subsequence $(z_{k_i})_{i\in\mathbb{N}}$ of $(z_k)_{k\in\mathbb{N}}$ whose image $A \equiv \{z_{k_i} : i \in \mathbb{N}\}$ is closed in $\mathbb{M}^t$ but not in $\mathbb{M}^E$. Since $A \subseteq C$, A^t is compact and hence A^E is compact. This proves that A is closed in $\mathbb{M}^E$, a contradiction. $\square$

Proposition 6.1.4. *Let $\mathbb{M}^*$ be the n-dimensional Minkowski space $\mathbb{M}$ with s or f or time or space topology and C be a nonempty subset of $\mathbb{M}$ such that C^* is compact. Then C does not contain image of a Zeno sequence in $\mathbb{M}^*$.*

Proof. It can be similarly proved as that of Proposition 6.1.3, in view of Proposition 5.2.2. $\square$

Proposition 6.1.5. *Let $\mathbb{M}^t$ be the n-dimensional Minkowski space $\mathbb{M}$ with t topology and C be a nonempty subset of $\mathbb{M}$ such that C^t is compact. Then C does not contain any open set in $\mathbb{M}^t$. Consequently, C does not contain any open set in $\mathbb{M}^E$, $\mathbb{M}$ with Euclidean topology.*

Proof. Suppose to the contrary that C contains an open subset G of $\mathbb{M}^t$. Then from Proposition 5.2.6, Part (i), it follows that G and hence C contains image of a Zeno sequence in $\mathbb{M}^t$, a contradiction to Proposition 6.1.3. That C does contain an open set in M^E, follows from the fact that the t topology is finer than the Euclidean topology. $\square$

Proposition 6.1.6. *Let $\mathbb{M}^*$ be the n-dimensional Minkowski space $\mathbb{M}$ with s or f or time or space topology and C be a nonempty subset of $\mathbb{M}$ such that C^* is compact. Then C does not contain any set open in $\mathbb{M}^*$. Consequently, C does not contain any open set in $\mathbb{M}^E$, $\mathbb{M}$ with Euclidean topology.*

Proof. It can be analogously proved as that of Proposition 6.1.5, in view of Propositions 5.2.6 (ii), 5.2.6 (iii), 5.2.8, 5.2.9, 6.1.4 and the fact that s, f, time and space topologies on $\mathbb{M}$ are finer than the Euclidean topology. $\square$

The following remark provides an alternate proof for the non-local compactness of the n-dimensional Minkowski space with each of the non-Euclidean topologies undertaken.

Remark 6.1.7. *Alternate proof of the non-local compactness of the n-dimensional Minkowski space $\mathbb{M}$ with each of the t, s, f, time and space topologies:*

Let $x \in \mathbb{M}$. Then in view of Proposition 6.1.5, there is no compact subspace of $\mathbb{M}^t$ that contains an open neighbourhood of x in $\mathbb{M}^t$. This proves that $\mathbb{M}^t$ is not locally compact at x. Hence $\mathbb{M}^t$ is not locally compact. The proof for the other topologies follow in a similar way, in view of Proposition 6.1.6. This proves the result.

6.2 Sufficient Conditions

In the following section, we determine sufficient conditions for a set to be compact in the n-dimensional Minkowski space with each of the non-Euclidean topologies under study.

Proposition 6.2.1. *Let $\mathbb{M}^t$ be the n-dimensional Minkowski space $\mathbb{M}$ with t topology and C be a nonempty subset of $\mathbb{M}$ such that C^E is compact and C does not contain completed image of a Zeno sequence in $\mathbb{M}^t$. Then C^t is compact.*

Proof. Since C does not contain completed image of a Zeno sequence in $\mathbb{M}^t$, it follows from Proposition 5.2.3 that $C^t = C^E$. This completes the proof. □

Proposition 6.2.2. *Let $\mathbb{M}^*$ be the n-dimensional Minkowski space $\mathbb{M}$ with s or f or time or space topology and C be a nonempty subset of $\mathbb{M}$ such that C^E is compact and C does not contain completed image of a Zeno sequence in $\mathbb{M}^*$. Then C^* is compact.*

Proof. It can be proved similar to Proposition 6.2.1, in view of Proposition 5.2.4. □

6.3 Analogue of Heine-Borel Theorem

In this section, a characterization of compact sets in $\mathbb{M}^t$, $\mathbb{M}^s$, $\mathbb{M}^f$, $\mathbb{M}^T$ and $\mathbb{M}^S$ is obtained.

Proposition 6.3.1. *Let $\mathbb{M}^t$ be the n-dimensional Minkowski space with t topology and C be a nonempty subset of $\mathbb{M}$. Then C^t is compact if and only if C^E is compact and C does not contain completed image of any Zeno sequence in $\mathbb{M}^t$.*

Proof. It follows from Propositions 6.1.1 , 6.1.3 and 6.2.1. □

Proposition 6.3.2. *Let $\mathbb{M}^s$ be the n-dimensional Minkowski space $\mathbb{M}$ with s topology and C be a nonempty subset of $\mathbb{M}$. Then C^s is compact if and only if C^E is compact and C does not contain completed image of any Zeno sequence in $\mathbb{M}^s$.*

Proof. It follows from Propositions 6.1.2 , 6.1.4 and 6.2.2. □

Proposition 6.3.3. *Let $\mathbb{M}^f$ be the n-dimensional Minkowski space $\mathbb{M}$ with f topology and C be a nonempty subset of $\mathbb{M}$. Then C^f is compact if and only if C^E is compact and C does not contain completed image of any Zeno sequence in $\mathbb{M}^f$.*

Proof. It follows from Propositions 6.1.2 , 6.1.4 and 6.2.2. □

Proposition 6.3.4. *Let $\mathbb{M}^T$ be the n-dimensional Minkowski space $\mathbb{M}$ with time topology and C be a nonempty subset of $\mathbb{M}$. Then C^T is compact if and only if C^E is compact and C does not contain completed image of any Zeno sequence in $\mathbb{M}^T$.*

Proof. It follows from Propositions 6.1.2 , 6.1.4 and 6.2.2. □

Proposition 6.3.5. *Let $\mathbb{M}^S$ be the n-dimensional Minkowski space $\mathbb{M}$ with space topology and C be a nonempty subset of $\mathbb{M}$. Then C^S is compact if and only if C^E is compact and C does not contain completed image of any Zeno sequence in $\mathbb{M}^S$.*

Proof. It follows from Propositions 6.1.2 , 6.1.4 and 6.2.2. □

Discussed below are some examples of compact and non-compact subspaces of the n-dimensional Minkowski space with various non-Euclidean topologies undertaken in the present work, using the preceding results obtained in this chapter: Example 6.3.6 (i) is discussed in detail.

Examples 6.3.6. (i) Let I^n denote the unit n-cube. Then since I^n contains an open set in $\mathbb{M}^E$, from Propositions 6.1.5 and 6.1.6, it follows that I^n is not a compact subspace of $\mathbb{M}^t$, $\mathbb{M}^s$, $\mathbb{M}^f$, $\mathbb{M}^T$ and $\mathbb{M}^S$. It may be noted that I^n is a compact subspace of the Euclidean n-space.

(ii) Any interval on a lightlike straight line is not compact in any of $\mathbb{M}^s$, $\mathbb{M}^f$, $\mathbb{M}^S$, $\mathbb{M}^t$ and $\mathbb{M}^T$.

(iii) A closed and bounded interval on a timelike straight line is compact in $\mathbb{M}^t$, $\mathbb{M}^f$ and $\mathbb{M}^T$, but not in $\mathbb{M}^s$ or $\mathbb{M}^S$.

(iv) A closed and bounded interval on a spacelike straight line is compact in $\mathbb{M}^s$, $\mathbb{M}^f$ and $\mathbb{M}^S$, but not in $\mathbb{M}^t$ or $\mathbb{M}^T$.

(v) A closed and bounded ball on a spacelike hyperplane is compact in $\mathbb{M}^s$, $\mathbb{M}^f$ and $\mathbb{M}^S$, but not in $\mathbb{M}^t$ or $\mathbb{M}^T$.

(vi) Let $Z \cup \{z\}$ be the completed image of the sequence constructed in Examples 5.1.4. Then $Z \cup \{z\}$ is compact in $\mathbb{M}^t$ and $\mathbb{M}^f$ but not in $\mathbb{M}^s$ or $\mathbb{M}^T$ or $\mathbb{M}^S$.

(vii) Let Z be the completed image of the sequence constructed in Examples 5.1.5. Then $Z \cup \{z\}$ is compact in $\mathbb{M}^s$ and $\mathbb{M}^f$ but not in $\mathbb{M}^t$ or $\mathbb{M}^T$ or $\mathbb{M}^S$.

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Chapter 7

Applications

In this chapter, the study carried out in the previous chapters is used to investigate the preservation of continuity by a map when the codomain $\mathbb{M}^E$ is considered with a finer topology and simple connectedness of $\mathbb{M}^t$, $\mathbb{M}^s$, $\mathbb{M}^f$, $\mathbb{M}^T$ and $\mathbb{M}^S$.

7.1 Continuity of Maps

Let h be a continuous map from a topological space X to $\mathbb{M}^E$, the n-dimensional Minkowski space $\mathbb{M}$ with Euclidean topology. Then h may not be continuous if $\mathbb{M}$ is considered with a finer topology. In this section, a sufficient condition is obtained for h to be continuous when $\mathbb{M}$ is considered with any of the non-Euclidean topologies under present study.

Proposition 7.1.1. *Let $\mathbb{M}^t$ be the n-dimensional Minkowski space with t topology, X be a topological space and $h : X \longrightarrow \mathbb{M}^E$ be a continuous map such that $h(X)$ does not contain the completed image of any Zeno sequence* *in* $\mathbb{M}^t$. *Then* $h : X \longrightarrow \mathbb{M}^t$ *is continuous.*

Proof. . Since $h : X \longrightarrow \mathbb{M}^E$ is continuous, hence $h : X \longrightarrow h(X)^E$ is continuous. By Proposition 5.2.3, $h(X)^t = h(X)^E$, hence $h : X \longrightarrow h(X)^t$ is continuous. This proves that $h : X \longrightarrow \mathbb{M}^t$ is continuous. □

Proposition 7.1.2. *Let* $\mathbb{M}^*$ *be the* n*-dimensional Minkowski space with* t *or* s *or* f *or time or space topology,* X *be a topological space and* $h : X \longrightarrow \mathbb{M}^E$ *be a continuous map such that* $h(X)$ *does not contain the completed image of any Zeno sequence in* $\mathbb{M}^*$. *Then* $h : X \longrightarrow \mathbb{M}^*$ *is continuous.*

Proof. It is similar to that of Proposition 7.1.1, in view of Proposition 5.2.4. □

7.2 Simple Connectedness

Through out this section, we use the symbol $\mathbb{M}_2$ to denote the 2-dimensional Minkowski space. Using the study of Zeno sequences and compact sets carried out in Chapters 5 and 6, it is proved that each of $\mathbb{M}_2^t$, $\mathbb{M}_2^s$, $\mathbb{M}_2^f$, $\mathbb{M}_2^T$ and $\mathbb{M}_2^S$ has a non-trivial fundamental group and are therefore non-simply connected. Further, the case $n = 2$, has been exploited to prove the non-simply connectedness of $\mathbb{M}^t$ and $\mathbb{M}^T$, for $n > 2$.

It is well know that the 2-dimensional Minkowski space $\mathbb{M}_2$ with Euclidean topology has trivial fundamental group. On the other hand that $\mathbb{M}_2$ with each of the non-Euclidean topologies undertaken in the present study has a non-trivial fundamental group, is proved below.

Proposition 7.2.1. *Let $\mathbb{M}_2^t$ be the 2-dimensional Minkowski space $\mathbb{M}_2$ with t topology and 0 be the zero vector of $\mathbb{M}_2$. Then $\mathbb{M}_2^t$ has a non-trivial fundamental group at 0.*

Proof. The result is proved by constructing two loops in $\mathbb{M}_2^t$ based at 0, which are not path homotopic.

Construction of loops:

For $i = 1, 2$, let γ_i be the join of $\gamma_{0u_i}, \gamma_{u_iv_i}$ and γ_{v_i0} and (u_i, v_i)'s are distinct pairs of timelike vectors such that $u_i - v_i$ is a timelike vector and $0, u_i, v_i$ are non-collinear. Then by Proposition 4.4.1, γ_i being the join of continuous maps is continuous. Further since γ_i's begin and end at 0, it follows that γ_i's are loops in $\mathbb{M}_2^t$ based at 0.

γ_1 and γ_2 are not path homotopic:

We assert that γ_1 and γ_2 are not path homotopic. To prove the assertion, suppose to the contrary, γ_1 and γ_2 are path homotopic. Let $H : [0,1] \times [0,1] \longrightarrow \mathbb{M}_2^t$ be a path homotopy between γ_1 and γ_2. Notice that, either $IntT_1 - T_2 \neq \phi$ or $IntT_2 - T_1 \neq \phi$, where T_1 and T_2 are respectively the triangles $\gamma_1([0,1])$ and $\gamma_2([0,1])$ together with their insides and $IntT_1$, $IntT_2$ are respectively the interiors of T_1 and T_2 in $\mathbb{M}_2^E$. Further, $Int(T_1) - T_2 \subseteq H([0,1] \times [0,1])$, for if not, there exists $p \in IntT_1 - T_2$ such that $p \notin H([0,1] \times [0,1])$, thereby proving that $H : [0,1] \times [0,1] \longrightarrow M^E - \{p\}$ is a path homotopy between γ_1 and γ_2 in $\mathbb{M}_2^E - \{p\}$ which is not possible as γ_1 winds around p while γ_2 does not. Similarly it can be proved that $IntT_2 - T_1 \subseteq H([0,1] \times [0,1])$. Thus $H([0,1] \times [0,1])$ contains a nonempty open set of $\mathbb{M}_2^E$. This is not possible in view of Proposition 6.1.5, because H being continuous and $[0,1] \times [0,1]$ being a compact subspace of $\mathbb{M}_2^E$, $H([0,1] \times [0,1])$ is a compact subspace of $\mathbb{M}_2^t$. This completes the proof. $\square$

Proposition 7.2.2. *The 2-dimensional Minkowski space $\mathbb{M}_2$ with f or time topology has a non-trivial fundamental group at 0, where 0 is the zero vector of $\mathbb{M}_2$.*

Proof. For $i = 1, 2$, let γ_i be as constructed in the proof of Proposition 7.2.1. Then γ_i's are loops in $\mathbb{M}_2^f$ and $\mathbb{M}_2^T$ in view of Proposition 4.4.3 and Proposition 4.4.4 respectively. The remaining part of the proof can be obtained from Proposition 7.2.1, by making suitable changes. $\square$

Proposition 7.2.3. *The 2-dimensional Minkowski space $\mathbb{M}_2$ with s or space topology has a non-trivial fundamental group at 0, where 0 is the zero vector of $\mathbb{M}_2$.*

Proof. Consider γ_i's, for $i = 1, 2$, as constructed in the proof of Proposition 7.2.1 by replacing timelike vectors by spacelike vectors. Then γ_i's are loops in $\mathbb{M}_2^s$ and $\mathbb{M}_2^S$ in view of Proposition 4.4.2 and Proposition 4.4.5 respectively. The remaining part of the proof can be obtained from Proposition 7.2.1, by making suitable changes. $\square$

Corollary 7.2.4. *The 2-dimensional Minkowski space $\mathbb{M}$ with each of the t, s, f, time and space topologies is not simply connected.*

Proof. Notice that each of $\mathbb{M}_2^t$, $\mathbb{M}_2^s$, $\mathbb{M}_2^f$, $\mathbb{M}_2^T$ and $\mathbb{M}_2^S$ is path connected as proved in Chapter 4. Further, each has a non-trivial fundamental group in view of the preceding Propositions. This completes the proof. $\square$

Remark 7.2.5. In the forthcoming results of this section, cones, neighbourhoods, 0 etc. are used in different spaces, for instance $\mathbb{M}_2$, $\mathbb{M}$ and $\mathbb{R}^{n-2}$. To avoid complexity of notation, the same notation is used in all the spaces. However, the parent space will become clear from the context.

Proposition 7.2.6. *Let $\mathbb{M}^t$ be the n-dimensional Minkowski space $\mathbb{M}$ with t topology, for $n > 2$. Then the subspace topology induced on $\mathbb{R}^2 \times \{0\}$ from t topology on $\mathbb{M}$ is same as that of the product topology on $\mathbb{M}_2^t \times \{0\}$, where 0 denotes the zero vector of the real vector space $\mathbb{R}^{n-2}$.*

Proof. Notice that for $x \equiv (x^0, x^1, \ldots, x^{n-1}) \in \mathbb{M}$,

$$N_\epsilon^t(x) \cap [\mathbb{R}^2 \times \{0\}] = \begin{cases} N_\epsilon^t(x) - \{x\} \cap [\mathbb{R}^2 \times \{0\}], & \text{if } x \notin \mathbb{R}^2 \times \{0\} \\ N_\epsilon^t((x^0, x^1)) \times \{0\}, & \text{if} x \in \mathbb{R}^2 \times \{0\} \end{cases}$$

By Proposition 3.2.7, $N_\epsilon^t(x) - \{x\} \cap [\mathbb{R}^2 \times \{0\}]$ is open in $\mathbb{M}_2^E \times \{0\}$ and hence in $\mathbb{M}_2^t \times \{0\}$. This proves that the subspace topology on $\mathbb{R}^2 \times \{0\}$ is coarser than the topology on $\mathbb{M}_2^t \times \{0\}$. Since $\{N_\epsilon^t((x^0, x^1)) \times \{0\} : \epsilon > 0, (x^0, x^1) \in \mathbb{M}_2\}$ is a basis for the topology on $\mathbb{M}_2^t \times \{0\}$, the other containment follows by noting that $N_\epsilon^t((x^0, x^1)) \times \{0\} = N_\epsilon^t(y) \cap [\mathbb{R}^2 \times \{0\}]$, where $y \equiv (x^0, x^1, 0 \ldots, 0)$ □

Proposition 7.2.7. *Let $\mathbb{M}^T$ be the n-dimensional Minkowski space $\mathbb{M}$ with time topology. Then for $n > 2$, the subspace topology induced on $\mathbb{R}^2 \times \{0\}$ from time topology on $\mathbb{M}$ is same as that of the product topology on $\mathbb{M}_2^T \times \{0\}$, where 0 denotes the zero vector of the real vector space $\mathbb{R}^{n-2}$.*

Proof. Since the subspace topology induced on $\mathbb{R}^2 \times \{0\}$ from time topology on $\mathbb{M}$ induces Euclidean topology on each timelike straight line in $\mathbb{R}^2 \times \{0\}$, it is coarser than the topology on $\mathbb{M}_2^T \times \{0\}$. To see that it is indeed equal, let $G \times \{0\}$ be open in $\mathbb{M}_2^T \times \{0\}$, where G is open in $\mathbb{M}_2^T$. Then $(G \times \mathbb{R}^{n-2}) \cap (\mathbb{R}^2 \times \{0\}) = G \times \{0\}$. Since $(G \times \mathbb{R}^{n-2}) \cap \tau$ is open in τ^E, for any timelike straight line τ, $G \times \mathbb{R}^{n-2}$ is open in $\mathbb{M}^T$. This completes the proof. $\square$

Proposition 7.2.8. *Let $\mathbb{M}^t$ denote the n-dimensional Minkowski space with t topology, for $n > 2$. Then $\mathbb{M}^t$ has a non-trivial fundamental group.*

Proof. Let 0 denote the zero vector of the real vector space $\mathbb{R}^{n-2}$. Consider the map $r : \mathbb{M}^t \longrightarrow \mathbb{M}_2^t \times \{0\}$ defined by $r(y^0, y^1, \ldots, y^{n-1}) = (y^0, y^1, 0, \ldots, 0)$, for $y \in \mathbb{M}$. Then in view of Proposition 7.2.6, $\mathbb{M}_2^t \times \{0\}$ is a subspace of $\mathbb{M}^t$. We assert that r is a retraction. To see that it is continuous, it is sufficient to check that, for $(x^0, x^1) \in \mathbb{M}_2$, $r^{-1}(C^T((x^0, x^1)) \times \{0\})$ is open in $\mathbb{M}^t$ because the just defined map $r : \mathbb{M}^E \longrightarrow \mathbb{M}_2^E \times \{0\}$ is continuous and the collection $\{N_\epsilon^t((x^0, x^1)) \times \{0\} : \epsilon > 0, (x^0, x^1) \in \mathbb{M}_2\}$ forms a basis for the product topology on $\mathbb{M}_2^t \times \{0\}$. Since $[C^T((x^0, x^1)) - \{(x^0, x^1)\}] \times \{0\}$ is open in $\mathbb{M}_2^E \times \{0\}$, it follows that each point of $r^{-1}([C^T((x^0, x^1)) - \{(x^0, x^1)\}] \times \{0\})$ is its interior point in $\mathbb{M}^t$. Further for $y \in r^{-1}(C^T((x^0, x^1)) \times \{0\})$ such that $r(y) = (x^0, x^1, 0 \ldots, 0)$, $C^T(y) \subseteq r^{-1}(C^T((x^0, x^1)) \times \{0\})$. This proves that $r : \mathbb{M}^t \longrightarrow \mathbb{M}_2^t \times \{0\}$ is continuous. Further since $r|_{\mathbb{R}^2 \times \{0\}}$ is equal to the identity map on $\mathbb{M}_2 \times \{0\}$, r is a retraction.

Since the natural homomorphism induced on the fundamental groups by a retraction is surjective [14], by Proposition 7.2.1, the fundamental group at 0 of $\mathbb{M}_2^t \times \{0\}$ and hence of $\mathbb{M}^t$ is non-trivial. $\square$

Proposition 7.2.9. *Let $\mathbb{M}^T$ denote the n-dimensional Minkowski space with time topology, for $n > 2$. Then $\mathbb{M}^T$ has a non-trivial fundamental group.*

Proof. In view of Proposition 7.2.7, it follows that $\mathbb{M}_2^T \times \{0\}$ is identical with the subspace $\mathbb{R}^2 \times \{0\}$ of $\mathbb{M}^T$, where 0 denotes the zero vector of the real vector space $\mathbb{R}^{n-2}$, for $n > 2$. Consider the map $r : \mathbb{M}^T \longrightarrow \mathbb{M}_2^T \times \{0\}$ defined by $r(x_0, x_1, \ldots, x_{n-1}) = (x_0, x_1, 0, \ldots, 0)$, for $(x_0, x_1, \ldots, x_{n-1}) \in \mathbb{M}$. Then r is continuous for if $G \times \{0\}$ is open in $\mathbb{M}_2^T \times \{0\}$, then $r^{-1}(G \times \{0\}) = G \times \mathbb{R}^{n-2}$, which is open in $\mathbb{M}^T$. Further, since $r|_{\mathbb{R}^2 \times \{0\}}$ is equal to the identity map on $\mathbb{M}_2 \times \{0\}$, r is a retraction.

Since the natural homomorphism induced on fundamental groups by retraction is surjective [14], by Proposition 7.2.2, the fundamental group at 0 of $\mathbb{M}_2^T \times \{0\}$ and hence of $\mathbb{M}^T$ is non-trivial. Hence the result. $\square$

Corollary 7.2.10. *The n-dimensional Minkowski space* $\mathbb{M}$ *with each of the t and time topologies is not simply connected.*

Proof. Notice that each of $\mathbb{M}^t$ and $\mathbb{M}^T$ is path connected as proved in Chapter 4. Further, each has a non-trivial fundamental group in view of the preceding Propositions. This completes the proof. $\square$

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Chapter 8

Conclusion

In this chapter, summarizing the main results obtained in the thesis, its salient features are highlighted. New direction of work is also proposed.

8.1 Main Results

The present thesis is focused on a topological study of the n-dimensional Minkowski space $\mathbb{M}$. Separation axioms, countability axioms, separability, metrizability, Lindelöfness, local compactness, connectedness, compact sets, fundamental group, etc. of $\mathbb{M}$ with various non-Euclidean topologies are explored.

Main results obtained are summarized below:

(i) The time topology on the n-dimensional Minkowski space $\mathbb{M}$ is strictly finer than the t and fine topologies on $\mathbb{M}$, the space topology on $\mathbb{M}$ is strictly finer than the s and fine topologies on $\mathbb{M}$, the fine topology on $\mathbb{M}$ is strictly finer than the f topology on $\mathbb{M}$, the t topology on

$\mathbb{M}$ is strictly finer than the f topology on $\mathbb{M}$ and the s topology on $\mathbb{M}$ is strictly finer than the f topology on $\mathbb{M}$ while the t, s and fine topologies are non-comparable and time and space topologies are also non-comparable.

(ii) The t and time topologies induce Euclidean topology on timelike straight lines while discrete topology on light rays and space axes.

(iii) The s and space topologies induce Euclidean topology on a space axis while discrete topology on a light rays and timelike straight line.

(iv) The f and fine topologies induce Euclidean topology on timelike straight lines and space axes while discrete topology on light rays.

(v) Topological properties, namely path connectedness, separability are commonly shared and regularity, local compactness, metrizability, paracompactness, Lindelöfness, second countability are commonly not shared by the n-dimensional Minkowski space $\mathbb{M}$ with each of the t, f, s, time, fine and space topologies while first countability is enjoyed by $\mathbb{M}$ with t, f and s topologies but not by $\mathbb{M}$ with time and space topologies.

(vi) A Zeno sequence in the n-dimensional Minkowski space $\mathbb{M}$ with t topology or f topology or s topology or time topology or space topology admits a subsequence whose image is closed in it but not closed in $\mathbb{M}$ with Euclidean topology.

(vii) For a subset C of the n-dimensional Minkowski space $\mathbb{M}$, the subspace topologies induced on C from Euclidean topology and from t topology or f topology or s topology or time topology or space topology, are same if and only if C does not contain completed image of any Zeno sequence in $\mathbb{M}$ with the corresponding topology.

(viii) A nonempty open set in the n-dimensional Minkowski space $\mathbb{M}$ with t topology or f topology or s topology or time topology or space topology, contains completed image of a Zeno sequence in the corresponding space.

(ix) An analogue of the Heine-Borel theorem for the n-dimensional Minkowski space $\mathbb{M}$ with non-Euclidean topologies is obtained.

(x) The n-cube, which is known to be compact in the n-dimensional Minkowski space $\mathbb{M}$ with Euclidean topology, is not compact in $\mathbb{M}$ with any of the t, f, s, time and space topologies.

(xi) A continuous map f from a space X to the n-dimensional Minkowski space with Euclidean topology such that $f(X)$ does not contain completed image of a Zeno sequence in $\mathbb{M}$ with t topology or f topology or s topology or time topology or space topology, continues to be continuous when $\mathbb{M}$ is considered with the corresponding topology

(xii) The 2-dimensional Minkowski space with any of the f, t, s, time and space topologies is not simply connected unlike the Euclidean 2-space.

(xiii) Generalization of the preceding result to the n-dimensional case has been obtained for t and time topologies (case $n = 2$ has been exploited to prove the result for the general case).

8.2 Concluding Remark

The present thesis, focused on a detailed topological study of the n-dimensional Minkowski space $\mathbb{M}$, forms a part of the study of compactification in superstring theories, $\mathbb{M}$ being one of the essential component of this compactification.

It has been revealed that various topological properties studied for $\mathbb{M}$ with different non-Euclidean topologies do not follow from the finer coarser relationship of topologies. Rather, independent proofs are required which are not straight forward. While studying analogue of the Heine-Borel theorem in $\mathbb{M}$ with a non-Euclidean topology, the notion of Zeno sequences has emerged as an important tool, and provides a new technique to characterize the compact sets in Minkowski space with a non-Euclidean topology. This characterization of compact sets has further led to the investigation of some more topological properties, such as simple connectedness. The techniques are sometimes same to prove a result for different topologies but sometimes differ tremendously. Even if the technique is same for different topologies, an independent visualization is required for each of them.

This non-triviality indicates that the study of each of the non-Euclidean topologies on the n-dimensional Minkowski space has its own independent existence from mathematical viewpoint also besides having the well established physical relevance.

8.3 Proposed Directions

Mentioned below are two research problems for possible consideration in future.

(i) Is n-dimensional Minkowski space $\mathbb{M}$ homeomorphic to the product space $X \times Y$, where X is k-dimensional Minkowski space, Y is m-dimensional Minkowski space and $k + m = n$, when $\mathbb{M}$, X and Y are with the same non-Euclidean topology?

(ii) Is n-dimensional Minkowski space with a non-Euclidean topology homeomorphic to the m-dimensional Minkowski space with the same topology, for $n \neq m$?

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List of Papers

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Conference Presentations

1. *Topological Properties of Minkowski Space with s-Topology* (Jointly with Gunjan Agrawal), International Conference on Analysis and its Applications, Aligarh Muslim University, Aligarh (India), November 3-5, 2008.

2. *A Non-Euclidean Topology Proposed by Zeeman* (Jointly with Gunjan Agrawal), Joint International Conference on Applied Systems Research and XXXIII National Systems Conference, Dayalbagh Educational Institute (Deemed University), Agra (India), November 27-29, 2009.

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