More on Galilean Electromagnetism

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Abstract

The subject called "Galilean electromagnetism" was created by Le Bellac and Lévy-Leblond [1] who had proved that there exist two non-relativistic limits of Maxwell's equations. We have a specific reason to reexamine these limits which is connected with recently obtained completed description of decomposable vector representations of the homogeneous Galilei group [2, 3]. Our knowledge of these representations makes it possible to refine and generalize the results of Le Bellac and Lévy-Leblond. It is shown that the collection of non-equivalent Galilei-invariant wave equations for massless fields with spin equal 1 and 0 is very broad. There exist a huge number of such equations for massless fields which correspond to various contractions of representations of the Lorentz group to those of the Galilei one. Moreover, it is possible to find essentially coupled systems of Galilei invariant equations via contraction of decoupled relativistic systems.

1. Introduction

Consistent physical models as a rule are characterized by nice symmetrical properties. There are many various symmetries, but two of them are the most fundamental ones. They are relativistic invariance and invariance w.r.t. the Galilei transformations. The relativistic invariance is seem to be an universal low of nature, the Galilean invariance change the Lorentz one whenever we deal with phenomena with velocities much smaller than the velocity of light.

Relativistic theories in principle are more complicated then non-relativistic ones. On the other hand, the structure of subgroups of the Galilei group

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and of its representations are in many respects more complex than those of the Poincaré group and therefore it is perhaps not so surprising that the representations of the Poincaré group were described by Wigner in 1939, almost 15 years earlier than the representations of the Galilei group (Bargman, 1954) in spite of the fact that the relativity principle of classical physics was formulated by Galilei about three centuries prior to that of relativistic physics formulated by Einstein.

An excellent review of the representation theory of the Galilei group was written by Lévy-Leblond in 1964. It appears that, as opposed to the Poincaré group, the Galilei group has the ordinary as well as the projective representations. Its subgroup - the homogeneous Galilei group $HG(1,3)$ which plays in non-relativistic physics the role of the Lorentz group in relativistic case, has a much more sophisticated structure so that *its finite-dimensional indecomposable representations and not classifiable.*

2. Galilei group.

The Galilei group $G(1,3)$ is a group of transformations in $R_3 \oplus R_1$: :

$$\begin{aligned} &t \to t' = t + a, \\ &\mathbf{x} \to \mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{v}t + \mathbf{b}, \end{aligned} \tag{1}$$

where $a, \mathbf{b}$ and $\mathbf{v}$ are real parameters, $\mathbf{R}$ is a rotation matrix.

The homogeneous Galilei group $HG(1,3)$ a subgroup of $G(1,3)$ leaving invariant $\mathbf{x} = (0,0,0)$ at $t = 0$ and formed by:

$$\begin{aligned} &t \to t' = t, \\ &\mathbf{x} \to \mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{v}t \end{aligned} \tag{2}$$

This group is noncompact and not semi-simple. Its maximal compact subgroup is $SO(3)$.

Lie algebra $\mathbf{hg(1,3)}$ is spanned on six basis elements: 3 rotation generators $S_a, a = 1,2,3$ and three generators of Galilean boosts η_a, with the commutation relations

$$\begin{aligned} &[S_a, S_b] = \mathrm{i}\varepsilon_{abc} S_c, \\ &[\eta_a, S_b] = \mathrm{i}\varepsilon_{abc} \eta_c, \end{aligned} \tag{3}$$

$$[\eta_a, \eta_b] = 0. \tag{4}$$

Classification of finite–dimensional representations of algebra (3) is wild algebraic problem. Nevertheless, it appears to be possible to describe indecomposable representations of $hg(1,3)$ which, however, when restricted to $so(3)$ are decomposed to direct sums of spin 0 and 1 representations.

3. Vector representations

Let us examine the finite-dimensional indecomposable representations of the algebra $hg(1,3)$ defined on vector and scalar, or spin-one and spin-zero, representation spaces. It is convenient to search for these representations in $so(3)-$basis, i.e., choose such basis in the carrier space in which the Casimir operators of the maximal compact subalgebra $so(3)$ are diagonal. The corresponding matrices S_a can be expressed as direct sums of spin-one and spin-zero matrices:

$$S_a = \begin{pmatrix} I_{n\times n}\otimes s_a & \cdot \\ \cdot & \mathbf{0_{m\times m}} \end{pmatrix}. \tag{5}$$

The symbols $I_{n\times n}$ and $\mathbf{0_{m\times m}}$ denote the $n\times n$ unit matrix and $m\times m$ zero matrix respectively, s_a $(a=1,2,3)$ are 3×3 matrices of spin equal to one for which we choose the following realization:

$$s_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6}$$

The general form of matrices η_a which satisfy relations (3) with matrices (6) is given by the following formulae (see, e.g., Gelfand et al)

$$\eta_a = \begin{pmatrix} A\otimes s_a & B\otimes k_a^\dagger \\ C\otimes k_a & \mathbf{0_{m\times m}} \end{pmatrix} \tag{7}$$

A, B and C are matrices of dimension $n\times n$, $n\times m$ and $m\times n$ respectively, k_a are 1×3 matrices of the form

$$k_1 = (\mathrm{i},0,0), \qquad k_2 = (0,\mathrm{i},0), \qquad k_3 = (0,0,\mathrm{i}). \tag{8}$$

The matrices (5) and (7) satisfy conditions (9) with any A, B and C. Substituting (7) into (10) we obtain the following equations for matrices A, B and C:

$$A^2 + BC = 0, \tag{9}$$

$$CA = 0, \quad AB = 0. \tag{10}$$

Up to equivalence, the problem of description of finite-dimensional indecomposable representations of algebra $hg(1.3)$ for vector and scalar fields is deduced to finding the general solution of the matrix problem (9), (10).

The solution of the nice matrix problem defined by equations (9) and (10) is relatively easily to handle. Namely, there exist ten non-equivalent indecomposable sets of matrices $\{A, B, C\}$, which can be labelled by triplets of numbers n, m, λ where n and m take the values

$$-1 \le (n-m) \le 2, \;\; n \le 3 \tag{11}$$

and define dimensions of these matrices as in eq.(7), λ =Rank$B$, whose values are

$$\lambda = \begin{cases} 0 \texttt{ if } m = 0, \\ 1 \texttt{ if } m = 2 \texttt{ or } n - m = 2, \\ 0, 1 \texttt{ if } m = 1, n \neq 3. \end{cases} \tag{12}$$

These sets and the corresponding IDR will be denoted by $D(m, n, \lambda)$ with n, m, λ satisfying the conditions given above. In accordance with (11) there exist ten non-equivalent indecomposable representations $D(m, n, \lambda)$.

Thus as distinct to the relativistic case where are only three Lorentz quantities (four-vectors, antisymmetric tensors of second order and scalars) which transforms as vectors or scalars under rotations, there are 10 IDR $D(m, n, \lambda)$ of $HG(1, 3)$.

There exist the following spaces of indecomposable representations: scalar (for $D(0, 1, 0)$), 3-vector (for $D(1, 0, 0)$), two four-vectors (for $D(1, 1, 0)$) and $D(1, 1, 1)$), a five-vector (for $D(1, 2, 1)$),..., and even a "ten-vector" (for $D(3, 1, 1)$).

The most complicated example: IDR $D(3, 1, 1)$, the representation space is formed by three rotation vectors $\mathbf{N}, \mathbf{W}, \mathbf{R}$ and a scalar A, with transformation lows:

$$\begin{aligned} &A \to A, \quad \mathbf{R} \to \mathbf{R}' = \mathbf{R}, \\ &\mathbf{W} \to \mathbf{W}' = \mathbf{W} + \mathbf{v} \times \mathbf{R}, \\ &\mathbf{N} \to \mathbf{N}' = \mathbf{N} + \mathbf{v} \times \mathbf{W} + \mathbf{v}B + \mathbf{v}(\mathbf{v} \cdot \mathbf{R}) - \tfrac{1}{2}\mathbf{v}^2\mathbf{R}. \end{aligned}$$

4. Contractions of representations of the Lorentz algebra

It is well known that $hg(1, 3)$ can be obtained from $so(1, 3)$ by the Inönü-Wigner contraction.

In the simplest case a *contraction* is a limit procedure which transforms an N-dimensional Lie algebra $\mathcal{L}$ into an non-isomorphic Lie algebra $\mathcal{L}'$, also with N dimensions. The commutation relations of a *contracted Lie algebra* $\mathcal{L}'$ are given by:

$$[x, y]' \equiv \lim_{\varepsilon \to \varepsilon_0} W_\varepsilon^{-1}([W_\varepsilon(x), W_\varepsilon(y)], \tag{13}$$

where $W_\varepsilon \in \mathrm{GL}(N, k)$ is a non-singular linear transformation of $\mathcal{L}$, with ε_0 being a singularity point of W or its inverse W_ε^{-1}.

Representations of these algebras also can be obtained by contractions, however, by more complicated ways.

In [2, 3] representations of $so(1, 3)$ which can be contracted to $D(m, n, \lambda)$ of $hg(1, 3)$ found and appropriate contractions specified. Here we present only few examples.

Let us start with representation $D(\frac{1}{2},\frac{1}{2})$ of $so(1,3)$. Its carrier space is formed by four-vectors and its basis is given by:

$$S_{ab}=\varepsilon_{abc}\begin{pmatrix} s_c & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & 0\end{pmatrix},\quad S_{0a}=\begin{pmatrix} \mathbf{0}_{3\times3} & -k_a^\dagger\\ k_a & 0\end{pmatrix}. \tag{14}$$

Here $(s_a)_{bc}=i\varepsilon_{abc}$ are matrix elements of spin 1, k_a are 1×3 matrices of the form

$$k_1=(\mathtt{i},0,0)\,,\qquad k_2=(0,\mathtt{i},0)\,,\qquad k_3=(0,0,\mathtt{i})\,.$$

The Inönü-Wigner contraction consists of transformation to a new basis

$$S_{ab}\to VS_{ab}V^{-1},\qquad S_{0a}\to\varepsilon VS_{0a}V^{-1}$$

(with a matrix V depending on contraction parameter ε) and passing $\varepsilon\to0$. Moreover, $V=V(\varepsilon)$ in a tricky way, so that all the transformed generators kept non-trivial and non-singular when $\varepsilon\to0$.

There exist two matrices V for representation (14), namely

$$V_1=\begin{pmatrix} \varepsilon I_{3\times3} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & 1\end{pmatrix},\ \texttt{and}\ V_2=\begin{pmatrix} I_{3\times3} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & \varepsilon\end{pmatrix}. \tag{15}$$

Using V_1 we obtain

$$\begin{aligned} S'_{ab}&=V_1S_{ab}V_1^{-1}=S_{ab},\\ S'_{0a}&=\varepsilon V_1S_{0a}V_1^{-1}=\begin{pmatrix} \mathbf{0}_{3\times3} & -\varepsilon^2k_a^\dagger\\ k_a & 0\end{pmatrix}. \end{aligned}\tag{16}$$

Then, passing ε to zero, we come to the following matrices

$$\begin{aligned} S_a&=\tfrac{1}{2}\varepsilon_{abc}S_{bc}=\begin{pmatrix} s_a & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & 0\end{pmatrix},\\ \eta_a&=\lim S'_{0a}|_{\varepsilon\to0}=\begin{pmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1}\\ k_a & 0\end{pmatrix}. \end{aligned}\tag{17}$$

Analogously, using matrix V_2 instead of V_1 we obtain

$$S_a=\begin{pmatrix} s_a & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & 0\end{pmatrix},\qquad \eta_a=\begin{pmatrix} \mathbf{0}_{3\times3} & -k_a^\dagger\\ \mathbf{0}_{1\times3} & 0\end{pmatrix}. \tag{18}$$

Matrices (17) and (18) realize representations $D(1,1,0)$ and $D(1,1,1)$ respectively.

To obtain five-dimensional representation $D(1,2,1)$ we start with a direct sum of representations $D(\frac{1}{2},\frac{1}{2})$ and $D(0,0)$ of $so(1,3)$. The corresponding generators of $so(1,3)$ have the form

$$\hat{S}_{\mu\nu} = \begin{pmatrix} S_{\mu\nu} & \cdot \\ \cdot & 0 \end{pmatrix} \tag{19}$$

where $S_{\mu\nu}$ are matrices (14) and the dots denote zero matrices of appropriate dimensions. The corresponding similarity transformation matrices are:

$$V_3 = \begin{pmatrix} I_{3\times3} & 0_{3\times1} & 0_{3\times1} \\ 0_{1\times3} & \frac{1}{2}\varepsilon & \frac{1}{2}\varepsilon \\ 0_{1\times3} & -\varepsilon^{-1} & \varepsilon^{-1} \end{pmatrix}, \ V_3^{-1} = \begin{pmatrix} I_{3\times3} & 0_{3\times1} & 0_{3\times1} \\ 0_{1\times3} & \varepsilon^{-1} & -\frac{1}{2}\varepsilon \\ 0_{1\times3} & \varepsilon^{-1} & \frac{1}{2}\varepsilon \end{pmatrix}. \tag{20}$$

As a result we obtain the following basis elements of representation $D(1,2,1)$ of algebra $hg(1,3)$:

$$S_a = \begin{pmatrix} s_a & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{3\times1} & 0 & 0 \\ \mathbf{0}_{3\times1} & 0 & 0 \end{pmatrix}, \ \eta_a = \begin{pmatrix} \mathbf{0}_{3\times3} & k_a^\dagger & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 0 & 0 \\ k_a & 0 & 0 \end{pmatrix}. \tag{21}$$

5. Galilean massless fields

Galilean massless equations can be obtained using different approaches. We will use contractions of relativistic wave equations.

1. Galilean limits of Maxwell's equations

There are two Galilean limits of Maxwell's equations. In the so called "magnetic" Galilean limit we receive a pre-Maxwellian electromagnetism with equations:

$$\begin{aligned} &\nabla\times\mathbf{E}_m - \tfrac{\partial\mathbf{H}_m}{\partial t} = 0, \quad \nabla\cdot\mathbf{E}_m = e j_m^0, \\ &\nabla\times\mathbf{H}_m = e\mathbf{j}_m, \qquad\quad \nabla\cdot\mathbf{H}_m = 0. \end{aligned} \tag{22}$$

Equations (22) are invariant with respect to the Galilei transformations (1) provided $\mathbf{H}_m,\ \mathbf{E}_m$ and j cotransform as

$$\begin{aligned} &\mathbf{H}_m \to \mathbf{H}_m,\ \mathbf{E}_m \to \mathbf{E}_m - \mathbf{v}\times\mathbf{H}_m, \\ &\mathbf{j}_m \to \mathbf{j}_m, \quad\ j_m^0 \to j_m^0 + \mathbf{v}\cdot\mathbf{j}_m. \end{aligned} \tag{23}$$

Introducing Galilean vector-potential $A = (A^0, \mathbf{A})$ such that

$$\mathbf{H}_m = \nabla\times\mathbf{A}, \quad \mathbf{E}_m = -\frac{\partial\mathbf{A}}{\partial t} - \nabla A^0 \tag{24}$$

we find that A transforms as:

$$A^0 \to A^0 + \mathbf{v}\cdot\mathbf{A}, \qquad \mathbf{A} \to \mathbf{A}. \tag{25}$$

For second, "electric" Galilean limit of Maxwell's equations we get

$$\begin{array}{ll} \nabla\times\mathbf{H}_e + \frac{\partial \mathbf{E}_e}{\partial t} = e\mathbf{j}_e, & \nabla\cdot\mathbf{E}_e = ej_e^4, \\ \nabla\times\mathbf{E}_e = 0, & \nabla\cdot\mathbf{H}_e = 0, \end{array} \tag{26}$$

with the Galilean transformation:

$$\begin{array}{ll} \mathbf{H}_e \to \mathbf{H}_e + \mathbf{v}\times\mathbf{E}_e, & \mathbf{E}_e \to \mathbf{E}_e, \\ \mathbf{j}_e \to \mathbf{j}_e + \mathbf{v}j_e^4, & j_e^4 \to j_e^4. \end{array} \tag{27}$$

$\mathbf{H}_e$ and $\mathbf{E}_e$ can be expressed as

$$\mathbf{H}_e = \nabla\times\mathbf{A}, \quad \mathbf{E}_e = -\nabla A^4 \tag{28}$$

with the Galilei transformations of A:

$$A^4 \to A^4, \ \mathbf{A} \to \mathbf{A} + \mathbf{v}A^4. \tag{29}$$

These results (obtained by Le Bellac and Levi-Leblond) can be clearly interpreted using representations and contractions discussed previously. Indeed, there exist exactly two non-equivalent representations of $HG(1,3)$ the carrier spaces of which are four-vectors – the representations $D(1,1,0)$ and $D(1,1,1)$. Equations for massless fields invariant with respect to these transformations are given by relations (22) and (26) respectively.

Both representations, i.e., $D(1,1,0)$ and $D(1,1,1)$, can be obtained via contractions of the representation $D(1/2,1/2)$ of $so(1,3)$ whose carrier space is formed by relativistic four-vectors. Each of these contractions generates the Galilean limit of Maxwell's equations which we considered in the above.

2. Extended Galilean electromagnetism

Thus, Maxwell's electrodynamics can be contracted either to the magnetic (22) or to the electric limit (26). Each of them corresponds to a particular IDR of $HG(1,3)$.

But we know that there are 9 such representations for vector fields. Maybe, the number of possible Galilei invariant equations for such fields is more than 2 ?

This conjecture is correct, there are many of them. And they can be obtained via contraction of relativistic equations.

Let us start with relativistic equations for the vector-potential A^μ

$$p^\mu p_\mu A^\nu = -ej^\nu \tag{30}$$

in the Lorentz gauge $p_\mu A^\mu = 0$ or $p_0 A^0 = \mathbf{p}\cdot\mathbf{A}$.

Consider also the inhomogeneous d'Alembert equation for a relativistic scalar field which we denote as A^4:

$$p^\mu p_\mu A^4 = ej^4. \tag{31}$$

Introducing the related vectors of the field strengthes in the standard form

$$\mathbf{H} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial x_0} - \nabla A^0, \ \mathbf{F} = \nabla A^4, \ F^0 = \frac{\partial A^4}{\partial x_0} \tag{32}$$

we come to Maxwell's equations for $\mathbf{E}$ and $\mathbf{H}$:

$$\begin{aligned} &\nabla \times \mathbf{E} - \frac{\partial \mathbf{H}}{\partial x_0} = 0, \quad \nabla \cdot \mathbf{H} = 0, \\ &\nabla \times \mathbf{H} + \frac{\partial \mathbf{E}}{\partial x_0} = e\mathbf{j}, \qquad \nabla \cdot \mathbf{E} = ej^0 \end{aligned} \tag{33}$$

and the following equations for $\mathbf{F}$ and F^0

$$\begin{aligned} &\frac{\partial F^0}{\partial x_0} + \nabla \cdot \mathbf{F} = ej^4, \\ &\nabla \times \mathbf{F} = 0, \quad \frac{\partial \mathbf{F}}{\partial x_0} = \nabla F^0. \end{aligned} \tag{34}$$

Surely the system of equations (33) and (34) is completely decoupled. Rather surprisingly its Galilean counterpart which we obtain using the Inönü-Wigner contraction appears to be coupled. This contraction, i.e., $D(\frac{1}{2},\frac{1}{2}) \oplus D(0,0) \rightarrow D(1,2,1)$, was considered in the above. The contracted equations are:

$$\begin{aligned} &\nabla \cdot \mathbf{N} - \frac{\partial}{\partial t}B - ej^0 = 0, \qquad \nabla \times \mathbf{W} + \nabla B - e\mathbf{j} = 0, \\ &\nabla \cdot \mathbf{R} - ej^4 = 0, \qquad \frac{\partial}{\partial t}\mathbf{W} + \nabla \times \mathbf{N} = 0, \\ &\frac{\partial}{\partial t}\mathbf{R} - \nabla B = 0, \qquad \boldsymbol{\mathcal{R}} \equiv \nabla \times \mathbf{R} = 0, \qquad \nabla \cdot \mathbf{W} = 0, \end{aligned} \tag{35}$$

where

$$\mathbf{W} = \nabla \times \mathbf{A}', \quad \mathbf{N} = -\frac{\partial \mathbf{A}'}{\partial t} - \nabla A'^0, \quad \mathbf{R} = \nabla A'^4, \quad B = \frac{\partial A'^4}{\partial t} \tag{36}$$

and $A'^0, A'^4, \mathbf{A}'$ are components of contracted relativistic potentials.

The system of equations (35) is invariant w.r.t. the Galilei group provided the current $j = (j^0, \mathbf{j}, j^4)$ co-transforms in accordance with representation $D(1,2,1)$.

3. Reduced Galilean electromagnetism

The finite dimensional representations of $HG(1,3)$ are indecomposable but reducible. Thus, in contrast with the relativistic case the Galilei invariant approach makes it possible to reduce the number of field variables. For example, considering the magnetic limit (22) of the Maxwell equations it is possible to restrict ourselves to the case $\mathbf{H}_m = 0$ in as much as this condition is invariant with respect to the Galilei transformations (23). Notice that in the relativistic theory such condition can be imposed only in a particular frame of references and will be affected by the Lorentz transformation.

Staring with equations (35) and considering its possible reductions one can find many other Galilei invariant equations for massless vector fields. Let me present the completed list of them. In addition to equations presented above this list includes the following systems:

$$\begin{aligned} &\frac{\partial}{\partial t}\tilde{\mathbf{H}} + \nabla\times\tilde{\mathbf{E}} = 0, \qquad \nabla\times\tilde{\mathbf{H}} = e\mathbf{j}, \qquad \nabla\cdot\tilde{\mathbf{H}} = 0, \\ &\nabla\cdot\tilde{\mathbf{E}} = \frac{\partial}{\partial t}S + ej^0, \qquad \nabla S = 0; \end{aligned} \tag{37}$$

$$\begin{aligned} &\nabla\times\mathbf{H} + \frac{\partial}{\partial t}\mathbf{E} - e\mathbf{j} = 0, \quad \nabla\cdot\mathbf{E} - ej^4 = 0, \quad \frac{\partial}{\partial t}\mathbf{E} - \nabla B = 0, \\ &\nabla\times\mathbf{E} = 0, \qquad\qquad\qquad \nabla\cdot\mathbf{H} = 0; \end{aligned} \tag{38}$$

$$\nabla\cdot\mathbf{E} - ej^4 = 0, \qquad \frac{\partial}{\partial t}\mathbf{E} - \nabla B = 0, \qquad \nabla\times\mathbf{E} = 0; \tag{39}$$

$$\nabla\times\widehat{\mathbf{E}} = 0, \qquad \nabla\cdot\widehat{\mathbf{E}} = ej^4; \tag{40}$$

$$\nabla\times\hat{\mathbf{H}} = e\mathbf{j}, \qquad \nabla\cdot\hat{\mathbf{H}} = 0 \tag{41}$$

We find all non-equivalent reductions of equations (22), (26) and also equations of "extended Galiean electromagnetism". In this way we find all possible Galilei invariant equations for massless vector fields. The numbers of field components described by these equations are 3, 4, 5, 6, 7, 8 and 10, which is in accordance with the dimensions of IDR discussed in the above.

4. Non-linear equations for vector fields

Starting with the found indecomposable representations for $HG(1,3)$ it is possible to find all possible PDE admitting these symmetries. In particular – equations for massive fields and also non-linear equations.

An example of a nonlinear equation is the following system:

$$\begin{aligned}
&\frac{\partial \mathbf{V}}{\partial t} = \nabla B,\\
&\frac{\partial \mathbf{W}}{\partial t} = -\nabla \times \mathbf{N},\\
&\frac{\partial B}{\partial t} = -\nabla \cdot \mathbf{N} + \mathbf{W} \cdot \mathbf{N},\\
&\nabla \times \mathbf{V} = 0, \qquad \nabla \cdot \mathbf{V} = -\mathbf{V} \cdot \mathbf{W},\\
&\nabla \cdot \mathbf{W} = 0, \qquad \nabla \times \mathbf{W} = -\nabla B + B\mathbf{W} + \mathbf{V} \times \mathbf{N}
\end{aligned} \tag{42}$$

which is nothing but a Galilei-invariant analogue of the Carroll-Field-Jackiw model which was formulated with a view to examine the possibility of Lorentz and CPT violations in Maxwell's electrodynamics and is invariant neither w.r.t. Lorentz nor w.r.t. Galilei transformations.

6. Discussion

Our knowledge of indecomposable representations of homogeneous Galilei group defined in vector and scalar fields [2] made it possible to complete the results of Le Bellac and Lévy-Leblond [1] and present a complete class of Galilei-invariant equations for massless vector fields.

It is necessary to stress that the majority of obtained equations admit clear physical interpretations. Thus equation (40) and (41) are applied in electro- and magnetostatics respectively.

We see that the number of Galilean wave equations for massless vector fields is rather extended, and so there are many possibilities to describe interaction of non-relativistic charged particles with external gauge fields. Starting with the found equations and using the list of functional invariants for Galilean vector fields presented in [3] it is easy to construct nonlinear models invariant with respect to the Galilei group, including its supersymmetric extensions.

Invariance of any particular found equation with respect to the Galilei transformations can be verified by direct calculation. The main result presented in this topic is the completed description of all such equations.

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