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for my family

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CONTENTS

Chapter 1

Prelude

In the beginning of the twentieth century two fundamental discoveries in theoretical physics completely revolutionized our understanding of the physical world around us. The first, Einstein's theory of relativity, led to a deeper understanding of the nature of space and time, the concept of mass and ultimately, the general theory of relativity provided a beautiful geometrical description of gravity, generalizing Newton's theory. The second, quantum mechanics, provided an extremely successful description of the subatomic world, explaining previously puzzling properties of light and atoms. However, that could not be the end of the story. Theoretical physicists have always sought to find the ultimate theory of nature describing all phenomena, from the subatomic to the extra-galactic scale. This instinct had led Maxwell in the late ninetienth century to unify electricity and magnetism in his mathematically beautiful theory of electromagnetism. So theorists were soon trying to put relativity and quantum mechanics into one packet, a quantum theory of gravity.

The quest for unification dominated most of twentieth century theoretical physics. By the 1950s, Richard Feynman, Julian Schwinger and Tomonaga Shin'ichiro, based on earlier work by Paul Dirac, Wolfgang Pauli and others, had successfully unified the classical theory of electromagnetism - a theory which is inherently special relativistic - with quantum mechanics. The offspring of this merging was the theory known as quantum electrodynamics (QED). It was the first example of a consistent quantum field theory (QFT), a theory which forced us to replace the notion of Newtonian particles with that of fields pervading spacetime. What we call particles, became now the local excitations, or ripples, of fields which can propagate in space and time.

By that time, however, two more fundamental forces of nature, namely the weak and the strong nuclear forces, had been discovered and a plethora of new particles was being detected in cosmic ray experiments and later in accelerators. After decades of intense experimental and theoretical work by many physicists, Sheldon Glashow, Abdus Salam, and Steven Weinberg proposed in the late 1960s the electroweak theory, a theory that unifies the weak interaction with quantum electrodynamics. This was the first example of a new type of a quantum field theory, known as Yang-Mills theory or non-abelian gauge theory. However, it was not until 1971, when Gerard 't Hooft proved the renormalizability of Yang-Mills theory - an essential property of any meaningful QFT - that the theory of electroweak interactions was accepted as a viable quantum field theory. This, together with further work by Frank Wilczek, David Gross and David Politzer and by Gerard 't Hooft, paved the way for the formulation of quantum chromodynamics (QCD) a few years later, another Yang-Mills theory that describes the strong nuclear force. The culmination of all this theoretical work, and decades of brilliant experimental discoveries, was what is known today as the 'standard model of particle physics', which effectively describes the interactions between all known fundamental particles and forces in nature - except gravity.

Various attempts at quantizing the general theory of relativity ended in failure. Unlike Maxwell's classical theory of the electromagnetic field or classical Yang-Mills theory, the perturbative quantization of general relativity leads to uncontrollable infinities which render the resulting quantum theory useless. Technically, one says that Einstein's gravity is not a 'renormalizable' field theory. Even when supergravity - a generalized theory of gravity that possesses a powerful symmetry between bosons and fermions known as 'supersymmetry' - was discovered in 1976 by Daniel Freedman, Peter van Nieuwenhuizen and Sergio Ferrara, the problem of infinities in the perturbative quantization of gravity was not resolved. Nevertheless, Stephen Hawking succeeded in 1974 to combine general relativity and quantum field theory in a semiclassical calculation and predicted that black holes are thermodynamic objects and radiate as black bodies.

In the same way that quantum field theory, the successful merging of special relativity and quantum mechanics, taught us that we should not think of particles as tiny billiard balls but as localized excitations of fields, a quantum theory of gravity would await an even more radical revision of the concept of a 'particle': there are no particles as such - they are all different vibration modes of one-dimensional extended objects, known as 'strings'.[1] Even though this idea might sound crazy and arbitrary at first, it is precisely what is needed in order to cure the undesirable infinities in the perturbative quantization of general relativity. The fundamental reason why these infinities arise in the first place is that 'gravitons', the quanta of the gravitational force, interact at a single point in spacetime. This is no longer the case in string theory, however. Gravitons arise as certain vibration modes of strings, which are extended objects, and so they no longer interact at a single point. This is one of the main reasons why string theory provides a consistent quantum theory of gravity. Moreover, it turns out that replacing point particles by strings, and not by higher-dimensional extended objects, is not arbitrary at all. The perturbative quantization of higher-dimensional extended objects is simply not consistent - remarkably, for the same reason that gravity cannot be quantized perturbatively. Higher-dimensional extended objects, such as membranes and the so-called 'D-branes', do arise in string theory however, but (at least so far) not as the fundamental degrees of freedom.

It is amusing that, historically, string theory did not first arise as a quantum theory of gravity. Its roots in fact go back to the mid 1960s when physicists were trying to develop a theory of the strong intercations. The so-called 'dual string models' were put forward to explain the huge number of hadrons - strongly interacting particles - that were being detected in particle accelerators. However, it was soon realized that these string models were not the correct description of the strong interaction and they were abandoned when QCD was discovered. In 1974, however, Joel Scherk

[1] In fact, these one dimensional extended objects can be thought of as the localized excitations of a 'string field', in the same way that point particles arise as the local excitations of fields in QFT, but there is no complete formulation of string field theory yet.

and John Schwarz suggested that string theory could provide a theory of quantum gravity. Although, this idea was not considered seriously for many years, it initiated a tremendous effort in theoretical physics and mathematics, which has resulted in the mathematically beautiful modern string theory and which is still continuing undiminished.

In 1974, however, there was another important development that connected the strong interaction with string theory in a completely new way. It was QCD itself however that was being connected with string theory this time. Soon after its discovery, it was realized that very little could be said about the low energy regime of quantum chromodynamics. In this limit QCD is strongly coupled and the perturbative calculations one most often relies on cannot be trusted. Since QCD is a highly non-linear theory, however, there is hardly any alternative analytic method of calculation besides perturbation theory. Gerard 't Hooft then suggested an alternative line of attack, relying on the fact that QCD is an $SU(3)$ gauge theory - this means that particles that interact via the strong interaction carry three types of 'charge', or three 'colors'. He observed that if one considers an $SU(N)$ gauge theory instead and allows the number of colors N to become very large, even infinite, then the theory simplifies significantly. If one can calculate the properties of the theory for $N \to \infty$, then, instead of the standard perturbation expansion in the QCD coupling constant, one can use a perturbative expansion in $1/N$ around the $N \to \infty$ limit. Although, admittedly, the hope of approximating $N = 3$ by $N = \infty$ in this fashion would seem to defy any attempt at a logical justification, this approximation remarkably does capture some of the features of QCD. What is even more remarkable though is that the expansion in $1/N$ turns out to be a topological expansion in the genus of compact Riemann surfaces. This is precisely the sort of expansion that appears in perturbative string theory as well! Since the $N \to \infty$ limit of QCD is still too difficult to solve, the corresponding string theory is still missing.

For many years this bold idea of 't Hooft remained largely vague, mainly because, even in the large N limit, gauge theories are still very complicated theories. Two very different lines of research, however, merged when Juan Maldacena conjectured in 1997 that a supersymmetric version of QCD, the $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory in four dimensions is dual to Type IIB superstring theory. The latter is the offspring of many years of research on string theory as a theory of quantum gravity - such theories had been put forward precisely because conventional QFT could not accommodate gravity. Suddenly, a consistent theory of quantum gravity was conjectured to be equivalent to a QFT without gravity! This was then the first concrete realization of the *holographic principle*, introduced by 't Hooft in 1993 and substantiated by Leonard Susskind in 1994. According to this principle, quantum gravity requires that all dynamical degrees of freedom of a gravitational theory in D-dimensions are localized in a $(D-1)$-dimensional space without gravity - in a way analogous to the way a two-dimensional holographic image encodes information about a three-dimensional object.

The above conjectured duality between $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory and Type IIB superstring theory, known as the *AdS/CFT correspondence*, if true, is a very useful tool, both from the point of view of quantum gravity, but also from the QFT perspective. First, although superstring theory as we know it is a consistent theory of quantum gravity, at present we only have a perturbative definition of string theory. The AdS/CFT correspondence, however, identifies a particular string theory with a gauge theory, of which we have a full non-perturbative definition. In this sense then the AdS/CFT conjecture provides a non-perturbative definition of string theory, and hence, of quantum gravity.

On the other hand, strongly coupled gauge theories are very difficult to study. Remarkably, the AdS/CFT correspondence relates the strongly coupled regime of the gauge theory to the low energy limit of string theory, which is classical supergravity - a much more tractable theory! This is precisely the point of view we adopt in this thesis. We will study how information about the strongly coupled gauge theory is encoded in classical gravity and we will develop techniques that allow one to extract this information in the most efficient way.

ORGANIZATION OF THE THESIS

This thesis begins with an introduction to the basic ideas of superstring theory and the AdS/CFT correspondence in Chapter 2. I have decided to include this material in the hope that it will provide a reasonably self-contained but at the same time succinct introduction to the subject of the gauge/gravity duality. Of course, in no way do I claim to have succeeded in reaching this goal. Given the vast and ever growing literature on the subject, however, and being strongly influenced by my own frustration at trying to navigate through this volume of information, I could not resist the temptation of taking some extra time to write this introductory material. I feel my effort will be justified if one or two graduate - and why not undergraduate - students find it useful.

Chapter 3 is a significantly revised and expanded version of the paper [1] with Kostas Skenderis. After reviewing the concept of 'asymptotically locally AdS spaces' and their relevance for the AdS/CFT correspondence in Section 3.1, I present systematically the method of holographic renormalization for the computation of general renormalized correlation functions of the gauge theory using classical supergravity, both in its original incarnation (Section 3.2) and in the Hamiltonian formalism developed in [1] (Section 3.3). The method is then applied to some examples in Section 3.4 and some general results valid in any dimension are derived. Moreover, a section on AdS_3 has been included, where some results that were not published elsewhere are presented.

Chapter 4 is based on the paper [2] with Kostas Skenderis and it concerns the evaluation of correlation functions in holographic renormalization group flows. First, Poincaré domain walls are considered in Section 4.1 and their field theory interpretation in terms of deformations by marginal operators or vacuum expectation values of scalar operators is substantiated. In the beginning of the section on Poincaré domain walls I have included some unpublished results on a domain wall solution that was discovered in [1]. AdS-sliced domain walls are then discussed in Section 4.2. The rest of the chapter is devoted to an extensive analysis of the geometry and the holographic correlation functions of the Janus solution, a non-supersymmetric but stable dilatonic AdS-sliced domain wall solution of gauge supergravity.

Finally, in Chapter 5, I present results reported in [3] with Kostas Skenderis. This chapter concerns certain properties of asymptotically AdS black holes. After some preliminary considerations, the variational problem for AdS gravity with Dirichlet boundary conditions is formulated in Section 5.1.3. General derivations of the conserved charges of asymptotically locally AdS black holes are presented in Section 5.2, followed by a general proof of the first law of black hole mechanics for such black holes in Section 5.3. The chapter concludes with some applications in Section 5.4.

Omissions

Regretably, the time frame for writing this thesis, as well as, the distinct nature of the content made me decide not to include here my work with Professor M. Cvetič on supersymmetric standard-like model building and Yukawa couplings calculations in the context of orientifold compactifications of Type IIA string theory. However, details of this work can be found in the three publications [4, 5, 6] with Professor Cvetič.

Chapter 2

Superstring Theory & the AdS/CFT Correspondence

This chapter is intended as a pedagogical introduction to the subject of gauge/gravity dualities. Of course, it is by no means an authoritative or thourough review of this vast subject. My aim is to present, from my limited point of view, the basic background necessary to understand the subject of the rest of this thesis. With this in mind, I have tried to present the material in a self-contained manner, to the extent this was possible. Inevitably, many important aspects of the story are not even mentioned, and I often had to refer to other sources for material I decided not to include. However, I have made no attempt to cite the original papers or even to cite any of the relevant papers in this chapter, as this would be an almost impossible task. Instead, I cite various reviews or textbooks, and occasionally some papers, where I think this is useful.

I begin in Section 2.1 with a review of string theory, with emphasis on Type IIB superstrings, which is the string theory relevant for the AdS/CFT duality. This section ends with a discussion of the low energy limit of this string theory, namely Type IIB supergravity, as well as, its various p-brane solutions and their modern understanding as D-branes. Some aspects of the maximally supersymmetric Yang-Mills gauge theory in four dimensions are then presented in Section 2.2. The AdS/CFT correspondence is discussed in Section 2.3, with particular emphasis on the supergravity approximation of the duality and on the calculation of correlation functions. Various technical results are collected in the appendices.

2.1 Type IIB Superstrings

Strings are one-dimensional extended objects which move in a D-dimensional ambient spacetime, M. As they move they span a two-dimensional surface, Σ, which is referred to as the 'world-sheet'. This is the analogue of the 'world-line' that is traversed by a zero-dimensional object, or a particle. The dynamics of such a particle is equivalent to the statement that its world-line between any two fixed points has minimum proper length. Analogously, the dynamics of a string follow from a 'least area principle'. If we take for now the 'target' spacetime M to be flat D-dimensional Minkowski space, and

$$X : \Sigma \hookrightarrow M \tag{2.1}$$

is the embedding map of the world-sheet into M, then the proper area of Σ is given by

$$\sqrt{-\det \partial_a X^\mu \partial_b X_\mu}, \tag{2.2}$$

where $a, b = 0, 1$ run over the world-sheet coordinates $\{\sigma^0, \sigma^1\}$, with $-\infty < \sigma^0 < +\infty,\ 0 \le \sigma^1 < 2\pi$, and target space indices $\mu = 0, \dots, D-1$ are lowered with the flat Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}\,(-1, 1, \cdots, 1)$. So, the world-sheet trajectory,

$X^\mu(\sigma^0, \sigma^1)$, is required to be a local minimum of the *Nambu-Goto action*

$$S_{NG}[X] = -T \int_\Sigma d^2\sigma \sqrt{-\det \partial_a X^\mu \partial_b X_\mu}. \tag{2.3}$$

The constant T is the string tension and is related to the Regge slope[1] α' by

$$T = \frac{1}{2\pi\alpha'}. \tag{2.4}$$

We now want to describe the *quantum* dynamics of strings. One would ideally want to have a *second* quantized formulation of string theory in terms of string *fields*, analogous to the ordinary quantum field theoretic description of particles. A second quantized description would allow for a deeper understanding of the off-shell and non-perturbative properties of strings, such as dynamical symmetry breaking and vacuum selection. However, it has proved particularly difficult to formulate such a description of string theory. An extensive discussion of many efforts in this direction can be found in [7]. Lacking a general second quantized formulation, we resort to a *first* quantized formulation, where the embedding map $X^\mu(\sigma^0, \sigma^1)$ of a single string is quantized.

The Nambu-Goto action (2.3) would be the natural starting point for such a first quantized description of strings were it not for the non-polynomial dependence on the embedding X^μ and its derivatives. Although some effort has been put into quantizing the Nambu-Goto action, it is customary to use a classically equivalent action which is polynomial in X^μ as the starting point for a quantum description of strings. The *Polyakov action*

$$S_P[X,\gamma] = -\frac{T}{2} \int_\Sigma d^2\sigma \sqrt{-\gamma}\gamma^{ab} \partial_a X^\mu \partial_b X_\mu \tag{2.5}$$

is seen to be equivalent to (2.3) upon eliminating the world-sheet metric γ_{ab} using its equation of motion. Nevertheless, there is no guarantee that the quantum theories following from the Nambu-Goto and the Polyakov actions are equivalent, but it is believed that this is the case, at least in certain special cases.

2.1.1 LOCAL SYMMETRIES

The action (2.5) has a number of local as well as global symmetries which play a crucial role in the quantization of the system. Starting with the local symmetries,

[1]This terminology originates in the early days of string theory, when it was put forward as an effective description of certain aspects of hadronic physics. In particular, a multitude of hadronic resonances were discovered and it was observed that the lightest particle with a given spin J satisfied the relation $m^2 = J/\alpha'$ for some constant α'. It later emerged that such behavior is characteristic of spinning strings with tension given by (2.4).

the Polyakov action is invariant under world-sheet diffeomorphisms

$$\sigma^a \longmapsto f^a(\sigma), \tag{2.6}$$

under which the world-sheet metric γ_{ab} transforms as a second-rank tensor and the embedding map X^μ as a scalar. Moreover, local Weyl rescalings of the world-sheet metric

$$\gamma_{ab} \longmapsto e^{2\omega(\sigma)}\gamma_{ab}, \tag{2.7}$$

are also a symmetry of (2.5). Although world-volume diffeomorphism invariance is a symmetry of the generalization of the Polyakov action for higher-dimensional extended objects such as membranes, local Weyl invariance is unique to strings. To understand the consequences of this symmetry we start with the observation that the Polyakov action contains no derivatives of the world-sheet metric, which is therefore non-dynamical. In fact this remains true even if we add an Einstein-Hilbert term to the Polyakov action, since such a term is topological, namely the Euler characteristic of the world-sheet Σ. The integration over the world-sheet metric γ_{ab} in the path integral then simply imposes the constraint

$$T_{ab} = -\frac{2}{T}\frac{1}{\sqrt{-\gamma}}\frac{\delta S_P}{\delta\gamma^{ab}} = 0, \tag{2.8}$$

where

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X^\mu \partial_d X_\mu \tag{2.9}$$

is the stress tensor of the embedding X^μ. It is traceless as a consequence of Weyl invariance of the Polyakov action. Another special property of any two-dimensional (pseudo)-Riemannian manifold Σ is that it is *conformally flat*. In other words, the space of metrics on Σ, $\mathrm{Met}(\Sigma)$, *locally* takes the form $\mathrm{Met}(\Sigma) \approx \mathrm{Diff}(\Sigma) \times \mathrm{Weyl}(\Sigma)$. Indeed, the world-sheet metric γ_{ab} has three independent components and so it is always possible to transform it locally to the flat Minkowski metric η_{ab} by means of a diffeomorphism (two independent functions) and a Weyl transformation (one independent function). It follows that we can *gauge-fix* the world-sheet metric, γ_{ab}, to the flat metric η_{ab} and then replace the path integral over γ_{ab} by the volume of the gauge (i.e. local symmetry) group, namely $\mathrm{Diff}(\Sigma) \times \mathrm{Weyl}(\Sigma)$, as long as we impose the constraint (2.8) on the Hilbert space of X^μ. Of course $\mathrm{Met}(\Sigma) \approx \mathrm{Diff}(\Sigma) \times \mathrm{Weyl}(\Sigma)$ does not hold globally and the path integral contains a sum over world-sheet topologies as well as an integral over the *moduli space* of Riemann surfaces

$$\mathcal{M}_\Sigma = \mathrm{Met}(\Sigma)/\mathrm{Diff}(\Sigma) \times \mathrm{Weyl}(\Sigma). \tag{2.10}$$

A careful analysis of the Jacobian resulting from the replacement of the path integral over $\mathrm{Met}(\Sigma)$ with an integral over the moduli space $\mathcal{M}_\Sigma$ leads to the covariant BRST quantization of the Polyakov string. A clear exposition of this procedure can be found in D'Hoker's lectures in [8].

From now on we will consider Euclidean world-sheets Σ and we will gauge-fix the metric to the flat metric δ_{ab}. It will also be convenient to introduce complex coordinates

$$z = \sigma^1 + i\sigma^2, \qquad \bar{z} = \sigma^1 - i\sigma^2, \tag{2.11}$$

where $\sigma^2 = i\sigma^0$. The gauge-fixed Euclidean version of the action (2.5) is

$$S = \frac{1}{2\pi\alpha'} \int_\Sigma d^2z \partial_z X^\mu \partial_{\bar{z}} X_\mu. \tag{2.12}$$

The reason why the perturbative quantization of string theory is tractable is precisely that this gauge-fixed action still possesses a large local symmetry group. The *conformal transformation*

$$z \longmapsto f(z), \qquad \bar{z} \longmapsto \bar{f}(\bar{z}), \tag{2.13}$$

where $f(z)$ is an arbitrary analytic function, leaves (2.12) invariant. The Hilbert space of the Polyakov string is therefore described by a two-dimensional *conformal field theory* (CFT). Since the conformal group in two dimensions is infinite-dimensional, these are highly constrained quantum field theories which in many cases can be solved exactly. We will not discuss the CFT description or the spectrum of the bosonic string that we have studied so far, but instead we will later discuss the CFT description and the spectrum of the superstring, which will be directly relevant to the subject of this thesis.

2.1.2 GLOBAL SYMMETRIES

Having discussed the local symmetries of the Polyakov action, let us now examine its *global symmetries*. Global symmetries are the 'internal' symmetries acting on the embedding X^μ but *not* on the world-sheet coordinates or metric. The gauge-fixed action (2.12) then has the same global symmetries as the Polyakov action (2.5), namely translations and rotations in D-dimensional Minkowski spacetime, which together make up the Poincaré group in D dimensions.

So far we have assumed that the target space manifold M is flat Minkowski spacetime but this is not necessary. Indeed the field theory defined by (2.12) makes sense in an arbitrary background M with a Lorentzian metric $G_{\mu\nu}(X)$ or even with an antisymmetric tensor field $B_{\mu\nu}(X)$. As we shall see, such background fields arise as coherent states of the low energy string spectrum. We therefore consider the *non-linear sigma model*

$$S = \frac{1}{2\pi\alpha'} \int_\Sigma d^2z \left(G_{\mu\nu}(X) + B_{\mu\nu}(X)\right) \partial_z X^\mu \partial_{\bar{z}} X^\nu. \tag{2.14}$$

The fact that this action leads to a consistent quantum field theory is another miraculous property of the two-dimensional world-sheet. Such a non-linear sigma model on the world-volume of a higher-dimensional object, such as a membrane, leads to a non-renormalizable quantum field theory, which is therefore meaningless. The global symmetry group of the action (2.14) is not necessarily the Poincaré group but rather the *isometry* group of the background metric and antisymmetric B-field.

2.1.3 World-sheet versus target space supersymmetry

Bosonic string theory in flat Minkowski spacetime as described by the CFT defined by the action (2.12) has a number of drawbacks. Most importantly, its spectrum contains a *tachyon*, i.e. a state of negative mass, which means that flat space is an *unstable* vacuum of bosonic string theory. Although such an instability could be addressed in the context of a second quantized formulation of string theory, it renders the perturbative first quantized theory completely meaningless. Moreover, the *perturbative* spectrum of bosonic string theory contains no fermionic states. Even though this is not an inconsistency of the bosonic theory in itself, it shows that bosonic string theory cannot possibly provide a description of the fermions, such as electrons and quarks, that we observe in the real world. Again, in the context of a second quantized string theory, this would not necessarily rule out the bosonic string which could possibly arise as a particular vacuum of the theory, while the theory possesses other vacua too whose perturbative spectrum does contain fermions.

Since we are lacking a satisfactory second quantized formulation of string theory, we will have to *guess* other possible vacua of the theory which do contain fermions. In practice this means that we will try to modify the string action (2.14) in such a way that the theory can still be quantized perturbatively, it is perturbatively stable - i.e. no tachyons - and the spectrum contains fermions. There are two common approaches to this problem. First, the so-called *Green-Schwarz* (GS) superstring theory, introduces *target space* fermions on the world-sheet, which transform under the spinor representation of the Lorentz group $SO(1,D-1)$, in addition to the target space vector X^μ. It turns out that the maximal number of independent spinors one can add and still have a consistent theory is two. The corresponding theory then has $\mathcal{N}=2$ *target space* supersymmetry. The *Neveu-Schwarz-Ramond* (RNS) formulation of superstrings, on the other hand, introduces *world-sheet* fermions on the world-sheet. The presence of fermions in the perturbative spectrum in this formulation is less obvious, but it can be shown that the two formulations, at least within the framework of the so-called 'light-cone quantization', are equivalent and indeed lead to precisely the same theories. However, the GS superstring turns out to be very difficult - or impossible - to quantize in a manifestly covariant way. We will therefore follow the RNS formulation below to derive the essential features of superstrings that we will need later.

The first step is to generalize the bosonic action (2.14). Recall that a crucial property of this action is that it is invariant under conformal transformations on the plane. It is then only reasonable to require that we maintain this property while introducing in addition world-sheet supersymmetry. It turns out that these symmetries combine to form a larger symmetry known as 'superconformal' symmetry. We explain how this symmetry arises as a generalization of conformal symmetry in two dimensions in Appendix 2.A.1. Roughly speaking, a superconformal transformation is an analytic diffeomorphism on the super-complex plane, or superspace, parameterized by two commuting coordinates $z, \bar{z}$ and two anticommuting coordinates $\theta, \bar{\theta}$. Demanding that the world-sheet action is invariant under superconformal transformations and its bosonic part is given by (2.14), determines its form completely.

To see this first note that superconformal invariance requires that the bosonic fields X^μ and their supersymmetry partners we are about to introduce must transform consistently under superconformal transformations, i.e. they must form a so-called 'superconformal multiplet'. This is ensured if we combine these fields into a *superconformal tensor*, which is defined in Appendix 2.A.1. In particular, the most general superconformal tensor which contains X^μ as its bosonic part takes the form

$$\mathbb{X}^\mu = \sqrt{\frac{2}{\alpha'}} X^\mu + i\theta\psi_+^\mu + i\bar{\theta}\psi_-^\mu + \theta\bar{\theta}F^\mu, \tag{2.15}$$

where $\psi_\pm^\mu(z,\bar{z})$ are anticommuting functions on the world-sheet and $F^\mu(z,\bar{z})$ is a commuting *auxiliary* field. The fields $D_+\mathbb{X}^\mu$ and $D_-\mathbb{X}^\mu$ then transform as superconformal tensors of weight $(1,0)$ and $(0,1)$ respectively. It is then easy to show that the action

$$S = \frac{1}{4\pi}\int_\Sigma d^2z d^2\theta \left(G_{\mu\nu}(\mathbb{X}) + B_{\mu\nu}(\mathbb{X})\right) D_-\mathbb{X}^\mu D_+\mathbb{X}^\nu \tag{2.16}$$

is invariant under superconformal transformations. Moreover, carrying out the integrations over the anticommuting coordinates $\theta, \bar{\theta}$ we find (after eliminating the auxiliary field using its equation of motion)

$$\begin{aligned} S = \frac{1}{4\pi}\int_\Sigma d^2z \bigg\{ &\frac{2}{\alpha'}\left(G_{\mu\nu} + B_{\mu\nu}\right)\partial_z X^\mu \partial_{\bar{z}} X^\nu + G_{\mu\nu}\psi_+^\mu \mathcal{D}_{\bar{z}}\psi_+^\nu \\ &- G_{\mu\nu}\psi_-^\mu \mathcal{D}_z \psi_-^\nu + \frac{1}{2}\mathcal{R}_{\mu\nu\rho\sigma}\psi_+^\mu\psi_+^\nu\psi_-^\rho\psi_-^\sigma \bigg\}, \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} \mathcal{D}_{\bar{z}}\psi_+^\mu &= \partial_{\bar{z}}\psi_+^\mu + \left(\Gamma^\mu_{\rho\sigma} + \frac{1}{2}H^\mu{}_{\rho\sigma}\right)\sqrt{\frac{2}{\alpha'}}\partial_{\bar{z}}X^\rho\psi_+^\sigma, \\ \mathcal{D}_z\psi_-^\mu &= \partial_z\psi_-^\mu + \left(\Gamma^\mu_{\rho\sigma} - \frac{1}{2}H^\mu{}_{\rho\sigma}\right)\sqrt{\frac{2}{\alpha'}}\partial_z X^\rho\psi_-^\sigma, \end{aligned} \tag{2.18}$$

$$\mathcal{R}_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{2}\nabla_\rho H_{\mu\sigma\nu} - \frac{1}{2}\nabla_\sigma H_{\mu\rho\nu} + \frac{1}{4}H_{\lambda\mu\sigma}H^\lambda{}_{\nu\rho} - \frac{1}{4}H_{\lambda\mu\rho}H^\lambda{}_{\nu\sigma}, \tag{2.19}$$

and

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu}, \tag{2.20}$$

with $\Gamma^\mu_{\rho\sigma}$ and $R_{\mu\nu\rho\sigma}$ denoting respectively the Christoffel symbol and the Riemann tensor of the metric $G_{\mu\nu}$. Note that after the θ and $\bar\theta$ integrations, the background metric and antisymmetric B-field, as well as their curvatures, are evaluated at $\sqrt{\frac{2}{\alpha'}}X$. The action (2.17) is the desired generalization of (2.14) since it has the same bosonic part[2] and it possesses superconformal symmetry.

2.1.4 Free superstrings

We will soon return to the supersymmetric non-linear sigma model action (2.17) and its significance for the low energy effective description of the string dynamics. However, let us now examine the superstring spectrum in flat Minkowski background with vanishing B-field. Although the superspace formulation is compact and powerful, we will use the component formulation here to make the discussion more transparent. The superstring action then takes the form

$$S = \frac{1}{4\pi}\int_\Sigma d^2z \left(\frac{2}{\alpha'}\partial_z X^\mu \partial_{\bar z} X_\mu + \psi^\mu_+ \partial_{\bar z}\psi_{+\mu} - \psi^\mu_- \partial_z \psi_{-\mu}\right). \tag{2.21}$$

As we already know, it is invariant under conformal transformations as well as the supersymmetry transformation

$$\delta_\epsilon X^\mu = -i\sqrt{\frac{\alpha'}{2}}(\epsilon\psi^\mu_+ + \bar\epsilon\psi^\mu_-), \quad \delta_\epsilon\psi^\mu_+ = i\epsilon\sqrt{\frac{2}{\alpha'}}\partial_z X^\mu, \quad \delta_\epsilon\psi^\mu_- = -i\bar\epsilon\sqrt{\frac{2}{\alpha'}}\partial_{\bar z} X^\mu, \tag{2.22}$$

where $\epsilon(z)$ is an infinitesimal anticommuting analytic function. The conserved Noether currents for these symmetries are respectively

$$T(z) = -\frac{1}{\alpha'}\partial_z X^\mu \partial_z X_\mu - \frac{1}{2}\psi^\mu_+ \partial_z \psi_{+\mu}, \qquad T_F(z) = i\sqrt{\frac{2}{\alpha'}}\psi^\mu_+ \partial_z X_\mu, \tag{2.23}$$

together with their antiholomorphic counterparts. These currents generate the full $\mathcal{N}=1$ superconformal algebra in two dimensions.

At this point we should recall that conformal symmetry appeared in the gauge-fixed bosonic action (2.12) as a left-over symmetry from the bigger $\mathrm{Diff}(\Sigma)\times\mathrm{Weyl}(\Sigma)$ invariance of the Polyakov action (2.5). As a consequence, we concluded that one should impose the constraint (2.8) on the Hilbert space. In complex coordinates, $z, \bar z$, the tracelessness of the stress tensor implies $T_{z\bar z} = 0$, while the components

[2] In (2.12) we have omitted the factors $\sqrt{2/\alpha'}$ in the argument of the background fields to simplify the notation.

T_{zz} and $T_{\bar{z}\bar{z}}$ are respectively the Noether currents, $T(z)$ and $\bar{T}(\bar{z})$, that generate conformal transformations. Therefore, we must impose the constraints

$$T(z) = 0, \qquad \bar{T}(\bar{z}) = 0 \tag{2.24}$$

on the Hilbert space of the superstring. As we will see, these constraints ensure that the negative-norm states in the Hilbert space of the bosonic fields X^μ decouple. Although this would be sufficient for the bosonic string, in the superstring there are more negative-norm states in the Hilbert space of the fermionic fields $\psi^\mu_\pm$. In order for these negative-norm states to decouple it turns out that one should impose the constraints

$$T_F(z) = 0, \qquad \bar{T}_F(\bar{z}) = 0 \tag{2.25}$$

on the Hilbert space of the superstring. However, constraints cannot be imposed at one's will. They must be imposed by a path integral over an auxiliary field such as the world-sheet metric γ_{ab}, in the case of the Virasoro constraint (2.8). Indeed, the superconformal invariant action (2.21) turns out to be the gauge-fixed version of an action with $s\mathrm{Diff}(\Sigma) \times s\mathrm{Weyl}(\Sigma)$ invariance, i.e. with *local* world-sheet supersymmetry, or *supergravity*. The fermionic constraints (2.25) are then imposed by the path integral over the supersymmetric partner of the world-sheet metric, namely the world-sheet *gravitino*, χ^α_a, where α is an index in the Dirac spinor representation of the world-sheet $SO(2)$ local frame group, while a is the usual world-sheet index. However, we will not need the explicit form of the world-sheet supergravity action here. It is sufficient to know that such an action exists and hence, the constraint (2.25) can be imposed consistently.

OPEs

Let us now return to the gauge-fixed action. The equations of motion are

$$\partial_z \partial_{\bar{z}} X^\mu = 0, \qquad \partial_{\bar{z}} \psi^\mu_+ = 0, \qquad \partial_z \psi^\mu_- = 0, \tag{2.26}$$

so that $\psi^\mu_+(z)$ is holomorphic, $\psi^\mu_-(\bar{z})$ is antiholomorphic and X^μ is the sum of a holomorphic and antiholomorphic part, namely $X^\mu(z,\bar{z}) = X^\mu_+(z) + X^\mu_-(\bar{z})$. Since the action is quadratic in the world-sheet fields we can immediately obtain the two-point functions

$$\langle X^\mu_+(z) X^\nu_+(w)\rangle = -\frac{\alpha'}{2}\eta^{\mu\nu}\log(z-w), \qquad \langle \psi^\mu_+(z)\psi^\nu_+(w)\rangle = \frac{\eta^{\mu\nu}}{z-w}, \tag{2.27}$$

and analogously for the antiholomorphic parts, while all other two-point functions vanish. This implies that the superstring Hilbert space will take the form

$$\mathcal{H}_{\text{superstring}} \approx \mathcal{H}_{X_+} \otimes \mathcal{H}_{\psi_+} \otimes \bar{\mathcal{H}}_{X_-} \otimes \bar{\mathcal{H}}_{\psi_-}. \tag{2.28}$$

Since the antiholomorphic part $\bar{\mathcal{H}}_{X_-} \otimes \bar{\mathcal{H}}_{\psi_-}$ is essentially a copy of the holomorphic part $\mathcal{H}_{X_+} \otimes \mathcal{H}_{\psi_+}$, we will focus for the time being on the holomorphic part alone. Moreover, the X_+ and ψ_+ Hilbert spaces can be discussed separately.

The above two-point functions can be encoded in the so-called 'Operator Product Expansions' (OPEs)

$$X_+^\mu(z) X_+^\nu(w) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\log(z-w), \qquad \psi_+^\mu(z)\psi_+^\nu(w) \sim \frac{\eta^{\mu\nu}}{z-w}, \tag{2.29}$$

where X^μ and ψ_+^μ are now treated as quantum operators. The OPEs reflect the singularity structure of the product of two operators when they approach each other, i.e. as $z \to w$. In order to make sense of *composite operators*, such as the Noether currents (2.23), at the quantum level we must introduce a procedure for removing these singularities. Such a procedure is called 'normal ordering' and is denoted by $:\ \ :$. For example,

$$T(z) = -\frac{1}{\alpha'} : \partial_z X_+^\mu(z) \partial_z X_{+\mu}(z) : -\frac{1}{2} : \psi_+^\mu(z)\partial_z \psi_{+\mu}(z) :, \tag{2.30}$$

where

$$\begin{aligned} : \partial_z X_+^\mu(z) \partial_z X_{+\mu}(z) &:= \lim_{w\to z} \left\{ \partial_w X_+^\mu(w) \partial_z X_{+\mu}(z) - \langle \partial_w X_+^\mu(w) \partial_z X_{+\mu}(z) \rangle \right\}, \\ : \psi_+^\mu(z)\partial_z \psi_{+\mu}(z) &:= \lim_{w\to z} \left\{ \psi_+^\mu(w)\partial_z \psi_{+\mu}(z) - \langle \psi_+^\mu(w)\partial_z \psi_{+\mu}(z) \rangle \right\}. \end{aligned} \tag{2.31}$$

From the OPEs (2.29) of the fundamental fields and the explicit form of the superconformal Noether currents one can derive the following OPEs:

$$\begin{aligned} T(z)\partial_z X_+^\mu(w) &\sim \frac{1}{(z-w)^2}\partial_w X_+^\mu(w) + \frac{1}{z-w}\partial_w^2 X_+^\mu(w) + \cdots, \\ T(z)\psi_+^\mu(w) &\sim \frac{1/2}{(z-w)^2}\psi_+^\mu(w) + \frac{1}{z-w}\partial_w \psi_+^\mu(w) + \cdots, \end{aligned} \tag{2.32}$$

$$\begin{aligned} T_F(z) X_+^\mu(w) &\sim -\frac{1}{z-w} i\sqrt{\frac{\alpha'}{2}}\psi_+^\mu(w) + \cdots, \\ T_F(z)\psi_+^\mu(w) &\sim \frac{1}{z-w} i\sqrt{\frac{2}{\alpha'}}\partial_w X_+^\mu(w) + \cdots, \end{aligned} \tag{2.33}$$

$$\begin{aligned} T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + \cdots, \\ T(z)T_F(w) &\sim \frac{3/2}{(z-w)^2}T_F(w) + \frac{1}{z-w}\partial_w T_F(w) + \cdots, \\ T_F(z)T_F(w) &\sim \frac{2c/3}{(z-w)^3} + \frac{2}{z-w}T(w) + \cdots, \end{aligned} \tag{2.34}$$

where $c = 3D/2$.

The first two OPEs simply reflect the fact that $\partial_z X^\mu_+(z)$ and $\psi^\mu_+(z)$ are conformal tensors (see (2.197)), or *conformal primary fields*, of conformal weight 1 and $1/2$ respectively. The second two OPEs are precisely the supersymmetry transformations (2.22) since, for example,

$$\delta_\epsilon \psi^\mu_+(z) = \{Q_F, \psi^\mu_+(z)\} = \oint \frac{dw}{2\pi i}\epsilon(w) T_F(w)\psi^\mu_+(z) = i\epsilon(z)\sqrt{\frac{2}{\alpha'}}\partial_z X^\mu_+(z), \quad (2.35)$$

where the last equality results from the contour integral around the simple pole in the $T_F\psi^\mu_+$ OPE. Together these OPEs are equivalent to the statement that the superfield $D_+\mathbb{X}^\mu = i\psi^\mu_+(z) + \theta\sqrt{\frac{2}{\alpha'}}\partial_z X^\mu_+(z)$ is a superconformal tensor (see (2.198)), or *superconformal primary field*, of weight $1/2$ (of course this refers to the holomorphic weight. We are supressing the antiholomorphic weight, which is zero in this case.)

The last three OPEs, however, contain crucial new information on the string dynamics. Since $T(z)$ and $T_F(z)$ are the Noether currents of the superconformal symmetry, we expect that their OPEs are equivalent to the $\mathcal{N} = 1$ superconformal algebra in two dimensions. This is indeed the case as we will see soon, with one caveat though. The algebra one would obtain at the classical level by extending the super-Euclidean algebra of Appendix 2.A.1 to include arbitrary conformal transformations corresponds the OPEs (2.34) but with the parameter c equal to zero. Such OPEs would imply that the Noether currents $T(z)$ and $T_F(z)$ are conformal primary fields of weight 2 and $3/2$ respectively, or equivalently, that the supercurrent $\mathbb{T}(z,\theta) = -\frac{1}{2}T_F(z) + \theta T(z)$ is a superconformal primary field of weight $3/2$. However, a non-zero c means that the conformal transformation of the stress tensor $T(z)$ and the supersymmetry transformation of the fermionic current $T_F(z)$ are *anomalous*. In particular, integrating the infinitesimal conformal transformation of the stress tensor

$$\delta_\epsilon T(z) = -\oint \frac{dw}{2\pi i}\epsilon(w)T(w)T(z) = -\frac{c}{12}\partial_z^3\epsilon(z) - 2T(z)\partial_z\epsilon(z) - \epsilon(z)\partial_z T(z), \quad (2.36)$$

one finds that under a finite conformal transformation $z \longmapsto z'(z)$

$$T'(z') = (\partial_z z')^{-2}\left(T(z) - \frac{c}{12}\{z';z\}\right), \quad (2.37)$$

where

$$\{z';z\} = \frac{\partial_z^3 z'}{\partial_z z'} - \frac{3}{2}\frac{(\partial_z^2 z')^2}{(\partial_z z')^2} \quad (2.38)$$

is known as the 'Schwarzian derivative' of z' with respect to z. This transformation rule is precisely the transformation of a conformal primary field of weight $(2,0)$ except for the term involving c, which is therefore anomalous.

The conformal anomaly

To understand the significance of this anomalous term in the two-point function of the stress tensor we must revisit the gauge-fixing of the Polyakov action (2.5). Namely, we argued that one can use the world-sheet diffeomorphism and Weyl invariance of the classical action to make the world-sheet metric flat. However, it is not guaranteed that we will be able to maintain these symmetries at the quantum level since a regulator that preserves both these symmetries may not exist. Indeed, in order to be able to consistently integrate over all world-sheet metrics, the sigma model must still make sense as a quantum theory for any *fixed* world-sheet metric, γ_{ab}. Consider then the correlation function

$$\langle \cdots \rangle_\gamma = \int [dX] \cdots e^{-S[X,\gamma]}, \tag{2.39}$$

where $S[X,\gamma]$ is the Euclidean version of the Polyakov action (2.5) and γ is an arbitrary but fixed world-sheet metric. By the definition of the stress tensor we have

$$\delta_\gamma \langle \cdots \rangle_\gamma = \frac{1}{4\pi} \int d^2\sigma \sqrt{\gamma} \delta\gamma^{ab}(\sigma) \langle T_{ab}(\sigma) \cdots \rangle_\gamma. \tag{2.40}$$

Using this identity we can expand the partition function

$$Z[\gamma] = \int [dX] e^{-S[X,\gamma]} \tag{2.41}$$

to second order around the flat metric δ_{ab}. The result is (see Polchinski [9], eq. (3.4.22))

$$\log \frac{Z[\delta + \delta\gamma]}{Z[\delta]} \approx \frac{1}{8\pi} \int d^2 z \int d^2 z' \delta\gamma_{\bar{z}\bar{z}}(z,\bar{z}) \delta\gamma_{\bar{z}\bar{z}}(z',\bar{z}') \langle T_{zz}(z) T_{zz}(z') \rangle_\delta, \tag{2.42}$$

where only terms quadratic in $\delta\gamma_{\bar{z}\bar{z}}$ have been kept. Inserting the anomalous two-point function of the stress tensor and integrating this expression one obtains, in covariant form,

$$Z[\gamma] = Z[\delta] \exp\left(\frac{c}{96\pi} \int d^2\sigma \sqrt{\gamma} R_\gamma \Box_\gamma^{-1} R_\gamma \right), \tag{2.43}$$

where $c = D$ for the X^μ action alone. However this result is general and holds for any covariant world-sheet action that defines a (free) CFT. However, the value of the parameter c does depend on the particular theory. The exponent on the right hand side is known as the 'Polyakov non-local action', not to be confused with the Polyakov string action (2.5). It is a direct manifestation of the fact that Weyl invariance (and hence conformal invariance) is broken at the quantum level. To see this we first evaluate

$$\delta\left(\sqrt{\gamma} R_\gamma \Box_\gamma^{-1} R_\gamma\right) = 2\delta\gamma_{ab} \mathcal{T}^{ab}, \tag{2.44}$$

where $\mathcal{T}^{ab}$ is the so-called Liouville stress tensor and is given by

$$\mathcal{T}_{ab} \equiv \frac{1}{2}\nabla_a(\Box_\gamma^{-1}R_\gamma)\nabla_b(\Box_\gamma^{-1}R_\gamma) - \nabla_a\nabla_b(\Box_\gamma^{-1}R_\gamma) - \frac{1}{2}\gamma_{ab}\left[\frac{1}{2}\nabla_c(\Box_\gamma^{-1}R_\gamma)\nabla^c(\Box_\gamma^{-1}R_\gamma) - 2R_\gamma\right]. \tag{2.45}$$

Since $\mathcal{T}^a_a = R_\gamma$, it follows that under an infinitesimal Weyl transformation, $\gamma_{ab} = 2\delta\omega(\sigma)\delta_{ab}$,

$$\delta_\omega Z[\delta] = \frac{c}{24\pi}\int d^2\sigma\delta\omega(\sigma)\langle R_\gamma\rangle_\delta. \tag{2.46}$$

Comparing with (2.40), we conclude

$$\langle T^a_a\rangle_\gamma = -\frac{c}{12}\langle R_\gamma\rangle_\gamma. \tag{2.47}$$

This now tells us directly that conformal invariance is broken at the quantum level and the parameter c, which is known as the *central charge* for reasons we will discuss shortly, measures this breaking. Although we have derived this result in two dimensions, it is actually quite universal. Throughout this thesis we will see many aspects of the conformal anomaly in various dimensions.

Although this would not be disastrous if we were just interested in the CFT on a curved world-sheet, it is unacceptable if we want to integrate over arbitrary world-sheet metrics. Since we have shown above that for the superstring $c = 3D/2$, we seem to be in trouble. What saves the day is the way the gauge-fixing is implemented at the quantum level via the covariant BRST quantization procedure. Namely, the gauge fixing introduces Faddeev-Popov ghosts for both the X^μ and $\psi^\mu_\pm$ CFTs. These unphysical degrees of freedom define a superconformal field theory themselves with central charge $c_{\text{ghosts}} = -15$. We will not need the explicit structure of these sCFTs for our discussion. The only piece of information we need is the fact that the total central charge of the superstring CFT is

$$c_{\text{total}} = \frac{3D}{2} + c_{\text{ghosts}} = \frac{3}{2}(D-10). \tag{2.48}$$

Hence, the superstring makes sense only in ten spacetime dimensions! Even though this result may not be very encouraging for someone who wants to see superstring theory as a theory describing our four-dimensional physical world, let us accept it at this point and see how far we can get.

MODE EXPANSIONS & MODE ALGEBRAS

Let us now construct explicitly the string Hilbert space. To this end we first need to impose suitable *boundary conditions* on the dynamical fields. Here we only discuss the *closed* string, that is we consider world-sheets that are topologically a cylinder, which is conformally equivalent to the complex plane with the origin removed. The bosonic fields, X^μ, then must be single-valued on the complex plane, which implies that we should impose the periodic boundary condition

$$X^\mu(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = X^\mu(z,\bar{z}). \tag{2.49}$$

However, the fermionic fields $\psi^\mu_+(z)$ and $\psi^\mu_-(\bar{z})$ are only required to be *doubly* periodic. This allows for two possible boundary conditions, namely

$$\psi^\mu_+(e^{2\pi i}z) = \pm\psi^\mu_+(z). \tag{2.50}$$

Each of these boundary conditions leads to a different sector of the ψ^μ_+ Hilbert space. The sector with periodic boundary conditions on the plane is known as the *Neveu-Schwarz* or NS sector, while antiperiodic fermions on the plane lead to the *Ramond* or R sector. The fermion Hilbert space is then the direct sum of these two sectors

$$\mathcal{H}_{\psi_+} = \mathcal{H}^{NS}_{\psi_+} \oplus \mathcal{H}^{R}_{\psi_+}. \tag{2.51}$$

Since $\partial_z X^\mu$ and ψ^μ_+ are holomorphic functions on the annulus $\mathbb{C} - \{0\}$ by the equations of motion, they admit Laurent expansions in the vicinity of the origin. Taking into account the possible boundary conditions we arrive at the *mode expansions*

$$\begin{aligned} X^\mu_+(z) &= \frac{1}{2}x^\mu_+ - i\sqrt{\frac{\alpha'}{2}}\alpha^\mu_0 \log z + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}-\{0\}}\frac{1}{n}\frac{\alpha^\mu_n}{z^n}, \\ \psi^\mu_+(z) &= \sum_{r\in\mathbb{Z}+\nu}\frac{\psi^\mu_r}{z^{r+\frac{1}{2}}}, \qquad \begin{cases} \nu = \frac{1}{2}, & \text{for NS} \\ \nu = 0, & \text{for R} \end{cases} \end{aligned} \tag{2.52}$$

where

$$\alpha^\mu_n = \sqrt{\frac{2}{\alpha'}}\oint\frac{dz}{2\pi}z^n\partial_z X^\mu_+(z), \qquad \psi^\mu_r = \oint\frac{dz}{2\pi i}z^{r-\frac{1}{2}}\psi^\mu_+(z). \tag{2.53}$$

Note that $X^\mu_+(z)$ is not single-valued due to the logarithmic term, but the sum $X^\mu(z,\bar{z}) = X^\mu_+(z) + X^\mu_-(\bar{z})$ is single-valued provided $\alpha^\mu_0 = \tilde{\alpha}'^\mu_0$, where $\tilde{\alpha}'^\mu_0$ is the zero mode of the antiholomorphic field $X^\mu_-(\bar{z})$. In fact

$$p^\mu = \frac{1}{2}(p^\mu_+ + p^\mu_-) = \frac{1}{2\pi\alpha'}\oint\left(dz\partial_z X^\mu - d\bar{z}\partial_{\bar{z}}X^\mu\right), \tag{2.54}$$

where $p^\mu_+ = \sqrt{\frac{2}{\alpha'}}\alpha^\mu_0$ and $p^\mu_- = \sqrt{\frac{2}{\alpha'}}\tilde{\alpha}^\mu_0$, is the total momentum carried by the closed string. Similarly, $x^\mu = \frac{1}{2}(x^\mu_+ + x^\mu_-)$ is the center of mass position of the string. Using the OPEs (2.29) one can now determine the mode algebra, which is

$$[x^\mu, p^\mu] = i\eta^{\mu\nu}, \qquad [\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu}, \qquad \{\psi^\mu_r, \psi^\mu_s\} = \delta_{r+s,0}\eta^{\mu\nu}. \tag{2.55}$$

The Noether currents (2.23) also admit Laurent expansions of the form

$$T(z) = \sum_{m\in\mathbb{Z}} \frac{L_m}{z^{m+2}}, \qquad T_F(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{G_r}{z^{r+\frac{3}{2}}}, \tag{2.56}$$

where ν is the *same* ν that appears in the mode expansion of ψ_+^μ in (2.52), i.e. $\nu = 1/2$ for the NS sector and $\nu = 0$ for the R sector. From the current OPEs (2.34) we now derive the mode algebra

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \\
[L_m, G_r] &= \frac{1}{2}(m-2r)G_{m+r}, \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{c}{12}(4r^2-1)\delta_{r+s,0}.
\end{aligned} \tag{2.57}$$

This algebra is the infinite dimensional $\mathcal{N} = 1$ superconformal (or super-Virasoro) algebra in two dimensions. It is a so-called 'central extension' of the classical superconformal algebra corresponding to $c = 0$. Thus the name 'central charge' for the parameter c.

NAIVE SPECTRUM & SUPER-VIRASORO CONSTRAINTS

NS sector

We are now in a position to define the Hilbert space $\mathcal{H}_{X_+} \otimes \mathcal{H}_{\psi_+}^{NS}$. The vacuum of $\mathcal{H}_{X_+}$ is labeled by the eigenvalue of the zero mode of X_+^μ which is the momentum operator p^μ. Since there are no fermionic zero modes in the NS sector we define the vacuum by

$$p^\mu \left|0,k\right\rangle_{NS} = k^\mu \left|0,k\right\rangle_{NS}, \quad \alpha_m^\mu \left|0,k\right\rangle_{NS} = 0, \quad \psi_r^\mu \left|0,k\right\rangle_{NS} = 0, \quad \text{for} \quad m,r>0. \tag{2.58}$$

The full Fock space is now constructed by acting with the negative modes on the vacuum in all possible ways.

R sector

Consider next $\mathcal{H}_{X_+} \otimes \mathcal{H}_{\psi_+}^{R}$. Of course $\mathcal{H}_{X_+}$ remains the same but now there are fermionic zero modes satisfying the $SO(1, D-1)$ *Clifford algebra*

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}, \tag{2.59}$$

where $\Gamma^\mu \equiv \sqrt{2}\psi_0^\mu$. The Ramond sector vacuum then carries a representation of this Clifford algebra, namely

$$\psi_0^\mu \left|0,\alpha,k\right\rangle_R = \frac{1}{\sqrt{2}}(\Gamma^\mu)^\beta{}_\alpha \left|0,\beta,k\right\rangle_R, \tag{2.60}$$

where $(\Gamma^\mu)^\beta{}_\alpha$ belong to the usual $2^{D/2}$-dimensional matrix representation of the Clifford algebra. It follows that the Ramond vacuum is a *Dirac spinor* of $SO(1, D-1)$, for it transforms under the Lorentz generators, $J_0^{\mu\nu} = -\frac{i}{4}[\Gamma^\mu, \Gamma^\nu]$, as

$$J_0^{\mu\nu} |0, \alpha, k\rangle_R = (\Sigma^{\mu\nu})^\beta{}_\alpha |0, \beta, k\rangle_R . \tag{2.61}$$

Otherwise, the Ramond vacuum is defined as usual by

$$p^\mu |0, \alpha, k\rangle_R = k^\mu |0, \alpha, k\rangle_R , \quad \alpha_m^\mu |0, \alpha, k\rangle_R = 0, \quad \psi_r^\mu |0, \alpha, k\rangle_R = 0, \quad \text{for} \quad m, r > 0, \tag{2.62}$$

and the Fock space is built by acting on the vacuum by the negative modes in all possible ways.

Physical states

In order to construct the physical Hilbert space we need to impose the super-Virasoro constraints

$$T(z) = 0, \qquad T_F(z) = 0 \tag{2.63}$$

on the Hilbert space. However, It is not possible to impose these as operator equations and still get a non-trivial theory. Instead we will require that

$$\langle\phi| T(z) |\chi\rangle = 0, \qquad \langle\phi| T_F(z) |\chi\rangle = 0 \tag{2.64}$$

hold for any *physical* states $|\phi\rangle, |\chi\rangle$. We need to keep in mind though that $T(z)$ transforms anomalously under conformal transformations and so the first condition does not have an invariant meaning. So we should specify whether we impose this on the cylinder or the annulus, the two being conformally related. It turns out that in order for all negative norm states to decouple we need to impose this condition on the *cylinder*.

To impose the physical state conditions on the Hilbert space we first express the superconformal current modes in terms of the modes of the fundamental fields. Using the definitions (2.23) we easily find

$$\begin{aligned} L_m &= \frac{1}{2} \sum_{n \in \mathbb{Z}} {}^\circ_\circ \alpha^\mu_{m-n} \alpha_{n\mu} {}^\circ_\circ + \frac{1}{4} \sum_{r \in \mathbb{Z}+\nu} (2r - m) {}^\circ_\circ \psi^\mu_{m-r} \psi_{r\mu} {}^\circ_\circ + a(\nu) \delta_{m,0}, \\ G_r &= \sum_{m \in \mathbb{Z}} \psi^\mu_{r-m} \alpha_{m\mu}, \end{aligned} \tag{2.65}$$

where ${}^\circ_\circ \quad {}^\circ_\circ$ denotes 'mode normal ordering' (i.e. negative modes to the left) and the ordering constant $a(1/2) = 0$, $a(0) = D/16$ is simply the vacuum energy

$$a(\nu) = z^2 \langle T(z) \rangle_\nu = \langle L_0 \rangle_\nu . \tag{2.66}$$

The physical state conditions (2.64) now translate into

$$L_n^{\text{cylinder}}\,|\phi\rangle = 0, \qquad G_r\,|\phi\rangle = 0, \qquad \text{for} \quad n, r \geq 0. \tag{2.67}$$

From the anomalous transformation (2.37) of the stress tensor one can derive the relation

$$L_m^{\text{cylinder}} = L_m - \frac{c}{24}\delta_{m,0}, \tag{2.68}$$

between the Virasoro modes on the cylinder and on the annulus, where $c = 3D/2$. However, it turns out that in order for all negative-norm states to decouple we need to use $c' = 3(D-2)/2$ in this relation. This can be understood by going to the lightcone gauge which shows that only $D-2$ spacetime dimensions contribute to the 'physical' degrees of freedom. Since $D = 10$,[3] we have

$$L_m^{\text{cylinder}} = L_m - \frac{1}{2}\delta_{m,0}. \tag{2.69}$$

For the NS vacuum these conditions reduce to

$$(L_0 - \frac{1}{2})\,|0,k\rangle_{NS} = \left(\frac{\alpha'}{4}k^2 - \frac{1}{2}\right)|0,k\rangle_{NS} = 0, \tag{2.70}$$

which means that it has a *negative* mass squared $M^2 \equiv -k^2 = -2/\alpha'$. Since the NS vacuum is also a scalar under the spacetime Lorentz group, this is a tachyon! The existence of a tachyon in the bosonic string spectrum was one of the major motivations for studying the superstring, but it seems this has not solved the problem. However, we will see soon that it is possible to consistently project out the tachyon of the superstring spectrum, which was not possible for the bosonic string.

For the R vacuum the physical state conditions reduce to

$$G_0\,|0,\alpha,k\rangle_R = 0 \iff k_\mu \Gamma^\mu\,|0,\alpha,k\rangle_R = k_\mu (\Gamma^\mu)^\beta{}_\alpha\,|0,\beta,k\rangle_R = 0, \tag{2.71}$$

which is precisely the massless Dirac equation. The Ramond ground state is therefore a massless Dirac spinor.

If we include the antiholomorphic sectors then we can summarize the low energy spectrum as follows:

- $\mathcal{H}_{X_+} \otimes \mathcal{H}^{NS}_{\psi_+} \otimes \bar{\mathcal{H}}_{X_-} \otimes \bar{\mathcal{H}}^{NS}_{\psi_-}$

  The ground state, $|0,k\rangle_{NS} \otimes \overline{|0,k\rangle}_{NS}$, is a tachyon with mass $M^2 = -2/\alpha'$. The first excited states are massless and take the form $\epsilon_{\mu\nu}(k)\psi^\mu_{-1/2}\,|0,k\rangle_{NS} \otimes \bar{\psi}^\nu_{-1/2}\overline{|0,k\rangle}_{NS}$. These can be decomposed into states that transform irreducibly under $SO(1,9)$, namely the *graviton* corresponding to the symmetric traceless part of $\epsilon_{\mu\nu}$, the *antisymmetric B-field* corresponding to the antisymmetric part of $\epsilon_{\mu\nu}$ and the *dilaton* corresponding to the trace of the polarization tensor.

[3] We know this from our previous discussion of the ghost sCFT, but it can also be derived it the present context by the requirement that all negative-norm states decouple.

- $\mathcal{H}_{X_+} \otimes \mathcal{H}^R_{\psi_+} \otimes \bar{\mathcal{H}}_{X_-} \otimes \bar{\mathcal{H}}^{NS}_{\psi_-}$ or $\mathcal{H}_{X_+} \otimes \mathcal{H}^{NS}_{\psi_+} \otimes \bar{\mathcal{H}}_{X_-} \otimes \bar{\mathcal{H}}^R_{\psi_-}$
  These sectors contain spacetime *fermions*. The ground state takes the form $|0, \alpha, k\rangle_R \otimes \zeta_\mu(k)\bar{\psi}^\mu_{-1/2}\overline{|0, k\rangle}_{NS}$, is massless, and can be decomposed into an irreducible spinor, the *dilatino*, and an irreducible vector-spinor, the *gravitino*. Analogously for the NS-R ground state.

- $\mathcal{H}_{X_+} \otimes \mathcal{H}^R_{\psi_+} \otimes \bar{\mathcal{H}}_{X_-} \otimes \bar{\mathcal{H}}^R_{\psi_-}$
  This sector contains again spacetime *bosons*. The ground state $|0, \alpha, k\rangle_R \otimes \overline{|0, \beta, k\rangle}_R$ is massless and decomposes into a direct sum of antisymmetric tensor representations of $SO(1, 9)$.

This spectrum has many of the desirable features, namely it contains a graviton, an antisymmetric B-field and a dilaton, which in fact also appear in the spectrum of the bosonic string, but also spacetime fermions and various antisymmetric tensor fields. However, there is still a tachyon and also, a theory which contains a massless gravitino and is not supersymmetric is ill-defined (see e.g. D'Hoker's lectures in [8]). Since there is a massless gravitino in the spectrum, the only way to make the theory consistent is to project out certain states so that the spectrum becomes supersymmetric. This will automatically project out the tachyon since it obviously has no supersymmetric partner. The Gliozzi-Scherk-Olive (GSO) projection of the RNS superstring spectrum does exactly this!

GSO PROJECTION

To implement the GSO projection we define the world-sheet fermion number operator, F, by the property

$$\{(-1)^F, \psi^\mu_r\} = 0, \quad \forall\, r \tag{2.72}$$

in both NS and R sectors. This determines F uniquely up to an overall sign corresponding to the F-assignment of the vacuum in each sector. Making such a choice for the NS and R vacua, the world-sheet fermion number operator takes the form

$$(-1)^F = \begin{cases} -(-1)^{\sum_{r>0} \psi^\mu_{-r}\psi_{r\mu}}, & \text{for} \quad NS, \\ \Gamma(-1)^{\sum_{r>0} \psi^\mu_{-r}\psi_{r\mu}}, & \text{for} \quad R, \end{cases} \tag{2.73}$$

where $\Gamma = \Gamma^0\Gamma^1 \cdots \Gamma^9$ is the chirality matrix in ten dimensions. The *GSO projection* then amounts to the projections

$$\mathcal{H}^{NS}_{\psi_+} \longmapsto \frac{1}{2}\left(1 + (-1)^F\right)\mathcal{H}^{NS}_{\psi_+}, \qquad \mathcal{H}^R_{\psi_+} \longmapsto \frac{1}{2}\left(1 \pm (-1)^F\right)\mathcal{H}^R_{\psi_+}, \tag{2.74}$$

and similarly for the antiholomorphic sectors. For the Ramond sector there is a freedom to choose the chirality of the ground state and this can be done independently in the holomorphic and antiholomorphic sectors. It is important that this projection is compatible with the physical state conditions since

$$[(-1)^F, L_m] = 0, \qquad \{(-1)^F, G_r\} = 0, \quad \forall\, m, r, \tag{2.75}$$

which follows trivially from the explicit form of $(-1)^F$.

In the NS sector, the GSO projection removes the tachyon while it keeps all massless states. In the R sector, it reduces the R vacuum from a Dirac spinor to a Weyl spinor. In ten dimensions it is possible to impose the Majorana condition together with the Weyl condition, in which case the Ramond ground state becomes a *Majorana-Weyl* spinor, i.e. a Weyl spinor with *real* components. Such a spinor has eight real degrees of freedom after imposing the massless Dirac equation, which precisely matches the eight real degrees of freedom of a massless $SO(1,9)$ vector in the NS sector! This, together with the fact that we have removed the tachyon form the NS sector, suggests that we have succeeded in constructing a supersymmetric spectrum. Indeed, choosing the *same* chirality for the vacuum of the holomorphic and antiholomorphic Ramond sectors, we obtain the massless spectrum of *Type IIB* superstring theory, which is shown in Table 2.1 and is manifestly supersymmetric. It is nothing more than the Clebsch-Gordan decomposition of the tensor product $V \otimes \bar{V}$ of the $SO(1,9)$ representations V of the holomorphic and $\bar{V}$ of the antiholomorphic sectors. In the NS-NS sector this involves the tensor product of two vector representations, in the R-NS and NS-R sectors it involves the tensor product of a vector and a Majorana-Weyl spinor representation, while in the R-R sector one needs to evaluate the tensor product of two Majorana-Weyl spinors of the *same* chirality. The fact that the gravitinos and the dilatinos have opposite chirality and that the four-form $C^{(4)+}_{\mu\nu\rho\sigma}$ has a self-dual field strength can be easily seen from the Clebsch-Gordan decomposition. However, we should point out that it is not the forms $C^{(p)}$ that appear directly in this decomposition, but rather their field strengths $F^{(p+1)}$. Through the Clebsch-Gordan decomposition, the Dirac equation satisfied by the Majorana-Weyl spinors translates into the equations

$$k_{[\mu} F^{(p+1)}_{\nu_1 \ldots \nu_{p+1}]} = 0, \qquad k^\mu F^{(p+1)}_{\mu\nu_2 \ldots \nu_{p+1}} = 0, \tag{2.76}$$

for the antisymmetric tensor fields $F^{(p+1)}$, which are precisely the Fourier-transformed field equations for an abelian higher spin gauge field, namely

$$\mathrm{d}F^{(p+1)} = 0, \qquad *\mathrm{d}F^{(p+1)} = 0. \tag{2.77}$$

The Bianchi identity $\mathrm{d}F^{(p+1)} = 0$ is then solved by putting $F^{(p+1)} = \mathrm{d}C^{(p)}$ locally, which results in the form fields appearing in table 2.1. As we will discuss in the next section, the fact that only the field strengths $F^{(p+1)}$, and not the forms $C^{(p)}$, appear in the perturbative superstring spectrum has physical significance and led to

sector	field		chirality	degrees of freedom
NS-NS	graviton	$G_{\mu\nu}$	-	35
	B-field	$B_{\mu\nu}$	-	28
	dilaton	Φ	-	1
R-NS	gravitino	χ^{α}_{μ}	$+$	56
	dilatino	λ^{α}	$-$	8
NS-R	gravitino	χ'^{α}_{μ}	$+$	56
	dilatino	λ'^{α}	$-$	8
R-R	axion	$C^{(0)}$	-	1
	2-form	$C^{(2)}_{\mu\nu}$	-	28
	4-form	$C^{(4)+}_{\mu\nu\rho\sigma}$	-	35

Table 2.1: The massless spectrum of Type IIB superstring theory.

the discovery of D-branes.[4]

Finally, although we have only seen that the *massless* spectrum of the Type IIB superstring is supersymmetric, it can be shown that this actually holds at each mass level. Indeed, a simple counting of the bosonic and fermionic states at each mass level confirms this.

2.1.5 IIB SUPERGRAVITY

In the previous section we obtained the massless spectrum of the Type IIB superstring and we connected the various states with target space fields, such as the target space *metric* $G_{\mu\nu}$, the *gravitino* χ^{α}_{μ} etc. However, it seems a rather non-trivial statement to say that, for example, the symmetric traceless tensor we obtained in the NS-NS sector is somehow related to the metric $G_{\mu\nu}$ appearing in the non-linear sigma model (2.17)! We will now try to justify why this is in fact true. The requirement that this non-linear sigma model (or a generalization thereof) can be consistently coupled to quantum gravity in two dimensions, i.e. that superconformal invariance survives at the *quantum* level, will then determine the dynamics of these background fields! In fact we got a glimpse of this dynamics in the previous section when we argued that the Dirac equation for the Ramond ground states implies that the RR fields satisfy the field equations (2.77) in flat space. It will turn out that the low energy dynamics of the massless degrees of freedom of Type IIB superstring theory is described by *Type IIB supergravity*.

[4] I am grateful to Professor Massimo Bianchi for bringing to my attention the reference [10], where a R-R 'vertex operator' that couples directly to the scalar potential $C^{(0)}$ of Type IIB string theory was proposed.

So let us first motivate the connection between the superstring spectrum and the background fields in the non-linear sigma model. We will be rather sketchy here since a careful analysis of this connection requires some rather technical tools that we have not explained so far and which we will not need later, such as the ghost sCFT that as we mentioned above arises from the covariant BRST quantization of the superstring, 'vertex operators', 'picture changing' etc. Details about these concepts can be found in standard texts such as Polchinski [9]. For our purposes it suffices to know that *states* in a CFT are in one-to-one correspondence with *operators*. An operator $\phi(z, \bar{z})$, inserted at the origin of the complex plane, 'creates' an asymptotic state $|\phi\rangle$ on the cylinder via

$$|\phi\rangle \equiv \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})\,|0\rangle\,, \tag{2.78}$$

where $|0\rangle$ is the vacuum. An important special case of this operator-state correspondence is the correspondence between conformal primary fields and the so-called 'highest weight states'. In this sense then, the massless states of the superstring spectrum correspond to insertions of certain operators on the world-sheet. These operators, which are known as 'physical vertex operators', can have various different representations in terms of the fundamental world-sheet fields (including the ghosts), each representation corresponding to a so-called different 'picture'. In a certain picture then, the vertex operator that corresponds to the graviton takes roughly the form

$$\mathcal{V}(k,\epsilon) \sim \epsilon_{\mu\nu}(k) \int_{\Sigma} d^2z d^2\theta D_- \mathbb{X}^{\mu} D_+ \mathbb{X}^{\nu} e^{ik\cdot\mathbb{X}}, \tag{2.79}$$

where $\mathbb{X}^{\mu}$ is the world-sheet superfield[5] (2.15) and $\epsilon_{\mu\nu}(k)$ is symmetric and traceless.

Ignoring the B-field in the non-linear sigma model (2.16), take the metric to be infinitesimally close to the flat Minkowski metric

$$G_{\mu\nu}(\mathbb{X}) = \eta_{\mu\nu} + \epsilon_{\mu\nu}(\mathbb{X}), \tag{2.80}$$

and Fourier-transform the linear perturbation so that

$$\epsilon_{\mu\nu}(\mathbb{X}) = \int d^{10}k \epsilon_{\mu\nu}(k) e^{ik\cdot\mathbb{X}}. \tag{2.81}$$

We can now expand the sigma model action (2.16) as

$$e^{-S} = e^{-S_0}\left(1 - \frac{1}{4\pi}\int d^{10}k\epsilon_{\mu\nu}(k)\int_{\Sigma} d^2zd^2\theta D_-\mathbb{X}^{\mu}D_+\mathbb{X}^{\nu}e^{ik\cdot\mathbb{X}} + \cdots\right), \tag{2.82}$$

where S_0 is the world-sheet action corresponding to the flat metric $\eta_{\mu\nu}$. But this shows that fluctuations of the background metric $G_{\mu\nu}$ couple to the graviton vertex operator (2.79)! In other words, the background metric can be viewed as a

[5]We have set $\alpha' = 2$ temporarily to avoid combersome notation.

coherent state of gravitons. Moreover, it can be shown that the physical state conditions that must be imposed on the polarization tensor $\epsilon_{\mu\nu}(k)$, namely tracelessness, transversality $k^\mu\epsilon_{\mu\nu} = 0$, and masslessness $k^2 = 0$, imply Einstein's equations for the background metric $G_{\mu\nu}$! A similar connection between states and background fields can be established for the rest of the massless states in the Type IIB string spectrum.

Now that we have seen some evidence for the relation between the massless superstring spectrum and the background fields in the non-linear sigma model, let us try to understand how the string dynamics determines the dynamics of the background fields. However, there are more background fields coming from the superstring spectrum than just the metric and B-field that appear in the sigma-model action (2.16). Indeed, one can relatively easily generalize the sigma model action to incorporate the dilaton, Φ, but this involves a term proportional to the world-sheet curvature, R, which vanishes for the flat metric we use in (2.16). To write this term properly we would need to develop an $\mathcal{N} = 1$ world-sheet *local* superspace as opposed to the rigid superspace we developed in Appendix 2.A.1, which would take us too far astray. We will therefore content ourselves with the statement that the dilaton coupling is roughly of the form

$$S_{\text{dilaton}} \sim \int_\Sigma d^2z d^2\theta R\Phi(\mathbb{X}), \tag{2.83}$$

while its bosonic part is (exactly)

$$S_{\text{dilaton}} = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{\gamma}R_\gamma\Phi(X). \tag{2.84}$$

Nevertheless, to include the background fields coming from the R-NS, NS-R and R-R sectors in the sigma model action turns out to be even harder due to subtleties relating to the vertex operator that creates the Ramond vacuum. A relatively easier way to approach this problem is to use the Green-Schwarz formulation of the superstring which we will not describe in this thesis. In fact, even if we could write down the full non-linear sigma model action involving all background fields, it would not help our discussion significantly since to determine the dynamics of the background fields one must do a two-loop calculation using this action! This calculation, first done in [11], is far too technical to be included even in most textbooks, but a significant part of the analogous calculation for the *bosonic* string can be found in D'Hoker's lectures [8].

Let us then just describe the basic idea of such a calculation and state the result. We start with the observation that α' has units of $(\text{length})^2$. For backgrounds that have a typical radius of curvature $r_c >> \sqrt{\alpha'}$, the parameter $\sqrt{\alpha'}/r_c$ is then small and can be used in a perturbation expansion. This is analogous to the $\hbar$ (or loop) expansion in standard quantum field theory. Since r_c essentially sets the 'unit of length' in spacetime, a spacetime derivative is of order $1/r_c$ and so the perturbation expansion is actually a *derivative* expansion, i.e. an expansion in $\sqrt{\alpha'}\partial_\mu$, which is dimensionless. The non-linear sigma model can then be quantized perturbatively. Such a quantization involves as usual a regularization scheme, which will break the Weyl invariance of the classical action (since we insist on a $\mathrm{Diff}(\Sigma)$-invariant scheme). As we know this leads to a non-zero trace for the world-sheet stress tensor. On general grounds, the trace of the stress tensor will be proportional to the vacuum expectation values of the relevant and marginal operators.[6] Since there is no tachyon, we are left only with the marginal operators which correspond precisely to the massless spectrum. That is

$$\langle T^a_a \rangle = \Sigma_i \beta^i \langle \mathcal{O}_i \rangle, \tag{2.85}$$

where $\mathcal{O}_i$ are the graviton, B-field, dilaton and other massless vertex operators. The 'beta functions' β^i can now be computed perturbatively in the α'-expansion. For the NS-NS fields the result is [11]

$$\begin{aligned}
\beta^G_{\mu\nu} &= \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} + \mathcal{O}(\alpha'^2), \\
\beta^B_{\mu\nu} &= -\frac{\alpha'}{2} \nabla^\rho H_{\rho\mu\nu} + \alpha' \nabla^\rho \Phi H_{\rho\mu\nu} + + \mathcal{O}(\alpha'^2), \\
\beta^\Phi &= -\frac{\alpha'}{2} \Box \Phi + \alpha' \nabla^\mu \Phi \nabla^\mu \Phi + \frac{\alpha'}{24} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{\alpha'}{2} R + \mathcal{O}(\alpha'^2).
\end{aligned} \tag{2.86}$$

The condition that the sigma model is Weyl-invariant at the quantum level is then

$$\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0. \tag{2.87}$$

The equation $\beta^\Phi = 0$ follows from the other two, which can be viewed as the *dynamical equations for the background fields*! They can be derived from the action

$$S^{NS}_{IIB} = \frac{1}{2\kappa^2_{10}} \int d^{10}x \sqrt{-G} e^{-2\Phi} \left(R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \mathcal{O}(\alpha') \right), \tag{2.88}$$

where $\kappa^2_{10} = 8\pi G_{10}$ and G_{10} it Newton's constant in ten dimensions. Inclusion of the other massless fields leads to the full Type IIB supergravity which we will now summarize.

IIB SUPERGRAVITY AND ITS SYMMETRIES

There is a unique supergravity in ten dimensions with the spectrum given in table 2.1, namely Type IIB supergravity with $\mathcal{N} = 2$ supersymmetry, which is described in

[6] There is also a pure anomaly term which arises even when the vevs of these operators vanish, as is the case for a flat target-space background. As we saw above, this is a constant multiple of the world-sheet Ricci scalar which can be absorbed as a constant shift in the coefficient of the dilaton vev. However, if we include the contribution of the ghosts such a term does not arise in the critical dimension.

chapter 12 of Polchinski [9]. For completeness and since IIB supergravity will play an important role in the rest of this thesis, we repeat its main features here.

We will only consider the bosonic fields, setting the gravitinos and the dilatinos to zero. The dynamics of the NS-NS fields is given by the action (2.88) above, so it remains to describe the dynamics of the R-R forms $C^{(0)}, C^{(2)}, C^{(4)+}$. Quite naturally this is described by a Maxwell-type kinetic term and a Chern-Simons coupling:

$$S^R_{IIB} = -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left(F_{(1)}^2 + \tilde{F}_{(3)}^2 + \frac{1}{2}\tilde{F}_{(5)}^{+2}\right) - \frac{1}{4\kappa_{10}^2}\int C^{(4)+}\wedge H\wedge F_{(3)}, \tag{2.89}$$

where

$$\begin{aligned}\tilde{F}_{(3)} &= \mathrm{d}C^{(2)} - C^{(0)}\wedge H,\\ \tilde{F}_{(5)}^{+2} &= \mathrm{d}C^{(4)+} - \frac{1}{2}C^{(2)}\wedge H + \frac{1}{2}B\wedge F_{(3)},\end{aligned} \tag{2.90}$$

and ${F_{(p)}}^2 \equiv \frac{1}{p!}F_{\mu_1\ldots\mu_p}F^{\mu_1\ldots\mu_p}$. However, this action must be supplemented by the self-duality condition

$$*\tilde{F}_{(5)}^{+} = \tilde{F}_{(5)}^{+}, \tag{2.91}$$

which cannot be deduced from a covariant action. The self-duality condition must be imposed on the *solutions* though and not at the level of the action since this would result in incorrect equations of motion.

The above action of Type IIB supergravity has an important $SL(2,\mathbb{R})$ symmetry. Using the field redefinitions

$$\begin{aligned}&G_{E\mu\nu} = e^{-\Phi/2}G_{\mu\nu}, \qquad \tau = C^{(0)} + ie^{-\Phi},\\ &\mathcal{M}_{ij} = \frac{1}{\mathrm{Im}\,\tau}\begin{pmatrix} |\tau|^2 & -\mathrm{Re}\,\tau \\ -\mathrm{Re}\,\tau & 1\end{pmatrix}, \qquad F^i_{(3)} = \begin{pmatrix} H \\ F_{(3)}\end{pmatrix},\end{aligned} \tag{2.92}$$

the action becomes

$$S_{IIB} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_E}\left(R_E - \frac{\partial_\mu\bar{\tau}\partial^\mu\tau}{2(\mathrm{Im}\,\tau)^2} - \frac{1}{2}\mathcal{M}_{ij}F^i_{(3)}\cdot F^j_{(3)} - \frac{1}{4}\tilde{F}_{(5)}^{+2}\right) - \frac{1}{8\kappa_{10}^2}\epsilon_{ij}\int C^{(4)+}\wedge F^i_{(3)}\wedge F^j_{(3)}. \tag{2.93}$$

$SL(2,\mathbb{R})$ now acts as

$$\begin{aligned}&\tau \mapsto \frac{a\tau+b}{c\tau+d}, \qquad F^i_{(3)} \mapsto \Lambda^i_j F^j_{(3)}, \qquad \Lambda^i_j = \begin{pmatrix} d & c \\ b & a\end{pmatrix},\\ &\tilde{F}^+_{(5)} \mapsto \tilde{F}^+_{(5)}, \qquad G_{E\mu\nu} \mapsto G_{E\mu\nu},\end{aligned} \tag{2.94}$$

where $a, b, c, d \in \mathbb{R}$ with $ad-bc = 1$, and leaves the action (2.93) invariant. Although $SL(2,\mathbb{R})$ is a symmetry of the low energy effective theory, only the discrete subgroup $SL(2,\mathbb{Z})$ is a symmetry of the full Type IIB string theory, known as *S-duality*.

2.1.6 P-BRANES VERSUS D-BRANES

The supergravity description of the low energy string dynamics is a powerful tool which can be used in order to extract new information about the structure of the theory. An obvious question is what are the classical solutions of the supergravity field equations and what is their significance in the full string theory? However, given the complexity of the supergravity equations and the variety of solutions one expects for such a system of non-linear partial differential equations, it might be useful to sit back for a moment and look at the *symmetries* of the supergravity action.

We already pointed out in the last section that Type IIB supergravity possesses an $SL(2,\mathbb{R})$ symmetry under which the dilaton mixes with the axion and the NS-NS B-field mixes with the R-R two-form $C^{(2)}$. Let us specifically consider the special $SL(2,\mathbb{R})$ transformation

$$\tau \mapsto -\frac{1}{\tau}, \qquad \left(\begin{array}{c} H \\ F_{(3)} \end{array} \right) \mapsto \left(\begin{array}{c} F_{(3)} \\ -H \end{array} \right), \tag{2.95}$$

which interchanges the NS-NS B-field with the R-R two-form $C^{(2)}$. This result should come as a complete surprise on the basis of our perturbative treatment of the superstring! Somehow the low energy effective description of the theory does not distinguish between B and $C^{(2)}$, although we know that these two fields have completely different origin in perturbative string theory. In particular, we know that the B-field couples *minimally* to the string world-sheet Σ, i.e. its world-sheet action is (cf. (2.14))

$$S_B \sim \int_\Sigma B, \tag{2.96}$$

where B is the pull-back of the form $B_{\mu\nu}\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu$ onto Σ, whereas the R-R vertex operators only involve the field strength $F_{(3)} = \mathrm{d}C^{(2)}$. Borrowing the terminology from classical electromagnetism, we say that the string world-sheet is *electrically charged* under the B-field, while it is neutral under the R-R two-form since only the field strength $F_{(3)} = \mathrm{d}C^{(2)}$ couples to it. In fact we have found no object in perturbative string theory that is 'electrically charged' under any of the R-R forms. The fact that the low energy effective action does not differentiate between B and $C^{(2)}$, however, strongly suggests that there must exist objects which are electrically charged under $C^{(2)}$ and indeed under all R-R potentials!

So how do we go about finding these objects? The supergravity action comes to the rescue again. We will look for solutions of the supergravity equations of motion which are electrically charged under a single of the R-R potentials $C^{(p+1)}$. Such solutions should describe $p+1$-dimensional extended objects and will be *singular* in the sense that they will be solutions of the equations of motion with a *delta function source* at the location of the extended object.

Indeed, making an ansatz that possesses an $SO(9-p) \times$ Poincaré$(1,p)$ isometry and preserves half of the original 32 supersymmetries (recall that in Type IIB string theory there are two Majorana-Weyl supercharges of the same chirality coming from the holomorphic and antiholomorphic Ramond vacua, each with 2^4 real components), it is possible to find such solutions. The role of supersymmetry here is crucial. On the practical side, restricting to purely bosonic backgrounds, supersymmetry requires that we put the gravitino and dilatino variations to zero. This leads to *first order* equations in contrast to the second order supergravity equations of motion. Moreover, in most cases a solution to the first order supersymmetry equations is also a solution of the supergravity equations. Although this is not automatic, it is a very powerful tool for simplifying the second order supergravity equations. On the physical side, the fact that these solutions preserve half of the supersymmetries means that they are so-called *BPS* solutions (after Bogomol'nyi, Prasad and Sommerfeld). We will discuss the significance of this property below, but let us first see how these solutions look like.

Since we are looking for purely bosonic solutions which are charged under a single R-R potential $C^{(p+1)}$, we consider the action

$$S_{\text{string}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left[e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi\right) - \frac{1}{2}F_{(p+2)}^2\right], \tag{2.97}$$

or in the *Einstein frame* defined by the Weyl transformation in (2.92)

$$S_E = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_E}\left(R_E - \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}e^{-\frac{1}{2}(p-3)\Phi}F_{(p+2)}^2\right). \tag{2.98}$$

The elementary (electric) *p-brane* solution to the equations of motion that follow from this action takes the form

$$\begin{aligned} ds_E^2 &= H_p^{-(7-p)/8}d\vec{x}^2 + H_p^{(p+1)/8}d\vec{y}^2, \\ C_{01\ldots p}^{(p+1)} &= H_p^{-1} - 1, \\ e^\Phi &= H_p^{-(p-3)/4}, \end{aligned} \tag{2.99}$$

where $x^{\bar{\mu}}$, $\bar{\mu} = 0,\ldots,p$, and y^m, $m = p+1,\ldots,9$, parameterize respectively the directions along and perpendicular to the brane, and

$$H_p = 1 + \frac{k_p}{r^{7-p}}, \qquad p < 7, \tag{2.100}$$

where $r = \sqrt{\vec{y}^2}$, is a harmonic function in the transverse directions, i.e.

$$\Box_\perp H_p = 0. \tag{2.101}$$

k_p is a constant related to the 'electric' charge and ADM mass of the p-brane and we will come back to it shortly.

In writing the p-brane solution we have assumed that the asymptotic value of the dilaton vanishes. However, this need not be the case and indeed the asymptotic value of the dilaton plays a very important role in string theory. From the way the dilaton couples to the world-sheet (see (2.84)) we see that a constant dilaton multiplies the Euler characteristic of the world-sheet[7]

$$\chi \equiv \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{\gamma}R_\gamma = 2-2h, \tag{2.102}$$

where h is the genus of the compact Riemann surface Σ. So it leads to a factor $e^{-\chi\Phi_0}$ in the string partition function which involves a sum over world-sheet topologies. This sum then becomes an expansion in

$$g_s \equiv e^{\Phi_0}, \tag{2.103}$$

which therefore has an interpretation as the *string coupling constant*. If we now include the string coupling in Newton's constant in front of the supergravity action (so that $\kappa_{10} \sim g_s$), then the p-brane solution we presented above remains valid. k_p can then be shown to depend linearly on g_s, while the ADM mass (or tension) and R-R charge of the p-brane, which are equal, turn out to be *inversely* proportional to the string coupling

$$T_p \sim 1/g_s. \tag{2.104}$$

This is indicative of non-perturbative or soliton-like behavior and implies that a p-brane becomes very massive at weak string coupling. Indeed, we already know from the form of the Polyakov action that the tension of the fundamental string is independent of the string coupling. This can also be confirmed by looking for a supergravity solution which is charged under the NS-NS B-field instead of the R-R two-form. Such a solution can in fact be obtained from the $p=1$ brane solution in (2.99) by replacing $C^{(2)} \mapsto B$ and $\Phi \mapsto -\Phi$ and one can easily verify that its ADM mass is independent of the string coupling. p-branes therefore become arbitrarily massive compared to the fundamental strings at weak coupling and one expects that they completely decouple from the perturbative sector. This is in agreement with the fact that we did not find any object carrying RR charge in string perturbation.

As we mentioned above, p-branes are also BPS objects, i.e. they preserve half supersymmetries. BPS objects correspond to very special massive representations of the supersymmetry algebra, called 'short' or BPS multiplets, that have only *half* of the expected number of states. The reason is that such multiplets satisfy the so-called 'BPS bound', which means that the mass is equal to some central charge (e.g. the R-R charge for the p-brane) in the supersymmetry algebra. This in turn implies that half of the supercharges can be consistently set to zero as operators, which leaves us with only half creation operators and hence half of the states. Since relaxing

[7] For unoriented world-sheets or world-sheets with boundaries this formula should be modified.

the BPS mass/charge equality would lead automatically to an abrupt doubling of the states in the supermultiplet, it follows that the BPS condition is robust under adiabatic changes of any continuous parameters in the theory. In particular, the mass/charge equality does not receive any quantum corrections - an example of a 'non-renormalization theorem'.

Another consequence of the BPS property of p-branes, is that they can be superimposed. Indeed, replacing the harmonic function (2.100) in the p-brane solution (2.99) by the multi-center harmonic function

$$H(\vec{r}) = 1 + \sum_i \frac{k_p^i}{|\vec{r} - \vec{r}_i|^{7-p}}, \tag{2.105}$$

leads to an equally good solution of the supergravity equations. Such a solution represents a number of parallel p-branes, each located at position $\vec{r}_i$ in the transverse space. One can even put a number N_i of p-branes on top of each other at each location $\vec{r}_i$ simply by multiplying k_p^i by N_i.

D-BRANES

A major breakthrough in our understanding of these extended objects came with the discovery of Polchinski [12] that, although p-branes decouple from the perturbative sector of closed strings, they *do* appear in perturbation theory as boundary conditions for *open* strings. We have not discussed open strings so far and we will not go into the details of their rich physics since we will not need it directly in this thesis. Nevertheless, it is easy to generalize our discussion of closed strings to include open strings. The main difference is that open strings have 'open' ends and therefore they span a world-sheet Σ that has a *boundary* $\partial\Sigma$. The classical string equations of motion must then be supplemented by appropriate boundary conditions. There are two types of possible boundary conditions, namely *Neumann* boundary conditions allowing the open string end to move freely, or *Dirichlet* boundary conditions, which correspond to fixing the end of the string.

Imagine now that we impose Neumann boundary conditions on the coordinates $X^{\bar{\mu}}$, $\bar{\mu} = 0, \ldots, p$ (which include time), and Dirichlet boundary conditions on the rest of the coordinates X^m, $m = p+1, \ldots, 9$. This defines a $p+1$-dimensional hyperplane which preserves an $SO(9-p) \times$ Poincaré$(1,p)$ subgroup of the full Poincaré$(1,9)$ isometry group of ten-dimensional Minkowski spacetime. Moreover, it can be shown to preserve half supersymmetries, exactly as the p-brane solution did. In fact these hyperplanes are *dynamical* objects: open string excitations correspond to deformations of the hyperplane in exactly the same way that closed string excitations correspond to deformations of the background geometry. The dynamics of the low energy excitations and the way these couple to the closed string fields is encoded in the effective action (see Chapter 13 of Polchinski [9] and the lectures [13])

$$S_{Dp} = -\mu_p \int d^{p+1}\xi \mathrm{Tr} \left\{ e^{-\Phi} \left[-\det \left(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab} \right) \right]^{1/2} + \mathcal{O}([X,X]) \right\} + i\mu_p \int_{p+1} \mathrm{Tr} \left[\exp \left(B + 2\pi\alpha' F \right) \wedge \sum_k C^{(k)} \right]. \quad (2.106)$$

Here F_{ab} is the field-strength of the gauge field that comes from the open string sector. When N such 'planes', or *D-branes*, are placed on top of each other this gauge field becomes non-abelian (i.e. matrix-valued) and transforms according to the adjoint representation of $U(N)$ (or a subgroup thereof). The trace in the D-brane action refers to the trace of these $N \times N$ matrices. Besides the gauge field, the embedding fields X^μ become non-abelian as well and the action will receive corrections involving their commutator, as indicated. The rest of the fields are the closed string fields pulled-back onto the world-volume of the D-brane. Finally, the exponential in the last term should be understood as a power series using the wedge product of forms. The integral over the world-volume of the D-brane then picks the appropriate power, n, such that $2n + k = p + 1$. This low energy effective action of the D-brane dynamics, which is a generalization of the so-called 'Dirac-Born-Infeld action', is the analog of the Nambu-Goto action (2.3) for the fundamental string.

Using the low energy string dynamics it is now possible to evaluate the D-brane tension, which, quite remarkably, turns out to be inversely proportional to the string coupling, exactly as the p-brane tension in (2.104). Isometries, supersymmetry and tension then all suggest that p-branes and D-branes look very similar. In fact they are just different manifestations of one and the same extended object, the 'Dp-brane'!

D-branes are therefore a fundamental constituent of the theory. They are the elementary carriers of the R-R charge. (See, however, footnote 4.) Although we started with a theory of closed strings, we see that the theory must contain open strings as well, which are confined on the world-volume of D-branes. Type IIB string theory contains $D(-1)$, $D1$, and $D3$-branes, as well as $D5$ and $D7$-branes, which are the duals[8] of the $D1$ and $D(-1)$-branes respectively. In the next section we will focus on the special properties of the $D3$-brane and its low energy world-volume theory.

2.2 $\mathcal{N} = 4$ SUPER YANG-MILLS

The $D3$-brane of Type IIB string theory is a very special object which is at the core of the AdS/CFT correspondence to be discussed in the next section. To set the scene,

[8]These couple minimally to the R-R potentials $C^{(6)}$ and $C^{(8)}$, whose field strengths are the Hodge duals of the field strengths of $C^{(2)}$ and $C^{(0)}$ respectively. These are singular (i.e. electric) solutions of the supergravity equations with a delta function source. There also exist 'magnetic dual' solutions, such as the NS5 brane which is the magnetic dual of the fundamental string, but these solutions are *non-singular*. We will not discuss these solitonic solutions here.

we will now take a closer look at the $D3$-brane, both as a supergravity solution and its world-volume theory.

Starting with the supergravity description, we see that the $D3$-brane solution has a constant dilaton and, although we did not include it in our general discussion of p-branes above, it admits a constant axion as well. As we have seen, the constant dilaton defines the dimensionless string coupling, $g_s = e^{\Phi}$. Similarly, the constant axion defines a second dimensionless coupling constant, $\theta = 2\pi C^{(0)}$, whose physical significance will emerge below. Using the convention that the string coupling is absorbed in Newton's constant, the $D3$-brane metric is the same in the string and Einstein frames. The full solution is

$$\begin{aligned} &ds^2 = H_3^{-1/2} dx^\mu dx_\mu + H_3^{1/2}(dr^2 + r^2 d\Omega_5^2),\\ &C^{(4)+}_{0123} = H_3^{-1} - 1, \quad \text{with} \quad * F^+_{(5)} = F^+_{(5)},\\ &\Phi,\ C^{(0)} \quad \text{constant} \end{aligned} \tag{2.107}$$

where we have written the transverse metric in radial coordinates and x^μ, $\mu = 0,\ldots,3$, are coordinates along the brane. For a general multicenter solution the harmonic function $H_3(\vec{r})$ takes the form

$$H_3(\vec{r}) = 1 + \sum_{i=1}^{N} \frac{L_i^4}{|\vec{r}-\vec{r}_i|^4}, \tag{2.108}$$

where $L_i^4 = 4\pi g_s N_i \alpha'^2$.

On the other hand, the low energy effective action on a stack of N $D3$-branes in a flat gravitational background is the $\mathcal{N}=4$ $SU(N)$ super Yang-Mills action

$$\begin{aligned} S = -\frac{1}{2g_{YM}^2}\int d^4x \mathrm{Tr}\Big\{ & F_{\mu\nu}F^{\mu\nu} + 2D_\mu X^i D^\mu X^i - [X^i,X^j]^2 - 4i\bar{\chi}^\alpha P_- \not{D}\chi_\alpha \\ & +2\hat{\Gamma}^i_{\bar{\beta}\alpha}\bar{\chi}^\alpha [X_i, P_-\tilde{\chi}^{\bar{\beta}}] - 2\hat{\Gamma}^{i\beta\bar{\alpha}}\bar{\tilde{\chi}}_{\bar{\alpha}}[X_i, P_+\chi_\beta]\Big\} + \frac{\theta_I}{8\pi^2}\int d^4x \mathrm{Tr}\,(F_{\mu\nu}\tilde{F}^{\mu\nu}). \end{aligned} \tag{2.109}$$

To understand the structure of this action let us start by pointing out the isomorphism $SO(6) \approx SU(4)$. In particular, the **6** of $SO(6)$ is identified with the antisymmetric tensor representation of $SU(4)$, while the two Weyl spinor representations of $SO(6)$ are identified respectively with the **4** and $\bar{\mathbf{4}}$ of $SU(4)$. The six scalars X^i, $i = 1,\ldots,6$, transform under the **6** of $SO(6)$ and the four $SO(1,3)$ Dirac fermions χ^α, $\alpha = 1,\ldots,4$, transform under the **4** of $SU(4)$. Moreover, $\tilde{\chi}^\alpha = -C_4\bar{\chi}_\alpha^T$, where C_4 is the $SO(1,3)$ (antisymmetric) charge conjugation matrix and $\hat{\Gamma}^i$ are the 8×8 $SO(6)$ gamma matrices with components

$$\hat{\Gamma}^i = \begin{pmatrix} (\hat{\Gamma}^i)^\beta{}_\alpha & (\hat{\Gamma}^i)^{\beta\bar{\alpha}} \\ (\hat{\Gamma}^i)_{\bar{\beta}\alpha} & (\hat{\Gamma}^i)_{\bar{\beta}}{}^{\bar{\alpha}} \end{pmatrix}. \tag{2.110}$$

Finally, $P_\pm = (1 \pm \gamma^5)/2$ are the $SO(1,3)$ chirality projection operators, D_μ is the covariant derivative with respect to the $SU(N)$ gauge field A_μ, and the trace is over the $SU(N)$ indices.

The bosonic part of this action can be deduced by expanding the low energy effective action (2.106) around a flat close string background, although one would first need to specify the $\mathcal{O}([X,X])$ terms in (2.106). The last term in (2.109) arises then from the Chern-Simons term in (2.106) with the constant axion becoming the instanton angle $\theta = 2\pi C^{(0)} = \theta_I$. The Yang-Mills coupling can also be related to the string coupling this way with the result [9] $g_{YM}^2 = 4\pi g_s$. However, a more practical way to deduce the action (2.109) is to use the symmetries that the low energy world-volume field theory must possess to conclude that it can be no other than the unique super Yang-Mills $SU(N)$ gauge theory with $\mathcal{N} = 4$ supersymmetry. The full action can then be obtained by so-called dimensional reduction of the $\mathcal{N} = 1$ super Yang-Mills theory in ten dimensions on a six-torus. This is the way the action (2.109) was derived, but we need not go into the details of this derivation here.

The fact that $\mathcal{N} = 4$ super Yang-Mills in four dimensions can be obtained by dimensional reduction of $\mathcal{N} = 1$ super Yang-Mills in ten dimensions, however, helps identify its symmetries. First of all we have seen that the action (2.109) has an $SO(6)$ global symmetry under which the scalars transform in the **6** and the fermions in the **4**. Obviously it is also Poincaré(1,3)-invariant. Perhaps more surprising is the fact that it is *scale invariant* as well. This enhances the Poincaré symmetry group to the full *conformal group* $SO(2,4) \approx SU(2,2)$. But we know that the action (2.109) possesses $\mathcal{N} = 4$ supersymmetry. Indeed, one can verify directly that the supersymmetry transformations

$$\begin{aligned}
\delta_\zeta A_\mu &= i\bar{\zeta}^\alpha P_- \gamma_\mu \chi_\alpha + i\bar{\tilde{\zeta}}_{\bar{\alpha}} P_+ \gamma_\mu \tilde{\chi}^{\bar{\alpha}}, \\
\delta_\zeta X^i &= (\hat{\Gamma}^i)_{\bar{\beta}\alpha} \bar{\zeta}^\alpha P_- \tilde{\chi}^{\bar{\beta}} - (\hat{\Gamma}^i)^{\beta\bar{\alpha}} \bar{\tilde{\zeta}}_{\bar{\alpha}} P_+ \chi_\beta, \qquad (2.111)\\
\delta_\zeta (P_+ \chi_\alpha) &= \frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu} P_+ \zeta_\alpha + (D_\mu X_i)(\hat{\Gamma}^i)_{\bar{\beta}\alpha} \gamma^\mu P_- \tilde{\zeta}^{\bar{\beta}} + \frac{i}{2} [X_i, X_j] (\hat{\Gamma}^{ij})^\beta{}_\alpha P_+ \zeta_\beta, \\
\delta_\zeta (P_- \tilde{\chi}^{\bar{\alpha}}) &= \frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu} P_- \tilde{\zeta}^{\bar{\alpha}} - (D_\mu X_i)(\hat{\Gamma}^i)^{\beta\bar{\alpha}} \gamma^\mu P_+ \zeta_\beta + \frac{i}{2} [X_i, X_j] (\hat{\Gamma}^{ij})_{\bar{\beta}}{}^{\bar{\alpha}} P_- \tilde{\zeta}^{\bar{\beta}},
\end{aligned}$$

leave the action invariant. The conformal group combined with these supersymmetries form the maximal *superconformal group* $PSU(2,2|4)$ in four dimensions.[9] A detailed construction of this superalgebra can be found in Appendix 2.A.2.

The superconformal global symmetry of $\mathcal{N} = 4$ super Yang-Mills is in fact an exact symmetry even at the quantum level! It has been shown that there are no ultraviolet divergences in the correlation functions of the fundamental fields and therefore the perturbative renormalization group β-function vanishes. There are no

[9] I am grateful to Professor Massimo Bianchi for pointing out to me that the superconformal group is $PSU(2,2|4)$ and not $SU(2,2|4)$.

instanton corrections to this result and it is believed that the β-function vanishes identically, which ensures scale invariance, and hence superconformal invariance, in the full quantum theory.

There is also another discrete global symmetry of this theory which is in fact analogous to S-duality in Type IIB string theory. Recall that S-duality acts on the dilaton and the axion as the Möbius transformation

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}, \tag{2.112}$$

where $\tau = C^{(0)} + ie^{-\Phi}$ and $ad-bc=1$, $a,b,c,d \in \mathbb{Z}$. From the low energy world-volume theory of the $D3$-brane we have identified the dilaton (or string coupling) with the Yang-Mills coupling and the axion with the instanton angle, namely

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}. \tag{2.113}$$

Although there is no proof, this $SL(2,\mathbb{Z})$ 'duality' acting on the couplings of $\mathcal{N}=4$ super Yang-Mills is believed to be an exact symmetry of the full quantum theory, known as *Montonen-Olive* duality. The AdS/CFT correspondence maps this duality to the S-duality of Type IIB string theory.

2.2.1 SUPERCONFORMAL MULTIPLETS

Any local, gauge-invariant operator in $d=4$, $\mathcal{N}=4$ super Yang-Mills of definite scaling dimension is a polynomial in the gauge-invariant fundamental fields X^i, χ^α and $F_{\mu\nu}$, whose scaling dimensions are respectively 1, $3/2$ and 2, as well as their covariant derivatives. In particular, all gauge-invariant operators have positive dimension. As in two-dimensional superconformal field theory, all such operators can be grouped into *superconformal multiplets* which are derived from a set of *superconformal primary operators*. In two dimensions these would be the operators we introduced in (2.198). In the present context, superconformal primary operators are defined as the local gauge-invariant operators $\mathcal{O}$ which (anti)commute with the superconformal generators $S^I_{+\tilde{\alpha}}$

$$[S^I_{+\tilde{\alpha}}, \mathcal{O}\} = 0. \tag{2.114}$$

Since $S^I_{+\tilde{\alpha}}$ has dimension $-1/2$, successive application of these supercharges on a state created by an operator of definite dimension must eventually annihilate the state, or else states of negative dimension would be created, which violates unitarity. This shows that there must exist operators that satisfy (2.114). All local gauge-invariant operators which are not conformal primaries can be derived from one of the conformal primaries by successive (anti)commutation with $Q^I_{+\alpha}$, i.e.

$$\mathcal{O}' = [Q,[Q,\ldots,[Q,\mathcal{O}\}\ldots\}\}. \tag{2.115}$$

Such operators are said to be *superconformal descendants* of the superconformal primary operator $\mathcal{O}$. Note that there are also *conformal primary* operators which also have a definite conformal dimension and are defined as gauge-invariant operators that commute with the special conformal generators K_μ. Since $\{S, S\} \sim K$, a superconformal primary operator is also a conformal primary operator but the converse is not true.

It turns out that in $\mathcal{N} = 4$ super Yang-Mills all superconformal primary operators (also known as *chiral primary operators*) can be built out of symmetrized traces of the scalar operators X^i. We will not go into the details of the construction and classification of these operators here since we will not need them in this thesis. More details can be found, for example, in the review [14]. There is a special class of operators, however, that we will focus on in our subsequent discussion of the AdS/CFT correspondence. These are the single-trace 1/2-BPS operators[10]

$$\mathcal{O}_k \sim s\mathrm{Tr}\, X^{\{i_1} \ldots X^{i_k\}}, \tag{2.116}$$

where $s\mathrm{Tr}$ stands for the symmetrized trace and $\{\}$ stands for the traceless (with respect to the $SO(6)$ indices) part. The BPS property means that their dimension does not renormalize and is therefore equal to k. Moreover, these operators transform under the $(k{+}1)(k{+}2)^2(k{+}3)/12$-dimensional representation of the R-symmetry group $SU(4)$ with Dynkin labels $(0, k, 0)$. Of those, the most relevant to our discussion is the scalar operator with $k = 2$, which is the lowest weight state of the so-called 'supergraviton' multiplet. This multiplet contains only relevant and marginal operators (i.e. with conformal dimension $\Delta \leq 4$), while all multiplets built on the $k > 2$ chiral primary operators contain irrelevant operators. In particular, the supergraviton multiplet contains the 15 currents of the $SO(6)$ R-symmetry group (these are conformal primaries of dimension 3) as well as the stress tensor (dimension 4).

2.2.2 SUPERSYMMETRIC VACUA

In order to find supersymmetric vacua we look for zeros of the potential

$$\mathrm{Tr}\,([X^i, X^j]^2), \tag{2.117}$$

which is positive definite. Any such vacuum is in fact a solution of the stronger condition

$$[X^i, X^j] = 0. \tag{2.118}$$

There are two classes of solutions to this equation, corresponding to the two types of possible supersymmetric vacua. The first is the *superconformal phase*, where $\langle X^i \rangle = 0$ for all $i = 1, \ldots, 6$ and the full $SU(N)$ gauge group, as well as superconformal

[10] There are also multi-trace 1/2-BPS operators, but we will not consider them here.

invariance remain unbroken. The second is the *Coulomb phase*, where at least one of the vevs $\langle X^i \rangle$ is non-zero. In this case the gauge group is spontaneously broken to a subgroup, corresponding to a number of the N $D3$-branes moving away from the rest, and superconformal symmetry is also broken spontaneously by the scale introduced by the vev $\langle X^i \rangle$.

2.2.3 Implications of conformal invariance

Finally let us consider the implications of (super)conformal invariance for the field theory correlation functions. An extensive analysis of this problem can be found in [15, 16]. It is a standard result in quantum field theory that the classical conservation equations for the Noether currents associated with a given global symmetry give rise to the so-called *Ward identities* for the correlation functions. In the case of the (super)conformal group the Ward identities impose very strong restrictions on the form of the correlation functions of (super)conformal primary operators. In fact, conformal invariance alone determines completely the form of the one, two, and three-point functions of conformal primary operators. Namely, if $\mathcal{O}_{\Delta_i}(x)$ are conformal primary operators of dimension Δ_i, conformal invariance determines

$$\begin{aligned}
\langle \mathcal{O}_{\Delta_i}(x) \rangle &= \delta_{\Delta_i,0}, \\
\langle \mathcal{O}_{\Delta_1}(x) \mathcal{O}_{\Delta_2}(x) \rangle &= \frac{c(\Delta_1,\Delta_2)}{|x_1-x_2|^{2\Delta_1}} \delta_{\Delta_1\Delta_2}, \\
\langle \mathcal{O}_{\Delta_1}(x) \mathcal{O}_{\Delta_2}(x) \mathcal{O}_{\Delta_3}(x) \rangle &= \frac{c(\Delta_1,\Delta_2,\Delta_3)}{|x_1-x_2|^{\Delta-2\Delta_3}|x_2-x_3|^{\Delta-2\Delta_1}|x_3-x_1|^{\Delta-2\Delta_2}},
\end{aligned} \tag{2.119}$$

where $c(\Delta_1, \Delta_2)$ and $c(\Delta_1, \Delta_2, \Delta_3)$ are constants (possibly depending on the couplings of the theory) and $\Delta = \Delta_1 + \Delta_2 + \Delta_3$. We will see more examples of the constraints imposed on correlation functions by conformal invariance in Chapter 4.

2.3 The AdS/CFT Correspondence

Now that we have some idea of the basics of both Type IIB string theory and $d = 4$ $\mathcal{N} = 4$ super Yang-Mills theory, we turn to the celebrated 'AdS/CFT correspondence' which relates these two theories in a very non-trivial and constructive way. Let us start by reviewing the argument of Maldacena [17] motivating this correspondence. See also [18, 19] and the reviews [20, 14].

Consider N parallel $D3$-branes placed on top of (or very close to) each other in flat ten-dimensional Minkowski spacetime. The degrees of freedom of this system are the closed strings in the bulk as well as the open strings attached to the $D3$-branes. In general these are highly interacting, but as we have seen, at low energy only the massless modes survive and their dynamics can be described by an effective action. This action contains two pieces, namely the low energy Type IIB supergravity action describing the dynamics of the massless closed string modes and the Dirac-Born-Infeld-type action (2.106) describing the low energy dynamics of open string modes on the branes and their interactions with the closed string modes. Expanding this effective action around a flat background (e.g. for the metric we write $G_{\mu\nu} = \eta_{\mu\nu} + \kappa_{10} h_{\mu\nu}$, where $\kappa_{10} \sim g_s \alpha'^2$) and taking the low energy limit (which can be made more explicit by taking $\alpha' \to 0$, while keeping g_s and N fixed), one sees that the closed string modes completely decouple from the open string modes on the branes. Moreover, the closed string interactions describe free supergravity away from the brane, while the dynamics of the open strings on the flat branes reduces exactly to $\mathcal{N} = 4$ super Yang-Mills gauge theory.

Let us now look at exactly the same system from a different perspective. We have seen that a stack on N $D3$-branes can be described in supergravity by the solution (2.107) whose metric reads

$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{L^4}{r^4}\right)^{1/2} (dr^2 + r^2 d\Omega_5^2), \qquad (2.120)$$

where $L^4 = 4\pi g_s N \alpha'^2$. Far away from the $D3$-branes, i.e. $r \to \infty$, this metric reduces to the flat ten-dimensional Minkowski metric. On the other hand, if we introduce the new radial coordinate $u = L^2/r$, the $D3$-brane metric takes the form

$$ds^2 = \left(1 + \frac{L^4}{u^4}\right)^{-1/2} \frac{L^2}{u^2} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{L^4}{u^4}\right)^{1/2} L^2 \left(\frac{du^2}{u^2} + d\Omega_5^2\right). \qquad (2.121)$$

Taking the limit $u \to \infty$, this metric becomes

$$ds^2 = \frac{L^2}{u^2} \left(du^2 + \eta_{\mu\nu} dx^\mu dx^\nu\right) + L^2 d\Omega_5^2, \qquad (2.122)$$

which we recognize as the metric on $AdS_5 \times S^5$ (see Appendix 2.A.3), where AdS_5 and S^5 have the same radius L.

Consider then perturbative string theory around the $D3$-brane background. Clearly, away from the branes we have free Type IIB string theory in flat target space. To see what happens in the second limit, which is known as the 'near horizon limit', we look at the Polyakov action (2.5) with the background metric (2.121) (we ignore the other background fields). The overall factor L^2 of the metric combines with the factor in front of the string action to an overall factor

$$\frac{L^2}{4\pi\alpha'} = \sqrt{\frac{\lambda}{4\pi}}, \qquad (2.123)$$

where we have introduced the dimensionless coupling $\lambda \equiv g_s N$ and the relation $L^4 = 4\pi g_s N \alpha'^2$ was used. After removing the factor L^2 from the background metric, it admits a smooth non-trivial limit as $L \to 0$. In fact it reduces to the $AdS_5 \times S^5$ metric (2.122) where both factors now have unit radius. We now take the limit $\alpha' \to 0$ in such a way that L^2/α', and hence λ, remain fixed. This is precisely the decoupling limit we considered before and it leads to string theory on $AdS_5 \times S^5$ with the string tension replaced by $\sqrt{\lambda/4\pi}$.

Therefore, from both points of view, the limit $\alpha' \to 0$, while keeping all dimensionless couplings fixed, leads to two decoupled systems, one of which is Type IIB string theory (supergravity) in flat Minkowski spacetime. Identifying the second component in the two different approaches leads to the AdS/CFT conjecture:

Type IIB superstring theory on $AdS_5 \times S^5$, both factors having radius $L^4 = 4\pi g_s N \alpha'^2$, where $N = \int_{\Sigma^5} F^+_{(5)}$ is the integer flux of the self-dual five-form through S^5 and g_s is the string coupling, is equivalent (dual) to $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions in its superconformal phase, with gauge group $SU(N)$, Yang-Mills coupling $g^2_{YM} = 4\pi g_s$ and instanton angle, $\theta = 2\pi C^{(0)}$, given by the expectation value of the Type IIB axion.

This duality relates two theories we can say very little about. On one hand quantizing string theory on a curved background has proven very difficult. Indeed this is an area of intense current research. On the other hand, very little is known about non-abelian gauge theories like $\mathcal{N} = 4$ super Yang-Mills, even though one would hope that this particular theory is more tractable due to its high symmetry. Nevertheless, at present we can only get a glimpse of each of these theories in certain limits. What makes the AdS/CFT conjecture so remarkable, useful and difficult to prove is that the limits we can access in each of the two sides are (almost) mutually exclusive! In other words, the limit of Type IIB string theory that we can handle corresponds to a completely inaccessible limit of the gauge theory and vice versa. In that sense then, the correspondence, if true, provides a handle to the previously inaccessible regions of each of these theories.

One limit that is well-defined on both sides of the theory is the *'t Hooft limit*. This consists in keeping the *'t Hooft coupling* $\lambda = g^2_{YM} N = 4\pi g_s N$ constant while taking $N \to \infty$. On the gauge theory side only certain Feynman diagrams, the so-called 'planar diagrams', survive in this limit. The $1/N$ expansion then corresponds to an expansion where the Feynman diagrams are rearranged according to their topology. Even though, the 't Hooft limit simplifies the gauge theory dynamics to some extent, it is still very difficult to solve the theory in this limit. On the string theory side, the $1/N$ expansion corresponds to the string perturbation expansion since $g_s = \lambda/4\pi N$ and λ is kept fixed.

The 't Hooft limit alone is not sufficient to render either of the two sides of the correspondence tractable. However, after taking the 't Hooft limit we are left with a free parameter, namely the 't Hooft coupling λ, which we can tune at our will. An obvious limit is $\lambda \to 0$ which corresponds to weakly coupled gauge theory. String theory, however, in this limit is strongly coupled. On the other hand, we have seen that $1/\sqrt{\lambda}$ has taken the role of α' in the string non-linear sigma model action. The supergravity approximation, which was equivalent to the $\alpha' \to 0$ limit in the original string action, therefore corresponds to the $\lambda \to \infty$ limit. This is a *strong coupling* limit on the gauge theory side and hence very little, if anything, can be said about it directly. If the correspondence holds, then supergravity on $AdS_5 \times S^5$ can be used to study the properties of the strongly coupled gauge theory! In the rest of this thesis we will investigate various aspects of this 'gauge/gravity' duality, thus justifying the title of the thesis.

2.3.1 THE ADS/CFT DICTIONARY

To convince ourselves of the plausibility of the AdS/CFT conjecture, let us check that the global symmetries of the two theories are the same. We saw earlier that $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions has an $PSU(2,2|4)$ global symmetry, whose maximal bosonic subgroup is $SU(2,2) \times SU(4) \approx SO(2,4) \times SO(6)$. String theory on $AdS_5 \times S^5$ must therefore have this global symmetry as well. Indeed, $SO(2,4) \times SO(6)$ is precisely the isometry group of $AdS_5 \times S^5$ (see Appendix 2.A.3 for a discussion of the isometries of AdS). Moreover, it can be shown that this background preserves exactly the same number of supersymmetries as ten-dimensional flat space, that is 32. Hence, the full symmetry group of $AdS_5 \times S^5$ is the full superconformal group $PSU(2,2|4)$ as well.

We also saw above that $\mathcal{N} = 4$ super Yang-Mills has another global symmetry, namely the Montonen-Olive duality. The AdS/CFT conjecture maps the Yang-Mills coupling and the instanton angle respectively to the vacuum expectation values of the dilaton and the axion of Type IIB string theory. The Montonen-Olive duality is therefore directly mapped to the S-duality of Type IIB.

A less trivial task is to match the $PSU(2,2|4)$ multiplets on the two sides of the correspondence. A large number of operators in the gauge theory have indeed been identified with various states on the string theory side. (For a discussion of this matching we refer to the review [14] and references therein.) Since the full spectrum of Type IIB string theory on $AdS_5 \times S^5$ is not known, however, the only states we know on the string theory side are those obtained by the Kaluza-Klein reduction of Type IIB supergravity on S^5. This procedure consists in expanding the ten dimensional (massless) supergravity fields in S^5 spherical harmonics as

$$\varphi = \sum_k \varphi_k Y_k(\Omega_5), \tag{2.124}$$

where φ_k is a field on AdS_5 which transforms under the $(0,k,0)$ representation of the $SU(4)$ R-symmetry group. These fields are generically massive. For scalar fields, for example, the Kaluza-Klein reduction determines the mass of φ_k to be $m_k^2 = L^{-2}k(k-4)$, where L is the common radius of AdS_5 and S^5. The AdS/CFT correspondence relates the $1/2$-BPS operators (2.116) with (a linear combination of) the fields φ_k on AdS_5. In particular, the operators in the supergraviton multiplet, which is built on the $k = 2$ chiral primary operator, are in one-to-one correspondence with the field content of $D = 5$, $\mathcal{N} = 8$ gauged supergravity. This includes 15 gauge fields and the five-dimensional metric, which are respectively the duals of the 15 R-symmetry currents and of the stress tensor in the supergraviton multiplet. Although there is no proof so far, $\mathcal{N} = 8$ gauged supergravity in five dimensions is believed to be a *consistent truncation* of Type IIB supergravity in ten dimensions. This means that any solution of $\mathcal{N} = 8$ gauged supergravity in five dimensions can be lifted to a solution of the full Type IIB supergravity in ten dimensions. Since the supergraviton multiplet contains only relevant or marginal operators, the $D = 5$, $\mathcal{N} = 8$ gauged supergravity fields are all dual to such operators. In the rest of this thesis we work within the framework of $D = 5$, $\mathcal{N} = 8$ gauged supergravity and its dual relevant and marginal operators.

2.3.2 CORRELATION FUNCTIONS

We have seen that the AdS/CFT conjecture relates the large N, large 't Hooft coupling limit of $\mathcal{N} = 4$ super Yang-Mills to Type IIB supergravity on $AdS_5 \times S^5$. Moreover, as we discussed in the previous section, the Kaluza-Klein reduction of Type IIB supergravity on S^5 gives rise to a set of five-dimensional fields which are in one-to-one correspondence with certain operators in $\mathcal{N} = 4$ super Yang-Mills gauge theory. We now wish to understand in more detail how this correspondence arises and how it can be used in order to extract information about the strongly coupled gauge theory.

To this end we restrict ourselves to the field content of $D = 5$, $\mathcal{N} = 8$ gauged supergravity and the dual relevant or marginal operators. To be concrete, consider a scalar gauge-invariant operator $\mathcal{O}_\Delta$ of dimension Δ. The dynamics of such an operator is encoded in the generating functional

$$Z_{\mathrm{CFT}}[\bar{\varphi}] \equiv e^{W_{\mathrm{CFT}}[\bar{\varphi}]} = \left\langle e^{-\int d^4x \bar{\varphi}(x)\mathcal{O}_\Delta(x)} \right\rangle . \tag{2.125}$$

By taking successive functional derivatives of the generating functional $W_{CFT}[\bar{\varphi}]$ with respect to the source $\bar{\varphi}(x)$ one can compute (connected) correlation functions with arbitrary number of $\mathcal{O}_\Delta(x)$ insertions

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\ldots\mathcal{O}_\Delta(x_n)\rangle_c = (-1)^n \frac{\delta}{\delta\bar{\varphi}(x_1)}\frac{\delta}{\delta\bar{\varphi}(x_2)}\cdots\frac{\delta}{\delta\bar{\varphi}(x_n)} W_{CFT}[\bar{\varphi}]\bigg|_{\bar{\varphi}=0} . \tag{2.126}$$

For every such operator there exists a dual bulk scalar field φ in AdS_5 (this would be the field φ_k with $k = \Delta$ in the notation of the previous section, but here we drop the subscript.) However, since a similar operator/bulk field correspondence exists in the context of other AdS/CFT-type dualities as well, we will not restrict ourselves to AdS_5 here. Instead we will consider $(d+1)$-dimensional AdS space which has a d-dimensional conformal boundary (see Appendix 2.A.3), where the field theory and the operator $\mathcal{O}_\Delta$ live. It will also be convenient to work with the Euclidean version of AdS_{d+1}, i.e. the hyperbolic space $\mathbb{H}_{d+1}$.

Ignoring for the moment any possible interactions with other fields, including the bulk metric, the dynamics of the scalar field φ is described by an action of the form

$$S = \frac{1}{2}\int d^{d+1}x\sqrt{g}\left(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + m^2\varphi^2 + \cdots\right), \tag{2.127}$$

where the dots stand for non-linear terms which we are going to ignore initially. φ then satisfies the linearized equation of motion

$$\left(-\Box_g + m^2\right)\varphi = 0. \tag{2.128}$$

In the upper half plane coordinates (2.220) this equation reads

$$\left(-z_0^2\partial_{z_0}^2 + (d-1)z_0\partial_{z_0} - z_0^2\Box + m^2\right)\varphi = 0, \tag{2.129}$$

where $\Box$ is the Laplacian on the boundary, which is located at $z_0 = 0$. We will solve this equation later, but for now we are just interested in the asymptotic behavior of its solutions as $z_0 \to 0$. It is not difficult to see that there are two possible asymptotic behaviors, namely

$$\varphi(z_0, \vec{z}) \sim z_0^{\alpha_\pm}, \qquad \alpha_\pm = \frac{d}{2} \pm \sqrt{(d/2)^2 + m^2}. \tag{2.130}$$

Assuming $\alpha_- < \alpha_+$, the linearly independent solutions behaving asymptotically as $z_0^{\alpha_+}$ and $z_0^{\alpha_-}$ are known respectively as the *normalizable* and the *non-normalizable* modes. The general solution then will be a linear combination of these two modes and hence, the leading asymptotic behavior of the solution will generically be

$$\varphi(z_0, \vec{z}) \sim z_0^{\alpha_-}\varphi_{(0)}(\vec{z}), \tag{2.131}$$

where $\varphi_{(0)}(\vec{z})$ is some arbitrary function on the boundary.

A statement of the AdS/CFT correspondence, in the supergravity approximation, is that the generating functional of connected correlators of the gauge-invariant operator $\mathcal{O}_\Delta$ is given by the on-shell action $S_{\text{on-shell}}[\varphi_{(0)}]$, with the arbitrary function $\varphi_{(0)}(\vec{z})$ that multiplies the leading asymptotic behavior of the bulk field φ being identified with the *source* of the dual operator:

$$\boxed{W_{\text{CFT}}[\varphi_{(0)}] = -S_{\text{on-shell}}[\varphi_{(0)}].} \tag{2.132}$$

On dimensional grounds then $\alpha_- = d - \Delta$, which implies $m^2 = \Delta(\Delta - d)$ and $\alpha_+ = \Delta$. In fact, a similar statement can be made for the strong form of the AdS/CFT conjecture relating the full Type IIB string theory on $AdS_5 \times S^5$ with $SU(N)$ $\mathcal{N} = 4$ super Yang-Mills at finite N and any value of the 't Hooft coupling $\lambda = g_{YM}^2 N$. The statement is that the string partition function Z_{string} is identified with the partition function Z_{CFT} of $\mathcal{N} = 4$ super Yang-Mills living on the conformal boundary of AdS_5:

$$Z_{\text{CFT}} = Z_{\text{string}}. \tag{2.133}$$

In the saddle-point approximation, the string partition function is replaced by the exponential of the on-shell supergravity action, $Z_{\text{string}} \approx e^{-S_{\text{SUGRA}}}$. In general there will be many saddle points, corresponding to various backgrounds with the same conformal boundary, and so the CFT partition function will involve a sum over these saddle points.

This is a very useful formulation of the AdS/CFT correspondence since it allows one to compute correlation functions of the strongly coupled gauge theory using classical supergravity calculations! As it stands, however, the identification (2.132) is not well-defined. The reason is that the on-shell action diverges because $\mathbb{H}_{d+1}$ (and AdS_{d+1}) is non-compact and hence it has an infinite volume. This divergence can also be understood from the field theory point of view as well. Being a conformal field theory, the correlation functions of the fundamental fields of $\mathcal{N} = 4$ super Yang-Mills are ultraviolet finite. Indeed, this is necessary for the renormalization group β-function to vanish. The correlation functions of *composite* operators such as $\mathcal{O}_\Delta$, however, will generically contain ultraviolet divergences. The AdS/CFT correspondence relates these divergences to the divergences of the on-shell supergravity action. Since these are long distance, or infrared, divergences from the supergravity point of view, the relation between the divergences of the two theories is a UV/IR relation.

To make sense of the identification (2.132) then we must find a consistent way of removing these divergences, i.e. we should determine the correct *renormalized* version of (2.132). On the field theory side this can be achieved by the usual renormalization procedure. On the supergravity side, renormalization corresponds to adding suitable boundary covariant counterterms to the on-shell supergravity action. In the next chapter we will describe in detail how these covariant counterterms can be determined systematically in a very general setting, a procedure known as *holographic renormalization*. Before we delve into the technicalities of the general case, however, it is instructive to illustrate the general idea in the simple example we considered above. In fact, many of the important ideas will be present already in this simple example.

CFT CALCULATION

The quantity that we want to evaluate is the renormalized two-point function $\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(y)\rangle_{\rm ren}$. Let us start with the field theory calculation since it is rather standard. We have seen that conformal invariance completely determines the form of the two-point function at separated points, namely

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(y)\rangle = \frac{c(g,\Delta)}{|x-y|^{2\Delta}}, \tag{2.134}$$

where $c(g,\Delta)$ is a constant that depends on the coupling constant of the theory g and the conformal dimension Δ of the operator. One may set it to one by a choice of normalization of O_Δ but we shall not do so. Depending on the conformal dimension, Δ, this correlator may suffer from short distance singularities. Consider the case $\Delta = d/2 + k + \epsilon$, where ϵ is an infinitesimal parameter and k is a non-negative integer. Iterating the identity

$$\frac{1}{|x-y|^{2\Delta}} = \frac{1}{2(\Delta-1)(2\Delta-d)}\Box\frac{1}{|x-y|^{2\Delta-2}}, \qquad |x-y|\neq 0, \tag{2.135}$$

where $\Box = \delta^{ij}\partial_i\partial_j$, $k+1$ times, we find

$$\begin{aligned}\frac{1}{|x-y|^{2\Delta}} &= \frac{1}{2\epsilon}\frac{\Gamma(1+\epsilon)\Gamma(d/2+\epsilon)}{2^{2k}\Gamma(k+1+\epsilon)\Gamma(d/2+k+\epsilon)}\frac{1}{d-2+2\epsilon}\Box^{k+1}\frac{1}{|x-y|^{d-2+2\epsilon}}\\ &\sim \frac{-1}{2\epsilon}\frac{\omega_{d-1}\Gamma(d/2)}{2^{2k}\Gamma(k+1)\Gamma(d/2+k)}\Box^k\delta^{(d)}(x-y),\end{aligned} \tag{2.136}$$

where $\omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of the unit $(d-1)$-sphere and we have used the identity $\Box(x^2)^{-d/2+1} = -(d-2)\omega_{d-1}\delta^{(d)}(x)$. We thus find that there is a pole at $\Delta = d/2+k$, or $\epsilon = 0$. To produce a well-defined distribution we use differential regularization [21] to subtract the pole and define [22, 23]

$$\begin{aligned}\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_{\rm ren} &= c(g,\Delta)\lim_{\epsilon\to 0}\left\{\frac{1}{2\epsilon}\frac{\Gamma(1+\epsilon)\Gamma(d/2+\epsilon)}{2^{2k}\Gamma(k+1+\epsilon)\Gamma(d/2+k+\epsilon)}\right.\\ &\qquad\left.\frac{1}{d-2+2\epsilon}\Box^{k+1}\frac{1}{|x|^{d-2}}\left(\frac{1}{|x|^{2\epsilon}}-\mu^{2\epsilon}\right)\right\}\\ &= \frac{-c_k}{2(d-2)}\Box^{k+1}\frac{1}{|x|^{d-2}}\left\{\log\left(\mu^2x^2\right)+a(k)\right\},\end{aligned} \tag{2.137}$$

where

$$c_k \equiv c(g,\Delta)\frac{\Gamma(d/2)}{2^{2k}\Gamma(k+1)\Gamma(d/2+k)}. \tag{2.138}$$

The constant $a(k)$ reflects the scheme dependence in the subtraction of the pole. Here we have defined the subtraction in such a way so that $a=0$, but other subtraction schemes, such as minimal subtraction, lead to a non-zero a [23]. The renormalized correlator agrees with the bare one away from coincident points but is also well-defined at $x^2 = 0$. To allow a direct comparison of the renormalized two-point function with the result we will obtain below from the bulk calculation, it is useful to write down its Fourier transform. Using the identity [21]

$$\int d^d x e^{ip\cdot x} \frac{1}{|x|^{d-2}} \log\left(\mu^2 x^2\right) = -\frac{4\pi^{d/2}}{\Gamma(d/2-1)} \frac{1}{p^2} \log(p^2/\bar{\mu}^2), \tag{2.139}$$

where $\bar{\mu} = 2\mu/\gamma$ and $\gamma = 1.781072\ldots$ is the Euler constant, we obtain

$$\langle \mathcal{O}_\Delta(p)\mathcal{O}_\Delta(-p)\rangle_{\text{ren}} = c_k \frac{(-1)^{k+1}}{2(d-2)} \frac{4\pi^{d/2}}{\Gamma(d/2-1)} p^{2k} \log(p^2/\bar{\mu}^2). \tag{2.140}$$

BULK CALCULATION

Next, we calculate the same two-point function using the AdS/CFT prescription that we described above. By the change of coordinates $z_0 = e^{-r}, z^i = x^i$, the upper half plane metric (2.220) takes the form (we set the radius $l = 1$ here)

$$ds^2 = dr^2 + \gamma_{ij} dx^i dx^j, \qquad \gamma_{ij} = e^{2r}\delta_{ij}. \tag{2.141}$$

Fourier-transforming in the transverse coordinates, the equation of motion (2.128) becomes

$$\left(\partial_r^2 + d\partial_r - p^2 e^{-2r} - m^2\right)\tilde{\varphi} = 0. \tag{2.142}$$

As we saw above, there are two linearly independent solutions to this equation, say $\tilde{\varphi}_\pm$, which behave asymptotically as $\tilde{\varphi}_\pm \sim e^{-\alpha_\pm r}$, with $\alpha_- = d - \Delta$ and $\alpha_+ = \Delta$. Shifting the radial coordinate as $\bar{r} = r - \log|p|$, the general solution then takes the form

$$\tilde{\varphi} = a_-(p)\tilde{\varphi}_-(\bar{r}) + a_+(p)\tilde{\varphi}_+(\bar{r}), \tag{2.143}$$

where $a_\pm(p)$ are arbitrary functions. To completely specify the solution we need to provide two boundary conditions, one at the boundary $\bar{r} \to \infty$ and one at the interior $\bar{r} \to -\infty$. As $\bar{r} \to \infty$, $\tilde{\varphi} \sim a_-(p)\tilde{\varphi}_-(\bar{r})$, and hence specifying $a_-(p)$ corresponds to specifying the source $\varphi_{(0)}(x)$ of the dual operator and imposes a Dirichlet boundary condition on φ. Since we want to keep the source arbitrary, we will keep $a_-(p)$ arbitrary. As is well-known from the Laplace problem, there is no freedom of specifying another function in the interior. However, the solution is required to be smooth in the interior. At fixed p, there is a unique linear combination of $\tilde{\varphi}_-$ and $\tilde{\varphi}_+$ that is non-singular as $\bar{r} \to -\infty$. But in order for the solution to be smooth for *arbitrary* p, $a_+(p)$ must be proportional to $a_-(p)$. This means that a necessary, but not sufficient, condition for the solution to be smooth in the interior is that it takes the form

$$\tilde{\varphi} = a_-(p)\left\{\tilde{\varphi}_-(\bar{r}) + \xi\tilde{\varphi}_+(\bar{r})\right\}, \tag{2.144}$$

where ξ is an arbitrary constant *independent* of p. It follows that for smooth solutions

$$\dot{\tilde{\varphi}} = f(\bar{r})\tilde{\varphi}, \tag{2.145}$$

where $f(\bar{r})$ is an arbitrary function. Note in particular that even though (2.145) holds up to a normalizable mode for *any* solution, that is even for singular solutions, imposing (2.145) exactly is equivalent to restricting to a subspace of the solution space that contains all non-singular solutions. Unless ξ is fixed though, this space will still contain singular solutions as well.

This observation is at the very center of the Hamiltonian version of holographic renormalization that we will discuss in the next chapter. In the Hamiltonian language, with the radial coordinate r as the 'time' coordinate, the above expression is equivalent to the following statement:

In the space of smooth solutions, the radial momentum $\dot{\varphi}$ of the field φ is proportional to the field φ itself.

This statement survives even when the full non-linear equations are considered, but in that case the radial momentum becomes an arbitrary functional of the dynamical field, instead of just being proportional to it. As long as we are interested in smooth solutions then, solving the equations of motion is equivalent to solving for the momenta as functionals of the dynamical fields.

Inserting (2.145) into the equation of motion (2.142) we obtain a *first* order equation for $f(\bar{r})$. This reflects the fact that we have eliminated the function $a_+(p)$ by restricting to non-singular solutions. The resulting equation reads

$$f' + f^2 + df - e^{-2\bar{r}} - m^2 = 0. \tag{2.146}$$

Changing variables to $\chi = e^{-\bar{r}}$, $y(\chi) = \exp\int^{\chi} d\log\chi' \,(d/2 - f(-\log\chi'))$, this equation becomes a standard Bessel equation, namely

$$\chi^2 \frac{d^2 y}{d\chi^2} + \chi\frac{dy}{d\chi} - (\chi^2 + k^2)y = 0, \tag{2.147}$$

where $k = \Delta - d/2$ as above, and we have used $m^2 = \Delta(\Delta - d)$. Fixing the single integration constant by requiring that the solution is not singular in the interior and a sign ambiguity by imposing the asymptotic behavior $f \sim -(d - \Delta)$, we obtain the unique solution

$$\begin{aligned} f(-\log\chi) &= -\frac{d}{2} - \frac{d\log K_k(\chi)}{d\log\chi} \\ &\sim -(d-\Delta) + \frac{\chi^2}{(2\Delta - d - 2)} - \frac{\chi^4}{(2\Delta - d - 2)^2(2\Delta - d - 4)} + \cdots \\ &\quad + \frac{(-1)^k}{2^{2k-1}\Gamma(k)^2}\chi^{2k}\log\chi^2 + a(k)\chi^{2k} + \cdots \end{aligned} \tag{2.148}$$

where $a(k)$ is a constant whose explicit value we will not need.

Evaluating the action (2.127) on-shell, we get

$$S_{\text{reg}} = \frac{1}{2}\int_{r_o} d^d x\sqrt{\gamma}\varphi\dot{\varphi}, \tag{2.149}$$

since only the boundary term from the integration by parts contributes. Here we have introduced a radial cut-off r_o to regulate the action. This can be evaluated directly by Fourier-transforming (2.145) using the above solution for f. The resulting regulated action is

$$\begin{aligned} S_{\text{reg}} = \tfrac{1}{2}\textstyle\int_{r_o} d^d x\sqrt{\gamma}\varphi \Big(&-(d-\Delta) + \tfrac{-\Box_\gamma}{(2\Delta-d-2)} - \tfrac{(-\Box_\gamma)^2}{(2\Delta-d-2)^2(2\Delta-d-4)} + \cdots \\ &+ \tfrac{(-1)^k}{2^{2k-1}\Gamma(k)^2}(-\Box_\gamma)^k\log(-\Box_\gamma) + a(k)(-\Box_\gamma)^k + \cdots\Big)\varphi, \end{aligned} \tag{2.150}$$

where the dots stand for terms that do not survive as the regulator is removed. Since the volume element contains a factor of e^{dr_o} and the scalar field behaves as $\varphi \sim e^{-(d-\Delta)r_o}$ asymptotically, this regulated action contains a number of boundary covariant *local* power-law singular terms which can therefore be removed by local boundary covariant counterterms. The fact that the divergences are organized in boundary covariant terms is very important here. It guarantees that the regularization and renormalization scheme preserves the Ward identities of the boundary field theory. We will return to this point below. However, the action contains a logarithmic term as well, which can be written as

$$(-\Box_\gamma)^k\log(-\Box_\gamma) = (-\Box_\gamma)^k\log(e^{-2r_o}\bar{\mu}^2) + (-\Box_\gamma)^k\log(-\Box/\bar{\mu}^2), \tag{2.151}$$

where $\Box = \delta^{ij}\partial_i\partial_j$ and we have introduced a scale $\bar{\mu}$ on dimensional grounds. The first of these two terms gives a local logarithmically divergent term, while the second is a *non-local* but non-divergent term. The logarithmic divergent term can again be removed by a local boundary counterterm. This counterterm however breaks partly covariance on the regulating surface since it depends on the cut-off explicitly and not merely through the cut-off dependence of covariant quantities, such as the induced metric γ_{ij} and the field φ, on the regulating surface. We will see below that this will lead to a violation of the Ward identity associated with scale invariance of the boundary field theory, which is known as the *conformal anomaly*. Finally, the term proportional to the constant $a(k)$ in the action is covariant, local and finite and so we have the freedom to remove it or not by adding the corresponding finite local boundary counterterm. This freedom is precisely the scheme dependence we met in the CFT calculation above.

The *renormalized* action now is defined as

$$S_{\text{ren}} \equiv \lim_{r_o\to\infty}\left(S_{\text{reg}} + S_{\text{ct}}\right), \tag{2.152}$$

where the counterterm action is defined as the negative of the divergent local part of the regulated action. In this case then, ignoring the scheme dependent contact terms, we have

$$S_{\text{ren}} = \frac{(-1)^k}{2^{2k}\Gamma(k)^2} \int d^dx \varphi_{(0)} (-\Box)^k \log(-\Box/\bar{\mu}^2) \varphi_{(0)}, \tag{2.153}$$

which leads, via the AdS/CFT prescription (2.132) to the two-point function

$$\langle \mathcal{O}_\Delta(p) \mathcal{O}_\Delta(-p) \rangle_{\text{ren}} = \frac{(-1)^{k+1}}{2^{2k-1}\Gamma(k)^2} p^{2k} \log(p^2/\bar{\mu}^2), \tag{2.154}$$

which is exactly of the form (2.140) as calculated from the CFT. Comparing the coefficients, we determine

$$c(g, \Delta) = \frac{2k\Gamma(d/2+k)}{\pi^{d/2}\Gamma(k)}, \tag{2.155}$$

which is precisely the correct coefficient consistent with the Ward identities [24].

We have now seen concretely how the identification (2.132) can be understood in a precise sense and how it can be employed in order to compute renormalized CFT correlation functions from supergravity. However, we discovered that such correlation functions will generically depend explicitly on some energy scale μ. For example, in the case we considered above

$$\mu \frac{\partial}{\partial \mu} \langle \mathcal{O}_\Delta(x) \mathcal{O}_\Delta(y) \rangle_{\text{ren}} = \frac{1}{2^{2k-2}\Gamma(k)^2} \Box^k \delta^{(d)}(x-y). \tag{2.156}$$

As we will now discuss, this violation of scale invariance in the correlation functions leads to a non-vanishing trace for the stress tensor of the boundary field theory.

WARD IDENTITIES AND THE CONFORMAL ANOMALY

In a general Lagrangian field theory, symmetries of the classical action[11] lead via Noether's theorem to conserved currents. For example, Poincaré invariance of the classical action implies that the stress-energy tensor, T_{ij}, is conserved, i.e.

$$\partial^i T_{ij} = 0. \tag{2.157}$$

Similarly, global internal symmetries lead to conserved currents J^i,

$$\partial_i J^i = 0. \tag{2.158}$$

At the quantum level these currents become quantum operators and their classical conservation laws imply relations among certain correlation functions that involve

[11] Here, of course, we refer to the classical action of the Lagrangian field theory, not to be confused with the supergravity action.

these currents. These identities relating various correlation functions as a result of the classical Noether theorem are known as *Ward identities*.

It is often the case, however, that some of the classical symmetries are broken at the quantum level. This happens because in a quantum field theory various quantities contain ultraviolet divergences which must be regulated and renormalized to yield a well-defined quantity. However, there may not exist a regulator that preserves all of the classical symmetries of the theory, which leads to the breaking of some symmetries at the quantum level. We have already seen this effect in the CFT two-point function calculation above. There, the two-point function of certain composite operators was singular at short distances and required renormalization. This in turn introduced a scale μ in the renormalized correlation function. Thus even though the two-point function is scale invariant at long distances, short distance effects break this scale invariance. This breaking of the classical symmetries at the quantum level leads to the so-called *quantum anomalies* in the Ward identities.

An elegant way to write down the Ward identities of a quantum field theory is in terms of the generating functional of correlation functions of certain operators. For concreteness, we will consider the generating functional for correlators of the stress tensor, T_{ij}, a symmetry current, J^i, and a scalar composite operator $\mathcal{O}$. The generating functional then takes the form

$$\begin{aligned} Z[g_{(0)}, A_{(0)}, \varphi_{(0)}] \;=\; & \int [d\phi^A] \exp\bigg\{ - S_{\mathrm{CFT}}[\phi^A; g_{(0)}, A_{(0)}] \\ & - \int d^d x \sqrt{g_{(0)}} \left(A_{(0)i} J^i(\phi^A) + \varphi_{(0)} \mathcal{O}(\phi^A) \right) \bigg\}, \end{aligned} \tag{2.159}$$

where ϕ^A represents collectively all fundamental fields of the theory, the background metric $g_{(0)ij}$ serves as a source for the stress-energy tensor, the background gauge field $A_{(0)i}$ is the source for the current J^i and $\varphi_{(0)}$ is a source for the scalar operator $\mathcal{O}$. Connected correlation functions can now be computed by differentiating $W[g_{(0)}, A_{(0)}, \varphi_{(0)}] \equiv \log Z[g_{(0)}, A_{(0)}, \varphi_{(0)}]$ successively with respect to the sources and then setting $A_{(0)i}$ and $\varphi_{(0)}$ to zero and $g_{(0)ij}$ to the flat metric δ_{ij}. The same information is contained in the one-point functions in the presence of sources, namely

$$\begin{aligned} \langle T_{ij}(x) \rangle_s &= -\frac{2}{\sqrt{g_{(0)}}} \frac{\delta W}{\delta g_{(0)}^{ij}(x)}, \\ \langle J^i(x) \rangle_s &= -\frac{1}{\sqrt{g_{(0)}}} \frac{\delta W}{\delta A_{(0)i}(x)}, \\ \langle \mathcal{O}(x) \rangle_s &= -\frac{1}{\sqrt{g_{(0)}}} \frac{\delta W}{\delta \varphi_{(0)}(x)}, \end{aligned} \tag{2.160}$$

where the subscript s in the correlation functions indicates that the sources are non-zero. Correlation functions are then computed by further differentiating with respect to the sources and setting the sources to zero.

In general, however, such correlation functions will contain ultraviolet divergences. To obtain well-defined correlation functions then, we need to renormalize the generating functional $W[g_{(0)}, A_{(0)}, \varphi_{(0)}]$. We have already seen how this can be done in the context of the AdS/CFT correspondence in a simple example and we will consider the general case in the next chapter. So we will take here $W[g_{(0)}, A_{(0)}, \varphi_{(0)}]$ to be the *renormalized* generating functional of connected correlation functions.

The Ward identities can be compactly expressed in terms of the one-point functions in the presence of sources. Let us first consider $U(1)$ gauge transformations which, when the sources are set to their flat values, correspond to the global internal symmetry transformation generated by the current J^i. Under such a gauge transformation the sources transform as

$$\delta g_{(0)ij} = 0, \quad \delta A_{(0)i} = D_i\alpha(x), \quad \delta\varphi_{(0)} = 0. \tag{2.161}$$

Invariance under this gauge transformation means that

$$\delta_{g_{(0)}} W + \delta_{A_{(0)}} W + \delta_{\varphi_{(0)}} W = 0, \tag{2.162}$$

which, using the one-point functions (2.160), leads to the Ward identity

$$\boxed{D_i\langle J^i(x)\rangle_s = 0.} \tag{2.163}$$

Next we consider diffeomorphisms under which the sources transform as

$$\delta\gamma_{(0)}^{ij} = -(D^i\xi^j + D^j\xi^i), \quad \delta A_{(0)i} = A_{(0)j}D_i\xi^j + \xi^j D_j A_{(0)i}, \quad \delta\varphi_{(0)} = \xi^j D_j\varphi_{(0)}. \tag{2.164}$$

Invariance under diffeomorphisms (which becomes Poincaré invariance in flat space) then implies the Ward identity

$$\boxed{D^i\langle T_{ij}(x)\rangle_s - \langle J^i(x)\rangle_s F_{(0)ij} + \langle\mathcal{O}(x)\rangle_s D_j\varphi_{(0)}(x) = 0,} \tag{2.165}$$

where $F_{(0)ij} = \partial_i A_{(0)j} - \partial_j A_{(0)i}$ is the field strength of the gauge field $A_{(0)i}$ and we have used the one-point functions (2.160) as well as the Ward identity (2.163).

Finally, let us consider Weyl transformations under which the sources transform as [25]

$$\delta g_{(0)ij} = 2\delta\sigma(x)g_{(0)ij}, \quad \delta A_{(0)i} = 0, \quad \delta\varphi_{(0)} = -(d-\Delta)\delta\sigma(x)\varphi_{(0)}. \tag{2.166}$$

However, the generating functional of renormalized correlation functions will not be in general invariant under such a Weyl transformation. Indeed, we have seen this already in the quantization of the superstring. The variation of the generating functional with respect to Weyl transformations defines the *conformal anomaly*

$$\delta_\sigma W \equiv A = \int d^dx\sqrt{g_{(0)}}\delta\sigma(x)\mathcal{A}, \tag{2.167}$$

where the anomaly density, $\mathcal{A}$ is a local function of the sources. Using the above transformation of the sources, this then leads to the trace Ward identity

$$\boxed{\langle T_i^i(x)\rangle_s = -(d-\Delta)\varphi_{(0)}\langle\mathcal{O}(x)\rangle_s + \mathcal{A}.} \tag{2.168}$$

In fact, the renormalized generating functional is not invariant even under *constant* Weyl rescalings, since the renormalization procedure introduces an explicit scale dependence. The fact that the theory is conformal, however, means that we can still write down an equation that must be satisfied by constant Weyl rescalings [26, 22], namely

$$\mu\frac{\partial}{\partial\mu}W = \int d^dx \left(2g_{(0)ij}\frac{\delta}{\delta g_{(0)ij}} + (\Delta-d)\varphi_{(0)}\frac{\delta}{\delta\varphi_{(0)}}\right)W, \tag{2.169}$$

where the energy scale $\log\mu = \sigma$, corresponds to a constant Weyl factor σ. This is precisely the Callan-Symanzik equation for the generating functional of renormalized connected correlation functions of the conformal field theory. In a non-conformal renormalizable field theory, this equation would involve additional terms proportional to the beta functions of the various couplings in the theory, as well as terms proportional to the anomalous dimensions of the operators. In a conformal field theory, the beta functions vanish, while the BPS property of the operators we are considering means that their anomalous dimensions vanish too.

From the definition (2.167) of the conformal anomaly we also find

$$\mu\frac{\partial}{\partial\mu}W = \int d^dx\sqrt{g_{(0)}}\mathcal{A}. \tag{2.170}$$

This relation allows us to evaluate the conformal anomaly, once we know the scale dependence of the renormalized correlation functions. To see this, recall that the generating functional can be written in terms of all correlation functions as

$$W = \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\sum_{\{I_1,\dots I_n\}}\prod_{m=1}^{n}\left(\int d^dx_m\sqrt{g_{(0)}}j_{I_m}(x_m)\right)\langle\mathcal{O}_{I_1}(x_1)\cdots\mathcal{O}_{I_n}(x_n)\rangle, \tag{2.171}$$

where I_m runs over all operators in the theory, in this case T_{ij}, J^i and $\mathcal{O}$, and j_{I_m} denotes the corresponding sources. Note that at each level n, one should sum over all possible n-tuples of operator insertions. Combining these two identities then we arrive at

$$\int d^dx\sqrt{g_{(0)}}\mathcal{A} = \tag{2.172}$$
$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\sum_{\{I_1,\dots I_n\}}\prod_{m=1}^{n}\left(\int d^dx_m\sqrt{g_{(0)}}j_{I_m}(x_m)\right)\mu\frac{\partial}{\partial\mu}\langle\mathcal{O}_{I_1}(x_1)\cdots\mathcal{O}_{I_n}(x_n)\rangle.$$

As an example, consider the scalar operator $\mathcal{O}_\Delta$ in a flat background metric and gauge field, which we considered earlier. Inserting (2.156) into this identity, we obtain

$$\mathcal{A} = \frac{1}{2^{2k-2}\Gamma(k)^2}\varphi_{(0)}\Box^k\varphi_{(0)} + \mathcal{O}(\varphi^3_{(0)}). \tag{2.173}$$

If a non-trivial background metric $g_{(0)ij}$ is turned on, this result generalizes to

$$\mathcal{A} = \frac{1}{2^{2k-2}\Gamma(k)^2}\varphi_{(0)}P_k\varphi_{(0)} + \mathcal{O}(\varphi^3_{(0)}) + \left(a\mathcal{E} + \sum_\alpha c_\alpha \mathcal{W}^\alpha\right) + D_i\mathcal{J}^i, \tag{2.174}$$

where P_k is a covariant differential operator that reduces to $\Box^k$ for a flat metric and transforms covariantly under Weyl transforms $g_{(0)} \to e^{2\sigma}g_{(0)}$

$$P_k \to e^{-(d/2+k)\sigma}P_k e^{(d/2-k)\sigma}. \tag{2.175}$$

For instance, for $k = 1$,

$$P_1 = \Box_{g_{(0)}} - \frac{d-2}{4(d-1)}R[g_{(0)}], \tag{2.176}$$

which is known as the *conformal Laplacian*. The two terms inside the parenthesis in (2.174) are purely gravitational and are present only when d is even. $\mathcal{E}$ is the Euler density, $\mathcal{W}^\alpha$ is a basis of Weyl invariants of dimension d and a and c_α are numerical constants that depend on the field content of the theory. For instance, in $d = 4$ there is one Weyl invariant (the square of the Weyl tensor), in $d = 6$ there are three such tensors, etc. The last term in (2.174) is scheme dependent, i.e. it can be modified by local finite counterterm terms in the action. The structure of (2.174) is dictated by the fact that the integrated conformal anomaly is itself conformally invariant [27, 28], which follows from the Wess-Zumino type consistency condition

$$(\delta_{\sigma_1}\delta_{\sigma_2} - \delta_{\sigma_2}\delta_{\sigma_1})W = 0. \tag{2.177}$$

In the next chapter we explain how, in the context of the AdS/CFT duality, all this information about a quantum field theory is encoded in classical supergravity and we develop a systematic approach for extracting this information.

2.A APPENDIX

2.A.1 $D = 2$ $\mathcal{N} = 1$ SUPERSPACE

SUPER-EUCLIDEAN ALGEBRA

The isometry group of two-dimensional Euclidean space $\mathbb{R}^2$ is the Euclidean group $\mathbf{E}_2 \approx SO(2) \ltimes \mathbb{R}^2$, which is the semidirect product of rotations and translations in the plane. It is generated by the two momenta P_1, P_2, and the angular momentum $J \equiv M_{21}$, which satisfy the algebra

$$[P_i, P_j] = 0, \qquad [J, P_1] = -iP_2, \qquad [J, P_2] = iP_1. \tag{2.178}$$

We want to describe the $\mathcal{N} = 1$ superalgebra which has this algebra as a bosonic subalgebra. To this end, recall that $\mathfrak{so}(2)$ has two one-dimensional irreducible Weyl spinor representations, D and $\bar{D}$. The Dirac spinor representation $D_D = D \oplus \bar{D}$ is then two-dimensional and can be built as usual from a representation of the Clifford algebra

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}. \tag{2.179}$$

A representation of this algebra is provided by the Pauli σ-matrices

$$\Gamma_i = \sigma^i, \quad i = 1, 2, \tag{2.180}$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.181}$$

The generator of $SO(2)$ in the Dirac spinor representation can now be written as usual as

$$\Sigma_{ij} = -\frac{i}{4}[\Gamma_i, \Gamma_j] = \frac{1}{2}\epsilon_{ij}\sigma^3, \tag{2.182}$$

or

$$J \equiv \Sigma_{21} = -\frac{1}{2}\sigma^3, \tag{2.183}$$

where ϵ_{ij} is the totally antisymmetric symbol and

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.184}$$

is the third Pauli matrix. We now introduce a two-component fermionic generator Q_α transforming under the Dirac representation of $SO(2)$, namely

$$[J, Q_\alpha] = J^\beta{}_\alpha Q_\beta. \tag{2.185}$$

Writing

$$Q_\alpha = \begin{pmatrix} Q_- \\ Q_+ \end{pmatrix}, \tag{2.186}$$

and

$$P_1 = P_+ + P_-, \qquad P_2 = i(P_+ - P_-), \tag{2.187}$$

we then have

$$[J, P_\pm] = \pm P_\pm, \qquad [J, Q_\pm] = \pm\frac{1}{2}Q_\pm. \tag{2.188}$$

The graded Jacobi identity, or equivalently the Clebsch-Gordan decomposition of the tensor product of the two Weyl spinor representations, determine

$$\{Q_\pm, Q_\pm\} = \mp 2iP_\pm, \qquad \{Q_\pm, Q_\mp\} = 0, \tag{2.189}$$

up to an overall normalization of the fermionic generators. These commutators/ anticommutators define the superalgebra $\widetilde{s\mathfrak{e}}_2$, the complexification of the super-Euclidean algebra $s\mathfrak{e}_2$.

$\mathcal{N} = 1$ SUPERSPACE REPRESENTATION

This algebra admits an elegant representation on the space of functions on $\mathcal{N} = 1$ *superspace*, $s\mathbb{C}$, which can be defined as the quotient

$$s\mathbb{C} \approx \widetilde{s\mathbf{E}}_2/SO(2). \tag{2.190}$$

This space can be parametrized by the usual complex coordinates, z and $\bar{z}$, on $\mathbb{C}$ together with two anticommuting coordinates $\theta, \bar{\theta}$. By considering the left action of $\widetilde{s\mathbf{E}}_2$ on $s\mathbb{C}$ we find that the super-Euclidean group generators take the form

$$\begin{aligned} &P_+ = i\partial_z, \qquad P_- = i\partial_{\bar{z}}, \qquad J = \bar{z}\partial_{\bar{z}} - z\partial_z + \frac{1}{2}\bar{\theta}\partial_{\bar{\theta}} - \frac{1}{2}\theta\partial_\theta, \\ &Q_+ = i\partial_\theta - i\theta\partial_z, \qquad Q_- = -i\partial_{\bar{\theta}} - i\bar{\theta}\partial_{\bar{z}}. \end{aligned} \tag{2.191}$$

By considering the right action of $\widetilde{s\mathbf{E}}_2$ on $s\mathbb{C}$ we also find the covariant derivatives

$$\begin{aligned} &\nabla_+ = \partial_z, \qquad \nabla_- = \partial_{\bar{z}}, \\ &D_+ = \partial_\theta + \theta\partial_z, \qquad D_- = -\partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}, \end{aligned} \tag{2.192}$$

which satisfy

$$D_+^2 = \partial_z, \qquad D_-^2 = -\partial_{\bar{z}}. \tag{2.193}$$

SUPERCONFORMAL TRANSFORMATIONS

The superspace representation of the super-Euclidean algebra is particularly useful because it allows us to define the notion of *superconformal* transformations as super-analytic diffeomorphisms on $s\mathbb{C}$ in the same way that conformal transformations are defined as analytic diffeomorphisms on $\mathbb{C}$. In particular, a diffeomorphism

$$z \mapsto z'(z, \bar{z}, \theta, \bar{\theta}), \qquad \theta \mapsto \theta'(z, \bar{z}, \theta, \bar{\theta}), \tag{2.194}$$

is said to be a superconformal transformation if

$$D_+ = (D_+\theta')D'_+. \tag{2.195}$$

It follows that

$$z' = f + \theta g\sqrt{\partial_z f + g\partial_z g}, \qquad \theta' = g + \theta\sqrt{\partial_z f + g\partial_z g}, \tag{2.196}$$

where $f(z)$, $g(z)$ are respectively commuting and anticommuting analytic functions.

We can now generalize the notion of *conformal tensors* to *superconformal tensors*. Under a conformal transformation, a conformal tensor $\phi(z,\bar{z})$ of weight $(h,\bar{h})$ transforms as

$$\phi'(z',\bar{z}') = (\partial_z z')^{-h}(\partial_{\bar{z}}\bar{z}')^{-\bar{h}}\phi(z,\bar{z}). \tag{2.197}$$

It is natural then to define a superconformal tensor $\phi(z,\bar{z},\theta,\bar{\theta})$ of weight $(h,\bar{h})$ by the transformation rule

$$\phi'(z',\bar{z}',\theta',\bar{\theta}') = (D_+\theta')^{-2h}(D_-\bar{\theta}')^{-2\bar{h}}\phi(z,\bar{z},\theta,\bar{\theta}) \tag{2.198}$$

under superconformal transformations.

2.A.2 Superconformal algebra in four dimensions

The conformal group of $\mathbb{R}^{p,q}$, $p+q=d>2$, is defined as the group of diffeomorphisms $\delta x^\mu = \xi^\mu$ that leave the flat metric $\eta = \mathrm{diag}\,(\underbrace{-\ldots-}_{p},\underbrace{+\ldots+}_{q})$ invariant up to a local factor, namely

$$\partial_\mu\xi_\nu + \partial_\nu\xi_\mu = \frac{2}{d}\eta_{\mu\nu}\partial_\rho\xi^\rho. \tag{2.199}$$

The solutions to these equations are the conformal transformations:

Infinitesimal	Finite	
$\delta x^\mu = a^\mu$,	$x^\mu \mapsto x^\mu + a^\mu$,	translations
$\delta x^\mu = \omega^\mu{}_\nu x^\nu$, $\omega_{\mu\nu} = -\omega_{\nu\mu}$,	$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$, $\Lambda^\mu{}_\nu \in SO(p,q)$	rotations
$\delta x^\mu = \lambda x^\mu$,	$x^\mu \mapsto \lambda x^\mu$,	dilatations
$\delta x^\mu = -b^\mu x^2 + 2x^\mu b\cdot x$,	$x^\mu \mapsto \frac{x^\mu - b^\mu x^2}{1-2b\cdot x+b^2x^2}$,	special conformal

On the space of functions on $\mathbb{R}^{p,q}$ the generators of these transformations admit respectively the representation

$$P_\mu = i\partial_\mu, \quad L_{\mu\nu} = -i(x_\mu\partial_\nu - x_\nu\partial_\mu), \quad D = ix^\mu\partial_\mu, \quad K_\mu = -i(x^2\partial_\mu - 2x_\mu x\cdot\partial). \tag{2.200}$$

One can now deduce the conformal algebra

$$\begin{aligned}
&[L_{\mu\nu}, L_{\rho\sigma}] = -i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}),\\
&[L_{\mu\nu}, P_\rho] = -i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu),\\
&[L_{\mu\nu}, K_\rho] = -i(\eta_{\nu\rho}K_\mu - \eta_{\mu\rho}K_\nu),\\
&[D, P_\mu] = -iP_\mu,\\
&[D, K_\mu] = iK_\mu,\\
&[P_\mu, K_\nu] = 2i(L_{\mu\nu} + \eta_{\mu\nu}D),
\end{aligned} \tag{2.201}$$

with all other commutators vanishing. Regrouping the generators into an antisymmetric tensor J_{ab}, $a,b=-1,0,1,\ldots,d$, $\mu=0,\ldots,d-1$, as

$$J_{\mu\nu}=L_{\mu\nu},\qquad J_{-1\mu}=\frac{1}{2}(P_\mu-K_\mu),\qquad J_{d\mu}=\frac{1}{2}(P_\mu+K_\mu),\qquad J_{-1d}=D, \tag{2.202}$$

this algebra becomes

$$[J_{ab},J_{cd}]=-i(\eta_{bc}J_{ad}+\eta_{ad}J_{bc}-\eta_{ac}J_{bd}-\eta_{bd}J_{ac}), \tag{2.203}$$

where $\eta_{-1\mu}=\eta_{-1,d}=\eta_{d\mu}=0$ and $\eta_{dd}=-\eta_{-1,-1}=1$. The conformal algebra of $\mathbb{R}^{p,q}$ is therefore identified as the Lie algebra $\mathfrak{so}(p+1,q+1)$.

We can now extend this algebra by introducing *fermionic* generators transforming under the Dirac spinor representation of $\mathfrak{so}(p,q)$.[12] To construct this superconformal algebra we start by embedding the spinor representation of $\mathfrak{so}(p,q)$ into the spinor representation of $\mathfrak{so}(p+1,q+1)$. Given a representation of the Clifford algebra

$$\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}, \tag{2.204}$$

we construct a representation of the Clifford algebra

$$\{\Gamma^a,\Gamma^b\}=2\eta^{ab}, \tag{2.205}$$

by defining

$$\Gamma^\mu=\begin{pmatrix}\gamma^\mu & 0\\ 0 & -\gamma^\mu\end{pmatrix},\quad \Gamma^{-1}=-\Gamma_{-1}=\begin{pmatrix}0 & -\mathbf{1}\\ \mathbf{1} & 0\end{pmatrix},\quad \Gamma^d=\Gamma_d=\begin{pmatrix}0 & \mathbf{1}\\ \mathbf{1} & 0\end{pmatrix}. \tag{2.206}$$

For even d there is also a chirality matrix γ which leads to the chirality matrix

$$\Gamma=\begin{pmatrix}\gamma & 0\\ 0 & -\gamma\end{pmatrix}. \tag{2.207}$$

The Lorentz generators in the spinor representation of $\mathfrak{so}(p+1,q+1)$ take the form

$$\Sigma^{ab}=-\frac{i}{4}[\Gamma^a,\Gamma^b]. \tag{2.208}$$

Using the relation (2.202) of the generators of the conformal algebra to the $\mathfrak{so}(p+1,q+1)$ generators, we obtain the following matrix representation of the conformal algebra:

$$\begin{aligned}\Sigma_{\mu\nu}&=\begin{pmatrix}-\frac{i}{4}[\gamma_\mu,\gamma_\nu] & 0\\ 0 & -\frac{i}{4}[\gamma_\mu,\gamma_\nu]\end{pmatrix},\qquad D=-\frac{i}{2}\begin{pmatrix}\mathbf{1} & 0\\ 0 & -\mathbf{1}\end{pmatrix},\\ P_\mu&=i\begin{pmatrix}0 & \gamma_\mu\\ 0 & 0\end{pmatrix},\qquad K_\mu=-i\begin{pmatrix}0 & 0\\ \gamma_\mu & 0\end{pmatrix}.\end{aligned} \tag{2.209}$$

[12] Since the properties of the spinor representation depend crucially on the signature of the metric, our results here strictly apply only to the physically relevant case $p=1$.

A Dirac spinor of $\mathfrak{so}(p+1,q+1)$ then takes the form

$$V_{\tilde{\alpha}} = \begin{pmatrix} Q_\alpha \\ (C^{-1}\bar{S}^T)_{\tilde{\alpha}} \end{pmatrix} \equiv \begin{pmatrix} Q_\alpha \\ \bar{S}_{\tilde{\alpha}} \end{pmatrix}, \tag{2.210}$$

where Q and S are Dirac spinors of $\mathfrak{so}(p,q)$ and C is the charge conjugation matrix. It follows that $C^{-1}\bar{S}^T$ also transforms as a Dirac spinor under $\mathfrak{so}(p,q)$. The $\mathcal{N}$-extended superconformal algebra is now obtained by introducing $\mathcal{N}$ such $\mathfrak{so}(p+1,q+1)$ spinor generators, which must transform under the superconformal group as

$$[J_{ab}, V^I_{\tilde{\alpha}}] = (\Sigma_{ab})^{\tilde{\beta}}{}_{\tilde{\alpha}} V^I_{\tilde{\beta}}, \qquad I = 1,\ldots,\mathcal{N}. \tag{2.211}$$

Using the above form of the superconformal generators these commutation relations translate to

$$\begin{aligned}
&[J_{\mu\nu}, Q^I_\alpha] = -\tfrac{i}{4}[\gamma_\mu,\gamma_\nu]^\beta{}_\alpha Q^I_\beta, && [J_{\mu\nu}, \bar{S}^I_{\tilde{\alpha}}] = -\tfrac{i}{4}[\gamma_\mu,\gamma_\nu]^{\tilde{\beta}}{}_{\tilde{\alpha}} \bar{S}^I_{\tilde{\beta}}, \\
&[P_\mu, Q^I_\alpha] = 0, && [P_\mu, \bar{S}^I_{\tilde{\alpha}}] = i(\gamma_\mu)^\beta{}_{\tilde{\alpha}} Q^I_\beta, \\
&[K_\mu, Q^I_\alpha] = -i(\gamma)^{\tilde{\beta}}{}_\alpha \bar{S}^I_{\tilde{\beta}}, && [K_\mu, \bar{S}^I_{\tilde{\alpha}}] = 0, \\
&[D, Q^I_\alpha] = -\tfrac{i}{2} Q^I_\alpha, && [D, \bar{S}^I_{\tilde{\alpha}}] = \tfrac{i}{2}\bar{S}^I_{\tilde{\alpha}}.
\end{aligned} \tag{2.212}$$

It remains to determine the anticommutators of the fermionic generators. Since this depends on the spacetime dimension d, we focus on $d = 4$ below. The Dirac spinor of $\mathfrak{so}(1,3)$ is the direct sum of two complex conjugate Weyl spinors. We can then take as independent irreducible generators the spinors $Q^I_{+\alpha}$, $\bar{Q}^I_{+\alpha}$, $\bar{S}^I_{+\tilde{\alpha}}$ and $S^I_{+\tilde{\alpha}}$, where the $+$ subscript indicates that they are the non-trivial eigenvectors of the chirality projection operator $P_+ = (1+\gamma)/2$. The Clebsch-Gordan decomposition of the product of two Weyl spinors, together with the super-Jacobi identity then determine

$$\begin{aligned}
&\{Q^I_{+\alpha}, \bar{Q}^J_{+\beta}\} = \delta^{IJ} P_\mu (CP_+\gamma^\mu)_{\alpha\beta}, \qquad \{S^I_{+\tilde{\alpha}}, \bar{S}^J_{+\tilde{\beta}}\} = \delta^{IJ} K_\mu (CP_+\gamma^\mu)_{\tilde{\alpha}\tilde{\beta}}, \\
&\{Q^I_{+\alpha}, Q^J_{+\beta}\} = \{S^I_{+\tilde{\alpha}}, S^J_{+\tilde{\beta}}\} = \{Q^I_{+\alpha}, \bar{S}^J_{+\tilde{\beta}}\} = 0, \\
&\{Q^I_{+\alpha}, S^J_{+\tilde{\beta}}\} = (a_1\delta^{IJ} D + a_2 T^{IJ})(CP_+)_{\alpha\tilde{\beta}} + a_3 \delta^{IJ} J_{\mu\nu} (CP_+\gamma^{\mu\nu})_{\alpha\tilde{\beta}},
\end{aligned} \tag{2.213}$$

and the automorphism - or R-symmetry - group is $U(\mathcal{N})$. For the $\mathcal{N} = 4$ extended superalgebra then the automorphism group is $U(4)$ but the superconformal group involves only the $SU(4)$ part. The constants a_1, a_2, a_3 and the matrix T^{IJ} can be determined by further use of the Jacobi identity involving the generators of the automorphism group.

2.A.3 AdS geometry

$(d+1)$-dimensional anti de Sitter space, AdS_{d+1}, is the homogeneous space

$$AdS_{d+1} \cong SO(d,2)/SO(d,1), \tag{2.214}$$

and can be represented as the hyperboloid

$$-Y_{-1}^2 - Y_0^2 + \sum_{i=1}^{d} Y_i^2 = -l_{d+1}^2, \tag{2.215}$$

embedded in $\mathbb{R}^{2,d}$, where l_{d+1} is the AdS_{d+1} radius of curvature. The flat metric

$$d\bar{s}^2 = -dY_{-1}^2 - dY_0^2 + \sum_{i=1}^{d} dY_i^2. \tag{2.216}$$

on $\mathbb{R}^{2,d}$ induces a Lorentzian metric on the hyperboloid which solves Einstein's equations in $d+1$ dimensions with a negative cosmological constant

$$\Lambda_{d+1} = -\frac{d(d-1)}{2l_{d+1}^2}. \tag{2.217}$$

This can be easily deduced from the fact that the flat metric (2.216) solves the vacuum Einstein equations in $d+2$ dimensions.

The hyperboloid (2.215) can be parametrized in various ways. In Table 2.2 we give the three parameterizations that will be most relevant for our discussion. The Poincaré coordinates cover only half of the hyperboloid, while the global coordinates cover the entire space. For the AdS_d slicing parametrization this question depends on the coordinates chosen for the slice. In global coordinates the AdS_{d+1} metric is conformal to (half of) the Einstein static universe

$$d\bar{s}^2 = d\theta^2 - dt^2 + \sin^2\theta d\Omega_{d-1}^2. \tag{2.218}$$

The boundary is located at $\theta = \pi/2$ and has the topology $S^1 \times S^{d-1}$, since $0 \leq t \leq 2\pi$. However, a compact time coordinate leads to closed time-like curves. To avoid these, one considers instead the universal cover of AdS_{d+1} with $-\infty \leq t \leq \infty$, whose boundary has the topology $\mathbb{R}^1 \times S^{d-1}$.

For the greater part of this thesis, however, we work instead with the *hyperbolic space*

$$\mathbb{H}_{d+1} \cong SO(d+1,1)/SO(d+1), \tag{2.219}$$

which is obtained from (2.215) by analytically continuing $Y_0 \to iY_0$ and whose boundary is topologically the sphere S^d. The analogue of the Poincaré parametrization of AdS leads to the *upper half plane* metric

$$ds^2 = \frac{l_{d+1}^2}{z_0^2}(dz_0^2 + dz_i^2), \qquad i = 1, \ldots, d. \tag{2.220}$$

In this coordinate system the conformal boundary corresponds to the one-point compactification of the hyperplane $z_0 = 0$, $S^d \cong \mathbb{R}^d \cup \{\infty\}$. Since this is the most convenient coordinate system for the calculation of correlation functions of the boundary CFT, it is appropriate to discuss the isometries and the geodesics of $\mathbb{H}_{d+1}$ in this parametrization.

Poincaré patch	
$Y_{-1} + Y_d = \frac{l_{d+1}}{z_0}$ $Y_{-1} - Y_d = \frac{l_{d+1}}{z_0}(z_0^2 - t^2 + z_i z^i)$ $Y_0 = \frac{l_{d+1}}{z_0} t$ $Y_i = \frac{l_{d+1}}{z_0} z_i, \quad i = 1, \ldots, d-1$	$ds^2 = \frac{l_{d+1}^2}{z_0^2}(dz_0^2 - dt^2 + dz_i^2)$ $z_0 \geq 0, \quad t, z_i \in \mathbb{R}$
Global coordinates	
$Y_{-1} = l_{d+1} \sec\theta \cos t$ $Y_0 = l_{d+1} \sec\theta \sin t$ $Y_i = l_{d+1} \tan\theta n^i, \quad \sum_{i=1}^d n^i n^i = 1$	$ds^2 = \frac{l_{d+1}^2}{\cos^2\theta}\left(d\theta^2 - dt^2 + \sin^2\theta d\Omega_{d-1}^2\right)$ $0 \leq \theta \leq \pi/2$
AdS_d slicing of AdS_{d+1}	
$Y_d = l_{d+1} \tan\mu$	$ds^2(AdS_{d+1}) = \frac{l_{d+1}^2}{\cos^2\mu}\left[d\mu^2 + \frac{1}{l_d^2} ds^2(AdS_d)\right]$ $-\pi/2 \leq \mu \leq \pi/2$

Table 2.2: AdS coordinate systems.

ISOMETRIES

If we let $z^\mu = (z_0, z^i)$, with $i = 1, \ldots, d$, then the Killing vectors of the upper half plane metric (2.220) take the form

$(0, a^i)$		translations
$(0, \omega^i{}_j z^j)$,	$\omega_{ij} = -\omega_{ji}$	rotations
$(\lambda z_0, \lambda z^i)$		dilatations
$(2\vec{c}\cdot\vec{z}z_0, 2\vec{c}\cdot\vec{z}z^i - c^i\vec{z}^2)$		special conformal

and hence the isometry group of hyperbolic space is identified with the conformal group $SO(1, d+1)$ in one dimension less. However, there are also the following *conformal* Killing vectors

$$(\alpha, 0), \qquad (\vec{\gamma}\cdot\vec{z}, -\gamma^i), \qquad (\varepsilon(z_0^2 - \vec{z}^2), 2\varepsilon z_0 z^i), \tag{2.221}$$

which together with the above Killing vectors make up the conformal isometry group $SO(1, d+2)$.

The upper half plane metric has in addition an important discrete isometry, namely the *inversion*

$$z^\mu \longmapsto z^\mu / z^2, \tag{2.222}$$

where $z^2 = z_o^2 + \vec{z}^2$. In fact, special conformal transformations can be written as a translation preceded and followed by an inversion. This discrete isometry imposes very strong constraints on the form of the boundary CFT correlation functions.

GEODESICS

The geodesics of the upper half plane metric are semicircles of radius R, centered at a point $\vec{c}$ on the boundary $z_0 = 0$:

$$z_0^2 + (\vec{z}^2 - \vec{c}^2)^2 = R^2. \tag{2.223}$$

They can be parametrized as

$$\begin{cases} z_0(\tau) = R\,\mathrm{sech}\tau, \\ z^i(\tau) = Rn^i \tanh\tau + c^i, \qquad \vec{n}^2 = 1. \end{cases} \tag{2.224}$$

The geodesic distance between two points z and w is

$$d(z, w) = \log\left(\frac{1 + \sqrt{1 - \xi^2}}{\xi}\right), \tag{2.225}$$

where ξ is the $SO(1, d+1)$-invariant length

$$\xi = \frac{2z_0 w_0}{z_0^2 + w_0^2 + (\vec{z} - \vec{w})^2}. \tag{2.226}$$

Finally, the geodesic through two given points z and w has parameters

$$R = \frac{\sqrt{1-\xi}}{\xi}\frac{z_0 w_0}{|\vec{z}-\vec{w}|}, \qquad \vec{n} = \frac{\vec{z}-\vec{w}}{|\vec{z}-\vec{w}|},$$
$$2\vec{c} = \left(1 + \frac{z_0^2 - w_0^2}{(\vec{z}-\vec{w})^2}\right)\vec{z} + \left(1 + \frac{w_0^2 - z_0^2}{(\vec{z}-\vec{w})^2}\right)\vec{w}. \tag{2.227}$$

Chapter 3

Holographic Renormalization

We have seen that the AdS/CFT correspondence relates string theory on $AdS_5 \times S^5$ with $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory 'residing' on the boundary of AdS_5. In particular, the supergravity approximation of string theory corresponds to the large N, large 't Hooft coupling limit of the gauge theory. Although this is the most studied and best understood example of an AdS/CFT-type duality, a plethora of other examples exists where, generically, string theory (M-theory) on an *asymptotically locally* AdS_{d+1} space times a compact manifold X_{9-d} (X_{10-d}) is related to a quantum field theory on the boundary of the asymptotically locally AdS space. However, not always does a well-defined supergravity approximation exist.

Moreover, even in the context of the original correspondence between $AdS_5 \times S^5$ and $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills, one can break the conformal invariance of the boundary theory either by a deformation by a relevant operator or by giving a vacuum expectation value (vev) to a scalar operator. In either case, the non-conformal theory will be dual to string theory on an *asymptotically locally* AdS_5 space and *not* exactly AdS_5.

Finally, in order to calculate correlation functions of the boundary quantum field theory via the AdS/CFT prescription, we are forced to consider solutions of the supergravity equations with arbitrary Dirichlet boundary conditions, since these play the role of sources for the dual gauge-invariant operators. In particular, one should consider arbitrary boundary metrics $g_{(0)ij}$.

If we want to have a general method for calculating correlation functions in the context of a generalized AdS/CFT duality, all the aforementioned reasons lead us to consider supergravity in *asymptotically locally* AdS space of arbitrary dimension. We therefore start in this chapter, which is an expanded version of the paper [1], with a precise definition of what we mean by an asymptotically locally AdS space. The method of holographic renormalization [29, 30, 31, 32] (for a review see [33]; for related work see [34, 35] – a more complete list of references can be found in the review) for calculating renormalized correlation functions of the boundary quantum field theory is then presented. This formalism automatically incorporates the 'kinematic' constraints, i.e. the Ward identities and their anomalies, and identifies the part of the geometry where the 'dynamical' information, i.e. the correlation functions, is encoded. A key ingredient of this method is the asymptotic expansion of all bulk fields in the radial distance from the boundary of AdS [36] (for relevant math reviews see [37, 38]). As we have seen, this radial distance corresponds to the energy scale of the dual field theory and hence, from the point of view of the dual field theory we expand around a UV fixed point of the boundary field theory. Correlation functions are encoded in specific coefficients in the asymptotic expansion of the bulk fields and Ward identities and anomalies originate in certain relations that these coefficients satisfy.

Subsequently we explain why it is much more efficient to replace the asymptotic expansions of the bulk fields with *covariant expansions in the eigenvalue of the di-*

latation operator. This allows us to develop an elegant 'Hamiltonian' version of the method of holographic renormalization, where the radial coordinate plays the role of time. Our Hamiltonian method builds on earlier Hamiltonian approaches to the holographic renormalization group using the Hamilton-Jacobi equation [39, 40, 41] or the Gauss-Codazzi equations [35]. In the new method the focus is shifted from the on-shell supergravity action to the canonical momenta of the bulk fields. The latter are associated with the (regularized) one-point functions of gauge-invariant operators in the presence of sources [39]. This leads to a faster algorithm for determining the covariant counterterms and the correlation functions. Moreover, since the method involves covariant expansions in the dilatation weight, the Ward identities are manifest throughout the analysis.

We conclude this chapter with some applications. First, we consider pure AdS gravity and we obtain universal recursion relations for the asymptotic solutions and the counterterms that are valid in all dimensions. Special attention is paid to the case of AdS_3, where the radial equations can be solved exactly. We then further demonstrate the method by considering gravity coupled to two active scalar fields in five dimensions with an arbitrary potential.

Throughout this chapter we work with Euclidean signature, but we give the Lorentzian version of some important formulas in Appendix 3.A.3 since they will be relevant for the last chapter of this thesis.

3.1 ASYMPTOTICALLY LOCALLY AdS SPACETIMES

As we discuss in Appendix 2.A.3, AdS_{d+1} is the maximally symmetric solution of Einstein's equations with negative cosmological constant, $\Lambda = -d(d-1)/2l^2$, where l is the radius of AdS_{d+1} (we set $l = 1$ in this chapter; one can easily reinstate this factor in all equations by dimensional analysis). Being maximally symmetric, its Riemann and Ricci curvature tensors are given respectively by

$$R_{\mu\nu\rho\sigma} = (g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}), \qquad R_{\mu\nu} = -dg_{\mu\nu}. \tag{3.1}$$

AdS_{d+1} has a conformal boundary with topology $\mathbb{R} \times S^{d-1}$, where $\mathbb{R}$ corresponds to the time direction. A precise definition of what one means by 'conformal boundary' will be given below.

We will define *asymptotically locally AdS* (AlAdS) spacetimes as solutions of Einstein's equations with a negative cosmological constant whose Riemann tensor approaches (3.1) asymptotically, in a sense that we will specify shortly. We warn the reader, however, that there is no consensus in the literature as to what is meant by 'asymptotically locally AdS' or 'asymptotically AdS' spacetimes. Various authors, by 'asymptotically AdS', refer to spacetimes that asymptotically become exactly AdS spacetime (e.g. [42, 43]). Moreover, many authors use the term 'asymptotically locally AdS' for spacetimes that are, or asymptotically become, exactly quotients of AdS by a discrete subgroup of its isometry group. We emphasize that all these spaces are special cases of our definition. To add to the confusion, what we call here 'asymptotically locally AdS' spaces, were referred to as 'asymptotically AdS' in [33]. Given the lack of a ubiquitous name in the physics terminology, we resort to the mathematics literature and call the manifolds we are interested in *conformally compact Einstein manifolds* [33].

Let us first define conformally compact manifolds following [44] (see also [37, 38]). Let $\mathcal{M}$ be the interior of a compact $(d+1)$-dimensional manifold $\overline{\mathcal{M}}$ with boundary $\partial\mathcal{M}$. A (pseudo)Riemannian metric g on $\mathcal{M}$ is conformally compact if there is a defining function z on $\overline{\mathcal{M}}$, that is a smooth, non-negative function on $\overline{\mathcal{M}}$ with $z(\partial\mathcal{M})=0$ and $dz(\partial\mathcal{M})\neq 0$, such that the conformally related metric

$$\tilde{g}=z^2 g \tag{3.2}$$

extends smoothly to a non-degenerate metric on $\overline{\mathcal{M}}$. The boundary metric $g_{(0)}=\tilde{g}|_{\partial\mathcal{M}}$ is uniquely specified by the conformal compactification $\tilde{g}$. However, there are many defining functions and hence many conformal compactifications of the metric g. This means that the pair $(\overline{\mathcal{M}},g)$ determines only the conformal class $[g_{(0)}]$ of the boundary metric, which is known as the *conformal infinity* of $(\overline{\mathcal{M}},g)$. A particular conformal compactification $\tilde{g}$ then determines a representative $g_{(0)}$ of the conformal class $[g_{(0)}]$.

Given a conformal compactification $\tilde{g}$, one easily sees that the Riemann tensor of the bulk metric g takes asymptotically the form

$$R_{\mu\nu\rho\sigma}=|dz|^2_{\tilde{g}}\left(g_{\mu\sigma}g_{\nu\rho}-g_{\mu\rho}g_{\nu\sigma}\right)+\mathcal{O}(z^{-3}), \tag{3.3}$$

where $|dz|^2_{\tilde{g}}=\tilde{g}^{\mu\nu}\partial_\mu z\partial_\nu z$ and hence the first term is $\mathcal{O}(z^{-4})$ as $z\to 0$. Requiring that g satisfies Einstein's equations with a negative cosmological constant $\Lambda=-d(d-1)/2$ determines

$$|dz|^2_{\tilde{g}}=1. \tag{3.4}$$

Therefore, the Riemann curvature tensor of a conformally compact manifold that is also Einstein approaches asymptotically that of exact AdS space (3.1). This is why we refer to such manifolds as 'asymptotically locally AdS' spacetimes.

The most general asymptotics of such spacetimes was determined in [45] for pure gravity and their analysis extends straightforwardly to include matter with soft enough behavior at infinity, see for instance [30, 46, 32, 1]. Near the boundary, one can always choose coordinates in which the metric takes the form,[1]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{dz^2}{z^2} + \frac{1}{z^2}g_{ij}(z,x)dx^i dx^j,$$
$$g(z,x) = g_{(0)} + zg_{(1)} \cdots + z^d g_{(d)} + h_{(d)} z^d \log z^2 + \cdots \tag{3.5}$$

In these coordinates the conformal boundary is located at $z = 0$ and $g_{(0)}$ is a representative of the conformal structure. The asymptotic analysis reveals that all coefficients shown above except the traceless and divergenceless part of $g_{(d)}$ are locally determined in terms of the boundary data $g_{(0)}$. The logarithmic term appears only in even (boundary) dimensions (for pure gravity; if matter fields are included, then a logarithmic term can appear in odd dimensions as well [30]) and is proportional [30] to the metric variation of the integrated holographic conformal anomaly [29]. Since (3.5) is a conformally compact Einstein metric, its Riemann tensor takes the form (3.1) up to a correction of order z. This continues to be true in the presence of matter if the cosmological constant remains asymptotically the dominant term in the stress-energy tensor. This is true for matter that corresponds to marginal or relevant operators of the dual theory in the AdS/CFT duality.

Exact AdS_{d+1} space is conformally flat and this implies [47] that $g_{(0)}$ is conformally flat as well. The asymptotic expansion (3.5) then terminates at order z^4 with

$$g_{(2)ij} = -\frac{1}{d-2}(R_{ij} - \frac{1}{2(d-1)}Rg_{(0)ij}), \qquad g_{(4)} = \frac{1}{4}(g_{(2)})^2, \tag{3.6}$$

where R_{ij} is the Ricci tensor of $g_{(0)}$ ($d = 2$ is a special case, see [47] for the expression of $g_{(2)}$) and $g_{(0)}$ may be chosen to be the standard metric on $\mathbb{R} \times S^{d-1}$. As we mentioned earlier, many authors refer to 'asymptotically AdS' spacetimes as spacetimes whose metric becomes asymptotically exactly that of AdS. In our language this means that they take the conformal representative $g_{(0)}$ to be exactly that of AdS. With an appropriate choice of a defining function then, an 'asymptotically AdS' metric differs from the metric of exact AdS_{d+1} at most at order $\mathcal{O}(z^d)$ since all lower order coefficients in the expansion (3.5) depend locally on the boundary metric $g_{(0)}$. In particular, as for exact AdS, the logarithmic term vanishes for such spacetimes.

AlAdS spacetimes have an arbitrary conformal structure $[g_{(0)}]$ and a general $g_{(d)}$, the logarithmic term is in general non-vanishing, and there is no *a priori* restriction on the topology of the conformal boundary. The mathematical structure of these spacetimes (or their Euclidean counterparts) is under current investigation in the mathematics community, see [38] and references therein. For instance, it is has not yet been established how many, if any, global solutions exist given a conformal

[1]In most examples in the literature the odd coefficients $g_{(2k+1)}$ vanish (except when $2k+1 = d$, the boundary dimension). In such cases, it is more convenient [29] to use instead of z a new radial coordinate $\rho = z^2$.

structure, although given (sufficiently regular) $g_{(0)}$ and $g_{(d)}$, a unique solution exists in a thickening $\partial\mathcal{M} \times [0, \epsilon)$ of the boundary $\partial\mathcal{M}$. On the other hand, interesting examples of such spacetimes exist, see [38] for a collection of examples.

There is an important difference between even and odd dimensions. When the spacetime is odd dimensional, there is a conformally invariant quantity $A[g_{(0)}]$ one can construct using the boundary conformal structure $[g_{(0)}]$, namely the integral of the holographic conformal anomaly [29] (called renormalized volume in the math literature [37]).[2] The holographic conformal anomaly was found in [29] by considering the response of the *renormalized* on-shell supergravity action to Weyl transformations. As we saw in the example in the previous chapter, in order to render finite the on-shell gravitational action (which diverges due to the infinite volume of the AlAdS spacetime) one is forced to add a certain number of boundary covariant counterterms and the latter induce an anomalous Weyl transformation. It was shown in [29] that this anomaly precisely matches the conformal anomaly in the boundary field theory.

In the last chapter we will argue that the covariant counterterms are also a direct consequence of the requirement that the variational problem for the supergravity action with arbitrary Dirichlet boundary conditions in a general AlAdS spacetime is well-posed. Hence the conformal anomaly is shown to be a genuine property of the variational problem for AdS gravity.

3.2 HOLOGRAPHIC RENORMALIZATION

We devote this section to an exposition of the original method of holographic renormalization [29, 30, 31, 32] (see [33] for a review). The new Hamiltonian version of the method will be then presented in the next section.

To illustrate the method we consider a single massive scalar field coupled to gravity. More fields can be easily included, but the analysis becomes considerably more elaborate. The general form of the bulk action is then[3]

$$S = \int d^{d+1}x\sqrt{g}\left(-\frac{1}{2\kappa^2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + V(\varphi) + \cdots\right), \tag{3.7}$$

where $\kappa^2 = 8\pi G_{d+1}$ is proportional to Newton's constant G_{d+1}, and the dots indicate potential contributions from additional fields such as gauge fields, fermions, and antisymmetric tensors. Restricting to the gravity-scalar sector means that we only

[2]When certain matter fields are present one has additional conformal invariants in all dimensions which can contribute to the matter conformal anomalies [23]. We have already seen an example of such an anomaly earlier when we discussed the scalar field in AdS space.

[3]One should include a Gibbons-Hawking boundary term [48] to ensure that the variational problem leads to the equations of motion (3.9). We assume here that such a term has been included and we postpone a more careful treatment until the next section.

study correlation functions of the stress-energy tensor and a scalar operator. The potential has the form,

$$V(\varphi) = \frac{\Lambda}{\kappa^2} + \frac{1}{2}m^2\varphi^2 + b\varphi^3 + \cdots \,, \tag{3.8}$$

where Λ is the cosmological constant, b is a constant and the mass m^2 of the scalar field is related to the dimension Δ of the dual operator by $m^2 = (\Delta - d)\Delta$. The bulk field equations are given by

$$G_{\mu\nu} = \kappa^2 \tilde{T}_{\mu\nu}(\varphi), \qquad \Box_g \varphi = \partial V/\partial\varphi, \tag{3.9}$$

where $G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}R/2$ is the Einstein tensor, the covariant Laplacian is given by $\Box_g\varphi = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\varphi)$ and $\tilde{T}_{\mu\nu}(\varphi)$ is the stress-energy tensor associated with the scalar field φ (see (3.20) below).

The method of holographic renormalization now consists in the following steps:

Asymptotic solutions

In the first step one works out the most general asymptotic solutions with given Dirichlet data

$$\begin{aligned} ds^2 &= \frac{dz^2}{z^2} + \frac{1}{z^2} g_{ij}(z,x)dx^i dx^j, \\ \varphi(z,x) &= z^{(d-\Delta)}\phi(z,x), \end{aligned} \tag{3.10}$$

where[4]

$$g_{ij}(z,x) = g_{(0)ij} + z^2 g_{(2)ij} + \cdots + z^d(g_{(d)ij} + \log z^2 h_{(d)ij}) + \cdots \,,$$

$$\phi(z,x) = \phi_{(0)} + z^2\phi_{(2)} + \cdots + z^{2\Delta-d}(\phi_{(2\Delta-d)} + \log z^2 \psi_{(2\Delta-d)}) + \cdots \,. \tag{3.11}$$

In this expansion, $g_{(0)ij}$ and $\phi_{(0)}$ are identified with the QFT sources that couple to the dual operators, as discussed in the previous chapter.

Inserting these expansions in the bulk field equations (3.9) one obtains a set of algebraic equations for the coefficients $g_{(n)}$ and $\phi_{(n)}$. These equations determine recursively all coefficients, except for $\phi_{(2\Delta-d)}$ and the traceless transverse part of $g_{(d)ij}$, as local functionals of the sources $g_{(0)ij}$ and $\phi_{(0)}$ [36, 29, 30]. The precise form of the coefficients depends on the spacetime dimension d and on the scaling

[4]As we will see explicitly in the examples considered at the end of this chapter, depending on the dimension d and the scaling dimension Δ of the scalar operator, these expansions may contain odd powers of the radial coordinate z as well. This does not affect our qualitative discussion here. See footnote 14.

dimension Δ of the scalar operator. For example, if the scalar field is turned-off, one finds for $d > 2$ [30]

$$g_{(2)ij}[g_{(0)}] = \frac{1}{d-2}\left(R_{ij}[g_{(0)}] - \frac{1}{2(d-1)}R[g_{(0)}]g_{(0)ij}\right), \qquad (3.12)$$

but in general $g_{(2)}$ and all other coefficients will be functions of *both* sources $g_{(0)ij}$ and $\phi_{(0)}$. As we will see below, the undetermined coefficients $g_{(d)ij}$ and $\phi_{(2\Delta-d)}$, which we call the *response functions*, correspond to the one-point functions of the dual operators in the presence of sources. The parts of $g_{(d)ij}$ that are determined, that is $D^i g_{(d)ij}$ and $\mathrm{Tr}\, g_{(d)}$, encode respectively the Ward identities (2.165) and (2.168). The logarithmic terms appear only in special cases: $h_{(d)}$ only in even dimensions and $\psi_{(2\Delta-d)}$ only when $\Delta - d/2$ is an integer. Both of them are directly related to the conformal anomalies discussed at the end of the previous chapter. Namely, $h_{(d)}$ is the metric variation of the gravitational part of the conformal anomaly and $\psi_{(2\Delta-d)}$ is the variation with respect to $\phi_{(0)}$ of the matter part of the conformal anomaly [30].

Divergences of the on-shell action

Having obtained the asymptotic solutions, one introduces a radial cut-off $z \geq \epsilon$ with $\epsilon > 0$, and evaluates the supergravity action (3.7) on the regulating surface $z = \epsilon$. The resulting regulated action contains generically a number of power law divergent terms as well as a logarithmically divergent term. Its general form is then

$$S_{\mathrm{reg}}[g_{(0)}, \phi_{(0)}; \epsilon] = \frac{1}{2\kappa^2}\int d^d x \sqrt{g_{(0)}}\left(\frac{a_{(0)}}{\epsilon^d} + \frac{a_{(1)}}{\epsilon^{d-1}} + \cdots + a_{(d)}\log\epsilon^2 + \mathcal{O}(\epsilon^0)\right). \qquad (3.13)$$

It turns out all coefficients $a_{(n)}$ depend locally only on $g_{(0)}$ and $\phi_{(0)}$ but not on the undetermined coefficients $g_{(d)}$ and $\phi_{(2\Delta-d)}$. The coefficient $a_{(d)}$ of the logarithmic divergence is the conformal anomaly of the dual CFT [29]. For illustrative purposes, let us give the values of the coefficients in (3.13) for pure gravity in $d = 4$ [30]:

$$a_{(0)} = -6, \quad a_{(2)} = 0, \quad a_{(4)} = \frac{1}{2}\left[\mathrm{Tr}\,(g_{(0)}^{-1}g_{(2)})^2 - (\mathrm{Tr}\,(g_{(0)}^{-1}g_{(2)}))^2\right], \qquad (3.14)$$

where $g_{(2)}$ is given in terms of $g_{(0)}$ in (3.12) and all odd coefficients vanish.

Counterterms and renormalized action

To obtain a well-defined on-shell action we should subtract the infinities and then remove the regulator. However this does *not* mean that one should simply subtract the divergent terms shown in (3.13) since this would generically break diffeomorphism covariance on the regulating surface and, as a result, the correlation functions calculated with such a renormalization scheme would violate the Ward identities. To ensure that the Ward identities are satisfied we must remove the divergences of (3.13) by adding *covariant counterterms*. To do this we first express the divergent terms in (3.13) in terms of induced fields at the hypersurface $z = \epsilon$. This entails inverting the asymptotic series (3.11) in order to express the sources $g_{(0)}$ and $\phi_{(0)}$ in terms of the induced metric $\gamma_{ij}(\epsilon, x) = g_{ij}(\epsilon, x)/\epsilon^2$ and the scalar field $\varphi(\epsilon, x)$ on the regulating surface. Inserting then these inverted series in the divergent terms (3.13), one obtains the divergences of the regulated action in *covariant form*. For pure gravity in $d = 4$ one finds

$$S_{\text{reg}} = -\frac{3}{\kappa^2}\int_{z=\epsilon} d^4x\sqrt{\gamma}\left[1 + \frac{1}{12}R[\gamma] - \frac{1}{48}\left(R[\gamma]^{kl}R[\gamma]_{kl} - \frac{1}{3}R[\gamma]^2\right)\log\epsilon^2 + \cdots\right]. \tag{3.15}$$

Inverting the asymptotic expansions in order to write the regulated action in this covariant form is one of the most laborious steps of the procedure. As we show in the next section, however, it is completely redundant. We will see that it is possible to replace the asymptotic expansions with covariant expansions in terms of the induced fields on the regulating surface, which will lead immediately to the covariant form of the regulated action, bypassing (3.13) and the tedious inversion of the asymptotic expansions.

Once we have the regulated action in covariant form, we can define the covariant counterterm action, S_{ct}, as the negative of the divergent part of the regulated action. The renormalized action is then obtained by

$$S_{\text{ren}} = \lim_{\epsilon\to 0} S_{\text{sub}}, \qquad S_{\text{sub}} = S_{\text{reg}} + S_{\text{ct}}. \tag{3.16}$$

One-point functions in the presence of sources

We can now differentiate the renormalized action to obtain the 1-point functions in the presence of sources [30],

$$\begin{aligned}
\langle T_{ij}(x)\rangle_s \equiv \frac{2}{\sqrt{g_{(0)}(x)}}\frac{\delta S_{\text{ren}}}{\delta g^{ij}_{(0)}(x)} &= \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^{d-2}}\frac{2}{\sqrt{\gamma(\epsilon,x)}}\frac{\delta S_{\text{sub}}}{\delta\gamma^{ij}(\epsilon,x)}\right) \\
&= \frac{d}{2\kappa^2}g_{(d)ij} + X_{ij}[g_{(0)}, \phi_{(0)}], \\
\langle \mathcal{O}(x)\rangle_s \equiv \frac{1}{\sqrt{g_{(0)}(x)}}\frac{\delta S_{\text{ren}}}{\delta\phi_{(0)}(x)} &= \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^{\Delta}}\frac{1}{\sqrt{\gamma(\epsilon,x)}}\frac{\delta S_{\text{sub}}}{\delta\varphi(\epsilon,x)}\right) \\
&= (d-2\Delta)\phi_{(2\Delta-d)} + Y[g_{(0)}, \phi_{(0)}],
\end{aligned} \tag{3.17}$$

where $X[g_{(0)}, \phi_{(0)}]$ and $Y[g_{(0)}, \phi_{(0)}]$ are certain computable local functionals of the sources. Their form depends on the dimension and the field content, however. In the next section, we derive a general expression for the one-point functions valid in any dimension and for arbitrary field content. The first equality in the above expressions for the one-point functions is a definition. In the second equality we expressed the renormalized one-point function as a limit of the regulated one-point function. The regulated one-point function can be computed in all generality (in a given dimension and field content) and the limit can be taken explicitly. This is a straightforward but rather tedious computation. The result is the one shown above.

This computation shows that the renormalized one-point functions are related to the coefficients that the asymptotic analysis left undetermined. As discussed above however, the near boundary analysis does determine the divergence and trace of $g_{(d)ij}$, and hence the divergence and trace of $\langle T_{ij}(x)\rangle_s$ can be determined. This yields the Ward identities (2.165) and (2.168), including the conformal anomaly. However, since one cannot write in this formalism an explicit expression for the one-point functions which is valid in general, one has to show that the trace and divergence of $g_{(d)ij}$ lead to the Ward identities for every case individually. In the next section, however, we will derive the Ward identities in full generality using the Hamiltonian version of holographic renormalization.

Correlation functions

To obtain higher-point functions one further differentiates (3.17) with respect to the sources. The expressions $X[g_{(0)}, \phi_{(0)}]$ and $Y[g_{(0)}, \phi_{(0)}]$ lead to contact terms. The (non-local) n-point function is thus encoded in the dependence of $g_{(d)}$ and $\phi_{(2\Delta-d)}$ on the sources. Hence,

The theory is solved if we determine the response functions in terms of the sources.

To obtain such a relation we need a regular exact, as opposed to merely asymptotic, solution of the bulk equations with boundary conditions specified by the (arbitrary) sources. Such a solution effectively requires solving a set of coupled first order functional differential equations for the response functions as functionals of the sources, a problem which is rarely amenable to present techniques. One then proceeds by linearizing the bulk equations around a background solution [31, 32]. As we will see in the next chapter, the background solution specifies the vacuum of the dual QFT (see also section 6.1 of [33]). Higher-point functions can be computed by solving the bulk equations perturbatively around the particular background. We will discuss extensively the calculation of correlation functions in the next chapter.

3.3 HAMILTONIAN HOLOGRAPHIC RENORMALIZATION

Having summarized the main steps involved in the original method of holographic renormalization, we now develop a new Hamiltonian version of the method which involves an elegant and more efficient algorithm. See [34, 35, 39, 40, 41] for related work.

Let $\overline{\mathcal{M}}$ be a conformally compact, Riemannian (d+1)-manifold, $\mathcal{M}$ its interior and $\partial\mathcal{M}$ its boundary. We will consider the following action for the Riemannian metric $g_{\mu\nu}$ on $\overline{\mathcal{M}}$

$$S_{\rm gr}[g] = -\frac{1}{2\kappa^2}\left[\int_{\mathcal{M}} d^{d+1}x\sqrt{g}R + \int_{\partial\mathcal{M}} d^d x\sqrt{\gamma}2K\right], \tag{3.18}$$

where $\kappa^2 = 8\pi G_{d+1}$, γ is the induced metric on $\partial\mathcal{M}$ and K is the trace of the extrinsic curvature of the boundary. This is the standard Einstein-Hilbert action with the Gibbons-Hawking boundary term which ensures that the variational problem is well-defined.[5] The overall sign is chosen so that the action is positive definite when evaluated on a classical solution in the vicinity of (Euclidean) AdS.

The supergravity action will also include a contribution from matter fields whose action takes the form

$$S_{\rm m} = \int_{\mathcal{M}} d^{d+1}x\sqrt{g}\mathcal{L}_{\rm m}, \tag{3.19}$$

where $\mathcal{L}_{\rm m}$ is a generic matter field Lagrangian density. The variation of this action with respect to the bulk metric defines the stress tensor

$$\delta_g S_{\rm m} \equiv \frac{1}{2}\int_{\mathcal{M}} d^{d+1}x\sqrt{g}\tilde{T}_{\mu\nu}\delta g^{\mu\nu}. \tag{3.20}$$

The Euler-Lagrange equations of the total action $S = S_{\rm gr} + S_{\rm m}$ are Einstein's equations

$$G_{\mu\nu} = \kappa^2\tilde{T}_{\mu\nu}, \tag{3.21}$$

together with the matter field equations, whose explicit form depends of course on the field content.

3.3.1 ADM FORMALISM AND THE GAUSS-CODAZZI EQUATIONS

In order to formulate a Hamiltonian version of holographic renormalization we will need the so-called ADM formalism (after R. Arnowitt, S. Deser and C.W. Misner) for gravity as well as the Gauss-Codazzi equations. We will now briefly review these standard results so that we can concentrate on the new method in the next subsection. The ADM formalism (see, for instance, [49] for a more extensive discussion) is a Hamiltonian formulation of Einstein gravity in a pseudo-Riemannian manifold. It relies on the existence of a *global* time function t which is used to foliate spacetime into diffeomorphic hypersurfaces of constant t (Cauchy surfaces). The existence of such a function requires the manifold to be 'globally hyperbolic' (see [49] for a definition of global hyperbolicity), which is a condition on the global structure of spacetime.

[5] We will show in the last chapter, however, that the variational problem is not completely well-defined unless *all* covariant counterterms are included too.

We will be interested, however, in the foliation of an AlAdS manifold in slices of constant 'radial distance from the boundary'. Close to the boundary, it is always possible to define a radial coordinate r that is 'normal' to a small patch of the boundary and we can take constant values of this radial distance to define locally our hypersurface. For AlAdS manifolds there always exists a radial function normal to the boundary which can be used to foliate the space in radial slices diffeomorphic to the boundary, at least in a neighborhood of the boundary[6] [36] (see also the review [38] and references therein). The question of if and where this radial coordinate emanating from the boundary ceases to be well-defined depends on the global properties of the manifold and does not affect the asymptotic analysis, which only requires the radial foliation of the manifold in a thickening of the boundary. It is vital, however, for the correct evaluation of correlation functions, which will be addressed in the next chapter.

Let r be the radial coordinate emanating from the boundary of the Riemannian manifold $(\mathcal{M}, g)$ in the way described above and consider the hypersurfaces Σ_r defined by $r(x) = \text{constant}$. The unit normal to Σ_r, pointing in the direction of increasing r, is given by $n^\mu = \frac{1}{|dr|_g} g^{\mu\nu} \frac{\partial r}{\partial x^\nu} |_\Sigma$. The induced metric on the hypersurfaces can now be expressed in a coordinate independent fashion as[7] $\hat{\gamma}_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$. The tensor $\hat{\gamma}^\mu_\nu \equiv g^{\mu\rho}\hat{\gamma}_{\rho\nu}$ acts as a projection operator onto the tangent space $T\Sigma_r$ of the hypersurface Σ_r. Let us now define the radial flow vector field $r^\mu(x)$ by the relation $r^\mu \partial_\mu r = 1$. The components of r^μ tangent and normal to Σ_r define respectively the *shift* and and *lapse* functions

$$r^\mu_\parallel = \hat{\gamma}^\mu_\nu r^\nu \equiv N^\mu, \qquad r^\mu_\perp = (r, n)_g n^\mu \equiv N n^\mu, \tag{3.22}$$

where $(\cdot, \cdot)_g$ denotes the inner product with respect to the metric $g_{\mu\nu}$. We will see later that these correspond to non-dynamical degrees of freedom which will be 'gauge-fixed'. Geometrically they measure how 'normal' the coordinate r is to the hypersurfaces: the choice $N = 1$, $N^\mu = 0$ makes r a Gaussian normal coordinate, in which case n^μ becomes tangent to *geodesics* normal to the hypersurfaces Σ_r. We can now construct a basis of one-forms on Σ_r without having to introduce a particular coordinate system on the hypersurfaces. Indeed, it can be easily checked that

$$d\hat{x}^\mu \equiv dx^\mu - r^\mu dr = dx^\mu - (N^\mu + N n^\mu) dr, \tag{3.23}$$

is a basis for the cotangent space $T^*\Sigma_r$. The metric on $\mathcal{M}$ is then decomposed as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = (N^2 + N_\mu N^\mu) dr^2 + 2N_\mu d\hat{x}^\mu dr + \hat{\gamma}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu. \tag{3.24}$$

[6] If the boundary consists of multiple disconnected components, then a radial function exists in the vicinity of each boundary. We assume, however, that the boundary is connected here.

[7] We use a hat to denote tensors that are purely transverse to the unit normal, i.e. quantities which vanish when contracted with n^μ.

A quantity that will be of central importance in our analysis is the extrinsic curvature of the hypersurfaces

$$\hat{K}_{\mu\nu} = \hat{\gamma}^{\rho}_{\mu}\nabla_{\rho}n_{\nu} = \frac{1}{2}\pounds_{n}\hat{\gamma}_{\mu\nu}, \tag{3.25}$$

where $\pounds_n$ denotes the Lie derivative with respect to the unit normal n^{μ}. Thus, the extrinsic curvature measures the radial evolution of the induced metric and hence encapsulates all dynamical information of the geometry of the hypersurfaces. In fact, the Riemann tensor of the (d+1)-dimensional manifold $\mathcal{M}$ can be expressed entirely in terms of the intrinsic (i.e. Riemannian) and extrinsic curvatures of the hypersurfaces Σ_r via the so-called *Gauss-Codazzi* equations

$$\begin{aligned}
&\hat{\gamma}^{\alpha}_{\mu}\hat{\gamma}^{\beta}_{\nu}\hat{\gamma}^{\gamma}_{\rho}\hat{\gamma}^{\delta}_{\sigma}R_{\alpha\beta\gamma\delta} = \hat{R}_{\mu\nu\rho\sigma} + \hat{K}_{\mu\sigma}\hat{K}_{\nu\rho} - \hat{K}_{\mu\rho}\hat{K}_{\nu\sigma},\\
&\hat{\gamma}^{\rho}_{\nu}n^{\sigma}R_{\rho\sigma} = \hat{D}_{\mu}\hat{K}^{\mu}_{\nu} - \hat{D}_{\nu}\hat{K}^{\mu}_{\mu},\\
&n^{\rho}n^{\sigma}R_{\mu\rho\nu\sigma} = -n^{\rho}\nabla_{\rho}\hat{K}_{\mu\nu} - \hat{K}_{\mu\rho}\hat{K}^{\rho}{}_{\nu}.
\end{aligned} \tag{3.26}$$

Here $\hat{D}_{\mu}$ is the covariant derivative with respect to the induced metric $\hat{\gamma}_{\mu\nu}$ on Σ_r and is defined by [49]

$$\hat{D}_{\mu}T^{\rho_1\ldots\rho_k}{}_{\sigma_1\ldots\sigma_l} \equiv \hat{\gamma}^{\rho_1}_{\nu_1}\ldots\hat{\gamma}^{\rho_k}_{\nu_k}\hat{\gamma}^{\tau_1}_{\sigma_1}\ldots\hat{\gamma}^{\tau_l}_{\sigma_l}\hat{\gamma}^{\lambda}_{\mu}\nabla_{\lambda}T^{\nu_1\ldots\nu_k}{}_{\tau_1\ldots\tau_l}, \tag{3.27}$$

for any tensor $T^{\rho_1\cdots\rho_k}{}_{\sigma_1\ldots\sigma_l}$. In particular $\hat{D}_{\mu}\hat{\gamma}_{\rho\sigma} = 0$.

A little manipulation of the Gauss-Codazzi equations brings them in the following form, which will be particularly useful for our purposes:

$$\begin{aligned}
&\hat{K}^2 - \hat{K}_{\mu\nu}\hat{K}^{\mu\nu} = \hat{R} + 2G_{\mu\nu}n^{\mu}n^{\nu},\\
&\hat{D}_{\mu}\hat{K}^{\mu}_{\nu} - \hat{D}_{\nu}\hat{K}^{\mu}_{\mu} = G_{\rho\sigma}\hat{\gamma}^{\rho}_{\nu}n^{\sigma},\\
&\pounds_{n}\hat{K}_{\mu\nu} + \hat{K}\hat{K}_{\mu\nu} - 2\hat{K}_{\mu}{}^{\rho}\hat{K}_{\rho\nu} = \hat{R}_{\mu\nu} - \hat{\gamma}^{\rho}_{\mu}\hat{\gamma}^{\sigma}_{\nu}R_{\rho\sigma},
\end{aligned} \tag{3.28}$$

where $G_{\mu\nu}$ is the Einstein tensor of the bulk metric $g_{\mu\nu}$. We emphasize that these equations are purely geometrical. They simply relate the geometry of the manifold $\mathcal{M}$ to the geometry of the hypersurfaces Σ_r. If one inserts Einstein's equations (3.21) in these equations however, they become, after gauge-fixing the shift and lapse functions, completely equivalent to Einstein's equations. The difference of course is that they are dynamical equations for the induced metric $\hat{\gamma}_{\mu\nu}$ on Σ_r instead of the metric $g_{\mu\nu}$, which is an advantage for the asymptotic analysis of AlAdS manifolds.

In order to provide a Hamiltonian description of the dynamics, we use the first equation in (3.28) in order to express the supergravity action purely in terms of quantities on Σ_r[8]

$$S = -\frac{1}{2\kappa^2}\int_{\mathcal{M}} d^{d+1}x\sqrt{\hat{\gamma}}N(\hat{R} + \hat{K}^2 - \hat{K}_{\mu\nu}\hat{K}^{\mu\nu} - 2\kappa^2\mathcal{L}_{\rm m}). \tag{3.29}$$

[8]Inserting the first equation in (3.28) in the Einstein-Hilbert action (3.18) one gets a boundary term which cancels the Gibbons-Hawking term.

Moreover, a small calculation shows that the extrinsic curvature (3.25) can be expressed as

$$\hat{K}_{\mu\nu} = \frac{1}{2N}\left(\dot{\hat{\gamma}}_{\mu\nu} - \hat{D}_\mu N_\nu - \hat{D}_\nu N_\mu\right), \tag{3.30}$$

and hence, the action is expressed in terms of the fields $\hat{\gamma}_{\mu\nu}$, N^μ and N, as well as the matter fields collectively denoted by f, and their derivatives. The canonical momenta conjugate to these fields are then given by

$$\pi^{\mu\nu} \equiv \frac{\delta L}{\delta\dot{\hat{\gamma}}_{\mu\nu}} = -\frac{1}{2\kappa^2}\sqrt{\hat{\gamma}}\left(\hat{K}\hat{\gamma}^{\mu\nu} - \hat{K}^{\mu\nu}\right), \qquad \pi_f \equiv \frac{\delta L}{\delta\dot{f}}, \tag{3.31}$$

where the Lagrangian L is defined as usual by $S = \int drL$ and the momenta conjugate to the lapse and shift functions vanish identically. This means that the corresponding equations of motion in the canonical formalism become constraints. These constraints are precisely the equations obtained by inserting Einstein's equations (3.21) in the first two equations in (3.28) and are known respectively as the Hamilton and momentum constraints. The Hamilton equations for the induced metric $\hat{\gamma}_{\mu\nu}$ are (3.30) and the equation obtained by varying (3.29) with respect to the induced metric. We will not need this last equation, since it is equivalent to the third equation in (3.28) after inserting Einstein's equations.

Consider now the regulated manifold $\mathcal{M}_{r_o}$ defined as the submanifold of $\mathcal{M}$ bounded by the hypersurface Σ_{r_o}. The values of the induced fields $\hat{\gamma}_{\mu\nu}$ and f on Σ_{r_o} then become boundary conditions for the action (3.29). As is well-known from the Hamilton-Jacobi formalism of classical mechanics, this means that the momenta on the regulating surface can be obtained as the variations of the on-shell action with respect to the boundary values of the induced fields, namely

$$\pi^{\mu\nu}(r_o, x) = \frac{\delta S_{\text{on-shell}}}{\delta\hat{\gamma}_{\mu\nu}(r_o, x)}, \qquad \pi_f(r_o, x) = \frac{\delta S_{\text{on-shell}}}{\delta f(r_o, x)}. \tag{3.32}$$

It is not difficult to check by direct calculation that these identities hold (see (5.25)). In fact, since r_o is arbitrary, the same relations hold for any r, as long as this coordinate is well-defined.

Finally, the regulated on-shell action can be evaluated by inserting Einstein's equations (3.21) in the first of the Gauss-Codazzi equations in (3.28). One finds

$$S_{\text{on-shell}} = -\frac{1}{\kappa^2}\int_{\mathcal{M}_{r_o}} d^{d+1}x\sqrt{\hat{\gamma}}N\left[\hat{R} + \kappa^2\left(n^\mu n^\nu\tilde{T}_{\mu\nu} - \mathcal{L}_{\text{m}}\right)\right]. \tag{3.33}$$

The derivative of the on-shell action with respect to r_o, namely

$$\dot{S}_{\text{on-shell}} = -\frac{1}{\kappa^2}\int_{\Sigma_{r_o}} d^d x\sqrt{\hat{\gamma}}N\left[\hat{R} + \kappa^2\left(n^\mu n^\nu\tilde{T}_{\mu\nu} - \mathcal{L}_{\text{m}}\right)\right], \tag{3.34}$$

will play an important role in the formulation of the new Hamiltonian method.

GAUGE-FIXING

Before we turn to the exposition of the method, let us fix the gauge freedom associated with the shift and lapse functions by setting $N^\mu = 0$ and $N = 1$. In this gauge the bulk metric (3.24) takes the form[9]

$$ds^2 = dr^2 + \gamma_{ij}(r,x)dx^i dx^j, \tag{3.35}$$

where $i,\, j = 1,\ldots,d$ are indices along the hypersurface Σ_r and we take $x^{d+1} = r$. The extrinsic curvature (3.30) now becomes

$$K_{ij} = \frac{1}{2}\dot{\gamma}_{ij}, \tag{3.36}$$

where the dot denotes differentiation with respect to r. The non-vanishing components of the Christoffel symbol of the metric $g_{\mu\nu}$ are

$$\Gamma^{d+1}_{ij} = -K_{ij}, \qquad \Gamma^{i}_{d+1j} = K^i_j, \qquad \Gamma^i_{jk}[g] = \Gamma^i_{jk}[\gamma]. \tag{3.37}$$

As we discussed above, the dynamical equations for the induced metric are obtained by inserting Einstein's equations (3.21) into the Gauss-Codazzi identities (3.28). The resulting equations in this gauge take the form

$$\begin{cases} K^2 - K_{ij}K^{ij} = R + 2\kappa^2 \tilde{T}_{d+1d+1}, \\ D_i K^i_j - D_j K = \kappa^2 \tilde{T}_{jd+1}, \\ \dot{K}^i_j + K K^i_j = R^i_j - \kappa^2\left(\tilde{T}^i_j - \frac{1}{d-1}\tilde{T}^\sigma_\sigma \delta^i_j\right), \end{cases} \tag{3.38}$$

where $\dot{K}^i_j$ stands for $\partial_r(\gamma^{ik}K_{kj})$.

The radial derivative of the on-shell action (3.34) now becomes

$$\dot{S}_{\mathrm{on-shell}} = -\frac{1}{\kappa^2}\int_{\Sigma_r} d^d x \sqrt{\gamma}\left[R + \kappa^2(\tilde{T}_{d+1d+1} - \mathcal{L}_{\mathrm{m}})\right]. \tag{3.39}$$

This allows us to write the regulated on-shell action, which will be denoted by I_{r_o} from now on, in a very useful form. Namely, we can introduce a covariant variable λ and write the action as[10]

$$\boxed{I_r = -\tfrac{1}{\kappa^2}\int_{\Sigma_r} d^d x\sqrt{\gamma}(K-\lambda),} \tag{3.40}$$

[9]All tensors are transverse and so we drop the hats form now on.

[10]We have explicitly included the Gibbons-Hawking boundary term so that λ corresponds to the on-shell value of the bulk integral in the supergravity action. Moreover, since the regulator r_o is arbitrary, we can evaluate the action on an arbitrary hepresurface Σ_r. Its functional derivatives will then give the momenta on Σ_r and not just on the regulating surface Σ_{r_o}.

provided λ satisfies the equation

$$\boxed{\dot{\lambda} + K\lambda - \kappa^2 \left(\mathcal{L}_{\rm m} + \tfrac{1}{d-1} \tilde{T}^\sigma_\sigma \right) = 0,} \tag{3.41}$$

which can be derived by taking the trace of the third equation in (3.38). Note that since Σ_r is compact, (3.40) defines λ up to a total divergence. This ambiguity can be utilized by making a judicious choice that will simplify the analysis below. Namely, since the canonical momenta are given by

$$\pi^{ij} = -\frac{1}{2\kappa^2}\sqrt{\gamma}\left(K\gamma^{ij} - K^{ij}\right) = \frac{\delta I_r}{\delta \gamma_{ij}}, \qquad \pi_f = \frac{\delta I_r}{\delta f}, \tag{3.42}$$

it follows that

$$\boxed{\pi^{ij}\delta\gamma_{ij} + \textstyle\sum_f \pi_f \delta f = -\frac{1}{\kappa^2}\delta\left[\sqrt{\gamma}(K-\lambda)\right],} \tag{3.43}$$

up to a total divergence. Therefore, the total divergence ambiguity in λ can be used to ensure that (3.43) holds *without* the integral over Σ_r. This can always be achieved by the following procedure.[11] Take first any λ satisfying the definition (3.41). The variation $\delta\left[\sqrt{\gamma}(K-\lambda)\right]$ will then generically produce terms with derivatives acting on the variations of the induced fields $\delta\gamma_{ij}$, δf. These derivatives can be moved to the coefficients of the field variations by integration by parts. When all derivatives acting on the field variations are removed, (3.42) guarantees that the coefficients of the field variations are precisely the radial momenta. Now, the total derivative terms which are produced by this procedure can be absorbed into λ. In writing (3.43), we assume that such a procedure has been performed.

3.3.2 The method

We have now the necessary material in order to formulate a Hamiltonian method of holographic renormalization. For concreteness, we will consider a massless[12] abelian gauge field A_μ and a number of scalar fields φ^I with the action

$$S_{\rm m}[g, A, \varphi] = \int_{\mathcal{M}} d^{d+1}x\sqrt{g}\left(\frac{1}{4}U(\varphi)F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}G_{IJ}(\varphi)\partial_\mu\varphi^I\partial^\mu\varphi^J + V(\varphi)\right). \tag{3.44}$$

[11] This argument assumes that λ is local. As we will see below, the divergent part of λ which is determined by the asymptotic analysis is local. As it is expected though, the finite part of λ will be non-local in the sources and hence, only the integrated version of (3.43) holds for this part. We will return to this point below.

[12] The method is easily generalized to include massive gauge fields using the Stückelberg formalism. See e.g. [32].

The function $U(\varphi)$, the metric $G_{IJ}(\varphi)$ on the scalar manifold and the potential $V(\varphi)$ are completely arbitrary, subject to the requirement that the action admits AdS space as a critical point. The equations of motion following from this action are given in gauge-fixed form in Appendix 3.A.2 (and in covariant form in 5.4) , but we will not need them explicitly here.

Gauge-fixing the metric as in (3.35) and choosing the gauge $A_r = 0$ for the vector field, the canonical momenta are given by

$$\pi^{ij} = -\frac{1}{2\kappa^2}\sqrt{\gamma}(K\gamma^{ij} - K^{ij}), \quad \pi^i = \sqrt{\gamma}U(\varphi)\dot{A}^i, \quad \pi_I = \sqrt{\gamma}G_{IJ}(\varphi)\dot{\varphi}^I, \qquad (3.45)$$

where $\dot{A}^i \equiv \gamma^{ij}\dot{A}_j$. Since the regulated on-shell action I_r is a functional of the induced fields on the hypersurface Σ_r, so are the momenta which are related to the on-shell action by (3.42). We have seen this before. In Section 2.3.2 of the previous chapter, we found that requiring the scalar field φ to be regular in the interior of AdS forced the momentum $\dot{\varphi}$ to be proportional to the scalar field itself (see (2.145). In the present case, regularity of the solution in the interior is assumed implicitly since it is necessary for the evaluation of the regulated on-shell action.[13] Therefore, for regular solutions, the on-shell momenta are functionals of the induced fields:

$$\boxed{\pi^{ij}[\gamma, A, \varphi], \qquad \pi^i[\gamma, A, \varphi], \qquad \pi_I[\gamma, A, \varphi].} \qquad (3.46)$$

In the Hamiltonian version of holographic renormalization then, one uses the equations of motion in order to determine the asymptotic form of the momenta as functionals of the induced fields. Since both the momenta and the induced fields transform covariantly under diffeomorphisms on the slice Σ_r, the method is manifestly covariant at all stages, which ensures that the Ward identities are manifest as well. This is in contrast with the original method of holographic renormalization where one solves asymptotically for the bulk fields as functions of the sources, a procedure that is not manifestly covariant with respect to diffeomorphisms on the hypersurface Σ_r.

Asymptotic analysis

In the original method of holographic renormalization, the asymptotic analysis starts by expanding the bulk fields in the radial distance $z = e^{-r}$ from the conformal boundary. However, we have now formulated the problem in a manifestly covariant language and so we would like to have a covariant way of carrying out the asymptotic analysis.

[13]To be more precise, one must impose *some* condition in the interior such that the on-shell action is well-defined. It is not known in general if this requires the solution itself to be regular in the interior, although this is assumed in most examples in the literature.

To this end, we observe that the non-normalizable modes of the induced fields behave asymptotically as

$$\gamma_{ij} \sim e^{2r} g_{(0)ij}(x), \qquad A_i \sim A_{(0)i}(x), \qquad \varphi^I \sim e^{-(d-\Delta_I)r} {\phi_{(0)}}^I(x). \tag{3.47}$$

These asymptotic relations can be written in covariant form as

$$\dot{\gamma}_{ij} \sim 2\gamma_{ij}, \qquad \dot{A}_i = \mathcal{O}(e^{-r}), \qquad \dot{\varphi}^I \sim -(d-\Delta_I)\varphi^I. \tag{3.48}$$

Moreover, the fact that the momenta are functionals of the induced fields means that the radial derivative can be represented on the solution space by the functional differential operator

$$\boxed{\partial_r = \int d^d x \left(2K_{ij}[\gamma, A, \varphi] \frac{\delta}{\delta \gamma_{ij}} + \dot{A}_i[\gamma, A, \varphi] \frac{\delta}{\delta A_i} + \dot{\varphi}^I[\gamma, A, \varphi] \frac{\delta}{\delta \varphi^I} \right).} \tag{3.49}$$

It follows that the radial derivative is identified asymptotically with the dilatation operator, δ_D, namely

$$\partial_r = \int d^d x \left(2\gamma_{ij} \frac{\delta}{\delta \gamma_{ij}} + \sum_I (\Delta_I - d)\varphi^I \frac{\delta}{\delta \varphi^I} \right) + \mathcal{O}(e^{-r}) \equiv \delta_D + \mathcal{O}(e^{-r}). \tag{3.50}$$

Since we have identified the radial coordinate with the energy scale of the dual field theory, this relation provides a holographic derivation of the Callan-Symanzik equation (2.169).

This observation is exactly what we need in order to formulate the asymptotics in a covariant manner. Namely, since the momenta and the on-shell action are functionals of the induced fields, one expects that they admit asymptotic expansions in *eigenfunctions of the dilatation* operator δ_D, namely[14]

$$\begin{aligned}
\pi^i_j &= \sqrt{\gamma} \left({\pi_{(0)}}^i_j + {\pi_{(2)}}^i_j + \cdots + {\pi_{(d)}}^i_j + {\tilde{\pi}_{(d)}}^i_j \log e^{-2r} + \cdots \right), \\
\pi^i &= \sqrt{\gamma} \left({\pi_{(3)}}^i + {\pi_{(4)}}^i + \cdots + {\pi_{(d)}}^i + {\tilde{\pi}_{(d)}}^i \log e^{-2r} + \cdots \right), \\
\pi_I &= \sqrt{\gamma} (\sum_{d-\Delta_I \leq s < \Delta_I} \pi_{(s)I} + \pi_{(\Delta_I)} + \tilde{\pi}_{(\Delta_I)I} \log e^{-2r} + \cdots), \\
\lambda &= \lambda_{(0)} + \lambda_{(2)} + \cdots + \lambda_{(d)} + \tilde{\lambda}_{(d)} \log \mathrm{e}^{-2r} + \cdots,
\end{aligned} \tag{3.51}$$

[14]Provided the scaling dimensions of the scalar fields are rational numbers, such expansions always exist. Note that the dilatation weight of the leading and logarithmic terms in these expansions is *universal* - i.e. independent of the value of the scalar dimensions. The 'step' however of the expansions is *not* universal and it depends crucially on the dimensions Δ_I. We consider here the most common case, but in general additional terms with odd or even fractional dilatation weight can appear in all these expansions. We emphasize that the method applies with no difficulty to the general case where such terms are present. An example where this is the case will be discussed in the next section.

where all terms, except for ${\pi_{(d)}}^i_j$, ${\pi_{(d)}}^i$, $\pi_{I(\Delta_I)}$ and $\lambda_{(d)}$, are assumed to transform homogeneously under the dilatation operator (i.e. under constant Weyl transformations) according to their scaling dimension:

$$\begin{aligned} &\delta_D {\pi_{(n)}}^i_j = -n {\pi_{(n)}}^i_j, && n < d, && \delta_D {\tilde\pi_{(d)}}^i_j = -d {\tilde\pi_{(d)}}^i_j, \\ &\delta_D {\pi_{(n)}}^i = -n {\pi_{(n)}}^i, && 3 \le n < d, && \delta_D {\tilde\pi_{(d)}}^i = -d {\tilde\pi_{(d)}}^i, \\ &\delta_D \pi_{I(s)} = -s \pi_{I(s)}, && d - \Delta_I \le s < \Delta_I, && \delta_D \tilde\pi_{I(\Delta_I)} = -\Delta_I \tilde\pi_{I(\Delta_I)}, \\ &\delta_D \lambda_{(n)} = -n \lambda_{(n)}, && 0 \le n < d, && \delta_D \tilde\lambda_{(d)} = -d \tilde\lambda_{(d)}. \end{aligned} \tag{3.52}$$

As is expected, these terms are related to the *local* coefficients in the asymptotic expansions of the original method of holographic renormalization. We will work out the precise relation for a particular example in the next section.

Requiring the tilded terms that multiply the logarithms to transform homogeneously under the dilatation operator and identifying the dilatation operator with the radial derivative as above leads to the transformation law of ${\pi_{(d)}}^i_j$, ${\pi_{(d)}}^i$, $\pi_{I(\Delta_I)}$ and $\lambda_{(d)}$. Namely, we find

$$\begin{aligned} &\delta_D {\pi_{(d)}}^i_j = -d {\pi_{(d)}}^i_j - 2 {\tilde\pi_{(d)}}^i_j, \\ &\delta_D {\pi_{(d)}}^i = -d {\pi_{(d)}}^i - 2 {\tilde\pi_{(d)}}^i, \\ &\delta_D \pi_{I(\Delta_I)} = -\Delta_I \pi_{I(\Delta_I)} - 2 \tilde\pi_{I(\Delta_I)}, \\ &\delta_D \lambda_{(d)} = -d \lambda_{(d)} - 2 \tilde\lambda_{(d)}. \end{aligned} \tag{3.53}$$

These transformations contain the expected homogeneous term, but they also contain an inhomogeneous term, which indicates that these terms depend *non-locally* on the induced fields. Indeed, we will soon identify these terms with the renormalized one-point functions and on-shell action, which are non-local functionals of the sources. As we will see below, the inhomogeneous terms, which are precisely the coefficients of the logarithms in the above expansions, correspond to the conformal anomalies.

Counterterms

There are now two possible ways to evaluate the local terms in the expansions (3.51), both of which are useful, although, depending on the case at hand, one might be more efficient than the other. We therefore discuss both here.

I. The first algorithm is very similar in spirit to the algorithm that determines the coefficients in the asymptotic expansions of the original method of holographic renormalization. The difference, however, is that the on-shell action is now treated on the same footing as the momenta, which, combined with the fact that all the expansions are covariant, leads to a much faster determination of the counterterm action.

The algorithm relies on the observation that by inserting the expansions (3.51) in the expression (3.49) for the radial derivative, one obtains a covariant expansion for the radial derivative in the form

$$\partial_r = \delta_D + \delta_{(1)} + \cdots + \log e^{-2r} \tilde{\delta}_{(d)} + \cdots , \tag{3.54}$$

where $\delta_{(n)}$ are covariant functional operators of successively higher dilatation weight.

Now, the dynamical equations for the induced fields, namely equations (3.38) together with the equations for the matter fields, are second order in the radial derivative. Writing these equations in terms of the momenta, one of the radial derivatives is absorbed in the momenta (see for example the third equation in (3.38)). Similarly, the equation (3.41) for the on-shell action λ contains one radial derivative. One then inserts the expansions (3.54) for the radial derivative and (3.51) for the momenta and the on-shell action into the equations of motion. Matching terms of equal dilatation weight leads to a set of (coupled) *algebraic* equations for the local coefficients. These equations are then solved iteratively for the coefficients. As in the previous method of holographic renormalization, the non-local terms in the expansions are not determined by the recursion relations, although the trace and divergence of $\pi_{(d)}{}^i_j$ is determined, as was the trace and divergence of $g_{(d)ij}$. However, the recursion relations now determine directly the coefficients $\lambda_{(n)}$ as covariant functionals of the induced fields, which as we will see, is equivalent to determining the counterterm action in the covariant form (see e.g. (3.15)).

Although this algorithm is often the fastest for relatively simple examples, it is not suited for proving general statements that are independent of the particular theory at hand, such as the Ward identities. Such general properties, are very nicely revealed by the second algorithm that we will now describe.

II. The second algorithm relies heavily on the Hamilton-Jacobi relations (3.42) for the momenta as derivatives of the regulated on-shell action. In particular, at the first stage, one uses the identity (3.43) to express all local coefficients in the expansion of λ in terms of the momenta. This is easily done by inserting the expansions (3.51) and collecting terms of equal dilatation weight. Inserting then these expressions for $\lambda_{(n)}$ into (3.40) and using the Hamilton constraint (first equation in (3.38)), all local terms in the expansions of the momenta are determined recursively from the Hamilton-Jacobi relations (3.42).

Either algorithm determines directly the covariant counterterm action, which is given by

$$I_{\rm ct} = \frac{1}{\kappa^2} \int_{\Sigma_{r_o}} d^d x \sqrt{\gamma} \left(\sum_{n=0}^{d-1} (K_{(n)} - \lambda_{(n)}) + (\tilde{K}_{(d)} - \tilde{\lambda}_{(d)}) \log e^{-2r_o} \right) . \tag{3.55}$$

Through the Hamilton-Jacobi relations (3.42), this leads to the covariant counterterms of the momenta, which are precisely the local terms in the expansions (3.51) and have already been determined as part of the algorithm leading to the counterterm action. There is therefore no need to differentiate this local action to obtain the momentum counterterms.

Renormalized action and one-point functions

The renormalized action is now defined as

$$I_{\rm ren}[g_{(0)}, A_{(0)}, \phi_{(0)}] = \lim_{r_o\to\infty}(I_{r_o} + I_{\rm ct}) = -\frac{1}{\kappa^2}\int_{\partial\mathcal{M}} d^dx\sqrt{\gamma}(K_{(d)} - \lambda_{(d)}). \quad (3.56)$$

The AdS/CFT prescription identifies this with the generating functional of renormalized connected correlation functions in the dual quantum field theory. In particular, the first derivatives of the renormalized action with respect to the sources correspond to the one-point functions of the dual operators. But the Hamilton-Jacobi relations (3.42) identify the first derivatives of the renormalized action with the non-local terms in the expansions of the momenta (3.51). Hence, we obtain the very general result[15]

$$\boxed{\langle T_{ij}\rangle_{\rm ren} = -\frac{1}{\kappa^2}\left(K_{(d)ij} - K_{(d)}\gamma_{ij}\right), \quad \langle J^i\rangle_{\rm ren} = {\pi_{(d)}}^i, \quad \langle \mathcal{O}_I\rangle_{\rm ren} = \pi_{I(\Delta_I)}.} \quad (3.57)$$

These expressions should be compared with the corresponding expressions (3.17) that were obtained from the original method of holographic renormalization. In (3.17) the one-point functions are expressed in terms of the non-local coefficients $g_{(d)}$ and $\phi_{(2\Delta-d)}$ of the asymptotic expansions, but also in terms of some local functionals, X_{ij} and Y, of the sources, which do not admit an obvious geometric interpretation and whose particular form depends on the case under consideration. In (3.57), however, we have determined the exact one-point functions completely in terms of geometric quantities. Moreover, these expressions are valid for any dimension and for any field content. One of the advantages of this formulation is that we can now prove in full generality that the holographic one-point functions satisfy the Ward identities (2.163), (2.165) and (2.168).

Ward identities

The Ward identities (2.163) and (2.165) follow immediately from the equations of motion for the induced fields. More precisely, they are a consequence of the

[15] Strictly speaking, the one-point functions are obtained after evaluating the limit $r_o \to \infty$. However, provided there is no risk of confusion, we will often use the quantities before and after the limit is taken interchangeably.

contsraint equations that follow by gauge-fixing part of the $U(1)$ gauge freedom and part of the bulk diffeomorphisms.

Let us consider first the first of the two equations (3.121) for the gauge field A_i. It can be written compactly as

$$D_i(\pi^i/\sqrt{\gamma}) = 0, \tag{3.58}$$

where $\pi^i = \sqrt{\gamma}U(\varphi)F^{ri}$ is the canonical momentum conjugate to A_i. Expanding this momentum in eigenfunctions of the dilatation operator as above, we conclude that this identity must hold for each term separately, since they all have different dilatation weight. In particular,

$$D_i\pi_{(d)}{}^i = 0, \tag{3.59}$$

which leads, via the identification (3.57) to the Ward identity (2.163).

The Ward identity (2.165) follows in a similar way from the second equation in (3.38), which takes the form

$$-\frac{1}{\kappa^2}D_i\left(K^i_j - K\delta^i_j\right) = -G_{IJ}(\varphi)\dot{\varphi}^I\partial_j\varphi^J + U(\varphi)F^{ri}F_{ij}. \tag{3.60}$$

Writing this in terms of the momenta (3.45) and inserting the expansions (3.51), leads to

$$-\frac{1}{\kappa^2}D_i\left(K_{(d)}{}^i_j - K_{(d)}\delta^i_j\right) = -\pi_{I(\Delta_I)}\partial_j\varphi^I + \pi_{(d)}{}^iF_{ij}, \tag{3.61}$$

which, via (3.57), is equivalent to the Ward identity (2.165).

Finally, let us derive the trace Ward identity and the conformal anomaly. To this end, consider an infinitesimal Weyl transformation of the renormalized action (3.56):

$$\delta_\sigma I_{\rm ren} = \frac{2}{\kappa^2}\int_{\partial\mathcal{M}}\sqrt{\gamma}(\tilde{K}_{(d)} - \tilde{\lambda}_{(d)})\delta\sigma. \tag{3.62}$$

But from the renormalized version of (3.42) we also have:

$$\delta_\sigma I_{\rm ren} = \int_{\partial\mathcal{M}} d^dx\sqrt{\gamma}\left[2\pi_{(d)}{}^i_i + (\Delta_I - d)\pi_{(\Delta_I)I}\varphi^I\right]\delta\sigma. \tag{3.63}$$

Since $\delta\sigma$ is arbitrary, we can equate the integrands to obtain

$$2\pi_{(d)}{}^i_i + (\Delta_I - d)\pi_{(\Delta_I)I}\varphi^I = \frac{2}{\kappa^2}(\tilde{K}_{(d)} - \tilde{\lambda}_{(d)}). \tag{3.64}$$

Note that this result, at least formally, follows also from the identity (3.43) once it is restricted to constant Weyl transformations, despite the fact that in general only the integrated form of (3.43) holds for the non-local term $\lambda_{(d)}$. The identifications (3.57) then lead to the anomalous trace Ward identity (2.168), with the conformal anomaly given by

$$\boxed{\mathcal{A} = -\tfrac{2}{\kappa^2}(\tilde{K}_{(d)} - \tilde{\lambda}_{(d)}).} \tag{3.65}$$

Therefore, as expected, the conformal anomaly is given by the coefficient of the logarithm in the covariant expansion (3.51) for λ. In particular, it is a local functional of the sources, as required by the Wess-Zumino consistency condition.

3.4 EXAMPLES

In order to illustrate the formal discussion of our Hamiltonian method of holographic renormalization, we now consider two examples. We discuss first the special but important case of pure AdS gravity in arbitrary dimension. We show in this case explicitly that the two methods of holographic renormalization are equivalent by deriving a one-to-one map between the coefficients of the asymptotic expansion of the metric and the coefficients in the covariant expansion of the extrinsic curvature. Moreover, the new method allows us to derive general recursion relations for the extrinsic curvature coefficients, valid in any dimension. We conclude the section on pure gravity with a few results for the spacial case of AdS_3. As a second example, we consider two scalar fields coupled to gravity in five dimensions with an arbitrary potential.

3.4.1 ADS GRAVITY

For pure gravity with a negative cosmological constant $\Lambda = d(1-d)/2$, the equations of motion (3.38) reduce to

$$\begin{aligned} &K^2 - K_{ij}K^{ij} = R + d(d-1), \\ &D_i K^i_j - D_j K = 0, \\ &\dot{K}^i_j + K K^i_j = R^i_j + d\delta^i_j, \end{aligned} \tag{3.66}$$

and the on-shell action is determined from the equation

$$\dot{\lambda} + K\lambda = d. \tag{3.67}$$

The first step is to expand the extrinsic curvature, which is related to the momentum conjugate to the induced metric via (3.45), as well as the on-shell action λ in eigenfunctions of the dilatation operator, which now takes the form

$$\delta_D = \int d^d x 2\gamma_{ij} \frac{\delta}{\delta\gamma_{ij}}. \tag{3.68}$$

We then have the expansions (see (3.51)),

$$K^i_j[\gamma] = K_{(0)}{}^i_j + K_{(2)}{}^i_j + \cdots + K_{(d)}{}^i_j + \tilde{K}_{(d)}{}^i_j \log e^{-2r} + \cdots ,$$

$$\lambda[\gamma] = \lambda_{(0)} + \lambda_{(2)} + \cdots + \lambda_{(d)} + \tilde{\lambda}_{(d)} \log e^{-2r} + \cdots . \tag{3.69}$$

Equivalence of asymptotic expansions

Before we proceed with the algorithm to determine the local terms in these covariant expansions, let us demonstrate the equivalence of this covariant expansion in eigenfunctions of the dilatation operator to the asymptotic expansion of the induced metric in the standard holographic renormalization method.

Recall that in the original method of holographic renormalization, one expands the induced metric in an asymptotic expansion in the radial coordinate $z = \exp(-r)$ as

$$\gamma_{ij} = \frac{1}{z^2}\left(g_{(0)ij} + z^2 g_{(2)ij} + \cdots + z^d g_{(d)ij} + z^d \log z^2 h_{(d)ij} + \cdots\right). \tag{3.70}$$

Differentiating this expansion with respect to r, gives

$$\begin{aligned} K_{ij} = \ \tfrac{1}{2}\dot{\gamma}_{ij} = \ & \frac{1}{z^2} g_{(0)ij} - z^2 g_{(4)ij} + \cdots + z^{d-2}\left((1-d/2)g_{(d)ij} - h_{(d)ij}\right) \\ & + z^{d-2}\log z^2 (1-d/2) h_{(d)ij} + \cdots . \end{aligned} \tag{3.71}$$

However, each term in the covariant expansion of the extrinsic curvature is a functional of the induced metric γ_{ij}. Using the expansion (3.70) of the metric we can then functionally expand the eigenfunctions of the dilatation operator as

$$\begin{aligned} K_{(0)ij}[\gamma] &= \gamma_{ij} = \frac{1}{z^2}\left(g_{(0)ij} + z^2 g_{(2)ij} + \cdots + z^d g_{(d)ij} + z^d \log z^2 h_{(d)ij} + \cdots\right), \\ K_{(2)ij}[\gamma] &= K_{(2)ij}[g_{(0)}] + z^2 \int d^d x g_{(2)kl} \frac{\delta K_{(2)ij}}{\delta g_{(0)kl}} + \cdots, \\ &\vdots \\ K_{(d)ij}[\gamma] &= z^{d-2} K_{(d)ij}[g_{(0)}] + \cdots, \\ \tilde{K}_{(d)ij}[\gamma] &= z^{d-2} \tilde{K}_{(d)ij}[g_{(0)}] + \cdots . \end{aligned} \tag{3.72}$$

Inserting these expressions in the covariant expansion for K_{ij} and comparing with (3.71) we determine

$$\begin{aligned} K_{(0)ij}[g_{(0)}] &= g_{(0)ij}, \\ K_{(2)ij}[g_{(0)}] &= -g_{(2)ij}[g_{(0)}], \\ &\vdots \\ K_{(n)ij}[g_{(0)}] &= -\frac{n}{2} g_{(n)ij}[g_{(0)}] + \text{lower}, \\ &\vdots \\ K_{(d)ij}[g_{(0)}] &= -\frac{d}{2} g_{(d)ij}[g_{(0)}] - h_{(d)ij}[g_{(0)}] + \text{lower}, \\ \tilde{K}_{(d)ij}[g_{(0)}] &= -\frac{d}{2} h_{(d)ij}[g_{(0)}], \end{aligned} \tag{3.73}$$

where 'lower' stands for terms involving functional derivatives with respect to $g_{(0)ij}$ of lower order coefficients $g_{(k)ij}[g_{(0)}]$. For d=4, for example,

$$K_{(4)ij}[g_{(0)}] = -2g_{(4)ij}[g_{(0)}] - h_{(4)ij}[g_{(0)}] + \int d^4x g_{(2)kl} \frac{\delta g_{(2)ij}[g_{(0)}]}{\delta g_{(0)kl}}. \qquad (3.74)$$

We therefore conclude that there is a one-to-one correspondence between the terms in the asymptotic expansion of the metric and the covariant expansion of the extrinsic curvature in eigenfunctions of the dilatation operator. In particular, the non-local terms, $K_{(d)ij}$ and $g_{(d)ij}$, are proportional to each other up to *local* terms, whereas the coefficients of the logarithms, which are related to the conformal anomaly in both formalisms, are just proportional to each other. The two methods are therefore equivalent. Indeed, as we will see below, they lead precisely to the same covariant counterterms. The new formulation, however, is more efficient and leads to a general expression for the counterterm action in terms of the extrinsic curvature coefficients, for arbitrary dimension. Furthermore, as we have seen, the one-point function of the stress tensor in the presence of sources is also expressed simply in terms of the non-local term in the expansion of the extrinsic curvature. In particular, the asymptotic analysis for pure gravity is done once, for all d, resulting in generic recursion relations for the extrinsic curvature coefficients.

In order to derive these dimension-independent results for pure gravity, we will follow the second algorithm (II) above which makes use of the Hamilton-Jacobi relations (3.42). In the present context, these lead to the following functional relation between the extrinsic curvature and the on-shell action:

$$K\gamma^{ij} - K^{ij} = \frac{2}{\sqrt{\gamma}} \frac{\delta}{\delta \gamma_{ij}} \int_{\Sigma_r} d^dx \sqrt{\gamma}(K - \lambda). \qquad (3.75)$$

Inserting the covariant expansions for K^i_j and λ we can relate the coefficients of the on-shell action to those of the extrinsic curvature as

$$K_{(2n)}{}^i_j = \lambda_{(2n)}\delta^i_j - \frac{2}{\sqrt{\gamma}} \int d^dx \sqrt{\gamma}\gamma_{kj} \frac{\delta}{\delta \gamma_{ik}} \left(K_{(2n)} - \lambda_{(2n)}\right), \qquad 0 \le n \le \frac{d}{2},$$
$$\tilde{K}_{(d)}{}^i_j = \tilde{\lambda}_{(d)}\delta^i_j - \frac{2}{\sqrt{\gamma}} \int d^dx \sqrt{\gamma}\gamma_{kj} \frac{\delta}{\delta \gamma_{ik}} \left(\tilde{K}_{(d)} - \tilde{\lambda}_{(d)}\right). \qquad (3.76)$$

Moreover, applying the identity (3.43) to dilatations (in which case it also applies to the non-local term $\lambda_{(d)}$ as we have discussed), one obtains

$$(1 + \delta_D) K_{(2n)} = (d + \delta_D) \lambda_{(2n)}, \quad 0 \le n \le \frac{d}{2}, \quad (1 + \delta_D) \tilde{K}_{(d)} = (d + \delta_D) \tilde{\lambda}_{(d)}. \qquad (3.77)$$

Since we know how the coefficients transform under the dilatation operator, these relations completely determine λ in terms of the trace of the extrinsic curvature.

Namely we obtain the significant result

$$\lambda_{(2n)} = \frac{(2n-1)}{(2n-d)} K_{(2n)}, \qquad 0 \leq n \leq \frac{d}{2} - 1, \qquad \tilde{\lambda}_{(d)} = \frac{d-1}{2} K_{(d)}, \qquad \tilde{K}_{(d)} = 0. \tag{3.78}$$

The coefficients ${K_{(2n)}}^i_j$ are only determined for $n < d/2$. If one does the computation for general d then the corresponding expression has a first order pole at $d = 2n$. A short computation using (3.76) shows that the residue of the pole is exactly ${\tilde{K}_{(d)}}^i_j$, i.e. the coefficient of the logarithmic term in d dimensions,

$${\tilde{K}_{(d)}}^i_j = \lim_{n \to d/2} \left((n - \frac{d}{2}) {K_{(2n)}}^i_j \right). \tag{3.79}$$

In practice one can also use this result in order to compute ${K_{(d-2)}}^i_j$ in d dimensions from ${\tilde{K}_{(d-2)}}^i_j$ in $d-2$ dimensions.

This result leads to a general closed form expression for the covariant counterterm action that renders the on-shell action finite:

$$I_{\rm ct} = \frac{(1-d)}{\kappa^2} \int_{\Sigma_{r_o}} d^d x \sqrt{\gamma} \left(\sum_{m=0}^{\frac{d}{2}-1} \frac{1}{(2m-d)} K_{(2m)} + \frac{1}{2} K_{(d)} \log e^{-2r_o} \right). \tag{3.80}$$

Therefore, the problem is now reduced to determining the coefficients in the covariant expansion of the extrinsic curvature. To determine these coefficients, we substitute the expansion of the extrinsic curvature into the Hamilton constraint (first equation in (3.66)). This leads to a recursive relation for the traces of the extrinsic curvature coefficients, namely

$$K_{(2)} = \frac{R}{2(d-1)}, \tag{3.81}$$

$$K_{(2n)} = \frac{1}{2(d-1)} \sum_{m=1}^{n-1} \left({K_{(2m)}}^i_j {K_{(2n-2m)}}^j_i - K_{(2m)} K_{(2n-2m)} \right), \quad 2 \leq n \leq \frac{d}{2}.$$

Using these expressions for the trace of the extrinsic curvature coefficients, together with the expressions (3.78) for $\lambda_{(2n)}$ in terms of these traces, in the functional relation (3.76) one can now evaluate all local coefficients ${K_{(2n)}}^i_j$ recursively. Some useful identities required for evaluating the functional derivatives in (3.76) are presented in Appendix 3.A.1). Note that the second equation in (3.66) implies that these terms satisfy the relations

$$D_i {K_{(2n)}}^i_j - D_j K_{(2n)} = 0, \quad 0 \leq n \leq \frac{d}{2}, \qquad D_i {\tilde{K}_{(d)}}^i_j - D_j \tilde{K}_{(d)} = 0. \tag{3.82}$$

In particular, the first of these equations for $n = d/2$ leads to the diffeomorphism Ward identity as we discussed above. Moreover, observe that although ${K_{(d)}}^i_j$ is non-local in general, (3.81) ensures that its trace is local since it is expressed in terms of the lower order coefficients, which are local. This is in agreement with the requirement that the conformal anomaly be local in the sources.

Carrying out the above procedure is straightforward but, due to the functional derivatives involved in evaluating the coefficients of the extrinsic curvature, it becomes of forbidding complexity as one goes up in dimension. The algorithm, however, could be implemented in a computer code which would in principle calculate the counterterms and the holographic Weyl anomaly for any dimension.

For illustrative purposes we give here the results for up to four boundary dimensions. As claimed, they are in perfect agreement with the results obtained by the original method of holographic renormalization (see e.g. [30]).

$$
\begin{aligned}
\mathbf{d=2}\quad & K^i_j[\gamma] = \delta^i_j + K_{(2)}{}^i_j + \cdots\,, \\
& K[\gamma] = d + P + \cdots\,, \\
& I_{\rm ct} = \frac{(d-1)}{\kappa^2}\int d^2x\sqrt{\gamma}\left(1 - \frac{1}{4}R\log\epsilon^2\right), \\
\mathbf{d=4}\quad & K^i_j[\gamma] = \delta^i_j + P^i_j + \frac{1}{2}\left(\frac{1}{2}\left(P^{kl}P_{kl} - P^2\right)\delta^i_j\right. \\
& \qquad \left. - \frac{1}{(d-2)}\left(2R^i{}_{kjl}P^{kl} - PR^i_j + \Box P^i_j - D^iD_jP\right)\right)\log\epsilon^2 + K_{(4)}{}^i_j + \cdots\,, \\
& K[\gamma] = d + P + \frac{1}{2(d-1)}\left(P^{kl}P_{kl} - P^2\right) + \cdots\,, \\
& I_{\rm ct} = \frac{(d-1)}{\kappa^2}\int d^4x\sqrt{\gamma}\left(1 + \frac{1}{(d-2)}P - \frac{1}{4(d-1)}\left(P^{kl}P_{kl} - P^2\right)\log\epsilon^2\right).
\end{aligned}
\tag{3.83}
$$

Here, $\epsilon = e^{-r_o}$ and the tensor P_{ij} is defined in (3.116) in Appendix 3.A.1.

THE SPECIAL CASE OF AdS_3

For any three-dimensional Riemannian manifold the Weyl tensor (3.115) vanishes identically. This fact, combined with the vacuum Einstein equations with a negative cosmological constant, implies that the Riemann tensor takes the form

$$R_{\mu\nu\rho\sigma} = \left(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}\right), \tag{3.84}$$

which is precisely the curvature tensor (3.1) of AdS space.

The Gauss-Codazzi equations (3.26) then imply that the extrinsic curvature satisfies the equation

$$\dot{K}^i_j + K^i_k K^k_j - \delta^i_j = 0, \tag{3.85}$$

which, in contrast to the equations of motion (3.66), involves only the extrinsic curvature and hence, it can be solved exactly.

To this end, let us write this equation in matrix notation as

$$\dot{\mathcal{K}} = 1 - \mathcal{K}^2. \tag{3.86}$$

Assuming that the matrix on the right hand side is invertible, we can write

$$2(1-\mathcal{K}^2)^{-1} = (1-\mathcal{K})^{-1} + (1+\mathcal{K})^{-1}. \tag{3.87}$$

Equation (3.86) can now be immediately integrated to give

$$\mathcal{K} = \frac{1-e^{-2r}\mathcal{C}(x)}{1+e^{-2r}\mathcal{C}(x)}, \tag{3.88}$$

where $\mathcal{C}(x)$ is an arbitrary matrix that depends only on the transverse coordinates. Expanding this in e^{-2r} we get

$$\mathcal{K} = 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-2nr}\mathcal{C}^n. \tag{3.89}$$

However, the Cayley-Hamilton theorem

$$\mathcal{C}^2 - \operatorname{tr}\mathcal{C}\mathcal{C} + \det\mathcal{C}1 = 0, \tag{3.90}$$

implies that the extrinsic curvature can in fact be written in the form

$$\mathcal{K} = a(r)1 + b(r)\mathcal{C}, \tag{3.91}$$

for some coefficients a and b that depend on $\mathcal{C}$. Inserting this in (3.86) and solving the resulting equations for a and b gives the extrinsic curvature in the closed form

$$\boxed{\mathcal{K} = (1+e^{-2r}\operatorname{tr}\mathcal{C} + e^{-4r}\det\mathcal{C})^{-1}\left[(1+e^{-2r}\operatorname{tr}\mathcal{C} - e^{-4r}\det\mathcal{C})1 - 2e^{-2r}\mathcal{C}\right].} \tag{3.92}$$

Recall now that the extrinsic curvature is related to the induced metric by

$$K^i_j = \frac{1}{2}\gamma^{ik}\dot{\gamma}_{kj} = \frac{1}{2}\partial_r(\ln\gamma)^i_j. \tag{3.93}$$

We can therefore integrate (3.89) to obtain

$$\gamma = e^{2r}\mathcal{B}(x)(1+e^{-2r}\mathcal{C})^2, \tag{3.94}$$

where $\mathcal{B}(x)$ is another integration function. This is then the most general AlAdS metric in three dimensions! This is precisely what it was found in [47], where it was shown that the Fefferman-Graham expansion terminates at quadratic order. Writing

$$\gamma_{ij} = e^{2r}\left(g_{(0)ij} + e^{-2r}g_{(2)ij} + e^{-4r}g_{(4)ij}\right), \tag{3.95}$$

we identify $g_{(0)ij} = \mathcal{B}_{ij}$, $g_{(2)ij} = 2(\mathcal{B}\mathcal{C})_{ij}$ and $g_{(4)ij} = (\mathcal{B}\mathcal{C}^2)_{ij}$, or $g_{(4)ij} = (g_{(2)}g_{(0)}{}^{-1}g_{(2)})_{ij}/4$, in complete agreement with [47].

As expected, the metric involves two arbitrary matrices, which are functions of the transverse coordinates. These can be interpreted respectively as the normalizable and non-normalizable modes of the full non-linear equations. However, the Hamilton and the momentum constraints (respectively the first and second equations in (3.66)) do relate these matrices in a non-local and non-linear way.

First, the Hamilton constraint together with the expression (3.92) for the extrinsic curvature allows us to evaluate the Ricci scalar of the induced metric in terms of $\mathcal{C}$, namely

$$R = 2(\det\mathcal{K} - 1) = \frac{-4e^{-2r}\mathrm{tr}\,\mathcal{C}}{1 + e^{-2r}\mathrm{tr}\,\mathcal{C} + e^{-4r}\det\mathcal{C}}. \tag{3.96}$$

Taking the limit $r \to \infty$ gives the constraint

$$\boxed{R[g_{(0)}] = -4\mathrm{tr}\,\mathcal{C},} \tag{3.97}$$

where $R[g_{(0)}]$ is the Ricci scalar of the boundary metric.

Moreover, a short calculation shows that the vector $\xi_i \equiv D_j K_i^j - D_i K$ satisfies the differential equation

$$\dot{\xi}_i + K\xi_i = 0, \tag{3.98}$$

or, equivalently, $\partial_r(\sqrt{\gamma}\xi_i) = 0$. The solution of this equation is

$$\xi_i = \frac{-2e^{-2r}}{1 + e^{-2r}\mathrm{tr}\,\mathcal{C} + e^{-4r}\det\mathcal{C}}\left(D_j^{(0)}C_i^j - D_i^{(0)}C\right), \tag{3.99}$$

where $D_j^{(0)}$ denotes the covariant derivative with respect to the boundary metric $g_{(0)}$ and $C = \mathrm{tr}\,\mathcal{C}$. Hence, the momentum constraint in (3.66) is equivalent to

$$\boxed{D_j^{(0)}C_i^j - D_i^{(0)}C = 0.} \tag{3.100}$$

These constraints impose a non-local relation between C_j^i and the boundary metric and they are solved by identifying the tensor

$$\mathcal{T}_j^i \equiv 4(C_j^i - C\delta_j^i), \tag{3.101}$$

with the Liouville stress tensor (2.45). This non-local tensor then becomes the exact one-point function of the stress tensor of the dual CFT in a curved background. Successive derivatives with respect to the boundary metric then compute all correlation functions of the CFT stress tensor.

3.4.2 Gravity coupled to scalars

Having carried out in detail the near boundary analysis for pure AdS gravity in our formalism, we will now briefly describe how the analysis can be generalized to include scalars. In this case the matter action takes the form[16]

$$S_m = \int_{\mathcal{M}} d^{d+1}x\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi^I\partial_\nu\varphi^I + V(\varphi^I)\right). \tag{3.102}$$

The the gravitational field equations (3.38) now become

$$\begin{aligned}
&K^2 - K_{ij}K^{ij} = R + \kappa^2\left(\dot{\varphi}^I\dot{\varphi}^I - \gamma^{ij}\partial_i\varphi^I\partial_j\varphi^I - 2V(\varphi)\right),\\
&D_iK^i_j - D_jK = \kappa^2\dot{\varphi}^I\partial_j\varphi^I,\\
&\dot{K}^i_j + KK^i_j = R^i_j - \kappa^2\left(\partial^i\varphi^I\partial_j\varphi^I + \frac{2}{d-1}V(\varphi)\delta^i_j\right),
\end{aligned} \tag{3.103}$$

while equation (3.41) for the on-shell action now reads

$$\dot{\lambda} + K\lambda + \frac{2\kappa^2}{d-1}V(\varphi) = 0. \tag{3.104}$$

Moreover, we now have the equations of motion for the scalar fields, namely

$$\ddot{\varphi}^I + K\dot{\varphi}^I + \Box\varphi^I - \frac{\partial}{\partial\varphi^I}V(\varphi) = 0. \tag{3.105}$$

Again, the asymptotic analysis starts by expanding the canonical momenta and λ in eigenfunctions of the dilatation operator

$$\delta_D = \int d^dx 2\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}} + \int d^dx(\Delta_I - d)\varphi^I\frac{\delta}{\delta\varphi^I}, \tag{3.106}$$

as in (3.51). Note however that, depending on the scaling dimension Δ_I of the scalar operators, the eigenvalues of the dilatation operator for the momenta of the scalar fields may not be integers anymore.

We then proceed exactly as for the pure gravity case, using the identity (3.43) in order to eliminate the coefficients $\lambda_{(n)}$ in favor of the coefficients of the momenta. Namely, inserting the expansions (3.51) into

$$(1+\delta_D)K + \kappa^2(\Delta_I - d)\dot{\varphi}^I\varphi^I = (d+\delta_D)\lambda, \tag{3.107}$$

and matching terms of equal dilatation weight determines all local terms in the expansion of λ in terms of the local terms in the expansions of the momenta.

At the next step, one uses the Hamilton contsraint (first equation in (3.103)) to derive a recursion relation for the traces of the extrinsic curvature. This relation is

[16] We consider a flat metric $G_{IJ}(\varphi) = \delta_{IJ}$ on the scalar manifold here.

the analogue of (3.81) for pure gravity, but it will now involve the coefficients of the scalar momenta as well. The precise form of such a recursion relation depends crucially on the dimension d as well as the scaling dimensions Δ_I of the scalar operators.

Finally, inserting the coefficients $\lambda_{(n)}$ as well as the trace of the extrinsic curvature coefficients into the Hamilton-Jacobi identities (3.42), all momenta can be determined iteratively.

As an illustration, let us quote the results of this procedure for the case of two scalar fields, φ and σ, both of scaling dimension $\Delta = 3$ in $d = 4$ and with a potential that has a critical point at $\varphi = \sigma = 0$. The most general potential compatible with these requirements is

$$V(\varphi,\sigma) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \kappa^{n-2} V_{(m,n-m)}\varphi^m\sigma^{n-m}, \tag{3.108}$$

where $V_{(0,0)} = \Lambda/\kappa^2$ is the cosmological constant, $V_{(0,1)} = V_{(1,0)} = 0$, i.e. there are no linear couplings, $V_{(1,1)} = 0$ and $V_{(2,0)} = V_{(0,2)} = -3$, i.e. the quadratic terms are diagonal in φ and σ and both have mass $m^2 = \Delta(\Delta - d) = -3$. All other couplings $V_{(m,m-n)}$ are arbitrary.

The iterative approach determines the following on-shell action:

$$\begin{aligned} I_{r_o} \;=\; & -\frac{1}{\kappa^2}\int_{\Sigma_{r_o}} d^4x\sqrt{\gamma}\left\{\frac{d-1}{d-2}P - \frac{1}{4}\left[P_{ij}P^{ij} - P^2 - \kappa^2\varphi(\Box + P)\varphi\right.\right. \\ & \left.\left. -\kappa^2\sigma(\Box + P)\sigma\right]\log e^{-2r_o} + K_{(4)} - \lambda_{(4)} + \cdots\right\} - \int_{\Sigma_{r_o}} d^4x\sqrt{\gamma}W(\varphi,\sigma), \end{aligned} \tag{3.109}$$

where the 'superpotential' $W(\varphi,\sigma)$ is given by

$$\begin{aligned} W(\varphi,\sigma) \;=\; & \frac{1}{\kappa^2}(d-1) + \frac{1}{2}\left(\varphi^2 + \sigma^2\right) \\ & + \frac{\kappa}{d-3}\left(V_{(3,0)}\varphi^3 + V_{(2,1)}\varphi^2\sigma + V_{(1,2)}\varphi\sigma^2 + V_{(0,3)}\sigma^3\right) \\ & + \kappa^2\left[\left(\frac{1}{2}V_{(4,0)} - \frac{1}{4(d-3)^2}(9{V_{(3,0)}}^2 + {V_{(1,2)}}^2) + \frac{d}{16(d-1)}\right)\varphi^4\right. \\ & + \left(\frac{1}{2}V_{(3,1)} - \frac{1}{(d-3)^2}(3V_{(3,0)}V_{(2,1)} + V_{(1,2)}V_{(0,3)})\right)\varphi^3\sigma \\ & + \left(\frac{1}{2}V_{(2,2)} - \frac{1}{2(d-3)^2}(3V_{(3,0)}V_{(1,2)} + 3V_{(0,3)}V_{(2,1)}\right. \\ & \left. + 2{V_{(2,1)}}^2 + 2{V_{(1,2)}}^2) + \frac{d}{8(d-1)}\right)\varphi^2\sigma^2 + \end{aligned} \tag{3.110}$$

$$
\begin{aligned}
&+\left(\frac{1}{2}V_{(0,4)} - \frac{1}{4(d-3)^2}(9V_{(0,3)}{}^2 + V_{(2,1)}{}^2) + \frac{d}{16(d-1)}\right)\sigma^4 \\
&+\left(\frac{1}{2}V_{(1,3)} - \frac{1}{(d-3)^2}(3V_{(0,3)}V_{(1,2)} + V_{(2,1)}V_{(3,0)})\right)\varphi\sigma^3\Bigg]\log e^{-2r_o} \\
&+W_{(4)} + \cdots . \qquad (3.111)
\end{aligned}
$$

A direct computation shows that W satisfies the differential equation

$$
V(\varphi) = \frac{1}{2}\left[\left(\frac{\partial W}{\partial \varphi^I}\right)^2 - \frac{d\kappa^2}{d-1}W^2\right]. \qquad (3.112)
$$

We emphasize, however, that the above algorithm determines W *without* making use of this equation, which arises as *consequence* of our formalism.[17] In particular, our algorithm determines the overall sign of W as well, in contrast to (3.112).

Equation (3.112) motivates the term 'superpotential' for W, by analogy to the superpotential obtained from the gauged supergravity action and which obeys a BPS equation similar to (3.112). We emphasize, however that (3.112) does not arise from any BPS condition. However, (3.112) and the fact that the AdS critical point of V is also a critical point of W guarantee the gravitational stability of the AdS critical point [50, 51].

Given a potential V, one may view (3.112) as a differential equation that can be solved to determine the 'superpotential'. This is potentially interesting because, as we will see in the next chapter, such a 'superpotential' W automatically provides a non-trivial (but non necessarily supersymmetric) domain wall solution to the supergravity equations, which, via the AdS/CFT dictionary, provides a holographic description of certain renormalization group flows of the dual quantum field theory. In the next chapter we study extensively these interesting solutions and describe how the Hamiltonian method of holographic renormalization can be used to efficiently compute correlation functions of the dual quantum field theory.

3.A Appendix

3.A.1 Conventions and useful formulas

Riemann tensor

We define the Riemann tensor as

$$
R^{\mu}{}_{\rho\nu\sigma} = \partial_\nu \Gamma^{\mu}_{\rho\sigma} + \Gamma^{\mu}_{\lambda\nu}\Gamma^{\lambda}_{\rho\sigma} - (\nu \leftrightarrow \sigma). \qquad (3.113)
$$

[17] (3.112) can indeed be derived in general in this formalism, i.e. for arbitrary dimension and scalar fields [39].

Then, for any vector v^μ,

$$[\nabla_\mu, \nabla_\nu]v^\rho = R_{\mu\nu}{}^\rho{}_\sigma v^\sigma. \tag{3.114}$$

This differs by an overall sign from the conventions used in [29, 30].

WEYL TENSOR

The Weyl tensor is defined for $D > 2$ by

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + g_{\mu\sigma}P_{\nu\rho} + g_{\nu\rho}P_{\mu\sigma} - g_{\mu\rho}P_{\nu\sigma} - g_{\nu\sigma}P_{\mu\rho}, \tag{3.115}$$

where

$$P_{\mu\nu} = \frac{1}{D-2}\left(R_{\mu\nu} - \frac{1}{2(D-1)}Rg_{\mu\nu}\right). \tag{3.116}$$

Under a Weyl transformation the Weyl tensor transforms homogeneously

$$C_{\mu\nu\rho\sigma}[e^{2\omega}g] = e^{2\omega}C_{\mu\nu\rho\sigma}[g]. \tag{3.117}$$

A Riemannian manifold $(\mathcal{M}, g)$ of dimension $D \geq 4$ is conformally flat if and only if its Weyl tensor vanishes. However, the Weyl tensor vanishes identically for any three-dimensional Riemannian manifold. Instead, for $D = 3$, a Riemannian manifold is conformally flat if and only if the Weyl-Schouten tensor

$$W_{\mu\nu\rho} \equiv \nabla_\nu P_{\mu\rho} - \nabla_\rho P_{\mu\nu}, \tag{3.118}$$

vanishes. The Weyl-Schouten tensor is invariant under Weyl transformations

$$W_{\mu\nu\rho}[e^{2\omega}g] = W_{\mu\nu\rho}[g]. \tag{3.119}$$

METRIC VARIATIONS

$$\begin{aligned}
&\delta_g\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}(\nabla_\rho\delta g_{\sigma\nu} + \nabla_\sigma\delta g_{\rho\nu} - \nabla_\nu\delta g_{\rho\sigma}),\\
&\delta_g R^\mu{}_{\nu\rho\sigma} = \nabla_\rho\delta_g\Gamma^\mu_{\nu\sigma} - \nabla_\sigma\delta_g\Gamma^\mu_{\nu\rho},\\
&\delta_g R_{\mu\nu} = R_{(\mu}{}^\rho\delta g_{\rho\nu)} - R_\mu{}^\rho{}_\nu{}^\sigma\delta g_{\rho\sigma} + \nabla_{(\mu}\nabla^\rho\delta g_{\rho\nu)} - \frac{1}{2}(\Box\delta g_{\mu\nu} + g^{\rho\sigma}\nabla_\mu\nabla_\nu\delta g_{\rho\sigma}),
\end{aligned} \tag{3.120}$$

where the indices inside the parentheses are symmetrized with weight one.

3.A.2 GAUGE-FIXED MATTER FIELD EQUATIONS

$$\begin{aligned}
&\textbf{Vector}:\\
&D_i(U(\varphi)F^{ri}) = 0,\\
&\partial_r(U(\varphi)F^{rj}) + KU(\varphi)F^{rj} + D_i(U(\varphi)F^{ij}) = 0.
\end{aligned} \tag{3.121}$$

$$\begin{aligned}
&\textbf{Scalar}: \\
&\partial_r(G_{IJ}(\varphi)\dot{\varphi}^J) + KG_{IJ}(\varphi)\dot{\varphi}^J + D^i(G_{IJ}(\varphi)\partial_i\varphi^J) \\
&-\frac{1}{2}\frac{\partial G_{JK}}{\partial\varphi^I}(\dot{\varphi}^J\dot{\varphi}^K + \partial_i\varphi^J\partial^i\varphi^K) - \frac{\partial V}{\partial\varphi^I} \\
&-\frac{1}{4}\frac{\partial U}{\partial\varphi^I}(2F^r{}_iF^{ri} + F_{ij}F^{ij}) = 0.
\end{aligned} \tag{3.122}$$

Here, K is the trace of the extrinsic curvature of the radial slices Σ_r, $F^r{}_i = \dot{A}_i$ and the gauge $A_r = 0$ has been used.

3.A.3 Lorentzian signature

Thoughout this chapter we have used exclusively Euclidean signature for the metric $g_{\mu\nu}$ on $\mathcal{M}$ and the induced metric γ_{ij} on the hypersurfaces Σ_r. In this appendix we provide the Lorentzian version of some of the important formulas that appear in the Hamiltonian version of holographic renormalization and which will be needed in the last chapter.

The Lorentzian supergravity action differs by an overall sign from its Euclidean version, namely

$$S_{\rm gr}[g] = \frac{1}{2\kappa^2}\left[\int_{\mathcal{M}} d^{d+1}x\sqrt{g}R + \int_{\partial\mathcal{M}} d^dx\sqrt{\gamma}2K\right], \tag{3.123}$$

and

$$S_{\rm m}[g,A,\varphi] = -\int_{\mathcal{M}} d^{d+1}x\sqrt{g}\left(\frac{1}{4}U(\varphi)F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}G_{IJ}(\varphi)\partial_\mu\varphi^I\partial^\mu\varphi^J + V(\varphi)\right). \tag{3.124}$$

The gauge-fixed canonical momenta are then given by

$$\pi^{ij} = -\frac{1}{2\kappa^2}\sqrt{-\gamma}(K^{ij} - K\gamma^{ij}), \quad \pi^i = -\sqrt{-\gamma}U(\varphi)\dot{A}^i, \quad \pi^I = -\sqrt{-\gamma}G_{IJ}(\varphi)\dot{\varphi}^I, \tag{3.125}$$

while the regulated on-shell action is expressed as

$$\int_{\mathcal{M}_{r_o}} \mathbf{L}_{\rm on-shell} = \int_{\mathcal{M}_{r_o}} d^{d+1}x\sqrt{-g}\left(\mathcal{L}_{\rm m} - \frac{1}{d-1}\tilde{T}^\sigma_\sigma\right) \equiv -\frac{1}{\kappa^2}\int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}\lambda, \tag{3.126}$$

where λ now satisfies the differential equation

$$\dot{\lambda} + K\lambda + \kappa^2\left(\mathcal{L}_{\rm m} - \frac{1}{d-1}\tilde{T}^\sigma_\sigma\right) = 0. \tag{3.127}$$

The regulated on-shell action (with the Gibbons-Hawking term included) is then given by

$$I_{r_o} = \frac{1}{\kappa^2} \int_{\Sigma_{r_o}} d^d x \sqrt{-\gamma}(K - \lambda), \tag{3.128}$$

and the momenta are again obtained by

$$\pi^{ij} = \frac{\delta I_{r_o}}{\delta \gamma_{ij}}, \qquad \pi^i = \frac{\delta I_{r_o}}{\delta A_i}, \qquad \pi_I = \frac{\delta I_{r_o}}{\delta \varphi^I}. \tag{3.129}$$

Chapter 4

Correlation Functions in Holographic RG Flows

One of the great successes of the gauge/gravity duality is that it allows one to compute correlation functions of a strongly coupled quantum field theory using classical supergravity. This statement was made concrete in the previous chapter, where we saw that the gauge/gravity correspondence identifies the generating functional of connected correlation functions of gauge-invariant operators with the on-shell supergravity action, with the arbitrary Dirichlet data of the supergravity fields acting as sources for the dual operators. We also saw that this identification is plundered by the infinities that appear in the on-shell supergravity action and we described how the method of holographic renormalization can be used to efficiently remove these infinities and consistently 'renormalize' the supergravity action. Finally, we discussed how the 'kinematics' of the correlation functions, such as Ward identities and anomalies, can be derived holographically.

What has not been discussed in detail so far though is the actual holographic computation of correlation functions. Contrary to the renormalization procedure and the kinematics, the evaluation of correlation functions requires *exact* solutions of the supergravity equations with arbitrary Dirichlet data instead of merely asymptotic solutions. Indeed, as we discussed in the previous chapter, the asymptotic analysis, either in the original or the covariant Hamiltonian formalism, does not determine the 'response' functions of the bulk fields, which are identified with the one-point functions of the dual operators in the presence of sources.

Solving the non-linear supergravity equations with arbitrary Dirichlet boundary conditions is too difficult a problem to be tackled with the present techniques, however. Instead, one linearizes the supergravity equations around some background and considers fluctuations around this background. The generating functional for correlation functions can then be computed perturbatively in the sources. In particular, the leading order approximation corresponding to the linearized equations is sufficient for the calculation of the two-point functions. Higher-point functions require a higher order calculation, however.

Let us first recall the computation of two-point functions as developed in the early period of the AdS/CFT correspondence [18, 24]. We will call this method the 'old approach'. To regulate the theory one imposes a cut-off at large radius and solves the linearized fluctuation equations with Dirichlet boundary conditions at the cut-off. The second variation of the regulated action is then computed in momentum space yielding an expression containing singular powers of the cut-off times integer powers of the momentum p plus non-singular terms in the cut-off which are non-analytic in p. The two-point function is then defined as the leading non-analytic term. The polynomial terms in p which are dropped are contact δ-function terms in the position space correlator, which are scheme dependent in field theory and largely unphysical. The non-analytic term has an absorptive part in p which correctly gives the two-point function for separated points in x-space. This method is quite efficient, but it is not fully satisfactory since it is not correct in general to simply drop the divergent contact terms in correlation functions. Since the correlation functions are related by Ward identities, one should make sure that contact terms are dropped in a way *consistent with the Ward identities.*

In holographic renormalization one replaces the above computation by a two-step procedure. Given a bulk action, the first step is to carry out the asymptotic analysis in order to construct a set of universal local covariant boundary counterterms that render the on-shell action finite on an arbitrary solution of the bulk equations of motion. This step was discussed in detail in the previous chapter using both the original and the Hamiltonian formalisms for holographic renormalization. With the general boundary counterterms at hand, one then solves the linearized supergravity equations with arbitrary Dirichlet boundary conditions and a radial cut-off. Evaluating then the regulated on-shell action and adding the covariant counterterms leads to the renormalized action. The two-point function is now obtained as the second variation of the *renormalized* action.

It is conceptually satisfying that the procedure of renormalization can be carried out in full generality without reference to a particular solution, i.e. that the counterterms are associated with a bulk action and not a particular solution. However, the above algorithm for calculating correlation functions in holographic renormalization means that every time one wants to compute, for example, a two-point function around some background field configuration, one must first carry out the general asymptotic analysis in order to construct the covariant boundary counterterms that render the particular action finite on any solution. Most often this requires a rather elaborate computation compared to solving the linearized equations which is required for the two-point function. Moreover, there are rarely more than one interesting exact solutions of a given action with a given field content, and so, in order to compute correlation functions in different RG-flows one would need to first complete the near-boundary analysis for each different action.

There is no reason, however, why the asymptotic analysis cannot be carried out at the linearized level. In other words, instead of solving the full non-linear supergravity equations asymptotically in order to determine the covariant boundary counterterms and then linearize the counterterms around the given background, one should be able to determine these linearized counterterms directly by solving asymptotically the *linearized* bulk equations. If this is possible, it is clearly a much more efficient route for computing correlation functions in holographic renormalization since one need only analyze the linearized equations. In this chapter, which is an expanded version of [2], we demonstrate that the Hamiltonian version of holographic renormalization indeed allows one to determine the linearized covariant boundary counterterms directly by solving asymptotically the linearized bulk equations.

The backgrounds we will be interested in are domain wall solutions of the bulk supergravity theory with a single active scalar field turned on. We consider first

Poincaré domain walls, which preserve Poincaré invariance in the transverse space. Such solutions correspond to either a deformation of the Lagrangian of the boundary CFT by a relevant operator, or to a vacuum of the dual CFT where conformal invariance is broken spontaneously by the vacuum expectation value of a scalar operator. We discuss how to distinguish between these two cases by computing the exact one-point functions for these backgrounds. Two-point functions are then calculated and we show explicitly how they can be renormalized without the need for a general near boundary analysis.

As a second class of interesting backgrounds, we study AdS-sliced domain walls and in particular the Janus solution [52], which is a particular non-supersymmetric but stable AdS domain wall [53]. The main difficulty in this example is that the boundary has a corner. However, we show that there exists a suitable Fefferman-Graham coordinate system which is well-defined everywhere in the neighborhood of the boundary except on the corner. This allows us to calculate the vevs of the background as well as some two-point functions. In particular, we show that the Ward identities associated with the symmetries of the background are satisfied.

We discuss here only two-point functions. Higher-point functions have been discussed in the context of holographic renormalization in [33, 54]. The procedure we describe for two-point functions here can be applied in such cases as well, significantly simplifying the process of renormalization. An effort to reproduce known results for higher-point functions using this procedure is presently under way.

4.1 POINCARÉ DOMAIN WALLS

In this section we will consider linear fluctuations around Poincaré domain wall solutions of the supergravity equations of motion. These take the generic form

$$ds_B^2 = dr^2 + e^{2A(r)}dx^i dx^i, \qquad \varphi = \phi_B(r). \tag{4.1}$$

Inserting this ansatz into the equations of motion (3.103) and (3.105) we find that $A(r)$ and $\phi_B(r)$ satisfy

$$\begin{aligned}
&\dot{A}^2 - \frac{\kappa^2}{d(d-1)}\left(\dot{\phi}_B^2 - 2V(\phi_B)\right) = 0,\\
&\ddot{A} + d\dot{A}^2 + \frac{2\kappa^2}{d-1}V(\phi_B) = 0,\\
&\ddot{\phi}_B + d\dot{A}\dot{\phi}_B - V'(\phi_B) = 0.
\end{aligned} \tag{4.2}$$

Moreover, (3.41) becomes

$$\dot{\lambda}_B + d\dot{A}\lambda_B + \frac{2\kappa^2}{d-1}V(\phi_B) = 0, \tag{4.3}$$

which immediately implies

$$\lambda_B = \dot{A} + \xi(x)e^{-dA} = \frac{1}{d}K_B + \frac{\xi(x)}{\sqrt{\gamma_B}}, \tag{4.4}$$

where ξ is an arbitrary integration function of the transverse coordinates. Therefore,

$$S^B_{\text{on-shell}} = -\frac{d-1}{d\kappa^2}\int_{\Sigma_r} d^dx\sqrt{\gamma_B}K_B + \frac{1}{\kappa^2}\int_{\Sigma_r} d^dx\xi(x). \tag{4.5}$$

The last term corresponds to finite local counterterms. As expected, there is an ambiguity in the on-shell value of the action corresponding to the renormalization scheme dependence of the dual field theory.

It is well known that the second order flat domain wall equations are solved by any solution of the first order flow equations

$$\begin{aligned} \dot{A} &= -\frac{\kappa^2}{d-1}W(\phi_B), \\ \dot{\phi}_B &= W'(\phi_B), \end{aligned} \tag{4.6}$$

provided the potential can be written in the form

$$V(\phi_B) = \frac{1}{2}\left[W'^2 - \frac{d\kappa^2}{d-1}W^2\right]. \tag{4.7}$$

The motivation for considering theories with such potentials stems from the fact that they guarantee gravitational stability of the AdS critical point and of associated domain-wall spacetimes, provided the AdS critical point is also a critical point of W [50, 51, 53]. This means, by the AdS/CFT duality, that the dual theory is unitary. Notice that (4.7) is identical to equation (3.112) which was deduced as a consequence of the Hamilton-Jacobi formulation of the asymptotic dynamics.

In general, the set of solutions of the second order equations (4.2) may include solutions which cannot be obtained from the first order flow equations. In this section we will restrict attention to solutions which can be derived from the flow equations. For this class of solutions we have

$$S^B_{\text{on-shell}} = \int_{\Sigma_r} d^dx\sqrt{\gamma_B}W(\phi_B) + \frac{1}{\kappa^2}\int_{\Sigma_r} d^dx\xi(x). \tag{4.8}$$

In principle it is always possible to write the potential in the form (4.7) if one views (4.7) as a differential equation for $W(\phi_B)$. The resulting W however may not have the original AdS spacetime as a critical point. Furthermore, in practice it is considerably difficult to solve (4.7). However, for certain potentials it is possible to find interesting solutions as we now demonstrate by the following example.

A toy domain wall solution

It was observed in [1] that (4.7) can be transformed to the form of Abel's equation [55]:

$$\boxed{y'(\psi) = \left(\frac{v'}{v}y \mp 1\right)(y^2-1),} \tag{4.9}$$

where $\psi = \sqrt{\frac{d\kappa^2}{d-1}}\varphi$, $y = \coth(u)$, $W = v\cosh(u)$, and v is related to the potential by $\frac{2(d-1)}{d\kappa^2}V = -v^2$. The general solution to this equation is not known, but it can be solved in special cases.

In [1] we solved (4.9) in arbitrary dimension for the potential

$$V(\psi) = -\frac{d(d-1)}{2\kappa^2}\cosh\left(2\psi/3\right). \tag{4.10}$$

This potential was later considered also in [56], where a black hole solution with a non-trivial scalar field in four dimensions was found. The authors of [56] also observed that, in four dimensions, this potential has a natural interpretation in the conformal frame defined by

$$\tilde{\psi}/3 = \tanh(\psi/3), \qquad \tilde{g}_{\mu\nu} = \cosh^2(\psi/3)g_{\mu\nu}, \tag{4.11}$$

where again $\tilde{\psi} = \sqrt{\frac{d\kappa^2}{d-1}}\tilde{\varphi}$. The bulk action then takes the form

$$S = \int_{\mathcal{M}} d^4x\sqrt{\tilde{g}}\left[-\frac{1}{2\kappa^2}\tilde{R} + \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\varphi}\partial_\nu\tilde{\varphi} + \frac{1}{12}\tilde{R}\tilde{\varphi}^2 + \tilde{V}(\tilde{\varphi})\right], \tag{4.12}$$

where

$$\tilde{V}(\tilde{\varphi}) = -\frac{d(d-1)}{2\kappa^2}\left(1-(\kappa^2/6)^2\tilde{\varphi}^4\right). \tag{4.13}$$

In this frame the scalar field is conformally coupled to gravity and the $\tilde{\varphi}^4$ potential ensures that the scalar field equations are conformally invariant. This property however is special to four dimensions.

Let us now review the solution of (4.9) obtained in [1]. Note that the two signs in (4.9) are related by a sign flip in y or ψ and so we will only consider the negative sign. Defining the new variables

$$s = 1/y, \qquad \varrho = \tanh(2\psi/3), \tag{4.14}$$

(4.9) takes the form

$$\frac{2ds}{1-s^2} + \frac{d\varrho}{1-\varrho^2}\left(\frac{\varrho}{s} - 3\right) = 0, \tag{4.15}$$

whose general solution is

$$s=\frac{\varrho}{1+(1-\varrho^2)(1+2\varrho\tanh\gamma+\varrho^2)^{-1/2}}, \tag{4.16}$$

where γ is an integration constant. It terms of the original variables, this means that the 'superpotential' takes the form [1]

$$W(\psi)=-\frac{(d-1)\cosh^{1/2}\left(\frac{2}{3}\psi\right)\left(\cosh\left(\frac{2}{3}\psi\right)+\cosh^{1/2}(\gamma)\text{sech}^{1/2}\left(\frac{4}{3}\psi+\gamma\right)\right)}{\kappa^2\sqrt{1+\cosh(\gamma)\text{sech}\left(\frac{4}{3}\psi+\gamma\right)+2\cosh\left(\frac{2}{3}\psi\right)\cosh^{1/2}(\gamma)\text{sech}^{1/2}\left(\frac{4}{3}\psi+\gamma\right)}}. \tag{4.17}$$

The full domain wall solution can in fact be expressed in closed form in terms of ϱ. Namely,

$$\begin{aligned} ds^2 &= \left(\frac{3}{d}\right)^2\left\{\frac{\left(1+\varrho\tanh\gamma+\sqrt{1+2\varrho\tanh\gamma+\varrho^2}\right)}{2\varrho^2\sqrt{1-\varrho^2}(1+2\varrho\tanh\gamma+\varrho^2)}d\varrho^2\right. \\ &\quad\left.+\left[\frac{\sqrt{1-\varrho^2}}{2\varrho^2}\left(1+\varrho\tanh\gamma+\sqrt{1+2\varrho\tanh\gamma+\varrho^2}\right)\right]^{3/d}dx^idx^i\right\}, \\ \phi_B &= \frac{3}{2}\sqrt{\frac{d-1}{d\kappa^2}}\tanh^{-1}\varrho. \end{aligned} \tag{4.18}$$

Note that the solution simplifies considerably in four dimensions (i.e. $d=3$), which is precisely the case where the scalar field action is conformally invariant in the conformal frame.

Expanding the potential (4.10) around the AdS critical point, we find that the mass of the scalar field is

$$m^2=-2(d/3)^2. \tag{4.19}$$

This satisfies the Breitenlohner-Freedman bound [57] $m^2\geq-(d/2)^2$ for all dimensions, which guarantees the stability of the above domain wall solution. The two solutions of the equation $m^2=\Delta(\Delta-d)$, relating the mass of the scalar field with the dimension of the dual operator are

$$\Delta_+=2d/3,\qquad \Delta_-=d/3. \tag{4.20}$$

It was shown in [58] that while only the choice $\Delta=\Delta_+$ is possible for the dimension of the dual operator when $m^2>-(d/2)^2+1$, when $-(d/2)^2<m^2<-(d/2)^2+1$ both Δ_+ and Δ_- are allowed dimensions for the dual operator, leading to two different CFTs on the boundary. The corresponding generating functionals for connected correlation functions are then the Legendre transform of each other. In the present case, the above condition for both $\Delta_\pm$ to be admissible as scaling dimensions of the dual operator translates into the condition $d<6$.

As we will explain in the next section, the expansion

$$W(\psi) = -\frac{d-1}{\kappa^2}\left(1 + \frac{1}{6}\psi^2 + \frac{1}{27}\tanh(\gamma)\psi^3 + \cdots\right) \tag{4.21}$$

of the 'superpotential' around the AdS critical point then implies that for $d > 6$ this domain wall solution describes the renormalization group flow of the dual field theory due to a *deformation* of the CFT Lagrangian by an operator of dimension $\Delta_+ = 2d/3$. For $d < 6$, however, further analysis, such as the computation of certain two-point functions in this background, is necessary in order to determine whether this domain wall describes a deformation of the CFT Lagrangian by an operator of dimension $\Delta_+ = 2d/3$ or a *vacuum expectation value* of an operator of dimension $\Delta_- = d/3$. The case $d = 6$ is special as $\Delta_- = 2$ saturates the unitarity bound $\Delta \geq d/2 - 1$ and further analysis is required in this case too. Finally, the evaluation of the two-point functions in this background is also likely to reveal the physical significance of the parameter γ which does not enter in the vev of the stress tensor or of the scalar operator, at least for $d > 6$. We will not evaluate these two-point functions here, however.

4.1.1 DEFORMATIONS VS VEVS

The domain wall solutions we have described correspond via the AdS/CFT duality to deformations of the boundary CFT by relevant operators, in which case the conformal invariance of the boundary theory is explicitly broken, or to a vacuum expectation value of a scalar operator which spontaneously breaks the conformal symmetry. Marginal deformations are also similarly described.

Locally asymptotically AdS metrics satisfy the asymptotic condition $\gamma_{ij}(r, x) \sim e^{2r} g_{(0)ij}(x)$ as $r \to \infty$, which requires $A(r) \sim r$. Moreover, a scalar field dual to an operator of dimension Δ behaves asymptotically as $\varphi(r, x) \sim e^{-(d-\Delta)r}\phi_{(0)}(x)$. These asymptotic conditions, together with the equations of motion, require that the scalar potential takes the form

$$V(\varphi) = -\frac{d(d-1)}{2\kappa^2} + \frac{1}{2}m^2\varphi^2 + \cdots, \tag{4.22}$$

where the mass is related to the dimension Δ of the dual operator by $m^2 = \Delta(\Delta - d)$. Solving the equation of motion for the scalar field with such a potential leads to a generic solution of the form

$$\varphi(r, x) = e^{-(d-\Delta)r}\left[\phi_{(0)}(x) + \cdots\right] + e^{-\Delta r}\left[\phi_{(2\Delta-d)}(x) + \cdots\right]. \tag{4.23}$$

We will consider operators for which $d - \Delta \geq \Delta$, or $\Delta \geq d/2$. Moreover, we are interested in relevant or marginal operators and so $\Delta \leq d$. In total then $d/2 \leq \Delta \leq d$. In this range the first term in the solution for φ is dominant asymptotically and corresponds to the source, while the second term is related to the one-point function of the dual operator. In the special case $\Delta = d/2$, which saturates the BF bound, the scalar field takes the form[1]

$$\varphi(r,x) = e^{-dr/2}\left[-2r\left(\phi_{(0)}(x)+\cdots\right)+\tilde{\phi}_{(0)}(x)+\cdots\right]. \tag{4.24}$$

Again, the first term is the source for the dual operator, while the second is related to its expectation value.

Depending on the form of the 'superpotential' $W(\phi_B)$ the domain wall solution can describe either a deformation of the dual CFT or a phase with spontaneously broken conformal symmetry as we now explain. A similar analysis can be found in [40]. Assuming the potential has a critical point at $\phi_B = 0$, equation (4.7) together with the flow equations (4.6) and the requirement that $\phi_B = 0$ is also a critical point of W, imply that W has an expansion around $\phi_B = 0$ of two possible forms:

$$\begin{aligned} W_+(\phi_B) &= -\frac{d-1}{\kappa^2} - \frac{1}{2}(d-\Delta)\phi_B^2 + \cdots \\ W_-(\phi_B) &= -\frac{d-1}{\kappa^2} - \frac{1}{2}\Delta\phi_B^2 + \cdots . \end{aligned} \tag{4.25}$$

Which of these cases is realized is purely a property of the background solution and we need to examine each case separately. Moreover, the extremal values $d/2$ and d of the scaling dimension Δ require special attention. We will now compute the vev for the stress energy tensor and dual scalar operator for all cases.

As we saw in the previous chapter, the part of the divergent part of the on-shell action involving only the scalar field is a function $U(\varphi)$, satisfying the equation for the 'superpotential' W (4.7), *and* having an expansion around $\varphi = 0$

$$U(\varphi) = -\frac{d-1}{\kappa^2} - \frac{1}{2}(d-\Delta)\varphi^2 + \cdots \tag{4.26}$$

Let us consider first the case W_+ is realized in the background. In this case we can choose a scheme where the counterterm action is W_+. To see this notice that any two solutions of (4.7) with identical expansions around $\varphi = 0$ up to order φ^2 can only differ at order $\varphi^{d/(d-\Delta)} \sim e^{-dr}$. This is easily proved by looking for the most general power series[2] solution of (4.7) with this particular form up to quadratic order in φ. One finds that all terms are uniquely determined up to order $d/(d-\Delta)$ where the recursion relations break down. This is precisely where an arbitrary integration constant appears and the two solutions could be potentially different. However, this is irrelevant for the purpose of removing the divergences of the on-shell action and so we can choose the renormalization scheme $U(\varphi) =$

[1] Note that the radial coordinate we use here is related to the Fefferman-Graham radial coordinate by $\rho = e^{-2r}$. The factor $-2r = \log\rho$ is chosen to match with the corresponding formulae in [31, 32].

[2] As usual one must include a logarithmic term at order $d/(d-\Delta)$ in the general case.

$W_+(\varphi)$, and set the integration function ξ to zero. This choice of counterterms corresponds to a supersymmetric renormalization scheme since it ensures $S^B_{\rm ren} = 0$ [31]. It follows that the background vevs of both the operator dual to φ and the stress tensor vanish identically and so this background describes a deformation of the boundary CFT by a relevant operator.

Let us now consider the case W_- is realized in the background. In this case we cannot choose W_- as the counterterm since it differs from $U(\varphi)$ at the quadratic order. In this case, however, $\phi_B \sim e^{-\Delta r}$ and so the on-shell action evaluated on the background contains only the volume divergence since, by the hypothesis, $2\Delta > d$. Hence, again, setting $\xi = 0$ corresponds to a supersymmetric renormalization scheme with $S^B_{\rm ren} = 0$. Accordingly, the background expectation value of the stress tensor vanishes, but not that of the scalar operator. In this case $W_-'(\phi_B) - U'(\phi_B) = -(2\Delta - d)\phi_B + \cdots$, and hence,

$$\langle \mathcal{O} \rangle^B_{\rm ren} = (d - 2\Delta)\phi_B, \tag{4.27}$$

which spontaneously breaks the conformal symmetry of the dual CFT. Here we used the fact that the regularized one-point function (=canonical momentum) is related to the superpotential via the first order equation (4.6). The renormalized one-point function is obtained by subtracting the contribution U' of the counterterm.

It remains to examine the two extremal cases $\Delta = d/2$ and $\Delta = d$. When $\Delta = d/2$ there is no distinction between W_+ and W_- as they are equal and $\phi_B \sim e^{-dr/2}$, i.e. it behaves asymptotically as the vev term in (4.24). However, the on-shell action function $U(\varphi)$ includes divergences coming from the source term in (4.24) and therefore it cannot be identified with $W(\varphi)$. It is straightforward to find the covariant counterterms for this case using the Hamiltonian method of holographic renormalization, but the singularity structure for the terms involving the scalar field are not the standard ones, as we now explain. This can be traced back to the fact that the source term for the scalar contains a logarithm, in contrast to the generic case. A simple calculation using the asymptotic form of the solution shows that the canonical momentum of the scalar field takes the form

$$\pi = \dot{\varphi} = \left(\frac{1}{r} - \frac{d}{2}\right)\varphi + \cdots \tag{4.28}$$

Hence, the on-shell action is

$$S_{\rm on-shell} = S^{\rm gr}_{\rm on-shell} + \int_{\Sigma_r} d^d x \sqrt{\gamma} \frac{1}{2}\left(\frac{1}{r} - \frac{d}{2}\right)\varphi^2 + \cdots \tag{4.29}$$

where $S^{\rm gr}_{\rm on-shell}$ is the on-shell action for pure gravity. We have therefore shown that

$$U(\varphi) = -\frac{d-1}{\kappa^2} + \frac{1}{2}\left(\frac{1}{r} - \frac{d}{2}\right)\varphi^2 + \cdots \tag{4.30}$$

which is the divergent part of the on-shell action and must be removed (of course there is another part coming from pure gravity). The same counterterms (for $d = 4$) were derived in [31, 32]. We now see that the scalar operator gets a vev since $W'(\phi_B) - U'(\phi_B) = -\frac{1}{r}\phi_B + \cdots$, i.e.

$$\langle \mathcal{O} \rangle^B_{\text{ren}} = 2\phi_B. \tag{4.31}$$

This also agrees with the results in [31, 32]. Again the difference between $W(\phi_B)$ and $U(\phi_B)$ is subleading and the stress tensor gets no vev since $S_{\text{ren}} = 0$.

Finally we consider the case $\Delta = d$, for which

$$\varphi(r, x) = \left[\phi_{(0)}(x) + \cdots\right] + e^{-dr}\left[\phi_{(d)}(x) + \cdots\right]. \tag{4.32}$$

The equations of motion require $V'(\varphi) = 0$ and so the potential is just the cosmological constant $V(\varphi) = -\frac{d(d-1)}{2\kappa^2}$. It follows that the on-shell function $U(\varphi)$ is also a constant, i.e. the first term of $W_\pm$. In this case, however, the general solution to (4.7) can be easily obtained. There are two distinct solutions (cf. eq. (2.10) in [53]),

$$W_+ = -\frac{d-1}{\kappa^2}, \qquad W_- = -\frac{d-1}{\kappa^2}\cosh\left(\sqrt{\frac{d\kappa^2}{d-1}}(\phi - \phi_o)\right), \tag{4.33}$$

in agreement with our general analysis. Notice that in this case the supergravity action and hence the second order field equations are invariant under constant shifts of the scalar field. Such a constant corresponds to the source term of the solution. One may use this symmetry to set ϕ_0 to zero in W_-. Then, exactly as for the case $d/2 < \Delta < d$, if W_+ is realized in the background, then neither the scalar operator nor the stress tensor acquire a vev, and if W_- is realized, the scalar operator gets a vev $-d\phi_B$, while the vev of the stress-energy tensor vanishes.

So finally we can summarize all possibilities for flat domain wall backgrounds in the following table:

Δ	W	$\langle \mathcal{O} \rangle^B_{\text{ren}}$	$\langle T^i_j \rangle^B_{\text{ren}}$
$d/2 < \Delta \leq d$	$+$	0	0
	$-$	$(d - 2\Delta)\phi_B$	0
$d/2$	$\pm$	$2\phi_B$	0

4.1.2 LINEARIZED EQUATIONS

Now let us consider fluctuations around the backgrounds we have described so far. We will only keep terms up to linear order in fluctuations, which suffices for the calculation of the two-point functions. The metric fluctuations take the form

$$\gamma_{ij} = \gamma^B_{ij}(r) + h_{ij}(r, x) = e^{2A(r)}\delta_{ij} + h_{ij}(r, x), \tag{4.34}$$

and the scalar field is

$$\varphi = \phi_B(r) + \phi(r,x). \tag{4.35}$$

The extrinsic curvature then becomes

$$K^i_j = \dot{A}\delta^i_j + \frac{1}{2}\dot{S}^i_j, \tag{4.36}$$

where $S^i_j \equiv \gamma^{ik}_B h_{kj}$. S^i_j can be decomposed into irreducible components as

$$S^i_j = e^i_j + \partial^i\epsilon_j + \partial_j\epsilon^i + \frac{d}{d-1}\left(\frac{1}{d}\delta^i_j - \frac{\partial^i\partial_j}{\Box_B}\right)f + \frac{\partial^i\partial_j}{\Box_B}S, \tag{4.37}$$

where $\partial_i e^i_j = e^i_i = \partial_i\epsilon^i = 0$ and indices are raised with the inverse background metric $e^{-2A}\delta^{ij}$. Conversely, each of the irreducible components can be expressed uniquely in terms of S^i_j as

$$e^i_j = \Pi^i{}_k{}^l{}_j S^k_l, \quad \epsilon_i = \pi^l_i \frac{\partial_k}{\Box_B} S^k_l, \quad f = \pi^l_k S^k_l, \quad S = \delta^l_k S^k_l, \tag{4.38}$$

where we have introduced the projection operators

$$\Pi^i{}_k{}^l{}_j = \frac{1}{2}\left(\pi^i_k\pi^l_j + \pi^{il}\pi_{kj} - \frac{2}{d-1}\pi^i_j\pi^l_k\right), \tag{4.39}$$

and

$$\pi^i_j = \delta^i_j - \frac{\partial^i\partial_j}{\Box_B}. \tag{4.40}$$

With this nomenclature we can now go on and derive the equations of motion for the linear fluctuations. The result is[3]

$$\begin{aligned}
&\left(\partial_r^2 + d\dot{A}\partial_r + e^{-2A}\Box\right)e^i_j = 0,\\
&\left(\partial_r^2 + [d\dot{A} + 2W\partial_\phi^2\log W]\partial_r + e^{-2A}\Box\right)\omega = 0,\\
&\dot{f} = -2\kappa^2\dot{\phi}_B\phi,\\
&\dot{S} = \frac{1}{(d-1)\dot{A}}\left[-e^{-2A}\Box f + 2\kappa^2\left(\dot{\phi}_B\dot{\phi} - V'(\phi_B)\phi\right)\right],
\end{aligned} \tag{4.41}$$

where

$$\omega \equiv \frac{W}{W'}\phi + \frac{1}{2\kappa^2}f, \tag{4.42}$$

and we have used the diffeomorphism invariance in the transverse space to set $\epsilon_i \equiv 0$. The last two equations give immediately the momenta dual to f and S and hence the corresponding one-point functions with linear sources. Moreover, since

[3] Note $\Box_B = e^{-2A}\Box = e^{-2A}\delta^{ij}\partial_i\partial_j$.

the canonical momenta are functionals of the bulk fields [1], to linear order in the fluctuations we must have

$$\dot{e}^i_j = E(A, \phi_B) e^i_j, \qquad \dot{\omega} = \Omega(A, \phi_B)\omega. \tag{4.43}$$

The first two equations then become first order equations for E and Ω:

$$\begin{aligned} &\dot{E} + E^2 + d\dot{A}E - e^{-2A}p^2 = 0, \\ &\dot{\Omega} + \Omega^2 + [d\dot{A} + 2W\partial^2_\phi \log W]\Omega - e^{-2A}p^2 = 0, \end{aligned} \tag{4.44}$$

where we have performed a Fourier transform in the transverse space. Given the solutions for E and Ω we can immediately write down all momenta, namely

$$\begin{aligned} &\dot{e}^i_j = E e^i_j, \\ &\dot{f} = -2\kappa^2 \dot{\phi}_B \phi, \\ &\dot{\phi} = (W'' + \Omega)\phi + \frac{1}{2\kappa^2}\frac{W'}{W}\Omega f, \\ &\dot{S} = -\frac{1}{\kappa^2}\left[\left(\frac{W'}{W}\right)^2 \Omega - \frac{e^{-2A}}{W}\Box\right] f - 2\frac{W'}{W}\left(\Omega + \frac{d\kappa^2}{d-1}W\right)\phi. \end{aligned} \tag{4.45}$$

To completely determine the one-point functions with linear sources we first need to obtain exact solutions for E and Ω and secondly, to determine the covariant counterterms for the momenta, *but* only to linear order in the fluctuations. Since e^{-2A} and $W(\phi_B)$ are already covariant functions of the background fields, it suffices to find covariant expansions for E and Ω in the background fields. These can be organized according to the dilatation operator for the background

$$\delta_D = \partial_A + (\Delta - d)\phi_B \partial_{\phi_B}. \tag{4.46}$$

More generally, the radial derivative is expanded in functional derivatives w.r.t. the background fields as

$$\partial_r = \dot{A}\partial_A + \dot{\phi}_B \partial_{\phi_B} = -\frac{\kappa^2}{d-1}W(\phi_B)\partial_A + W'(\phi_B)\partial_{\phi_B} \sim \delta_D + \cdots \tag{4.47}$$

Inserting the following expansions[4] for E and Ω in the first order equations (4.44),

$$\begin{aligned} &E = E_{(1)} + \cdots + \tilde{E}_{(d)} \log(e^{-2r}) + E_{(d)} + \cdots, \\ &\Omega = \Omega_{(0)} + \cdots + \tilde{\Omega}_{(2\Delta-d)} \log(e^{-2r}) + \Omega_{(2\Delta-d)} + \cdots, \end{aligned} \tag{4.48}$$

one determines all covariant counterterms which render all momenta finite to linear order in the sources. This procedure is substantially simpler than the general holographic renormalization required to determine the full non-linear counterterms and is a significant improvement over previous methods.

[4] These expansions are strictly correct for $d/2 < \Delta \leq d$, but we will deal with the special case $\Delta = d/2$ in the examples below.

A final simplification can be made for the case of backgrounds corresponding to deformations of the dual CFT. As we saw in the previous section, the 'superpotential' W of the background can be included in the counterterm action, corresponding to a supersymmetric renormalization scheme. After this counterterm is added to the on-shell action (we still have to determine the counterterms for E and Ω), the momenta take the simpler form

$$\begin{aligned}
\dot{e}^i_j &= E e^i_j, \\
\dot{f} &= 0, \\
\dot{\phi} &= \Omega\phi + \frac{1}{2\kappa^2}\frac{W'}{W}\Omega f, \\
\dot{S} &= -\frac{1}{\kappa^2}\left[\left(\frac{W'}{W}\right)^2\Omega - \frac{e^{-2A}}{W}\Box\right] f - 2\frac{W'}{W}\Omega\phi.
\end{aligned} \tag{4.49}$$

4.1.3 Examples

We will treat the two examples that have been the main testing ground for holographic computation of correlation functions, namely the GPPZ flow [59] and the Coulomb branch flow [60, 61]. The computation of certain two-point functions for the CB flow was first discussed in [60] and for the GPPZ flow in [62]. Two-point functions for both flows were systematically studied in [63, 31], see also [64, 65, 66] for earlier work. Since the results are known, the emphasis here will be in method rather than the correlators themselves. For a discussion of the physical properties of the correlators and comparison with the dual field theories we refer to [31].

GPPZ flow

The GPPZ flow describes a deformation by a supersymmetric mass term of $\mathcal{N}=4$ SYM. The bulk theory is that of a scalar field dual to an operator of dimension $\Delta = 3$ coupled to gravity in five dimensions. The background 'superpotential' is

$$W(\phi_B) = -\frac{3}{2\kappa^2}\left[1 + \cosh\left(\sqrt{\frac{2}{3}}\kappa\phi_B\right)\right] = -\frac{3}{\kappa^2} - \frac{1}{2}\phi_B^2 + \cdots \tag{4.50}$$

which is of the form W_+ and corresponds to a deformation of the boundary CFT by a relevant operator. The background solution takes the form

$$\phi_B = \frac{1}{\kappa}\sqrt{\frac{3}{2}}\log\left(\frac{1+\sqrt{1-u}}{1-\sqrt{1-u}}\right), \qquad e^{2A} = \frac{u}{1-u}, \qquad 1-u = e^{-2r}. \tag{4.51}$$

It is also useful to note the relations

$$W = -\frac{3}{\kappa^2}\frac{1}{u}, \qquad W' = -\frac{\sqrt{6}}{\kappa}\frac{\sqrt{1-u}}{u}, \qquad W'' = \frac{2\kappa^2}{3}W + 1. \tag{4.52}$$

Changing variable from r to u in (4.44) we obtain

$$2(1-u)E'(u)+E^2+\frac{4}{u}E-\frac{1-u}{u}p^2=0,$$
$$2(1-u)\Omega'(u)+\Omega^2+\left(\frac{4}{u}-2\right)\Omega-\frac{1-u}{u}p^2=0. \tag{4.53}$$

The solutions which are regular at $u=0$ are

$$E(u)=\frac{1}{4}p^2(1-u)\frac{F\left(1-\frac{ip}{2},1+\frac{ip}{2};3;u\right)}{F\left(-\frac{ip}{2},\frac{ip}{2};2;u\right)}, \tag{4.54}$$

and

$$\Omega(u)=\frac{1}{4}p^2(1-u)\frac{F\left(\frac{3-\alpha}{2},\frac{3+\alpha}{2};3;u\right)}{F\left(\frac{1-\alpha}{2},\frac{1+\alpha}{2};2;u\right)}, \tag{4.55}$$

where $\alpha=\sqrt{1-p^2}$.

Next we need to find covariant counterterms for E and Ω. Inserting

$$\partial_r=\delta_D+\frac{\kappa^2}{6}\phi_B^2\left(\partial_A-\frac{2}{3}\phi_B\partial_{\phi_B}\right)+\cdots \tag{4.56}$$

and the expansions (4.48) in (4.44) one very easily determines

$$E=\frac{p^2}{2}e^{-2A}+\frac{p^2}{4}e^{-2A}\left(\frac{p^2}{2}e^{-2A}+\frac{\kappa^2}{3}\phi_B^2\right)\log e^{-2r}+E_{(4)}+\cdots,$$
$$\Omega=-\frac{p^2}{2}e^{-2A}\log e^{-2r}+\Omega_{(2)}+\cdots. \tag{4.57}$$

Expanding the exact solution in $1-u$ and removing the covariant terms we have just determined allows for the evaluation of $E_{(4)}$ and $\Omega_{(2)}$, which are precisely the terms required to calculate the renormalized one-point functions. Putting everything together we find the following two-point functions:

$$\langle\mathcal{O}(p)\mathcal{O}(-p)\rangle=-\frac{1}{2}p^2\bar{J},\qquad \langle T_i^i(p)\mathcal{O}(-p)\rangle=\frac{\sqrt{6}}{2\kappa}p^2\bar{J},$$
$$\langle T_i^i(p)T_i^i(-p)\rangle=-\frac{3}{\kappa^2}p^2(\bar{J}+1),\qquad p^jp_i\langle T_j^i\rangle=0,$$
$$\langle T_{ij}(p)T_{kl}(-p)\rangle_{TT}=\frac{2}{\kappa^2}\Pi_{ijkl}[\frac{1}{16}p^2(p^2+4)\bar{K}+\frac{p^2}{8}]. \tag{4.58}$$

where

$$\bar{J}=2\psi(1)-\psi(\frac{3}{2}+\frac{1}{2}\sqrt{1-p^2})-\psi(\frac{3}{2}-\frac{1}{2}\sqrt{1-p^2}),$$
$$\bar{K}=\psi(1)+\psi(3)-\psi(2+\frac{ip}{2})-\psi(2-\frac{ip}{2}). \tag{4.59}$$

COULOMB BRANCH FLOW

The Coulomb branch flow is a solution of five dimensional AdS gravity coupled to a scalar field of mass $m^2 = -4$, which therefore saturates the BF bound. The solution describes the case where an operator of dimension 2 gets a vev. The superpotential is

$$W(\phi_B) = -\frac{2}{\kappa^2}\left[e^{-\kappa\phi_B/\sqrt{3}} + \frac{1}{2}e^{2\kappa\phi_B/\sqrt{3}}\right]. \tag{4.60}$$

The solution can be parametrized by $v \equiv e^{\sqrt{3}\kappa\phi_B}$ as

$$\dot{v} = 2v^{2/3}(1-v), \qquad e^{-2A} = v^{-2/3}(1-v), \qquad W = -\frac{1}{\kappa^2}v^{-1/3}(v+2). \tag{4.61}$$

The boundary is located at $v = 1$. In terms of v the first order equations (4.44) become

$$2(1-v)E'(v) + v^{-2/3}E^2 + \frac{4}{3}\left(1+\frac{2}{v}\right)E - p^2v^{-4/3}(1-v) = 0, \tag{4.62}$$

and

$$2(1-v)\Omega'(v) + v^{-2/3}\Omega^2 + \left[\frac{4}{3}\left(1+\frac{2}{v}\right) - \frac{12}{v+2}\right]\Omega - p^2v^{-4/3}(1-v) = 0. \tag{4.63}$$

The solution for E which is regular at $v = 0$ is

$$E(v) = 2a(1-v)v^{-1/3}\left[1 + \frac{av}{2(a+1)}\frac{F(a+1,a+1;2a+3;v)}{F(a,a;2a+2;v)}\right], \tag{4.64}$$

where $a = -\frac{1}{2} + \frac{1}{2}\sqrt{1+p^2}$. We will not give here explicitly the exact solution for Ω since it is rather complicated. To obtain such a closed form solution one must transform the above equation for Ω into a soluble form and then obtain Ω implicitly through the solution of the transformed equation. After obtaining covariant counterterms for E and Ω by the method we described above, we can write these in the desired form, namely

$$E = \frac{p^2}{2}e^{-2A} + \frac{p^4}{8}e^{-4A}\log e^{-2r} + \frac{p^2}{2}e^{-4A}\left[-\frac{1}{3} + \frac{p^2}{2}\left(\psi(a+1) - \psi(1)\right)\right] + \cdots, \tag{4.65}$$

$$\Omega = \frac{1}{r} + \frac{1}{r^2}\left(-\frac{4}{3p^2} + \psi(a+1) - \psi(1)\right) + \cdots. \tag{4.66}$$

Inserting these expansions into the expressions for the momenta, after taking into account the effect of the counterterm

$$U(\varphi) = -\frac{3}{\kappa^2} + \frac{1}{2}\left(\frac{1}{r} - \frac{d}{2}\right)\varphi^2 = W(\varphi) + \frac{1}{2r}\varphi^2 + \cdots, \tag{4.67}$$

we obtain the two-point functions

$$\langle\mathcal{O}(p)\mathcal{O}(-p)\rangle = \left(4\psi(1) - 4\psi(1+a) + \frac{16}{3p^2}\right), \quad \langle T_i^i(p)T_j^j(-p)\rangle = 0,$$
$$p^ip^j\langle T_{ij}(p)\mathcal{O}(-p)\rangle = -\frac{2}{\sqrt{3}\kappa}p^2, = \langle\mathcal{O}\rangle_B p^2, \quad \langle T_i^i(p)\mathcal{O}(-p)\rangle = -\frac{4}{\sqrt{3}\kappa} = 2\langle\mathcal{O}\rangle_B,$$
$$\langle T_{ij}(p)T_{kl}(-p)\rangle_{TT} = \Pi_{ijkl}\frac{p^2}{2\kappa^2}\left[\frac{1}{3} - \frac{p^2}{2}\left(\psi(a+1) - \psi(1)\right)\right]. \tag{4.68}$$

4.2 AdS-sliced Domain Walls

AdS-sliced domain walls have also been studied in the literature [67, 68, 69, 52, 53]. In this case the background is of the form

$$ds_B^2 = dr^2 + e^{2A(r)}g_{ij}(x)dx^idx^i, \qquad \varphi = \phi_B(r), \tag{4.69}$$

where $g_{ij}(x)$ is the metric of Euclidean AdS_d with radius l and we have set the radius of the bulk AdS_{d+1} equal to 1[5]. Inserting this ansatz into the bulk equations of motion leads to the following equations for $A(r)$ and $\phi_B(r)$

$$\dot{A}^2 - \frac{\kappa^2}{d(d-1)}\left(\dot{\phi}_B^2 - 2V(\phi_B)\right) + \frac{1}{l^2}e^{-2A} = 0,$$
$$\ddot{A} + d\dot{A}^2 + \frac{2\kappa^2}{d-1}V(\phi_B) + \frac{d-1}{l^2}e^{-2A} = 0,$$
$$\ddot{\phi}_B + d\dot{A}\dot{\phi}_B - V'(\phi_B) = 0. \tag{4.70}$$

Note that as $l \to \infty$ these reduce to the equations for flat domain walls, as they should. From now on we set $l^2 = 1$.

4.2.1 Janus solution

A particularly interesting AdS-sliced domain wall solution is the dilaton domain wall solution of type IIB supergravity of [52][6]. This is a non-supersymmetric regular solution. When reduced to five dimensions, it solves the field equations of AdS gravity coupled to a massless scalar with a constant potential. Similar solutions exist in all dimensions [53]. These solutions are of particular interest because they enjoy non-perturbative stability for a broad class of deformations [53]. This strongly suggests that they should have a well-defined QFT dual.

[5]In [53] a different convention was used: the radius of bulk AdS_{d+1} and of the AdS_d-slice were set equal to each other.

[6]Additional dilatonic deformations have been presented in [70].

Setting $V = -\frac{d(d-1)}{2\kappa^2}$, the equation for the scalar field can be trivially integrated to give

$$\dot{\phi}_B = ce^{-dA}, \tag{4.71}$$

where c is an arbitrary constant of integration. The remaining equations imply,

$$\dot{A}^2 = 1 - e^{-2A} + be^{-2dA} \tag{4.72}$$

where $b = \frac{c^2\kappa^2}{d(d-1)}$. The geometry is non-singular provided the parameter b is within the range,

$$0 \leq b < b_0 \equiv \frac{1}{d}\left(\frac{d-1}{d}\right)^{d-1}. \tag{4.73}$$

One can obtain an implicit solution of (4.72) as

$$r = \int_{A_0}^{A} \frac{dA}{\sqrt{1 - e^{-2A} + be^{-2dA}}} \tag{4.74}$$

where A_0 is the smallest zero of $P(u) \equiv bu^d - u + 1$, where $u \equiv e^{-2A}$. This defines half of the geometry, i.e. the region with $0 \leq r < \infty$. The other half is obtained by extending $A(r)$ to negative values of r as an even function, $A(-r) = A(r)$.

We can obtain an explicit expression for the bulk metric by changing variables from r to u. Using

$$\dot{A}^2 = 1 - u + bu^d, \tag{4.75}$$

we obtain[7]

$$ds_B^2 = \frac{du^2}{4u^2(1 - u + bu^d)} + \frac{1}{u} g_{ij}(x) dx^i dx^i. \tag{4.76}$$

Note that if $b = 0$ this is precisely the metric for AdS_{d+1} in the AdS_d-slicing parameterization. The range of the u-coordinate depends on the value of the parameter b, namely $0 \leq u \leq u_o$, where $u_o \geq 1$ with equality iff $b = 0$. We give the explicit form of u_o as a function of b in Appendix 4.A.1. In this parameterization the two halves of the space, i.e. $r > 0$ and $r < 0$, are not distinguished since u is an even function of r. In particular, the regions at $r \to \pm\infty$ are mapped to $u = 0$.

We discuss in Appendix 4.A.1 the conformal compactification of the solution. The conformal boundary consists of two half-spheres with angular excess joined along their equator [52, 53]. We will refer to the joining equator as 'corner'. In order to calculate correlation functions of the dual field theory we need to write the Janus metric in the Fefferman-Graham (FG) form. Provided the boundary metric is smooth, this is always possible in a neighborhood of the boundary but the FG radial coordinate may in general not be valid far away from the boundary. In the present case, the boundary metric is smooth except for the presence of corners. We therefore

[7] An equivalent form of this metric with A instead of u as a variable was found by C. Núñez (unpublished notes, July 2003).

except to be able to find a FG coordinates that are well defined in the neighborhood of the boundary except perhaps at the corner.

In Appendix 4.A.2 we construct the FG metric to all orders in b for the Janus geometry and determine the range of validity of the radial coordinate. We find that the FG coordinates are well-defined everywhere in a neighborhood of the boundary except on the corner where the two half-spheres of the boundary meet. In particular, the FG metric takes the form

$$ds_B^2 = \frac{1}{z_o^2}\left[dz_o^2 + (1 + bc_3(x) + \mathcal{O}(b^2))dz_d^2 + (1 + bc_4(x) + \mathcal{O}(b^2))dz_a^2\right], \qquad (4.77)$$

where $x \equiv z_d/z_o$ and z_a, $a = 1, \cdots, d-1$ are the standard transverse coordinates in the upper half plane parameterization of the AdS_d slice. The location of the corner is at $z_d = 0$. The functions $c_3(x)$ and $c_4(x)$ as well as the form of the FG metric to all orders in b are given in Appendix 4.A.2. As discussed there this coordinate system covers the region $|x| > x_o = b/\sqrt{2} + \mathcal{O}(b^2)$, so $z_d = 0$ only when $z_o = 0$. In other words, this coordinate system does not cover a (radially extended) neighborhood of $z_d = 0$.

In this coordinate system the background scalar takes the form

$$\phi_B(x) = \phi_o + cc_5(x) + \mathcal{O}(c^3), \qquad (4.78)$$

where again $c_5(x)$ is given in the appendix. It is significant to point out here that on the boundary, i.e. $z_o = 0$, the value of the scalar field is a step function in z_d, namely

$$\phi_B(z_d) = \phi_o + \text{sgn}(z_d)c, \qquad (4.79)$$

which implies that the coupling of the dual operator is different on the two sides of the corner, or 'wall', at $z_d = 0$. These results are sufficient for calculating correlation functions, which we do in the next section.

4.2.2 VEVs

Now that we have determined the appropriate FG coordinate system we can carry out the algorithm we described in Chapter 3 and evaluate the vevs of the stress tensor and the scalar operator dual to the dilaton, as well as, all two-point functions using perturbation theory in c. The first step is to define the radial coordinate[8] $r = -\log z_o$ which is used as the 'time' coordinate in the Hamiltonian formalism. Due to the fact that the background depends also on the transverse space coordinates, a full asymptotic analysis is required to determine the covariant counterterms. We will not give these here but they are easily determined following the procedure described in

[8]This radial coordinate is different from the original radial coordinate in (4.74) but we hope this causes no confusion.

the previous chapter. Evaluating these counterterms on the background using the following expressions for the non-vanishing components of the Christoffel symbol and Ricci tensor:

$$\begin{aligned}
&\Gamma^d_{dd} = \tfrac{b}{2}e^r c_3'(x) + \mathcal{O}(b^2), && \Gamma^a_{bd} = \tfrac{b}{2}e^r c_4'(x)\delta^a_b + \mathcal{O}(b^2),\\
&R_{dd} = -\tfrac{(d-1)b}{2}e^{2r}c_4''(x) + \mathcal{O}(b^2), && R_{ad} = R_{ab} = \mathcal{O}(b^2),\\
&R = -\tfrac{(d-1)b}{2}c_4''(x) + \mathcal{O}(b^2),
\end{aligned} \tag{4.80}$$

and adding them to the canonical momenta obtained directly by differentiating the background fields w.r.t. r one obtains the following expressions for the vevs of the scalar operator and the stress tensor:

$$\langle\mathcal{O}\rangle_B = c\frac{z_d}{|z_d|^{d+1}}\,, \qquad \langle T^i_j\rangle_B = 0. \tag{4.81}$$

Although the calculation has been done to leading order in c, it is not difficult to show that these results are in fact exact. The reason is that the coordinate transformation (4.110) ensures that for every power of b there is a factor of $z_o^{2(d-1)}$ which means that higher order in b terms are subleading and do not survive when the regulator is removed. This can also be seen from the exact expressions for the Fefferman-Graham metric and scalar background given in Appendix 4.A.2, which can be used to obtain the exact canonical momenta. Namely,

$$K^d_{Bd} = \left(1 + \frac{bu^d}{1-u}\right)^{1/2}, \qquad K^a_{Bb} = \left(u + \sqrt{(1-u)(1-u+bu^d)}\right)\delta^a_b, \tag{4.82}$$

and

$$\dot\phi_B(x) = \mathrm{sgn}(x)cu^{d/2}\sqrt{1-u}. \tag{4.83}$$

One immediately sees that the vevs given above are in fact exact, as claimed.

The form of the vacuum expectation values is the one required by the symmetries of the problem. As shown in [71], the one-point functions for a conformal field theory on a flat space with a boundary at $z_d = 0$ (which breaks the conformal group from $O(1, d+1)$ to $O(1, d)$) are precisely of the form (4.81). In the present case we consider the theory on both sides of the wall $z_d = 0$. The McAvity-Osborn result applies separately to the two regions, $z_d > 0$ and $z_d < 0$, and it gives

$$\langle\mathcal{O}\rangle_B = \frac{c_1}{|z_d|^d}, \quad z_d > 0, \qquad \langle\mathcal{O}\rangle_B = \frac{c_2}{|z_d|^d}, \quad z_d < 0. \tag{4.84}$$

In the present case $c_1 = -c_2 = c$.

These considerations suggest [52] that the dual field theory for $d = 4$ is $\mathcal{N} = 4$ SYM possibly coupled to non-supersymmetric conformal matter localized at $z_d = 0$

and with g_{YM} being different on the two sides of the wall (similar suggestions can be formulated in all dimensions). This is consistent with the symmetries of the model: the presence of the defect breaks the symmetries to $O(1,4)$ (i.e. the (Euclidean) conformal group in three dimensions). It would be interesting to investigate whether there is a classical solution of $\mathcal{N}=4$ SYM coupled to such defect that can reproduce (4.81), but we will not pursue this here.[9]

4.2.3 Two-point functions

Since the leading correction to the AdS_{d+1} metric is order c^2, while the leading corrections to the (off diagonal) two-point functions are order c, we can take the background to be exactly AdS and consider linear fluctuations driven by a source $\tilde{T}^i_j$ which is of order c. Decomposing the metric fluctuations as was done for flat domain walls above we derive the following equations for the irreducible components:

$$\begin{aligned}
-\Box_g e^i_j &= 2\kappa^2 \Pi^i{}_k{}^l{}_j \tilde{T}^k_l, \\
\dot{\epsilon}_j &= 2\kappa^2 \frac{\pi^k_j}{\Box} \tilde{T}_{kd+1}, \\
\dot{f} &= -2\kappa^2 \frac{\partial^k}{\Box} \tilde{T}_{kd+1}, \\
\dot{S} &= \frac{1}{d-1}\left(2\kappa^2 \tilde{T}_{d+1d+1} - e^{-2r}\Box f\right), \\
-\Box_g \phi &= \frac{1}{2}\dot{\phi}_B \dot{S} - e^{-2r}\left(S^i_j \partial_i \partial^j \phi_B + \partial_i S^i_j \partial^j \phi_B - \frac{1}{2}\partial_j S \partial^j \phi_B\right). \qquad (4.85)
\end{aligned}$$

Only the first and the last equations need further analysis as the rest give immediately the momenta as functions of the linear sources. The responses for both the transverse traceless metric fluctuation and the scalar field fluctuation can be obtained using the massless scalar bulk-to-bulk propagator

$$G(\xi) = \frac{c_d}{2^d d}\xi^d F\left(d, \frac{d+1}{2}; \frac{d}{2}+1; \xi^2\right) \qquad (4.86)$$

which satisfies

$$-\Box_g G(\xi) = \delta(z,w) = \frac{1}{\sqrt{g}}\delta(z-w). \qquad (4.87)$$

Here $c_d = \Gamma(d)/(\Gamma(d/2)\pi^{d/2})$ and $\xi = 2z_o w_o/(z_o^2 + w_o^2 + (\vec{z}-\vec{w})^2)$. As $z_o \to 0$

$$G(\xi) \sim \frac{z_o^d}{d} K_d(w, \vec{z}) \qquad (4.88)$$

[9] A precise proposal for the dual theory was made in [72]: they consider $\mathcal{N}=4$ SYM theory on two half-spaces separated by a planar interface that contains no matter and with a different coupling constant coupled to specific operator closely related to the $\mathcal{N}=4$ Lagrangian density. The field theory computations in [72] exactly agree (to the extend that they can be compared) with the holographic computations described in this and next subsection.

where

$$K_d(w,\vec{z}) = c_d\left(\frac{w_o}{w_o^2+(\vec{w}-\vec{z})^2}\right)^d \tag{4.89}$$

is the well-known bulk-to-boundary propagator. To complete the calculation then we need the source $\tilde{T}_{\mu\nu}$ which is

$$\begin{aligned}\tilde{T}_{\mu\nu} &= \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi \\ &= \partial_\mu\phi_B\partial_\nu\phi + \partial_\nu\phi_B\partial_\mu\phi - g^{AdS}_{\mu\nu}\dot{\phi}_B\left(\dot{\phi}+\frac{e^{-2r}}{z_d}\partial_{z_d}\phi\right) + \mathcal{O}(b). \end{aligned} \tag{4.90}$$

Here we have used the fact that the background scalar is a function of $x = z_d/z_o$ which implies $z_d\partial_{z_d}\phi_B = \dot{\phi}_B$. With a little more algebra the source can be cast in the form

$$\begin{aligned}&\tilde{T}_{d+1d+1} = \dot{\phi}_B\left(\dot{\phi}-\frac{e^{-2r}}{z_d}\partial_{z_d}\phi\right) + \mathcal{O}(b), \\ &\tilde{T}_{jd+1} = \dot{\phi}_B\left(\partial_j\phi - \frac{1}{z_d}\delta_{jd}\dot{\phi}\right) + \mathcal{O}(b), \\ &\tilde{T}_{ij} = \dot{\phi}_B\left[\frac{1}{z_d}\left(\delta_{id}\partial_j\phi + \delta_{jd}\partial_i\phi - \delta_{ij}\partial_{z_d}\phi\right) - e^{-2r}\dot{\phi}\delta_{ij}\right] + \mathcal{O}(b). \end{aligned} \tag{4.91}$$

Using these sources and the above bulk-to-bulk propagator we can now evaluate the canonical momenta which give the one-point functions with linear sources. It is not difficult to show that no counterterms contribute to the order c terms of the momenta. It turns out to be easier to obtain the two-point functions from the canonical momentum of the graviton by differentiating w.r.t. the scalar source rather than from the scalar momentum and so we only consider the graviton momentum here. Of course both calculations should give identical results and we show this explicitly in Appendix 4.A.3, where we calculate the two-point functions from the scalar momentum.

To obtain the canonical momentum of the transverse traceless component of the graviton we note that the inhomogeneous solution to its equation of motion is

$$e^i_j = 2\kappa^2\int d^{d+1}w\sqrt{g(w)}G(\xi)\Pi^i{}_k{}^l{}_j\tilde{T}^k_l(w), \tag{4.92}$$

so that asymptotically

$$\dot{e}^i_j \sim -2\kappa^2 e^{-dr}\int d^{d+1}w\sqrt{g(w)}K_d(w,\vec{z})\Pi^i{}_k{}^l{}_j\tilde{T}^k_l(w). \tag{4.93}$$

Substituting the above expressions for the source into the canonical momenta and differentiating w.r.t. the scalar source we arrive at the two-point functions

$$\Pi^i{}_k{}^l{}_j \langle T^k_l(\vec{z})\mathcal{O}(\vec{w})\rangle = -2(d+1)c\Pi^i{}_d{}^d{}_j I(\vec{z},\vec{w}), \tag{4.94}$$

$$\pi^l{}_j \frac{\partial_k}{\Box}\langle T^k_l(\vec{z})\mathcal{O}(\vec{w})\rangle = \frac{\pi^k{}_j}{\Box}\left(\langle\mathcal{O}(\vec{z})\rangle_B \partial_k \delta^{(d)}(\vec{z}-\vec{w})\right), \tag{4.95}$$

$$\pi^l{}_k \langle T^k_l(\vec{z})\mathcal{O}(\vec{w})\rangle = -\frac{\partial^k}{\Box}\left(\langle\mathcal{O}(\vec{z})\rangle_B \partial_k \delta^{(d)}(\vec{z}-\vec{w})\right), \tag{4.96}$$

$$\langle T^i_i(\vec{z})\mathcal{O}(\vec{w})\rangle = 0, \tag{4.97}$$

where the projection operators are acting on $\vec{z}$ and

$$I(\vec{z},\vec{w}) = c_d^2 \int d^d x dx_o x_o^{2d+1} \frac{x_d}{(x_o^2+x_d^2)^{(d+3)/2}} \frac{1}{[x_o^2+(\vec{x}-\vec{z})^2]^d} \frac{1}{[x_o^2+(\vec{x}-\vec{w})^2]^d}. \tag{4.98}$$

The last three correlators can be easily seen as a consequence of the Ward identities (2.165) and (2.168), which is a non-trivial consistency check of our calculation. In fact, since the vevs are exact, these two-point functions must also be exact in c, although our calculation of the two-point functions was done only to leading order in c.

The computation of the remaining of the two-point functions $\langle T_{ij}(x)T_{kl}(y)\rangle$ and $\langle\mathcal{O}(x)\mathcal{O}(y)\rangle$ requires an analysis to order c^2. This computation is rather complex since the background metric receives a correction at this order. This means that we need to linearize the bulk field equations around the corrected solution (4.118). The latter, however, is inhomogeneous and this complicates the analysis. Nevertheless, the conformal invariance of the boundary theory completely determines the two-point function of the scalar operator, while the two-point function of the stress tensor is determined up to a scalar function (except for $d=2$ where it is fully determined) [71]. It would be interesting to check that the holographic calculation reproduces these two-point functions as well.

4.2.4 JANUS TWO-POINT FUNCTIONS VS BOUNDARY CFT

Let us now take a closer look at the structure of the Janus two-point functions. We would like to show that the two-point functions are of the form required by conformal invariance for a CFT on a space with a wall at $z_d = 0$.[10] The subgroup of the conformal group $O(1,d+1)$ that leaves $z_d = 0$ invariant is $O(1,d)$. This is precisely the isometry group of the Janus metric. McAvity and Osborn [71] have

[10]We are grateful to the authors of [72] for pointing us to the work of McAvity and Osborn and for prompting us to check that our calculation is consistent with their results.

given explicitly the form of this two-point function in such a CFT. It is given by

$$\langle T^i_j(\vec{z})\mathcal{O}(\vec{w})\rangle = -\text{sgn}(z_d)c\frac{2^{d-1}d^2\Gamma\left(\frac{d}{2}\right)}{(d-1)\pi^{d/2}}\left(\frac{v}{\vec{s}^2}\right)^d\left(X^iX_j - \frac{1}{d}\delta^i_j\right), \tag{4.99}$$

where[11]

$$v^2 = \frac{\xi}{\xi+1}, \qquad \xi = \frac{\vec{s}^2}{4z_d w_d}, \qquad \vec{s} = \vec{z} - \vec{w}, \tag{4.100}$$

and

$$X_i = z_d\frac{v}{\xi}\partial_i\xi = v\left(\frac{2z_d}{\vec{s}^2}s_i - n_i\right), \tag{4.101}$$

with $n_i = \delta_{id}$. The normalization of the two-point function is fixed by the normalization of the vev of the scalar operator [71]. It is clear from (4.84) that this normalization has opposite signs for $z_d > 0$ and $z_d < 0$, which is the origin of the $\text{sgn}(z_d)$ factor. This expression applies for $z_d w_d > 0$, i.e. both points on the same side of the wall, but not for $z_d w_d < 0$. The holographic expression (4.153) applies to both cases, however. Under conformal transformations that leave the hyperplane $z_d = 0$ invariant we have

$$\vec{s}^2 \to \frac{\vec{s}^2}{\Omega(\vec{z})\Omega(\vec{w})}, \qquad z_d \to \frac{z_d}{\Omega(\vec{z})}, \qquad w_d \to \frac{w_d}{\Omega(\vec{w})}. \tag{4.102}$$

It follows that ξ is a conformal invariant while X_i transforms as a vector. In particular, under inversion $\vec{z} \to \vec{z}/\vec{z}^2$, $\vec{w} \to \vec{w}/\vec{w}^2$,

$$X_i \to I_{ij}(\vec{z})X_j, \tag{4.103}$$

where $I_{ij}(\vec{z}) = \delta_{ij} - 2\frac{z_i z_j}{\vec{z}^2}$. It is easy then to see that the two-point function given above transforms correctly under inversion, namely

$$\langle T^i_j(\vec{z}')\mathcal{O}(\vec{w}')\rangle = \vec{z}^{2d}\vec{w}^{2d}I^i_k(\vec{z})I^l_j(\vec{z})\langle T^k_l(\vec{z})\mathcal{O}(\vec{w})\rangle. \tag{4.104}$$

One can show in general, using the fact that the background has the correct isometries, that the holographic two-point functions transform as they should. Since the results of [71] follow from the same symmetries, this argument shows that our results are consistent with that of [71]. It is, however, a rather non-trivial exercise to explicitly demonstrate that the correlator is of the form given in [71], mainly because of the integral representation of the transverse-traceless part of the correlator. The integral that appears in the transverse traceless part of the holographic two-point function is not easy to evaluate in general, and evaluating the projection operator acting on it is not straightforward either. This makes a direct comparison

[11] We use ξ here to conform with the notation of [71]. This should not be confused with the argument of the bulk-to-bulk propagator used earlier.

of the two results rather non-trivial. Instead, we will expand both results in a short distance expansion and compare them term by term. We do this for the first three orders in the expansion and we find complete agreement.

To facilitate the comparison we first expand the above result of McAvity and Osborn. Of course, this expansion is valid only when $z_d w_d > 0$, which is also the condition for the validity of the McAvity-Osborn expression. After some algebra we get

$$\begin{aligned}\langle T^i_j(\vec{z})\mathcal{O}(\vec{w})\rangle \;=\;& -c\frac{2^{d-1}d^2\Gamma\left(\frac{d}{2}\right)}{(d-1)\pi^{d/2}}\frac{2w_d}{|2w_d|^{d+1}}\frac{1}{(\vec{s}^2)^{d/2}}\left\{\frac{s^is_j}{\vec{s}^2}-\frac{1}{d}\delta^i_j\right.\\ &-\frac{1}{2w_d}\left(n^is_j+n_js^i-\vec{n}\cdot\vec{s}\delta^i_j+(d-2)\vec{n}\cdot\vec{s}\frac{s^is_j}{\vec{s}^2}\right)\\ &+\frac{1}{(2w_d)^2}\left[\frac{1}{2}\left(d(d-2)(\vec{n}\cdot\vec{s})^2-(d+2)\vec{s}^2\right)\frac{s^is_j}{\vec{s}^2}+\vec{s}^2n^in_j\right.\\ &\left.\left.+2d(\vec{n}\cdot\vec{s})n^{(i}s_{j)}-\frac{1}{2}\left((d+2)(\vec{n}\cdot\vec{s})^2-\vec{s}^2\right)\delta^i_j\right]+\mathcal{O}(s^3)\right\}.\end{aligned} \tag{4.105}$$

The holographic result can be easily evaluated to this order too. First, using

$$\delta^{(d)}(\vec{s}) = -\frac{\Gamma\left(\frac{d}{2}\right)}{2(d-2)\pi^{d/2}}\Box\frac{1}{(\vec{s}^2)^{\frac{d-2}{2}}}, \tag{4.106}$$

we find that the longitudinal part of the holographic two-point function reproduces precisely the first two orders of the short distance expansion of the McAvity and Osborn result. The transverse traceless part is then evaluated by acting with the projection operator on

$$I(\vec{z},\vec{w}) = \frac{\Gamma\left(\frac{d}{2}-1\right)d^2}{8(d+1)\pi^{d/2}}\frac{w_d}{|w_d|^{d+3}}\frac{1}{(\vec{s}^2)^{\frac{d-2}{2}}}\left(1+\mathcal{O}(s)\right), \tag{4.107}$$

and it reproduces exactly the third order term. Some details of this calculation are presented in Appendix 4.A.4. Therefore, at least to this order in the short distance expansion, we have shown that the holographic two-point function is exactly what one expects for a CFT with a wall at $z_d = 0$.

4.A APPENDIX

4.A.1 CONFORMAL COMPACTIFICATION OF THE JANUS SOLUTION

In order to determine the conformal compactification of the Janus geometry we introduce a new radial coordinate

$$z = \dot{A} = \pm\sqrt{1-u+bu^d}. \tag{4.108}$$

This coordinate has the range $-1 \leq z \leq 1$ for *any* value of b and the $u = 0$ region is mapped to $z = \pm 1$. u can be determined as a function of z by solving the algebraic equation

$$u - bu^d = 1 - z^2 \tag{4.109}$$

as a power series in b. The relevant solution is the smallest real positive root which is given for arbitrary d and to all orders in b by

$$\begin{aligned} u(z; b, d) &= (1 - z^2) \sum_{n=0}^{\infty} \frac{\Gamma(nd + 1)}{\Gamma(n + 1)\Gamma(n(d - 1) + 2)} b^n (1 - z^2)^{n(d-1)} \\ &= {}_{(d-1)}F_{(d-2)} \left[\left(\frac{1}{d}, \frac{2}{d}, \cdots, \frac{d-1}{d} \right), \left(\frac{2}{d-1}, \frac{3}{d-1}, \cdots, \frac{d-2}{d-1}, \frac{d}{d-1} \right), \right. \\ &\qquad \left. \frac{d^d}{(d-1)^{d-1}} b (1 - z^2)^{d-1} \right], \end{aligned} \tag{4.110}$$

where ${}_pF_q$ is the generalized hypergeometric function. Note that the bound of the u coordinate mentioned above is just $u_o = u(0; b, d)$. It is not possible to express $u(z)$ in terms of elementary functions except for the cases $d = 2$ and $d = 3$. We have respectively,

$$\begin{aligned} u(z; b, 2) &= \frac{1}{2b} \left(1 - \sqrt{1 - 4b(1 - z^2)} \right), \\ u(z; b, 3) &= \frac{2}{\sqrt{3b}} \sin \left[\frac{1}{3} \arcsin \left(\frac{3}{2} \sqrt{3b} (1 - z^2) \right) \right]. \end{aligned} \tag{4.111}$$

If we now write the (Euclidean) AdS_d-slice metric in global coordinates and set $z = \sin\theta$, the metric (4.76) becomes

$$\begin{aligned} ds_B^2 = & \frac{1}{u(\sin\theta)\cos^2\lambda} \times \\ & \left[d\lambda^2 + \cos^2\lambda \left(1 + (2d - 1) b (\cos^2\theta)^{d-1} + \mathcal{O}(b^2) \right) d\theta^2 + d\tau^2 + \sin^2\lambda d\Omega_{d-2}^2 \right], \end{aligned} \tag{4.112}$$

where $0 \leq \lambda \leq \pi/2$ and $-\pi/2 \leq \theta \leq \pi/2$. A few comments are in order here. First, the transformation (4.110) implies that every power of b in the coefficient of $d\theta^2$ comes with a factor of $(\cos^2\theta)^{d-1}$ and hence the metric inside the square brackets is non-singular for any θ. Second, note that the (λ, θ) part of the metric can be transformed into the standard metric on S^2 by introducing the angular coordinate

$$\begin{aligned} \mu &= \int_0^{\sin\theta} \frac{dz}{\sqrt{u(z)}\,(1 - bdu(z)^{d-1})} \\ &= \theta + \left(d - \frac{1}{2} \right) b \sin\theta F \left(\frac{1}{2}, \frac{3}{2} - d; \frac{3}{2}; \sin^2\theta \right) + \mathcal{O}(b^2). \end{aligned} \tag{4.113}$$

This is precisely the angular coordinate introduced in [52, 53] and it takes values in $[-\mu_o, \mu_o]$, where $\mu_o \geq \pi/2$ is given in equation (B.8) of [53]. Because of the excess angle the compact metric has a corner at $\lambda = \pi/2$, as is discussed in [53].

4.A.2 FEFFERMAN-GRAHAM COORDINATES FOR JANUS METRIC

To construct the Fefferman-Graham metric we start with (4.76) and the coordinate transformation (4.110) and write the AdS_d-slice metric in the upper-half plane coordinates. Then

$$\begin{aligned} ds_B^2 &= \frac{dz^2}{(1-z^2)^2}\left[1+2(d-1)b(1-z^2)^{d-1}+\mathcal{O}(b^2)\right] \\ &\quad +\frac{1}{1-z^2}\left[1-b(1-z^2)^{d-1}+\mathcal{O}(b^2)\right]\frac{1}{\tilde{z}_o^2}(d\tilde{z}_o^2+dz_a^2), \end{aligned} \tag{4.114}$$

where $a=1,\cdots,d-1$. For $b=0$ the coordinate transformation

$$z=\frac{z_d}{\sqrt{z_o^2+z_d^2}}, \qquad \tilde{z}_o=\sqrt{z_o^2+z_d^2}, \tag{4.115}$$

brings this metric into the upper half plane metric with radial coordinate z_o. To determine the Fefferman-Graham form of the Janus metric we need to obtain appropriate b-dependent corrections to this transformation. We can determine these as a Taylor series in b by introducing two arbitrary functions at each order in b and solving the differential equations that result by requiring that the transformed metric is of the Fefferman-Graham form. The unique transformation which ensures that the metric remains asymptotically AdS independent of b is to linear order in b

$$z=\frac{z_d}{\sqrt{z_o^2+z_d^2}}+bf_1(x)+\mathcal{O}(b^2), \qquad \tilde{z}_o=\sqrt{z_o^2+z_d^2}+bz_of_2(x)+\mathcal{O}(b^2), \tag{4.116}$$

where $x\equiv z_d/z_o$ and

$$\begin{aligned} f_1(x) &= \frac{x}{2(1+x^2)^{3/2}}\left[\left(1-\frac{1}{2d}\right)\frac{1}{x^{2d}}F\left(d,d;d+1;-\frac{1}{x^2}\right)\right. \\ &\quad \left.+\frac{1}{2(d+1)x^{2(d+1)}}F\left(d+1,d;d+2;-\frac{1}{x^2}\right)+\frac{1}{(1+x^2)^{d-1}}\right], \\ f_2(x) &= \frac{1}{2\sqrt{1+x^2}}\left[\frac{1+2dx^2}{2dx^{2d}}F\left(d,d;d+1;-\frac{1}{x^2}\right)\right. \\ &\quad \left.+\frac{1}{2(d+1)x^{2d}}F\left(d+1,d;d+2;-\frac{1}{x^2}\right)-\frac{1}{(1+x^2)^{d-1}}\right]. \end{aligned} \tag{4.117}$$

The metric then takes the from

$$ds_B^2=\frac{1}{z_o^2}\left[dz_o^2+(1+bc_3(x)+\mathcal{O}(b^2))dz_d^2+(1+bc_4(x)+\mathcal{O}(b^2))dz_a^2\right], \tag{4.118}$$

where

$$\begin{aligned} c_3(x) &= \frac{(2d-1)}{2dx^{2d}}F(d,d;d+1;-\frac{1}{x^2}) + \frac{1}{2(d+1)x^{2(d+1)}}F(d+1,d;d+2;-\frac{1}{x^2}) \\ &\quad -\frac{1}{x^2(1+x^2)^{d-1}}, \\ c_4(x) &= -\frac{1}{2dx^{2d}}F(d,d;d+1;-\frac{1}{x^2}). \end{aligned} \tag{4.119}$$

Note that the derivatives of these functions have a much simpler form:

$$\begin{aligned} c_3'(x) &= \frac{1}{x^3(1+x^2)^d}\left[-1-(2d-1)x^2+2\frac{1+(d+2)x^2}{(1+x^2)^2}\right], \\ c_4'(x) &= \frac{1}{x(1+x^2)^d}. \end{aligned} \tag{4.120}$$

The metric in (4.118) is manifestly invariant under translations and rotations of the z_a coordinates and scale transformations (the x coordinate is invariant under scale transformations). The original metric (4.76) however was invariant under the larger group $O(1,d)$ associated with the AdS slice metric. We now show that the metric (4.118) is also invariant under a discrete inversion isometry to order b which enhances the isometry group to the full $O(1,d)$. Actually, we will see that the inversion symmetry can be used to obtain the Fefferman-Graham form of the metric to all orders in b.

Let us write the AdS slice metric in (4.76) in the upper half plane coordinates so that

$$ds_B^2 = \frac{du^2}{4u^2(1-u+bu^d)} + \frac{1}{u\tilde{z}_o^2}(d\tilde{z}_o^2 + dz^a dz^a). \tag{4.121}$$

This form is invariant under the discrete isometry

$$\tilde{z}_o \to \frac{\tilde{z}_o}{\tilde{z}_o^2 + z_a^2}, \qquad z_a \to \frac{z_a}{\tilde{z}_o^2 + z_a^2}. \tag{4.122}$$

We now bring this metric into the Fefferman-Graham form by means of a coordinate transformation[12]

$$z = s(x;b), \qquad \tilde{z}_o = z_o t(x;b), \tag{4.123}$$

where $x = z_d/z_o$. We point out that this is precisely the form of the coordinate transformation (4.116), but we now treat the b-dependence non-perturbatively. This allows us to express the above discrete isometry in terms of the Fefferman-Graham coordinates $z^\mu = (z_o, z_a, z_d)$. We find

$$z^\mu \to \frac{z^\mu}{z_o^2 t(x;b)^2 + z_a^2}. \tag{4.124}$$

[12]Note that $u = u(z;b)$ is given in eq. (4.110).

Now, the Fefferman-Graham metric (4.118) takes the form

$$ds_B^2 = \frac{1}{z_o^2}\left[dz_o^2 + \lambda(x;b)dz_d^2 + \mu(x;b)dz_a^2\right]. \tag{4.125}$$

The requirement that this is invariant under inversion uniquely determines the functions $\lambda(x;b)$ and $\mu(x;b)$ in terms of $t(x;b)$. Namely we find the exact FG metric

$$\boxed{ds_B^2 = \frac{1}{z_o^2}\left[dz_o^2 + \frac{\partial_x t}{x(t - x\partial_x t)}dz_d^2 + \frac{1}{t(t - x\partial_x t)}dz_a^2\right].} \tag{4.126}$$

Requiring further that this is equal to the Janus metric above uniquely fixes the transformation functions $s(x;b)$ and $t(x;b)$. In particular we obtain the system of coupled equations

$$u = 1 - x\frac{\partial_x t}{t}, \qquad (\partial_x s)^2 = \frac{1}{x^2}u^2(1-u)(1-bdu^{d-1})^2, \tag{4.127}$$

where $u = u(s(x))$ is given by (4.110).

In order to solve these equations we use (4.108) to trade $s(x)$ for $u(x)$ in the second equation, which gives

$$\int^{u(x)} \frac{du'}{u'\sqrt{(1-u')(1-u'+bu'^d)}} = -\log x^2. \tag{4.128}$$

The sign and the integration constant are chosen so that $u(x) \sim 1/x^2$ as $x \to \infty$, independent of b. Unfortunately it seems rather difficult to do this integral explicitly for arbitrary dimension d. Instead, one can expand the integrand in b and integrate term by term. This gives

$$u(x) = \frac{1}{1+x^2} + \frac{b}{2d}\frac{x^2}{(1+x^2)^{d+2}}F(d,2;d+1;\frac{1}{1+x^2}) + \mathcal{O}(b^2). \tag{4.129}$$

The transformation functions $s(x;b)$ and $t(x;b)$ are now determined from

$$\begin{aligned} s(x) &= 1 - u(x) + bu(x)^d \\ t(x) &= \exp\left[\frac{1}{2}\int_{u(x)}^1 \frac{du'}{u'}\left(\frac{1-u'}{1-u'+bu'^d}\right)^{1/2}\right]. \end{aligned} \tag{4.130}$$

Inserting the above expansion for $u(x)$ we reproduce (after some manipulation of the hypergeometric functions) precisely the coordinate transformation (4.116).

Moreover, inserting (4.129) in

$$\phi_B(x) = \phi_o + c\int_0^x \frac{dx'}{|x'|}u(x')^{d/2}\sqrt{1-u(x')}, \tag{4.131}$$

gives

$$\phi_B(x) = \phi_o + cc_5(x) + \mathcal{O}(c^3), \tag{4.132}$$

where ϕ_o is a constant and

$$c_5(x) = \frac{x}{\sqrt{1+x^2}} F\left(\frac{1}{2}, 1 - \frac{d}{2}; \frac{3}{2}; \frac{x^2}{1+x^2}\right). \tag{4.133}$$

Again, this has a simple derivative:

$$c_5'(x) = \frac{1}{(1+x^2)^{(d+1)/2}}. \tag{4.134}$$

Notice that as $z_o \to 0$ with all other coordinates fixed (i.e. as we approach the conformal boundary) $c_5(x) = \mathrm{sgn}(z_d)$ while higher order terms do not contribute. So at the boundary

$$\phi_B(z_d) = \phi_o + \mathrm{sgn}(z_d) c. \tag{4.135}$$

This implies that the coupling of the dual operator is different on the two sides of the wall.

Finally, let us examine the range of validity of the coordinate transformation (4.123). The Jacobian of the transformation is equal to $J = t\partial_x s$. Now $J = 0$ implies $\partial_x s = 0$ since $t(x)$ is positive definite, as can be seen from (4.130). It follows that the coordinate transformation breaks down at $u = 1$. Note that the zero of $(1 - bdu^{d-1})$ occurs at $u = 1/(bd)^{1/(d-1)} > 1$, where the inequality follows from (4.73). We conclude that the Fefferman-Graham coordinates are valid in the range $0 < u < 1$ although, in general $0 \leq u \leq u_o$ with $u_o \geq 1$. Recall that in general the Fefferman-Graham coordinate system is only guaranteed to exist in a neighborhood of the boundary, and here we see an explicit illustration of this.

Recall that the Fefferman-Graham coordinate system [45] is obtained as follows (see Section 3 of [33] for a review). One considers Gaussian normal coordinates centered at the boundary and the radial coordinate is identified with the affine parameter of the geodesics emanating perpendicularly from the boundary. Clearly the region of validity of this coordinate system depends on the behavior of the radial geodesics. We therefore need to analyze such geodesics, and we will do this in the $(u, \tilde{z}_o, z_a)$ coordinate system which is well-defined everywhere.

One easily shows that there are geodesics with z_a constant. The geodesic equations for the remaining coordinates lead to the following two equations

$$\begin{aligned} &\frac{d\log\tilde{z}_o}{d\tau} = a_1 u, \\ &\ddot{u} - \left(\frac{1}{u} + \frac{-1 + bdu^{d-1}}{2(1-u+bu^d)}\right)\dot{u}^2 + 2a_1^2 u^2(1 - u + bu^d) = 0, \end{aligned} \tag{4.136}$$

where a_1 is an integration constant. If $a_1 \neq 0$ the second equation can be integrated once to get

$$\dot{u} = \pm 2a_1 u\sqrt{(a_2 - u)(1 - u + bu^d)}, \tag{4.137}$$

for some constant $0 < a_2 \leq u_o$. If $a_1 = 0$ one gets instead

$$\dot{u} = \pm a_3 u\sqrt{1 - u + bu^d}, \tag{4.138}$$

where a_3 is again a constant. Now, depending on the values of the parameters a_1 and a_2, we can identify three qualitatively different types of geodesics as shown in fig.4.1.

Consider now the radial geodesics defined by $\dot{z}_d = \dot{z}_a = 0$ in the Fefferman-Graham coordinates, where the dot stands for the derivative w.r.t. the affine parameter $\tau = \log z_o$. Since $z_d = \text{constant}$ along these geodesics we will take $\tau = \log(z_o/|z_d|) = -\log|x|$ for later convenience. The transformation (4.123) immediately gives

$$\frac{d\log\tilde{z}_o}{d\tau} = u, \tag{4.139}$$

while (4.128) implies

$$\dot{u} = 2u\sqrt{(1-u)(1-u+bu^d)}. \tag{4.140}$$

The Fefferman-Graham radial geodesics therefore correspond to radial geodesics with $a_1 = a_2 = 1$. In particular, they are geodesics of type (i) if $b > 0$ but they are type (ii) if $b = 0$. This is an important qualitative difference between the FG coordinates for the Janus geometry and pure AdS. This is in fact why the FG coordinates cover the whole of AdS but only part of the Janus geometry.

It is now clear why the FG coordinate system for $b > 0$ breaks down at $u = 1$. Namely, the radial FG coordinate corresponds to geodesics which do not reach beyond $u = 1$. If one continues to affine parameter values greater than $\tau*$, where $u(\tau*) = 1$, the geodesics bounce back and they cannot be used to define a coordinate system since they doubly cover the region $u < 1$ as is shown in fig.4.2. Therefore the FG coordinates are well-defined for affine parameter values $\tau < \tau*$. This means that $|x| = e^{-\tau}$ must be bounded below. Another way to see this is to observe that (4.128) implies that x^2 is a monotonically decreasing function of u. Hence the upper bound $u < 1$ on u implies a lower bound on x^2. Setting $u = 1$ in (4.129) and solving for x to leading order in b we find[13] $|x| > x_o = b/\sqrt{2} + \mathcal{O}(b^2)$. Therefore $x = 0$ is *not* part of the manifold and hence the metric (4.118) (and (4.126)) is *non-singular* in the region it is well-defined. (To cover the entire spacetime one would have to use another coordinate patch that covers the deep interior region $1 \leq u \leq u_o$, but this is irrelevant for our holographic computations.) Notice that the bound on x translates

[13] One must be careful since the hypergeometric function $F(d, 2; d+1; \frac{1}{1+x^2})$ is singular at $x^2 = 0$. See eq. 15.3.12 in [73].

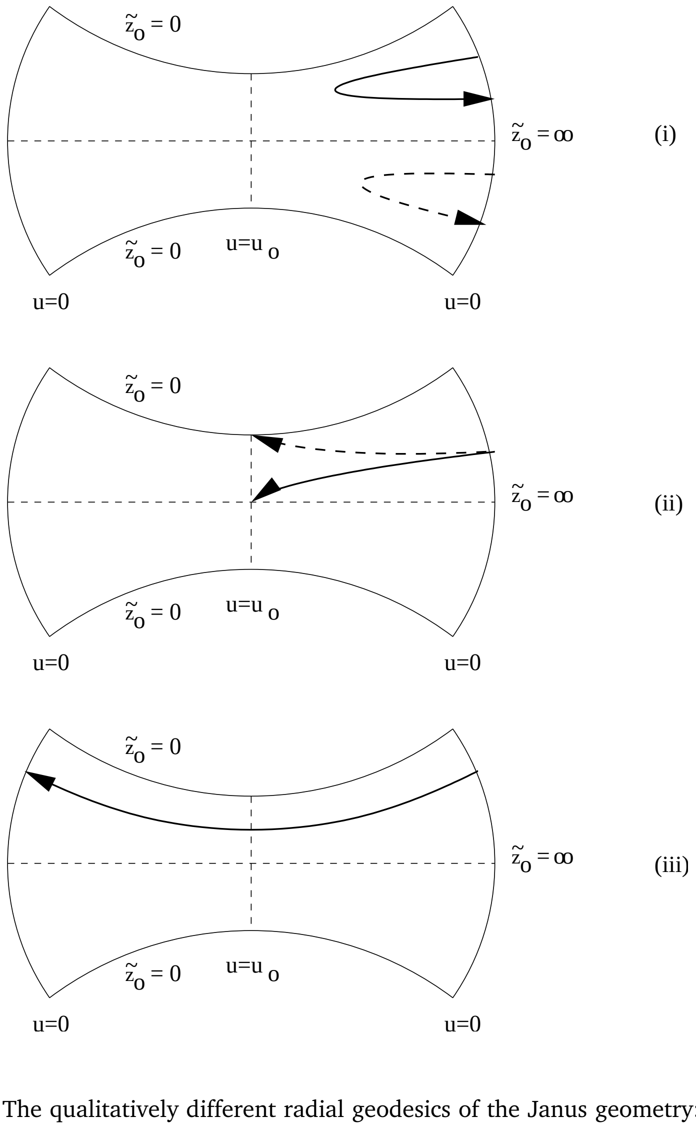

Figure 4.1: The qualitatively different radial geodesics of the Janus geometry: (i) $a_2 < u_o$ with $a_1 > 0$ (solid arrow) or $a_1 < 0$ (broken arrow), (ii) $a_2 = u_o$ with $a_1 > 0$ (solid arrow) or $a_1 < 0$ (broken arrow) and (iii) $a_1 = 0$. These qualitative features are insensitive to the value of b.

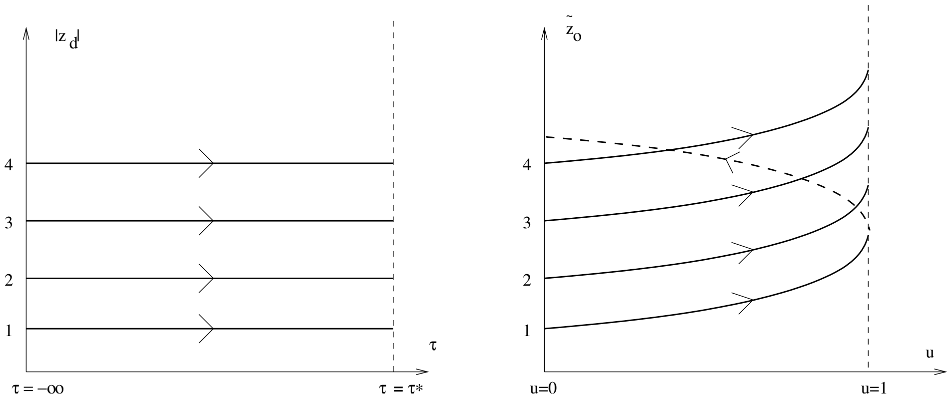

Figure 4.2: A radial geodesic in the Fefferman-Graham coordinates (left) is not defined beyond $u = 1$ due to the failure of the coordinate system at this point. In the coordinate system which extends beyond $u = 1$ this geodesic bounces back at $u = 1$. Only the branch before the bounce corresponds to the geodesic on the left.

into $|z_d| > x_o z_o$ and so only at the boundary $z_o \to 0$ does z_d cover the entire real line. More crucially, the bound $z_o < |z_d|/x_o$ means that the radial coordinate, z_o, is well-defined everywhere *except* at $z_d = 0$, which precisely corresponds to the corner where the two-halves of the boundary meet. This can also be seen directly from the properties of type (i) geodesics. Since $\tilde{z}_o \to |z_d|$ as $\tau \to -\infty$ with z_d constant along the geodesics, we have

$$\tilde{z}_o(\tau) = |z_d| \exp \int_{-\infty}^{\tau} d\tau' u(\tau'). \tag{4.141}$$

So for $b > 0$ the FG geodesics hit again the boundary $u = 0$ at $\tilde{z}_o = |z_d|\alpha$, where $\alpha = \exp \int_0^1 \frac{du}{\sqrt{(1-u)(1-u+bu^d)}}$. At $z_d = 0$ these geodesics become degenerate and they stay along $\tilde{z}_o = 0$. Since the entire $\tilde{z}_o = 0$ subspace is mapped to $(z_o, z_d) = (0, 0)$ in the FG coordinates, the FG geodesics do not leave the origin once $z_d = 0$. Hence, the FG radial coordinate is not defined at $z_d = 0$.

Finally let us discuss the possibility to use the geodesics of type (ii) in order to define FG coordinates. This is also a natural choice as these geodesics are the obvious generalization of the pure AdS case. Following radial type (ii) geodesics for the Janus geometry, the exact FG metric is

$$ds^2 = \frac{dz_o^2}{z_o^2} + \frac{u_o(u_o - u)}{u z_d^2} dz_d^2 + \frac{dz_a^2}{u \tilde{z}_o^2}, \tag{4.142}$$

where

$$\log\left(\frac{z_o^{2/\sqrt{u_o}}}{z_d^2}\right) = \int^u \frac{du}{u\sqrt{(u_o-u)(1-u+bu^d)}}, \tag{4.143}$$

and

$$\tilde{z}_o^2 = z_d^{2u_o} \exp\int_0^u \frac{du}{\sqrt{(u_o-u)(1-u+bu^d)}}. \tag{4.144}$$

Asymptotically, i.e. as $z_o \to 0$,

$$ds^2 \sim \frac{1}{z_o^2}\left[dz_o^2 + (\tilde{z}_d^2)^{\frac{1}{\sqrt{u_0}}-1}(d\tilde{z}_d^2 + dz_a^2)\right], \tag{4.145}$$

where $\tilde{z}_d = z_d^{u_0}$. So when $u_o = 1$, i.e. the pure AdS case, we get the standard result but for the Janus solution these geodesics lead to a non-flat representative of the conformally flat conformal structure. One can perform the additional change of variables given in (8)-(9) of [74] to change the representative to the flat metric. Notice however that the conformal factor is singular at $z_d = 0$ so the corresponding coordinate transformation is singular there. We thus arrive again at the conclusion that the FG coordinates are not well-defined at the corner.

We have therefore determined the exact form of the Fefferman-Graham metric for the Janus geometry and we have shown that is well-defined everywhere except on the defect where the two half-boundaries are joined and it is non-singular where it is defined. Moreover, we have shown that the FG metric possesses an inversion isometry which enhances the isometry group to the full $O(1,d)$ isometry group of the original Janus metric. This is reflected in the fact that the holographic calculation gives a zero vev for the stress tensor, which is consistent with a boundary QFT invariant under conformal transformations leaving the plane $z_d = 0$ invariant.

4.A.3 Two-point functions for Janus from scalar momentum

Here we give some details of the calculation of the two-point functions for the Janus background based on the scalar equation of motion (4.85). The non-trivial part of the calculation consists in casting the source in a form which significantly reduces the amount of work required. To this end we again use the fact that $\phi_B(x)$ is a function of $x = z_d/z_o$ only and it satisfies

$$\ddot{\phi}_B + d\dot{\phi}_B + e^{-2r}\Box\phi_B = \mathcal{O}(b), \tag{4.146}$$

to write

$$\partial_j\phi_B(x) = \delta_{jd}\frac{1}{z_d}\dot{\phi}_B, \qquad \partial_i\partial_j\phi_B = -(d+1)\delta_{id}\delta_{jd}\frac{\dot{\phi}_B}{z_o^2+z_d^2} + \mathcal{O}(b). \tag{4.147}$$

Decomposing S^i_j into irreducible components then (4.85) becomes

$$\begin{aligned} -\Box_g \phi \quad = \quad & -cz_o \partial_i \left[\frac{1}{(1+x^2)^{\frac{d+1}{2}}} e^i_d \right] + \frac{cx}{(1+x^2)^{\frac{d+1}{2}}} \left\{ -\frac{z_o^2}{2(d-1)} \Box f \right. \\ & + \frac{(d+1)}{1+x^2} \left[2\partial_{z_d}\epsilon_d + \frac{d}{d-1}\left(\frac{1}{d} - \frac{\partial^2_{z_d}}{\Box}\right) f + \frac{\partial^2_{z_d}}{\Box} S \right] \\ & \left. - \frac{z_o^2}{z_d}\left[\Box\epsilon_d - \partial_{z_d} f + \frac{1}{2}\partial_{z_d} S \right] \right\} + \mathcal{O}(b). \end{aligned} \tag{4.148}$$

Quite remarkably, this can be cast in the form

$$-\Box_g \phi = -cz_o \partial_i \left[\frac{1}{(1+x^2)^{\frac{d+1}{2}}} e^i_d \right] - \Box_g \tilde{J}_\phi + \mathcal{O}(b), \tag{4.149}$$

where

$$\tilde{J}_\phi = \frac{cz_o^d}{(z_o^2+z_d^2)^{\frac{d+1}{2}}} \left[\alpha + \epsilon_d + \frac{1}{2}\frac{\partial_{z_d}}{\Box} S + \frac{1}{2(d-1)} \left(z_d - d\frac{\partial_{z_d}}{\Box} \right) f \right], \tag{4.150}$$

and α is a constant. Hence, the inhomogeneous solution is

$$\phi = \tilde{J}_\phi - c \int d^{d+1}w \sqrt{g(w)} G(\xi) \partial_i \left[\frac{z_o}{(1+x^2)^{\frac{d+1}{2}}} e^i_d \right], \tag{4.151}$$

where e^i_d is given by the zero-order solution

$$e^i_j(z) = \int d^d y K_d(z, \vec{y}) e_{(0)}{}^i_j(\vec{y}). \tag{4.152}$$

This expression for ϕ immediately gives the canonical momentum from which we obtain the two-point function by differentiating w.r.t. $S_{(0)}{}^i_j$.[14] The result is

$$\begin{aligned} & \langle T^i_j(\vec{z}) \mathcal{O}(\vec{w}) \rangle = -2(d+1)c\Pi^i{}_d{}^d{}_j I(\vec{z},\vec{w}) \\ & -\frac{cd}{|w_d|^{d+1}} \left[2\pi_d^{(i}\frac{\partial_{j)}}{\Box} + \delta^i_j \frac{\partial_d}{\Box} + \frac{1}{d-1}\left(w_d - d\frac{\partial_d}{\Box} \right)\pi^i_j - \frac{1}{d} w_d \delta^i_j \right]_w \delta^{(d)}(\vec{z}-\vec{w}). \end{aligned} \tag{4.153}$$

It is a straightforward exercise to verify that this is equivalent to the two-point functions given above, as calculated from the graviton momentum.

[14] Note that

$$\begin{aligned} \langle T_{ij}(x)\mathcal{O}(x')\rangle &= \left(\frac{-1}{\sqrt{g_{(0)}(x')}} \frac{\delta}{\delta\phi_{(0)}(x')} \right) \left(\frac{-2}{\sqrt{g_{(0)}(x)}} \frac{\delta}{\delta g_{(0)}{}^{ij}(x)} \right) W \\ &= \frac{-1}{\sqrt{g_{(0)}(x')}} \frac{\delta}{\delta\phi_{(0)}(x')} \langle T_{ij}(x)\rangle = \frac{-2}{\sqrt{g_{(0)}(x)}} \frac{\delta}{\delta g_{(0)}{}^{ij}(x)} \langle \mathcal{O}(x')\rangle + \delta^{(d)}(x,x') g_{(0)ij}(x) \langle \mathcal{O}(x')\rangle, \end{aligned}$$

and so if one starts the computation from the scalar one-point functions, one should remember to include the contact term given above to obtain the full expression.

4.A.4 Short distance expansion of the holographic two-point function $\langle T^i_j(\vec{z})\mathcal{O}(\vec{w})\rangle$

For the convenience of the reader we will give here the essential steps required to evaluate the short distance expansion of the transverse traceless part of the holographic two-point function $\langle T^i_j(\vec{z})\mathcal{O}(\vec{w})\rangle$, namely $\Pi^i{}_d{}^d{}_j I(\vec{z},\vec{w})$, where $I(\vec{z},\vec{w})$ is given in (4.98).[15]

First, after a shift and rescaling of the integration variables, $I(\vec{z},\vec{w})$ can be written as

$$I(\vec{z},\vec{w}) = \frac{c_d^2}{(\vec{s}^2)^{d/2-1}}\int_0^\infty dx_o x_o^{2d+1} \times \int d^d x \frac{w_d + |\vec{s}|x_d}{[x_o^2\vec{s}^2 + (w_d+|\vec{s}|x_d)^2]^{\frac{d+3}{2}}}\frac{1}{[x_o^2+\vec{x}^2]^d}\frac{1}{[x_o^2+(\vec{x}-\hat{s})^2]^d}, \tag{4.154}$$

where $\hat{s} = \vec{s}/\vec{s}^2$. This form is suitable for a short distance expansion in $|\vec{s}|$. Each term in the expansion can be explicitly evaluated using the standard Feynman parameters technique. The result to leading order is given in (4.107).

To evaluate the projection operator on this expression we use the fact that

$$\frac{1}{(\vec{s}^2)^\alpha} = -\frac{1}{2(\alpha-1)(d-2\alpha)}\Box\frac{1}{(\vec{s}^2)^{\alpha-1}}, \tag{4.155}$$

for any power $\alpha \neq d/2$ in order to cancel the $1/\Box$ factors in the projection operator. It is then straightforward to evaluate the derivatives in the numerator of the projection operator to obtain the short distance expansion of the transverse traceless part. The result is given in section 4.2.4.

[15] Incidentally, this integral transforms under inversion as $I(\vec{z'},\vec{w'}) = \vec{z}^{2d}\vec{w}^{2d}I(\vec{z},\vec{w})$ and hence it must be of the form $f(v)/(\vec{s}^2)^d$ for some function $f(v)$, where v is defined in (4.100). However, we have not succeeded in determining this function so far.

Chapter 5

The First Law for AlAdS Black Holes

In this chapter we consider black hole solutions of the supergravity equations. Black hole spacetimes have a horizon and are characterized by a number of asymptotic charges, such as mass, electric and magnetic charges. The AdS/CFT duality implies that the field theory living on the conformal boundary of such spacetimes is a field theory at non-zero temperature [19, 75]. The thermodynamics of the black hole is then mapped to the thermodynamics of the strongly coupled thermal field theory.

Moreover, we have seen that in the saddle point approximation the string partition function involves a sum over all AlAdS manifolds with the same conformal boundary (see the recent review [76] for an extensive discussion and concrete examples of bulk manifolds with a given boundary). Via the AdS/CFT dictionary, different bulk manifolds with the same conformal boundary correspond to different vacua of the field theory residing on the common boundary. Which vacuum dominates the path integral is then dictated by the value of the on-shell Euclidean action.

When applied to spacetimes with a non-zero temperature (i.e. with periodic Euclidean time), this observation leads to the possibility of thermal phase transitions between different vacua. For example, it was argued in [19] that the Hawking-Page transition [77] between thermal AdS_{d+1} at low temperature and the Schwarzchild-AdS_{d+1} black hole at high temperature, both of which have a boundary with topology $S^1 \times S^{d-1}$, implies a confining/deconfining phase transition for the boundary strongly coupled field theory. Although the field theory lives on a finite volume space, namely $S^1 \times S^{d-1}$, such a phase transition is possible in the $N \to \infty$ limit.

While the AdS/CFT conjecture has been the driving force behind the recent interest in asymptotically AdS spacetimes, such spaces have been long studied on their own merit. In particular, an important aspect of asymptotically AdS spacetimes that has attracted considerable attention over the years is the definition of conserved charges associated with the asymptotic symmetries of such spacetimes [78, 42, 43, 79, 80], see also [81] and references therein.

Even though most of the literature studies 'asymptotically AdS spacetimes' in the restricted sense, that is spacetimes which asymptotically become exactly AdS with the standard conformal boundary $\mathbb{R} \times S^{d-1}$, it has proved difficult to define asymptotic charges for such spacetimes. The main obstruction in defining such conserved charges is the fact that the infinite volume of these spacetimes causes various 'natural candidates' for conserved quantities, such as Komar integrals, to diverge [82]. One is then forced to introduce some regularization procedure, which is inherently ambiguous.

Various approaches have been suggested to circumvent this difficulty. Some of them exploit the special properties of AdS to construct conserved quantities which are manifestly finite, e.g. [42, 43], while others embed the asymptotically AdS spacetime into a spacetime with the same asymptotics (most often exact AdS) and then define manifestly finite conserved quantities relative to the ambient spacetime, e.g. [83, 84]. Although the philosophy and the precise definition of the conserved charges varies among these methods, they all implement some form of 'background subtraction'. However, not all asymptotically locally AdS spacetimes can be embedded in a suitable ambient spacetime and, therefore, it is desirable that one has a background independent definition of the conserved charges of any AlAdS space.

It is known that holographic renormalization does provide such a background independent definition of the conserved charges for any AlAdS space. As we will review below, the renormalized one-point function of the stress tensor of the dual field theory, which is identified with the non-local term ${\pi_{(d)}}^{ij}$ in the covariant expansion of the canonical momentum conjugate to the induced metric γ_{ij} (see (3.57)), leads to a well-defined and general expression for the conserved charges associated with the asymptotic symmetries of an AlAdS space.

Nevertheless, there has long existed a debate about the connection between the holographic charges and the various alternative definitions of conserved charges and, in particular, regarding the question of whether the holographic charges of AlAdS black holes satisfy the first law of black hole mechanics [85]. One of the aims of this chapter, which is a mildly revised version of the paper [3], is to clarify the concept of the holographic charges (see also[86]) and, in particular, to prove in general that *all* AlAdS black holes satisfy the first law of black hole mechanics and the charges entering this law are the holographic charges.

We start in this chapter by formulating the variational problem with Dirichlet boundary conditions for AdS gravity. As we have discussed in Chapter 3, the bulk metric induces only a conformal structure - and not a metric - on the boundary and so, the Dirichlet problem for AdS gravity requires that a *conformal structure* is kept fixed on the boundary and *not* a metric. Any other choice of Dirichlet boundary conditions breaks part of the bulk diffeomorphisms, namely the ones that induce a Weyl transformation at the boundary. We then show that the variational problem for such Dirichlet boundary conditions is well-posed provided the conformal anomaly $\mathcal{A}$ is zero and a set of new covariant boundary terms (in addition to the Gibbons-Hawking term) is added to the action. These new boundary terms are precisely the boundary counterterms introduced in [29, 34] in order to achieve finiteness of the on-shell action and of the holographic stress energy tensor as we discussed in Chapter 3. If the conformal anomaly is non-zero, however, one has to choose a specific representative of the boundary conformal structure to make the variational problem well-posed, thus breaking part of the bulk diffeomorphisms. In this case the boundary counterterms guarantee that the on-shell action has a well-defined transformation under the broken diffeomorphisms, the transformation rule being determined by the conformal anomaly. In other words, we need to pick a reference representative in this case, but the change from one representative to another is essentially determined by the conformal class of the boundary metric via the conformal anomaly.

We then use Noether's theorem to derive the conserved charges for AlAdS spacetimes that possess asymptotic symmetries and we show that they are precisely the holographic charges (see also [86] where the same conclusion was drawn by different methods.) The holographic charges were originally derived [34, 30, 74] using the Brown-York prescription [87] supplemented by appropriate boundary counterterms [29]. Finally, we show that the covariant phase space method of Wald *et al* [88, 89, 90] also reproduces the holographic charges.

These results allow us then to prove that the holographic charges for general stationary, axisymmetric, charged AlAdS black holes in any dimension satisfy the first law of black hole mechanics, *provided* the variations that enter in the first law respect the boundary conditions. These variations need not respect any of the symmetries of the solution however. This resolves a puzzle in the literature where it seemed that only the charges relative to exact AdS satisfy the first law [18]. The key observation is that in some cases, such as the Kerr-AdS black holes in Boyer-Lindquist coordinates, the conformal representative on the boundary depends on the parameters of the black hole, in this case the mass and the rotational parameters. Since these parameters correspond essentially to the conserved charges of the solution, varying these charges will generically change the representative of the conformal structure. While this poses no problem in the absence of a conformal anomaly, when there is a conformal anomaly it leads to the violation of the allowed boundary conditions. In this case, to preserve the boundary conditions one must undo the change in the conformal representative by a compensating bulk diffeomorphism. It is then found that the *combined* variations resulting from the variation of the black hole parameters *and* the compensating diffeomorphism do satisfy the first law.

We finally illustrate this mechanism in the context of the four-dimensional Kerr-Newman-AdS and the five-dimensional Kerr-AdS black holes. Several technical results are collected in the appendices. In particular, in Appendix 5.A.4 we comment on the connection between the 'conformal mass' of Ashtekar and Magnon [42] and the holographic mass.

In this chapter we work with Lorentzian signature. The relevant formulas for holographic renormalization in Lorentzian signature can be found in Appendix 3.A.3.

5.1 COUNTERTERMS AND THE VARIATIONAL PROBLEM FOR ADS GRAVITY

5.1.1 THE THEORY

We will consider in this section the variational problem for AdS gravity coupled to scalars and a Maxwell field. Other matter fields, like forms and non-abelian gauge fields, can be easily incorporated in the analysis, but for simplicity we do not include them. Moreover, to keep the analysis general we do not include any Chern-Simons terms since their particular form depends on the spacetime dimension. Within this framework we consider the most general Lagrangian consistent with the fact that the field equations admit a solution that is asymptotically locally AdS. The Lagrangian D-form ($D{=}d+1$) is given by

$$\mathbf{L} = \left(\frac{1}{2\kappa^2}R - V(\varphi)\right) * \mathbf{1} - \frac{1}{2}G_{IJ}(\varphi)\mathrm{d}\varphi^I \wedge *\mathrm{d}\varphi^J - \frac{1}{2}U(\varphi)\mathbf{F} \wedge *\mathbf{F}, \tag{5.1}$$

where we use mostly plus signature and $\mathbf{F} = \mathrm{d}\mathbf{A}$ and $V(\varphi)$, $U(\varphi)$ and $G_{IJ}(\varphi)$ are only constrained by the requirement that the field equations admit AlAdS solutions. The exact conditions follow from the asymptotic analysis discussed in the next subsection, but we will not need the detailed form of the conditions here.

The variation of the Lagrangian with respect to arbitrary field variations takes the form

$$\delta\mathbf{L} = \mathbf{E}\delta\psi + \mathrm{d}\mathbf{\Theta}(\psi, \delta\psi), \tag{5.2}$$

where we use $\psi = (g_{\mu\nu}, A_\mu, \varphi^I)$ to denote collectively all fields and $\mathbf{E}$ is the equations of motion D-form. More specifically, we have

$$\delta\mathbf{L} = \mathbf{E}^{\mu\nu}{}_{(1)}\delta g_{\mu\nu} + \mathbf{E}^{\mu}{}_{(2)}\delta A_\mu + \mathbf{E}^{(3)}_I\delta\varphi^I + \mathrm{d}\mathbf{\Theta}(\psi, \delta\psi), \tag{5.3}$$

where

$$\begin{aligned}
\mathbf{E}_{(1)}{}^{\mu\nu} &= -\frac{1}{2\kappa^2}\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} - \kappa^2\tilde{T}^{\mu\nu}\right) * \mathbf{1}, \\
\mathbf{E}_{(2)}{}^{\nu} &= \nabla_\mu(U(\varphi)F^{\mu\nu}) * \mathbf{1}, \\
\mathbf{E}^{(3)}_I &= \left(\nabla^\mu(G_{IJ}(\varphi)\partial_\mu\varphi^J) - \frac{1}{2}\frac{\partial G_{JK}}{\partial\varphi^I}\partial_\mu\varphi^J\partial^\mu\varphi^K - \frac{\partial V}{\partial\varphi^I} - \frac{1}{4}\frac{\partial U}{\partial\varphi^I}F_{\mu\nu}F^{\mu\nu}\right) * \mathbf{1},
\end{aligned} \tag{5.4}$$

and the matter stress tensor is given by

$$\tilde{T}_{\mu\nu} = G_{IJ}(\varphi)\partial_\mu\varphi^I\partial_\nu\varphi^J + U(\varphi)F_{\mu\rho}F_\nu{}^\rho - g_{\mu\nu}\mathcal{L}_{\mathrm{m}}, \tag{5.5}$$

with $\mathcal{L}_{\mathrm{m}}$ denoting the matter part of the Lagrangian. Moreover,

$$\mathbf{\Theta}(\psi, \delta\psi) = - * v(\psi, \delta\psi), \tag{5.6}$$

where

$$v^\mu = -\frac{1}{2\kappa^2}(g^{\mu\rho}\nabla^\sigma\delta g_{\rho\sigma} - g^{\rho\sigma}\nabla^\mu\delta g_{\rho\sigma}) + G_{IJ}(\varphi)\delta\varphi^I\nabla^\mu\varphi^J + U(\varphi)F^{\mu\nu}\delta A_\nu. \tag{5.7}$$

5.1.2 GAUGE INVARIANCE OF THE RENORMALIZED ACTION

In this section we first determine the most general bulk diffeomorphisms and $U(1)$ gauge transformations which preserve the gauge-fixed metric (3.35) as well as the gauge condition $A_r = 0$ for the Maxwell field. We note that this gauge need only be preserved up to terms of next-to-normalizable mode order, i.e. up to order $e^{-(d-1)r}$. Such transformations leave invariant the functional form of the boundary conditions, of the asymptotic solutions, and of the counterterm action on the regulated boundary Σ_{r_o}. Subsequently, we derive the maximal subset of gauge-preserving transformations that leave the renormalized action invariant, where only the functional form of the boundary conditions is imposed, namely we require that

$$\gamma_{ij}(r,x) \sim e^{2r} g_{(0)ij}(x), \qquad A_i(x,r) \sim A_{(0)i}(x), \qquad \varphi^I(r,x) \sim \phi^I_{(0)}(x) e^{-(d-\Delta_I)r}, \tag{5.8}$$

but no conditions are imposed on $g_{(0)ij},\ A_{(0)i},\ \phi^I_{(0)}$. Notice that the transformations below do act on these coefficients.

In the gauge (3.35), the Lie derivative, $\mathcal{L}_\xi$, of the bulk fields w.r.t. a bulk vector field ξ^μ is given by

$$\begin{aligned} &\mathcal{L}_\xi g_{rr} = \dot{\xi}^r, \\ &\mathcal{L}_\xi g_{ri} = \gamma_{ij}(\dot{\xi}^j + \partial^j \xi^r), \\ &\mathcal{L}_\xi g_{ij} = L_\xi \gamma_{ij} + 2K_{ij}\xi^r \sim L_\xi \gamma_{ij} + 2\gamma_{ij}\xi^r, \end{aligned} \tag{5.9}$$

$$\begin{aligned} &\mathcal{L}_\xi A_r = A_j \dot{\xi}^j, \\ &\mathcal{L}_\xi A_i = L_\xi A_i + \xi^r \dot{A}_i \sim L_\xi A_i, \end{aligned} \tag{5.10}$$

$$\mathcal{L}_\xi \varphi^I = L_\xi \varphi^I + \xi^r \dot{\varphi}^I \sim L_\xi \varphi^I + (\Delta_I - d)\xi^r \varphi^I, \tag{5.11}$$

where L_ξ is the Lie derivative w.r.t. the transverse components ξ^i of the bulk vector field ξ. This bulk diffeomorphism, combined with a $U(1)$ gauge transformation, preserves the gauge fixing (up to the desired order; see (5.22) below) provided $\mathcal{L}_\xi g_{rr} = \mathcal{L}_\xi g_{ri} = \mathcal{O}(e^{-dr})$ and $\mathcal{L}_\xi A_r + \dot{\alpha} = \mathcal{O}(e^{-(d+2)r})$. Integrating these conditions gives:

$$\begin{aligned} &\xi^r = \delta\sigma(x) + \mathcal{O}(e^{-dr}), \\ &\xi^i = \xi^i_o(x) + \partial_j \delta\sigma(x) \int_r^\infty dr' \gamma^{ji}(r',x) + \mathcal{O}(e^{-(d+2)r}), \\ &\alpha = \alpha_o(x) + \partial_i \delta\sigma(x) \int_r^\infty dr' A^i(r',x) + \mathcal{O}(e^{-(d+2)r}), \end{aligned} \tag{5.12}$$

where $\delta\sigma(x)$ and $\alpha_o(x)$ are arbitrary functions of the transverse coordinates and $\xi^i_o(x)$ is an arbitrary transverse vector field. For $\xi_o = 0$, this bulk diffeomorphism is precisely the 'Penrose-Brown-Henneaux (PBH) transformation' [44, 91] which induces a Weyl transformation on the conformal boundary [92, 30, 74]. Here, we will call a 'PBH transformation' the combined bulk diffeomorphism with $\xi_o = 0$ *and* the gauge transformation with $\alpha_o = 0$, which is required in order to preserve the gauge of the Maxwell field.

Next we determine which subset of (5.12) leaves invariant the renormalized action

$$I_{\rm ren} = \int_{\mathcal{M}_{r_o}} \mathbf{L} + \frac{1}{\kappa^2}\int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}K + I_{\rm ct}, \tag{5.13}$$

where

$$\mathbf{L} = \left(\frac{1}{2\kappa^2}R[g] + \mathcal{L}_{\rm m}\right) * \mathbf{1}, \tag{5.14}$$

and $I_{\rm ct}$ is given by (3.55). Since $\mathbf{L}$ is covariant under diffeomorphisms and gauge invariant, we have

$$\delta_\xi \mathbf{L} = \mathcal{L}_\xi \mathbf{L} = {\rm d}i_\xi \mathbf{L}, \quad \delta_\alpha \mathbf{L} = 0, \tag{5.15}$$

where we have used the identity $\mathcal{L}_\xi = i_\xi {\rm d} + {\rm d}i_\xi$ for the Lie derivative on forms. Hence,

$$\delta_{\xi,\alpha} I_{\rm ren} = \int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}\xi^r\left(\frac{1}{2\kappa^2}R[g] + \mathcal{L}_{\rm m}\right) + \frac{1}{\kappa^2}\delta_\xi\int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}K + \delta_{\xi,\alpha}I_{\rm ct}. \tag{5.16}$$

Now, in the gauge we are using, the Ricci scalar of the bulk metric can be expressed as

$$R[g] = R + K^2 - K_{ij}K^{ij} - \frac{2}{\sqrt{-\gamma}}\partial_r(\sqrt{-\gamma}K), \tag{5.17}$$

Moreover, for the diffeomorphisms given by (5.12) a short computation gives

$$\delta_\xi \int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}K = \int_{\Sigma_{r_o}} d^dx\xi^r\partial_r(\sqrt{-\gamma}K), \tag{5.18}$$

and hence

$$\delta_{\xi,\alpha}I_{\rm ren} = \frac{1}{2\kappa^2}\int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}\xi^r\left(R + K^2 - K_{ij}K^{ij} + 2\kappa^2\mathcal{L}_{\rm m}\right) + \delta_{\xi,\alpha}I_{\rm ct}. \tag{5.19}$$

The last term takes the form

$$\delta_{\xi,\alpha}I_{\rm ct} = \int_{\Sigma_{r_o}} d^dx\left(\hat\pi^{ij}_{\rm ct}\delta_\xi\gamma_{ij} + \hat\pi_{\rm ct\,I}\delta_\xi\varphi^I + \hat\pi^i_{\rm ct}(\delta_\xi A_i + \partial_i\alpha)\right), \tag{5.20}$$

where we put hats on the counterterm momenta to emphasize that they should be viewed as predetermined local functionals of the induced fields as opposed to the asymptotic behavior of the radial derivative of the on-shell induced fields. Inserting now the transformation (5.12) and using the second equation in (3.38) and the first equation in (3.121), which the counterterms satisfy by construction, we are left with

$$\delta_{\xi,\alpha}I_{\rm ren} = \int_{\Sigma_{r_o}} d^dx\xi^r\left\{\frac{1}{2\kappa^2}\sqrt{-\gamma}\left(R + K^2 - K_{ij}K^{ij} + 2\kappa^2\mathcal{L}_{\rm m}\right)\right.$$
$$\left. + \left(\hat\pi^{ij}_{\rm ct}2K_{ij} + \hat\pi_{\rm ct\,I}\dot\varphi^I + \hat\pi^i_{\rm ct}\dot A_i\right)\right\}. \tag{5.21}$$

Using the form of the boundary conditions (5.8) one finds that the leading order divergent term cancels and the terms inside the curly brackets are of order $e^{(d-1)r}$. We therefore conclude that a transformation (5.12) that leaves the renormalized action invariant must have

$$\xi^r = \mathcal{O}(e^{-dr}), \qquad \dot{\xi}^i = -\partial^i \xi^r + \mathcal{O}(e^{-(d+2)r}) = \mathcal{O}(e^{-(d+2)r}). \tag{5.22}$$

This leaves us with $\xi^i = \xi^i_o(x)$ and $\alpha = \alpha_o(x)$, up to sufficiently high order in e^{-r} as $r \to \infty$.

In fact, as is well known, the PBH transformation, i.e. the part of the transformation (5.12) that is driven by $\delta\sigma(x)$, induces a Weyl transformation on the boundary and even the *on-shell* renormalized action is not invariant under such transformations unless the anomaly vanishes. To see this let us first rewrite the Hamilton constraint (first equation in (3.38)) as

$$\frac{1}{2\kappa^2}\sqrt{-\gamma}\left(R + K^2 - K_{ij}K^{ij} + 2\kappa^2\mathcal{L}_{\rm m}\right) = \pi^{ij}2K_{ij} + \pi_I\dot{\varphi}^I + \pi^i\dot{A}_i. \tag{5.23}$$

Then, using the trace Ward identity (2.168), (5.21) becomes on-shell

$$\delta_{\xi,\alpha} I_{\rm ren}^{\rm on-shell} = \int_{\Sigma_{r_o}} d^d x \sqrt{-\gamma}\xi^r \mathcal{A}. \tag{5.24}$$

5.1.3 Variational problem

We investigate in this section under which conditions the variational problem is well-posed, i.e. under which conditions the boundary terms in the variation of the action cancel so that $\delta I = 0$ (under generic variation) implies the field equations and vice versa.

Let n^μ be the outward unit normal to the hypersurfaces Σ_r. Using (5.7) and the definition of the radial momenta (3.125) one easily finds that the pullback of $\mathbf{\Theta}$ onto Σ_r is given by[1]

$$\begin{aligned} \mathbf{\Theta} &= -n_\mu v^\mu *_\Sigma \mathbf{1} \\ &= \left(-\frac{1}{\kappa^2}\delta(\sqrt{-\gamma}K) + \pi^{ij}\delta\gamma_{ij} + \pi^i\delta A_i + \pi_I\delta\varphi^I\right) d\mu, \end{aligned} \tag{5.25}$$

where $\sqrt{-\gamma}d\mu \equiv *_\Sigma\mathbf{1}$, and $*_\Sigma$ denotes the Hodge dual w.r.t. Σ_r. We thus arrive at the well-known fact [48] that the Gibbons-Hawking term is sufficient to render the variational problem well-defined when all induced fields at the boundary are kept fixed, i.e.

$$\delta\gamma_{ij} = 0, \qquad \delta A_i = 0, \qquad \delta\varphi^I = 0, \qquad \text{on} \quad \Sigma_{r_o}. \tag{5.26}$$

[1] Up to an exact term $\mathrm{d}(*\mathbf{y})$, where $y^\mu = \frac{1}{2\kappa^2}n^\rho g^{\mu\sigma}\delta g_{\rho\sigma}$ vanishes for variations that preserve the gauge fixing.

These boundary conditions are perfectly acceptable for the regulated manifold with boundary Σ_{r_o} at finite r_o, since the bulk fields uniquely induce fields on Σ_{r_o}. However, as $\Sigma_{r_o} \to \partial\mathcal{M}$ this is no longer the case. The induced fields generically diverge (or vanish) in this limit and the bulk fields only determine the *conformal class* of the boundary fields. It is therefore not possible to impose the above boundary conditions on the conformal boundary. At most, one can demand that the boundary fields are kept fixed up to a Weyl transformation, namely

$$\delta\gamma_{ij} = 2\gamma_{ij}\delta\sigma, \qquad \delta A_i = 0, \qquad \delta\varphi^I = (\Delta_I - d)\varphi^I\delta\sigma, \qquad \text{on} \quad \partial\mathcal{M}. \tag{5.27}$$

To implement these weaker boundary conditions we insert the expansions (3.51) into (5.25) and use (3.43) to get

$$\begin{aligned}\boldsymbol{\Theta} &= \left\{-\frac{1}{\kappa^2}\delta(\sqrt{-\gamma}K) - (\pi_{\rm ct}^{ij}\delta\gamma_{ij} + \pi_{\rm ct}^{i}\delta A_i + \pi_{{\rm ct}I}\delta\varphi^I)\right.\\ &\qquad \left. + \sqrt{-\gamma}(\pi_{(d)}{}^{ij}\delta\gamma_{ij} + \pi_{(d)}{}^{i}\delta A_i + \pi_{(\Delta_I)I}\delta\varphi^I) + \cdots\right\} d\mu \\ &= \left\{\delta\left(-\frac{1}{\kappa^2}\sqrt{-\gamma}[K-(K-\lambda)_{\rm ct}]\right)\right.\\ &\qquad \left. + \sqrt{-\gamma}(\pi_{(d)}{}^{ij}\delta\gamma_{ij} + \pi_{(d)}{}^{i}\delta A_i + \pi_{(\Delta_I)I}\delta\varphi^I) + \cdots\right\} d\mu. \end{aligned} \tag{5.28}$$

Hence,

$$\begin{aligned}\int_{\Sigma_{r_o}}\boldsymbol{\Theta} &= \delta\left(-\frac{1}{\kappa^2}\int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}K - I_{\rm ct}\right) \\ &\quad + \int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}\left[\pi_{(d)}{}^{ij}\delta\gamma_{ij} + \pi_{(d)}{}^{i}\delta A_i + \pi_{(\Delta_I)I}\delta\varphi^I + \cdots\right],\end{aligned} \tag{5.29}$$

where the dots denote terms of higher dilatation weight which do not survive after the regulator is removed and $I_{\rm ct}$ is *local* in the boundary fields. Finally we insert the boundary conditions (5.27) and use the diffeomorphism and trace Ward identities (2.165) and (2.168) to arrive at

$$\int_{\Sigma_{r_o}}\boldsymbol{\Theta} = \delta\left(-\frac{1}{\kappa^2}\int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}K - I_{\rm ct}\right) + \int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}\mathcal{A}\delta\sigma. \tag{5.30}$$

It follows that

$$\delta I_{\rm ren}^{\rm on-shell} = \int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}\mathcal{A}\delta\sigma. \tag{5.31}$$

Notice that $\mathcal{A}$ is uniquely determined from boundary data. Furthermore, its integral is conformally invariant. It follows that $\mathcal{A}$ is a conformal density of weight d modulo total derivatives.

There are three cases to discuss now.

1. The unintegrated anomaly vanishes identically:

$$\mathcal{A} \equiv 0. \tag{5.32}$$

This is the case, for instance, for pure AAdS gravity in even dimensions. Our analysis shows that the variational problem in this case is well-posed, provided we augment the Gibbons-Hawking term by the usual counterterms.

2. The integrated anomaly vanishes for a particular conformal class $[g_{(0)}]$,

$$A[g_{(0)}] \equiv \int_{\partial\mathcal{M}} d^d x \sqrt{-g_{(0)}} \mathcal{A}[g_{(0)}] = 0. \tag{5.33}$$

This is the case, for instance, for pure AAdS gravity in odd dimensions with the conformal class represented by the standard metric on the boundary $\mathbb{R} \times S^{d-1}$. When (5.33) holds the anomaly density does not necessarily vanish and so the variational problem with the boundary conditions (5.27) is not well-defined in general since the variation of the action generically contains a non-vanishing boundary term. Nevertheless, the vanishing of the integrated anomaly guarantees that there exists a representative $g_{(0)}$ of the conformal class $[g_{(0)}]$ for which the anomaly density, $\mathcal{A}$, is zero. For instance, for pure AAdS gravity in odd dimensions such a representative is the standard metric on $\mathbb{R} \times S^{d-1}$. Hence one can pick a suitable defining function which induces this particular representative. In practice this means that we want to perform a PBH transformation such that the resulting radial coordinate acts as a defining function which induces the desired representative. We then consider the variational problem *around this gauge* that corresponds to the privileged representative of the conformal structure at the boundary for which the anomaly density vanishes. However, this ensures only that the first order variation of the action will contain no boundary terms. To make the variational problem well-defined to all orders one is forced to break the bulk diffeomorphisms which induce a Weyl transformation on the boundary and consider variations of the bulk fields which preserve a particular representative of the conformal class. In other words, in this case, in order to make the variational problem well-defined to all orders we *must* impose the boundary conditions,

$$\delta g_{(0)ij} = 0, \qquad \delta A_{(0)i} = 0, \qquad \delta\phi_{(0)} = 0, \tag{5.34}$$

where $g_{(0)ij}(x)$ is the chosen representative of the conformal structure and $A_{(0)i}(x)$ and $\phi_{(0)}(x)$ are the leading terms in the asymptotic expansion of the bulk gauge and scalar fields, respectively,

$$A_i(r,x) = A_{(0)i}(x)(1 + \mathcal{O}(e^{-r})), \quad \varphi(r,x) = \phi_{(0)}(x) e^{-(d-\Delta_I)r}(1 + \mathcal{O}(e^{-r})). \tag{5.35}$$

As we have seen, however, this is only possible if one breaks certain bulk diffeomorphisms.

3. The integrated anomaly is non-zero. In this case, to ensure that the variational problem is well defined already at leading order, we have to pick a representative and allow only variations that preserve the corresponding gauge.

To summarize, we have seen that bulk covariance in AlAdS spaces requires that we formulate the variational problem with the boundary conditions (5.27) instead of the stronger (5.26). The counterterms are *essential* in making the variational problem well-defined with such boundary conditions and are exactly on the same footing with the Gibbons-Hawking term. However, when the unintegrated anomaly does not vanish identically, the variational problem can only be well-defined (to all orders) with the boundary conditions (5.34), which can only be imposed if certain bulk diffeomorphisms are broken. The counterterms in this case guarantee that the on-shell action has a well-defined transformation under the broken diffeomorphisms. The transformation is given precisely by the anomaly.

5.2 THE HOLOGRAPHIC CHARGES ARE NOETHER CHARGES

5.2.1 CONSERVED CHARGES ASSOCIATED WITH ASYMPTOTIC SYMMETRIES

We have seen in section 5.1.2 that the renormalized action is invariant under bulk diffeomorphisms and $U(1)$ gauge transformations that asymptotically take the form (5.12) provided $\xi^r = \mathcal{O}(e^{-dr})$. Moreover, requiring that such transformations preserve the boundary conditions (5.27) constrains ξ^i to be an asymptotic conformal Killing vector, i.e. to asymptotically approach a boundary conformal Killing vector (see Appendix 5.A.1 for the precise definition). When the anomaly does not vanish, however, we impose the boundary conditions (5.34) which are only preserved if ξ^i is a boundary *Killing* vector (as opposed to asymptotic conformal Killing vector).

We now apply Noether's theorem to extract the conserved currents and charges associated with these asymptotic symmetries. To this end we first consider the following field variations:

$$\delta_1\psi = f_1(r,x)\mathcal{L}_\xi\psi, \quad \delta_2\psi = f_2(r,x)\delta_\alpha\psi, \tag{5.36}$$

where $f_1(r,x),\ f_2(r,x)$ are arbitrary functions on $\mathcal{M}$ which reduce to functions $\bar{f}_1(x)$ and $\bar{f}_2(x)$ respectively on $\partial\mathcal{M}$, ξ^i is an asymptotic conformal Killing vector of the induced fields on Σ_r and α is a gauge parameter which asymptotically tends to a constant. These, transformations are not a symmetry of the renormalized action unless f_1 and f_2 are constants, but they preserve the boundary conditions (5.27) for arbitrary f_1, f_2. Varying the renormalized action, whose general variation is given by

$$\delta I_{\rm ren} = \int_{\mathcal{M}_{r_o}} \mathbf{E}\delta\psi + \int_{\Sigma_{r_o}} d^d x\sqrt{-\gamma}\left(\pi_{(d)}{}^{ij}\delta\gamma_{ij} + \pi_{(d)}{}^{i}\delta A_i + \pi_{(\Delta_I)I}\delta\varphi^I\right), \qquad (5.37)$$

with respect to such field variations we will now derive the conserved Noether charges.

Electric charge

Let us first consider the transformation $\delta_2\psi$ and derive the corresponding conserved current. Since $\delta_\alpha \mathbf{L} = 0$, we have from (5.2)

$$\mathbf{E}\delta_\alpha\psi = -\mathrm{d}\mathbf{\Theta}(\psi, \delta_\alpha\psi). \qquad (5.38)$$

Hence,

$$\delta_2 I_{\rm ren} = -\int_{\mathcal{M}_{r_o}} f_2 \mathrm{d}\mathbf{\Theta}(\psi, \delta_\alpha\psi) + \int_{\Sigma_{r_o}} d^d x\sqrt{-\gamma} f_2 \pi_{(d)}{}^{i}\partial_i\alpha, \qquad (5.39)$$

But α is asymptotically constant and so the boundary term vanishes. Hence, on-shell the bulk integral on the RHS must vanish for arbitrary f_2, which leads to the conservation law for the $U(1)$ current

$$\mathbf{J}_\alpha \equiv \mathbf{\Theta}(\psi, \delta_\alpha\psi). \qquad (5.40)$$

Since on-shell $\mathbf{J}_\alpha$ is closed, it is locally exact. In fact one easily finds

$$\mathbf{J}_\alpha = \mathrm{d}\mathbf{Q}_\alpha, \qquad (5.41)$$

where $\mathbf{Q}_\alpha = -\alpha * \mathcal{F}$ and $\mathcal{F}_{\mu\nu} = U(\varphi)F_{\mu\nu}$. Then, given a Cauchy surface C, the conserved Noether charge is given by[2]

$$Q = \int_C \mathbf{J}_\alpha = -\int_{\partial\mathcal{M}\cap C} *\mathcal{F}, \qquad (5.42)$$

where we have assumed without loss of generality that $\alpha \to 1$ on $\partial\mathcal{M}$. One can check that this charge is conserved, i.e. independent of the Cauchy surface C, which follows immediately from the field equation

$$\mathrm{d} * \mathcal{F} = 0. \qquad (5.43)$$

Charges associated with boundary conformal isometries

[2] Throughout this chapter we use the convention about the (relative) orientation $\epsilon_{rti_2\cdots i_d} \equiv \epsilon_{ti_2\cdots i_d} = +1$. The minus sign in the definition of the electric charge is included to compensate for this choice of orientation, which is opposite from the conventional one.

The same argument can be applied to derive the conserved currents and Noether charges associated with asymptotic conformal isometries of the induced fields. Again from (5.2) we have

$$\mathbf{E}\mathcal{L}_\xi\psi = \mathrm{d}\left(i_\xi\mathbf{L} - \mathbf{\Theta}(\psi,\mathcal{L}_\xi\psi)\right). \tag{5.44}$$

Hence, defining the current

$$\mathbf{J}[\xi] \equiv \mathbf{\Theta}(\psi,\mathcal{L}_\xi\psi) - i_\xi\mathbf{L}, \tag{5.45}$$

we get

$$\delta_1 I_{\rm ren} = -\int_{\mathcal{M}_{r_o}} f_1\mathrm{d}\mathbf{J}[\xi] + \int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}f_1\left({\pi_{(d)}}^{ij}L_\xi\gamma_{ij} + \pi_{(d)}L_\xi A_i + \pi_{(\Delta_I)I}L_\xi\varphi^I\right). \tag{5.46}$$

Since ξ^i is an asymptotically conformal Killing vector, it follows that

$$\delta_1 I_{\rm ren} = -\int_{\mathcal{M}_{r_o}} f_1\mathrm{d}\mathbf{J}[\xi] + \int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}f_1\left(2{\pi_{(d)}}^i_i + (\Delta_I - d)\pi_{(\Delta_I)I}\varphi^I\right)\frac{1}{d}D_i\xi^i. \tag{5.47}$$

Now, evaluating the LHS using (5.31) and the RHS using the trace Ward identity (2.168), we deduce that on-shell the bulk integral vanishes, which leads to the conservation law

$$\mathrm{d}\mathbf{J}[\xi] = 0. \tag{5.48}$$

Hence, $\mathbf{J}[\xi]$ is locally exact, $\mathbf{J}[\xi] = \mathrm{d}\mathbf{Q}[\xi]$, and it is easily shown that

$$\mathbf{Q}[\xi] = -\frac{1}{\kappa^2} * \mathbf{\Xi}[\xi], \tag{5.49}$$

where the 2-form $\mathbf{\Xi}$ is given by

$$\Xi_{\mu\nu} = \nabla_{[\mu}\xi_{\nu]} + \kappa^2 U(\varphi)F_{\mu\nu}A_\rho\xi^\rho. \tag{5.50}$$

However, $\mathbf{J}[\xi]$ is not the full Noether current in this case as there is an extra contribution with support on Σ_{r_o}. To derive the correct form of the current we use (5.28) to rewrite (5.46) as

$$\begin{aligned}\delta_1 I_{\rm ren} &= \int_{\mathcal{M}_{r_o}} \mathrm{d}f_1\wedge\mathbf{J}[\xi] - \int_{\Sigma_{r_o}} f_1\mathbf{J}[\xi] \\ &\quad + \int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}f_1\left({\pi_{(d)}}^{ij}L_\xi\gamma_{ij} + \pi_{(d)}L_\xi A_i + \pi_{(\Delta_I)I}L_\xi\varphi^I\right) \\ &= \int_{\mathcal{M}_{r_o}} \mathrm{d}f_1\wedge\mathbf{J}[\xi] + \int_{\Sigma_{r_o}} f_1 i_\xi\mathbf{L} + \int_{\Sigma_{r_o}} d^dx f_1\delta_\xi\left(\frac{1}{\kappa^2}\sqrt{-\gamma}[K-(K-\lambda)_{\rm ct}]\right).\end{aligned} \tag{5.51}$$

Since ξ is tangent to Σ_{r_o}, the second term vanishes. To put the last term in the desired form, we define the d-form

$$\mathbf{B} \equiv -\frac{1}{\kappa^2}[K-(K-\lambda)_{\rm ct}] *_\Sigma \mathbf{1} \tag{5.52}$$

on Σ_r which is covariant w.r.t. diffeomorphisms within Σ_r. Using the identity $L_\xi = \bar{\mathrm{d}} i_\xi + i_\xi \bar{\mathrm{d}}$ on forms, $\bar{d}$ being the exterior derivative on Σ_r, we obtain

$$\begin{aligned}
\delta_1 I_{\rm ren} &= \int_{\mathcal{M}_{r_o}} \mathrm{d}f_1 \wedge \mathbf{J}[\xi] - \int_{\Sigma_{r_o}} f_1 \delta_\xi \mathbf{B} \\
&= \int_{\mathcal{M}_{r_o}} \mathrm{d}f_1 \wedge \mathbf{J}[\xi] - \int_{\Sigma_{r_o}} f_1 \bar{\mathrm{d}} i_\xi \mathbf{B} \\
&= \int_{\mathcal{M}_{r_o}} \mathrm{d}f_1 \wedge \mathbf{J}[\xi] + \int_{\Sigma_{r_o}} \bar{\mathrm{d}} f_1 \wedge i_\xi \mathbf{B} \\
&= \int_{\mathcal{M}_{r_o}} \mathrm{d}f_1 \wedge \mathbf{J}[\xi] + \int_{\mathcal{M}_{r_o}} \rho(\Sigma_{r_o}) \wedge \mathrm{d}f_1 \wedge i_\xi \mathbf{B},
\end{aligned} \tag{5.53}$$

where $\rho(\Sigma_r)$ is a one-form with delta function support on Σ_r, known as the Poincaré dual of Σ_r in $\mathcal{M}$. Therefore, the full Noether current is

$$\tilde{\mathbf{J}}[\xi] \equiv \mathbf{J}[\xi] - \rho(\Sigma_{r_o}) \wedge i_\xi \mathbf{B}. \tag{5.54}$$

Given a Cauchy surface C, we now define the Noether charge

$$Q[\xi] \equiv \int_C \tilde{\mathbf{J}}[\xi] = \int_{\partial\mathcal{M}\cap C} \left(\mathbf{Q}[\xi] - i_\xi \mathbf{B}\right). \tag{5.55}$$

If C and C' are two Cauchy surfaces whose intersection with $\partial\mathcal{M}$ bounds a domain $\Delta \subset \partial\mathcal{M}$, then Stokes' theorem and the conservation law (5.48) imply

$$\begin{aligned}
Q_C[\xi] - Q_{C'}[\xi] &= \int_{\Delta\subset\partial\mathcal{M}} \left(\mathbf{J}[\xi] - \mathrm{d} i_\xi \mathbf{B}\right) \\
&= \int_{\partial\mathcal{M}} d^d x \sqrt{-\gamma} \left(\pi_{(d)}{}^{ij} L_\xi \gamma_{ij} + \pi_{(d)} L_\xi A_i + \pi_{(\Delta_I)I} L_\xi \varphi^I\right) \\
&= \int_{\partial\mathcal{M}} d^d x \sqrt{-\gamma} \left(\pi_{(d)}{}^i_i + (\Delta_I - d)\pi_{(\Delta_I)I}\varphi^I\right) \frac{1}{d} D_i \xi^i \\
&= \int_{\partial\mathcal{M}} d^d x \sqrt{-\gamma} \mathcal{A} \frac{1}{d} D_i \xi^i.
\end{aligned} \tag{5.56}$$

Therefore, if the anomaly vanishes, this charge is conserved for any asymptotic conformal Killing vector. However, if the anomaly is non-zero, it is only conserved for symmetries associated with boundary Killing vectors.

5.2.2 Holographic charges

Let us now derive an alternative form of the conserved charges by considering instead of (5.36) the following variations:

$$\delta'_1 \psi = \mathcal{L}_{\epsilon\xi} \psi, \quad \delta'_2 \psi = \delta_\alpha \psi, \tag{5.57}$$

where ξ^i is again an asymptotic conformal Killing vector but now α and ϵ reduce to arbitrary functions on Σ_{r_o}. In contrast to (5.36), these field variations *are* a symmetry of the action, but they violate the boundary conditions (5.27).

Since these are symmetries of the renormalized action we have

$$\begin{aligned} 0 = \delta_2' I_{\text{ren}} &= \int_{\mathcal{M}_{r_o}} \mathbf{E}\delta_2'\psi + \int_{\Sigma_{r_o}} d^d x\sqrt{-\gamma}\pi_{(d)}{}^i\partial_i\alpha \\ &= \int_{\mathcal{M}_{r_o}} \mathbf{E}\delta_2'\psi - \int_{\Sigma_{r_o}} d^d x\sqrt{-\gamma}\alpha\partial_i\pi_{(d)}{}^i. \end{aligned} \tag{5.58}$$

But now α is arbitrary and so we conclude that on-shell we must have

$$\partial_i\pi_{(d)}{}^i = 0, \tag{5.59}$$

which also follows immediately from the first equation in (3.121). Hence the quantity

$$\mathcal{Q} \equiv -\int_{\partial\mathcal{M}\cap C} d\sigma_i\pi_{(d)}{}^i, \tag{5.60}$$

defines a conserved charge, namely the *holographic* electric charge.

Similarly,

$$\begin{aligned} 0 &= \delta_1' I_{\text{ren}} \\ &= \int_{\mathcal{M}_{r_o}} \mathbf{E}\delta_1'\psi + \int_{\Sigma_{r_o}} d^d x\sqrt{-\gamma}\left(\pi_{(d)}{}^{ij}L_{\epsilon\xi}\gamma_{ij} + \pi_{(d)}{}^i L_{\epsilon\xi}A_i + \pi_{(\Delta_I)I}L_{\epsilon\xi}\varphi^I\right) \\ &= \int_{\mathcal{M}_{r_o}} \mathbf{E}\delta_1'\psi + \int_{\Sigma_{r_o}} d^d x\sqrt{-\gamma}\epsilon\left(\pi_{(d)}{}^{ij}L_{\xi}\gamma_{ij} + \pi_{(d)}{}^i L_{\xi}A_i + \pi_{(\Delta_I)I}L_{\xi}\varphi^I\right) \\ &\quad + \int_{\Sigma_{r_o}} d^d x\sqrt{-\gamma}\left(2\pi_{(d)}{}^i_j + \pi_{(d)}{}^i A_j\right)\xi^j D_i\epsilon. \end{aligned} \tag{5.61}$$

Therefore, after integration by parts in the last term and using the fact that ϵ is arbitrary, we conclude that on-shell we must have

$$\begin{aligned} D_i\left[(2\pi_{(d)}{}^i_j + \pi_{(d)}{}^i A_j)\xi^j\right] &= \pi_{(d)}{}^{ij}L_\xi\gamma_{ij} + \pi_{(d)}{}^i L_\xi A_i + \pi_{(\Delta_I)I}L_\xi\varphi^I \\ &= \left(2\pi_{(d)}{}^i_i + (\Delta_I - d)\pi_{(\Delta_I)I}\varphi^I\right)\frac{1}{d}D_i\xi^i \\ &= \mathcal{A}\frac{1}{d}D_i\xi^i, \end{aligned} \tag{5.62}$$

where we have used the trace Ward identity (2.168) in the last step. Hence the quantity

$$\mathcal{Q}[\xi] \equiv \int_{\partial\mathcal{M}\cap C} d\sigma_i\left(2\pi_{(d)}{}^i_j + \pi_{(d)}{}^i A_j\right)\xi^j, \tag{5.63}$$

defines a holographic conserved charge associated with every asymptotic conformal Killing vector, if the anomaly vanishes, or every boundary Killing vector, if the anomaly does not vanish.

From the above analysis we have obtained two apparently different expressions for the conserved charges associated with every asymptotic symmetry. However, we show in the following lemma that the two expressions for the conserved charges are equivalent.

Lemma 5.2.1 *Let ψ denote an AlAdS solution of the bulk equations of motion possessing an asymptotic timelike Killing vector k and possibly a set of $N-1$ asymptotic spacelike Killing vectors m_α with closed orbits, forming a maximal set of commuting asymptotic isometries. In adapted coordinates such that $k=\partial_t$ and $m_\alpha=\partial_{\phi^\alpha}$ the background ψ is independent of the coordinates $x^a=\{t,\phi^\alpha\}$. Then,*
i)

$$\int_{\partial\mathcal{M}\cap C} d\sigma_i \pi_{(d)}{}^i = \int_{\partial\mathcal{M}\cap C} *\mathcal{F}. \tag{5.64}$$

ii) If in addition the background metric and gauge field take asymptotically the form

$$d\bar{s}^2 \equiv \gamma_{ij}dx^i dx^j = \tau_{ab}dx^a dx^b + \sigma_{ij}d\tilde{x}^i d\tilde{x}^j, \quad \mathbf{A}\equiv A_i dx^i = A_a dx^a, \tag{5.65}$$

where τ_{ab}, σ_{ij} and A_a depend only on the rest of the transverse coordinates $\tilde{x}^i$ as well as the radial coordinate r, then, for any asymptotic conformal Killing vector ξ,

$$\int_{\partial\mathcal{M}\cap C} d\sigma_i \left(2\pi_{(d)}{}^i_j + \pi_{(d)}{}^i A_j\right)\xi^j = -\int_{\partial\mathcal{M}\cap C} \left(\mathbf{Q}[\xi] - i_\xi \mathbf{B}\right). \tag{5.66}$$

A proof of this lemma can be found in Appendix 5.A.2. However, a few comments are in order regarding the condition (5.65) we have assumed in order to prove the second part of the lemma. Firstly, as can be seen from the explicit proof, it is only required in order to show equivalence of the charges for true asymptotic *conformal* isometries, i.e. with *non-zero* conformal factor. Otherwise, this condition is not used in the proof. Secondly, in certain special cases the fact that the background takes the form (5.65) turns out to be a consequence of the existence of the set of commuting isometries and the field equations.

More specifically, the condition that the background takes the form (5.65) is closely related to the integrability of the $D-N$-dimensional submanifolds orthogonal to k and m_α. In particular, it was shown in [49], Theorem 7.7.1, using Frobenius' theorem, that for pure gravity in four dimensions, the 2-planes orthogonal to a timelike isometry k and a rotation m are integrable, and hence the metric takes the form (5.65). This result can be easily extended to include a Maxwell field as well as scalar fields in four dimensions [93, 94]. More recently, this result was generalized for pure gravity in D dimensions and $D-2$ orthogonal (non-orthogonal) commuting isometries in [95] ([96]). It appears that these results cannot be generalized in a straightforward way to include gauge fields for $D>4$, or for less than $D-2$

commuting isometries in D dimensions. Obviously the restriction to $D-2$ commuting isometries is too strong for our purposes since even AdS_D only has $[(D+1)/2]$ commuting isometries, which is less than $D-2$ for $D>5$.

Despite the fact that we lack a general proof of (5.65) as a consequence of the presence of the commuting isometries and the field equations, this condition is satisfied by a very wide range of AlAdS spacetimes, including Taub-Nut-AdS and Taub-Bolt-AdS [97, 98]. It would be very interesting to determine what are the most general conditions so that (5.65) holds.

COMMENT ON SCHEME-DEPENDENCE

An important consequence of the above lemma is that the electric charge, as well as, the conserved holographic charges associated with an asymptotic conformal Killing vector ξ which is tangent to $\partial\mathcal{M}\cap C$ are *scheme-independent* since they can be expressed entirely in terms of scheme-independent bulk quantities. By following the argument in the proof of this lemma in Appendix 5.A.2 one can easily see that indeed any local finite counterterm does not affect the value of such charges. The only charges which are scheme-dependent, therefore, are those associated with an asymptotic conformal Killing vector ξ such that

$$\int_{\partial\mathcal{M}\cap C} i_\xi \mathbf{B} \neq 0. \tag{5.67}$$

Within the maximal set of commuting asymptotic isometries, the only such charge is the mass, which involves the asymptotic timelike Killing vector $k=\partial_t$. As we will see below, the only other thermodynamic quantity which is scheme-dependent is the on-shell action, whose scheme-dependence precisely counteracts that of the mass to ensure that the first law is scheme-independent.

The same conclusions follow also from the identification of the holographic charges with the Wald Hamiltonians which we now discuss.

5.2.3 WALD HAMILTONIANS

We now give a third derivation of the conserved charges as 'Hamiltonians' on the covariant phase space [99, 100, 90]. Some results relevant to this section are collected in Appendix 5.A.3.

Let ξ be an asymptotic conformal Killing vector and α an asymptotically constant gauge transformation, namely

$$\mathcal{L}_\xi\psi = \mathcal{L}_{\hat{\xi}}\psi + \delta_{\hat{\alpha}}\psi + \mathcal{O}(e^{-s_+ r}), \quad \delta_\alpha\psi = \mathcal{O}(e^{-s^+ r}), \tag{5.68}$$

where $\hat{\alpha}$, $\hat{\xi}$ and s^+ are given in Appendix 5.A.1. The 'Hamiltonians' which generate these symmetries in phase space must satisfy Hamilton's equations, which in the covariant phase space formalism take the form

$$\delta H_\xi = \Omega_C(\psi, \delta\psi, \mathcal{L}_\xi \psi), \quad \delta H_\alpha = \Omega_C(\psi, \delta\psi, \delta_\alpha \psi), \tag{5.69}$$

where the pre-symplectic form Ω_C is defined in (5.205). The Hamiltonians exist if these equations can be integrated in configuration space to give H_ξ and H_α. As is discussed in Appendix 5.A.3, the symplectic form is independent of the Cauchy surface used to define it if the anomaly vanishes or if the variations are associated with boundary Killing vectors. It follows that the corresponding Hamiltonians are conserved, provided the 'integration' constant is also independent of the Cauchy surface. We further discuss this issue below.

Let us first consider H_α which can be obtained very easily. Using the result for the symplectic form in (5.198) we have

$$\delta H_\alpha = \int_{\partial\mathcal{M}\cap C} \delta \mathbf{Q}_\alpha, \tag{5.70}$$

and hence, up to a constant,

$$H_\alpha = \int_{\partial\mathcal{M}\cap C} \mathbf{Q}_\alpha = -\int_{\partial\mathcal{M}\cap C} *\mathcal{F}, \tag{5.71}$$

taking $\alpha \to 1$ asymptotically. Therefore, once again, we have derived the conserved electric charge.

Consider next H_ξ. From (5.197) we have

$$\delta H_\xi = \int_{\partial\mathcal{M}\cap C} \left(\delta \mathbf{Q}[\xi] - i_\xi \mathbf{\Theta}\right). \tag{5.72}$$

This equation has a non-trivial integrability condition. Applying a second variation and using the commutativity of two variations, $\delta_1\delta_2 - \delta_2\delta_1 = 0$, we obtain the integrability condition [90]

$$\int_{\partial\mathcal{M}\cap C} i_\xi \omega(\psi, \delta\psi_1, \delta\psi_2) = 0. \tag{5.73}$$

Since ξ is tangent to Σ_r, from (5.208) follows that this is equivalent to

$$\int_{\partial\mathcal{M}\cap C} d^{d-1}x \xi^t \left\{\delta_2(\sqrt{-\gamma}\mathcal{A})\delta_1\sigma - 1 \leftrightarrow 2\right\} = 0. \tag{5.74}$$

Therefore, if the anomaly vanishes, a Hamiltonian associated to any asymptotic CKV ξ exists. However, if there is an anomaly and $\xi^t \neq 0$, then a Hamiltonian for ξ exists only if the stronger boundary condition (5.34) is used - i.e. a particular representative of the conformal class is kept fixed - in agreement with the analysis of the variational problem.

The same conclusion can be drawn by trying to integrate (5.72) directly. This is possible provided we can find a d-form $\mathbf{B}$ such that

$$\int_{\partial\mathcal{M}\cap C} i_\xi \mathbf{\Theta} = \delta \int_{\partial\mathcal{M}\cap C} i_\xi \mathbf{B}. \tag{5.75}$$

Once such a form is found, then H_ξ exists and it is given by

$$H_\xi = \int_{\partial\mathcal{M}\cap C} (\mathbf{Q}[\xi] - i_\xi \mathbf{B}). \tag{5.76}$$

However, since ξ is tangent to Σ_r, we can use (5.28), the boundary conditions (5.27) and the trace Ward identity (2.168) to obtain

$$\int_{\Sigma_{r_o}\cap C} i_\xi \mathbf{\Theta} = \frac{1}{\kappa^2}\delta \int_{\Sigma_{r_o}\cap C} d\sigma_i \xi^i [K - (K-\lambda)_{\rm ct}] - \int_{\Sigma_{r_o}\cap C} d\sigma_i \xi^i \mathcal{A}\delta\sigma. \tag{5.77}$$

Therefore, if $\xi^t \neq 0$, then such a form exists for the boundary conditions (5.27) provided the anomaly vanishes, in complete agreement with the conclusion from the integrability condition. Moreover, (5.77) shows that when such a $\mathbf{B}$ exists it coincides with $\mathbf{B}$ in (5.52) and hence, the corresponding Hamiltonian is precisely the Noether charge (5.55).

Notice that the Wald Hamiltonians are only defined up to quantities in the kernel of the variations. In particular, when integrating (5.75) to obtain (5.76), one can add to H_ξ an integral of a local density constructed only from boundary data and the asymptotic conformal Killing vector ξ. This 'integration constant' is constrained by the fact that the Hamiltonians should be conserved. In particular, if H_ξ is a Wald Hamiltonian, so is

$$H'_\xi = H_\xi + \int_{\partial\mathcal{M}\cap C} d\sigma_i H^i_j \xi^j, \tag{5.78}$$

provided H^i_j is constructed locally from boundary data, has dilatation weight d, and it is covariantly conserved.

In fact such ambiguity is present in $AAdS_{2k+1}$ spacetimes and has caused some confusion in the literature. $AAdS_{2k+1}$ spacetimes are special in that the boundary is conformally flat, and even-dimensional conformally flat spacetimes admit local covariantly conserved stress energy tensors. This is true in all even dimensions, as we discuss in Appendix 5.A.4. The best known case is the four dimensional one: the tensor

$$H^i_j \equiv \frac{1}{4}\left(-R^i_k R^k_j + \frac{2}{3} R R^i_j + \frac{1}{2} R^k_l R^l_k \delta^i_j - \frac{1}{4} R^2 \delta^i_j\right) \tag{5.79}$$

is covariantly conserved provided the metric is conformally flat. This tensor is well-known from studies of quantum field theories in curved backgrounds, see [101] (where it is called ${}^3H_{\mu\nu}$). It has been called 'accidentally conserved' in [101] because it is not the limit of a local tensor that is conserved in non-conformally flat spacetimes and cannot be derived by varying a local term. The same tensor is the holographic stress energy tensor of[3] AdS_5 [30]!

This tensor also appeared recently in comparisons between the 'conformal mass' of [102] and the holographic mass, see [102, 81] and Appendix 5.A.4. It follows that the conserved charges according to both definitions are Wald Hamiltonians. It also follows from this discussion that the conformal mass is the mass of the spacetime *relative* to the AdS background. Furthermore, we conclude that the definition of [42] does not extend to general AlAdS spacetimes since H_{ij} is not covariantly conserved when the boundary metric is not conformally flat and we already know that the holographic charges are conserved for general AlAdS (and as shown in this section are also Wald Hamiltonians).

5.3 THE FIRST LAW OF BLACK HOLE MECHANICS

The above detailed description of the conserved charges allows us to study the thermodynamics of AlAdS black hole spacetimes quite generically. In particular, we will consider a black hole solution of (5.1) possessing a timelike Killing vector k and possibly a set of spacelike isometries with closed orbit forming a maximal set of commuting isometries as in lemma 5.2.1, but here we require that these isometries be exact and not just asymptotic. The form (5.65) of the metric then implies that these bulk isometries correspond to boundary isometries and not merely asymptotic boundary conformal isometries. Moreover, we will assume that the event horizon, $\mathcal{N}$, of the black hole is a non-degenerate bifurcate Killing horizon of a timelike (outside the horizon) Killing vector χ such that the surface gravity, $\hat{\kappa}$, of the horizon is given by

$$\hat{\kappa}^2 = -\frac{1}{2}\nabla^\mu\chi^\nu\nabla_\mu\chi_\nu|_{\mathcal{N}}. \tag{5.80}$$

The inverse temperature, β, then is

$$\beta = T^{-1} = \frac{2\pi}{\hat{\kappa}}. \tag{5.81}$$

Let us begin with a lemma which is central to our analysis.

Lemma 5.3.1 *Let ξ be a bulk Killing vector, I the renormalized on-shell Euclidean action and $\mathcal{H} = \mathcal{N} \cap C$ the intersection of the horizon with the Cauchy surface. Let also t be the adapted coordinate to the timelike isometry k so that $k = \partial_t$.*

i) If $\xi^t = 1$, then[4]

$$\beta\mathcal{Q}[\xi] - I = -\beta\int_{\mathcal{H}} \mathbf{Q}[\xi]. \tag{5.82}$$

[3]More precisely, (5.79) is the holographic stress energy tensor associated to a bulk solution that is conformally flat, see (3.20) of [30]; all such solutions are locally isometric to AdS_5.

[4]Note that the integrals over $\mathcal{H}$ should be done with an *inward*-pointing unit vector.

ii) If $\xi^t = 0$, then

$$\mathcal{Q}[\xi] = -\int_{\mathcal{H}} \mathbf{Q}[\xi]. \tag{5.83}$$

Proof:
By Stokes' theorem

$$\begin{aligned}
\int_{\partial\mathcal{M}\cap C} \mathbf{Q}[\xi] &= \int_C \mathrm{d}\mathbf{Q}[\xi] + \int_{\mathcal{H}} \mathbf{Q}[\xi] \\
&= \int_C (\boldsymbol{\Theta}(\psi, \mathcal{L}_\xi \psi) - i_\xi \mathbf{L}) + \int_{\mathcal{H}} \mathbf{Q}[\xi] \\
&= -\int_C i_\xi \mathbf{L} + \int_{\mathcal{H}} \mathbf{Q}[\xi].
\end{aligned} \tag{5.84}$$

Now, (3.41) and the fact that the background is stationary allow us to write

$$\int_C i_\xi \mathbf{L} = -\int_{\Sigma_{r_o}\cap C} d\sigma_i \xi^i \lambda, \tag{5.85}$$

where the minus sign arises due to our choice of orientation (see footnote 2). Hence,

$$\begin{aligned}
\int_{\Sigma_{r_o}\cap C} (\mathbf{Q}[\xi] - i_\xi \mathbf{B}) &= \int_{\mathcal{H}} \mathbf{Q}[\xi] - \int_C i_\xi \mathbf{L} - \frac{1}{\kappa^2} \int_{\Sigma_{r_o}\cap C} d\sigma_i \xi^i \left(K_{(d)} + \lambda_{\mathrm{ct}}\right) \\
&= \int_{\mathcal{H}} \mathbf{Q}[\xi] - \frac{1}{\kappa^2} \int_{\Sigma_{r_o}} d\sigma_i \xi^i \left(K_{(d)} - \lambda_{(d)}\right).
\end{aligned} \tag{5.86}$$

For $\xi^t = 0$ the last term vanishes. If however $\xi^t = 1$, then we can use the fact that the background is stationary to obtain

$$\frac{\beta}{\kappa^2} \int_{\partial\mathcal{M}\cap C} d\sigma_i \xi^i \left(K_{(d)} - \lambda_{(d)}\right) = -\frac{1}{\kappa^2} \int_{\partial\mathcal{M}} d^d x \sqrt{\gamma_E} \left(K_{(d)} - \lambda_{(d)}\right) \equiv I, \tag{5.87}$$

where γ_{Eij} is the Euclidean metric. This completes the proof.

□

Since the right hand sides of (5.82) and (5.83) are manifestly scheme-independent, this lemma implies that the left hand sides of these expressions, under the corresponding conditions, are scheme-independent too. As we will see in the next section, this implies that all charges associated with bulk Killing vectors, except for the mass, are scheme-independent. Moreover, the scheme-dependence of the mass is now seen to exactly cancel that of the on-shell action, as it was advertised earlier.

5.3.1 Black hole thermodynamics

This lemma, besides relating the conserved charges to local integrals over the horizon, leads immediately to the quantum statistical relation [48]

$$I = \beta G\left(T, \Omega_i, \Phi\right), \tag{5.88}$$

where

$$G\left(T, \Omega_i, \Phi_i\right) \equiv M - TS - \Omega_i J_i - \Phi\, Q, \tag{5.89}$$

is the Gibbs free energy. (5.88) follows trivially from lemma 5.3.1 provided

$$\mathcal{Q}[\chi] + \int_{\mathcal{H}} \mathbf{Q}[\chi] = M - TS - \Omega_i J_i - \Phi\, Q, \tag{5.90}$$

where χ is the null generator of the horizon, normalized such that $\chi^t = 1$. To show that this is the case we need a precise definition of the thermodynamic variables appearing in the Gibbs free energy.

Electric charge

Using Stokes' theorem, the electric charge (5.42) is also given by

$$Q \equiv -\int_{\partial\mathcal{M}\cap C} *\mathcal{F} = -\int_{\mathcal{H}} *\mathcal{F}. \tag{5.91}$$

Electric potential

We define the electric potential, Φ, conjugate to the charge Q, by

$$\Phi \equiv -A_\mu \chi^\mu|_{\mathcal{H}}. \tag{5.92}$$

This is well-defined, for $A_\mu \chi^\mu$ is constant on $\mathcal{H}$. To see this consider a vector field t tangent to the horizon. Then,

$$t \cdot \partial(A_\mu \chi^\mu) = t^\rho(\chi^\mu F_{\rho\mu} + \mathcal{L}_\chi A_\rho) = t^\rho \chi^\mu F_{\rho\mu}. \tag{5.93}$$

But since t is tangent to $\mathcal{H}$, $t|_{\mathcal{H}} \propto \chi$ and hence

$$t \cdot \partial(A_\mu \chi^\mu)|_{\mathcal{H}} = 0. \tag{5.94}$$

Mass

In order to define the mass we have to supply an asymptotic timelike Killing vector. In contrast to asymptotically flat spacetimes, in AlAdS spacetimes there is an additional subtlety in that there can be a non-zero angular velocity, Ω_i^∞, at spatial infinity. This is the case, for example, for the Kerr-AdS black holes in Boyer-Lindquist coordinates as we will see below. In such a rotating frame, there are many timelike Killing vectors obtained by appropriate linear combinations of ∂_t and ∂_{ϕ_i}. Using a general timelike Killing vector will result in a conserved quantity that is linear combination of the true mass and the angular momenta. To resolve this issue we first go to a non-rotating frame by the coordinate transformation

$$t' = t, \quad \phi_i' = \phi_i - \Omega_i^\infty t. \tag{5.95}$$

In this frame there is no such ambiguity and one can define the mass, as usual, using the Killing vector $\partial_{t'}$. In terms of the original coordinates we have

$$\partial_{t'} = \frac{\partial t}{\partial t'}\partial_t + \frac{\partial \phi_i}{\partial t'}\partial_{\phi_i} = \partial_t + \Omega_i^\infty \partial_{\phi_i}. \tag{5.96}$$

Therefore, the mass is defined as

$$M \equiv \mathcal{Q}[\partial_t + \Omega_i^\infty \partial_{\phi_i}]. \tag{5.97}$$

Angular velocities

Let $\chi = \partial_t + \Omega_i^H \partial_{\phi_i}$ be the null generator of the horizon. This defines the angular velocities, Ω_i^H, of the horizon. We define the angular velocities, Ω_i, by

$$\Omega_i \equiv \Omega_i^H - \Omega_i^\infty. \tag{5.98}$$

Angular momenta

We define the angular momenta, J_i, by

$$J_i \equiv -\mathcal{Q}[\partial_{\phi_i}] = \int_{\mathcal{H}} \mathbf{Q}[\partial_{\phi_i}], \tag{5.99}$$

where the second equality follows from lemma 5.3.1.

Entropy

Finally, using Wald's definition of the entropy [88] (see also [103]) we get

$$-\beta \int_{\mathcal{H}} \mathbf{Q}[\chi] = S + \beta\Phi\, Q. \tag{5.100}$$

With these definitions it is now straightforward to see that (5.90), and hence (5.88) hold.

5.3.2 FIRST LAW

To derive the first law we consider variations that satisfy our boundary conditions. Namely, if the anomaly vanishes, then the boundary conditions (5.27) should be satisfied, otherwise (5.34) should hold, i.e. a representative of the conformal class should be kept fixed. We will discuss the significance of this in the next section. In other words, we only vary the normalizable mode of the solutions, as one might have anticipated on physical grounds.[5]

We now show, following Wald et al. [88, 89], that these variations satisfy the first law. From equation (5.197) we have

$$\mathrm{d}\left(\delta\mathbf{Q}[\chi] - i_\chi\mathbf{\Theta}\right) = \omega(\psi, \delta\psi, \mathcal{L}_\chi\psi). \tag{5.101}$$

Hence,

$$\begin{aligned}\int_C \mathrm{d}\left(\delta\mathbf{Q}[\chi] - i_\chi\mathbf{\Theta}\right) &= \int_{\partial\mathcal{M}\cap C}\left(\delta\mathbf{Q}[\chi] - i_\chi\mathbf{\Theta}\right) - \int_{\mathcal{H}}\left(\delta\mathbf{Q}[\chi] - i_\chi\mathbf{\Theta}\right) \\ &= \int_C \omega(\psi, \delta\psi, \mathcal{L}_\chi\psi) = 0, \end{aligned} \tag{5.102}$$

since χ is a Killing vector. However, χ is tangent to $\mathcal{H}$ and so we arrive at

$$\int_{\partial\mathcal{M}\cap C}\left(\delta\mathbf{Q}[\chi] - i_\chi\mathbf{\Theta}\right) = \int_{\mathcal{H}}\delta\mathbf{Q}[\chi]. \tag{5.103}$$

Consider first the left hand side. Writing $\chi = \partial_t + \Omega_i^\infty\partial_{\phi_i} + \Omega_i\partial_{\phi_i}$ and using the fact that ∂_{ϕ_i} is tangent to $\partial\mathcal{M}$ we get

$$\begin{aligned}\int_{\partial\mathcal{M}\cap C}\left(\delta\mathbf{Q}[\chi] - i_\chi\mathbf{\Theta}\right) &= \int_{\partial\mathcal{M}\cap C}\left(\delta\mathbf{Q}[\partial_t + \Omega_i^\infty\partial_{\phi_i}] - i_t\mathbf{\Theta}\right) + \Omega_i\int_{\partial\mathcal{M}\cap C}\delta\mathbf{Q}[\partial_{\phi_i}] \\ &= \delta\int_{\partial\mathcal{M}\cap C}\left(\mathbf{Q}[\partial_t + \Omega_i^\infty\partial_{\phi_i}] - i_t\mathbf{B}\right) + \Omega_i\delta\int_{\partial\mathcal{M}\cap C}\mathbf{Q}[\partial_{\phi_i}] \\ &= -(\delta M - \Omega_i\delta J_i). \end{aligned} \tag{5.104}$$

In order to evaluate the right hand side of (5.103) we need to match the horizons of the perturbed and unperturbed solutions [88], the unit surface gravity generators, $\tilde{\chi} \equiv \frac{1}{\tilde{\kappa}}\chi$, of the horizons and the electric potentials. From (5.100) then we immediately get

$$-\int_{\mathcal{H}}\delta\mathbf{Q}[\chi] = T\delta S + \Phi\delta Q. \tag{5.105}$$

Therefore, (5.103) is a statement of the first law, namely

$$\delta M = T\delta S + \Omega_i\delta J_i + \Phi\delta Q. \tag{5.106}$$

[5] Note that the non-normalizable mode determines the conformal class at the boundary. The non-normalizable mode *together* with a defining function specify a representative of the conformal class.

However, we emphasize that the variations in this expression must satisfy the appropriate boundary conditions that make the variational problem well-defined. Namely, if the anomaly vanishes, then the boundary conditions (5.27) should be satisfied, but if there is a non-zero anomaly, then (5.34) must be satisfied instead, i.e. the representative of the conformal class should be kept fixed. We will discuss the significance of this in the next section.

5.3.3 DEPENDENCE ON THE REPRESENTATIVE OF THE CONFORMAL CLASS

Let us now discuss how the thermodynamic variables defined above depend on the representative of the conformal class at the boundary.

To this end we recall that a Weyl transformation on the boundary is induced by a PBH transformation, i.e. a combined bulk diffeomorphism and a compensating gauge transformation, given by (5.12) after setting $\xi_o = 0$ and $\alpha_o = 0$. However, in order to be able to compare the mass and angular momenta for the two representatives of the conformal class we require that the two representatives have the same maximal set of commuting isometries, i.e. we restrict to Weyl factors $\delta\sigma$ which are independent of the coordinates $t,\ \phi_\alpha$ adapted to the asymptotic isometries.

It is now straightforward to see that all intensive thermodynamic variables, namely the temperature T, the angular velocities Ω_i and the electric potential Φ are invariant under such diffeomorphisms. The same holds for the entropy S, the angular momenta J_i, and the electric charge Q, since, as we saw above, they can be expressed as local integrals over the horizon. Therefore, the only quantities which could potentially transform non-trivially are the mass M and the on-shell Euclidean action I. However, their transformations are not independent since they are constrained by the quantum statistical relation (5.88), namely

$$\delta_\sigma I = \beta \delta_\sigma M. \tag{5.107}$$

This is a significant result which cannot be seen easily otherwise. We know that

$$\delta_\sigma I = -\int_{\partial\mathcal{M}} d^d x \sqrt{\gamma_E} \mathcal{A} \delta\sigma, \tag{5.108}$$

while

$$\delta_\sigma M = -2 \int_{\partial\mathcal{M}\cap C} d\sigma_i \{ (2\tilde{\pi}_{(d)}{}^i_j + \tilde{\pi}_{(d)}{}^i A_j) \tilde{k}^j \delta\sigma + \cdots \}, \tag{5.109}$$

where $\tilde{k} = \partial_t + \Omega_i^\infty \partial_{\phi_i}$ and the dots stand for terms involving derivatives of the Weyl factor $\delta\sigma$. One can check this explicitly in certain examples by directly evaluating the transformation of the renormalized stress tensor under a PBH transformation [30, 74].

As a final point let us consider how (5.106) would be modified if there is a non-vanishing anomaly and we allow for variations which keep fixed only the conformal class and not a particular representative. In this case, (5.77) implies that

$$-\int_{\partial\mathcal{M}\cap C}(\delta\mathbf{Q}[\chi]-i_\chi\mathbf{\Theta})=-T\delta_\sigma I+\delta M-\Omega_i\delta J_i, \tag{5.110}$$

and the first law should be modified to

$$\begin{aligned}\delta M &= T\delta_\sigma I+T\delta S+\Omega_i\delta J_i+\Phi\delta Q\\ &= \delta_\sigma M+T\delta S+\Omega_i\delta J_i+\Phi\delta Q,\end{aligned} \tag{5.111}$$

where $\delta\sigma$ is the Weyl factor by which the representative of the conformal class is changed due to the variation δ and the second equality follows from (5.107).

We can now state precisely how the first law works in the presence of a non-vanishing anomaly. A generic variation δ will not keep the conformal representative fixed and it will induce a Weyl transformation $\delta\sigma$. We should then undo this Weyl transformation by a PBH transformation with Weyl factor $-\delta\sigma$. Then, (5.111) ensures that the combined variation, which *does* keep the conformal representative fixed, satisfies the usual first law. The general Kerr-AdS black hole in five dimensions provides a clear illustration of this.

5.4 EXAMPLES

In this section we will demonstrate our analysis by two examples, the Kerr-Newman-AdS black hole in four dimensions [104, 105] and the general Kerr-AdS black hole in five dimensions [106]. The second example provides a clear illustration of the role of the conformal anomaly in the thermodynamics.

Before focusing on the specific examples however we discuss the steps and subtleties involved in the computation. Recall that the defining feature of the counterterm method is that the on-shell action of AdS gravity can be rendered finite on *any* solution by adding to the action a set of *local covariant* boundary counterterms. One should not forget, however, that the precise form of the counterterms depends on the regularization/renormalization scheme. The counterterms used in the literature were derived using as regulator a cut-off in the Fefferman-Graham radial coordinate z [29], or equivalently in the radial coordinate r we use here. The cut-off hypersurface $r=r_o$ is in general different from the hypersurfaces $\tilde{r}=const.$, where $\tilde{r}$ is another radial coordinate that might appear naturally in the bulk metric. So, to evaluate correctly the counterterm contribution to the on-shell action, one should transform asymptotically the solution to Fefferman-Graham coordinates and then evaluate the counterterm action (or equivalently transform the hypersurface $r=r_o$ and the counterterm action in the new coordinates). Of course, it is always possible to work with a different regulator but then the counterterm action should be worked out from scratch.

Let us discuss now the evaluation of the conserved charges. Given that the asymptotics and counterterms are universal, one can work out in full generality the explicit form of the renormalized stress energy tensor in terms of the metric coefficients $g_{(m)}$ that appear in the asymptotic expansion of the solutions of a given action. This is done for pure gravity in [30] and for gravity coupled to certain matter in [32]. To evaluate the holographic stress tensor on a specific solution one could thus simply read off the metric coefficients from the asymptotic expansion of the metric and plug them in the general formula. This is the simplest way to proceed if the explicit expression for the holographic stress energy tensor is known. If this is not the case, it is simpler to just compute from the asymptotics of the given solution the contribution of the bulk and counterterm actions to the holographic stress energy tensor and add them up to produce a finite answer. To evaluate the conserved charges we finally integrate the holographic stress energy tensor contracted with the appropriate asymptotic conformal Killing vector over the appropriate domain. The only remaining subtlety is the choice of a timelike Killing vector to be used in the definition of mass when the boundary metric is in a rotating frame. In this case we choose the Killing vector that corresponds to the standard timelike Killing vector $\partial/\partial t$ is the corresponding non-rotating frame.

Below we describe our calculation for the four dimensional Kerr-Newman-AdS black hole in considerable detail, mainly in order to emphasize the role of the Fefferman-Graham coordinate system in the asymptotic analysis, which is not fully appreciated in the literature. We then turn to the five dimensional Kerr-AdS black hole, emphasizing the role of the anomaly and its relation to the Casimir energy. Previous work on the thermodynamics of these black holes includes [43, 107, 106, 108, 109, 110, 111, 85, 83, 84, 112].

5.4.1 D=4 KERR-NEWMAN-ADS BLACK HOLE

The metric of the Kerr-Newman-AdS black hole in Boyer-Lindquist coordinates reads [104, 105, 110]

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(adt - \frac{r^2+a^2}{\Xi}d\phi\right)^2, \tag{5.112}$$

where

$$\rho^2 = r^2 + a^2 \cos^2\theta,$$
$$\Delta_r = (r^2 + a^2)\left(1 + \frac{r^2}{l^2}\right) - 2mr + q^2,$$
$$\Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta, \quad \Xi = 1 - \frac{a^2}{l^2}. \tag{5.113}$$

The gauge potential in this coordinate system is given by

$$\mathbf{A} = -\frac{2qr}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right). \tag{5.114}$$

This metric and gauge field solve the Einstein-Maxwell equations which follow from the action (omitting the boundary terms)

$$I_{\text{Lorentzian}} = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{4}F^2\right). \tag{5.115}$$

The event horizon is located at $r = r_+$, where r_+ is the largest root of $\Delta_r = 0$, and its area is

$$A = \frac{4\pi(r_+^2 + a^2)}{\Xi}. \tag{5.116}$$

The analytic continuation of the Lorentzian metric (5.112) by $t = -i\tau$, $a = i\alpha$ develops a conical singularity unless we periodically identify $\tau \sim \tau + \beta$ and $\phi \sim \phi + i\beta\Omega_H$, where

$$\beta = \frac{4\pi(r_+^2 + a^2)}{r_+\left(1 + \frac{a^2}{l^2} + \frac{3r_+^2}{l^2} - \frac{a^2+q^2}{r_+^2}\right)}, \tag{5.117}$$

is the inverse temperature and the angular velocity of the horizon, Ω_H, is given by

$$\Omega_H = \frac{a\Xi}{r_+^2 + a^2}. \tag{5.118}$$

However, in this coordinate system there is a non-zero angular velocity at infinity, namely

$$\Omega_\infty = -\frac{a}{l^2}. \tag{5.119}$$

Following our prescription (5.98), we define the angular velocity relevant for the thermodynamics as the difference (see also [110, 85])

$$\Omega = \Omega_H - \Omega_\infty = \frac{a(1 + r_+/l^2)}{r_+^2 + a^2}. \tag{5.120}$$

Finally, if $\chi = \partial_t + \Omega_H\partial_\phi$ is the null generator of the Killing horizon, the electric potential is given by

$$\Phi \equiv -A_\mu\chi^\mu|_{r_+} = \frac{2qr_+}{r_+^2 + a^2}. \tag{5.121}$$

Next we determine the electric charge, angular momentum and mass, as well as the Euclidean on-shell action of the Kerr-Newman-AdS solution. Our general analysis of the charges in section 5.2 showed that the counterterms do not contribute to the value of the electric charge or the angular momentum (lemma 5.2.1). However, the counterterms are essential for evaluating the mass and the on-shell action. Starting with the electric charge we easily find

$$Q \equiv -\frac{1}{2\kappa^2}\int_{\partial\mathcal{M}\cap C} *\mathrm{d}\mathbf{A} = \frac{4\pi q}{\kappa^2\Xi}. \tag{5.122}$$

The angular momentum can be evaluated equally easily as

$$J \equiv \int_{\partial\mathcal{M}\cap C} \mathbf{Q}[\partial_\phi] = \frac{8\pi m a}{\kappa^2\Xi^2}. \tag{5.123}$$

Before we can calculate the mass and the on-shell Euclidean action, we must first carry out the asymptotic analysis and determine the counterterms. Expanding the metric (5.112) for large r we get

$$\begin{aligned} ds^2 \;=\; & -\frac{r^2}{l^2}\left[1+\left(1+\frac{a^2}{l^2}\sin^2\theta\right)\frac{l^2}{r^2}-\frac{2ml^2}{r^3}+\mathcal{O}\left(\frac{1}{r^4}\right)\right]\left(dt-\frac{a\sin^2\theta}{\Xi}d\phi\right)^2 \\ & +\frac{l^2}{r^2}\left[1-\left(1+\frac{a^2}{l^2}\sin^2\theta\right)\frac{l^2}{r^2}+\frac{2ml^2}{r^3}+\mathcal{O}\left(\frac{1}{r^4}\right)\right]dr^2 \\ & +\frac{r^2}{\Delta_\theta}\left(1+\frac{a^2}{r^2}\cos^2\theta\right)d\theta^2 \\ & +\frac{r^2\Delta_\theta\sin^2\theta}{\Xi^2}\left[d\phi^2+\frac{a^2}{r^2}\left((1+\sin^2\theta)d\phi^2-\frac{2\Xi}{a}d\phi dt\right)+\mathcal{O}\left(\frac{1}{r^4}\right)\right]. \end{aligned} \tag{5.124}$$

This metric is not of the standard form since the coefficient of the radial line element depends on the angle θ. Indeed the standard counterterms are derived using a Fefferman-Graham coordinate system of the form (3.35) [29, 30, 33, 1]. These counterterms, defined on hypersurfaces of *constant Fefferman-Graham radial coordinate*, are not necessarily the correct counterterms on the hypersurfaces of constant Boyer-Lindquist radial coordinate, as is widely assumed in the literature. Of course, it is in principle possible to choose a gauge which asymptotes to the Boyer-Lindquist form of the Kerr-AdS metric and carry out the asymptotic analysis from scratch using a regulator of constant Boyer-Lindquist radial coordinate and rederive the appropriate counterterms for such a regulator. However, it is much more efficient to bring the metric (5.124) into the Fefferman-Graham form and use the standard counterterms.

To this end we introduce new coordinates

$$\begin{aligned} \bar{r} &= r+\frac{1}{r}f(\theta)+\mathcal{O}\left(\frac{1}{r^3}\right), \\ \bar{\theta} &= \theta+\frac{1}{r^4}h(\theta)+\mathcal{O}\left(\frac{1}{r^6}\right), \end{aligned} \tag{5.125}$$

or

$$
\begin{aligned}
r &= \bar{r}\left[1 - \frac{1}{\bar{r}^2} f(\bar{\theta}) + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right], \\
\theta &= \bar{\theta} - \frac{1}{\bar{r}^4} h(\bar{\theta}) + \mathcal{O}\left(\frac{1}{\bar{r}^6}\right).
\end{aligned} \tag{5.126}
$$

Requiring that the coefficient of the new radial line element has no angular dependence and that there is no mixed term $d\bar{r}d\bar{\theta}$ in the metric uniquely fixes the functions $f(\bar{\theta})$ and $h(\bar{\theta})$ to be

$$
\begin{aligned}
f(\bar{\theta}) &= -\frac{a^2}{4}\cos^2\bar{\theta}, \\
h(\bar{\theta}) &= \frac{1}{8} l^2 a^2 \Delta_{\bar{\theta}} \sin\bar{\theta}\cos\bar{\theta}.
\end{aligned} \tag{5.127}
$$

In the new coordinate system the asymptotic form of the metric (5.112) becomes

$$
\begin{aligned}
ds^2 \;=\;& \frac{l^2}{\bar{r}^2}\left[1 - \left(1 + \frac{a^2}{l^2}\right)\frac{l^2}{\bar{r}^2} + \frac{2ml^2}{\bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right] d\bar{r}^2 \\
& + \frac{\bar{r}^2}{\Delta_{\bar{\theta}}}\left[1 + \frac{3}{2}\frac{a^2}{\bar{r}^2}\cos^2\bar{\theta} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right] d\bar{\theta}^2 \\
& - \frac{\bar{r}^2}{l^2}\left[1 + \left(1 + \frac{a^2}{l^2} - \frac{a^2}{2l^2}\cos^2\bar{\theta}\right)\frac{l^2}{\bar{r}^2} - \frac{2ml^2}{\bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right] \times \\
& \left(dt - \frac{a\sin^2\bar{\theta}}{\Xi} d\phi\right)^2 \\
& + \frac{\bar{r}^2 \Delta_{\bar{\theta}} \sin^2\bar{\theta}}{\Xi^2}\left[d\phi^2 + \frac{a^2}{\bar{r}^2}\left((2 - \frac{1}{2}\cos^2\bar{\theta})d\phi^2 - \frac{2\Xi}{a} d\phi dt\right) + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right],
\end{aligned} \tag{5.128}
$$

which is almost of the desired form. For later convenience let us write explicitly the components of the induced metric:

$$
\begin{aligned}
\gamma_{\bar{\theta}\bar{\theta}} &= \frac{\bar{r}^2}{\Delta_{\bar{\theta}}}\left[1 + \frac{3}{2}\frac{a^2}{\bar{r}^2}\cos^2\bar{\theta} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right], \\
\gamma_{tt} &= -\frac{\bar{r}^2}{l^2}\left[1 + \left(1 + \frac{a^2}{l^2} - \frac{a^2}{2l^2}\cos^2\bar{\theta}\right)\frac{l^2}{\bar{r}^2} - \frac{2ml^2}{\bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right], \\
\gamma_{t\phi} &= \frac{\bar{r}^2 a \sin^2\bar{\theta}}{l^2\Xi}\left[1 + \left(1 + \frac{1}{2}\cos^2\bar{\theta}\right)\frac{a^2}{\bar{r}^2} - \frac{2ml^2}{\bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right], \\
\gamma_{\phi\phi} &= \frac{\bar{r}^2\sin^2\bar{\theta}}{\Xi}\left[1 + \left(1 + \frac{1}{2}\cos^2\bar{\theta}\right)\frac{a^2}{\bar{r}^2} + \frac{2ma^2\sin^2\bar{\theta}}{\bar{r}^3\Xi} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right].
\end{aligned} \tag{5.129}
$$

We can now introduce a cut-off at $\bar{r} = \bar{r}_o$ and proceed with the asymptotic analysis in the standard fashion. Note that the regulating surface $\bar{r} = \bar{r}_o$ becomes angle-dependent in the Boyer-Lindquist coordinates, namely

$$
r_o(\theta) = \bar{r}_o\left[1 + \frac{a^2}{4\bar{r}_o^2}\cos^2\theta + \mathcal{O}\left(\frac{1}{\bar{r}_o^4}\right)\right]. \tag{5.130}
$$

This is precisely the reason why the counterterms on a regulating surface defined by $r_o = \text{constant}$ are not necessarily the same as the counterterms on a regulating surface defined by $\bar{r}_o = \text{constant}$.

Finally, to bring the metric in the form (3.35) we define the canonical radial coordinate

$$dr_* = l\left[1 - \frac{1}{2}\left(1 + \frac{a^2}{l^2}\right)\frac{l^2}{\bar{r}^2} + \frac{ml^2}{\bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right]\frac{d\bar{r}}{\bar{r}}. \tag{5.131}$$

Counterterms[6]

Following the standard algorithm for the asymptotic analysis we find that the counterterm action for the Maxwell-AdS gravity system in four dimensions is

$$I_{\text{ct}} = \frac{1}{\kappa^2}\int_{\Sigma_{\bar{r}_o}} d^3x\sqrt{\gamma_E}\left(\frac{2}{l} + \frac{l}{2}R\right). \tag{5.132}$$

On-shell Euclidean action

We are now ready to evaluate the renormalized on-shell Euclidean action

$$I_{\text{ren}} = -\frac{1}{2\kappa^2}\int_{\mathcal{M}_{\bar{r}_o}} d^4x\sqrt{g_E}\left(R[g_E] + \frac{6}{l^2} - \frac{1}{4}F^2\right) - \frac{1}{\kappa^2}\int_{\Sigma_{\bar{r}_o}} d^3x\sqrt{\gamma_E}\left(K - \frac{2}{l} - \frac{l}{2}R\right). \tag{5.133}$$

Since the background is stationary, the bulk integral gives

$$\frac{\beta}{2\kappa^2}\int_0^{2\pi} d\phi\int_0^{\pi} d\theta\int_{r_+}^{r_o(\theta)} dr\sqrt{g_E}\left(\frac{6}{l^2} + \frac{1}{4}F^2\right) = \frac{4\pi\beta}{\kappa^2 l^2\Xi}\left[\bar{r}_o\left(\bar{r}_o^2 + \frac{5}{4}a^2\right) - r_+(r_+^2 + a^2) - \frac{q^2l^2r_+}{r_+^2 + a^2} + \mathcal{O}\left(\frac{1}{\bar{r}_o}\right)\right]. \tag{5.134}$$

Moreover, the boundary term is

$$-\frac{1}{\kappa^2}\int_{\Sigma_{\bar{r}_o}} d^3x\sqrt{\gamma_E}\left(K - \frac{2}{l} - \frac{l}{2}R\right) = -\frac{4\pi\beta}{\kappa^2 l^2\Xi}\left[\bar{r}_o(\bar{r}_o^2 + \frac{5}{4}a^2) - ml^2 + \mathcal{O}\left(\frac{1}{\bar{r}_o}\right)\right]. \tag{5.135}$$

Hence,

$$I_{\text{ren}} = \frac{4\pi\beta}{\kappa^2 l^2\Xi}\left[ml^2 - r_+(r_+^2 + a^2) - \frac{q^2l^2r_+}{r_+^2 + a^2}\right]. \tag{5.136}$$

Renormalized stress tensor and conserved charges

[6] We give the counterterms for the Euclidean action which we want to evaluate. The counterterms for the Lorentzian action are easily obtained by analytic continuation.

We need now to evaluate the renormalized stress tensor

$$T_{(3)}{}^{i}_{j} = -\frac{l}{\kappa^2}\left(K_{(3)}{}^{i}_{j} - K_{(3)}\delta^i_j\right). \tag{5.137}$$

This can be done either by first writing the renormalized stress tensor in terms of the coefficients in the Fefferman-Graham expansion of the metric [30, 32, 1] and then reading off the coefficients from (5.129), or by first evaluating the extrinsic curvature using

$$K_{ij} = \frac{1}{2}\frac{d\bar{r}}{dr_*}\frac{d}{d\bar{r}}\gamma_{ij}, \tag{5.138}$$

and then subtracting the appropriate counterterms, namely

$$T_{(3)}{}^{i}_{j} = -\frac{l}{\kappa^2}\left(K^i_j - K\delta^i_j + \frac{2}{l}\delta^i_j - lR^i_j + \frac{1}{2}lR\delta^i_j\right) + \mathcal{O}\left(\frac{1}{\bar{r}_o^4}\right). \tag{5.139}$$

In any case we find (in agreement with [110])

$$\begin{aligned}
T_{(3)}{}^{t}_{t} &= -\frac{2m}{\kappa^2}\frac{l^3}{\bar{r}_o^3} + \mathcal{O}\left(\frac{1}{\bar{r}_o^4}\right), \\
T_{(3)}{}^{\bar{\theta}}_{\bar{\theta}} = T_{(3)}{}^{\phi}_{\phi} &= \frac{m}{\kappa^2}\frac{l^3}{\bar{r}_o^3} + \mathcal{O}\left(\frac{1}{\bar{r}_o^4}\right), \\
T_{(3)}{}^{t}_{\phi} &= \frac{3m}{\kappa^2}\frac{a\sin^2\bar{\theta}}{l\Xi}\frac{l^3}{\bar{r}_o^3} + \mathcal{O}\left(\frac{1}{\bar{r}_o^4}\right), \\
T_{(3)}{}^{\phi}_{t} &= \mathcal{O}\left(\frac{1}{\bar{r}_o^4}\right).
\end{aligned} \tag{5.140}$$

For this solution one can easily show that the gauge field momentum does not contribute to the holographic charge (5.63) and so, for any boundary conformal Killing vector, ξ, we have

$$\mathcal{Q}[\xi] = -\int_0^{2\pi} d\phi \int_0^{\pi} d\bar{\theta}\sqrt{-\gamma}T_{(3)}{}^{t}_{j}\xi^j. \tag{5.141}$$

As a check, we evaluate

$$\mathcal{Q}[-\partial_\phi] = \frac{8\pi m a}{\kappa^2\Xi^2}, \tag{5.142}$$

in complete agreement with (5.123).

To obtain the mass now we first need to identify the correct timelike Killing vector. This can be done unambiguously as follows. From the asymptotic form of the metric in Boyer-Lindquist coordinates we see that the corresponding boundary metric is not the standard metric on $\mathbb{R}\times S^2$ even for $m = q = 0$, since there is a non-zero angular velocity $\Omega_\infty = -\frac{a}{l^2}$. However, as it is discussed e.g. in [85], this boundary metric is *conformal* to the standard boundary metric of AdS_4. To see this we perform a coordinate transformation from the coordinates $(t, \bar{\theta}, \phi)$ to $(t', \bar{\theta}', \phi')$, given by

$$t' = t, \quad \phi' = \phi + \frac{a}{l^2}t, \quad \Xi \tan^2 \bar{\theta}' = \tan^2 \bar{\theta}. \tag{5.143}$$

The resulting boundary metric in the new coordinates is the standard metric on $\mathbb{R} \times S^2$ up to the conformal factor $\cos^2 \bar{\theta} / \cos^2 \bar{\theta}'$. It follows that the correct timelike Killing vector that defines the mass is

$$\partial_{t'} = \frac{\partial t}{\partial t'}\partial_t + \frac{\partial \phi}{\partial t'}\partial_\phi = \partial_t - \frac{a}{l^2}\partial_\phi, \tag{5.144}$$

in agreement with (5.96). Hence,

$$M \equiv \mathcal{Q}[\partial_t - \frac{a}{l^2}\partial_\phi] = \frac{8\pi m}{\kappa^2 \Xi^2}. \tag{5.145}$$

This is precisely the mass obtained in [85] by integrating the first law.[7] Finally, defining the entropy by

$$S = \frac{2\pi}{\kappa^2}A, \tag{5.146}$$

it can now be easily seen that the quantum statistical relation (5.88) as well as the first law (5.106) are satisfied.

5.4.2 D=5 KERR-ADS BLACK HOLE

As a second example we consider the general five dimensional Kerr-AdS solution [106], which illustrates the role of the conformal anomaly.

In Boyer-Lindquist coordinates the metric is[8]

$$\begin{aligned} ds^2 \;=\; & -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi_a}d\phi - \frac{b\cos^2\theta}{\Xi_b}d\psi\right)^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(adt - \frac{r^2+a^2}{\Xi_a}d\phi\right)^2 \\ & + \frac{\Delta_\theta \cos^2\theta}{\rho^2}\left(bdt - \frac{r^2+b^2}{\Xi_b}d\psi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 \\ & + \frac{(1+r^2l^{-2})}{r^2\rho^2}\left(abdt - \frac{b(r^2+a^2)\sin^2\theta}{\Xi_a}d\phi - \frac{a(r^2+b^2)\cos^2\theta}{\Xi_b}d\psi\right)^2, \end{aligned} \tag{5.147}$$

[7] Note that our timelike Killing vector is *different* from the Killing vector, $\partial_t + \frac{a}{l^2}\partial_\phi$, which the authors of [85] claim makes the conformal mass [42, 102, 108] equal to the mass obtained from the first law.

[8] Note that $0 \leq \theta \leq \pi/2$ in five dimensions, while $0 \leq \theta \leq \pi$ in four dimensions.

where

$$\begin{aligned}
\rho^2 &= r^2 + a^2 \cos^2\theta + b^2 \sin^2\theta, \\
\Delta_r &= \frac{1}{r^2}(r^2+a^2)(r^2+b^2)\left(1+\frac{r^2}{l^2}\right) - 2m, \\
\Delta_\theta &= 1 - \frac{a^2}{l^2}\cos^2\theta - \frac{b^2}{l^2}\sin^2\theta, \\
\Xi_a &= 1 - \frac{a^2}{l^2}, \quad \Xi_b = 1 - \frac{b^2}{l^2}.
\end{aligned} \tag{5.148}$$

The event horizon is located at $r = r_+$, where r_+ is the largest root of $\Delta_r = 0$, and its area is

$$A = \frac{2\pi^2(r_+^2+a^2)(r_+^2+b^2)}{r_+\Xi_a\Xi_b}. \tag{5.149}$$

The inverse temperature is given by

$$\beta = 2\pi\left[r_+\left(1+\frac{r_+^2}{l^2}\right)\left(\frac{1}{r_+^2+a^2}+\frac{1}{r_+^2+b^2}\right) - \frac{1}{r_+}\right]^{-1}. \tag{5.150}$$

The angular velocities relative to a non-rotating frame at infinity are

$$\Omega_a = \frac{a(1+r_+^2 l^{-2})}{r_+^2+a^2}, \quad \Omega_b = \frac{b(1+r_+^2 l^{-2})}{r_+^2+b^2}, \tag{5.151}$$

and the corresponding angular momenta are easily evaluated

$$J_a = \int_{\partial\mathcal{M}\cap C} \mathbf{Q}[\partial_\phi] = \frac{4\pi^2 ma}{\kappa^2\Xi_a^2\Xi_b}, \tag{5.152}$$

$$J_b = \int_{\partial\mathcal{M}\cap C} \mathbf{Q}[\partial_\psi] = \frac{4\pi^2 mb}{\kappa^2\Xi_a\Xi_b^2}. \tag{5.153}$$

As for the four-dimensional Kerr-Newman-AdS black hole, in order to bring the metric into the standard asymptotic form, we need to introduce the new coordinates

$$\begin{aligned}
r &= \bar{r}\left\{1+\frac{1}{4}\hat{\Delta}_{\bar\theta}\frac{l^2}{\bar{r}^2}+\frac{1}{16}\hat{\Delta}_{\bar\theta}(1+\hat{\Xi}_a+\hat{\Xi}_b-2\hat{\Delta}_{\bar\theta})\frac{l^4}{\bar{r}^4}+\mathcal{O}\left(\frac{1}{\bar{r}^6}\right)\right\}, \qquad (5.154)\\
\theta &= \bar\theta+\frac{1}{16}(1-\hat{\Delta}_{\bar\theta})\hat{\Delta}'_{\bar\theta}\frac{l^4}{\bar{r}^4}-\frac{1}{32}(1-\hat{\Delta}_{\bar\theta})\hat{\Delta}'_{\bar\theta}(1+\hat{\Xi}_a+\hat{\Xi}_b+3\hat{\Delta}_{\bar\theta})\frac{l^6}{\bar{r}^6}+\mathcal{O}\left(\frac{1}{\bar{r}^8}\right),
\end{aligned}$$

where, to simplify the notation, we have defined

$$\hat{\Delta}_\theta = 1-\Delta_\theta, \quad \hat{\Xi}_a = 1-\Xi_a, \quad \hat{\Xi}_b = 1-\Xi_b. \tag{5.155}$$

In the new coordinate system the induced metric, up to terms of order $1/\bar{r}^6$ inside the braces, takes the form

$$\gamma_{\bar{\theta}\bar{\theta}} = \frac{\bar{r}^2}{\Delta_{\bar{\theta}}} \left\{1 + \frac{3\hat{\Delta}_{\bar{\theta}}}{2}\frac{l^2}{\bar{r}^2} + \frac{1}{4}\left[(1 - \frac{3\hat{\Delta}_{\bar{\theta}}}{2})(\hat{\Xi}_a + \hat{\Xi}_b - \frac{3\hat{\Delta}_{\bar{\theta}}}{2}) + \hat{\Xi}_a\hat{\Xi}_b\right]\frac{l^4}{\bar{r}^4}\right\},$$

$$\gamma_{tt} = -\frac{\bar{r}^2}{l^2}\left\{1 + (1 + \hat{\Xi}_a + \hat{\Xi}_b - \frac{\hat{\Delta}_{\bar{\theta}}}{2})\frac{l^2}{\bar{r}^2} + \left[\frac{\hat{\Delta}_{\bar{\theta}}}{8}(1 + \hat{\Xi}_a + \hat{\Xi}_b - \frac{3\hat{\Delta}_{\bar{\theta}}}{2}) - \frac{2m}{l^2}\right]\frac{l^4}{\bar{r}^4}\right\},$$

$$\gamma_{t\phi} = \frac{\bar{r}^2}{l^2}\frac{a\sin^2\bar{\theta}}{\Xi_a}\left\{1 + (\hat{\Xi}_a + \frac{\hat{\Delta}_{\bar{\theta}}}{2})\frac{l^2}{\bar{r}^2} + \frac{1}{4}\left[(\hat{\Xi}_b - \frac{\hat{\Delta}_{\bar{\theta}}}{2})(\Xi_a - \frac{\hat{\Delta}_{\bar{\theta}}}{2}) + \hat{\Xi}_a\hat{\Xi}_b - \frac{8m}{l^2}\right]\frac{l^4}{\bar{r}^4}\right\},$$

$$\gamma_{t\psi} = \frac{\bar{r}^2}{l^2}\frac{b\cos^2\bar{\theta}}{\Xi_b}\left\{1 + (\hat{\Xi}_b + \frac{\hat{\Delta}_{\bar{\theta}}}{2})\frac{l^2}{\bar{r}^2} + \frac{1}{4}\left[(\hat{\Xi}_a - \frac{\hat{\Delta}_{\bar{\theta}}}{2})(\Xi_b - \frac{\hat{\Delta}_{\bar{\theta}}}{2}) + \hat{\Xi}_a\hat{\Xi}_b - \frac{8m}{l^2}\right]\frac{l^4}{\bar{r}^4}\right\},$$

$$\gamma_{\phi\phi} = \bar{r}^2\frac{\sin^2\bar{\theta}}{\Xi_a}\left\{1 + (\hat{\Xi}_a + \frac{\hat{\Delta}_{\bar{\theta}}}{2})\frac{l^2}{\bar{r}^2} + \frac{1}{4}\left[(\hat{\Xi}_b - \frac{\hat{\Delta}_{\bar{\theta}}}{2})(\Xi_a - \frac{\hat{\Delta}_{\bar{\theta}}}{2}) + \hat{\Xi}_a\hat{\Xi}_b + \frac{8m}{l^2}\frac{a^2\sin^2\bar{\theta}}{l^2\Xi_a}\right]\frac{l^4}{\bar{r}^4}\right\},$$

$$\gamma_{\psi\psi} = \bar{r}^2\frac{\cos^2\bar{\theta}}{\Xi_b}\left\{1 + (\hat{\Xi}_b + \frac{\hat{\Delta}_{\bar{\theta}}}{2})\frac{l^2}{\bar{r}^2} + \frac{1}{4}\left[(\hat{\Xi}_a - \frac{\hat{\Delta}_{\bar{\theta}}}{2})(\Xi_b - \frac{\hat{\Delta}_{\bar{\theta}}}{2}) + \hat{\Xi}_a\hat{\Xi}_b + \frac{8m}{l^2}\frac{b^2\cos^2\bar{\theta}}{l^2\Xi_b}\right]\frac{l^4}{\bar{r}^4}\right\},$$

$$\gamma_{\phi\psi} = \bar{r}^2\left\{\frac{2m}{l^2}\frac{a\cos^2\bar{\theta}}{l\Xi_a}\frac{b\sin^2\bar{\theta}}{l\Xi_b}\frac{l^4}{\bar{r}^4}\right\},$$

while the canonical radial coordinate r_* is given by

$$\begin{aligned} dr_* = l\Bigg\{&1 - \frac{1}{2}(1 + \hat{\Xi}_a + \hat{\Xi}_b)\frac{l^2}{\bar{r}^2} \\ &+ \left[\frac{m}{l^2} + \frac{1}{8}(1 + \hat{\Xi}_a + \hat{\Xi}_b)^2 + \frac{1}{4}(1 + \hat{\Xi}_a^2 + \hat{\Xi}_b^2)\right]\frac{l^4}{\bar{r}^4}\Bigg\}\frac{d\bar{r}}{\bar{r}}. \end{aligned} \tag{5.156}$$

On-shell Euclidean action

The renormalized Euclidean action in five dimensions is given by

$$\begin{aligned} I_{\rm ren} \quad = \quad & -\frac{1}{2\kappa^2}\int_{\mathcal{M}_{\bar{r}_o}} d^5x\sqrt{g_E}\left(R[g_E] + \frac{12}{l^2}\right) \\ & -\frac{1}{\kappa^2}\int_{\Sigma_{\bar{r}_o}} d^4x\sqrt{\gamma_E}\left(K - \frac{3}{l} - \frac{l}{4}R + \frac{l^3}{16}(R_{ij}R^{ij} - \frac{1}{3}R^2)\log e^{-2\bar{r}_o}\right). \end{aligned} \tag{5.157}$$

Evaluating this expression we obtain

$$I = \beta M_{\rm Casimir} + \frac{2\pi^2\beta}{\kappa^2 l^2\Xi_a\Xi_b}\left[ml^2 - (r_+^2 + a^2)(r_+^2 + b^2)\right], \tag{5.158}$$

where

$$M_{\rm Casimir} \equiv \frac{3\pi^2 l^2}{4\kappa^2}\left(1 + \frac{(\Xi_a - \Xi_b)^2}{9\Xi_a\Xi_b}\right). \tag{5.159}$$

This expression for the on-shell Euclidean action is precisely the expression obtained in [111] and it differs from that of [85] by the term involving the Casimir energy. Moreover, (5.159) is equal the Casimir energy of the field theory on the rotating Einstein universe [111].

Evaluating the holographic mass we find

$$M \equiv \mathcal{Q}[\partial_t - \frac{a}{l^2}\partial_\phi - \frac{b}{l^2}\partial_\psi] = M_{\rm Casimir} + \frac{2\pi^2 m(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b)}{\kappa^2\Xi_a^2\Xi_b^2}, \tag{5.160}$$

which again agrees with the mass obtained in [85] except for the Casimir energy part. However, except for the Casimir energy, this mass is not the same as the one given in [111]. The discrepancy arises presumably because [111] do not use the correct non-rotating timelike Killing vector to evaluate the mass.

With the expressions for the mass and on-shell action we have obtained, one can easily see that the quantum statistical relation (5.88) is satisfied, despite the presence of the Casimir energy. However, to show that our expressions do satisfy the first law, we need to examine the effect of an arbitrary variation of the parameters a, b and m on the representative of the conformal class at the boundary.

The boundary metric is

$$d\bar{s}^2 = -dt^2 + \frac{2a\sin^2\bar\theta}{\Xi_a}dtd\phi + \frac{2b\cos^2\bar\theta}{\Xi_b}dtd\psi + \frac{l^2}{\Delta_{\bar\theta}}d\bar\theta^2 + \frac{l^2\sin^2\bar\theta}{\Xi_a}d\phi^2 + \frac{l^2\cos^2\bar\theta}{\Xi_b}d\psi^2. \tag{5.161}$$

Under a variation of the angular parameters $a,\ b$, this metric is *not* kept fixed as is required by the variational problem. The conformal class however is kept fixed (up to a diffeomorphism). To see this first consider the variation of (5.161) w.r.t. a and b, and then perform the compensating infinitesimal diffeomorphism

$$t = t', \quad \tan^2\bar\theta = \left(1 + \frac{\delta\Xi_a}{\Xi_a} - \frac{\delta\Xi_b}{\Xi_b}\right)\tan^2\bar\theta', \quad \phi = \phi' - \frac{\delta a}{l^2}t', \quad \psi = \psi' - \frac{\delta b}{l^2}t'. \tag{5.162}$$

The result of the combined transformation is

$$d\bar{s}^2 \to \left(1 - \frac{\delta\Xi_a}{\Xi_a}\sin^2\bar\theta - \frac{\delta\Xi_b}{\Xi_b}\cos^2\bar\theta\right)d\bar{s}^2. \tag{5.163}$$

The variation of the on-shell action due to this Weyl factor is

$$\delta_\sigma I = -\int_{\partial\mathcal{M}} d^dx\sqrt{\gamma_E}\mathcal{A}\delta\sigma = \frac{\pi^2\beta l^2}{12\kappa^2}\delta\left(\frac{\Xi_a}{\Xi_b} + \frac{\Xi_b}{\Xi_a}\right) = \beta\delta M_{\rm Casimir} = \beta\delta_\sigma M, \tag{5.164}$$

where the last equality follows from (5.107). Therefore, as expected, only the Casimir energy part of the mass transforms non trivially under a Weyl transformation.

Summarizing, we have shown that under a generic variation of the parameters a, b and m

$$\delta M = \delta M_{\text{Casimir}} + T\delta S + \Omega_a \delta J_a + \Omega_b \delta J_b, \tag{5.165}$$

in complete agreement with (5.111). The first law then is satisfied once we accompany such a generic variation with a compensating PBH transformation which undoes the Weyl transformation of the representative of the conformal class.[9]

5.A APPENDIX

5.A.1 ASYMPTOTIC CONFORMAL KILLING VECTORS AND ASYMPTOTIC BULK KILLING VECTORS

We discuss in this appendix the connection between asymptotic bulk isometries and boundary *conformal* isometries. In this discussion we will need a well-known property of the linearized supergravity equations of motion, namely that for each bulk field they admit two linearly independent solutions, the *normalizable* and the *non-normalizable* modes, which near the boundary behave as e^{-s_+r} and e^{-s_-r} respectively. The exponents s_+, s_- are related to the scaling dimension of the dual operators and the spacetime dimension. Specifically, we have

$$\begin{aligned} s^+ &= d-2, \quad s^- = -2, \quad \text{for } \gamma_{ij}, \\ s^+ &= d-2, \quad s^- = 0, \quad \text{for } A_i, \\ s^+ &= \Delta_I, \quad s^- = d-\Delta_I, \quad \text{for } \varphi^I, \end{aligned} \tag{5.166}$$

with $\Delta_I \geq d - \Delta_I$.

Asymptotic conformal Killing vectors

Definition: We define an asymptotic conformal Killing vector (CKV) to be a bulk vector field ξ which is asymptotically equal to a boundary conformal Killing vector. The precise asymptotic conditions are

$$(i) \quad \xi^r = \mathcal{O}(e^{-dr}), \quad (ii) \quad \xi^i(x,r) = \zeta^i(x)(1+\mathcal{O}(e^{-(d+2)r})), \tag{5.167}$$

where $\zeta^i(x)$ is a conformal Killing vector of $g_{(0)}$.

The asymptotic conformal Killing vectors are in one-to-one correspondence with asymptotic bulk Killing vectors, for if ξ is an asymptotic CKV as defined above, then there exist $\hat{\xi}$, $\hat{\alpha}$, given in (5.172) below, such that $\xi - \hat{\xi}$ is an asymptotic bulk Killing

[9] Of course we should also perform a compensating diffeomorphism (5.162), but this does not affect the first law since all thermodynamic variables are invariant under such a diffeomorphism.

vector, up to a gauge transformation required to preserve the gauge fixing of the vector field, namely

$$\mathcal{L}_{\xi-\hat{\xi}}\psi = \delta_{\hat{\alpha}}\psi + \mathcal{O}(e^{-s+r}), \tag{5.168}$$

or equivalently

$$\mathcal{L}_{\xi}\psi = \mathcal{L}_{\hat{\xi}}\psi + \delta_{\hat{\alpha}}\psi + \mathcal{O}(e^{-s+r}). \tag{5.169}$$

To prove this we note that both $\mathcal{L}_{\xi}\psi$ and $\mathcal{L}_{\hat{\xi}}\psi + \delta_{\hat{\alpha}}\psi$ satisfy the linearized equations of motion. As noted above, a basis for solutions of the linearized equations of motion are the normalizable and non-normalizable solution. Since in (5.169) we require equality up to normalizable mode, a sufficient condition for proving (5.169) is that the leading asymptotics between the left and right hand side agree. To show this we note that condition (i) and (5.9)-(5.11) imply that in the gauge (3.35)

$$\mathcal{L}_{\xi}\psi = L_{\xi}\psi + \mathcal{O}(e^{-s+r}). \tag{5.170}$$

Furthermore, condition (ii) is equivalent to

$$L_{\xi}\psi = \frac{1}{d}D_i\xi^i\delta_D\psi(1 + \mathcal{O}(e^{-r})). \tag{5.171}$$

It follows that the leading asymptotics agree with a PBH transformation with parameters,

$$\begin{aligned}
\hat{\xi}^r &= \delta\sigma(x), \\
\hat{\xi}^i &= \partial_j\delta\sigma(x)\int_r^\infty dr'\gamma^{ji}(r',x), \\
\hat{\alpha} &= \partial_i\delta\sigma(x)\int_r^\infty dr' A^i(r',x),
\end{aligned} \tag{5.172}$$

where

$$\delta\sigma = \frac{1}{d}D_i\xi^i, \tag{5.173}$$

which proves our assertion.

Notice that the asymptotic fall-off of ξ^i in (ii) follows from the fact that in order for a vector field to preserve the gauge (3.35) we need

$$\dot{\xi}^i = -\partial^i\xi^r \qquad \Rightarrow \qquad \dot{\xi}^i = \mathcal{O}(e^{-(d+2)r}). \tag{5.174}$$

5.A.2 Proof of lemma 5.2.1

In this appendix we give a proof of lemma 5.2.1.

ELECTRIC CHARGE

To prove (5.64) we start with the identity

$$\int_{\Sigma_r \cap C} *\mathcal{F} = \int_{\Sigma_r \cap C} d\sigma_i \frac{1}{\sqrt{-\gamma}} \pi^i, \tag{5.175}$$

where $\pi^i = -\sqrt{-\gamma} U(\varphi) F^{ri}$ is the gauge field momentum. The second equation in (3.121) can now be written as

$$\dot{\pi}^i = -\partial_j (\sqrt{-\gamma} U(\varphi) F^{ij}). \tag{5.176}$$

The momentum π^i and the radial derivative ∂_r can be expanded in eigenfunctions of the dilatation operator as in (3.51) and (3.54) respectively. Moreover, by Taylor expanding $U(\varphi)$ one obtains such an expansion for the RHS of (5.176) too, which takes the form

$$U(\varphi) F^{ij} = \left\{ U(0) + \frac{\partial U}{\partial \varphi^I} \varphi^I + \frac{1}{2!} \frac{\partial^2 U}{\partial \varphi^I \partial \varphi^J} \varphi^I \varphi^J + \cdots \right\} F^{ij} \equiv \varphi_{(4)}{}^{ij} + \varphi_{(5)}{}^{ij} + \cdots . \tag{5.177}$$

Matching terms of the same dilatation weight on both sides of (5.176) then we obtain

$$\begin{aligned}
&\pi_{(3)}{}^i = 0, \\
&\sqrt{-\gamma}\pi_{(4)}{}^i = -\frac{1}{d-4} \partial_j (\sqrt{-\gamma} \varphi_{(4)}{}^{ij}), \\
&\sqrt{-\gamma}\pi_{(5)}{}^i = -\frac{1}{d-5} \partial_j \left[\sqrt{-\gamma} \varphi_{(5)}{}^{ij} - \frac{1}{d-4} \delta_{(1)} \left(\sqrt{-\gamma} \varphi_{(4)}{}^{ij} \right) \right], \\
&\quad \vdots \\
&\sqrt{-\gamma}\tilde{\pi}_{(d)}{}^i = \frac{1}{2} \partial_j \left(\sqrt{-\gamma} \varphi_{(d)}{}^{ij} + \cdots \right).
\end{aligned} \tag{5.178}$$

Therefore, all local terms in the momentum expansion are total derivatives while the non-local term $\pi_{(d)}{}^i$ is left undetermined by this iterative argument. Hence,

$$\int_{\Sigma_{r_o} \cap C} *\mathcal{F} = \int_{\Sigma_{r_o} \cap C} d\sigma_i \frac{1}{\sqrt{-\gamma}} \pi^i = \int_{\Sigma_{r_o} \cap C} d\sigma_i \pi_{(d)}{}^i + \cdots . \tag{5.179}$$

Taking the limit $\Sigma_{r_o} \to \partial\mathcal{M}$ then completes the proof of (5.64).

CHARGES ASSOCIATED WITH ASYMPTOTIC CONFORMAL ISOMETRIES

Applying a similar argument we now prove (5.66). Let, ξ be an asymptotic conformal Killing vector as defined in Appendix 5.A.1, i.e.

$$\mathcal{L}_\xi \psi = \mathcal{L}_{\hat{\xi}} \psi + \delta_{\hat{\alpha}} \psi + \mathcal{O}(e^{-s_+ r}), \tag{5.180}$$

where $\hat{\xi}$ and $\hat{\alpha}$, given in (5.172), generate a PBH transformation with conformal factor $\delta\sigma = \frac{1}{d} D_i \xi^i$. Then, using (5.49), (5.52) and the fact that in the gauge (3.35) one has

$$\begin{aligned}
\Xi^{ri} &= \nabla^{[r}\xi^{i]} + \kappa^2 U(\varphi) F^{ri} A_j \xi^j \\
&= \dot{\xi}^i + \Gamma^i_{rj}\xi^j - \frac{\kappa^2}{\sqrt{-\gamma}}\pi^i A_j \xi^j \\
&= \left(K^i_j - \frac{\kappa^2}{\sqrt{-\gamma}}\pi^i A_j\right)\xi^j + \mathcal{O}\left(e^{-(d+2)r}\right),
\end{aligned} \tag{5.181}$$

we can write

$$\begin{aligned}
\int_{\Sigma_{r_o}\cap C} (\mathbf{Q}[\xi] - i_\xi \mathbf{B}) &= \frac{1}{\kappa^2}\int_{\Sigma_{r_o}\cap C} d\sigma_i \left(K^i_j - \frac{\kappa^2}{\sqrt{-\gamma}}\pi^i A_j\right)\xi^j \\
&\quad - \frac{1}{\kappa^2}\int_{\Sigma_{r_o}\cap C} d\sigma_i \xi^i \left(K_{(d)} + \lambda_{\rm ct}\right) \qquad (5.182) \\
&= -\int_{\Sigma_{r_o}\cap C} d\sigma_i \left[\left(2\pi_{(d)}{}^i_j + \pi_{(d)}{}^i A_j\right)\xi^j + \mathcal{O}\left(e^{-(d+2)r}\right)\right] \\
&\quad + \frac{1}{\kappa^2}\int_{\Sigma_{r_o}\cap C} d\sigma_i \left(K^i_j - \frac{\kappa^2}{\sqrt{-\gamma}}\pi^i A_j - \lambda\delta^i_j\right)_{\rm ct}\xi^j.
\end{aligned}$$

Taking the limit $\Sigma_{r_o} \to \partial\mathcal{M}$ we see that (5.66) is equivalent to

$$\int_{\partial\mathcal{M}\cap C} d\sigma_i \left(K^i_j - \frac{\kappa^2}{\sqrt{-\gamma}}\pi^i A_j - \lambda\delta^i_j\right)_{\rm ct}\xi^j = 0, \tag{5.183}$$

which we now prove.

From Section 5.2.1 we know that on-shell

$$\mathrm{d}\mathbf{Q}[\xi] + i_\xi \mathbf{L} = \mathbf{\Theta}(\psi, \mathcal{L}_\xi\psi), \tag{5.184}$$

which, using (5.6) and (5.49), can be written as

$$\nabla_\mu \Xi^{\mu\nu} = \kappa^2\xi^\nu\left(-\mathcal{L}_{\rm m} + \frac{1}{d-1}\tilde{T}^\sigma_\sigma\right) - \kappa^2 v^\nu(\psi, \mathcal{L}_\xi\psi). \tag{5.185}$$

In the gauge (3.35) we can use (3.127) to get

$$\partial_r\left[\sqrt{-\gamma}(\Xi^{ri} - \xi^i\lambda)\right] = \partial_j(\sqrt{-\gamma}\Xi^{ij}) - \kappa^2\sqrt{-\gamma}v^i(\psi, \mathcal{L}_\xi\psi) + \mathcal{O}\left(e^{-2r}\right), \tag{5.186}$$

or, using (5.181),

$$\partial_r\left\{\left[\sqrt{-\gamma}\left(K^i_j - \lambda\delta^i_j\right) - \kappa^2\pi^i A_j\right]\xi^j\right\} = \partial_j(\sqrt{-\gamma}\Xi^{ij}) - \kappa^2\sqrt{-\gamma}v^i(\psi, \mathcal{L}_\xi\psi) + \mathcal{O}\left(e^{-2r}\right). \tag{5.187}$$

To prove (5.183) we only need the time component of this equation. In particular, if $v^t(\psi, \mathcal{L}_\xi\psi) = \mathcal{O}(e^{-(d+2)r})$, then we can expand both sides of (5.187) in eigenfunctions of the dilatation operator using (3.51), as was done for (5.176) in the previous section, and apply the same iterative argument to show that (5.183) holds. Therefore, the proof of (5.66) is complete once we show that $v^t(\psi, \mathcal{L}_\xi\psi) = \mathcal{O}(e^{-(d+2)r})$. As we now explain, this follows from (5.65).

From the explicit form of v^t, given in (5.7), we see that

$$v^t(\psi, \mathcal{L}_\xi\psi) = v^t(\psi, \mathcal{L}_{\hat{\xi}}\psi + \delta_{\hat{\alpha}}\psi + \mathcal{O}(e^{-s_+r})) = v^t(\psi, \mathcal{L}_{\hat{\xi}}\psi + \delta_{\hat{\alpha}}\psi) + \mathcal{O}\left(e^{-(d+2)r}\right). \tag{5.188}$$

Moreover,

$$\begin{aligned}
v^t(\psi, \mathcal{L}_{\hat{\xi}}\psi + \delta_{\hat{\alpha}}\psi) \quad = \quad & -\frac{1}{2\kappa^2}(\gamma^{ti}\gamma^{jk} - \gamma^{tk}\gamma^{ij})D_k\left(D_{(i}\hat{\xi}_{j)} + 2K_{ij}\delta\sigma\right) \\
& +U(\varphi)F^{tj}\left(L_{\hat{\xi}}A_j + \dot{A}_j\delta\sigma + \partial_j\hat{\alpha}\right) \\
& +G_{IJ}(\varphi)\partial^t\varphi^I\left(\hat{\xi}^i\partial_i\varphi^J + \dot{\varphi}^J\delta\sigma\right) \qquad (5.189) \\
= \quad & -\frac{1}{2\kappa^2}(\gamma^{ti}\gamma^{jk} - \gamma^{tk}\gamma^{ij})\left(D_kD_{(i}\hat{\xi}_{j)} + 2K_{ij}D_k\delta\sigma\right) \\
& +U(\varphi)F^{tj}\left(L_{\hat{\xi}}A_j + \partial_j\hat{\alpha}\right) + G_{IJ}(\varphi)\partial^t\varphi^I\hat{\xi}^i\partial_i\varphi^J \\
& -\frac{1}{\kappa^2}\Big\{D^jK^t_j - D^tK - \kappa^2U(\varphi)F^{tj}\dot{A}_j \\
& \qquad - \kappa^2G_{IJ}(\varphi)\partial^t\varphi^I\dot{\varphi}^J\Big\}\delta\sigma.
\end{aligned}$$

The last term inside the braces vanishes by the second equation in (3.38). From (5.65) and (5.172) now follows that $\hat{\alpha} = 0$ and $\hat{\xi}^i$ has no components along the isometry directions. Making repeated use of (5.65) it is then straightforward to show that $v^t(\psi, \mathcal{L}_{\hat{\xi}}\psi + \delta_{\hat{\alpha}}\psi) = 0$, which completes the proof.

5.A.3 SYMPLECTIC FORM ON COVARIANT PHASE SPACE

In this appendix we give the explicit form of the symplectic current on the covariant phase space as given by [99, 100] (see also [90]) and we show that the corresponding pre-symplectic form is well-defined with the boundary conditions (5.27), if there is no anomaly, or (5.34) when the anomaly is non-vanishing.

Symplectic current

The symplectic current $D-1$-form is defined by [100, 90]

$$\omega(\psi, \delta_1\psi, \delta_2\psi) = \delta_2\mathbf{\Theta}(\psi, \delta_1\psi) - \delta_1\mathbf{\Theta}(\psi, \delta_2\psi). \tag{5.190}$$

The explicit form of this for the Lagrangian (5.1) can be derived directly from (5.7). Writing

$$\omega(\psi,\delta_1\psi,\delta_2\psi) = -*\, w(\psi,\delta_1\psi,\delta_2\psi), \tag{5.191}$$

with $w^\mu = w^\mu_{\rm gr} + w^\mu_{\rm vec} + w^\mu_{\rm sc}$, we find

$$\begin{aligned} w^\mu_{\rm gr} &= \frac{1}{2\kappa^2}\left(g^{\mu\rho}g^{\nu\kappa}g^{\sigma\lambda} - \frac{1}{2}g^{\mu\nu}g^{\rho\kappa}g^{\sigma\lambda} - \frac{1}{2}g^{\mu\kappa}g^{\nu\lambda}g^{\sigma\rho} - \frac{1}{2}g^{\mu\rho}g^{\nu\sigma}g^{\kappa\lambda}\right. \\ &\quad \left. + \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}g^{\kappa\lambda}\right)\left(\delta_2 g_{\kappa\lambda}\nabla_\nu\delta_1 g_{\rho\sigma} - \delta_1 g_{\kappa\lambda}\nabla_\nu\delta_2 g_{\rho\sigma}\right), \end{aligned} \tag{5.192}$$

$$\begin{aligned} w^\mu_{\rm vec} &= U(\varphi)\left(\frac{1}{2}g^{\rho\sigma}F^{\mu\nu} - g^{\mu\sigma}F^{\rho\nu} - g^{\nu\sigma}F^{\mu\rho}\right)\left(\delta_2 g_{\rho\sigma}\delta_1 A_\nu - \delta_1 g_{\rho\sigma}\delta_2 A_\nu\right) \\ &\quad + \frac{\partial U(\varphi)}{\partial\varphi^I}F^{\mu\nu}\left(\delta_1 A_\nu\delta_2\varphi^I - \delta_2 A_\nu\delta_1\varphi^I\right) \\ &\quad + U(\varphi)(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})(\delta_1 A_\nu\nabla_\rho\delta_2 A_\sigma - \delta_2 A_\nu\nabla_\rho\delta_1 A_\sigma), \end{aligned} \tag{5.193}$$

$$\begin{aligned} w^\mu_{\rm sc} &= G_{IJ}(\varphi)\nabla_\rho\varphi^J\left(\frac{1}{2}g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right)\left(\delta_2 g_{\nu\sigma}\delta_1\varphi^I - \delta_1 g_{\nu\sigma}\delta_2\varphi^I\right) \\ &\quad + \left(\frac{\partial G_{IJ}(\varphi)}{\partial\varphi^K} - \frac{\partial G_{KJ}(\varphi)}{\partial\varphi^I}\right)\nabla^\mu\varphi^J\delta_1\varphi^I\delta_2\varphi^K \\ &\quad + G_{IJ}(\varphi)(\delta_1\varphi^I\nabla^\mu\delta_2\varphi^J - \delta_2\varphi^I\nabla^\mu\delta_1\varphi^J). \end{aligned} \tag{5.194}$$

For the reader's convenience we now compile a list of the most important properties of the symplectic current that we will need, along with the relevant proofs. Further details can be found in [100, 90].

I. If ψ satisfies the equations of motion and $\delta_1\psi$, $\delta_2\psi$ satisfy the linearized equations of motion, then ω is closed

$$\mathrm{d}\omega = 0. \tag{5.195}$$

Proof: Taking the second variation of (5.2) and using the fact that the functional derivatives of the Lagrangian commute we get

$$\begin{aligned} &\delta_2\delta_1\mathbf{L} = \delta_2\mathbf{E}\delta_1\psi + \mathrm{d}\delta_2\mathbf{\Theta}(\psi,\delta_1\psi) = \delta_1\mathbf{E}\delta_2\psi + \mathrm{d}\delta_1\mathbf{\Theta}(\psi,\delta_2\psi) = \delta_1\delta_2\mathbf{L} \Rightarrow \\ &\mathrm{d}\omega(\psi,\delta_1\psi,\delta_2\psi) = \delta_1\mathbf{E}\delta_2\psi - \delta_2\mathbf{E}\delta_1\psi. \end{aligned} \tag{5.196}$$

This completes the proof since $\delta_1\mathbf{E} = \delta_2\mathbf{E} = 0$, by the hypothesis.

II. For an arbitrary fixed vector field ξ on $\mathcal{M}$ and an arbitrary gauge transformation α, on-shell we have

$$\omega(\psi, \delta\psi, \mathcal{L}_\xi\psi) = \mathrm{d}\left(\delta\mathbf{Q}[\xi] - i_\xi\mathbf{\Theta}\right), \tag{5.197}$$

$$\omega(\psi, \delta\psi, \delta_\alpha\psi) = \mathrm{d}\delta\mathbf{Q}_\alpha. \tag{5.198}$$

Proof: The variation of the diffeomorphism current with respect to an arbitrary variation $\delta\psi$ of the fields (not necessarily satisfying the linearized equations of motion) is given by

$$\begin{aligned} \delta\mathbf{J}[\xi] &= \delta\mathbf{\Theta}(\psi, \mathcal{L}_\xi\psi) - i_\xi\delta\mathbf{L} \\ &= \delta\mathbf{\Theta}(\psi, \mathcal{L}_\xi\psi) - i_\xi\mathrm{d}\mathbf{\Theta}(\psi, \delta\psi) \\ &= \delta\mathbf{\Theta}(\psi, \mathcal{L}_\xi\psi) - \mathcal{L}_\xi\mathbf{\Theta}(\psi, \delta\psi) + \mathrm{d}(i_\xi\mathbf{\Theta}(\psi, \delta\psi)), \end{aligned} \tag{5.199}$$

where the equations of motion, $\mathbf{E} = 0$, have been used together with the identity $\mathcal{L}_\xi = i_\xi\mathrm{d} + \mathrm{d}i_\xi$ on forms. Since $\mathbf{\Theta}$ is covariant with respect to bulk diffeomorphisms we have $\mathcal{L}_\xi\mathbf{\Theta}(\psi, \delta\psi) = \delta'\mathbf{\Theta}(\psi, \delta\psi)$, where $\delta'\psi = \mathcal{L}_\xi\psi$. Hence,

$$\delta\mathbf{\Theta}(\psi, \mathcal{L}_\xi\psi) - \mathcal{L}_\xi\mathbf{\Theta}(\psi, \delta\psi) = \omega(\psi, \delta\psi, \mathcal{L}_\xi\psi), \tag{5.200}$$

and so

$$\omega(\psi, \delta\psi, \mathcal{L}_\xi\psi) = \delta\mathbf{J}[\xi] - \mathrm{d}(i_\xi\mathbf{\Theta}). \tag{5.201}$$

Specializing this to solutions, $\delta\psi$, of the linearized equations of motion completes the proof of (5.197).

Moreover,

$$\omega(\psi, \delta\psi, \delta_\alpha\psi) = \delta\mathbf{\Theta}(\psi, \delta_\alpha\psi) - \delta_\alpha\mathbf{\Theta}(\psi, \delta\psi). \tag{5.202}$$

Gauge invariance implies that the second term on the RHS vanishes and hence, on-shell, we obtain (5.198).

(5.203)

III. The pullback of the symplectic current on Σ_r takes the form

$$\begin{aligned} \omega(\psi, \delta_1\psi, \delta_2\psi) \;=\; & \Big\{\delta_2(\sqrt{-\gamma}{\pi_{(d)}}^{ij})\delta_1\gamma_{ij} + \delta_2(\sqrt{-\gamma}{\pi_{(d)}}^{i})\delta_1 A_i \\ & + \delta_2(\sqrt{-\gamma}\pi_{(\Delta_I)I})\delta_1\varphi^I - 1 \leftrightarrow 2\Big\}\, d\mu. \end{aligned} \tag{5.204}$$

Proof: This follows immediately from the form of the pullback (5.25) of $\mathbf{\Theta}$ on Σ_r together with the commutativity of the field variations, $\delta_2\delta_1 - \delta_1\delta_2 = 0$.

Pre-symplectic form

Having established the relevant properties of the symplectic current we now introduce the corresponding pre-symplectic 2-form on the field configuration space. Such a form induces a symplectic form on the solution submanifold of the configuration space [100]. Given a Cauchy surface C, the pre-symplectic form relative to C is defined by [100, 90]

$$\Omega_C(\psi, \delta_1\psi, \delta_2\psi) = \int_C \omega(\psi, \delta_1\psi, \delta_2\psi). \tag{5.205}$$

In order for this to be well-defined obviously the integral on the RHS of (5.205) must converge for all solutions ψ of the field equations and any solutions $\delta_1\psi,\ \delta_2\psi$ of the linearized equations of motion that satisfy the boundary conditions (5.27) - or (5.34) in the case of non-vanishing anomaly. These boundary conditions should also ensure that Ω_C is independent of the Cauchy surface C.

To address these questions we note that the most general solution of the linearized equations of motion satisfying the boundary conditions (5.27) takes the form

$$\delta\psi = \mathcal{L}_{\hat{\xi}}\psi + \delta_{\hat{\alpha}}\psi + \hat{\delta}\psi, \tag{5.206}$$

where $\hat{\xi}$, $\hat{\alpha}$, given by (5.172), generate a PBH transformation and $\hat{\delta}\psi = \mathcal{O}(e^{-s^+r})$ is an arbitrary *normalizable* solution. Since, as can be seen from (5.192), (5.193) and (5.194), the pullback of $\omega(\psi, \delta\psi, \hat{\delta}\psi)$ onto C is $\mathcal{O}(e^{-2r})$, the only contribution to the pre-symplectic form which could be divergent is the integral of $\omega(\psi, \mathcal{L}_{\hat{\xi}_1}\psi + \delta_{\hat{\alpha}_1}\psi, \mathcal{L}_{\hat{\xi}_2}\psi + \delta_{\hat{\alpha}_2}\psi)$. However, if the background, ψ, satisfies the conditions of lemma 5.2.1 and the Weyl factors $\delta\sigma_1$ and $\delta\sigma_2$ are independent of the coordinates adapted to the isometries, then the pullback of $\omega(\psi, \mathcal{L}_{\hat{\xi}_1}\psi + \delta_{\hat{\alpha}_1}\psi, \mathcal{L}_{\hat{\xi}_2}\psi + \delta_{\hat{\alpha}_2}\psi)$ onto the Cauchy surface C vanishes. Hence, the defining integral (5.205) of Ω_C is convergent.

Next, let C and C' be two Cauchy surfaces bounding a region $\Delta \subset \partial\mathcal{M}$ of the boundary. Using Stokes' theorem and the fact that ω is closed on-shell (property I), we get

$$\int_C \omega(\psi, \delta_1\psi, \delta_2\psi) - \int_{C'} \omega(\psi, \delta_1\psi, \delta_2\psi) = \int_{\Delta\subset\partial\mathcal{M}} \omega(\psi, \delta_1\psi, \delta_2\psi). \tag{5.207}$$

Property III together with the boundary conditions (5.27) and the trace Ward identity (2.168) now give

$$\omega(\psi, \delta_1\psi, \delta_2\psi) = \left\{\delta_2(\sqrt{-\gamma}\mathcal{A})\delta_1\sigma - 1 \leftrightarrow 2\right\} d\mu. \tag{5.208}$$

Therefore, Ω_C is independent of the Cauchy surface provided we use the boundary conditions (5.27) when the anomaly vanishes, and the boundary conditions (5.34) when there is a non-zero anomaly. This is in perfect agreement with our discussion of the variational problem.

5.A.4 ELECTRIC PART OF THE WEYL TENSOR AND THE ASHTEKAR-MAGNON MASS

In this appendix we briefly discuss the connection between the 'conformal mass' of [42] and our analysis. This issue is also discussed in the recent work of [81].

The authors of [42, 102] give a definition of the conserved charges for AAdS spacetimes in terms of the electric part of the Weyl tensor, which, in the gauge (3.35), and for arbitrary matter fields, takes the form

$$E^i_j = KK^i_j - K^i_k K^k_j - R^i_j + \frac{\kappa^2}{d-1}\left[(d-2)\tilde{T}^i_j - \left((d-2)\frac{1}{d}\tilde{T}^\sigma_\sigma + \tilde{T}_{d+1d+1}\right)\delta^i_j\right]. \tag{5.209}$$

This tensor is traceless due to the Hamilton constraint in (3.38)

$$E^i_i = K^2 - K_{ij}K^{ij} - R - 2\kappa^2\tilde{T}_{d+1d+1} = 0. \tag{5.210}$$

To make contact with their discussion let us specialize to pure gravity in five dimensions (the inclusion of matter in the discussion is completely straightforward). Expanding this tensor in eigenfunctions of the dilatation operator we immediately see that the term of weight 4 is given by

$$E_{(4)}{}^i_j = 2\left(K_{(4)}{}^i_j - K_{(4)}\delta^i_j\right) + 3K_{(4)}\delta^i_j + K_{(2)}K_{(2)}{}^i_j - K_{(2)}{}^i_k K_{(2)}{}^k_j. \tag{5.211}$$

Using now the expressions [29, 30, 1]

$$K_{(2)}{}^i_j = \frac{1}{2}\left(R^i_j - \frac{1}{6}R\delta^i_j\right), \quad K_{(4)} = \frac{1}{24}\left(R^{ij}R_{ij} - \frac{1}{3}R^2\right), \tag{5.212}$$

we obtain

$$E_{(4)}{}^i_j = -2\kappa^2 T_{(4)}{}^i_j + \frac{1}{4}\left(-R^i_k R^k_j + \frac{2}{3}RR^i_j + \frac{1}{2}R^k_l R^l_k \delta^i_j - \frac{1}{4}R^2\delta^i_j\right), \tag{5.213}$$

where

$$T_{(4)}{}^i_j \equiv -\frac{1}{\kappa^2}(K_{(4)}{}^i_j - K_{(4)}\delta^i_j) \tag{5.214}$$

is the renormalized stress tensor. Therefore, in agreement with Ashtekar and Das [42] and Hollands, Ishibashi and Marolf [81], the difference between the holographic conserved charges, defined using $T_{(4)}{}^i_j$, and the Ashtekar-Magnon charges, defined using $E_{(4)}{}^i_j$, is the tensor

$$H^i_j \equiv \frac{1}{4}\left(-R^i_k R^k_j + \frac{2}{3}RR^i_j + \frac{1}{2}R^k_l R^l_k \delta^i_j - \frac{1}{4}R^2\delta^i_j\right). \tag{5.215}$$

As discussed in the main text, this tensor is covariantly conserved and is equal to the holographic stress energy tensor of AdS_5 [30].

There is a similar local tensor that is covariantly conserved when the metric is conformally flat in all even dimensions: it is the holographic stress energy tensor of AdS_{2k+1}. As it was shown in [47], and reviewed in section 2, see (3.6), the Fefferman-Graham expansion of $AdS_{(2k+1)}$ terminates at order z^4 and all terms are locally related to $g_{(0)}$. It follows that the holographic stress energy tensor, which in general contains the non-local (w.r.t. $g_{(0)}$) term $g_{(d)}$, is local in this case. The explicit expression for $d = 6$ is given in (3.21) of [30].

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SAMENVATTING

Het ultieme doel van de moderne theoretische fysica is het unificeren van alle bekende fysische wetten in een overkoepelende theorie. Zo'n theorie moet alle fysische fenomenen op alle afstandsschalen beschrijven en uitleggen, van de Planckschaal ($1.6 \times 10^{-35}\mathrm{m}$) tot de schaal van het universum ($7.4 \times 10^{26}\mathrm{m}$).

Wat wordt er verstaan onder het begrip 'theorie'? In de fysica wordt met het begrip 'theorie' een wiskundig model bedoeld, dat alle bekende systematische observaties binnen het domein van geldigheid van het specifieke model - die voortkomen uit experimenten, beschrijft en ook de mogelijke observaties die in de toekomst zouden kunnen worden gemaakt binnen dit domein beschrijft. Een theorie verschilt daarom van een complete catalogus met observaties, omdat een theorie het resultaat kan voorspellen van experimenten die nog niet uitgevoerd zijn. Een karakteristiek voorbeeld is de theorie van de Newtoniaanse zwaartekracht die was voorgesteld als uitleg van de observaties van de beweging van de planeten van Tycho Brahe en Johannes Kepler. Diezelfde theorie was later de basis voor de calculaties die leidden tot bijvoorbeeld het zenden van satellieten naar de ruimte. Dit feit bewijst dat met behulp van deze theorie het mogelijk is om voorspellingen te doen. Een ander voorbeeld is James Clerk Maxwells theorie van het electromagnetisme, die het resultaat was van de poging om de toen bekende electromagnetische fenomenen te beschrijven en uit te leggen. De wiskundige structuur van de theorie voorspelde het bestaan van de electromagnetische golven. Tegelijkertijd speelde deze theorie een belankrijke rol bij de ontdekking van de speciale relativiteitstheorie.

Elke fysische theorie heeft een beperkt regime van geldigheid, dat voornamelijk wordt bepaald door de oorspronkelijke observaties die leiden tot de specifieke theorie. We weten bijvoorbeeld dat de mechanica van Newton niet geldt op microscopisch niveau, zoals op het atomair niveau ($0.5 \times 10^{-10}\mathrm{m}$), terwijl de Newtoniaanse zwaartekracht de verschillende fysische fenomenen niet kan beschrijven in een sterk zwaartekrachtsveld of op hogere niveaus dan het niveau van het zonnestelsel ($5.9 \times 10^{12}\mathrm{m}$). De moderne theoretische fysica bestaat daarom uit een collectie van fysische theorieën die gelden op verschillende niveaus of, meer in het algemeen, onder speciale omstandigheden. Het unificeren van alle beperkte theorieën in een theorie met een groter domein van geldigheid is de belangrijkste uitdaging voor de natuurwetenschap.

DE PIJLERS VAN DE MODERNE NATUURKUNDE

Een belangrijke ontwikkeling in de theoretische fysica was de ontdekking van de relativiteitstheorie en de quantummechanica in het begin van de twintigste eeuw. De speciale relativiteitstheorie leidde tot een fundamentele herziening van het denken over tijd, plaats en massa, die essentiële constituenten zijn van elke fysische theorie. Tien jaar later veranderde de algemene relativiteitstheorie het beeld van de 'ruimtetijd' nog meer en gaf die ons tegelijkertijd een geometrische zwaartekrachtstheorie die de theorie van Newton generaliseerde. De algemene relativiteitstheorie is tot op heden het enige complete wiskundige model voor het beschrijven en het begrijpen van alle zwaartekrachtsfenomenen op macroscopisch niveau. Het feit dat alle observaties tot nu toe precies overeenkomen met de voorspellingen van de algemene relativiteitstheorie en het feit dat de theorie een mooie en simpele structuur heeft, leidden tot de conclusie dat welke vorm de uiteindelijke theorie dan ook mag hebben, de algemene relativiteitstheorie in ieder geval een onderdeel zal zijn van de overkoepelende theorie.

In dezelfde periode droeg de quantummechanica bij aan het begrip van de structuur van materie op atomair niveau. Om succesvol de fysische fenomenen te kunnen beschrijven op atomair niveau is het nodig om afstand te nemen van de klassieke dynamica, die inhoudt dat verschillende deeltjes in bepaalde banen in de ruimte bewegen. Experimentele observaties stelden een statistische beschrijving van de werking van de microkosmos voor, die bijvoorbeeld toelaat dat een deeltje tussen twee punten allerlei trajecten kan volgen die een verschillende waarschijnlijkheid hebben. De klassieke beschrijving van de beweging van een deeltje, die vereist dat de plaats van een deeltje in de ruimte als functie van de tijd bepaald is, is vervangen door de 'golffunctie', die de waarschijnlijkheid van het deeltje bepaalt, dat zich op elk punt in de ruimte en op elk moment in de tijd kan bevinden. In tegenstelling tot wat men zou verwachten van het statistische karakter van de quantummechanica, bleek dat haar voorspelbaarheid ongekend was en werden er daarom nieuwe fenomenen ontdekt. Bovendien leerde de quantummechanica ons dat voor de hand liggende vragen over de werking van de natuur op het niveau van ons dagelijks leven, zoals de vraag wat de plaats is van een deeltje als functie van de tijd, niet altijd de juiste vragen zijn voor het bestuderen van de microkosmos. Net als de algemene relativiteitstheorie is ook de quantummechanica één van de meest succesvolle fysische theorieën die we tot op heden kennen en daarom is te verwachten dat ze een belangrijke rol zal spelen in de ontwikkeling van een overkoepelende theorie.

DE ZOEKTOCHT NAAR EEN OVERKOEPELENDE THEORIE

De originele vorm van de quantummechanica beschrijft de eigenschappen van materie op atomair niveau in snelheden die veel kleiner zijn dan de snelheid van het licht of, volgens de speciale relativiteitstheorie, in energieën die veel kleiner zijn dan de massa van atomische deeltjes. Deze restrictie van de energie is het gevolg van het feit dat de originele vorm van de quantummechanica niet overeenkomt met de speciale relativiteitstheorie, die de beweging in grote snelheden, vergelijkbaar met de snelheid van het licht, beschrijft.

Jarenlange pogingen om de quantummechanica en de speciale relativiteitstheorie te unificeren leidden in het midden van de twintigste eeuw tot de formulering van de zogenaamde 'quantumveldentheorie'. Deze unificatie vereist dat men afstand neemt van het idee van het geïsoleerde deeltje, dat ons bekend is uit de klassieke mechanica en dat tot op zekere hoogte in de non-relativistische formulering van de quantummechanica bleef. Volgens de quantumveldentheorie zijn juist de verschillende velden die zich uitstrekken over de gehele ruimtetijd de enige fundamentele objecten in de natuur, zoals bijvoorbeeld het bekende electromagnetische veld van Maxwell. De verschillende deeltjes corresponderen met de locale storingen van deze velden, die zich verbreiden en interageren in de ruimtetijd. Tijdens de interacties kunnen nieuwe deeltjes worden geproduceerd uit het vacuüm en andere deeltjes kunnen verdwijnen: fenomenen die niet kunnen worden beschreven door vroegere theorieën, die de verschillende deeltjes immers behandelden als fundamentele objecten die voor altijd bestaan.

De eerste complete quantumveldentheorie was de zogenaamde 'quantumelectrodynamica' (QED), die de samensmelting is van drie fundamentele theorieën, namelijk van de quantummechanica, van de speciale relativiteitstheorie en van Maxwells theorie van het electromagnetisme. Deze theorie beschrijft erg nauwkeurig alle eigenschappen van de quantumstoringen van het electromagnetische veld en van het 'electronveld' - die bekend staan als respectievelijk 'fotonen' en 'electronen' - en beschrijft ook hun onderlinge interacties. De ontdekking van de quantumelectrodynamica was ook belangrijk omdat het de eerste complete quantumtheorie was van één van de vier bekende fundamentele natuurkrachten: het electromagnetisme, de zwakke en de sterke nucleaire interacties en de zwaartekracht. Het was daarom te verwachten dat de quantumelectrodynamica het prototype zou worden in de systematische zoektocht naar een quantumtheorie voor de andere fundamentele krachten.

Dit gebeurde aan het eind van de jaren '60 en in het begin van de jaren '70, toen na vele jaren van experimenteren met cosmische stralen en deeltjesversnellers, maar ook na intensieve theoretische pogingen, de juiste quantumtheorieën voor de beschrijving van de zwakke en de sterke interacties ontdekt werden. Bovendien realiseerde men zich dat het bestaan van electromagnetisme vereist is voor een quantumtheorie van de zwakke interactie. De electromagnetische en de zwakke interacties zijn daarom verschillende facetten van één fundamentele kracht, van de zogenaamde ‘electrozwakke interactie’. De quantumtheorie die de electrozwakke interactie beschrijft was het eerste voorbeeld van een nieuw type van een quantumveldentheorie, die bekend staat als ‘ niet-abelse ijktheorie’. Het besef dat zo’n quantumveldentheorie wiskundig gezien acceptabel is - namelijk ‘renormaliseerbaar’ is - leidde erg snel tot de ontdekking van de juiste quantumbeschrijving van de sterke nucleaire interactie, van de ‘quantum chromodynamica’ (QCD), die, net als de electrozwakke theorie, een niet-abelse ijktheorie is. Deze groep van quantumveldentheorieën staat bekend als het ‘standaardmodel van de deeltjesfysica’ en is een complete en erg nauwkeurige beschrijving van alle bekende deeltjes en van alle fundamentele krachten, behalve van de zwaartekracht.

Hoewel het standaardmodel van de deeltjesfysica een belangrijke stap in de zoektocht naar een overkoepelende theorie is, impliceert het feit dat het geen zwaartekracht bevat dat het geen finale of complete theorie kan zijn. In tegenstelling tot de speciale relativiteitstheorie, die geïncorporeerd is in elke quantumveldentheorie, is de unificatie van de algemene relativiteitstheorie met de quantummechanica één van de moeilijkste en langdurigste problemen van de theoretische fysica. Het is niet moeilijk om te begrijpen waarom de formulering van een quantumzwaartekrachtstheorie zulke moeilijkheden laat zien, aangezien de algemene relativiteitstheorie en de quantumveldentheorie fenomenen beschrijven op twee totaal verschillende afstandsschalen. We weten erg goed dat de rol van de quantummechanica verwaarloosbaar is op de afstandsschaal in ons dagelijks leven en natuurlijk op macroscopisch niveau, zoals op het niveau van de omvang van het universum. Tegelijkertijd is de zwaartekracht erg zwak vergeleken met de andere drie fundamentele krachten op subatomair niveau. Een quantumzwaartekrachtstheorie moet daarom fenomenen met een zo groot mogelijk bereik over de afstandsschalen beschrijven en deze theorie zal dus dichtbij een overkoepelende theorie komen.

Aangezien de algemene relativiteitstheorie en de quantumveldentheorie perfect overeenkomen met de experimentele observaties in hun domein van geldigheid, moet elke quantumzwaartekrachtstheorie deze theorieën in de corresponderende limiet reproduceren. Om het echte karakter van een quantumzwaartekrachtstheorie, dat niet kan worden beschreven door de algemene relativiteitstheorie of de quantumveldentheorie, te isoleren, moet men zich dus concentreren op de afstandsschalen waarop de zwaartekracht en de quantummechanische fenomenen tegelijkertijd voorkomen. Dit gebeurt op niveaus die veel kleiner zijn dan het atomair niveau, namelijk op de Planckschaal, waarop de consequenties van de quantummechanica belangrijk blijven, maar waarop tegelijkertijd ook de zwaartekracht vergelijkbaar is met de andere fundamentele krachten. Het is erg onwaarschijnlijk dat we ooit in staat zullen zijn om fenomenen met behulp van experimenten direct te bestuderen op zulke kleine afstandsschalen, oftewel op zulke hoge energieën, omdat het praktisch onmogelijk is om de nodige apparatuur daarvoor te construeren op Aarde. We weten echter dat deze condities bestonden in de periode direct na het ontstaan van het universum, oftewel na de Oerknal, toen de omvang van het universum vergelijkbaar was met de Planckschaal. Dit feit laat de mogelijkheid open voor het bestaan van direct bewijs van quantumzwaartekracht op cosmologische schaal.

Vanuit technisch perspectief is het grootste probleem voor de formulering van een quantumzwaartekrachtstheorie, die gebaseerd is op de algemene relativiteitstheorie, dat het op dezelfde manier 'quantiseren' van het zwaartekrachtsveld als het quantiseren van de andere fundamentele krachten leidt tot wiskundige inconsistenties. Met name het feit dat de quanta van het zwaartekrachtsveld, de 'gravitonen', in de standaard veldentheorie interageren op één punt in de ruimtetijd, leidt tot de verschijning van verschillende oneindigheden die onacceptabel zijn. Dit probleem kan alleen voorkomen worden als de gravitonen interageren vanaf een minimale afstand in plaats van op één punt. De zoektocht naar een fundamentele theorie die dit idee realiseert leidde tot de ontdekking van de 'snaartheorie'.

DE SNAARTHEORIE

In de snaartheorie zijn de fundamentele vrijheidsgraden ééndimensionale uitgestrekte objecten, oftewel 'snaren', die bewegen in de ruimtetijd en over een tweedimensionaal oppervlak opspannen. De mogelijke snaarinteracties worden erg beperkt door de geometrie en de topologie van deze oppervlakken: ze corresponderen met de verschillende wijzen van knippen en vastlijmen van een tweedimensionaal oppervlak. Daarom hebben de interacties tussen de gravitonen - die bepaalde trillingstoestanden van de snaar zijn - een puur geometrische oorsprong. Het feit dat de snaren uitgestrekte objecten zijn betekent dus dat de gravitonen niet op één punt interageren en daarom bevat de theorie geen wiskundige problemen die de directe quantisatie van de algemene relativiteitstheorie onmogelijk maken. Tegelijkertijd impliceert de consistente quantisatie van de snaren dat de snaartheorie de algemene relativiteitstheorie exact reproduceert op erg lage energieën, oftewel op afstandsschalen die groter zijn dan de Planckschaal. Dit gedrag komt overeen met wat men zou verwachten van een quantumzwaartekrachtstheorie. Dit is inderdaad één van de belangrijkste redenen die ons laten geloven dat de snaartheorie tot op heden de meest geschikte kandidaat is voor een quantumzwaartekrachtstheorie.

De snaartheorie is niet alleen een consistente quantumzwaartekrachtstheorie, maar ook een prototype voor de gewenste overkoepelende theorie. Naast het feit dat de verschillende trillingstoestanden van de snaren resulteren in een heleboel deeltjes - inclusief de gravitonen - vereisen alle vijf mogelijke snaartheorieën dat de ruimtetijd tien dimensies heeft in tegenstelling tot de vier dimensies die het waarneembare universum heeft. Dit aanwijsbare nadeel kan omgezet worden in een voordeel voor de snaartheorie als we aannemen dat zes van de tien dimensies microscopisch - of ‘opgerold’ - zijn en daarom alleen waarneembaar zijn op de kleine afstandsschalen. Elk van de vele manieren waarop de extra zes dimensies kunnen worden opgerold correspondeert met een andere set van deeltjes en interacties in de vier macroscopische dimensies. De snaartheorie is in staat om via dit mechanisme alle fundamentele krachten en elementaire deeltjes in de waarneembare wereld te beschrijven. De vraag tot op welke hoogte de snaartheorie het standaardmodel in vier dimensies kan reproduceren is tegenwoordig een actief onderzoeksgebied.

DE ADS/CFT - CORRESPONDENTIE

We hebben gezien dat de snaartheorie werd ontwikkeld als een generalisatie van de quantumveldentheorie in de poging om een consistente quantumzwaartekrachtstheorie te vinden. De recente ontdekking van de zogenaamde ‘AdS/CFT - correspondentie’ (Anti-de Sitter/Conformal Field Theory) relateert echter op een hele andere manier de snaartheorie met de quantumveldentheorie. Met name impliceert de AdS/CFT-correspondentie dat een bepaalde snaartheorie in tien dimensies gelijk is aan een gewone quantumveldentheorie zonder zwaartekracht in vier dimensies. Deze correspondentie - die ons in staat stelt om via de snaartheorie een quantumveldentheorie te bestuderen en vice versa - is het onderwerp van dit proefschrift.

Περίληψη

Ο απώτερος σκοπός της σύγχρονης θεωρητικής φυσικής είναι η ενοποίηση όλων των γνωστών φυσικών νόμων σε μια 'θεωρία των πάντων' ή 'θεωρία μεγαλοενοποίησης'. Μια τέτοια θεωρία θα πρέπει να περιγράφει και να ερμηνεύει όλα τα φυσικά φαινόμενα σε όλες τις κλίμακες μεγέθους, από τη μικροσκοπική κλίμακα του Planck (1.6×10^{-35}m) μέχρι τη μακροσκοπική κλίμακα του σύμπαντος (7.4×10^{26}m).

Τι εννοεί όμως κανείς με τον όρο 'θεωρία'; Στο χώρο της φυσικής, ο όρος 'θεωρία' αναφέρεται σε ένα μαθηματικό μοντέλο το οποίο περιγράφει όλες τις γνωστές συστηματικές παρατηρήσεις στο χώρο ισχύος του συγκεκριμένου μοντέλου όπως αυτές προκύπτουν μέσω πειραμάτων, άλλα και όλες τις δυνατές παρατηρήσεις που θα μπορούσαν να γίνουν στο μέλλον στον χώρο αυτόν. Μια θεωρία διαφέρει επομένως από έναν εκτενή κατάλογο παρατηρήσεων στο ότι η θεωρία μπορεί να προβλέψει το αποτέλεσμα πειραμάτων τα οποία δεν έχουν πραγματοποιηθεί. Χαρακτηριστικό παράδειγμα είναι η θεωρία της Νευτώνειας βαρύτητας η οποία προτάθηκε ως ερμηνεία των παρατηρήσεων του Tycho Brahe και του Johannes Kepler για την κίνηση των πλανητών. Η ίδια θεωρία όμως αποτέλεσε πολύ αργότερα την βάση υπολογισμών που οδήγησαν για παράδειγμα στην αποστολή δορυφόρων στο διάστημα, αποδεικνύοντας με αυτόν τον τρόπο την προβλεπτική της ικανότητα. Εξίσου χαρακτηριστική είναι η περίπτωση της θεωρίας του James Clerk Maxwell για τον ηλεκτρομαγνητισμό η οποία διαμορφώθηκε στα τέλη του δέκατου ένατου αιώνα για να εξηγηθούν τα μέχρι τότε γνωστά ηλεκτρομαγνητικά φαινόμενα. Η μαθηματική δομή της θεωρίας όμως οδήγησε στην πρόβλεψη των ηλεκτρομαγνητικών κυμάτων ενώ παράλληλα έπαιξε καθοριστικό ρόλο στην ανακάλυψη της ειδικής θεωρίας της σχετικότητας.

Κάθε φυσική θεωρία ωστόσο έχει ένα περιορισμένο πεδίο εφαρμογής το οποίο υπαγορεύεται σε μεγάλο βαθμό από τις αρχικές παρατηρήσεις που συνέβαλαν στην διαμόρφωση της συγκεκριμένης θεωρίας. Γνωρίζουμε για παράδειγμα ότι η μηχανική του Νεύτωνα δεν ισχύει σε μικροσκοπικές κλίμακες μεγέθους όπως αυτή του ατόμου (0.5×10^{-10}m), ενώ η Νευτώνεια βαρύτητα δεν μπορεί να περιγράψει τα διάφορα φυσικά φαινόμενα στην παρουσία ισχυρού βαρυτικού πεδίου ή σε κλίμακες μεγαλύτερες αυτής του ηλιακού μας συστήματος (5.9×10^{12}m). Έτσι, η σύγχρονη θεωρητική φυσική διαθέτει μια συλλογή από φυσικές θεωρίες οι οποίες ισχύουν σε διαφορετικές κλίμακες μεγέθους ή πιο γενικά κάτω από συγκεκριμένες συνθήκες. Η ενοποίηση όλων αυτών των περιορισμένων θεωριών σε μια και μόνη θεωρία με μεγαλύτερο πεδίο εφαρμογής είναι η κυριότερη πρόκληση για την επιστήμη της φυσικής.

Οι πυλώνες της σύγχρονης φυσικής

Σημαντικότατη πρόοδος στον χώρο της θεωρητικής φυσικής σημειώθηκε στις αρχές του εικοστού αιώνα με την ανακάλυψη της θεωρίας της σχετικότητας και της κβαντομηχανικής. Η ειδική θεωρία της σχετικότητας οδήγησε σε μια ριζική αναθεώρηση των εννοιών του χρόνου, του χώρου και της μάζας, οι οποίες αποτελούν θεμελιώδη στοιχεία κάθε φυσικής θεωρίας. Λίγα χρόνια αργότερα, η γενική θεωρία της σχετικότητας άλλαξε περαιτέρω την εικόνα του χωροχρόνου, ενώ ταυτόχρονα παρείχε μια γεωμετρική θεωρία βαρύτητας, γενικεύοντας τη θεωρία του Νεύτωνα. Η γενική θεωρία της σχετικότητας αποτελεί μέχρι σήμερα το μόνο ολοκληρωμένο μαθηματικό πλαίσιο για την περιγραφή και την κατανόηση όλων των βαρυτικών φαινομένων σε μακροσκοπικές κλίμακες. Το γεγονός ότι όλες οι παρατηρήσεις μέχρι τώρα συμφωνούν με τις προβλέψεις της γενικής θεωρίας της σχετικότητας με εξαιρετική ακρίβεια, καθώς και η ιδιαίτερα απλή και όμορφη μαθηματική δομή της, οδηγούν στην πεποίθηση ότι οποιανδήποτε μορφή κι αν έχει μια πιθανή τελική θεωρία μεγαλοενοποίησης, θα περιέχει τη γενική θεωρία της σχετικότητας ως ειδική περίπτωση.

Παράλληλα, η ανάπτυξη της κβαντομηχανικής είχε ως αποτέλεσμα την κατανόηση της δομής της ύλης σε ατομική κλίμακα. Για να περιγράψει κανείς με επιτυχία τα φυσικά φαινόμενα στο ατομικό επίπεδο ήταν απαραίτητο να αποσυρθεί η εικόνα της κλασσικής δυναμικής σύμφωνα με την οποία τα διάφορα σωματίδια κινούνται σε καθορισμένες τροχιές μέσα στο χρόνο. Οι πειραματικές παρατηρήσεις υπαγόρευαν μια στατιστική περιγραφή της συμπεριφοράς του μικρόκοσμου, επιτρέποντας για παράδειγμα σε ένα σωματίδιο να ακολουθεί όλες τις δυνατές διαδρομές μεταξύ δυο σημείων, με διαφορετικές όμως πιθανότητες. Η κλασσική περιγραφή της κίνησης ενός σωματιδίου, που απαιτεί τον καθορισμό της θέσης του σωματιδίου στον χώρο ως συνάρτηση του χρόνου, αντικαταστάθηκε με την 'κυματοσυνάρτηση', η οποία καθορίζει την πιθανότητα το σωματίδιο να βρίσκεται σε κάποιο σημείο του χώρου σε κάθε χρονική στιγμή. Σε αντίθεση με ότι θα ανέμενε κανείς από τον στατιστικό της χαρακτήρα, η προβλεπτική ικανότητα της κβαντομηχανικής ήταν πρωτοφανής και οδήγησε στην ανακάλυψη μιας πληθώρας νέων φαινομένων, ενώ μας δίδαξε ότι οι προφανείς ερωτήσεις για τη συμπεριφορά της φύσης στην κλίμακα της καθημερινής μας ζωής, όπως το ποια είναι η θέση ενός σώματος ως συνάρτηση του χρόνου, δεν είναι απαραίτητα οι 'σωστές' ερωτήσεις για τη μελέτη των φαινομένων στην ατομική κλίμακα. Όπως η γενική θεωρία της σχετικότητας, έτσι και η κβαντομηχανική έχει αποδειχθεί μια από τις πλέον επιτυχείς θεωρίες φυσικής που γνωρίζουμε μέχρι σήμερα και επομένως αναμένεται να έχει καθοριστικό ρόλο στην διαμόρφωση μιας ενοποιημένης θεωρίας.

Η αναζήτηση μιας θεωρίας μεγαλοενοποίησης

Η αρχική μορφή της κβαντομηχανικής περιγράφει τις ιδιότητες της ύλης στην κλίμακα του ατόμου και σε ταχύτητες πολύ μικρότερες αυτής του φωτός ή ισοδύναμα σε σχετικά χαμηλές ενέργειες. Ο περιορισμός αυτός στην ενέργεια οφείλεται στο γεγονός ότι η αρχική μορφή της κβαντομηχανικής δεν είναι συμβατή με την ειδική θεωρία της σχετικότητας που διέπει την κίνηση σε μεγάλες ταχύτητες, συγκρίσιμες με την ταχύτητα του φωτός.

Μια μακροχρόνια προσπάθεια ενοποίησης της κβαντομηχανικής με την ειδική θεωρία της σχετικότητας οδήγησε στη διαμόρφωση της λεγόμενης 'κβαντικής θεωρίας πεδίων' στα μέσα του εικοστού αιώνα. Η ενοποίηση αυτή απαιτεί την εγκατάλειψη της έννοιας του μεμονωμένου σωματιδίου όπως αυτή μας είναι γνωστή από την κλασσική μηχανική και η οποία παρέμεινε σε μεγάλο βαθμό και στην μη-σχετικιστική μορφή της κβαντομηχανικής. Αντίθετα, σύμφωνα με την κβαντική θεωρία πεδίων, τα μόνα θεμελιώδη αντικείμενα στη φύση είναι διάφορα πεδία, όπως για παράδειγμα το γνωστό από την θεωρία του Maxwell ηλεκτρομαγνητικό πεδίο, τα οποία εκτείνονται σε όλο τον χωροχρόνο. Τα διάφορα σωματίδια αντιστοιχούν απλώς σε τοπικές διαταραχές των πεδίων αυτών οι οποίες ταξιδεύουν και αλληλεπιδρούν μέσα στον χωροχρόνο σύμφωνα με τους κανόνες της κβαντομηχανικής και της ειδικής θεωρίας της σχετικότητας. Κατά τη διάρκεια αυτών των αλληλεπιδράσεων, νέα σωματίδια μπορούν να παραχθούν από το κενό και άλλα σωματίδια να εξαφανιστούν, φαινόμενα που είναι αδύνατο να περιγραφούν στο πλαίσιο των προγενέστερων θεωριών οι οποίες αντιμετώπιζαν τα διάφορα σωματίδια ως θεμελιώδη αντικείμενα που υπάρχουν για πάντα.

Η πρώτη ολοκληρωμένη κβαντική θεωρία πεδίου ήταν η λεγόμενη 'κβαντική ηλεκτροδυναμική' (QED) η οποία αποτελεί το συγκερασμό τριών θεμελιωδών θεωριών: της κβαντομηχανικής, της ειδικής θεωρίας της σχετικότητας και της θεωρίας ηλεκτρομαγνητισμού του Maxwell. Η θεωρία αυτή περιγράφει με εξαιρετική ακρίβεια όλες τις ιδιότητες των κβαντικών διαταραχών του ηλεκτρομαγνητικού πεδίου και του 'πεδίου ηλεκτρονίων', γνωστών αντίστοιχα ως 'φωτόνια' και 'ηλεκτρόνια', καθώς και τις μεταξύ τους αλληλεπιδράσεις. Η ανακάλυψη όμως της κβαντικής ηλεκτροδυναμικής είχε ιδιαίτερη σημασία για τον επιπρόσθετο λόγο ότι ήταν η πρώτη ολοκληρωμένη κβαντική θεωρία μιας από τις τέσσερις γνωστές θεμελιώδεις δυνάμεις της φύσης, του ηλεκτρομαγνητισμού, της ασθενούς και της ισχυρής αλληλεπίδρασης και της βαρύτητας. Ήταν αναμενόμενο επομένως να αποτελέσει το πρότυπο στη συστηματική αναζήτηση μιας κβαντικής θεωρίας για τις υπόλοιπες θεμελιώδεις δυνάμεις.

Η προσδοκία αυτή επιτεύχθηκε στα τέλη της δεκαετίας του ΄60 και στις αρχές της δεκαετίας του ΄70, όταν μετά από χρόνια πειραμάτων με κοσμικές ακτίνες και επιταχυντές σωματιδίων, αλλά και μετά από εντονότατη θεωρητική προσπάθεια, ανακαλύφθηκαν οι σωστές κβαντικές θεωρίες για την περιγραφή της ασθενούς και της ισχυρής πυρηνικής αλληλεπίδρασης. Διαπιστώθηκε μάλιστα ότι μια κβαντική θεωρία για την ασθενή αλληλεπίδραση απαιτεί την ύπαρξη του ηλεκτρομαγνητισμού! Η ηλεκτρομαγνητική και η ασθενής αλληλεπιδράσεις είναι επομένως διαφορετικές εκφάνσεις μίας θεμελιώδους δύναμης, της λεγομένης 'ηλεκτροασθενούς αλληλεπίδρασης'. Η κβαντική θεωρία πεδίων που περιγράφει την ηλεκτροασθενή αλληλεπίδραση ήταν το πρώτο παράδειγμα ενός νέου τύπου κβαντικής θεωρίας πεδίου, γνωστής ως 'μη-αβελιανή θεωρία βαθμίδας'. Η διαπίστωση ότι αυτός ο τύπος θεωρίας αποτελεί μια μαθηματικά αποδεκτή - συγκεκριμένα 'επανακανονικοποιήσιμη' - κβαντική θεωρία πεδίου, οδήγησε πολύ γρήγορα στην ανακάλυψη της σωστής κβαντικής περιγραφής και της ισχυρής πυρηνικής αλληλεπίδρασης, τη λεγόμενη 'κβαντική χρωμοδυναμική' (QCD), η οποία, όπως η ηλεκτροασθενής θεωρία, είναι μια μη-αβελιανή θεωρία βαθμίδας. Το σύνολο αυτών των κβαντικών θεωριών πεδίου είναι γνωστό ως το 'καθιερωμένο μοντέλο της φυσικής σωματιδίων' και αποτελεί μια πλήρη και εξαιρετικής ακρίβειας περιγραφή όλων των γνωστών σωματιδίων και όλων των θεμελιωδών δυνάμεων - εκτός από τη βαρύτητα.

Παρόλο που το καθιερωμένο μοντέλο της φυσικής σωματιδίων είναι ένα σημαντικότατο βήμα στην αναζήτηση μίας ενοποιημένης θεωρίας, το γεγονός ότι δεν περιλαμβάνει την βαρύτητα οδηγεί αναπόφευκτα στο συμπέρασμα ότι δεν μπορεί να είναι μια τελική ή ολοκληρωμένη θεωρία. Σε αντίθεση με την ειδική θεωρία της σχετικότητας που είναι ενσωματωμένη σε κάθε κβαντική θεωρία πεδίου, ο συγκερασμός της γενικής θεωρίας της σχετικότητας με την κβαντομηχανική έχει αποδειχθεί ένα από τα πλέον δυσεπίλυτα και μακροχρόνια προβλήματα της θεωρητικής φυσικής. Δεν είναι δύσκολο να κατανοήσει κανείς γιατί η διαμόρφωση μιας κβαντικής θεωρίας βαρύτητας παρουσιάζει τέτοια δυσκολία, αφού η γενική θεωρία της σχετικότητας και η κβαντική θεωρία πεδίων περιγράφουν φαινόμενα σε δυο τελείως διαφορετικές κλίμακες μεγέθους. Γνωρίζουμε πολύ καλά ότι ο ρόλος της κβαντομηχανικής είναι αμελητέος στις κλίμακες μεγέθους της καθημερινής μας ζωής και φυσικά σε υπεργαλακτικές κλίμακες όπως αυτή της διάστασης του σύμπαντος. Ταυτόχρονα, η βαρυτική δύναμη είναι εξαιρετικά ασθενής σε σχέση με τις άλλες τρεις θεμελιώδεις δυνάμεις σε υποατομικές κλίμακες. Μια κβαντική θεωρία βαρύτητας επομένως θα πρέπει να περιγράφει φαινόμενα σε μια πολύ μεγάλη έκταση κλιμάκων μεγέθους και απαραίτητα θα βρίσκεται πολύ κοντά σε μια θεωρία μεγαλοενοποίησης.

Αφού η γενική θεωρία της σχετικότητας και η κβαντική θεωρία πεδίων βρίσκονται σε πλήρη συμφωνία με τις πειραματικές παρατηρήσεις στο πεδίο ισχύς τους, οποιαδήποτε κβαντική θεωρία βαρύτητας θα πρέπει να αναπαράγει τις θεωρίες αυτές στο ανάλογο όριο. Ο μόνος τρόπος επομένως για να απομονώσουμε τα γνήσια χαρακτηριστικά μιας κβαντικής θεωρίας βαρύτητας, τα οποία δεν μπορούν να περιγραφούν από την γενική θεωρία της σχετικότητας ή την κβαντική θεωρία πεδίων, είναι να εστιάσουμε το ενδιαφέρον μας στις κλίμακες μεγέθους όπου τα βαρυτικά και τα κβαντομηχανικά φαινόμενα εμφανίζονται ταυτόχρονα. Αυτό συμβαίνει σε κλίμακες πολύ μικρότερες από την ατομική κλίμακα και συγκεκριμένα στην κλίμακα του Planck, όπου οι συνέπειες της κβαντομηχανικής παραμένουν σημαντικές, αλλά ταυτόχρονα η βαρυτική έλξη γίνεται συγκρίσιμη με τις υπόλοιπες θεμελιώδεις δυνάμεις. Πειραματικά είναι πολύ απίθανο να μπορέσουμε ποτέ να εξετάσουμε άμεσα φαινόμενα σε τόσο μικρές κλίμακες μεγέθους ή ισοδύναμα σε τόσο υψηλές ενέργειες, αφού είναι πρακτικά αδύνατο να κατασκευαστεί ο απαραίτητος εξοπλισμός πάνω στη Γη. Γνωρίζουμε όμως ότι οι συνθήκες αυτές υπήρχαν τις πρώτες στιγμές μετά την γέννηση του σύμπαντος κατά τη 'μεγάλη έκρηξη', όταν το μέγεθος του σύμπαντος ήταν συγκρίσιμο με την κλίμακα του Planck! Το γεγονός αυτό αφήνει ανοιχτό το ενδεχόμενο να υπάρχουν άμεσες ενδείξεις κβαντικής βαρύτητας σε κοσμολογικές κλίμακες.

Από τεχνική άποψη, η βασική δυσκολία στη διαμόρφωση μιας κβαντικής θεωρίας βαρύτητας βασισμένης στην γενική θεωρία της σχετικότητας είναι το ότι η διαδικασία 'κβάντωσης' του βαρυτικού πεδίου με τρόπο ανάλογο αυτού που έχει χρησιμοποιηθεί για την κβάντωση των υπόλοιπων θεμελιωδών δυνάμεων οδηγεί σε μαθηματικά αδιέξοδα. Συγκεκριμένα, το γεγονός ότι τα κβάντα του βαρυτικού πεδίου, ή 'βαρυτόνια', αλληλεπιδρούν σύμφωνα με την καθιερωμένη θεωρία πεδίων σε ένα σημείο στον χωροχρόνο έχει ως αποτέλεσμα την εμφάνιση διάφορων μαθηματικών εκφράσεων των οποίων η αριθμητική τιμή τείνει στο άπειρο με μη αποδεκτό τρόπο. Το πρόβλημα αυτό μπορεί να αποφευχθεί μόνο αν τα βαρυτόνια είναι αναγκασμένα να αλληλεπιδρούν από κάποια ελάχιστη απόσταση αντί σε ένα σημείο. Η αναζήτηση μιας θεμελιώδους θεωρίας που υλοποιεί την ιδέα αυτή οδήγησε στην ανάπτυξη της 'θεωρίας χορδών'.

Η θεωρία χορδών

Στη θεωρία χορδών οι θεμελιώδεις 'βαθμοί ελευθερίας' είναι αντικείμενα με μία εκτεταμένη διάσταση, οι λεγόμενες 'χορδές', οι οποίες κινούνται μέσα στο χωροχρόνο καλύπτοντας έτσι μια δισδιάστατη επιφάνεια. Οι δυνατές αλληλεπιδράσεις των χορδών είναι πολύ περιορισμένες και υπαγορεύονται από την γεωμετρία και την τοπολογία αυτών των επιφανειών: αντιστοιχούν απλά στους διαφορετικούς τρόπους που μπορεί να κοπεί και να ενωθεί μια δισδιάστατη επιφάνεια. Συνεπώς, οι αλληλεπιδράσεις μεταξύ των βαρυτονίων, τα οποία εμφανίζονται ως συγκεκριμένοι τρόποι ταλάντωσης των χορδών, έχουν καθαρά γεωμετρική προέλευση. Το γεγονός όμως ότι οι χορδές είναι εκτεταμένα αντικείμενα έχει ως αποτέλεσμα τα βαρυτόνια να μην αλληλεπιδρούν σε ένα συγκεκριμένο σημείο, και επομένως η θεωρία αποφεύγει με αυτόν τον τρόπο τις άπειρες εκφράσεις που καθιστούν αδύνατη την απευθείας κβάντωση της γενικής θεωρίας της σχετικότητας. Παράλληλα, η συνεπής κβάντωση των χορδών συνεπάγεται ότι η θεωρία χορδών αναπαράγει ακριβώς την γενική θεωρία της σχετικότητας σε πολύ χαμηλές ενέργειες ή σε κλίμακες μεγέθους πολύ μεγαλύτερες της κλίμακας του Planck. Αυτή η συμπεριφορά είναι σύμφωνη με ό,τι αναμένει κανείς από μια θεωρία κβαντικής βαρύτητας. Πραγματικά, αυτός είναι ένας από τους βασικούς λόγους που μας κάνουν να πιστεύουμε ότι η θεωρία χορδών είναι η πλέον κατάλληλη υποψήφια θεωρία κβαντικής βαρύτητας που διαθέτουμε σήμερα.

Η θεωρία χορδών όμως δεν είναι μόνο μια συνεπής κβαντική θεωρία βαρύτητας, αλλά και ένα πρότυπο για την πολυπόθητη θεωρία μεγαλοενοποίησης. Εκτός από το γεγονός ότι οι διάφοροι τρόποι ταλάντωσης των χορδών αντιστοιχούν σε μια πληθώρα σωματιδίων μεταξύ των οποίων είναι φυσικά και τα βαρυτόνια, όλες οι πέντε δυνατές θεωρίες χορδών απαιτούν ότι ο χωροχρόνος έχει δέκα διαστάσεις σε αντίθεση με τις τέσσερις διαστάσεις του ορατού σύμπαντος. Αυτό το φαινομενικό μειονέκτημα μπορεί να μετατραπεί σε πραγματικό πλεονέκτημα της θεωρίας χορδών αν υποθέσουμε ότι οι έξι από τις δέκα διαστάσεις είναι μικροσκοπικές - ή 'συμπαγείς' - σε σχέση με τις υπόλοιπες τέσσερις και συνεπώς είναι παρατηρήσιμες μόνο σε πολύ μικρές κλίμακες μεγέθους. Καθένας από τους πολυάριθμους τρόπους να συμπαγοποιήσει κανείς τις έξι αυτές επιπλέον διαστάσεις αντιστοιχεί σε ένα διαφορετικό σύνολο σωματιδίων και αλληλεπιδράσεων μεταξύ τους στις τέσσερις μακροσκοπικές διαστάσεις. Η θεωρία χορδών διαθέτει μέσω αυτού του μηχανισμού τη δυνατότητα να περιγράφει όλες τις θεμελιώδεις δυνάμεις και όλα τα στοιχειώδη σωματίδια στο παρατηρήσιμο σύμπαν. Το κατά πόσο μπορεί η θεωρία χορδών να αναπαράγει το καθιερωμένο μοντέλο στις τέσσερις διαστάσεις αποτελεί σήμερα αντικείμενο εντατικής έρευνας.

Η αντιστοιχία AdS/CFT

Η θεωρία χορδών αναπτύχθηκε όπως είδαμε παραπάνω ως μια γενίκευση της κβαντικής θεωρίας πεδίων στη προσπάθεια ανεύρεσης μιας συνεπούς κβαντικής θεωρίας βαρύτητας. Η πρόσφατη ανακάλυψη της λεγόμενης 'αντιστοιχίας AdS/CFT' όμως συνέδεσε με έναν τελείως διαφορετικό τρόπο την θεωρία χορδών με την κβαντική θεωρία πεδίων. Συγκεκριμένα, η αντιστοιχία AdS/CFT αξιώνει ότι μια ορισμένη θεωρία χορδών στις δέκα διαστάσεις είναι ισοδύναμη με μια κβαντική θεωρία πεδίων χωρίς βαρύτητα στις τέσσερις διαστάσεις. Η αντιστοιχία - ή 'δυϊκότητα' - αυτή, η οποία μας δίνει τη δυνατότητα να μελετήσουμε ορισμένες κβαντικές θεωρίες πεδίων μέσω της θεωρίας χορδών και αντίστροφα, είναι το αντικείμενο αυτής της διατριβής.