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OPERATOR PRODUCT EXPANSIONS AND ANOMALOUS DIMENSIONS IN THE THIRRING MODEL*

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ABSTRACT

An example of an operator product expansion is worked out for the Thirring model. The Thirring model involves a two-dimensional zero mass Dirac field ψ interacting via the Fermi interaction. The model is scale invariant but the dimensions of local fields in the model vary with the coupling constant λ. It is shown that ψ has dimension $1/2 + \lambda^2/4\pi^2\,(1 - \lambda^2/4\pi^2)^{-1}$, while the composite fields $\overline{\psi}\psi$ and $\overline{\psi}\gamma_5\psi$, appropriately defined, have the dimension $(1 - \lambda/2\pi)\,(1+\lambda/2\pi)^{-1}$.

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I. INTRODUCTION

In a recent paper[1] several new hypotheses were proposed concerning the short distance behavior of strong interactions. One of the hypotheses was that products of currents (or other local fields) at short distances would have "operator product expansions" of the form

$$T\, j_\mu(x)\, j_\nu(y) = \sum_n C_{n\mu\nu}(x-y)\, O_n(y) \tag{I.1}$$

where the $O_n(y)$ are a complete, linearly independent set of local fields, and the functions $C_{n\mu\nu}(x-y)$ are functions which give the singularities of the current-current product when $x \rightarrow y$. Another hypothesis was that the strong interactions would become scale invariant at short distances;[2] in particular the functions $C_{n\mu\nu}(x-y)$ would reflect scale invariance when x-y is small except for small finite mass corrections. A third hypothesis was that the dimensions of the fields O_n would be different from the dimensions of fields in any free field model of current algebra. To be precise the dimension of the current j_μ would remain the same as the free field dimension (namely 3 in mass units) because this dimension is fixed by Gell-Mann's current algebra. However the dimension Δ of the pion field would differ from the dimension predicted by any free field model; this dimension was considered an arbitrary parameter since there is at present no way to compute it.

It should be helpful to see how these hypotheses work in a model field theory which can be solved explicitly. The Thirring model,[3-6] namely a Dirac field in one space and one time dimension interacting via the Fermi interaction, is a suitable example for this purpose. In this paper an example of an operator product expansion in the Thirring model is worked out. Also the dimensions of the field ψ, the current j_μ and the scalar and pseudoscalar fields $\bar{\psi}\psi$ and $\bar{\psi}\gamma_5\psi$ are computed. These dimensions indeed differ from free field dimensions, except for the current. There is a much more thorough discussion of the operator product expansion in Ref. 6, Section IV.

II. THE THIRRING MODEL

The Thirring model involves a Dirac field $\psi(x)$ in one space and one time dimension. The field is coupled to itself by the current-current interaction $\lambda j_\mu(x) j^\mu(x)$, where λ is the coupling constant and j_μ is the current $\bar{\psi}\gamma_\mu\psi$. Provided the mass of the field is zero, the model can be exactly solved. A transparent method for solving the theory is described by K. Johnson.[4] He uses the fact that in the zero mass theory both the vector and axial vector currents are conserved. He also needs the result (special to one space dimension) that the axial current is just $\epsilon_{\mu\nu}$ times the vector current, where $\epsilon_{\mu\nu}$ is the covariant antisymmetric tensor. From these results Johnson is able to reconstruct the two- and four-point Green's functions of the theory. Any 2n-point function can be derived by Johnson's method.[5]

The Thirring model is clearly a special theory, depending for its solution on special properties of two-dimensional space time. However any general feature of quantum field theory, which one expects to hold for all quantum field theories, must hold in particular for the Thirring model. The operator product expansion is a property which one would like to hold generally so it is worth investigating whether operator product expansions exist in the Thirring model. Furthermore, working with the explicit formulae of the Thirring model is one way to get experience with operator product expansions. Finally, the Thirring model is one of the sources for the idea that the dimension of a field is a dynamical quantity, i.e., dependent on the strength of the interactions of the field.

The two- and four-point functions obtained by Johnson are as follows:

$$G(x-y) = i\langle\Omega|T\,\psi(x)\bar{\psi}(y)|\Omega\rangle = \exp\left\{-i\lambda(a-\bar{a})\,D_0(x-y)\right\}\,G_0(x-y) \qquad (II.1)$$

$$G(xx'yy') = -\langle\Omega|T\,\psi(x)\,\psi(x')\,\bar{\psi}(y')\,\bar{\psi}(y)\,|\Omega\rangle$$

$$= \exp\left\{i\lambda\,(a-\bar{a}\,\gamma_{5x}\,\gamma_{5x'})\left[D_0(x-x') - D_0(x-y') + D_0(y-y') - D_0(y-x')\right]\right\}$$

$$\times\, G(x-y)\;G(x'-y') - (\text{term with } x\longleftrightarrow x') \qquad (II.2)$$

where $|\Omega\rangle$ is the vacuum state, T is the time-ordering symbol, $G_0(x-y)$ is the free Dirac propagator (zero mass) and $D_0(x-y)$ the free propagator of a zero mass scalar field. Also

$$a = (1 - \lambda/2\pi)^{-1} \qquad (II.3)$$

$$\bar{a} = (1 + \lambda/2\pi)^{-1} \qquad (II.4)$$

The spin matrix γ_{5x} multiplies G(x-y), the spin matrix $\gamma_{5x'}$ multiplies G(x'-y'). The exchange term in Eq. (II.2) is sufficient to make G(xx'yy') antisymmetric to either $x\longleftrightarrow x'$ or $y\longleftrightarrow y'$, as is required by Fermi statistics. Explicit formulae for the free propagators are

$$D_0(z) = (-i/4\pi)\,\ln\,(-z^2+i\epsilon) \qquad (II.5)$$

$$G_0(z) = +\,(1/2\pi)\,\gamma_\mu z^\mu\,(z^2-i\epsilon)^{-1} \qquad (II.6)$$

The function G(x-y) has been normalized arbitrarily. Customarily the normalization of G is fixed by the canonical commutation rules, but in the Thirring model with interaction, ψ does not satisfy canonical commutation rules[4] and can be normalized arbitrarily. One can also add an arbitrary constant to D_0 without affecting anything except the normalization of ψ; this fact can be used to replace $\ln(-z^2+i\epsilon)$ by $\ln\left[(-z^2+i\epsilon)/x_0^2\right]$ where x_0 is a constant length, thus making the argument of the logarithm dimensionless. The constant x_0 is put equal to 1 here. Johnson also obtains matrix elements of the current $j_\mu(x)$; in particular

$$i\langle\Omega | T j^\mu(y)\, \psi(x')\, \bar{\psi}(y') | \Omega\rangle = (g^{\mu\nu}\, a + \epsilon^{\mu\nu}\, \bar{a}\gamma_5)\, V_\nu^y \left[D_0(y-x') - D_0(y-y')\right] G(x'-y') \tag{II.7}$$

Explicit forms of $g^{\mu\nu}$, $\epsilon^{\mu\nu}$, and the γ matrices used here are as follows

$$(g^{00}, g^{11}) = (1, -1) \tag{II.8}$$

$$\epsilon^{01} = -1, \quad \epsilon^{10} = 1 \tag{II.9}$$

$$(\gamma^0, \gamma^1, \gamma^5) = (\sigma^2, i\sigma^1, \sigma^3) \tag{II.10}$$

where σ^1, σ^2, and σ^3 are the Pauli matrices.

From the Green's functions it can be seen that the field ψ is scale invariant, and the dimension of ψ can be determined. If ψ is a scale invariant field, then there exists a unitary transformation U(s) with the property

$$U^+(s)\, \psi(x)\, U(s) = s^d\, \psi(sx) \tag{II.11}$$

where d is the dimension of ψ in mass units. By conjugation one gets also

$$U^+(s)\, \bar{\psi}(x)\, U(s) = s^d\, \bar{\psi}(sx) \tag{II.12}$$

Assuming the vacuum to be invariant to scale transformations, one has

$$-i\, G(x-y) = \langle\Omega | T\, \psi(x)\, \bar{\psi}(y) | \Omega\rangle = \langle\Omega | U^+(s)\; T\, \psi(x)\, \bar{\psi}(y)\; U(s) | \Omega\rangle \tag{II.13}$$

Because U(s) is unitary ($U(s)\, U^+(s) = 1$) one has

$$U^+(s)\; T\, \psi(x)\, \bar{\psi}(y)\; U(s) = U^+(s)\; T\, \psi(x)\; U(s)\; U^+(s)\, \bar{\psi}(y)\; U(s) = s^{2d}\; T\, \psi(sx)\, \bar{\psi}(sy) \tag{II.14}$$

Hence scale invariance and an invariant vacuum imply that

$$G(x-y) = s^{2d}\, G(sx - sy) \tag{II.15}$$

Similarly

$$G(xx'yy') = s^{4d}\, G(sx, sx', sy, sy') \tag{II.16}$$

Both these equations are satisfied by Johnson's solution provided that

$$d = 1/2 + (\lambda^2/4\pi^2)\,(1-\lambda^2/4\pi^2)^{-1} \tag{II.17}$$

The scaling law for $G(x-y)$ follows from the fact that $D_0(x-y)$ is a logarithm in $(x-y)^2$ so the exponential of D_0 is a power of $(x-y)^2$. The scaling law for $G(xx'yy')$ follows from the fact that the exponential in Eq. (II.2) is independent of scale transformations (since the exponential involves differences of logarithms which can be combined to involve only dimensionless ratios); one is left with the product $G(x-y)\, G(x'-y')$ which scales as s^{4d}.

Similar arguments hold for the 2n-point functions. Hence all the Green's functions are consistent with scale invariance and an invariant vacuum. This means that the theory is scale invariant and has an invariant vacuum, unless there is some feature of the theory that cannot be determined from the Green's functions and is not invariant. I do not know of any such.

When $\lambda=0$ the dimension d is .5 which is the dimension of a free spinor field in one space, one time dimension. The dimension .5 is what one predicts for ψ using the canonical commutation relations. For nonzero λ, d is greater than .5, which is inconsistent with canonical commutation relations, but one already knows that the canonical commutators do not hold for $\lambda \neq 0$. As $\lambda \to 2\pi$, $d \to \infty$, so the departure from the free field dimension can be arbitrarily large. Furthermore d need not be an integer or half integer. Clearly one has to modify one's usual picture of what a dimension is in order to accept the dimension that ψ has in the presence of interaction.

Using Johnson's solutions for the two- and four-point Green's functions one can construct the leading terms in the expansion of $T\,\psi(x)\,\overline{\psi}(y)$ for x near y.[6] To be complete one must use all the 2n functions; this problem will not be discussed.

Consider first the free field limit ($\lambda=0$). In this limit one can express the T product in terms of a Wick product:

$$T\,\psi(x)\,\overline{\psi}(y) = -i\,G_0(x-y)\,I + :\psi(x)\,\overline{\psi}(y): \tag{II.18}$$

where I is the unit operator. To obtain an operator product expansion one must express the Wick product in terms of local operators of y. This is accomplished by expanding $:\psi(x)\,\overline{\psi}(y):$ in a Taylor's series in x-y:

$$:\psi(x)\,\overline{\psi}(y): = :\psi(y)\,\overline{\psi}(y): + (x-y)^{\mu} :\left[\nabla_{\mu}\,\psi(y)\right]\overline{\psi}(y): + \ldots \tag{II.19}$$

This expansion is legitimate for any given matrix element of the operator $:\psi(x)\,\overline{\psi}(y):$ because the x dependence of the matrix element depends only on the momenta of the states in the matrix element and is smooth as $x \to y$. In contrast one cannot expand $T\,\psi(x)\,\overline{\psi}(y)$ in powers of x-y because of the G_0 term which is singular when x=y. The operator product expansion for $T\,\psi(x)\,\overline{\psi}(y)$ is

$$T\,\psi(x)\,\overline{\psi}(y) = -i\,G_0(x-y)\,I + :\psi(y)\,\overline{\psi}(y): + \text{(terms of order } (x-y)) \tag{II.20}$$

In studying the generalization of this expansion to interacting fields the terms of order x-y will be ignored, to simplify the analysis. Also the operator $:\psi(y)\,\overline{\psi}(y):$ is actually four separate operators because ψ and $\overline{\psi}$ both have two components. It will be convenient to generalize each component separately to the case of interacting fields. A convenient separation of $:\psi(y)\,\overline{\psi}(y):$ into components is to define

$$\phi_{\pm}(x) = :\overline{\psi}(y)\;(1 \pm \gamma_5)\;\psi(y): \tag{II.21}$$

and

$$j_{\pm}(x) = :\overline{\psi}(y)\ (\gamma^1 \pm \gamma^0)\ \psi(y): \qquad (II.22)$$

The operators $j_{\pm}(x)$ are just the combinations $j^1(x) \pm j^0(x)$ of components of the current $j^{\mu}(x)$; the generalization to interacting fields is that $j_{\pm}(x)$ continue to be $j^1(x) \pm j^0(x)$. The fields $\phi_{\pm}(x)$ do not have an a priori generalization to the interacting case. The matrix elements of $\phi_{\pm}$ will have to be determined as part of the calculation which determines the generalization of Eq. (II.20).

The generalization of Eq. (II.20) which will be obtained here for interacting fields has the form

$$T\,\psi(x)\,\overline{\psi}(y) = -iG(x-y)\ I + C_1(x-y)\ \phi_+(y) + C_2(x-y)\ \phi_-(y)$$

$$+ C_3(x-y)\ j_+(x) + C_4(x-y)\ j_-(y) + \text{Remainder} \qquad (II.23)$$

where the "Remainder" includes terms which are smaller by at least one power of x-y than the terms $C_1 \ldots C_4$. The functions $C_1(x-y) \ldots C_4(x-y)$ are 2×2 matrices labeled by the spin indices of $\psi(x)$ and $\overline{\psi}(y)$. When this expansion is sandwiched between the operators $\psi(x')$ and $\overline{\psi}(y')$ one obtains the following:

$$-G(xx'yy') = -G(x-y)\ G(x'-y') + C_1(x-y) \langle\Omega|T\phi_+(y)\ \psi(x')\ \overline{\psi}(y')|\Omega\rangle$$

$$+ C_2(x-y) \langle\Omega|T\phi_-(y)\ \psi(x')\ \overline{\psi}(y')|\Omega\rangle$$

$$+ C_3(x-y) \langle\Omega|Tj_+(y)\ \psi(x')\ \overline{\psi}(y')|\Omega\rangle$$

$$+ C_4(x-y) \langle\Omega|Tj_-(y)\ \psi(x')\ \overline{\psi}(y')|\Omega\rangle$$

$$+ \text{Remainder} \qquad (II.24)$$

It is this formula that will actually be derived. It implies that when x-y is small, G(xx'yy') can be written as a sum of products of functions of x-y (C_1, etc.) times functions only of y, x', and y', apart from a small remainder term.

The calculation which gives Eq. (II.24) will now be laid out. It is simplest (in the author's experience) to work with spin components in an explicit representation of the γ matrices, rather than writing formulae in covariant form in terms of γ matrices. The representation has already been given (Eq. (II.10)). It is convenient to introduce the following definitions and formulae. For any space-time variable x let

$$x_\pm = x^1 \pm x^0 \qquad \text{(II.25)}$$

Then

$$x^2 = (x^0)^2 - (x^1)^2 = -x_+x_- \qquad \text{(II.26)}$$

$$x \cdot y = x^0y^0 - x^1y^1 = 1/2\left[x_+y_- + x_-y_+\right] \qquad \text{(II.27)}$$

$$\gamma_\mu x^\mu = -i\,(\sigma_+x_+ + \sigma_-x_-) \qquad \text{(II.28)}$$

where

$$\sigma_\pm = 1/2\,(\sigma^1 \pm i\sigma^2) \qquad \text{(II.29)}$$

Define

$$\xi = x-y \qquad \text{(II.30)}$$

$$z = x'-y \qquad \text{(II.31)}$$

$$z' = y'-y \qquad \text{(II.32)}$$

Define

$$\beta = \lambda(a+\bar{a})/4\pi = \lambda/2\pi \quad (1 - \lambda^2/4\pi^2)^{-1} \qquad \text{(II.33)}$$

$$\gamma = \lambda/4\pi\,(a-\bar{a}) = \lambda^2/4\pi^2\,(1 - \lambda^2/4\pi^2)^{-1} \qquad \text{(II.34)}$$

Note that

$$\gamma/\beta = \beta/(\gamma+1) = \lambda/2\pi \qquad \text{(II.35)}$$

$$d = 1/2 + \gamma \qquad \text{(II.36)}$$

Also

$$G(\xi) = (i/2\pi)\,(-\xi^2)^{-\gamma-1}\,(\xi_-\sigma_- + \xi_+\sigma_+) \qquad (\text{II}.37)$$

Now a whole sequence of formulae will be quoted giving explicitly various components of $G(xx'yy')$ and other matrix elements. These formulae are all straightforward to derive from Johnson's formulae (Eqs. (II.1), (II.2), and (II.7)). The formulae are separated by components of $\psi(x')$ and $\overline{\psi}(y')$, since for each component separately of $\psi(x')$ and $\overline{\psi}(y')$ one has a matrix element of $T\,\psi(x)\,\overline{\psi}(y)$ to study. In the following, "Remainder" means a term smaller by at least one power of ξ than any term given explicitly. The matrix elements supplied besides $G(xx'yy')$ are: matrix elements of $j_\pm(y)$, for all λ, and matrix elements of $\phi_\pm(y)$ for free fields. Only the nonzero matrix elements of these operators are listed. Note that in Eq. (II.2) for $G(xx'yy')$, the matrices γ_{5x} and $\gamma_{5x'}$ are diagonal because of the representation (II.10); γ_{5x} and $\gamma_{5x'}$ will be either +1 or -1 depending on what components of $\psi(x)$ and $\psi(x')$ are being considered. The first and second terms in both Eqs. (II.44) and (II.48) below are of order $\xi^{-2\gamma-1}$. These terms are expanded to order $\xi^{-2\gamma}$, i.e., terms of order $\xi^{-2\gamma-1}$ and $\xi^{-2\gamma}$ are kept in the expansion, the remainder being of order $\xi^{-2\gamma+1}$. For all other terms only the leading order in ξ is kept.

A. Matrix elements with $\psi_1(x')$ and $\overline{\psi}_1(y')$

$$<\Omega | T\, \psi_1(x')\, \psi(x)\, \overline{\psi}(y)\, \overline{\psi}_1(y') | \Omega>$$

$$= -\frac{1}{4\pi^2} \left[z^2 (z'-\xi)^2\right]^{-\gamma-1} \left[(z-\xi)^2 (z')^2\right]^{\beta} \left[(z-z')^2 \xi^2\right]^{-\beta} (\xi - z')_-\, z_+$$

$$\times 1/2\, (1-\sigma^3) \qquad (\text{II}.38)$$

$$= + \frac{1}{4\pi^2} \left[z^2 z'^2\right]^{\beta-\gamma-1} \left[-(z-z')^2\right]^{-\beta} z_+ z'_- \left[-\xi^2\right]^{-\beta} \times 1/2\, (1-\sigma^3)$$

$$+ \text{Remainder} \qquad (\text{II}.39)$$

$$<\Omega | T\, \psi_1(x')\, \phi_-(y)\, \overline{\psi}_1(y') | \Omega> = -\frac{1}{4\pi^2} (z^2 z'^2)^{-1} (2 z_+ z'_-) \quad (\text{for } \lambda = 0) \qquad (\text{II}.40)$$

B. Matrix elements with $\psi_2(x')$ and $\overline{\psi}_2(y')$

$$<\Omega | T\, \psi_2(x')\, \psi(x)\, \overline{\psi}(y)\, \overline{\psi}_2(y') | \Omega> = -\frac{1}{4\pi^2} \left[z^2 (z'-\xi)^2\right]^{-\gamma-1}$$

$$\times \left[(z-\xi)^2 z'^2\right]^{\beta} \left[(z-z')^2 \xi^2\right]^{-\beta} (\xi - z')_+\, z_- \times 1/2\, (1+\sigma^3) \qquad (\text{II}.41)$$

$$= + \frac{1}{4\pi^2} \left[z^2 z'^2\right]^{\beta-\gamma-1} \left[-(z-z')^2\right]^{-\beta} z'_+ z_- (-\xi^2)^{-\beta} \times 1/2\, (1+\sigma^3)$$

$$+ \text{Remainder} \qquad (\text{II}.42)$$

$$<\Omega | T\, \psi_2(x')\, \phi_+(y)\, \overline{\psi}_2(y') | \Omega> = -\frac{1}{4\pi^2} (z^2 z'^2)^{-1} (2 z_- z'_+) \quad (\lambda = 0) \qquad (\text{II}.43)$$

C. Matrix elements with $\psi_1(x')$ and $\overline{\psi}_2(y')$

$$\langle\Omega|T\,\psi_1(x')\,\psi(x)\,\overline{\psi}(y)\,\overline{\psi}_2(y')|\Omega\rangle$$

$$= \frac{1}{4\pi^2}\left[\xi^2(z-z')^2\right]^{-\gamma-1}\left[(z-\xi)^2(z')^2\right]^{\beta}\left[(z'-\xi)^2 z^2\right]^{-\beta}\xi_-(z-z')_+\sigma_-$$

$$+\frac{1}{4\pi^2}\left[\xi^2(z-z')^2\right]^{-\gamma-1}\left[(z-\xi)^2 z'^2\right]^{\gamma}\left[(z'-\xi)^2 z^2\right]^{-\gamma}\xi_+(z-z')_+\sigma_+$$

$$-\frac{1}{4\pi^2}\left[z^2(z'-\xi)^2\right]^{-\gamma-1}\left[(z-\xi)^2 z'^2\right]^{\gamma}\left[(z-z')^2\xi^2\right]^{-\gamma}z_+(\xi-z')_+\sigma_+ \qquad \text{(II.44)}$$

$$= +\frac{1}{4\pi^2}\left[-(z-z')^2\right]^{-\gamma-1}(z-z')_+(-\xi^2)^{-\gamma-1}\left[\xi_-\sigma_-+\xi_+\sigma_+\right]$$

$$+\frac{1}{4\pi^2}\left[-(z-z')^2\right]^{-\gamma-1}(z^2 z'^2)^{-1}$$

$$\times\left\{\beta(z-z')_+(z-z')_-z_+z'_+(-\xi^2)^{-\gamma-1}\left[\xi_-^2\sigma_-+(\gamma+1)\beta^{-1}\xi_+\xi_-\sigma_+\right]\right.$$

$$\left.+\gamma(z-z')_+^2 z_-z'_-(-\xi^2)^{-\gamma-1}\left[\beta\gamma^{-1}\xi_+\xi_-\sigma_-+\xi_+^2\sigma_+\right]\right\}$$

$$+\text{ Remainder} \qquad \text{(II.45)}$$

$$\langle\Omega|T\,\psi_1(x')\,j_+(y)\,\overline{\psi}_2(y')|\Omega\rangle$$

$$= -i(\pi\lambda)^{-1}\beta\left[-(z-z')^2\right]^{-\gamma-1}\left[z^2 z'^2\right]^{-1}(z-z')_+(z-z')_-z_+z'_+ \qquad \text{(II.46)}$$

$$\langle\Omega|T\,\psi_1(x')\,j_-(y)\,\overline{\psi}_2(y')|\Omega\rangle$$

$$= -i(\pi\lambda)^{-1}\gamma\left[-(z-z')^2\right]^{-\gamma-1}\left[z^2 z'^2\right]^{-1}(z-z')_+^2 z_-z'_- \qquad \text{(II.47)}$$

D. Matrix elements with $\psi_2(x')$ and $\overline{\psi}_1(y')$

$$\langle \Omega | T\, \psi_2(x')\, \psi(x)\, \overline{\psi}(y)\, \overline{\psi}_1(y') | \Omega \rangle$$

$$= \frac{1}{4\pi^2} \left[\xi^2 (z-z')^2\right]^{-\gamma-1} \left[(z-\xi)^2 (z')^2\right]^{\beta} \left[(z'-\xi)^2 z^2\right]^{-\beta} \xi_+ (z-z')_- \sigma_+$$

$$+ \frac{1}{4\pi^2} \left[\xi^2 (z-z')^2\right]^{-\gamma-1} \left[(z-\xi)^2 z'^2\right]^{\gamma} \left[(z'-\xi)^2 z^2\right]^{-\gamma} \xi_- (z-z')_- \sigma_-$$

$$- \frac{1}{4\pi^2} \left[z^2 (z'-\xi)^2\right]^{-\gamma-1} \left[(z-\xi)^2 z'^2\right]^{\gamma} \left[(z-z')^2 \xi^2\right]^{-\gamma} (\xi-z')_- z_- \sigma_- \quad \text{(II.48)}$$

$$= + \frac{1}{4\pi^2} \left[-(z-z')^2\right]^{-\gamma-1} (z-z')_- \left[-\xi^2\right]^{-\gamma-1} \left[\xi_- \sigma_- + \xi_+ \sigma_+\right]$$

$$+ \frac{1}{4\pi^2} \left[-(z-z')^2\right]^{-\gamma-1} \left[z^2 z'^2\right]^{-1}$$

$$\times \left\{ \gamma (z-z')_-^2 z_+ z'_+ (-\xi^2)^{-\gamma-1} \left[\xi_-^2 \sigma_- + \beta\gamma^{-1} \xi_- \xi_+ \sigma_+\right] \right.$$

$$\left. + \beta (z-z')_+ (z-z')_- z_- z'_- (-\xi^2)^{-\gamma-1} \left[(\gamma+1) \beta^{-1} \xi_+ \xi_- \sigma_- + \xi_+^2 \sigma_+\right] \right\}$$

$$+ \text{Remainder} \quad \text{(II.49)}$$

$$\langle \Omega | T\, \psi_2(x')\, j_+(y)\, \overline{\psi}_1(y') | \Omega \rangle$$

$$= - i (\pi\lambda)^{-1} \gamma \left[-(z-z')^2\right]^{-\gamma-1} \left[z^2 z'^2\right]^{-1} (z-z')_-^2 z_+ z'_+ \quad \text{(II.50)}$$

$$\langle \Omega | T\, \psi_2(x')\, j_-(y)\, \overline{\psi}_1(y') | \Omega \rangle$$

$$= - i (\pi\lambda)^{-1} \beta \left[-(z-z')^2\right]^{-\gamma-1} \left[z^2 z'^2\right]^{-1} (z-z')_+ (z-z')_- z_- z'_- \quad \text{(II.51)}$$

Given Eqs. (II.38) - (II.51), it is straightforward to verify the expansion (II.24). The first term $G(x-y)\ G(x'-y')$ is known explicitly and becomes the first term in the expansions (II.45) and (II.49). In the free field limit $\phi_+(y)$ has a nonzero matrix element only between $\psi_2(x')$ and $\overline{\psi}_2(y')$. Furthermore, in the free field limit the matrix elements of the other three operators (ϕ_-, j_+, and j_-) with $\psi_2(x')$ and $\overline{\psi}_2(y')$ all vanish. This turns out not to be an accident; it is a consequence of the conservation of axial charge, namely the charge whose current is the axial current $\epsilon^{\mu\nu} j_\nu$. From the commutation rules given by Johnson[4] ψ_1 and $\overline{\psi}_1$ have axial charge $\bar{a}$ while ψ_2 and $\overline{\psi}_2$ have axial charge $(-\bar{a})$. Hence, from Eqs. (II.21) and (II.22), $j_\pm$ have axial charge 0, ϕ_+ has axial charge $2\bar{a}$, and ϕ_- has axial charge $-2\bar{a}$. The total axial charge of all fields in a nonzero vacuum expectation value must add to 0. Thus ϕ_+ has nonzero matrix elements only with $\psi_2 \overline{\psi}_2$, ϕ_- with $\psi_1 \overline{\psi}_1$, and $j_\pm$ with $\psi_1 \overline{\psi}_2$ and $\psi_2 \overline{\psi}_1$. Let us assume that ϕ_+ and ϕ_- continue to have axial charge $2\bar{a}$ and $-2\bar{a}$ respectively for nonzero λ. Then only the C_1 term in Eq. (II.24) will occur in the expansion of the $\psi_2(x') \ldots \overline{\psi}_2(y')$ matrix element of $T\,\psi(x)\,\overline{\psi}(y)$. Comparing Eqs. (II.24) and (II.42), we see that they agree provided that

$$C_1(\xi) = b_1\,(-\xi^2)^{-\beta}\ 1/2\ (1+\sigma^3) \tag{II.52}$$

$$<\Omega\,|\,T\,\psi_2(x')\ \phi_+(y)\ \overline{\psi}_2(y')\,|\,\Omega> = (4\pi^2 b_1)^{-1}\left[z^2 z'^2\right]^{\beta-\gamma-1}\left[-(z-z')^2\right]^{-\beta} z'_+ z_- \tag{II.53}$$

where b_1 is an arbitrary constant. The value of b_1 is unimportant since it can always be changed by changing the normalization of ϕ_+. Since C_1 and the matrix element depend on different variables, both are determined from the single equation (II.42) except for the scale factor b_1. Apart from the scale factor

Eq. (II.53) reduces to the known free field matrix element of ϕ_+ (Eq. (II.43)) when $\lambda \to 0$.

An analogous argument gives

$$C_2(\xi) = b_2 \, (-\xi^2)^{-\beta} \, 1/2 \, (1-\sigma^3) \tag{II.54}$$

$$<\Omega | T \, \psi_1(x') \, \phi_-(y) \, \overline{\psi}_1(y') | \Omega> = (4\pi^2 b_2)^{-1} \left[z^2 z'^2\right]^{\beta-\gamma-1} \left[-(z-z')^2\right]^{-\beta} z_+ z'_- \tag{II.55}$$

from Eq. (II.39).

To determine the C_3 and C_4 terms in the expansion one can look at either the $\psi_1(x') \ldots \overline{\psi}_2(y')$ or the $\psi_2(x') \ldots \overline{\psi}_1(y')$ matrix elements. Consider first the $\psi_1(x') \ldots \overline{\psi}_2(y')$ matrix element (Eq. (II.45)). The first term in its expansion matches the G(x-y) G(x'-y') term in Eq. (II.24). The other term in Eq. (II.45) is a linear combination of j_+ and j_- matrix elements. This is easily seen since the matrix elements of j_+ and j_- are known explicitly. Comparing Eq. (II.24) with Eqs. (II.45) - (II.47), and using Eq. (II.35), one gets

$$C_3(\xi) = (+i\lambda/4\pi) \, (-\xi^2)^{-\gamma-1} \left[\xi_-^2 \, \sigma_- + (2\pi/\lambda) \, \xi_+ \xi_- \sigma_+\right] \tag{II.56}$$

$$C_4(\xi) = (+i\lambda/4\pi) \, (-\xi^2)^{-\gamma-1} \left[(2\pi/\lambda) \, \xi_+ \xi_- \sigma_- + \xi_+^2 \, \sigma_+\right] \tag{II.57}$$

The coefficients of these functions in Eq. (II.45) are precisely the matrix elements of j_+ and j_- given by Eqs. (II.46) and (II.47).

One can also determine $C_3(\xi)$ and $C_4(\xi)$ from the $\psi_2(x') \ldots \overline{\psi}_1(y')$ matrix element. Using the identity (II.35), the result is again Eqs. (II.56) and (II.57).

With $C_1 \ldots C_4$ given by Eqs. (II.52), (II.54), (II.56), and (II.57), and the nonzero matrix elements of $\phi_\pm$ and $j_\pm$ given by Eqs. (II.53), (II.55), (II.46), (II.47), (II.50), and (II.51), it is now seen that the expansion (II.24) holds with the remainder being smaller by one power of ξ than the terms kept for each axial charge component of $\psi(x)\,\overline{\psi}(y)$.

Given the matrix elements of $\phi_\pm$ and $j_\pm$ one can determine the dimensions of these fields. Using the same type of analysis as was used earlier for G(x-y), one finds that scale invariance implies

$$<\Omega\,|\,T\,\psi(x')\;\phi_\pm(y)\;\overline{\psi}(y')\,|\,\Omega> = s^{(d_\phi + 2d)}\;<\Omega\,|\,T\,\psi(sx')\;\phi_\pm(sy)\;\overline{\psi}(sy')\,|\,\Omega> \qquad (\text{II}.58)$$

where d_ϕ is the dimension of $\phi_\pm$ and d the dimension of ψ (given by Eq. (II.36)). Comparing this requirement with the explicit formulae (II.53) and (II.55), one gets

$$d_\phi = (1-\lambda/2\pi)\;(1+\lambda/2\pi)^{-1} \qquad (\text{II}.59)$$

The same analysis for $j_\pm$ gives its dimension as 1 always. This is required in any case if the equal time commutation rule for ψ with $j^{0\,(4)}$ is scale invariant.

While the dimension of ψ increases with λ, going to ∞ when $\lambda \to 2\pi$, the dimension of the composite field ϕ decreases with λ and goes to zero as $\lambda \to 2\pi$. In the free field limit $\phi_\pm$ has the same dimension as the product $\overline{\psi}\psi$; but this is no longer true in the presence of interaction. The current $j_\pm$ also does not have the dimension of $\overline{\psi}\gamma_\mu\psi$ in the presence of interaction, nor do $\phi_\pm$ and $j_\pm$ have the same dimension in the presence of interaction. So the dimensions of the fields ψ, $\phi_\pm$, and $j_\pm$ get almost totally scrambled by the interaction.

Scale invariance requires that the ξ dependence of $C_1(\xi)\ldots C_4(\xi)$ be such as to make dimensions match in all terms of the expansion (II.24).[1'] For example, from the dimensions of ψ, $\overline{\psi}$, and $\phi_\pm$ one deduces that C_1 must obey

$$C_1(\xi) = s^{2d-d_\phi}\;C_1(s\xi) \qquad (\text{II}.60)$$

This formula is easily verified using Eqs. (II.52), (II.36), and (II.59). C_2, C_3, and C_4 also can be shown to scale according to the analogous rules.

Thus we have the beginnings of an operator product expansion for $T\,\psi(x)\,\overline{\psi}(y)$ in the Thirring model. A complete analysis would require studying matrix elements of $T\,\psi(x)\,\overline{\psi}(y)$ with arbitrary many other fields, and expanding to all orders in x-y. But such an analysis would be more than an exercise. The above analysis should be sufficient to clarify somewhat the nature of an operator product expansion and to emphasize the dynamical character of dimensions of fields in the Thirring model.

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REFERENCES

1. K. Wilson, Phys. Rev. 179, 1499 (1969).

2. This idea is due to Kastrup and Mack; see G. Mack, Nuclear Physics B5, 499 (1968) and references cited therein.

3. W. Thirring, Ann. Phys. (N.Y.) 3, 91 (1958).

4. K. Johnson, Nuovo Cimento 20, 773 (1960).

5. Formulae for the 2n-point functions are given in B. Klaiber, Helv. Phys. Acta 37, 554 (1964). For more recent work on the Thirring model see Ref. 6 and B. Klaiber in Lectures in Theoretical Physics, Vol. X-A: Quantum Theory and Statistical Physics, A. O. Barut and W. E. Brittin, Editors (Gordon and Breach Science Publishers, Inc., New York, 1968) p. 141; A. Wightman in High Energy Electromagnetic Interactions and Field Theory, M. Levy, Editor (Gordon and Breach Science Publishers, Inc., New York, 1967) p. 218.

6. John Lowenstein, "Normal products in the Thirring model," Preprint from Departamento de Fisica, Universidade de São Paulo, São Paulo, Brasil, discusses operator product expansions using Klaiber's solution (Ref. 5). However, Lowenstein does not discuss scale invariance as it is proposed here. What he refers to is an earlier unpublished version of the author's formulation of scale invariance for operator product expansions. In this earlier version fields were assigned their free field dimensions, but the operator product expansions were allowed to violate scale invariance through logarithmic terms. Lowenstein correctly points out that this hypothesis is inadequate for the Thirring model in strong coupling. This is in fact the reason the earlier version was not published. A summary of the hypotheses of the earlier version appears in R. A. Brandt, Ann. Phys. (N.Y.) 44, 221 (1967), Appendix C. The

author apologizes to those who have been inconvenienced by the unavailability of the earlier work and the long delay in providing a substitute.