Imperial College of Science, Technology and Medicine
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Exotic Gauge Theories of Spin-2 Fields

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Abstract

There has been a resurgence of interest in theories of massive spin-2 fields, owing to the recent discovery of ghost-free self-interactions. In addition to reviewing the historical and recent progress in this subject, I discuss my contributions, including the derivation of the complete decoupling limit of dRGT ghost-free massive gravity, proving no-go theorems on ghost-free interactions for charged spin-2 fields, updating the method of Dimensional Deconstruction for fermions to obtain massive supersymmetric gauge theories, and my progress towards supergravity theories with non-zero graviton mass.

Statement of Originality

All of original research presented in this thesis are works that were done either by myself or alongside contributors. My original contributions to these works represent a substantial contribution. Any non-original works are appropriately cited to the best of the author's knowledge.

Publications

(*) "*The Complete Decoupling Limit of Ghost-free Massive Gravity*" by Nicholas A. Ondo, Andrew J. Tolley.

Published in **JHEP 1311 (2013) 059**. [ARXIV:1307.4769]

(*) "*Interactions of Charged Spin-2 Fields*" by Claudia de Rham, Andrew Matas, Nicholas A. Ondo, and Andrew J. Tolley.

Published in **Class.Quant.Grav. 32 (2015) no.17, 175008**. [ARXIV:1410.5422]

(*) "*Deconstructing Supergravity, I: Massive Supermultiplets*" by Nicholas A. Ondo, Andrew J. Tolley.

Submitted to *JHEP*. [ARXIV:1612.08752]

(*) "*Deconstructing Supergravity, II: Massive RS Fields*" by Nicholas Ondo, Andrew J. Tolley.

To appear.

Copyright Declaration

Dedication

I dedicate this work to my mother Deborah Ondo, my father John Ondo, and my brother Jeremy Ondo, all of whom have given me endless support and encouragement.

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- My supervisor, Andrew Tolley, my unofficial secondary supervisor Claudia de Rham, and my close friend Andrew Matas for the countless hours of conversation, tutelage, and intellectual engagement that they have given me. I would also like to thank Andrew Matas and Alexander Harrold for looking over an early manuscript of this thesis.

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Chapter 1

Introduction to IR-Modifications of Gravity

1.1 GR as a Quantum Theory

It has been over a century since Einstein and Hilbert first wrote down the field equations for General Relativity (GR) and the Einstein-Hilbert action. Since then, GR has contained many surprises; one prominent area of surprise is its close relationship to Quantum Field Theory (QFT). Ever since Wigner's classification [1], it has been known that all field theories are fundamentally characterised by their mass and their spin. Unexpectedly, many highly non-trivial aspects of GR can be predicted in the QFT framework by systematically answering the question, "What low-energy effective field theory (LEEFT) should describe a massless spin-2 field with local, Lorentz invariant interactions?"

The first published work speculating the connection between GR and massless spin-2 fields was given by Fierz and Pauli [2] in the 1930's. The later twentieth century works by Feynman, Weinberg, Deser, *et al* proved the full correspondence. It was shown that the unique LEEFT of a local, Lorentz-invariant, interacting massless spin-2 field is given precisely by General Relativity expanded around Minkowski spacetime [3–8]. Specifically, it was demonstrated that the Einstein-Hilbert action can be constructed as *the* consistent LEEFT action of a local, Lorentz-invariant theory containing a massless spin-2 mode; there can only be one self-interacting massless spin-2 mode in the spectrum [5, 9, 10]; and that the S-matrix elements for massless spin-2 fields require many non-trivial properties, such as the weak principle of equivalence. This has spurred many interesting developments into the field-theoretic study of GR, with early attempts [11] in both canonical quantisation [12–14] and covariant quantisation [15, 16]; exploring loop corrections (both in pure GR [8, 17–21] and in supergravity e.g. [22, 23] with interesting recent progress on $\mathcal{N} = 8$ supergravity [24]); semi-classical treatments of GR (see [25]) that leads to the information paradox [26]; a full treatment and understanding of GR as a consistent Wilsonian EFT [27, 28]. All of these developments lie outside of the full attempts to quantise, for instance via perturbative quantisation of new, distinct theories (e.g. string theory [29, 30]) or non-perturbative quantisation attempts (e.g. holography, for a review see [31], and other more speculative methods like [32]). Therefore, while a consistent, fully fleshed-out, UV-complete theory of General Relativity remains illusive even after decades of research, one can confidently state that there are many non-trivial things known about quantum gravity –even staying entirely within the framework of QFT! Given the incredible progress towards understanding the relationship between gravity and self-interacting massless spin-2 fields, an obvious, weighty question arises. Since Wigner's classification also alerts us to the possibility of massive spin-2 fields, one may naturally ask, "What happens for the case of *massive* spin-2 fields with local, Lorentz-invariant interactions?" I take this as the first critical reason to study massive theories of gravity: It is fundamentally interesting in its own right to study massive spin-2 fields as a QFT-inspired alteration of General Relativity. Crucially, one may wonder if these theories exhibit similar theorems and severe restrictions, if there are no-go theorems on how they can interact, and so forth, like there are for their massless cousins. Some of this is already well-known; they do have new non-trivial physics, which I broadly review in Chapter 2. Historically, the literature abounds with either no-go theorems for massive spin-2 theories [9, 10, 33–40], the most devastating of which have turned out to have important loopholes or evasions. Indeed, until very recently it was believed that there were insurmountable inconsistencies in massive gravity. For instance, it was incorrectly believed that self-interacting theories of massive spin-2 fields necessarily contained a ghostly scalar ([41–45] gave a counter-example, namely the dRGT theory of ghost-free massive gravity), and that massive gravity necessarily generates acausalities that prevent it from having a consistent UV-completion ([46–48] have demonstrated otherwise). Still, a full understanding is absent and a proof regarding the ultimate consistency of massive spin-2 fields remains unsolved; however, new research in this area leaves hope for the development of a consistent framework for massive spin-2 fields. In this thesis, I develop and explore theories of massive spin-2 fields, working towards a deeper understanding of these questions.

1.2 Cosmological Constant: Observation and Theory

There is a second area where massive gravity has arisen within modern physics, which has to do with the observed cosmic acceleration. Within the context of pure GR, the Cosmological Constant arises as an allowed contribution to the stress-energy that sources gravity. It enters into the theory in a special way, with a coupling,

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} + \kappa^2 \, \Lambda g_{\mu\nu} = 2\kappa^2 T^{\text{matter}}_{\mu\nu} \,, \tag{1.1}$$

with $\mathcal{R}_{\mu\nu}$ being the usual Ricci tensor, my conventions being set by $\kappa = \sqrt{4\pi G} = \frac{1}{M_{\text{Pl}}}$ and the cosmological constant's dimension is defined by $[\Lambda] = E^4$, in natural units where $c = \hbar = 1$. In terms of a variational principle, the Einstein-Hilbert action for GR with a cosmological constant is given by

$$\mathcal{S}_{\text{EH}}[g_{\mu\nu}] = \int \mathrm{d}^4 x \, \frac{1}{2\kappa^2} \sqrt{-g} \left(R - 2\kappa^2 \, \Lambda \right) + \mathcal{L}^{\text{matter}} \,, \tag{1.2}$$

and $\mathcal{R}$ is the usual Ricci scalar. The cosmic acceleration of the universe was famously, and perhaps shockingly, discovered at the end of the 1990's by Riess, *et al*, and Perlmutter, *et al*, [49, 50], which is interpreted as hinting at a type of Dark Energy. From the perspective of purely classical physics, this constant is merely a tunable parameter freely allowed in GR, and one can explain the observed Dark Energy by simply dialling this knob. Once measured, this parameter takes on the observed value, and there is very little else to be said. However, in the context of field theory, the concept of the cosmological constant becomes more problematic. In QFT, the issue of renormalisation arises, wherein the parameters of the QFT change as one flows from a UV theory into an IR theory. The cosmological constant is a relevant operator, and does not appear as though there can be a symmetry protecting its 'smallness', which is the main protection mechanism for the mass scales [51] entering into relevant operators. Without one, relevant operators grow as one flows into the IR. To explicitly see the effect of running, suppose some new physics enters at a scale Λ_{UV}. One can estimate that the cosmological constant will get quantum contributions of the form[1]

$$\langle \Lambda \rangle \sim \int_0^{\Lambda_{UV}} \mathrm{d}^4 k \sim \Lambda_{UV}^4 \tag{1.3}$$

If one takes the scale of new physics to be M_{Pl} and uses the observed value, Λ_{obs}, one then requires an *enormous* tuning between the bare CC and the quantum corrections:

$$\Lambda_{\mathrm{obs}} \approx M_{\mathrm{Pl}}^2 H_0^2 \approx 10^{-120} M_{\mathrm{Pl}}^4 \,. \tag{1.4}$$

This disconnect between the expectation from the running of quantum corrections and the observed value remains an oddity in modern theoretical physics. Sometimes the appearance of fine-tuning really indicates that there are hidden mechanisms behind the small size (or even being zero) of various parameters. For instance, some energy scales in marginal and relevant interactions are protected from quantum corrections via symmetries; these cases are called 't Hooftian technical naturalness [51] and related non-renormalisation theorems (see for example the Galileon non-renormalisation theorem [53]). But in principle, the universe may just fail to be technically natural, with alternative explanations coming from anthropic arguments (like those first discussed in [54]).

1.2.1 New IR Gravitational Physics from Non-Zero Graviton Mass

If one takes a cosmologist's interest in finding potential, non-anthropic explanations for the observed value of the cosmological constant, then the 't Hooftian technical naturalness argument is rather difficult. The absence of a known symmetry, which would need to appear when the cosmological constant is sent to zero, makes a technical naturalness argument difficult; indeed, there also is no known symmetry which makes the cosmological constant zero.[2] I will briefly mention two possibilities. Firstly, perhaps new physics could cause the cosmological constant to couple weakly through gravity through IR-modifications of gravity. I will review this more carefully in the next section. Secondly, one can ask a different, but still intriguing question: Perhaps other far-IR (cosmological scale) physics could give rise

[1] Of course, one should use a more careful, non-anomalous regulator, such as dimensional regularisation. There only the logarithmic contributions are real; however, this estimator gives the log-only contributions up to an $\mathcal{O}(1)$ factor. See [52] for further details on these quantum details.

[2] Outside of two cases: 1.) Supersymmetry, which is broken in Nature, and thus cannot aid us. 2.) The $\mathcal{U}(1)$ of partially massless gravity (see [55]), whose dynamical interactions are unknown (and possibly may not exist) and, in any case, is a massive spin-2 field.

to the same predictions as a Cosmological Constant (e.g. an expanding universe). In other words, perhaps the observed accelerated expansion could come from new IR physics rather than a CC; there were many early ideas for this. To reference but a few [56–59]; for a more complete list please see [55] and references therein. Many of these theories led directly back to theories of massive spin-2 fields. A comment on all of these is that in the limit that $m \to 0$ one regains diffeomorphism invariance, so one might hope that a 't Hooftian naturalness argument could apply directly to the mass term itself, giving a technically natural explanation of the accelerated expansion of the universe. Either or both of these phenomena could be considered a success for explaining major aspects of the cosmological constant problem! Thus, many straightforward attempts to modify gravity in the IR suggestively lead back to theories of massive spin-2 fields. I will take a moment to describe one of the most interesting ideas.

1.2.2 De-Electrification and Degravitation

During the period immediately following the discovery of cosmic acceleration, many different ideas spawned for how one might deal with a cosmological constant in a technically natural manner. One interesting possibility was developed by Dvali, *et al*, called "degravitation" [60, 61]. The quintessential idea is perhaps one should take the enormous bare cosmological constant that one gets from a naïve QFT calculation seriously, and instead ask a different question: "If there is an enormous cosmological constant with a small measured value, is it possible that gravity simply couples weakly to cosmological constants?"

In other words, is there a reason why gravity would couple weakly to constant sources of stress-energy (e.g. a cosmological constant)? At first glance, one might think that it is impossible to do so without appealing to some kind of non-local or similarly questionable physics. Such an intuition would presumably apply to a massless force-carrying field; however, no such restriction exists for massive force-carrying fields! Actually, this is already known to happen in Nature. In a superconductor [62], the photon spontaneously picks up a mass and the net affect is that it acts to screen all constant sheets of charge! This is sometimes called "de-electification" [52]. Given a Proca theory, one has the field equations

$$\begin{aligned} (\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)A_\nu - m^2 A_\nu &= J^\mu \qquad (1.5)\\ \Longrightarrow \partial_\mu((\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)A_\nu - m^2 A_\nu) &= \partial_\mu(J^\mu) \qquad (1.6)\\ \Longrightarrow \partial_\mu A^\mu &= 0\,. \qquad (1.7) \end{aligned}$$

The last line follows since on the LHS, the identity $\Box := \partial_\mu \partial^\mu$ causes the first term to drop out leaving only the non-zero mass term $-m^2 \partial_\mu A^\mu$. The RHS comes from $J^\mu = \text{const}$, then $\partial_\mu J^\mu = 0$.

Now, famously the field equations for a Proca theory become

$$\begin{aligned} \Box A_\mu - m^2 A_\mu &= J_\mu \,, \\ \partial_\mu A^\mu &= 0 \,. \end{aligned} \tag{1.8}$$

Notice, however that one can find a constant solution for this constant source, $A_\mu = \frac{-1}{m} J_\mu$ which satisfies the field equations. However, since these are linear differential equations, one can clearly see that all constant contributions to the stress-energy amount to constant shifts in the 4-vector potential A_μ.

However, since particles only couple to the electromagnetic fields, $F_{\mu\nu} \sim \partial_\mu A_\nu$, all of these constant shifts drop out of the physical forces and similar quantities and thus **the forces imparted on point particles are totally insensitive to constant sources**! This is why it is called "de-electrification". One could imagine similar physics occurring for gravity, where roughly $A_\mu \to h_{\mu\nu}$ and $J_\mu \to T_{\mu\nu}$.

Thus, if one can find a theory of massive gravity, one has good reason to hope that large cosmological sources could be screened by the presence of a non-zero graviton mass, and this would give a technically natural explanation for the smallness of the observed Dark Energy appealing to nothing but an IR modification to General Relativity! I will return to this in Chapter 2. This is one example, although as stated above, there are similar interesting physics related to cosmological constants/new ways to mimic Dark Energy when one gives the graviton a non-zero graviton mass.

1.3 Objectives

Thus, summarising the main issues:

1.) Massless spin-2 fields are well-known to have interesting restrictions on them, coming only from QFT. Massive spin-2 fields are a natural object within QFT, but the status of their interactions lead to interesting, still not fully understood, results. This merits further exploration.

2.) If one is interested in expanding the theoretical understanding of what types of new IR physics may be present, e.g. perhaps in cosmological context (and may not seen in physics at collider scales), then the theory space of massive gravity should be understood. Indeed, many attempts on the cosmology side of things lead one back to the study of massive spin-2 fields.

In this thesis, I will therefore work towards a further understanding of the dRGT theory of massive gravity. Specifically, I will explore the full limits of the theory, called the decoupling limit (Chapter 3); I will attempt to enhance the symmetries of the theory to include charged interactions (Chapter 5), and I will also enhance the theory to include potentially supersymmetric interactions for the theory (Chapter 6). I will also extend the current knowledge of Dimensional Deconstruction to include supersymmetry and broaden the conceptual development of this idea.

1.3.1 Outline

The outline of my thesis is as follows:

Chapter (2): I begin by reviewing the recent history and results of massive gravity. This begins with the free theory of a massive spin-2 field given by Fierz-Pauli theory and the new physics associated to this theory. From there, I give a toy model of an interacting gauge theory, namely Proca-Yang-Mills theory. After addressing the new physics found in massive gauge theories, I move back to discussing generic theories of self-interacting massive spin-2 fields. These theories can be shown to typically contain a ghostly scalar mode in their spectrum. In the final subsection, I introduce the recently discovered dRGT theory of ghost-free, self-interacting massive gravity. Additionally, I review this theory in the Einstein-Cartan formalism and show the existence of a Λ_3 decoupling limit.

Chapter (3): In the third section, I demonstrate my original proof of the full form and the full tower of interactions present of dRGT theory in the decoupling limit. I use novel methods to demonstrate the clean nature and relationship of dRGT massive gravity in metric formulation and in the vielbein variables. Using the Lorentz Stückelberg formalism, I obtain the complete form of the previously unknown helicity-1 interactions, which I prove are exclusively between the helicity-0 and the helicity-1 modes and can be placed into an integral form. Taken together with the decoupling limit of the action, I write down the action for complete decoupling limit of dRGT massive gravity.

Chapter (4.) I review a major discovery of dRGT massive gravity discovered by C. de Rham, *et al*, namely that it may be obtained from a higher dimensional theory of Einstein-Cartan (massless) gravity. I provide my perspective of the Dimensional Deconstruction procedure, a procedure which will be necessary for the following sections.

Chapter (5.) I discuss my original derivation of a no-go theorem regarding self-interacting charged spin-2 fields with a dRGT mass term. I briefly review the uses that such a field would have in modern holographic superconductors and DC conductivity, as well as giving a possible Lagrangian for charged spin-2 meson resonances in nuclear physics. To provide the no-go theorem, a novel technique is developed for checking for the existence of ghosts.

Chapter (6.) After reviewing supersymmetry, I give a discuss the relationship between dRGT massive gravity and Zinoviev theories (supersymmetric Fierz-Pauli theories), and I provide a new Dimensional Deconstruction prescription for obtaining massive supersymmetric theories from higher dimensional massless supersymmetric theories. I show how $\mathcal{N} = 1$ super-Proca and $\mathcal{N} = 1$ Zinoviev theory can be obtained from deconstructing supersymmetric Maxwell and linear supergravity theories, respectively.

Chapter (7.) I discuss my work towards developing a theory of a massive supergravity following the SUSY deconstruction procedure from Chapter 6.

Chapter 2

Introduction to Massive Gravity

2.1 Massive Spin-2 Theories?

As soon as one starts an exploration of theories of non-zero graviton mass, there are immediate structural changes to gravitation. The first kind of structural changes relate to new physics present around the scale of the graviton mass, m. The second kind of structural changes are related to the fact that massive representations have new physical degrees of freedom, which contain a rich set of physics not present in its massless cousin. The third kind relates to the stringent requirements when introducing interactions for massive gravitons, which is one of the major recent breakthroughs in the field. Before moving into the main body of this section, I pause to list some of my conventions. A detailed list of conventions can be found in Appendix A, but as a reminder to the reader, these are some of the salient choices:

1.) My metric convention is $(-,+,+,+)$ for $\mathbb{M}^{1,3}$. I follow standard supergravity conventions, so $\kappa = \sqrt{4\pi G}$ and a "reduced" Planck mass $M_{\mathrm{Pl}}^2 = \frac{1}{\kappa^2}$.

2.) I shall make constant usage of generalised Kronecker delta symbols, $\delta^{\mu_1 \cdots \mu_N}_{\nu_1 \cdots \nu_N}$, such that $N \leq D$. For a detailed discussion of these objects, see the Appendix B. These symbols will rapidly speed up calculations and manifest crucial gauge symmetries/constraint structures.

3.) A brief glossary of acronyms:

   - QFT = "Quantum Field Theory",
   - EFT = "Effective Field Theory",
   - LEEFT = "Low-Energy Effective Field Theory",
   - DL = "Decoupling Limit",

- GR = “General Relativity”,
- LGR = “Linearised General Relativity”,
- mGR = “Massive Gravity”,
- dRGT = “de Rham-Gabadadze-Tolley gravity”,
- YM = “Yang-Mills theory”,
- PYM = “Proca-Yang-Mills theory”
- FP = “Fierz-Pauli”,
- DOF = “Degree of freedom” (e.g. dimension of configuration space)
- PDF = “Physical Degree of Freedom” (polarisation; on-shell, local DOF),
- GKD = “Generalised Kronecker Delta symbol”,
- vDVZ = “van Dam-Veltman-Zakharov”.

This section proceeds as follows:

1.) In the first subsection 2.2, I review the details of a free (non-interacting) massive spin-2 field, called Fierz-Pauli theory. I discuss the new physics associated at the linear description of a massive spin-2 mode. Finally, I review the conclusion that a robust analysis of a fully interacting theory is necessary.

2.) In the second subsection 2.3, I pause from the case of a spin-2 field to discuss the more well-known case of an interacting, massive spin-1 gauge field (i.e. Proca-Yang-Mills), and I review some important features of this theory.

3.) In the third section 2.4, I discuss the naïve attempts to covariantise the Fierz-Pauli theory, and discover the return of a ghostly mode in the self-interacting theory.

4.) In the fourth subsection 2.5, I introduce the dRGT theory of massive gravity, discovered by de Rham, Gabadadze, and Tolley. It adds mass terms to GR that identically and uniquely vanquish the ghostly mode. I show how the theory can be reformulated in Einstein-Cartan formalism, and review the existence of a convergent Λ_3 decoupling limit.

2.2 Massive Gravity without Self-Interactions

2.2.1 Representation-Theoretic Aspects

For simplicity, I start with the linearised (weak gravitational coupling, $\frac{1}{M_{\rm Pl}} = \kappa \to 0$) theory in 4-D. This must be described by a classical field theory in Minkowski space. Before discussing the field-theoretic aspects, I begin with several comments about the group theory that comes to bare. Firstly, following Wigner's classification of the unitary representations of the Poincaré algebra, a massive spin-2 mode sits inside a $j = 2$ representation of the Wigner little group, SO(3) and has a mass $m \neq 0$, coming from the Casimir $P^2 = -m^2$ [1]. The dimension of this representation is 5, therefore the field theory must propagate 5 polarisations or PDF's. If I wish to move to arbitrary dimensions (i.e. D-dim Minkowski), the number of PDF's is given by

$$\text{Number of PDF's } = \frac{1}{2}(D^2 - D - 2)\,, \tag{2.1}$$

with $d = 4$ clearly giving 5 PDF's. Secondly, Wigner's classification of unitary representations of the Poincaré group for massive spin-2 state must be encoded into a local, symmetric 2-tensor field $h_{\mu\nu}$. In other words, the tensor structure is the same, but the PDF's are markedly increased!

These are the basic ingredients, from which I can begin the search for a linear theory of massive gravity on Minkowski.

2.2.2 Fierz-Pauli Theory

Having a definite Lorentz representation, $h_{\mu\nu}$, and a definite number of PDF's, one needs to find a Lagrangian to provide dynamics for a massive spin-2 field $h_{\mu\nu}$. The Lagrangian was originally developed by Fierz and Pauli [2] in 1939, with the eponymous action:

$$\mathcal{S}_{\rm FP} = \int \mathrm{d}^4x \left[-\frac{1}{2} h_\mu{}^\alpha \left(\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\, \partial_\nu \partial^\beta \right) h_\rho{}^\gamma + \frac{1}{2} m^2 h_\mu{}^\alpha \delta^{\mu\nu}_{\alpha\beta} h_\nu{}^\beta \right] . \tag{2.2}$$

An analysis of the equations of motion allows one to see that it propagates only 5 PDF's.

$$\delta^{\mu\nu\rho}_{\alpha\beta\gamma} \partial_\nu \partial^\beta h_\rho{}^\gamma + m^2 \delta^{\mu\nu}_{\alpha\beta} h_\nu{}^\beta = 0\,, \tag{2.3}$$

which when contracted into a derivative and traced over, respectively, yields the constraints

$$\begin{aligned}
\partial^\alpha \left(\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\partial_\nu\partial^\beta h_\rho{}^\gamma + m^2\delta^{\mu\nu}_{\alpha\beta}h_\nu{}^\beta\right) &= 0 \\
\Longrightarrow \partial^\mu h_\mu{}^\alpha = 0 & \qquad \text{(4 constraints)}
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
\delta_\mu{}^\alpha \left(\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\partial_\nu\partial^\beta h_\rho{}^\gamma + m^2\delta^{\mu\nu}_{\alpha\beta}h_\nu{}^\beta\right) &= 0 \\
\Longrightarrow h_\mu{}^\mu = 0 & \qquad \text{(1 constraint)}\,,
\end{aligned} \tag{2.5}$$

where I have algebraically simplified the final expressions of both constraints using the intermediate expressions. After subtracting these constraints from the original 10 symmetric tensor components, one is left with 5 unfixed, propagating degrees of freedom. Crucially, notice the importance of the relative coefficients in the mass terms, cf. $(h_{\mu\nu}h^{\mu\nu} - (h_\mu{}^\mu)^2)$ when deriving the trace constraint. There is no requirement from Poincaré invariance that the two allowed mass terms have this relative weighting. It turns out that the theory will have 6 PDF's for alternative weightings, and the 6th degree of freedom is unstable and violates the positivity of the Hamiltonian. Such degrees of freedom are called “ghosts” and will spoil unitarity of underlying quantum theories unless their masses are pushed above the cutoff of the theory. I return to this in subsection 2.2.6, for now one can rule the theory out just on the grounds of clashing with the desired PDF's. One can now do the usual calculations, such as computing the propagator, deriving at the profile of potentials sourced by a point mass, and so forth. Immediately, there is an odd puzzle with the massive spin-2 field. Naïvely, one expects for there to be continuity in parameters; for instance, as the gravitational coupling $\kappa = \sqrt{4\pi G} = \frac{1}{M_{\rm Pl}}$ is dialled down, one expects GR to ‘flow’ into linear GR or if one takes YM and dials down g, one gets back Maxwell theories. This begs an interesting question: If one dials the mass down in Fierz-Pauli theory, $m \to 0$, then naïvely one expects to recover the predictions of linear GR. For instance, if one computes the gravitational potential generated by FP theory by a point source M, one obtains

$$V_{\rm FP}(r) = \frac{4}{3}G\frac{M_1}{r^2}e^{-m\,r}\,. \tag{2.6}$$

The long-distance Yukawa suppression is not surprising, given that it is a massive particle, and represents the introduction of the mass into the theory. But if I compute the gravitational potential created by that field, the answer for LGR immediately differs from the FP potential when $m \to 0$:

$$\begin{aligned} V_{\text{LGR}}(r) &= G\frac{M}{r^2} && (2.7) \\ V_{\text{FP}}(r) &= \frac{4}{3}G\frac{M}{r^2}\,. && (2.8) \end{aligned}$$

Also, there are additional discrepancies in predictions, such as the lensing angles, so this cannot be fixed by doing a classical renormalisation of G. Stated more devastatingly, the PPN parameters of FP at $m = 0$ and LGR are different; thus, they are fundamentally two totally different Poincaré invariant theories. Therefore, it appears as though the scaling limit is discontinuous at $m = 0$. This discontinuity is referred to as the van Dam-Veltman-Zakharov discontinuity (i.e. the vDVZ discontinuity) after its discoverers [33, 34]. It is worth noting how this differs from the lower spin cases. For spin-0 and spin-$\frac{1}{2}$, the mass trivially may be scaled to zero in a continuous fashion. In the case of the massive spin-1 field, there is a new polarisation, but the electric potential (and related electromagnetic physics) created by a point charge in the massless field agrees with the massive theory's solution in the limit that $m \to 0$. This bizarre new phenomena for massive spin-2 fields seems like an immediate killer for any massive theory of gravity, since the $m = 0$ limit should dominate in the weak-coupling, small distance limits (i.e. $r \ll \frac{1}{m}$, while $r \gg \kappa$) where Newtonian gravity is known to be the dominant description of gravity. The physics of massive spin-2 fields, however, lies precisely in how one can address the issues outlined in this section, namely: consistently maintaining 5 PDF's while keeping a consistent high momentum limit (small distance) that flows to an ordinary massless gravity (LGR in the weak gravity limit, but ideally GR when graviton self-interactions are present). In order to accomplish this, I will need carefully explore these issues within the Stückelberg formalism. But first a short interlude.

2.2.3 An Aside on Degravitation

For the interested cosmologist, I briefly prove that this linear theory does indeed degravitate, as advertised. In other words, the massive spin-2 field will fail to respond to a uniform, static source. Here, a cosmological constant for the free theory is just a constant source, for example the leading order contribution goes as

$$T_{\mu\nu} = \Lambda\eta_{\mu\nu}\,, \tag{2.9}$$

where Λ is a parameter containing the scale of the cosmological constant. Given this source, one can vary the action to arrive equations of motion of the form

$$\delta^{\alpha\beta\gamma}_{\mu\nu\rho}\partial_\nu\partial^\beta h^\gamma_\rho + m^2\delta^{\alpha\gamma}_{\mu\nu}h_\rho{}^\gamma = -\Lambda\eta^\alpha_\mu\,. \tag{2.10}$$

I take as an ansatz that

$$h_\rho{}^\gamma = A\delta^\gamma_\rho\,. \tag{2.11}$$

Plugging in this solution, one generates a constraint on A, so that,

$$3\cdot 4A\eta_{\rho\gamma} = -\Lambda\eta_{\rho\gamma} \tag{2.12}$$

$$\implies A = -\frac{1}{12}\Lambda\,. \tag{2.13}$$

This is simply rescaling the $\eta_{\mu\nu}$ Minkowski background (which would ostensibly lead to something like $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = (1 - \frac{\Lambda}{12} + \cdots)\eta_{\mu\nu}$ in a covariant, self-interacting theory). In other words, even though one is adding a cosmological constant as a source, massive gravity simply returns a Minkowski spacetime as the solution! Note that this shift does not affect physical observables like forces imparted between particles. So long as matter are coupled gauge-invariantly to $h_{\mu\nu}$, they are only sourced by gauge-invariant $\sim \partial h$ quantities, which are invariant under constant shifts! Note that this is wildly different than the case for even linear GR, and it is entirely due to the new IR physics: The mass acts as a high-pass filter that screens constant sources, e.g. a cosmological constant!

2.2.4 Proca Theory in Stückelberg Formalism

Returning to the small distance physics of the massive spin-2 field, I now introduce a powerful formalism. It is quite convenient to explore both of these issues within the Stückelberg formalism, which I now define for the massive spin-1 case. It is well known that Maxwell, Yang-Mills, and General Relativity have gauge symmetries (the $\mathcal{U}(1)$ group; a compact, semi-simple Lie group G; and diffeomorphism group, respectively). However, it is often stated in introductory QFT textbooks (e.g. [63]) that theories with massive gauge particles break the gauge symmetries. This is an inaccurate statement, although it is a close proxy for the truth. A more accurate statement is that these massive theories are sitting in a very useful gauge-fixed form. The Stückelberg procedure is a method for deriving the overarching gauge theory of the massive gauge field. It begins by taking the original gauge symmetry for the massless theory; for the spin-1 case, one has

$$A_\mu \to A_\mu - \partial_\mu \xi \,, \tag{2.14}$$

and noting that while the kinetic term obeys gauge symmetry, the mass term manifestly violates it

$$\mathcal{S}_{\text{Proca}}[A] \;=\; \int \mathrm{d}^4x \, -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu \,. \tag{2.15}$$

Therefore, at the Lagrangian level, one should conclude that the new degrees of freedom must be associated with mass term. This can be explicitly seen by manually installing the gauge symmetry at the price of introducing a new field φ. If one substitutes

$$A_\mu \to A_\mu - \partial_\mu \varphi \tag{2.16}$$

into the action, it yields a new action

$$\begin{aligned} \mathcal{S}_{\text{Stu}}[A,\varphi] \;&=\; \int \mathrm{d}^4x \, -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 (A_\mu - \partial_\mu\varphi)(A^\mu - \partial^\mu\varphi) & (2.17) \\ &=\; \int \mathrm{d}^4x \, -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 \left(\partial_\mu\varphi\partial^\mu\varphi + 2A_\mu\partial^\mu\varphi - \frac{1}{2}A_\mu A^\mu\right) . & (2.18) \end{aligned}$$

It is useful to canonically normalise the scalar mode; this is achieved via a redefinition

$$\varphi \to \frac{1}{m}\varphi \tag{2.19}$$

after which the action takes the canonically-normalised form

$$\mathcal{S}_{\text{Norm-Stu}}[A,\varphi] \;=\; \int \mathrm{d}^4x \, -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + m\, A_\mu\partial^\mu\varphi - \frac{1}{2}m^2\, A_\mu A^\mu \,, \tag{2.20}$$

which has the $\mathcal{U}(1)$ gauge symmetry

$$\begin{aligned} A_\mu \;&\to\; A_\mu - \partial_\mu\xi & (2.21) \\ \varphi \;&\to\; \varphi - m\xi \,. & (2.22) \end{aligned}$$

This theory is physically equivalent to the first, and can be seen at two different levels. Firstly in terms of DOF, one has added a precise combination of one new DOF alongside one new gauge symmetry. This simultaneously adds and subtracts a DOF, thus keeping to the original number of PDF's. Secondly, one action directly comes from the other. One is clearly allowed to choose the "unitary gauge", i.e. $\varphi = 0$ in the Stückelberg action. This choice manifestly returns one to the original Proca theory, thus proving the physical equivalence of these actions. So I am really just expressing the same theory in two different forms. What is the upshot of re-writing the theory in this manner? This new action boasts many new useful features; the most important of which is that it manifests the degrees of freedom. Now one is free to smoothly vary m and keep the 3 PDF's of Proca manifest! Setting $m = 0$ into the Stückelberg Lagrangian simply yields the free helicity-(± 1) modes and helicity-0 mode Lagrangian description. Additionally, this limit of the theory happens to be the descriptor of Proca at high momentum, when the contribution of the mass is negligible. This technique will be very helpful for providing simpler effective descriptions of interacting massive gauge theories at high energies, $E \gg m$. Although not relevant for my thesis, it is noteworthy that this is the most natural connection to Higgs mechanisms, since this is Stückelberg mode is clearly the "eaten" degree of freedom from the Higgs sector.

2.2.5 FP Theory in Stückelberg Formalism

I will repeat this analysis for Fierz-Pauli theory. To do so, I must first restore linearised diffeomorphisms within the action by adding a vector mode B:

$$h_\mu{}^\alpha \to h_\mu{}^\alpha - \frac{1}{2m}\left(\partial_\mu B^\alpha + \partial^\alpha B_\mu\right) , \tag{2.23}$$

where I have already put in the obviously required power-counting factor of $\frac{1}{m}$. This generates a new action of the form

$$\begin{aligned}\mathcal{S}_{\text{FP-Stu}} &= \int \mathrm{d}^4x - \frac{1}{2}h_\mu{}^\alpha \left(\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\,\partial_\nu\partial^\beta\right) h_\rho{}^\gamma \qquad (2.24)\\ &\quad + \frac{1}{2}m^2\left(h_\mu{}^\alpha - \frac{1}{2m}\left(\partial_\mu B^\alpha + \partial^\alpha B_\mu\right)\right)\delta^{\mu\nu}_{\alpha\beta}\left(h_\nu{}^\beta - \frac{1}{2m}\left(\partial_\nu B^\beta + \partial^\beta B_\nu\right)\right)\\ &= \int \mathrm{d}^4x - \frac{1}{2}h_\mu{}^\alpha \left(\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\,\partial_\nu\partial^\beta\right) h_\rho{}^\gamma - \frac{1}{2}\partial_\mu B_\nu \delta^{\mu\nu}_{\alpha\beta}\partial^\alpha B^\beta\\ &\quad \frac{1}{2}m^2\delta^{\mu\nu}_{\alpha\beta}\Big(h_\mu{}^\alpha h_\nu{}^\beta - \frac{2}{m}\partial_\mu B^\alpha h_\nu{}^\beta\Big) , \qquad (2.25)\end{aligned}$$

where in the last line I have integrated by parts and exchanged indices. This theory possesses a linearised diffeomorphism gauge symmetry

$$
\begin{aligned}
h_\mu{}^\alpha &\to h_\mu{}^\alpha - \frac{1}{2}\left(\partial_\mu \xi^\alpha + \partial^\alpha \xi_\mu\right) \\
B_\mu &\to B_\mu - m\xi_\mu \,.
\end{aligned} \tag{2.26}
$$

Now, however, the new vector that I have introduced fails to have an associated $\mathcal{U}(1)$ symmetry. To manifest all of the gauge symmetries of the theory, one may then add in a $\mathcal{U}(1)$-restoring Stückelberg field, this time a scalar mode π:

$$
B_\mu \to B_\mu - \frac{1}{m}\partial_\mu \pi \,. \tag{2.27}
$$

This causes the action to take the form

$$
\begin{aligned}
\mathcal{S}_{\text{FP-Stu}} &= \int \mathrm{d}^4x - \frac{1}{2} h_\mu{}^\alpha \left(\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\,\partial_\nu \partial^\beta\right) h_\rho{}^\gamma - \partial_\mu B_\nu \delta^{\mu\nu}_{\alpha\beta} \partial^\alpha B^\beta \\
&\quad + \frac{1}{2} m^2 \delta^{\mu\nu}_{\alpha\beta} \Big(h_\mu{}^\alpha h_\nu{}^\beta - \frac{2}{m}\partial_\mu B^\alpha h_\nu{}^\beta + \frac{2}{m^2}\partial_\mu \partial^\alpha \pi h_\nu{}^\beta \Big) \,.
\end{aligned} \tag{2.28}
$$

It is now more difficult to canonically normalising the modes. The scalar mode clearly fails to have a canonical kinetic term, which can only be repaired by a diagonalising transformation; since the vector mode cannot give it one, one must guess something of the form $h_{\mu\nu} \to h_{\mu\nu} + \pi\eta_{\mu\nu}$. One may then plug in arbitrary coefficients rescaling the fields, and the fix them by diagonalising the interactions and canonically normalising. One will find that the field redefinitions

$$
\begin{aligned}
& h_\mu{}^\alpha \to h_\mu{}^\alpha - \frac{1}{\sqrt{6}}\pi \delta^\alpha_\mu \,, \\
& B_\mu \to \frac{1}{\sqrt{2}} B_\mu \,, \\
& \pi \to \sqrt{\frac{2}{3}}\pi \,,
\end{aligned} \tag{2.29}
$$

will diagonalise and canonically normalise the action into the form

$$
\begin{aligned}
\mathcal{S}_{\text{FP-Stu}}[h,B,\pi] &= \int \mathrm{d}^4x \, - \frac{1}{2} h_\mu{}^\alpha \delta^{\mu\nu\rho}_{\alpha\beta\gamma} \partial_\nu \partial^\beta h_\rho{}^\gamma - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} - \frac{1}{2}\partial_\mu \pi \partial^\mu \pi \\
&\quad + m \left[-\sqrt{2}\delta^{\mu\nu}_{\alpha\beta} h_\mu{}^\alpha \partial_\nu B^\beta + \sqrt{3}\pi \partial_\mu B^\mu \right] \\
&\quad + m^2 \left[\frac{1}{2} h_\mu{}^\alpha \delta^{\mu\nu}_{\alpha\beta} h_\nu{}^\beta + \left(\pi^2 - \sqrt{\frac{3}{2}}\pi h_\mu{}^\mu \right) \right] \,,
\end{aligned} \tag{2.30}
$$

with $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. This action now possesses the linearised diffeomorphism gauge symmetries, which are given by the infinitesimal transformations

$$\begin{aligned}\delta h_{\mu\nu} &= \partial_{(\mu}\xi_{\nu)} \\ \delta B_\mu &= m\sqrt{2}\xi_\mu\,, \\ \delta\pi &= 0\,. \end{aligned} \tag{2.31}$$

and a $\mathcal{U}(1)$ gauge symmetry with infinitesimal actions[1]

$$\begin{aligned}\delta h_{\mu\nu} &= \frac{m}{2}\eta_{\mu\nu}\xi\,, \\ \delta B_\mu &= \partial_\mu\xi\,, \\ \delta\pi &= m\sqrt{\frac{3}{2}}\xi\,. \end{aligned} \tag{2.32}$$

To conclude this derivation of the Stückelberg formulation of FP, I make a few cursory comments. First, one can now see that in the limit $m \to 0$, one manifestly obtains the action for a free massless spin-2 mode (2 PDF's) with helicity-(± 2), a free massless spin-1 mode (2 PDF's) with helicity-(± 1), and single free massless spin-0 (1 PDF) mode; thus, the action manifests the correct degrees of freedom (5 PDF's). Of course, the $m \to 0$ limit can also be thought of as a high momentum/energy limit. This limit of massive theories is called the "decoupling limit", where ones scales the theory to zoom onto the dominant classical contribution of the theory when probing energies well above the mass, $E \gg m$. I will return to the issue of decoupling limits many times, as they are helpful simplifications of massive gauge theories.

2.2.6 Fierz-Pauli Tuning and Ghostly Modes

It was claimed in subsection 2.2.2 that a deviation from the Fierz-Pauli tuning leads to a sixth PDF that is a ghost. I pause now to provide an elegant proof of this claim employing the Stückelberg formalism. This is doubly useful because it shows one how effective the Stückelberg formalism is at diagnosing the existence of PDF's and it will efficiently show the true origin of this PDF as being necessarily ghostly. To begin, I start by writing down the most general mixing of mass terms. One may

[1]Notice that due to the diagonalisation, the tensor mode also transforms.

generalise the FP action for a massive linear gravity theory to the following form

$$\mathcal{S}_{\text{mLGR}} = \int \mathrm{d}^4x \left(\frac{1}{2} h_\mu{}^\alpha \delta^{\mu\nu\rho}_{\alpha\beta\gamma} \partial_\rho \partial^\gamma h_\nu{}^\beta - \frac{1}{2} m^2 h_\mu{}^\alpha \left[\delta^{\mu\nu}_{\alpha\beta} + a \delta^\mu_\beta \delta^\nu_\alpha \right] h_\nu{}^\beta \right) , \tag{2.33}$$

with the special case of FP only when $a = 0$; however, by carefully tuning a and m one can get WLOG any linear combination of $(h_{\mu\nu})^2$ and $(h_\mu{}^\mu)^2$. I now apply the usual Stückelberg procedure by adding in a canonically-normalised vector Stückelberg mode, $h_{\mu\nu} \to h_{\mu\nu} - \frac{1}{2\sqrt{2}m}(\partial_\mu B_\nu + \partial_\nu B_\mu)$. This only affects the mass term since the kinetic term is gauge invariant. If one singles out the mass term, one sees it generates other terms of the form:

$$\begin{aligned} \mathcal{S}_{\text{mass term}} \;\; = & \int \mathrm{d}^4x \; -\frac{1}{2} m^2 h_\mu{}^\alpha \left[\delta^{\mu\nu}_{\alpha\beta} + a \delta^\mu_\beta \delta^\nu_\alpha \right] h_\nu{}^\beta \\ & + \frac{1}{16} \left(\partial_\mu B^\alpha + \partial^\alpha B_\mu \right) \left[\delta^{\mu\nu}_{\alpha\beta} + a \delta^\mu_\beta \delta^\nu_\alpha \right] \left(\partial_\nu B^\beta + \partial^\beta B_\nu \right) \\ & + \frac{1}{2\sqrt{2}} m h_\mu{}^\alpha \left[\delta^{\mu\nu}_{\alpha\beta} + a \delta^\mu_\beta \delta^\nu_\alpha \right] \left(\partial_\nu B^\beta + \partial^\beta B_\nu \right) . \end{aligned} \tag{2.34}$$

Now, I zoom onto the pure kinetic terms for B_μ, i.e. all terms of the form $B\partial^2 B$. After performing some integration by parts, one finds that absent the crucial anti-symmetry provided by the FP tuning, one instead ends up with kinetic terms

$$\begin{aligned} \mathcal{S}_{\text{B-kinetic}} \;\; & = \int \mathrm{d}^4x \; \frac{1}{16} \left(2\partial_\mu B^\alpha \right) \left[\delta^{\mu\nu}_{\alpha\beta} + a \delta^\mu_\beta \delta^\nu_\alpha \right] \left(\partial_\nu B^\beta + \partial^\beta B_\nu \right) \\ & = \int \mathrm{d}^4x \; -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{1}{2} a \left(\partial^\mu B_\mu \right)^2 , \end{aligned} \tag{2.35}$$

with the usual $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. Notice now that the action for the helicity-1 mode manifestly violates $U(1)$ invariance, and moreover explicitly provides a kinetic term for the B_0 mode. The appearance of a kinetic term for the time-component signals the existence of a ghost and unitarity violation. One can see this in an alternate way by finishing the Stückelberg procedure. If the Stückelberg procedure is applied for the $\mathcal{U}(1)$ symmetry, i.e. $B_\mu \to B_\mu - \frac{1}{m}\partial_\mu \pi$, one ends up with

$$\mathcal{S}_{\text{B-kinetic}} = \int \mathrm{d}^4x \; -\frac{1}{2} \left(G_{\mu\nu} \right)^2 + a \frac{1}{2} \left(\partial^\mu B_\mu - \frac{1}{m} \Box \pi \right)^2 \tag{2.36}$$

$$= \int \mathrm{d}^4x \; -\frac{1}{4} (G_{\mu\nu})^2 + a \frac{1}{2} (\partial^\mu B_\mu)^2 + \frac{a}{2m^2} \left(\Box \pi \right)^2 + \cdots . \tag{2.37}$$

In other words, the theory has a scalar with a kinetic term of the form $\pi \partial^4 \pi$. Due to the Ostragradsky theorem [64], this means it propagates two scalars, one with a valid kinetic interaction but the other with a kinetic term of the wrong sign (i.e. a ghost).

These theories are classically unstable because, once the ghost is propagating, they cause the Hamiltonian to unbounded from below owing to their negative kinetic energy (which leads to unitarity violations upon quantising the theory). Therefore, within the linear theory, it is easy to see that this mode is identically killed if and only if $a = 0$. To prove this without the use of Stückelberg modes is actually quite challenging, and typically requires a long, careful Dirac analysis of the Hamiltonian density associated to this theory [65]. As an added bonus, if one wishes to churn through the analysis of the previous two sections (checking what mass is created via the diagonalisation $h_{\mu\nu} \to h_{\mu\nu} + \frac{1}{\sqrt{6}}\pi\eta_{\mu\nu}$ field redefinition), one can derive the mass for the ghostly mode which acts as a cut-off if interpreted as an EFT (see, e.g. [55]). This is one of areas where the Stückelberg formalism proves very powerful.

2.2.7 Origin of the vDVZ Discontinuity

Suppose one introduces a source term to Fierz-Pauli theory; at the level of the action, one clearly has

$$\mathcal{S}_{\rm src} = \int \mathrm{d}^4x \; \kappa \, h_{\mu\nu} T^{\mu\nu} \, . \tag{2.38}$$

Notice that with the diagonalising transformations (2.29) in the Stückelberg formalism, the source terms get modified as

$$\begin{aligned} \mathcal{S}_{\rm src} &= \int \mathrm{d}^4x \; \kappa \, h_{\mu\nu} T^{\mu\nu} & (2.39)\\ &\to \int \mathrm{d}^4x \; \kappa \left(h_{\mu\nu} T^{\mu\nu} + \frac{1}{\sqrt{6}} \pi T_\mu{}^\mu \right) . & (2.40) \end{aligned}$$

Note that the derivative terms $\partial_\mu B_\nu$ and $\partial_\mu\partial_\nu\pi$ drop out due to conservation of $T^{\mu\nu}$, but crucially there is still a helicity-0 mode coupling to the trace of the stress-energy! The origin of the vDVZ discontinuity is now trivial to see. Firstly, in order to take the straightforward $m \to 0$ limit, one needs to be in the Stückelberg formalism, which smoothly interpolates the PDF's. Secondly, in particle language, one can see that an additional force is added between sources beyond the ones generated by the helicity-2 modes; there is an additional contribution to the gravitational potential from exchanging virtual helicity-0 modes. Alternatively, at the level of the action, FP does not turn into LGR in the $m \to 0$ limit, one can see that it turns into $\mathcal{S}_{\rm LGR}$ sourced by stress-energy, a free Maxwell theory $\mathcal{S}_{\rm Maxwell}$ with no source, and finally a helicity-0 mode $\mathcal{S}_{\rm KG}$ sourced by the trace of the stress-energy.[2] If one wishes to

[2]In other words, a (linearised) Brans-Dicke theory of gravity [66] (plus a decoupled vector). This has all of the obvious 5th forces implications, only the scalar-matter coupling cannot be dialled

restore purely LGR (plus a totally decoupled vector and scalar), one would have to remove the T term sourcing π. Unfortunately it is clear that, structurally, FP requires the presence of this term. Shortly after the 1976 papers on the vDVZ discontinuity, Vainshtein sketched out an important oversight of this analysis and a related possible resolution to this issue [67].

2.2.8 Self-Interactions and Vainshtein's Conjecture

Shortly after the vDVZ papers, it was pointed out that the vDVZ analysis radically changes if one makes a very innocent assumption: Suppose that the spin-2 field has self-interactions. More than being an innocent assumption, if one wants this to be a gravitational theory then one necessarily will require the existence of the self-interactions present in GR. Although the original Vainshtein analysis [67] and further clarifications answered this question by guessing covariantisations of the FP action and then analysed the solutions, I will give a simpler, more general argument. Here, I will purely analyse the power-counting analysis of what happens when self-interactions are present within massive gravity. This means that I will neglect the index structure. If one desires kinetic interactions like those present in GR, then there will be a tower of interactions like

$$\mathcal{S}_{\text{kinetic}} = \int \mathrm{d}^4x\, \partial^2 h^2 + (\kappa h)\partial^2 h^2 + (\kappa h)^2 \partial^2 h^2 + \cdots + (\kappa h)^n \partial^2 h^2 + \cdots . \tag{2.41}$$

Any covariantisation, naïve or otherwise, of the Fierz-Pauli action will generically have an infinite tower of interactions of the type[3]

$$\mathcal{S}_{\text{mass}} = \int \mathrm{d}^4x\, m^2 h^2 + (\kappa h) m^2 h^2 + (\kappa h)^2 m^2 h^2 + \cdots + (\kappa h)^n m^2 h^2 + \cdots . \tag{2.42}$$

Next, I remind the reader that the Stückelberg formalism relies on following the gauge invariance of the massless theory, which for GR is diffeomorphism invariance. This complicates the introduction of the Stückelberg fields, but at leading order, they will enter like the linearised diffeomorphisms in equation (2.23). So the Stückelberg procedure enters into the theory as

$$h \to \tilde{h} = h + \frac{1}{m}\partial B + \frac{1}{m^2}\partial^2 \pi + \mathcal{O}\left(\frac{\kappa}{m^n}\right) . \tag{2.43}$$

to a small value.

[3]For instance, expanding the requisite $\sqrt{-g} = \exp\left(\frac{1}{2}\mathrm{Tr}[\ln(I + \kappa h)]\right)$ yields such an infinite series.

Expanding the action to cubic order with the above Stückelberg procedure, one has the following mass term

$$\begin{aligned}\mathcal{S}_{\text{mass}} &= \int \mathrm{d}^4x\, m^2 \left(h + \frac{1}{m}\partial B + \frac{1}{m^2}\partial^2\pi + \mathcal{O}\left(\frac{\kappa}{m^n}\right)\right)^2 \\ &\quad + \kappa m^2 \left(h + \frac{1}{m}\partial B + \frac{1}{m^2}\partial^2\pi + \mathcal{O}\left(\frac{\kappa}{m^n}\right)\right) \\ &\quad \times \left(h + \frac{1}{m}\partial B + \frac{1}{m^2}\partial^2\pi + \mathcal{O}\left(\frac{\kappa}{m^n}\right)\right)^2 . \end{aligned} \tag{2.44}$$

From which, assuming the index structure does not make this identically zero, the generic cubic self-interactions for π can read off as

$$\begin{aligned}\mathcal{S}_{\text{mass}} &\supset \int \mathrm{d}^4x\, \kappa \left(\frac{1}{m^2}\partial^2\pi\right) m^2 \left(\frac{1}{m^2}\partial^2\pi\right)^2 . \\ &= \int \mathrm{d}^4x \left(\frac{1}{M_{\text{Pl}} m^4}\right) (\partial^2\pi)^3 . \end{aligned} \tag{2.45}$$

Therefore, typically one can see that π will enter in with derivative self-interactions suppressed by powers of the graviton mass m. **This is crucial, because if one tries to take the limit $m \to 0$, then π becomes strongly self-interacting and the dominant contributions for the scalar do not enter into the theory from the Fierz-Pauli terms.** Even worse, when one sends $m \to 0$, the scalar becomes infinitely strongly coupled; this will wash away the entire Fierz-Pauli contribution to the solution, including the predictions of the 5th force! The Vainshtein conjecture states that the complete theory of self-interacting massive gravity will allow one to make sense of these interactions, and non-linearly the theory will exhibit a smooth limit, i.e.

$$\lim_{m\to 0} [\mathcal{S}_{\text{mGR}}] = \mathcal{S}_{\text{GR}} + \mathcal{S}_{\text{V+S}} \tag{2.46}$$

where the action splits into GR plus a decoupled sector containing a vector/scalar theory. This was first conjectured in [67] and further expounded upon in [68–71].[4] Before moving to the task of constructing self-interactions for spin-2 fields, I pause to treat the case of a simpler massive, self-interacting gauge theory: Proca-Yang-Mills. It will illustrate several crucial features new to massive gauge theories.

[4]There is at least one counter example to Vainshtein's claim, called the 'minimal model' of dRGT massive gravity [72]; however, the purpose of the Vainshtein argument is to establish what is *typical* of a self-interacting massive gravity theory.

2.3 Self-Interacting Massive Gauge Theories

2.3.1 Implications of Nonabelian Gauge Redundancy

Although at first glance it may seem like a simple task to add in a set of desired interactions (in addition to a mass term) to a physical theory, it is well known that for gauge field theories this naïve assumption when born out is remarkably wrong. For massive scalars, this story is largely as simple as one would expect; however, for interacting bosons with a spin greater than zero, they also have nonabelian gauge symmetries. For spin-1 theories, they have YM interactions with, e.g., an SU(N) gauge group; for spin-2 theories, they have GR interactions and have a full, nonabelian diffeomorphism gauge symmetry (and also a local Lorentz group in Einstein-Cartan formalism). It is precisely this gauge structure plus the addition of a mass term that creates issues within the theory. To help unravel this physics, I will go through the toy example for the spin-1 case.

2.3.2 Proca-Yang-Mills

To create a self-interacting theory of a massive, spin-1 field, I start with 4-dim SU(N) Yang-Mills theory with a mass term. This called the Proca-Yang-Mills theory, henceforth denoted by "PYM". Firstly, the YM conventions are given by

$$a = 1, 2, \cdots, N^2 - 1 \tag{2.47}$$

$$A = A_\mu{}^a \mathrm{d}x^\mu T^a \tag{2.48}$$

$$[T^a, T^b] = i f^{abc} T^c \tag{2.49}$$

$$\mathrm{Tr}(T^a T^b) = \delta^{ab} \tag{2.50}$$

$$\mathcal{D}\Phi = \mathrm{d}\Phi - ig[A, \Phi] \tag{2.51}$$

$$[\mathcal{D}, \mathcal{D}]\Phi = -ig[F, \Phi]\,, \tag{2.52}$$

with some other definitions and useful derived relations given by

$$
\begin{aligned}
[T^a]^{bc} &= -if^{abc} && (2.53)\\
f^{abc} &= -i\left(\mathrm{Tr}\left(T^a[T^b,\,T^c]\right)\right) && (2.54)\\
F &= dA + \frac{1}{2}(-ig)[A,A] && (2.55)\\
&= (F_{\mu\nu}{}^a)\left(\frac{1}{2}\mathrm{d}x^\mu \mathrm{d}x^\nu\right)T^a \\
&= \left(\partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + gf^{abc}A_\mu{}^b A_\nu{}^c\right)\left(\frac{1}{2}\mathrm{d}x^\mu \mathrm{d}x^\nu\right)T^a && (2.56)\\
\mathcal{D}\Phi &= \left(\partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + gf^{abc}A_\mu{}^b\Phi^c\right)\mathrm{d}x^\mu T^a\,, && (2.57)
\end{aligned}
$$

where Φ is any adjoint field of $\mathrm{SU}(N)$ in the rep basis, $\Phi = \Phi^a T^a$, and wedge products are always implied, so $\mathrm{d}x^\nu \mathrm{d}x^\nu := \mathrm{d}x^\nu \wedge \mathrm{d}x^\nu$. Secondly, if I add the Proca mass term, I will obtain an action of the form

$$
\mathcal{S}_{\mathrm{PYM}} = \int \mathrm{d}^4x\,\mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu\right)\,. \tag{2.58}
$$

This mass term breaks gauge invariance and leads to a theory with $3\times(N^2-1)$ PDF's, as one would predict for a theory with (N^2-1) massive spin-1 fields. Naïvely, one would predict that this theory is power-counting renormalisable, since the YM interactions are renormalisable and $m^2(A_\mu)^2$ appears as though it is a relevant operator. But famously the Proca mass term renders the theory non-renormalisable; the problem stems from gauge symmetry and the fact that this action does not manifest the PDF's. In an important sense, this action should really be thought of as coming from a larger gauge theory, and (2.58) is a specific gauge-fixed version of that action. While it is often said that gauge symmetry is a sham and all gauge-fixed actions are equivalent, this statement ought to be handled with care, which I will now show. Why is the PYM action non-renormalisable? Crucially, when it is written in the form (2.58) it appears to have discontinuous PDF's if one applies the naïve power-counting rules. But this invalidates the conditions necessary for power-counting, as one cannot decrease the PDF's when one zooms onto a theory. To reiterate this, in 4-D, the gluon field obeys a scaling relation $[A_\mu] = E$. So at low energies, the theory has $3\times(N^2-1)$ PDF's of PYM. But if one (classically) flows the action to high energies, where $\Lambda \gg m$, then the relevant action would then be given by ordinary YM, with $2\times(N^2-1)$ PDF's. In order to be able to manifest all PDF's of the massive gauge theory at all scales, one must reintroduce the full $\mathrm{SU}(N)$ gauge redundancies –only then will it be safe to take $m\to 0$ limits of the theory! I now proceed to describe how to add Stückelberg fields to restore the nonabelian gauge symmetries. Once obtained, one immediately discovers that the theory contains an infinite tower of non-renormalisable interactions.

2.3.3 PYM in Stückelberg Formalism

I now turn to the task of restoring the SU(N) of PYM [73], like I restored the $\mathcal{U}(1)$ gauge symmetry for ordinary Proca in subsection 2.2.4. Firstly, I note that the full gauge transformation for YM is

$$\begin{aligned} A \to A' &= GAG^{-1} - \frac{i}{g} dGG^{-1} && (2.59)\\ \implies \delta A &= D\theta\,, && (2.60) \end{aligned}$$

for $G = \exp(ig\theta)$ (where the SU(N) gauge parameter is required to obey $\theta = \theta^\dagger$). The natural idea then is to place a Stückelberg field, π^a, into the action with the form

$$A_\mu \to \mathcal{U}A_\mu\mathcal{U}^{-1} - \frac{i}{g}\partial_\mu\mathcal{U}\mathcal{U}^{-1} \tag{2.61}$$

where $\mathcal{U} = \exp(ig\frac{\pi}{m})$ and $\pi = \pi^a T^a$. Now the action takes on the form,

$$\begin{aligned} \mathcal{S}_{\text{PYM}} &= \int \mathrm{d}^4x\, \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2\left(A_\mu\mathcal{U}^{-1} + \frac{i}{g}\partial_\mu\mathcal{U}^{-1}\right)\left(\mathcal{U}A^\mu - \frac{i}{g}\partial^\mu\mathcal{U}\right)\right)\\ &= \int \mathrm{d}^4x\, \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\frac{m^2}{g^2}\mathcal{D}_\mu\mathcal{U}^\dagger\mathcal{D}^\mu\mathcal{U}\right)\,, \end{aligned} \tag{2.62}$$

where I have made use of $\mathrm{Tr}[\mathcal{U}M\mathcal{U}^{-1}] = \mathrm{Tr}[M]$, and there is a right-covariant derivative $\mathcal{D}_\mu\mathcal{U} = \partial_\mu\mathcal{U} - ig\mathcal{U}A_\mu$, which manifests the Stückelberg symmetry. The Stückelberg gauge transformations are given by the right-group action,

$$\begin{aligned} A_\mu &\to GA_\mu G^{-1} - \frac{i}{g}\partial_\mu GG^{-1}\\ \mathcal{U} &\to \mathcal{U}G^{-1}\,, \end{aligned} \tag{2.63}$$

which the right-covariant derivative $\mathcal{D}_\mu\mathcal{U}$ transforms trivially under. Pausing for a moment, this convention is setup such that in the limit $g \to 0$, one obtains $(N^2 - 1)$ copies of pure Proca theories in the Stückelberg formalism, as one expects, with the action given in equation (2.17) and abelian Stückelberg gauge transformations in equation (2.14). I now move onto analysing the structure of the interactions present in PYM.

2.3.4 The Decoupling Limit of PYM

If one analyses the interactions of the PYM action in Stückelberg formalism, i.e. employing equation (2.62), one sees immediately that once one expands the Lie group element out as the exponential of the Lie algebra matrix, $pi = \pi^a T^a$, i.e.

$$\mathcal{U} = e^{ig\frac{\pi}{m}} \tag{2.64}$$

in the action, the mass term generates an infinite tower of irrelevant (perturbatively non-renormalisable) operators, which follow the form

$$\begin{aligned} S &\sim \int \mathrm{d}^4x \, \frac{m^2}{g^2} \left[\partial\left(e^{\frac{g}{m}\pi}\right) + gA\left(e^{\frac{g}{m}\pi}\right)\right]^2 \\ &\sim \int \mathrm{d}^4x \, \frac{m^2}{g^2} \left[\mathcal{O}\left(\frac{g}{m}\partial\pi\left(\pi\frac{g}{m}\right)^n\right) + \mathcal{O}\left(gA\left(\frac{g}{m}\pi\right)^n\right)\right]^2 . \end{aligned} \tag{2.65}$$

One can read off that the leading order interactions enter in as

$$\mathcal{O}_n = \left(\frac{\pi}{\Lambda_1}\right)^n \pi\partial^2\pi \,, \tag{2.66}$$

with $\Lambda_1 = \frac{m}{g}$. Note that this is the scale at which perturbative unitarity breaks down, essentially owing to the fact that resonances appear and/or non-perturbative effects (e.g. instantons) contribute at the same order as loop corrections to the theory, and so quantum mechanics completely washes away the classical theory. It is expected that most theories go bad during this transition, and thus the general expectation is that these theories should be UV-completed. In the case of the Standard Model, this is what precisely happens in the Higgs mechanism; there, these irrelevant operators in (2.66) explicitly can be seen as coming from integrating out the Higgs particle, and the PYM arises as a part of the LEEFT of the SM in the Wilsonian sense. Sticking to the classical theory, one can see that there is actually a regime where there is a valid classical theory, called the decoupling limit. In some sense, the decoupling limit is the high-energy effective action, where the theory is still weakly coupled, $g \ll 1$, and thus one has a regime within a hierarchy of energies: E s.t. $\Lambda_1 \gg E \gg m$.[5] The decoupling limit stems from excising any interactions

[5]It is worth reiterating, however, that this theory is not the Wilsonian LEEFT; the LEEFT naturally needs to include the mass terms for the fields. The theory cannot actually forget the existence of finite, non-zero mass terms in the quantum theory since they control the form of the propagator, and thus affect EFT arguments. This turns out to be an important distinction in massive gravity [46]. The decoupling limit, however, defines the dominant classical contribution in this regime.

suppressed by scales higher than Λ_1, since these are the sub-leading contributions in this regime. This can be accomplished by taking the following scaling limit of the theory

$$\begin{aligned} m\,,\, g &\rightarrow 0 \\ \frac{m}{g} &\rightarrow \Lambda_1 = \text{ constant}\,. \end{aligned} \tag{2.67}$$

In this limit, the theory reduces to a system of a self-interacting scalar and (N^2-1)-copies of Maxwell theory, where the two modes have completely decoupled from one another (A fact from which the term "decoupling limit" derives). Explicitly, the decoupling limit action is then simply,

$$\mathcal{S}_{\text{DL-PYM}} = \int \mathrm{d}^4x \,\mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}G_\mu{}^\dagger G^\mu\right)\,. \tag{2.68}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $G_\mu = i\Lambda_1\partial_\mu \mathcal{U}\mathcal{U}^{-1}$. This theory is invariant under a global $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ symmetry, for left and right group actions $\mathcal{U}$ to $\mathcal{U}G^{-1}$ and $G\mathcal{U}$, respectively. There are also $\mathcal{U}(1)^{N^2-1}$ gauge symmetries as well, for the Maxwell theories. I end by noting that there is a simple integral resummation of the decoupling limit interactions for PYM, using the standard derivative of a matrix-exponential formula, which results in

$$G_\mu = \int_0^1 \mathrm{d}u\; e^{i\,u\frac{1}{\Lambda_1}\pi}\,\partial_\mu\pi\, e^{-i\,u\frac{1}{\Lambda_1}\pi}\,. \tag{2.69}$$

In section 3, I will give my derivation of the complete decoupling limit of massive gravity; there I demonstrate that integral representations of decoupling limit interactions are required, as well. So integral representations of interactions are quite generic in the complete decoupling limits of massive gauge theories!

2.3.5 PYM and the vDVZ Discontinuity

It is often forgotten that the original vDV paper does not only analyse the vDVZ discontinuity of Fierz-Pauli theory; they also analyse the existence of a similar discontinuity at loop-level in PYM [33]. They note that the 1-loop corrected self-energy for the PYM theory enters as twice the amount from an ordinary theory of YM theory, which can be seen to come from a sum over helicity states. Here, the introduction of new PDF's causes a discontinuity in a physical observable, even when $m \rightarrow 0$! There are other issues worth mentioning beyond those initially noted by van Dam and Veltman. Actually, this discontinuity is obvious even at tree-level in the Stückelberg action, as seen above, and the appearance of these loop corrections come from counter-terms for the non-renormalisable interactions for the helicity-0 mode of PYM. But here it is interesting to note that the leading order operators in the theory enter as $\sim \frac{g}{m}\partial$ so they are suppressed by the energy scale $\Lambda_1 = \frac{m}{g}$, which implies that the scalar theory becomes infinitely strongly-coupled in this limit! Therefore, one should probably expect that the (presumably divergent) 2-loop corrections are far worse than the 1-loop discontinuity as $m \to 0$. However, none of this is a problem if these operators come from integrating out a massive field (e.g. the Higgs mechanism) or (more speculative, but still possible) this theory comes from a non-trivial UV fixed point. But in both cases, it involves new physics entering into the theory beyond what is seen at the classical level. But if one wishes to additionally discuss the loop-level calculations, this can also be seen to be (unsurprisingly) even more problematic. In the Stückelberg formulation, one can schematically see that there are (at least) generic ℓ-loop counter-terms for the non-renormalisable interactions of the form

$$\mathcal{O}_n \sim g^{\ell+F}\left(\frac{\pi}{m}\right)^q \left(\frac{A}{m}\right)^p \left(\frac{\partial}{m}\right)^n \partial^2\pi^2\,, \tag{2.70}$$

where I leave F to compensate all of the ways differing powers of g might enter into the loop calculation. The important fact of equation (2.70) is that one should not expect that they are able to take the $m \to 0$ limit at any order in the quantum effective action without causing the loop corrections to diverge, signalling a break down in the perturbative unitarity around $\Lambda_1 = \frac{m}{g}$.[6] Now that I have reviewed this well-known toy model and have demonstrated what happens in a massive gauge theory, I now return to the issue of the self-interacting, massive spin-2 fields.

[6] It is worth noting that Λ_1 is not necessarily the scale where the theory breaks down; for an interesting discussion on this point see [74]. More concretely, it is known that $\mathcal{N} = 2$ SYM in 5-D is not perturbatively renormalisable but is still a UV-complete theory with a non-trivial UV fixed point [75–77]. It remains an open question whether or not such an 'asymptotic safety' scenario [78] exists for massive spin-1 bosons, supersymmetric or otherwise. A framework for this flavor of idea for massive gravity was attempted by others in [79].

2.4 Self-Interacting Massive Gravity

2.4.1 Boulware-Deser Theories

In the previous two subsections 2.2 and 2.3, I discussed several bizarre phenomena that arise in theories of massive spin-2 fields and self-interacting massive spin-1 fields. Returning to the issues raised in subsection 2.2.8, I now discuss the naïve approaches for developing self-interacting theories of massive gravity. The basic conditions can be stated thus:

(1.) I assume that in order for self-interacting spin-2 fields to exist, they must couple to their own stress-energy since in the UV they ought to (classically) flow to ordinary gravity. Thus, **the kinetic term of self-interacting massive gravity must be precisely that of Einstein-Hilbert**, and the variational field should be $g_{\mu\nu}$. (An explicit, direct verification of this fact was proven recently for ghost-free massive gravity in [80].)

(2.) Massive gauge theories necessarily break the gauge symmetry (away from any Stückelberg formalism); **to break diffeomorphisms, I need an explicit "reference metric"** $f_{\mu\nu}$. Because I wish to maintain global LI, I take it to be given by the Minkowski metric $f_{\mu\nu} = \eta_{\mu\nu}$.

(3.) In the limit of zero gravitational coupling, the theory must return to Fierz-Pauli theory. Therefore, $\lim \mathcal{S}_{\rm mGR} \to \mathcal{S}_{\rm FP}$ as $\kappa \to 0$ (Equivalently, $M_{\rm Pl} \to \infty$).

Taken together, this leads to an action of the form,

$$\mathcal{S}_{\rm mGR} = \int \mathrm{d}^4x\, \frac{1}{2\kappa^2}\sqrt{-g}\left(\mathcal{R} + m^2\,\mathcal{U}(g_{\mu\nu}, \eta_{\mu\nu})\right) \tag{2.71}$$

where finally one must require that

$$\lim_{\kappa\to 0}\left[\sqrt{-g}\frac{1}{2\kappa^2}\mathcal{U}(g_{\mu\nu}, \eta_{\mu\nu})\right] = m^2 h_\mu{}^\alpha \delta^{\mu\nu}_{\alpha\beta} h_\nu{}^\beta\,. \tag{2.72}$$

Noting that $\sqrt{g} = 1 + \kappa \mathrm{Tr}(h) + \mathcal{O}(\kappa^2)$ and $g_{\mu\nu} - \eta_{\mu\nu} = \kappa h_{\mu\nu}$, one can see that an obvious candidate is

$$\mathcal{U}(g_{\mu\nu},\, \eta_{\mu\nu}) = \left[(g_{\mu\nu} - \eta_{\mu\nu})\eta^{\mu\alpha}\eta^{\nu\beta}(g_{\alpha\beta} - \eta_{\alpha\beta})\right]\,. \tag{2.73}$$

2.4.2 The Boulware-Deser Ghost

One will immediately discover that the action (2.73) is problematic, originally discovered by Boulware, *et al*, in [35] (and further discussed in [9, 10, 36]). For this section, I will use the notation

1.) $[M] := \mathrm{Tr}[M] = M^\mu{}_\mu$.

2.) The usual matrix multiplication notation $(M^2)^\mu{}_\nu = M^\mu{}_\alpha M^\alpha{}_\nu$.

3.) h means $h^\mu{}_\nu = \eta^{\nu\alpha} h_{\mu\alpha}$.

In non-interacting Fierz-Pauli theory, I had to choose the Fierz-Pauli tuning of the mass terms[7],

$$\mathcal{S}_{\text{FP mass}} = \int \mathrm{d}^4x \, \frac{1}{2} m^2 \left([h]^2 - [h^2]\right) , \tag{2.74}$$

on pain of generating a ghostly mode. At linear order, it turns out there is only one coefficient that needs to be fixed; however, as I add in interactions for the massive spin-2 field, I will have a growing number of coefficients with tensor-contraction combinatorics like

$$\begin{aligned} \mathcal{S}_{\text{mass}} \supset & \int \mathrm{d}^4x \, \kappa m^2 \left(a_1[h]^3 + a_2[h^2][h] + a_3[h^3]\right) \\ & + \kappa^2 m^2 \left(b_1[h]^4 + b_2[h^2]^2 + b_3[h][h^3] + b_4[h^4]\right) \\ & + \cdots . \end{aligned} \tag{2.75}$$

An immediate issue arises, namely that typically these interactions are ghostly. This can already been seen in the naïve cubic interactions from subsection 2.2.8. With indices re-introduced and including $h_{\mu\nu}$ terms, the cubic vertices looks like

$$\begin{aligned} \mathcal{S}_{\text{mGR}} \supset & \int \mathrm{d}^4x \left(\frac{1}{M_{\text{Pl}} m^2}\right) \mathcal{A}_1^{\mu\nu\rho\sigma\lambda\omega} h_{\lambda\omega} \partial_\mu \partial_\nu \pi \partial_\rho \partial_\sigma \pi \\ & + \left(\frac{1}{M_{\text{Pl}} m^3}\right) \mathcal{A}_1^{\mu\nu\rho\sigma\lambda\omega} B_{\lambda\omega} \partial_\mu \partial_\nu \pi \partial_\rho \partial_\sigma \pi \\ & + \left(\frac{1}{M_{\text{Pl}} m^4}\right) \mathcal{A}_2^{\mu\nu\rho\sigma\lambda\omega} \partial_\mu \partial_\nu \pi \partial_\rho \partial_\sigma \pi \partial_\lambda \partial_\omega \pi \, , \end{aligned} \tag{2.76}$$

when one adds Stückelberg fields $h_{\mu\nu} \to h_{\mu\nu} + \frac{1}{m} \partial_{(\mu} B_{\nu)} + \frac{1}{m^2} \partial_\mu \partial_\nu \pi$ into the cubic terms in (2.75), and restricts to the cubic terms for π. Short of a miracle, these interactions will generally give rise to higher-order equations of motion. The appearance of these

[7] Here for simplicity I have gauged away all Stückelberg fields.

higher-order equations signals the existence of a ghostly mode, called the Boulware-Deser mode, originally found in [35]. There is one other way to understand the BD ghost, so I will discuss that now. Note, however, that one can easily read off that this mode appears, it always enters through the 4th order derivatives on π![8] Thus, this mode is necessarily a ghost.

2.4.3 All 6 Helicities of a $D = 4$ Spin-2 Field

I give a separate discussion of the modes of a spin-2 field in 4 spacetime dimensions. One can do a Hamiltonian analysis of a generic mGR theory, where one may diagnose all possible PDF's. For concreteness, I shall work with the metric version of GR, so the main variable is $g_{\mu\nu}$. If one accepts the Einstein-Hilbert action for the kinetic term, plus potential terms to generate a mass, then after a (3+1)-split, one arrives at a schematic ADM formulation [81]

$$\begin{aligned} \mathcal{S}_{\mathrm{mGR}}[g_{\mu\nu}] &= \frac{1}{2\kappa^2}\int \mathrm{d}^4x\,\sqrt{-g}\left(\mathcal{R} + m^2\mathcal{U}(g_{\mu\nu},\eta_{\rho\sigma})\right) && (2.77)\\ &= \int \mathrm{d}t\mathrm{d}^3x\, K^{ij}\dot{g}_{ij} + \Pi^{\mu}\dot{g}_{0\mu} - \mathcal{H}_{\mathrm{mGR}}[g_{ij}, g_{0i}, g_{00}, K^{ij}, \Pi^i, \Pi]\,, && (2.78) \end{aligned}$$

where K_{ij}, Π^i and Π are given by

$$K^{ij} = \frac{\delta\mathcal{S}_{\mathrm{EH}}}{\delta\dot{g}_{ij}} \tag{2.79}$$

$$\Pi^{i} = \frac{\delta\mathcal{S}_{\mathrm{EH}}}{\delta\dot{g}_{0i}} \tag{2.80}$$

$$\Pi = \frac{\delta\mathcal{S}_{\mathrm{EH}}}{\delta\dot{g}_{00}}\,, \tag{2.81}$$

(since only Einstein-Hilbert term contains derivatives) with the dot operator is Lie derivatives WRT the foliated time, t. As is well-known for Einstein-Hilbert term from the ADM analysis, the fields $g_{0\mu}$ have no conjugate momentum

$$\Pi^i = 0 \tag{2.82}$$

$$\Pi = 0\,. \tag{2.83}$$

[8]It may look like you could get terms $\partial^3 B$, but crucially those always come in as $(\partial^2\pi)^n h^m \partial^2(\partial B)^k$ owing to the form of the Euler-Lagrange equations. With two integration by parts, that always turns into first/second-order h terms and third/fourth-order π terms.

If one takes stock of this fact, it means that one starts out with a naïve phase space of 10 DOF for a symmetric 2-tensor $g_{\mu\nu}$, but 4 of those modes do not propagate. This leaves a maximum of 6 possible PDF for a spin-2 field. How would one get only 5 PDF's from a variational principle? Here the Dirac analysis [82] tells one that: Whatever potential mass terms one adds to their action, $\sqrt{-g}\mathcal{U}(g_{\mu\nu}, \eta_{\rho\sigma})$, one needs a potential that generates two further second-class constraints. This is the only way to eliminate the unwanted PDF! Without a mass term, the constraints generated by (2.82) lead to secondary first-class constraints, which generate the diffeomorphism gauge symmetries via the Castellani algorithm [83, 84]. The mass term will necessarily breaks the gauge symmetries, so there are no first-class constraints. However, the mass term must have the property that it still keeps enough "nice" structure that it preserves 2 second-class constraints. This will be possible, for example, if upon integrating out g_{i0}, one obtains a Hamiltonian that is linear in g_{00} (which generates only a single pair of second-class constraints). This will kill one of the $(h_{ij},\, K^{mn})$ pairs, leading to 5 PDF's. Both of these analyses of this subsection and the previous are equivalent, but they both provide distinctly useful ways of understanding the origin and nature of the Boulware-Deser mode.

2.5 Ghost-Free, Self-Interacting Massive Gravity

2.5.1 Towards Ghost-Freedom

One can begin this process for discovering ghost-free self-interacting mass terms by starting with the equations (2.76). Here I have lowered half of the indices into the form

$$\begin{aligned}\mathcal{S}_{\rm mGR} \;\supset\; & \int \mathrm{d}^4x \left(\frac{1}{M_{\rm Pl}m^2}\right) \mathcal{A}_1{}^{\mu\nu\rho}_{\alpha\beta\gamma}\, h_\mu{}^\alpha \partial_\beta\partial_\beta\pi\partial_\rho\partial_\gamma\pi \\ & + \left(\frac{1}{M_{\rm Pl}m^3}\right) \mathcal{A}_2{}^{\mu\nu\rho}_{\alpha\beta\gamma}\, \partial_\mu B^\alpha \partial_\nu\partial^\beta\pi\partial_\rho\partial^\gamma\pi \\ & + \left(\frac{1}{M_{\rm Pl}m^4}\right) \mathcal{A}_3{}^{\mu\nu\rho}_{\alpha\beta\gamma}\, \partial_\mu\partial^\alpha\pi\partial_\nu\partial_\beta\pi\partial_\rho\partial_\gamma\pi\,. \end{aligned} \tag{2.84}$$

First, one should note that the third line with $\mathcal{A}_3$ will *necessarily* generate a fourth-order equation of motion short of two things happening:

1.) One requires $\mathcal{A}_3$ to be identically zero.

2.) One requires that the manner $\mathcal{A}_3$ enters the Euler-Lagrange equation is identically zero. In other words, one imposes

$$\partial_\alpha \partial^\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\alpha \partial^\mu \pi} \right) \propto \partial_\alpha \partial^\mu \left(\mathcal{A}_3{}^{\mu\nu\rho}_{\alpha\beta\gamma} \, \partial_\nu \partial^\beta \pi \partial_\rho \partial^\gamma \pi + \cdots \right) = 0 \,. \tag{2.85}$$

with the $\cdots$ indicating the permutations of the indices from removing the second and third $\partial^2\pi$ terms in the functional variation.

The first condition implies no mass terms at cubic order, which is not viable; thus, one must explore the second possible condition. Actually, the second case is quite simple to accommodate, as was discovered in [41, 42].[9] One just needs the condition

$$\mathcal{A}_3{}^{\mu\nu\rho}_{\alpha\beta\gamma} = \mathcal{A}_3{}^{\mu\nu\rho}_{[\alpha\beta\gamma]} = \mathcal{A}_3{}^{[\mu\nu\rho]}_{\alpha\beta\gamma} \,. \tag{2.86}$$

Combined with the requirement that these tensors are Lorentz invariant, a dedicated student of differential geometry will recall that there is a unique candidate, known as a "generalised Kronecker delta" tensor (henceforth, "GKD tensor"), which satisfies these conditions. See Appendix B for further discussion of these objects and their properties. One can see then that $h_\mu{}^\nu$ must enter into cubic combinations of the form,

$$\mathcal{S}^{(3)}_{\text{mGR mass}} \;=\; m^2 \int \mathrm{d}^4x \, \kappa \, \alpha \left(\delta^{\mu\nu\rho}_{\alpha\beta\gamma} \; h_\mu{}^\alpha h_\nu{}^\beta h_\rho{}^\gamma \right) \,, \tag{2.87}$$

where α is a free numerical coefficient. Once this choice is made, then

$$\mathcal{A}_1 \propto \mathcal{A}_2 \propto \mathcal{A}_3 \propto \delta_3 \,, \tag{2.88}$$

with δ_3 being the 3-index GKD tensor. From here, it is trivial to prove that the terms proportional to $\mathcal{A}_1$ and $\mathcal{A}_2$ in (2.84) are all ghost-free; thus, this renders the entire cubic action ghost-free! Also note that it turns the Stückelberg-only interactions into total divergences. Following the obvious pattern of needing the $(\partial^2\pi)^n$ terms to be total derivatives, one can see that a good quartic interaction, with a free coefficient β, is given by

$$\mathcal{S}^{(4)}_{\text{mGR mass}} = m^2 \int \mathrm{d}^4x \, \kappa^2 \beta \left(\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} \; h_\mu{}^\alpha h_\nu{}^\beta h_\rho{}^\gamma h_\sigma{}^\delta \right) \,. \tag{2.89}$$

This actually makes the Fierz-Pauli tuning's properties extremely manifest: The Fierz-Pauli tuning is simple the combination that leads to the GKD symbol with two indices, and this anti-symmetry is what exorsises the would-be ghost's kinetic

[9] They follow a different route to prove this, but the two approaches are equivalent.

term! Also, GKD symbols may not have indices greater than the dimensionality of the space (due to the antisymmetry), and thus in 4-D, one must stop here. No other terms may be present at these orders if one wants to keep ghost vanquished, but this leaves the higher-order interactions an open question.

2.5.2 dRGT Theory of a Ghost-Free Massive Graviton

Given the stringent requirements of ghost-freedom at cubic and quartic order in an mGR theory, one has arrived at the following criteria:

(1.) The theory is still in need of a covariantisation, $\mathcal{U}(g_{\mu\nu}, \eta_{\rho\sigma})$, that reproduces (2.89) only to cubic and quartic order when $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$.

(2.) Once this covariantisation is found, the higher order terms must fail to reintroduce the BD ghost.

In [43], an action was discovered which correctly solves the (1.), and gave arguments pointing towards (2.) being satisfied. In [44, 45], it was shown conclusively that this action solves (2.), and thus there exists a self-interacting, ghost-free massive theory of gravity that propagates 5 PDF's. This theory is called "dRGT massive gravity" after the authors of [43]. The action discovered takes the form,

$$\mathcal{S}_{\mathrm{dRGT}} = \int \mathrm{d}^4x \, \frac{1}{2\kappa^2}\sqrt{-g}\left(\mathcal{R} + m^2\mathcal{U}(g_{\mu\nu}, \eta_{\rho\sigma})\right) \,, \tag{2.90}$$

such that, given two free parameters α and β, one uses the combination

$$\mathcal{K}_\mu{}^\alpha = \quad \delta^\alpha_\mu - \sqrt{g^{\mu\rho}\eta_{\rho\nu}} \tag{2.91}$$

$$\begin{aligned} \mathcal{U}(g_{\mu\nu}, \eta_{\rho\sigma}) = \quad & \delta^{\mu\nu}_{\alpha\beta}\mathcal{K}_\mu{}^\alpha\mathcal{K}_\nu{}^\beta + \alpha\,\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\mathcal{K}_\mu{}^\alpha\mathcal{K}_\nu{}^\beta\mathcal{K}_\nu{}^\beta \\ & \beta\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}\mathcal{K}_\mu{}^\alpha\mathcal{K}_\nu{}^\beta\mathcal{K}_\nu{}^\beta\mathcal{K}_\sigma{}^\delta \,. \end{aligned} \tag{2.92}$$

The square root here is defined as a matrix square root of $g^{-1}\eta$ at the level of matrices, and the conditions of its general existence are argued in [85]. I should remark that there are now many proofs which show that this action is ghost-free, e.g. [86–91], including relaxing the $\eta_{\mu\nu}$ reference metric to more general reference metrics [92], and even allowing for a second dynamical metric (i.e. theories of "bigravity") [93].

2.5.3 dRGT in Einstein-Cartan Formalism

There is a considerably more elegant formulation of dRGT massive gravity when one re-writes the dRGT action into an Einstein-Cartan formulation; for a review of the Einstein-Cartan formalism, see Appendix C. This will be the predominant description of dRGT massive gravity that I will work with for the remainder of this text. Following earlier attempts [94, 95], it was shown by Hinterbichler and Rosen [90] that once one introduces the vielbein variables into the theory,

$$g_{\mu\nu} = e_\mu{}^a \eta_{ab} e_\nu{}^b \,, \tag{2.93}$$

there is a simple way to remove the square root structure present in dRGT. This substitution adds in a local Lorentz gauge symmetry,

$$e_\mu{}^a \to \Lambda^a{}_b e_\mu{}^b \,, \tag{2.94}$$

into the action. Then one can use these 6 gauge symmetries to fix the following 6 conditions

$$g^{\mu\nu} e_\mu{}^{[a} \delta_\nu{}^{b]} = 0 \,, \tag{2.95}$$

noting that $\delta_\mu{}^a$ is the flat space vielbein $\eta_{\mu\nu} = \delta_\mu{}^a \eta_{ab} \delta_\nu{}^b$. This is known as the Deser-van Nieuwenhuizen condition (which is also known as the "symmetric vielbein condition")[85, 90, 95–97]. Once this is plugged into the action, and noting that the GKD structure immediately extends to a p-form notation, one finds that the dRGT action turns into the form

$$\begin{aligned} \mathcal{S}_{\mathrm{dRGT}} \;=\; & \int \frac{1}{4\cdot 2!\,\kappa^2}\Big(\mathcal{R}^{ab} + (e^a-\delta^a)(e^b-\delta^b)\Big)e^c e^d \varepsilon_{abcd} \\ & + \alpha\frac{m^2}{4\cdot 1!\,\kappa^2}(e^a-\delta^a)(e^b-\delta^b)(e^c-\delta^c)e^d\varepsilon_{abcd} \\ & + \beta\frac{m^2}{4\cdot 0!\,\kappa^2}(e^a-\delta^a)(e^b-\delta^b)(e^c-\delta^c)(e^d-\delta^d)\varepsilon_{abcd} \end{aligned} \tag{2.96}$$

with wedge products left implied, with 1-forms given by $e^a := e_\mu{}^a \mathrm{d}x^\mu$ and $\delta^a = \delta_\mu{}^a \mathrm{d}x^\mu$, and $\mathcal{R}^{ab}$ is the usual curvature 2-form. The precise nature of how the square root structure falls out was given in [90], but also I discovered the connection between the two actions without needing to appeal to gauge-fixing and which shows a direct equivalence of the two actions via the use of Lorentz Stückelberg fields. I will discuss this proof in Chapter 3. The simplicity of this action is quite nice, and it makes it clear how to generalise to arbitrary dimensions [90], it makes the action is polynomial when using the vielbein and the symmetric-vielbein gauge choice.

2.5.4 Partial Decoupling Limit of dRGT

Recall from subsection 2.3.4 that there is a regime, assuming weak coupling and a small mass, where Proca-Yang-Mills' dominant classical contribution can be defined by a "decoupling limit theory." The crucial item here was identifying where the strong-coupling scale enters, and staying well below that but also well above the mass. If such a regime exists, it is dominated classically by the decoupling limit. This raises the question of which scale in dRGT is the lowest scale where irrelevant operators enter into the theory? This reflects the naïve strong coupling scale, where the theory will need to be treated more carefully. I will use the following notation to concisely see this:

$$(\Lambda_N)^N := M_{\rm Pl} m^{N-1}\,, \text{ s.t. } N \geq 1\,. \tag{2.97}$$

Recall that in my conventions, $M_{\rm Pl} = \frac{1}{\kappa}$. Λ_N is how generic scales enter into the denominators of the interactions of the dRGT action, which I will prove now. All operators in the theory with n-fields take on the form

$$\mathcal{O}_{n=p+q+r} = M_{\rm Pl}^2 m^2 \left(\frac{h}{M_{\rm Pl}}\right)^p \left(\frac{\partial B}{M_{\rm Pl} m}\right)^q \left(\frac{\partial^2 \pi}{M_{\rm Pl} m^2}\right)^r \tag{2.98}$$

$$= M_{\rm Pl}^{2-p-q-r} m^{2-q-2r} \left(h\right)^p \left(\partial B\right)^q \left(\partial^2\pi\right)^r\,. \tag{2.99}$$

Now, since for an $n^{\rm th}$-order interaction $n = p+q+r > 2$, the interactions are suppressed by an energy scale

$$\Lambda_{pqr}^{n-2} = \left(M_{\rm Pl} m^{\frac{q+2r-2}{p+q+r-2}}\right)^{p+q+r-2}\,. \tag{2.100}$$

For the Boulware-Deser-type theories of mGR [35, 67], it was found that there is a smallest scale, Λ_5 (cf. to the previous subsection). However, when one makes the tunings necessary for the dRGT action, one can check from the equations in (2.87) that are generated by (2.84) all fail to have interactions at the scale Λ_5^5 and even Λ_4, as was demonstrated by de Rham and Gabadadze in [41, 42]; all of these interactions being suppressed by Λ_5 and Λ_4 are total derivatives. This raises the smallest scale to $\Lambda_3 = \left(M_{\rm Pl} m^2\right)^{1/3}$! Ergo, the decoupling limit of dRGT must be given by scaling limit of the form

$$
\begin{aligned}
m &\rightarrow 0\,, \\
M_{\rm Pl} &\rightarrow \infty\,, \\
M_{\rm Pl} m^2 &\rightarrow \Lambda_3 = \text{const}\,.
\end{aligned} \tag{2.101}
$$

For now, I will only analyse what terms are generated in the decoupling limit that are generated from the helicity-0 sector.[10] To this end, I re-write the vielbein action in symmetric polynomial notation[11], which I first introduced in [98]. With this notation, the action takes on the form

$$
\mathcal{S}_{\rm dRGT} = \mathcal{S}_{EH} + \int \mathrm{d}^4x\, M_{\rm Pl}^2 m^2 \delta_4[(E-F)^2 e^2 + \alpha (E-F)^3 e + \beta (E-F)^4]\,, \tag{2.102}
$$

and expand the action in terms of the Stückelberg expansion[12]

$$
E \rightarrow I + \frac{1}{M_{\rm Pl}} h F \rightarrow I + \underbrace{\frac{\partial B}{M_{\rm Pl} m}}_{\text{neglecting}} + \Pi\,, \tag{2.103}
$$

where ${\Pi_\mu}^\nu = \frac{1}{m^2 M_{\rm Pl}} \partial_\mu \partial^\nu \pi$, and h is the usual ${h_\mu}^\nu$ owing to the symmetric vielbein condition. I ignore the helicity-1 mode B for now. Plugging this into the above action, noting that $M_{\rm Pl}^2 m^2 = M_{\rm Pl}(\Lambda_3)^3$,

$$
\begin{aligned}
\mathcal{S}_{\rm dRGT} = \frac{1}{4} \int \mathrm{d}^4x\, M_{\rm Pl}(\Lambda_3)^3 \delta_4 \Big[& \frac{1}{2} \left(\frac{1}{M_{\rm Pl}} h + -\Pi \right)^2 \left(I + \frac{1}{M_{\rm Pl}} h \right)^2 \\
& + \alpha \left(\frac{1}{M_{\rm Pl}} h - \Pi \right)^3 \left(I + \frac{1}{M_{\rm Pl}} h \right) \\
& + \beta \left(\frac{1}{M_{\rm Pl}} h - \Pi \right)^4 \Big]\,,
\end{aligned} \tag{2.104}
$$

one immediately finds that the only pieces that survive in the decoupling limit are the terms linear in $\frac{1}{M_{\rm Pl}}$. It may look like that this action is schematic, but the notation and the simplicity of dRGT in the vielbein formulation makes this the

[10] I return to this issue in Chapter 3. Here I go over my proof of the complete decoupling limit of dRGT, including the full tower of helicity-1 interactions.

[11] See Appendix B for further information on this notation.

[12] It is critical to note that I have not placed the Stückelberg fields into E as one might expect, but instead I have placed it into the background vielbein; in the interacting theory, I can choose either. For continuity with later results in Chapter 3, I choose to place the Stückelberg fields in the background vielbein. For a detailed discussion, including demonstrating the equivalence between where you place the diffeomorphism Stückelberg fields, see Chapter 3.

exact action. Notice that the **only interactions which survive are those linear in h**! Again, the terms containing only Π's are total derivatives, otherwise this limit would diverge as $\sim M_{\rm Pl}$. Noting this, the decoupling limit action becomes

$$\begin{aligned} \mathcal{S}_{\text{dRGT in DL}} &= \int \mathrm{d}^4x \; -\frac{1}{2}h_\mu{}^\alpha \delta^{\mu\nu\rho}_{\alpha\beta\gamma}\partial_\nu\partial^\beta h_\rho{}^\gamma \\ &\quad +\Lambda_3^3 h_\mu{}^\alpha \left(X_\alpha{}^\mu + (1+3\alpha)Y_\alpha{}^\mu + (\alpha+3\beta)Z_\alpha{}^\mu\right) , \end{aligned} \tag{2.105}$$

with the "transverse tensors" [41, 42]

$$\Pi_{ab} = \frac{\partial_a\partial_b\pi}{\Lambda_3^3} \tag{2.106}$$

$$X^\mu{}_\alpha = -\delta^{\mu\nu}_{\alpha\beta}\Pi_\nu{}^\beta$$

$$Y^\mu{}_\alpha = \frac{1}{2!}\,\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\Pi_\nu{}^\beta\Pi_\rho{}^\gamma \tag{2.107}$$

$$Z^\mu{}_\alpha = -\frac{1}{3!}\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}\Pi_\nu{}^\beta\Pi_\rho{}^\gamma\Pi_\sigma{}^\delta\,. \tag{2.108}$$

This theory has many interesting properties, and after a field redefinition, can be seen to be a theory containing one decoupled massless helicity-2 theory and a separate, self-interacting helicity-0 mode described through Galileon interactions [41, 42, 99].

2.5.5 Galileon Theory and the Vainshtein Mechanism

Before closing out this section, I briefly review the properties of the Galileon theory described above. First, one may use the diagonalising field redefinitions found in [41, 42]

$$\begin{aligned} \pi &\to \sqrt{\frac{2}{3}}\pi \\ h_{\mu\nu} &\to h_{\mu\nu} + \frac{1}{\sqrt{6}}\pi\eta_{\mu\nu} + \frac{1}{2\Lambda_3^3}\partial_\mu\pi\partial_\nu\pi + \mathcal{O}\left(\frac{1}{\Lambda_3^6}\partial^3\pi^3\right) , \end{aligned} \tag{2.109}$$

to turn this into an action of the form

$$\begin{aligned} \mathcal{S}_{DL} &= \mathcal{S}_{\rm LGR} + \int \mathrm{d}^4x \; -\frac{1}{2}\pi\Box\pi + \frac{A}{\Lambda_3^3}\pi\delta^{\mu\nu}_{\alpha\beta}(\partial_\mu\partial^\alpha\pi)(\partial_\nu\partial^\nu\pi) \\ &\quad + \frac{B}{\Lambda_3^6}\pi\delta^{\mu\nu\rho}_{\alpha\beta\gamma}(\partial_\mu\partial^\alpha\pi)(\partial_\nu\partial^\nu\pi)(\partial_\rho\partial^\gamma\pi)\,, \end{aligned} \tag{2.110}$$

where A and B depend on α and β (See [41] or the review [100] for details on this). These interactions are commonly known as "Galileon" interactions, owing to their shift symmetry and internal "Galilean" symmetries

$$\pi \to \pi + a + b_\mu x^\mu \,, \tag{2.111}$$

which comes from the fact that the EOM always contain two derivatives on π. I will not prove these claims in this text, but several interesting points can be shown within this theory:

1.) At the purely classical level, this theory manifests the Vainshtein mechanism. In other words, it becomes strongly self-interacting roughly at some scale called the Vainshtein radius, $r_V = \left(\frac{r_S}{m^2}\right)^{1/3}$ which can also depend on α and β; inside this radius the Galileon scalar mode gets screened due to self-interactions and there is no fifth force. Outside of this radius, the scalar mode is fully present and FP dominates the description of dRGT. This creates a continuity between LGR close to a point source and FP far away from the point source!

2.) These interactions do come with several concerns. Firstly, they are present when $E \sim \Lambda_3$, and thus raise the question of either needing a full quantum treatment and presumably a UV completion with new physics. For instance, these theories generically give rise to superluminal modes (issues raised in first for generic mGR in [9, 10], for Galileon-like interactions specifically in [37], and again for dRGT in [38, 39]). Such questions have been taken up in [37, 46–48, 79, 101]. Explicit quantum corrections to dRGT in and out of the decoupling limit have been taken up in [53, 102, 103]. This is a diverse topic with many different perspectives, and I could not do justice to all of the perspectives and the discussion on this topic. For the most recent review of dRGT massive gravity, which should give the reader an entry point into the literature, I recommend [55].

It is safe to say that there is much left to understand about dRGT massive gravity in terms of what is needed to make the theory completely consistent at or around the scale Λ_3, what the status of acausalities might be, and how to understand the theory in the UV. Thus far, it seems that dRGT can be thought of as an LEEFT well beneath this scale. What happens above this scale and if there is a full realisation of the Vainshtein mechanism remains a mystery until I have a more developed understanding of dRGT. I now turn to this task.

Chapter 3

The Complete Decoupling Limit of dRGT Massive Gravity

3.1 The Λ_3 Decoupling Limit

I now derive the complete decoupling limit of dRGT ghost-free massive gravity, following my work in [98]. In subsection 2.5.4), I discussed how there is a simple procedure for obtaining the decoupling limit of dRGT massive gravity. To recap, in the EC formulation of dRGT ghost-free massive gravity is given by the action

$$
\begin{aligned}
\mathcal{S}_{\mathrm{dRGT}} \;=\; & \int \frac{1}{4\cdot 2!}\Big(\mathcal{R}^{ab}+(e^a-I^a)(e^b-I^b)\Big)e^c e^d \varepsilon_{abcd} \\
& +\alpha\frac{m^2}{2\kappa^2}(e^a-I^a)(e^b-I^b)(e^c-I^c)e^d\varepsilon_{abcd} \\
& +\beta\frac{m^2}{2\kappa^2}(e^a-I^a)(e^b-I^b)(e^c-I^c)(e^d-I^d)\varepsilon_{abcd} \qquad (3.1)
\end{aligned}
$$

with $I^a = \delta_\mu{}^a\,\mathrm{d}x^\mu$. In massive gauge theories, there can exist a "decoupling limit". This is the dominant classical contribution to the theory, so long as there is a hierarchy of scales between the smallest scale that suppresses an irrelevant operator and the mass of the gauge field (cf. to the analyses in subsections 2.2.4, 2.3.4, and 2.5.4). Typically, such a hierarchy exists when the coupling constant (in gravity, this is given by $1/M_{\mathrm{Pl}}$) and the mass are sufficiently small. In dRGT, there is an essential feature that the irrelevant operators of dRGT mass terms are divided by a smallest energy scale Λ_3, and so a Λ_3 decoupling limit can consistently be defined as the scaling limit, i.e.

$$\begin{aligned} m &\to 0 \\ M_{\rm Pl} &\to \infty \\ M_{\rm Pl}m^2 &\to \Lambda_3 = \text{ const}\,. \end{aligned} \tag{3.2}$$

The first feature of the decoupling limit is that it greatly simplifies the interactions. The second feature is that in the high-momentum limit of dRGT, where its 5 PDF's decompose into 5 helicity states (± 2, ± 1, 0). This can only be seen in Stückelberg language, where I have

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{1}{m}\partial_{(\mu}B_{\nu)} + \frac{1}{m^2}\partial_\mu\partial_\nu\pi\,, \tag{3.3}$$

where these must be the fundamental variables in the decoupling limit.

It turns out, however, that taking the *full* decoupling limit is slightly more subtle than this. This Stückelberg procedure will allow for all $\pi - h$ interactions, but it fails to provide all of the $\pi - B$ interactions in any straightforward manner [104]. To get the complete decoupling limit systematically, I will work with the EC formulation and then add into my theory a Lorentz Stückelberg field, $\Lambda^a{}_b$, thus completing all of the gauge symmetries of the Einstein-Cartan formulation. The decoupling limit in this formalism is straightforward to derive, once a few clever observations are made.

3.2 Stückelberg Procedure for dRGT

3.2.1 dRGT Interactions as a Deformed Determinant

I begin by restructuring the dRGT action in a manner that elucidates some novel properties of the dRGT mass terms. I reshuffle the mass terms in (3.1) by directly expanding the quadratic, cubic, and quartic terms in $(e - I)$ into their constituent pieces

$$\begin{aligned} S_{\rm dRGT} &= \frac{M_{\rm Pl}^2}{4}\varepsilon_{abcd}\int m^2\Big[\frac{\beta_0}{4!}e^ae^be^ce^d + \frac{\beta_1}{3!}I^ae^be^ce^d \\ &\quad + \frac{\beta_2}{2!2!}I^aI^be^ce^d + \frac{\beta_3}{3!}I^aI^bI^ce^d\Big]\,, \end{aligned} \tag{3.4}$$

plus a pure number $\propto I^a I^b I^c I^d \varepsilon_{abcd}$, which one may discard. If one wishes to make explicit contact with the previous action, I have

$$\begin{aligned}\beta_0 &= -12 - 8\alpha_3 - 2\alpha_4 \\ \beta_1 &= 6 + 6\alpha_3 + 2\alpha_4 \\ \beta_2 &= -2 - 4\alpha_3 - 2\alpha_4 \\ \beta_3 &= 2\alpha_3 + 2\alpha_4\,. \end{aligned} \tag{3.5}$$

From here, there is a clever observation that these can be reformulated as a 'deformed determinant' [105]. A deformed determinant contains the same terms in the expansion of the determinant of a matrix, but each term is deformed away from an ordinary determinant with an arbitrary c_n coefficient. Explicitly, this is

$$\mathcal{L}_{\text{mass}}(E^a, F^b) = -\frac{1}{4} M_{\text{Pl}}^2 m^2 \widehat{\text{Det}}[\Lambda E - F] \tag{3.6}$$

$$\begin{aligned} &= c_0 \varepsilon_{abcd} (\Lambda E)^a (\Lambda E)^b (\Lambda E)^c (\Lambda E)^d \\ &+ c_1 \varepsilon_{abcd} F^a (\Lambda E)^b (\Lambda E)^c (\Lambda E)^d \\ &+ c_2 \varepsilon_{abcd} F^a F^b (\Lambda E)^c (\Lambda E)^d \\ &+ c_3 \varepsilon_{abcd} F^a F^b F^c (\Lambda E)^d \\ &+ c_4 \varepsilon_{abcd} F^a F^b F^c F^d\,, \end{aligned} \tag{3.7}$$

where from now on I drop off the term proportional to c_4 (it is just a constant), and for convenience I denote $\Lambda^a{}_b E^b := (\Lambda E)^a$. Note that to compare to the old form, the $\{c_i\}$ can be read off as

$$c_n = -\frac{1}{4\, n!(4-n)!} M_{\text{Pl}}^2 m^2 \beta_n\,. \tag{3.8}$$

This can be re-expressed as

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2} M_{\text{Pl}}^2 m^2 \widehat{\text{Det}}[E - I] \tag{3.9}$$

$$\begin{aligned} &= c_0 \varepsilon_{abcd} E^a E^b E^c E^d \\ &+ c_1 \varepsilon_{abcd} I^a E^b E^c E^d \\ &+ c_2 \varepsilon_{abcd} I^a I^b E^c E^d \\ &+ c_3 \varepsilon_{abcd} I^a I^b I^c E^d \\ &+ c_4 \varepsilon_{abcd} I^a I^b I^c I^d\,, \end{aligned} \tag{3.10}$$

using the fact that $\text{Det}[\Lambda] = 1$ for $\Lambda \in \text{SO}(1,3)$ and the identity $\varepsilon_{abcd} =$

$\Lambda_a{}^{a'}\Lambda_b{}^{b'}\Lambda_c{}^{c'}\Lambda_d{}^{d'}\varepsilon_{a'b'c'd'}$. Now, note the following simplified form of the dRGT action:

$$\mathcal{S}_{\text{mass}} = \int \sum_{n=0}^{4} \frac{(-1)^n}{n!} c_n \frac{\partial^n}{\partial \mu^n} \text{Det}[E - \mu I]\Big|_{\mu=0} . \tag{3.11}$$

It turns out that this action will exhibit some remarkable properties when dealing with the Lorentz Stückelberg field. I now turn to the issue of re-incorporating all of the EC symmetries to the dRGT action, including local Lorentz invariance.

3.2.2 Stückelberg Fields in Einstein-Cartan Variables

Following the Stückelberg procedure (cf. subsections 2.2.4, 2.2.5, and 2.3.3), I systematically re-incorporate the old gauge symmetry, which for the ordinary Einstein-Cartan action is diffeomorphisms **and** local Lorentz symmetry [106, 107]

$$E_\mu{}^a \to \frac{\partial y^\nu}{\partial x^\mu} E_\nu{}^a \qquad \text{(Diffeomorphism)} \tag{3.12}$$

$$E_\mu{}^a \to \Lambda^a{}_b E_\nu{}^b \qquad \text{(Local Lorentz).} \tag{3.13}$$

This can be done by manually installing them using a new field ϕ^μ (called a diffeomorphism Stückelberg field)[108] and a substitution of the form

$$E_\mu{}^a \to \frac{\partial x^\nu}{\partial \phi^\mu} E_\nu{}^b , \tag{3.14}$$

which will linearise like the vector mode in subsection 2.2.5[1]. I choose a diffeomorphism of that particular form because it will be useful later. Finally, I add in a Lorentz Stückelberg symmetry (following the prescient work of [94]) via

$$E_\mu{}^a \to \Lambda^a{}_b \frac{\partial x^\nu}{\partial \phi^\mu} E_\nu{}^b , \tag{3.15}$$

[1] A clever reader may notice that I have not yet included the scalar mode, π. Strictly speaking, the fully covariant $\mathcal{U}(1)$ symmetry remains elusive; this is discussed, for instance, in [86]. However, it is irrelevant for the decoupling limit analysis, since whatever such an analysis would yield, it must approach the ordinary $B \to B - \frac{1}{m}\partial\pi$ in the decoupling limit: in the limit $M_{\text{Pl}} \to \infty$, the infinitesimal $\mathcal{U}(1)$ linearises to the known form.

Then one may note that the action is invariant under a diffeomorphism by exactly ϕ^μ, which has the following effect on the variables

$$\begin{aligned} E^a_\mu &\to \Lambda^a{}_b E^b_\nu \\ I^a_\mu &\to F^a_\mu = \frac{\partial \phi^\nu}{\partial x^\mu} I^a_\nu = \partial_\mu(\phi^a)\,, \end{aligned} \tag{3.16}$$

since they are always contracted into $\varepsilon_{\mu\nu\rho\sigma}$ from the p-form structure, and a quick check shows the determinant factors cancel out. After performing this, one can see that the dRGT action equals

$$\begin{aligned} S_{\text{mass}} &= -\frac{1}{2} M_{\text{Pl}}^2 m^2 \varepsilon_{abcd} \int \left[\frac{\beta_1}{3!} F^a \left(\Lambda^b{}_{b'} E^{b'}\right) \left(\Lambda^c{}_{c'} E^{c'}\right) \left(\Lambda^d{}_{d'} E^{d'}\right) \right. \\ &+ \left. \frac{\beta_2}{2!2!} F^a F^b \left(\Lambda^c{}_{c'} E^{c'}\right) \left(\Lambda^d{}_{d'} E^{d'}\right) + \frac{\beta_3}{3!} F^a F^b F^c \left(\Lambda^d{}_{d'} E^{d'}\right) \right]. \end{aligned} \tag{3.17}$$

Here one can read off that **the Lorentz Stückelberg field is an auxiliary field** (it has no derivative interactions), which means that we can solve for its equations of motion explicitly and integrate it out if one chooses!

3.2.3 Auxiliary Equations for Lorentz Stückelberg Field

I now return to the issue of solving for the Lorentz Stückelberg field's equations of motion. To begin, I take the 'deformed determinant' action and with my Lorentz Stückelberg mode, I have an action of the form

$$\begin{aligned} \mathcal{S}_{\text{mass}} &= \int c_0 \varepsilon_{abcd} (\Lambda E)^a (\Lambda E)^b (\Lambda E)^c (\Lambda E)^d \\ &+ c_1 \varepsilon_{abcd} F^a (\Lambda E)^b (\Lambda E)^c (\Lambda E)^d \\ &+ c_2 \varepsilon_{abcd} F^a F^b (\Lambda E)^c (\Lambda E)^d \\ &+ c_3 \varepsilon_{abcd} F^a F^b F^c (\Lambda E)^d \\ &+ c_4 \varepsilon_{abcd} F^a F^b F^c F^d \,, \end{aligned} \tag{3.18}$$

$$\Longrightarrow \mathcal{S}_{\text{mass}} = \int \sum_{n=0}^{4} \frac{(-1)^n}{n!} c_n \frac{\partial^n}{\partial \mu^n} \operatorname{Det}[\Lambda E - \mu\, F] \Big|_{\mu=0} \tag{3.19}$$

again with $F^a = \mathrm{d}\phi^a$ and the coefficients $\{c_i\}$ defined by (3.8). The cleverness of using the deformed determinant formulation alongside the Lorentz Stückelberg field is that I will be able to simply derive that the equations of motion for Λ do **not depend upon** μ! From here, one can see that the equation of motion of Λ are totally independent of the values of c_n parameters (i.e. m, α, β). Thus, despite all of the different possible ghost-free mass terms of dRGT, I will demonstrate that the Lorentz Stückelberg fields always take on the same form. To prove this, I make an important observation about the deformed determinant: *if* one varies the ordinary determinant $\mathrm{Det}[\Lambda E - \mu\, F]$, and one obtains an equation for $\Lambda^a{}_b$ such that μ drops out, then this solution is independent of any values the c_n's take on, since

$$\mathcal{L}_{\text{mass}} = \frac{(-1)^n}{n!} c_n \frac{\partial^n}{\partial \mu^n} \mathrm{Det}[\Lambda E - \mu\, F]\Big|_{\mu=0} . \tag{3.20}$$

That means if I can find a solution that is independent of μ and sets the variation of $\mathrm{Det}[\Lambda E - \mu\, F]$ to zero, then that solution works for all values of c_n by construction.[2] First, I need to vary equation (3.18), which at the level of matrices yields

$$\delta\, \mathrm{Det}[\Lambda E - \mu\, F] \quad = \quad \mathrm{Det}[\Lambda E - \mu\, F]\, \mathrm{Tr}\left[\delta\Lambda E\, (\Lambda E - \mu\, F)^{-1}\right] , \tag{3.21}$$

using well known variation rules for matrices. I pause for a moment to note that at the level of matrices, by construction of $g = E^T \eta E$ crucially only allows for Lorentz indices to contract together and spacetime indices to contract together. Since the determinant for a generic vielbein $X_\mu{}^a$ takes on the form

$$\mathrm{Det}[X] = \frac{1}{4!} \delta^{\mu_1 \cdots \mu_4}_{a_1 \cdots a_4} X_{\mu_1}{}^{a_1} \cdots X_{\mu_4}{}^{a_4} , \tag{3.22}$$

this remains respected in this expression. Returning to the variation, one can see that the variational term

$$\mathrm{Det}[\Lambda E - \mu\, F]\, \mathrm{Tr}\left[\delta\Lambda E\, (\Lambda E - \mu\, F)^{-1}\right] \tag{3.23}$$

can be made equal, after an insertion of unity $I = (\Lambda^{-1}\eta)(\eta\Lambda)$, to the equation

$$\mathrm{Det}[\Lambda E - \mu\, F]\, \mathrm{Tr}\left[(\delta\Lambda\Lambda^{-1}\eta)\eta(\Lambda E)\, (\Lambda E - \mu\, F)^{-1}\right] . \tag{3.24}$$

From here, the well-known Lorentz identity $\Lambda\eta\Lambda^T = \eta$ can be varied to obtain the equation

$$\left((\delta\Lambda)\Lambda^{-1}\eta\right)^T = -\left((\delta\Lambda)\Lambda^{-1}\eta\right) . \tag{3.25}$$

[2] One might ask if the solution space is non-unique, and if there are alternative solutions for Lorentz Stückelberg field. This question was taken up in [91], and the answer is affirmative, though they are unphysical. However, *a fortiori*, the existence of such a solution at minimum entails there is a solution which works for all possible mass terms.

This means that in (3.23) is an equation of the form

$$\begin{aligned} A^T = -A \quad &, \quad \mathrm{Tr}[AB] = 0 \\ \Leftrightarrow B^T \quad &= \quad B\,. \end{aligned} \tag{3.26}$$

Thus, following equation (3.25), I can infer

$$\begin{aligned} &\mathrm{Tr}\Big[(\delta\Lambda\Lambda^{-1}\eta)\eta(\Lambda E)\,(\Lambda E - \mu\,F)^{-1}\Big] = 0 \\ \Longrightarrow\ &\eta\Lambda E\,(\Lambda E - \mu\,F)^{-1} = \big(\eta\Lambda E\,(\Lambda E - \mu\,F)^{-1}\big)^T\,. \end{aligned} \tag{3.27}$$

Expanding, one finds

$$\begin{aligned} \eta(\Lambda E)\,(\Lambda E - \mu\,F)^{-1} &= \big[\eta(\Lambda E)\,(\Lambda E - \mu\,F)^{-1}\big]^T \\ &= \big[(\Lambda E)^T - \mu\,F^T\big]^{-1}(\Lambda E)^T\eta\,, \end{aligned} \tag{3.28}$$

After cross-multiplying the matrices, one obtains

$$(\Lambda E)^T\eta(\Lambda E - \mu\,F) = ((\Lambda E)^T - \mu\,F^T)\eta(\Lambda E)\,. \tag{3.29}$$

Finally, one can see that the terms proportional to μ appear on both the left and right of the equation, and thus are completely cancelled out in the solution. This completes the proof of the independence of μ within this solution, ergo the resulting equation solves all possible dRGT mass terms! After cancelling the μ-dependent terms, one obtains the simple equation

$$(\Lambda E)^T\eta F = F^T\eta(\Lambda E)\,. \tag{3.30}$$

I now move out of matrix notation and back into component form. Here, this equation takes the form

$$(\Lambda E)_\mu{}^a\eta_{ab}F_\nu{}^b = (\Lambda E)_\nu{}^a\eta_{ab}F_\mu{}^b. \tag{3.31}$$

With little effort, this equation can be seen to be equal to

$$g^{\mu\nu}(\Lambda E)_\mu{}^{[a}F_\nu{}^{b]} = 0\,. \tag{3.32}$$

This is not a new equation. It was discovered as a useful LLT gauge-fixing condition in [96, 97] (whose existence was further discussed in [85]) with $\Lambda = 1$ and as an imposed condition on the background and fluctuations. As such, this is often called the **Deser-van Nieuwenhuizen condition** (It is also referred to as the "symmetric vierbein condition", e.g. [90], for obvious reasons).[3] With this equation in hand, I can now do several powerful things that were not capable of being done before. Firstly, one may derive the *complete* decoupling limit of massive gravity, since the Lorentz Stückelberg scales non-trivially in the Λ_3 decoupling limit. I will derive the missing tower of helicity-1 interactions, and I will show that they can be resummed in a tractable form in the next subsection. Before I move onto the derivation of this action, I pause to make two important point about dRGT: firstly, how this relates to the explicit equivalence to the metric formalism; secondly, I will give an original elegant proof for why the metric formulation turns into the EC formulation upon gauge-fixing.

3.2.4 Equivalence to Metric dRGT

In dRGT massive gravity in metric formulation famously appears with a curious, seemingly unaccountable square root structure [43]. One might naturally ask why this structure appears, and it owes its existence to the DvN condition. The relationship between the square root structure and the DvN condition was first discussed in [85, 90], but a precise, direct equivalence without appealing to gauge-fixing remained elusive. I will show that the square root structure is obtained by explicitly solving for Λ in equation (3.30), which will generate the square root structure. Then, upon substituting back into the Einstein-Cartan formulation of the dRGT metric, I conclusively and explicitly obtain the metric formulation of dRGT. First, I take the equation of motion (3.30) for the Lorentz Stückelberg field

$$(\Lambda E)^T \eta F = F^T \eta (\Lambda E) \,,$$

and note that, after multiplying $\eta(E)^{-T}$ on the left, it equals

$$\eta \Lambda^T \eta F E^{-1} = \eta (E)^{-T} F^T \eta \Lambda \,. \tag{3.33}$$

This may be simplified after: Firstly, taking this equation and performing left multiplication with the RHS expression; secondly making use of the $\Lambda \eta \Lambda^T = \eta$ identity for the Lorentz group. After which, one finds

$$(\eta (E^T)^{-1} F^T \eta \Lambda)^2 = \eta (E^T)^{-1} F^T \eta \Lambda \eta \Lambda^T \eta F E^{-1} \tag{3.34}$$

$$= \eta (E^T)^{-1} F^T \eta F E^{-1} \,. \tag{3.35}$$

[3] Henceforth, "DvN condition."

The general solution for Λ now obviously involves a matrix square root. In general, existence and uniqueness of a square root of a matrix is not guaranteed; in some cases, one can choose a non-positive matrix roots, as has been considered in, e.g. [85, 109]. But in order to have a well-formed Minkowski reference metric upon which to take a decoupling limit, I self-consistently impose that the square root be defined via as the one stemming from the diagonal basis with all positive eigenvalues. Given this, I may write down the (positive) matrix square root as

$$\eta(E^T)^{-1}F^T\eta\Lambda = \sqrt{\eta(E^T)^{-1}F^T\eta F E^{-1}}\,. \tag{3.36}$$

I may use a similarity transformation (existence is all that is required here) identity, i.e.

$$E^{-1}\sqrt{Y}E = \sqrt{E^{-1}YE}\,. \tag{3.37}$$

to simplify my expression into

$$\begin{aligned} E^{-1}(\eta(E^T)^{-1}F^T\eta\Lambda)E &= E^{-1}\sqrt{\eta(E^T)^{-1}F^T\eta F E^{-1}}E \\ &= \sqrt{E^{-1}\eta(E^T)^{-1}F^T\eta F} \\ &= \sqrt{g^{-1}f}\,, \end{aligned} \tag{3.38}$$

using $g^{-1} = (E^T\eta E)^{-1} = E^{-1}\eta E^{-T}$. This derives the usual dRGT action in metric formulation, showing explicitly how one bridges the EC and metric formulations, and gives a simple reason for the appearance of the square root structure owing to the DvN condition. To go beyond this statement, I can see that the non-linear variable for the massive spin-2 mode is given by $E^a - F^a$. Finally, I determine the explicit solution of Λ to be

$$\Lambda = \eta(F^T)^{-1}g\sqrt{g^{-1}f}E^{-1}\,. \tag{3.39}$$

Re-writing this back into the usual component notation,

$$\Lambda^a{}_b = \eta^{ac}F^\mu_c g_{\mu\alpha}(\sqrt{g^{-1}f})^\alpha{}_\nu E^\nu_b\,. \tag{3.40}$$

I may now take my solution for Λ and place it into the combination $\mathrm{Det}[\Lambda E - \mu F]$

$$\begin{aligned}
\mathrm{Det}[\Lambda E - \mu\, F] &= \mathrm{Det}[\eta(F^T)^{-1}g\sqrt{g^{-1}f} - \mu\, F] &(3.41)\\
&= \mathrm{Det}\left[F\Big(F^{-1}\eta(F^T)^{-1}g\sqrt{g^{-1}f} - \mu I\Big)\right] &(3.42)\\
&= \mathrm{Det}[F]\mathrm{Det}[f^{-1}g\sqrt{g^{-1}f} - \mu I] &(3.43)\\
&= \sqrt{-\mathrm{Det}[g]}\mathrm{Det}[I - \mu\sqrt{g^{-1}f}]\,, &(3.44)
\end{aligned}$$

repeatedly employing the relations $\mathrm{Det}[AB] = \mathrm{Det}[A]\mathrm{Det}[B]$ and $\mathrm{Det}[\sqrt{A}] = \sqrt{\mathrm{Det}[A]}$. When this is substituted back into the action, it yields

$$\begin{aligned}
\mathcal{S}_{\text{mass}} &= \int \mathrm{d}^4x \sum_{n=0}^{4} \frac{(-1)^n}{n!} c_n \frac{\partial^n}{\partial\mu^n} \mathrm{Det}[\Lambda E - \mu\, F]\Big|_{\mu=0} &(3.45)\\
&= \int \mathrm{d}^4x\, \sqrt{-g} \sum_{n=0}^{4} \frac{(-1)^n}{n!} c_n \frac{\partial^n}{\partial\mu^n} \mathrm{Det}[I - \mu\sqrt{g^{-1}f}]\Big|_{\mu=0}\,. &(3.46)
\end{aligned}$$

By construction, this immediately regenerates the symmetric polynomials of $I - \sqrt{g^{-1}f}$, which is the metric dRGT action (2.91), thus I have proven the equivalence constructively!

3.2.5 dRGT and the Deser-van Nieuwenhuizen Gauge

For completeness, I show how the DvN gauge-fixing, plus fixing diffs to unitary gauge, for veilbeins defined as $F^a = I^a$ and $\Lambda = 1$, returns one back to the Einstein-Cartan formulation of Hinterbichler and Rosen [90]. I fix unitary gauge (i.e. $\Lambda = 1$) and the DvN condition (3.30) now becomes a constraint imposed upon the theory of the form

$$(E^{-1}\eta F) = (E^{-1}\eta F)^T\,,$$

Once I fix the diffs to unitary gauge (i.e. $\phi^\mu = x^\mu$), I have $F^a = I^a$ and so

$$(\eta E)^T = (\eta E)\,. \tag{3.47}$$

This is the “symmetric vierbein” condition $E_{\mu a} = E_{a\mu}$ of [90]. This action obviously leads back the EC formulation. Alternatively (but ultimately equivalently), one could consider forcing the DvN gauge by fiat

$$(E^{-1}\eta F) = (E^{-1}\eta F)^T\,, \tag{3.48}$$

which trivially implies $\Lambda = 1$; however, one should worry whether it is consistent to set this gauge. To verify this at the level of the action, I take the following the usual definitions $g^{-1} = E^{-1}\eta E^{-T}$ and $f = F^T\eta F$, and the easily verifiable identities $E^{-1} = \eta E^{-T}\eta$ and $F = \eta F^T\eta$. Applying these to the square root structure, one can see that they entail:

$$\begin{aligned} \sqrt{g^{-1}f} &= \sqrt{E^{-T}\eta E^{-1}F\eta F^T} && (3.49)\\ &= \sqrt{E^{-T}\eta F^T E^{-T}\eta F^T} && (3.50)\\ &= E^{-T}\eta F^T\,. && (3.51) \end{aligned}$$

Obviously, for the same reason as the previous subsubsection, one can see that

$$\begin{aligned} S_{\text{metric}} &\equiv -\frac{M_{\text{Pl}}^2 m^2}{2}\int \sqrt{-g}\sum \beta_i\, e_i\left(\sqrt{g^{-1}f}\right) \\ &= -\frac{M_{\text{Pl}}^2 m^2}{2}\int \det E \sum \beta_i\, e_i(E^{-T}\eta F^T) \equiv S_{\text{vierbein}}\,. \end{aligned} \tag{3.52}$$

And thus I have provided explicit vielbein formulation from the metric formulation upon gauge-fixing, as promised. I reiterate that the Stückelberg formulation of the local Lorentz symmetry greatly clarifies the role between the DvN condition and the square root structure of dRGT massive gravity, and indeed the bridge between the theory in Einstein-Cartan variables and metric variables.

3.3 Surviving Helicity-1 Interactions

3.3.1 Scaling of Lorentz Stückelberg

Following the analysis in section 3.1 and the form of the Lorentz Stückelberg field from section 3.2, the fields obey scalings of the form

$$\begin{aligned} E_\mu{}^a &= \delta^a_\mu + \frac{1}{2M_{\text{Pl}}}e^a_\mu \\ \Lambda^a{}_b &= e^{\hat\omega^a{}_b} = I^a + \hat\omega^a{}_b + \frac{1}{2}\hat\omega^a{}_c\hat\omega^c{}_b + \cdots \\ \partial_\mu\phi^a &= \partial_\mu\left(x^a + \frac{B^a}{mM_{\text{Pl}}} + \frac{E^{a\nu}\partial_\nu\pi}{\Lambda_3^3}\right) \\ &= I^a + \frac{1}{mM_{\text{Pl}}}\mathrm{d}B^a + \Pi^a \end{aligned} \tag{3.53}$$

where $\hat{\omega}$ is the yet-to-be-fixed-scale for the Lorentz auxiliary field (which by construction must be anti-symmetric $\hat{\omega}_{ab} = -\hat{\omega}_{ba}$ to form a Lorentz transformation), and $\Pi^a = \frac{1}{\Lambda_3^3}\mathrm{d}(E^{a\nu}\partial_\nu \pi)$. To reiterate, the Λ_3 limit scales the parameters as

$$\begin{aligned} M_{\mathrm{Pl}} &\to \infty\,, \\ m &\to 0\,, \\ \text{while holding } \Lambda_3 &= (m^2 M_{\mathrm{Pl}})^{\frac{1}{3}} = \text{ const}\,. \end{aligned} \tag{3.54}$$

In order to evaluate the decoupling limit, I will need to fix the scaling properties of $\hat{\omega}$. By inspection and with some thought, one will find that the only non-pathological (i.e. forcing it to not have terms that diverge in the decoupling limit) is the choice

$$\hat{\omega}^a{}_b = \frac{\omega^a{}_b}{m M_{\mathrm{Pl}}}\,. \tag{3.55}$$

Specifically, I will have a single schematic operator[4] which diverges like $\sim \frac{1}{m}$ in the action

$$\mathcal{L}_{\mathrm{mass}} \supset m^2 M_{\mathrm{Pl}}^2 \left(\frac{1}{m M_{\mathrm{Pl}}}\omega\right)\left(\frac{\partial^2\pi}{m^2 M_{\mathrm{Pl}}}\right)\left(I + \frac{\partial^2\pi}{m^2 M_{\mathrm{Pl}}}\right)^2\,, \tag{3.56}$$

but is identically zero, owing to anti-symmetry in ω and symmetry in $\partial^2\pi$ as they sit inside a symmetric polynomial. From here, I have two avenues for determining the equation of motion for the Lorentz Stückelberg field: Firstly, by varying for ω its EOM in the decoupling limit action. The second is applying the decoupling limit to the non-linear equation of motion (i.e. the Deser-van Nieuwenhuizen condition) and consistently truncating to the leading order piece. Although the second is the most immediate result, I will show both since: Firstly, it gives a nice sanity check on both methods. Secondly, although taking the limit directly on the EOM is far simpler, one will need to see how to take the decoupling limit of the mass terms to construct the action anyway, it is good practice to see how ω enters into the action in the decoupling limit.

3.3.2 Solution via Decoupling Limit Action

Using the argument from subsection 3.2.3, I know that I can pick any of the mass term in dRGT to determine the equations of motion for the Lorentz Stückelberg field. For concreteness, I choose the term proportional to β_1 in symmetric polynomial

[4] If this is not manifestly clear to the reader, these terms will be explicitly derived in the following section.

notation

$$-\beta_1 \frac{m^2 M_{\rm Pl}^2}{12} \delta_4 \left(I + \frac{\partial B}{m M_{\rm Pl}} + \Pi\right) \left(I^3 + 3\left(\frac{I^2\omega}{m M_{\rm Pl}} + \frac{I\omega^2}{m^2 M_{pl}^2}\right) + 3\frac{I^2\omega\cdot\omega}{2m^2 M_{pl}^2}\right) \tag{3.57}$$

where I have chosen the notation $\Pi = \frac{\partial^2 \pi}{{\Lambda_3}^3}$, since none of this is affected by the decoupling limit procedure, and I is the Kronecker delta/identity matrix. Now, this action simplifies at leading order into the form

$$\begin{aligned} &= -\beta_1 \frac{m^2 M_{\rm Pl}^2}{12} \delta_4 \left(\partial B I^2 \omega + (I + \Pi)\left(I\omega^2 + \frac{I^2\omega\cdot\omega}{2}\right)\right) \\ &\quad + \mathcal{O}\left(\frac{1}{M_{\rm Pl}^{1/2}}\right) . \end{aligned} \tag{3.58}$$

For the reader's convenience, I note that this is equivalent to the component notation

$$= -\beta_1 \frac{1}{4} \delta^{\mu\nu\rho\sigma}_{abcd} \left(\partial_\mu B^a \delta^b_\nu \delta^c_\rho \omega^d{}_\sigma + (\delta + \Pi)^a_\mu \left(\delta^b_\nu \omega^c_\rho \omega^d_\sigma + \frac{\delta^b_\nu \delta^c_\rho \omega^d{}_c \omega^c{}_\sigma}{2}\right)\right) . \tag{3.59}$$

Here I also remind the reader that since the expansion is off of flat space, the vielbein a, b, ... are the same as μ, ν, ... and thus may be contracted. Now I vary the action and obtain the equation of motion for $\omega^a{}_b$. Firstly, since $\omega \in \mathfrak{so}(1,3)$, I have $\omega_{ab} = -\omega_{ba}$. Ergo, my equations of motions follow the anti-symmetrisation rule

$$\begin{aligned} \frac{\delta \mathcal{S}}{\delta \omega^{ab}} \delta\omega^{ab} &= A_{[ab]} \delta\omega^{ab} \\ \Longrightarrow \frac{\delta \omega_{ab}}{\delta \omega_{cd}} &= \frac{1}{2} \delta^{cd}_{ab} . \end{aligned} \tag{3.60}$$

Using this variational scheme, the equations of motion come out as a symmetric-polynomial equation of the form

$$\delta_4 \Big(GI\frac{\delta\omega}{\delta\omega}\eta + (I + \Pi)\Big(2\omega\frac{\delta\omega}{\delta\omega} + I\frac{\delta\omega}{\delta\omega}\cdot\omega + \omega\cdot\frac{\delta\omega}{\delta\omega}I\Big)\Big) = 0 , \tag{3.61}$$

with normal matrix notation $A \cdot B = A^a{}_b B^b{}_c$. Before reinserting the indices and evaluating, I break this up into specific terms and evaluate them separately for simplicity. I choose cut up the problem into these 4 parts:

$$\varepsilon\left(\underbrace{GI\frac{\delta\omega}{\delta\omega}\eta}_{(A)}+(I+\Pi)\Big(\underbrace{2\omega\frac{\delta\omega}{\delta\omega}}_{(B)}+\underbrace{I\frac{\delta\omega}{\delta\omega}\cdot\omega}_{(C)}+\underbrace{\omega\cdot\frac{\delta\omega}{\delta\omega}\delta}_{(D)}\Big)\right)=0\,. \tag{3.62}$$

A simple, direct evaluation leads to

$$\begin{aligned}
(A) &= \delta^{\mu\nu\rho}_{abc}\delta^a_\rho\partial_\nu B^b\delta^{a\mu'}_{\alpha\beta}\eta_{\mu'\mu} \\
&= 2G_{\alpha\beta} && (3.63)\\
(B) &= 2\delta^{\mu\nu\rho}_{abc}\omega^a{}_\mu\delta^{b\nu}_{\alpha\beta}\eta_{\nu\nu'}(\delta+\Pi)^c{}_\rho \\
&= 4\left[(2+[\Pi])\omega_{\alpha\beta}+\omega_{a\alpha}\Pi^a{}_\beta-\omega_{a\beta}\Pi^a{}_\alpha\right] && (3.64)\\
(C) &= \delta^{\mu\nu}_{ab}\delta^{ac}_{\alpha\beta}\omega_{c\mu}(\delta+\Pi)^b_\nu \\
&= -2\omega_{\alpha\beta}\left(3+[\Pi]\right)-\left(\omega_{\beta a}\Pi^a{}_\alpha-\omega_{\alpha\beta}\Pi^a_\beta\right) && (3.65)\\
(D) &= \delta^{\mu\nu}_{ab}\omega^a{}_\gamma\delta^{\gamma\mu'}_{\alpha\beta}\eta_{\mu\mu'}(\delta+\Pi)^b{}_\nu \\
&= (C) && (3.66)
\end{aligned}$$

where I have used the trace-notation of $[\Pi]=(\Pi^a{}_a)$, and the field strength notation $G_{\alpha\beta}=\partial_\alpha B_\beta-\partial_\beta B_\alpha$. Together, they sum to the full equation of motion:

$$\begin{aligned}
-(A) &= (B)+(C)+(D) \\
\Longrightarrow G_{ab} &= 2\omega_{ab}-\left(\omega_{ca}\Pi^c_b-\omega_{cb}\Pi^c_a\right). && (3.67)
\end{aligned}$$

Which gives the equation of motion for the Lorentz Stückelberg field in the decoupling limit.

3.3.3 Solution via Limit of DvN Condition

I now proceed with the method that essentially immediately determines the equation of motion. If I take the full equation of motion for Λ (3.31) that I discovered in subsection 3.2.3, and combine it with the scaling limits in equation (3.55). Then, finally, recalling that $F^a=\mathrm{d}\phi^a$, I finally arrive at the equation

$$E^\mu{}_a\left(e^{\hat{\omega}}\right)_{bc}\partial_\mu\phi^c=E^\mu{}_b\left(e^{\hat{\omega}}\right)_{ac}\partial_\mu\phi^c\,. \tag{3.68}$$

Expanding out with the decoupling parameters, at leading order to arrive at

$$\omega_b{}^c\left(\delta_{ca}+\Pi_{ca}\right)+\partial_a B_b=\omega_a{}^c\left(\delta_{cb}+\Pi_{cb}\right)+\partial_b B_a+\mathcal{O}(m)\,. \tag{3.69}$$

With a bit of trivial algebra, this equation can be seen as the final equation

$$\begin{aligned}\omega_{ba} + \omega_{bc}\Pi^c{}_a + \partial_a B_b &= \omega_{ab} + \omega_{ac}\Pi^c{}_b + \partial_b B_a \\ \implies G_{ab} &= 2\omega_{ab} - \left(\omega_{ca}\Pi^c{}_b - \omega_{cb}\Pi^c{}_a\right) .\end{aligned} \tag{3.70}$$

This gives one a faster, cleaner derivation of this equation of motion!

3.3.4 Integral Form of the Lorentz Stückelberg Field

Now that I have the equation of motion for the decoupling limit, I can solve for ω. This is one of the more non-trivial uses of the Lorentz Stückelberg field. In a sense, this variable contains all of the interactions between the helicity-0 mode π and the helicity-1 mode B_μ. The solution of this equation is guaranteed as it is 6 equations and linear (i.e. with a 6×6 matrix linear equation); however, it would be nice if there was a way to quickly get at an elegant, if not simple, solution to this equation. I will use the following observation: Since Π_{ab} is a symmetric, real matrix, then I can always use a global Lorentz transformation to diagonalise Π_{ab} *at one point*. This leads to

$$\Pi_{ab} = 0 \text{ if } \; a \neq b \, . \tag{3.71}$$

I may invert the equations at that point, since it has the much simplified form

$$G_{ab} = \left(2 + \Pi^a_a + \Pi^b_b\right)\omega_{ab} \tag{3.72}$$

$$\implies \omega_{ab} = \left(\frac{G_{ab}}{2 + \Pi^a_a + \Pi^b_b}\right) . \tag{3.73}$$

Note that I am explicitly **not** making use of Einstein summation convention. The second clever observation is to note that this solution followed from an equation who started out life in a totally covariant form. Thus, if I can find a formal solution which, upon diagonalization at any point in Minkowski, leads to the form above, I have found the complete solution. The task now becomes finding an expression that reduces to (3.72) after diagonalizing Π. A manifestly Lorentz-covariant solution can be found by using a usual Schwinger-parameterisation integral technique. Equation (3.72) is equivalent to the integral equation

$$\omega_{ab} = \int_0^\infty \mathrm{d}u \, e^{-u(2+\Pi_a{}^a+\Pi^b{}_b)} G_{ab} \, . \tag{3.74}$$

This can be easily re-expressed as a matrix equation

$$\omega_{ab} = \int_0^\infty \mathrm{d}u\, e^{-2u} e^{-u\Pi_a{}^{a'}} G_{a'b'} e^{-u\Pi^{b'}{}_b}\,, \tag{3.75}$$

which trivially recovers the old equation when Π is diagonal. Therefore, this is the formal solution for ω_{ab}! I now move onto a constructing the complete action of dRGT massive gravity in the decoupling limit.

3.4 The Complete dRGT Action in the Decoupling Limit

It the simplest way to derive the complete tower of interactions of ghost-free massive gravity in the decoupling limit is to return to the action for dRGT in symmetric polynomial form:

$$\mathcal{S}_{\text{mass}} \;=\; \int \mathrm{d}^4x\, M_{\text{Pl}}^2 m^2 \delta_4 \Big[(\Lambda E - F)^2(\Lambda E)^2 + \alpha(\Lambda E - F)^3(\Lambda E) + \beta(\Lambda E - F)^4\Big]\,. \tag{3.76}$$

Upon substituting in these terms (they are the highest order in terms that contribute to the action)

$$\begin{aligned} \Lambda^a{}_b E_\mu{}^b &= \delta^a_\mu + \frac{1}{M_{\text{Pl}}} h_\mu{}^a + \frac{1}{m M_{\text{Pl}}}\omega^a{}_b + \frac{1}{2m^2 M_{\text{Pl}}^2}\omega^a{}_c\omega^c{}_b \\ \Longrightarrow (\Lambda E) &= I + \frac{h}{M_{\text{Pl}}} + \frac{1}{m M_{\text{Pl}}}\omega + \frac{1}{m^2 M_{\text{Pl}}^2}\omega\cdot\omega \end{aligned} \tag{3.77}$$

$$\begin{aligned} F^a = &= \partial_\mu\left(x^a + \frac{B^a}{m M_{\text{Pl}}} + \frac{\partial^a \pi}{\Lambda_3^3}\right) \\ \Longrightarrow F &= I + \frac{1}{m M_{\text{Pl}}}\partial B + \Pi\,. \end{aligned} \tag{3.78}$$

Upon substituting these relations into the above action (3.76), one arrives at

$$\begin{aligned}\mathcal{S}_{\rm mass} \;=\; & \int \mathrm{d}^4x\,\frac{1}{2}M_{\rm Pl}^2 m^2\delta_4\Big[\left(\frac{h}{M_{\rm Pl}}-\Pi+\frac{(\omega-\partial B)}{mM_{\rm Pl}}+\frac{\omega\cdot\omega}{2m^2M_{\rm Pl}^2}\right)^2 \\ & \times\left(I+\frac{h}{M_{\rm Pl}}+\frac{\omega}{mM_{\rm Pl}}+\frac{\omega\cdot\omega}{2m^2M_{\rm Pl}^2}\right)^2\Big] \\ & +\alpha M_{\rm Pl}^2 m^2\delta_4\Big[\left(\frac{h}{M_{\rm Pl}}-\Pi+\frac{(\omega-\partial B)}{mM_{\rm Pl}}+\frac{\omega\cdot\omega}{2m^2M_{\rm Pl}^2}\right)^3 \\ & \times\left(I+\frac{h}{M_{\rm Pl}}+\frac{\omega}{mM_{\rm Pl}}+\frac{\omega\cdot\omega}{2m^2M_{\rm Pl}^2}\right)\Big] \\ & +\beta M_{\rm Pl}^2 m^2\delta_4\left[\left(\frac{h}{M_{\rm Pl}}-\Pi+\frac{(\omega-\partial B)}{mM_{\rm Pl}}+\frac{\omega\cdot\omega}{2m^2M_{\rm Pl}^2}\right)^4\right]\,. \end{aligned} \tag{3.79}$$

Although somewhat algebraically lengthy, the math here is simple owing to isomorphism between ordinary real algebra and variables inside of a symmetric polynomial; crucially, multinomial expansion rules are the same, e.g.

$$(a+b)^n = a^n + n\,a^{n-1}\,b + \cdots + n\,a\,b^{n-1} + b^n\,. \tag{3.80}$$

It is also useful to outright neglect any term which comes in at too high of an order (e.g. terms containing factors of ω^{2+n}, $h\omega^{1+n}$, h^{2+n} for $n \geq 1$) and to recall that the pure ∂B and pure Π terms are total derivatives. All together, one quickly arrives at a simple lemma that the only terms that need to be considered are the surviving terms, which given the overall common factor $m^2 M_{\rm Pl}^2$ are terms that inside the parenthesis

(A.) Have a scale factor $\frac{1}{m^2 M_{\rm Pl}^2}$.

(B.) Have a scale factor $\frac{1}{M_{\rm Pl}}$.

Noting all of those, one is quickly lead to a Lagrangian with a leading-order expansion of the form

$$
\begin{aligned}
&\frac{1}{2}M_{\rm Pl}^2 m^2\delta_4\Big[-4\Pi\frac{\omega^2-\partial B\omega}{m^2M_{\rm Pl}^2}I-\frac{\omega\cdot\omega}{m^2M_{\rm Pl}^2}\Pi I^2+\frac{\Pi^2(\omega^2+\omega\cdot\omega I)}{m^2M_{\rm Pl}^2}\\
&+2\frac{\omega^2-\partial B\omega}{m^2M_{\rm Pl}^2}I^2+2h\frac{-\Pi I^2+\Pi^2 I}{M_{\rm Pl}}\Big]\\
&+\alpha M_{\rm Pl}^2 m^2\delta_4\Big[\frac{-\Pi^3\omega\cdot\omega}{2m^2M_{\rm Pl}^2}+3\Pi^2\frac{\omega^2-\partial B\omega}{m^2M_{\rm Pl}}-3\Pi\frac{(\omega-\partial B)^2 I}{m^2M_{\rm Pl}^2}\\
&+3\frac{\omega\cdot\omega}{m^2M_{\rm Pl}^2}\Pi^2 I+h\frac{3\Pi^2 I-\Pi^3}{M_{\rm Pl}}\Big]\\
&+\beta M_{\rm Pl}^2 m^2\delta_4\Big[-\frac{3}{2}\Pi^3\frac{\omega\cdot\omega}{m^2M_{\rm Pl}^2}+6\Pi^2\frac{(\omega-\partial B)^2}{m^2M_{\rm Pl}^2}-3h\frac{\Pi^3}{M_{\rm Pl}}\Big]+\mathcal{O}\left(\frac{1}{M_{\rm Pl}}\right).
\end{aligned}
\tag{3.81}
$$

After canceling the common scale factors, I may simplify the derivatives further as follows: All interactions containing a single factor ∂B always come with a factor of anti-symmetric terms, and in this context $\partial_a B^b = \frac{1}{2}G_a{}^b$; all terms containing more than one factor of ∂B are total derivatives! Therefore, one may linearise ∂B and apply the aforementioned substitution. Upon which, one obtains the Lagrangian

$$
\begin{aligned}
&\frac{1}{2}\delta_4\Big[\Pi(-4\omega^2+2G\omega)I-\omega\cdot\omega\,\Pi I^2+\Pi^2\left(\omega^2+\omega\cdot\omega I\right)\\
&+(2\omega^2-G\omega)I^2+2\Lambda_3^3 h(-\Pi I^2+\Pi^2 I)\Big]\\
&+\alpha\delta_4\Big[-\Pi^3\omega\cdot\omega+3\Pi^2\left(\omega^2-\frac{1}{2}G\omega\right)-3\Pi(\omega^2-G\omega)I\\
&+3\omega\cdot\omega\Pi^2 I+\Lambda_3^3 h(3\Pi^2 I-\Pi^3)\Big]\\
&+\beta\delta_4\Big[-\frac{3}{2}\Pi^3\omega\cdot\omega+6\Pi^2(\omega^2-G\omega)-3\Lambda_3^3 h\Pi^3\Big].
\end{aligned}
\tag{3.82}
$$

Finally, one may collect like terms and re-insert indices to get

$$
\begin{aligned}
\mathcal{S}_{\mathrm{dRGT}} \;=\; & \int \mathrm{d}^4x \, -\frac{1}{2} h_\mu{}^\alpha \delta^{\mu\nu\rho}_{\alpha\beta\gamma} \partial_\nu \partial^\beta h_\rho{}^\gamma + \Lambda_3^3 h_\mu{}^\alpha \left(X_\alpha{}^\mu + (1+3\alpha) Y_\alpha{}^\mu + (\alpha+3\beta) Z_\alpha{}^\mu \right) \\
& + \frac{1}{2} \delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} \Big[\Pi_\mu{}^\alpha (-4\omega_\nu{}^\beta \omega_\rho{}^\gamma + 2G_\nu{}^\beta \omega_\rho{}^\gamma) \delta^\delta_\sigma - \omega_\mu{}^\lambda \omega_\lambda{}^\alpha \Pi_\nu{}^\beta \delta^\gamma_\rho \delta^\delta_\sigma \\
& + \Pi_\mu{}^\alpha \Pi_\nu{}^\beta \left(\omega_\rho{}^\gamma \omega_\sigma{}^\delta + \omega_\rho{}^\lambda \omega_\lambda{}^\gamma \delta_\sigma{}^\delta \right) + (2\omega_\rho{}^\gamma \omega_\sigma{}^\delta - G_\rho{}^\gamma \omega_\sigma{}^\delta) \delta^\gamma_\rho \delta^\delta_\sigma \Big] \\
& + \alpha \delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} \Big[-\Pi_\mu{}^\alpha \Pi_\nu{}^\beta \Pi_\rho{}^\gamma \omega_\sigma{}^\lambda \omega_\lambda{}^\delta + 3\Pi_\mu{}^\alpha \Pi_\nu{}^\beta \left(\omega_\rho{}^\gamma \omega^\delta_\sigma - \frac{1}{2} G_\rho{}^\gamma \omega_\sigma{}^\delta \right) \\
& - 3\Pi_\mu{}^\alpha \left(\omega_\nu{}^\beta \omega^\gamma_\rho - G_\nu{}^\beta \omega^\gamma_\rho \right) \delta^\delta_\sigma + 3\omega_\mu{}^\lambda \omega_\lambda{}^\alpha \Pi_\nu{}^\beta \Pi_\rho{}^\gamma \delta^\delta_\sigma \Big] \\
& + \beta \delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} \Big[-\frac{3}{2} \Pi_\mu{}^\alpha \Pi_\nu{}^\beta \Pi_\rho{}^\gamma \omega_\sigma{}^\lambda \omega_\lambda{}^\delta + 6\Pi_\mu{}^\alpha \Pi_\nu{}^\beta (\omega_\rho{}^\gamma \omega^\delta_\sigma - G_\rho{}^\gamma \omega^\delta_\sigma) \Big]
\end{aligned}
\tag{3.83}
$$

where $h_\mu{}^\alpha$ is the usual spin-2 graviton perturbation $e_\mu{}^\alpha = \delta^\alpha_\mu + \frac{1}{M_{\mathrm{Pl}}} h_\mu{}^\alpha$, and I remind the reader of the known relations

$$\omega_{ab} = \int_0^\infty \mathrm{d}u \; e^{-2u} e^{-u\Pi_a{}^{a'}} G_{a'b'} e^{-u\Pi^{b'}{}_b} \tag{3.84}$$

$$G_{ab} = \partial_a B_b - \partial_b B_a \tag{3.85}$$

$$\Pi_{ab} = \frac{\partial_a \partial_b \pi}{\Lambda_3^3} \, . \tag{3.86}$$

The "transverse tensors", which I denote as $X^\mu{}_\alpha$, $Y^\mu{}_\alpha$, and $Z^\mu{}_\alpha$ are defined via [42],

$$X^\mu{}_\alpha = -\delta^{\mu\nu}_{\alpha\beta} \Pi_\nu{}^\beta$$

$$Y^\mu{}_\alpha = \frac{1}{2!} \delta^{\mu\nu\rho}_{\alpha\beta\gamma} \Pi_\nu{}^\beta \Pi_\rho{}^\gamma \tag{3.87}$$

$$Z^\mu{}_\alpha = -\frac{1}{3!} \delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} \Pi_\nu{}^\beta \Pi_\rho{}^\gamma \Pi_\sigma{}^\delta \, . \tag{3.88}$$

3.5 Comments on the Helicity-1 Modes

Although my resummation for the decoupling limit Stückelberg mode given in (3.75) is not the simplest expression, it does simplify the infinite tower of interactions considerably and gives a systematic, order-by-order approach for constructing the interactions. This is considerably more straightforward than expanding a matrix square root (which was done with much greater effort in [104, 110]). The results in these papers follow from much simpler, more systematic nature of (3.75). With minimal effort, one can now go to much higher order. For instance, given a background that has $\bar{\Pi}$, I can first expand the expression as a series centred around some exact solution $\bar{G}_{ab}$, i.e.

$$\bar{G}_{ab} = 2\bar{\omega}_{ab} - (\bar{\omega}_{ca}\bar{\Pi}^c_b - \bar{\omega}_{cb}\bar{\Pi}^c_a)\,, \tag{3.89}$$

Then the interactions (or what will become interactions when this is substituted back into the decoupling limit action) for fluctuations may be systematically derived via the same Schwinger parameterisation technique

$$\begin{aligned} \delta G_{ab} &= 2\delta\omega_{ab} - (\delta\omega_{ca}\bar{\Pi}^c_b - \delta\omega_{cb}\bar{\Pi}^c_a) - (\bar{\omega}_{ca}\delta\Pi^c_b - \bar{\omega}_{cb}\delta\Pi^c_a) && (3.90)\\ \Longrightarrow \delta\omega &= \int_0^\infty \mathrm{d}u\, e^{-2u} e^{-u\bar{\Pi}_a{}^{a'}} (\delta G - \bar{\omega}\delta\Pi + \delta\Pi\bar{\omega})_{a'b'} e^{-u\bar{\Pi}^{b'}{}_b} && (3.91)\\ \delta\omega &= \sum_{n,m} \frac{(n+m)!}{2^{1+n+m}n!m!}(-1)^{n+m}\,\bar{\Pi}^n(\delta G - \bar{\omega}\delta\Pi + \delta\Pi\bar{\omega})\bar{\Pi}^m\,. && (3.92) \end{aligned}$$

Subsequent to this work[5], these kinds of constructions have been central for analysing the helicity-1 modes. This lead to various interesting breakthroughs about their properties, such as a further understanding of the superluminalities [111], extensions of the decoupling limit to other theories [112, 113], all of which was performatively unattainable prior to the development of my methods. Shortly after being published, my systematic analyses allowed for the discovery of the mysterious Galileon duality [112–116], which still remains poorly understood but seems likely to be important to massive gravity. Additionally, my work also sheds new light onto the no-go proof found in [117] for partially massless gravity, because the proof of the decoupling limit demonstrates that the helicity-0 and helicity-1 mode cannot be made to decouple if one picks the dRGT mass terms[6].

[5] Originally published in [98].
[6] I would like to thank C. de Rham for observing this.

Chapter 4

Review of Dimensional Deconstruction

4.1 Massive Theories from Higher-Dimensional Massless Theories

I now review Dimensional Deconstruction, originally developed in [108, 118] and further developed in [119–124]. It is a major technique that will allow one to create massive gauge theories. Although originally constructed for Yang-Mills theories and BSM physics, Dimensional Deconstruction (henceforth 'deconstruction') also shows how to derive dRGT gravity from Einstein-Cartan gravity in one dimension higher. In the following sections, I shall make repeated use of (and find extensions of) deconstruction; therefore, I shall carefully discuss the powers and limitations of this technique. To give a rough overview, the essential idea is to take a *massless* theory of a spin-J field living in $(d+1)$-dimensions, forcibly breaking the spacetime symmetry down to d-dimensions, to generate a theory of a *massive* spin-J in d-dimensions. One breaks the spacetime symmetry by demoting integrals and derivatives in this continuous dimension into linear maps on "site basis" by hand; one may geometrically interpret this as discretising a continuous dimension. The simplest case possible is to show how one may "deconstruct over 1-site", where one takes a massless 5-D scalar and deforms it into a massive 4-D scalar. Note that in the literature, typically a 1-site theory is taken, by definition, to be straight dimensional reduction and thus only contain a massless mode. I reserve the term "1-site deconstruction" to apply explicitly to generating a single massive mode. For now, suppose that I write down the action of a massless scalar in 5-D. At the level of the action, this is given by

$$\mathcal{S}[\varphi] = \int \mathrm{d}^5 X \; -\frac{1}{2}\partial_M \varphi \partial^M \varphi \,. \tag{4.1}$$

If one separates out one of the spatial dimensions, i.e. $X^M = (x^\mu,\, y)$, then one can split the action up into

$$\mathcal{S}[\varphi] = \int \mathrm{d}^4 x \mathrm{d}y \; -\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2}\partial_y \varphi \partial_y \varphi \,. \tag{4.2}$$

In turn, one may then explicitly break the structure of this extra dimension by

(1) Deforming the y-derivatives into $\partial_y \varphi \to m\varphi$.

(2) Deforming y-integrals to $\int \mathrm{d}y\, f(x^\mu, y) \cdots g(x^\mu, y) \to \frac{1}{m} f_n(x^\mu) \cdots g_n(x^\mu)$.

(3) I absorb this normalisation into the fields themselves, since bosons in 5-D have different scales than they do in 4-D, which this factor of $\frac{1}{m}$ accounts for. Thus I ignore it and give the fields their 4-D scaling dimension, which is equivalent to $\varphi \to \sqrt{m}\varphi = \varphi_{\text{4-D}}$.

The deconstucted action is now simply a 4-d massive scalar

$$\int \mathrm{d}^4 x \; -\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 \,. \tag{4.3}$$

At this level, this may appear just to be a cute trick; however, this procedure proves remarkably endurable, including for interacting gauge theories.

4.1.1 Embeddings of Reps

From a completely group-theoretic perspective, it is easy to see that the Poincaré groups $\mathrm{ISO}(1, d)$ for Minkowski spaces $\mathbb{M}^{1,d}$ of differing dimension are nested within each other:

$$\mathrm{ISO}(1,1) \subset \mathrm{ISO}(1,2) \subset \mathrm{ISO}(1,3) \subset \cdots \subset \mathrm{ISO}(1,d) \,. \tag{4.4}$$

Therefore, if one looks at representations of $\mathrm{ISO}(1, d+1)$, one expects that they necessarily contain all of information about the representations of $\mathrm{ISO}(1, d)$. For instance, this is famously how Kaluza-Klein compactification over an 1-dimensional internal manifold works [125, 126]. The compactification is realised by exactly decomposing an $\mathrm{ISO}(1, d)$ representation into an infinite tower of $\mathrm{ISO}(1, d-N)$ representations. Typically, it will contain a single massless rep combined with an infinite tower of massive modes; this forces the dynamics to be consistent. For example, purely at the level of representations, one then can decompose a 5-D massless, spin-J rep into a set of 4-D massive and massless spin-J reps, and since one is only at the level of representations there are no concerns over the dynamics or the Lagrangian. In a bra-ket notation, this can be seen through the Casimir invariant on the single-particle state

$$P_M P^M |k_N\rangle = \left(P_\mu P^\mu + P_y P^y\right)|k_\mu,\, k_y\rangle = 0\,. \tag{4.5}$$

If one takes the amount of momentum in P_y and define the state to have the $E = m$ and rotate the direction of motion to lie in the y direction, one can see that massive reps in 4-D naturally fall out of massless 5-D reps:

$$\begin{aligned} \left(P_\mu P^\mu + m^2\right)|k_\mu,\, m\rangle &= 0 \\ \Leftrightarrow P_\mu P^\mu |k_\mu\rangle &= -m^2 |k_\mu\rangle \end{aligned} \tag{4.6}$$

where the state $|k_\mu\rangle$ is now manifestly a massive single-particle state in 4-D with mass m. Following the obvious condition that $P_y^2 = m^2$ implies that one may generalise this to account for multiple sites, in which case the promotion is $P_y \to M_{IJ}$ (i.e. a linear operator on the site basis), and then one has (using Einstein summation convention for site basis indices)

$$\begin{aligned} \left(P_\mu P^\mu + P_y^2\right)|k_\mu,\, n\rangle &= 0\,, \\ \to P_\mu P^\mu |k_\mu,\, n\rangle &= -m^2 |k_\mu,\, n\rangle\,, \qquad (4.7) \\ \text{and } M_{IJ} M_{JK} |k_\mu,\, K\rangle &= \delta_{IK} m_K^2 |k_\mu,\, K\rangle\,, \qquad (4.8) \end{aligned}$$

which leads to the condition $M^2 = I$ in matrix notation. This can be translated into field-theoretic language. Then one obvious choice is to force the condition

$$\begin{aligned} \varphi\left(x^\mu,\, y\right) &\to \varphi_I\left(x^\mu\right) \\ \partial_y \varphi\left(x^\mu,\, y\right) &\to M_{IJ}\varphi_J\left(x^\mu\right) \\ \int \mathrm{d}y\, \varphi\left(x^\mu,\, y\right)\cdots\psi\left(x^\mu,\, y\right) &\to \sum_I \varphi_I\left(x^\mu\right)\cdots\psi_I\left(x^\mu\right) \end{aligned} \tag{4.9}$$

at the level of the fields.[1]

4.2 Deconstructing Free Theories

Returning to the general case of a scalar field, one can see that if the previous prescription is applied, one derives that

$$\mathcal{S}_{\text{(D+1)-dim}}[\varphi] = \int \mathrm{d}^{D+1}X \left(-\frac{1}{2}\partial_M\varphi\partial^M\varphi\right) \tag{4.10}$$

$$= \int \mathrm{d}^D x \mathrm{d}y \left(-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}(\partial_y\varphi)^2\right), \tag{4.11}$$

after doing a $(D+1)$-split and then deforming the action according to (4.9), one arrives at

$$\rightarrow \mathcal{S}_{\text{D-dim}}[\varphi] = \int \mathrm{d}^D x \sum_I \left(-\frac{1}{2}\partial_\mu\varphi_I\partial^\mu\varphi_I - \frac{1}{2}\varphi_J M_{IJ} M_{IK}\varphi_K\right). \tag{4.12}$$

Next, there is an eigenvalue decomposition (or a singular-value decomposition, if massless modes are present) that allows one to diagonalise the fields via a similarity transformation S

$$\varphi_I \rightarrow \tilde{\varphi} = S_{IJ}\varphi_N \Longrightarrow M_{IJ}\tilde{\varphi}_J = m_I\tilde{\varphi}_I\,, \tag{4.13}$$

so then

$$\mathcal{S}_{\text{D-dim}}[\tilde{\varphi}] = \int \mathrm{d}^D x \sum_N \left(-\frac{1}{2}\partial_\mu\tilde{\varphi}_I\partial^\mu\tilde{\varphi}_I - \frac{1}{2}m_I^2\tilde{\varphi}_I\tilde{\varphi}_I\right), \tag{4.14}$$

where in principle there might be zero-modes $m_I = 0$ for some I, representing the massless modes. In principle an arbitrary number of massless modes are allowed; however, in practice I will only allow at most a single massless mode. The masses then are simply the eigenvalues of the M_{IJ} matrix. This greatly generalises Kaluza-Klein compactification; it includes straight dimensional reduction, lattice discretisations like the kind put on computers, and it also includes totally new ways of breaking a 5-D theory down to a 4-D theory. (Note, however, that only KK compactifications form an equivalency with the original theory; otherwise one is only

[1]The clever reader may notice that $P_y = i\partial_y$ in the bra-ket notation, and thus rightfully ask about what happened to all of the factors of i. Perhaps unexpectedly, they are a choice in the procedure. Since once I deform the derivative, the Leibniz rule no longer applies (the crucial part of making P anti-Hermitian), one has to pin down a convention by hand. If one demands that only first derivatives on operators $\partial_y\varphi \rightarrow M_{IJ}\varphi_J$ are well-defined via the deformation procedure, then the requirement of integrating by parts before deforming removes the required factors of i.

extracting portions of the higher dimensional theory.)

4.2.1 Deconstructing Maxwell to Proca Theory

I now explore how deconstruction behaves on a gauge theory. For concreteness, I take a massless spin-1 theory (i.e. Maxwell) in 5-D and I focus on deconstructing over a single site. *A priori*, this ought to lead to a Proca theory in 4-D, which I will now show. Using the logic of the previous section, I take the 5-D Maxwell theory and do a $(4+1)$-split of the action

$$\begin{aligned} \mathcal{S}_{\mathrm{M}}[A_M] &= \int \mathrm{d}^{D+1}X \; -\frac{1}{4}\mathcal{F}_{MN}\mathcal{F}^{MN} \\ &= \int \mathrm{d}^{D}x\mathrm{d}y \; -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{2}{4}\mathcal{F}_{y\mu}\mathcal{F}^{y\mu} \\ &= \int \mathrm{d}^{D}x\mathrm{d}y \; -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}(\partial_y A_\mu)^2 \end{aligned} \tag{4.15}$$

where in the final line I have made use of $A_y = 0$. Applying the deconstruction prescription, the Proca action is obtained immediately

$$\mathcal{S} = \int \mathrm{d}^{D+1}x \; -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu \,. \tag{4.16}$$

One might ask what happens if one does **not** gauge-fix prior to applying the deformation. Things will become more complex when one does not gauge fix in interacting theories, however for quadratic Lagrangians there is an elegant result: If one has a gauge symmetry present when deconstructing a massless theory, one will generate the Stückelberg formulation of the massive theory! To see this, one can leave the gauge unfixed and leave in this mode free; in this case, the $(D+1)$-split takes the form

$$\mathcal{S}_{\mathrm{M}}[A_M] = \int \mathrm{d}^{D}x\mathrm{d}y \; -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}(\partial_y A_\mu - \partial_\mu A_y)^2 \,, \tag{4.17}$$

and I now choose to relabel $A_y = \varphi$. Now, applying the deformation, the action transforms into

$$\mathcal{S}[A_\mu, \varphi] = \int \mathrm{d}^{D}x \; -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}m^2 \left(A_\mu - \frac{1}{m}\partial_\mu\varphi\right)^2 \,, \tag{4.18}$$

which by sight is the Stückelberg action (cf. the formalism in subsection 2.2.4) and even enters with the correct normalisation factors. Note that for more complex theories, like spin-$\frac{3}{2}$ and spin-2 fields, this will not happen and more complex diagonalising transformations will be needed even when deconstructing over a single site. I finish the Stückelberg theory by generating the Stückelberg gauge symmetry from the higher dimensional gauge theory. The higher dimensional symmetry is generated by

$$\delta A_M = \partial_M \xi \Longrightarrow \begin{array}{l} \delta A_\mu = \partial_\mu \xi \\ \delta A_y = \partial_y \xi \end{array} . \tag{4.19}$$

Applying the deformation, one immediately generates the Stückelberg symmetry

$$\begin{aligned} \delta A_\mu &= \partial_\mu \xi \\ \delta \varphi &= m\xi \,. \end{aligned} \tag{4.20}$$

Even though this is only the linear theory, these correspondences are still quite remarkable!

4.2.2 Deconstructing LGR to FP Theory

One may repeat this process with linearised gravity. Starting with LGR in $(D+1)$-dimensions, I will deconstruct this theory and arrive at Fierz-Pauli theory in D-dimensions. Likewise, gauge-fixing will lead to ordinary FP theory, but leaving in the linearised diffeomorphisms gauge symmetries will generate the FP theory within the Stückelberg formulation. For now, I focus on the gauge-fixed version (which leads to an ordinary FP theory), setting $H_{\mu y} = H_{yy} = 0$. Plugging this into the action and doing a $(D+1)$-split you end up with

$$\mathcal{S}_{\text{LGR}}[H_M{}^A] = \int \mathrm{d}^{D+1}X \; -\frac{1}{2} H_M{}^A \delta^{MNR}_{ABC} \partial_N \partial^B H_R{}^C \tag{4.21}$$

$$= \int \mathrm{d}^D x \mathrm{d}y \; -\frac{1}{2} H_\mu{}^\alpha \delta^{\mu\nu\rho}_{\alpha\beta\gamma} \partial_\nu \partial^\beta H_\rho{}^\gamma + \frac{1}{2} \partial_y H_\mu{}^\alpha \delta^{\mu y \nu}_{\alpha y \beta} \partial_y H_\nu{}^\gamma \,, \tag{4.22}$$

were in the last line has made use of integration by parts. Then, making use of the fact that $\delta^{\mu y\nu}_{\alpha y\beta} = \delta^{\mu\nu y}_{\alpha\beta y} = \delta^{\mu\nu}_{\alpha\beta}$, one can see this naturally gives the required Fierz-Pauli tuning. We relabel $H_{\mu\nu} = h_{\mu\nu}$, and finish by deforming the y-derivatives. This gives rise to

$$\to \mathcal{S}_{\text{FP}} = \int \mathrm{d}^D x \; -\frac{1}{2} h_\mu{}^\alpha \delta^{\mu\nu\rho}_{\alpha\beta\gamma} \partial_\nu \partial^\beta h_\rho{}^\gamma + \frac{1}{2} m^2 h_\mu{}^\alpha \delta^{\mu y \nu}_{\alpha y \beta} h_\nu{}^\gamma \,, \tag{4.23}$$

which one may compare to (2.2) to see that this is the Fierz-Pauli theory. Therefore, deconstruction allows one to generate a massive spin-2 theory from a massless spin-2 theory in one dimension higher. I leave the Stückelberg formulation as an exercise for the reader, but I give a quick sketch of the proof of these statements. The B_μ Stückelberg mode is proportional to $H_{\mu y}$, and π is proportional to H_{yy}. Likewise, one can show that the $(D+1)$-dimensional gauge algebra of LGR, which I christen L-$\mathfrak{Diff}(D+1)$, generates the massive Stückelberg gauge algebra of $\mathfrak{u}(1)\times$L-$\mathfrak{Diff}(D)$ by deforming the gauge algebra from LGR! (Note that the gauge algebra of L-$\mathfrak{Diff}(D)$ is isomorphic to $\mathfrak{u}(1)^D$, since linearised diffeomorphisms are abelian.)

4.3 Deconstruction and Interactions

4.3.1 Obstructions to Interactions

The procedure for deconstruction that I outlined in sections 4.1 and 4.2 clearly plays nicely with quadratic Lagrangians. Now that I have deconstructed several bosonic theories[2] I note that for a generic interacting bosonic theory, several crucial relationships can be violated away from the free theories. The differences between the free and interacting theories can easily be seen at the schematic level since in the free action. To start with the quadratic terms in the Lagrangian are given by kinetic terms and mass terms of the form

$$\mathcal{S} \sim \int \varphi \mathcal{D}^2 \varphi \tag{4.24}$$

where $\mathcal{D}^2$ is a second-order differential operator potentially with index structure (including spacetime and internal symmetries). Crucially, however, these differential operators obey $\int A\mathcal{D}^2 B = \int -\mathcal{D}A\mathcal{D}B = \int B\mathcal{D}^2 A$ for any field A and B, modulo boundary terms. The $(D+1)$-split of the action then obeys a relationship like

$$\mathcal{S} \sim \int \varphi \mathcal{D}^2 \varphi - \partial_y \varphi \mathcal{O} \partial_y \varphi \,, \tag{4.25}$$

where $\mathcal{O}$ holds the indices for the spin reps, which for spin-2 fields already contains non-trivial information about ghostly modes (see subsection 2.2.6). As for the deconstruction procedure, the ambiguities in resolving ∂_y vs P_y then can be seen to be irrelevant at quadratic order, namely if I define $\partial_y^2\varphi \to -m^2\varphi$ or I force the derivative to only operate once on any field, $\partial_y\varphi = m\varphi$. Either substitution clearly results in a valid deconstruction procedure (although it will make extracting the

[2]I will discuss fermionic theories in Chapter 6, where I discuss my foundational work on deconstructing fermions while maintaining the SUSY required for long multiplets.

Stückelberg symmetries less clear, so there is at least a minor preference for the single-derivative method; however, it may seem more pleasing to follow the natural QM relation of $P_y^2|k\rangle = m^2|k\rangle$.) The major structural change underlying all of this is that **once the substitution $\partial_y \to M_{IJ}$ takes place, the Leibniz identity is lost**. So after the deformation, $\partial_y[\varphi(y)\psi(y)] \neq (\partial_y\varphi)\psi + \varphi(\partial_y\psi)$ [3] But what happens when we add in natural interactions found in nature, like those of the NLSM, YM, or GR? For instance, even at cubic order (subsuming all indices into a tensor $\mathcal{O}_3$) one has

$$\begin{aligned} \mathcal{S} &\sim \int \varphi\mathcal{D}^2\varphi + \mathcal{O}_3(\varphi^2\partial^2\varphi) \\ &\sim \int (\varphi\mathcal{D}^2\varphi - \partial_y\varphi\mathcal{O}_2\partial_y\varphi) + \underbrace{\mathcal{O}_3(\varphi^2\partial^2\varphi - \varphi^2\partial_y^2\varphi)}_{\text{Ambiguous!}}, \end{aligned} \tag{4.26}$$

since after the substitution relations, e.g.

$$\partial_y\varphi^2 \neq 2\varphi\partial_y\varphi \tag{4.27}$$

are fundamentally broken! With the criterion of gauge symmetries, this clearly becomes even worse in terms of the effects this can have on the theory, since gauge symmetries typically require delicate cancellations between interactions and often makes repeated use of the Leibniz rule of all of the derivatives. This is potentially a disastrous result for deconstruction, and it certainly makes working with deconstruction more challenging for interacting theories!

4.3.2 Deconstructing YM to PYM

Here I show that:

1.) Given a mode $X_{MN\cdots Q}$, I always fix axial gauge $X_{MN\cdots Q} = 0$, if M, N, $\cdots$, or $Q = y$.

2.) I always use the single derivative prescription, $\partial_y X \to M_{ij}X_j$.

provides a robust, albeit still not fully understood, procedure that plays well with gauge symmetries. To show this, I will now work with Yang-Mills, and deconstruct it into PYM. To start, one must write down Yang-Mills in $(D+1)$-dimensions with

[3] Notably, KK-compactification still maintains an algebriac identity similar to the Leibniz rule, owing to M_{IJ} being anti-symmetric, and in discretisations the Abel summation-by-parts identity also applies. These, however, do not contain the full algebraic properties of the Leibniz law!

gauge group SU(N); see section 2.3 for my conventions. I will choose the gauge-fixing condition such that $A_y = 0$, which at the moment is simply an *ad hoc* choice. The action after $(D+1)$-split, followed by the deformation, of the action yields the deconstructed action

$$\mathcal{S}_{\mathrm{YM}}[A_M] = \int \mathrm{d}^{D+1}X \; -\frac{1}{4}\mathrm{Tr}\left[\mathcal{F}_{MN}\mathcal{F}^{MN}\right] \tag{4.28}$$

$$= \int \mathrm{d}^D x \mathrm{d}y \; -\frac{1}{4}\mathrm{Tr}\left[\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + 2\mathcal{F}_{y\mu}\mathcal{F}^{y\mu}\right] \tag{4.29}$$

$$= \int \mathrm{d}^D x \mathrm{d}y \; -\frac{1}{4}\mathrm{Tr}\left[\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + 2\partial_y A_\mu \partial_y A^\mu\right] \tag{4.30}$$

$$\rightarrow \int \mathrm{d}^D x \, \mathrm{Tr}\left[-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu\right] \tag{4.31}$$

where in the last line I have used the usual deformation, $\partial_y A_\mu{}^a = m A_\mu{}^a$. This is Yang-Mills Proca theory. If one wishes to go back to adjoint indices, this is equal to

$$\mathcal{S}_{\mathrm{PYM}}[A_\mu{}^a] = \int \mathrm{d}^D x \; -\frac{1}{4}\mathcal{F}_{\mu\nu}{}^a \mathcal{F}^{\mu\nu\, a} - \frac{1}{2}m^2 \left(A_\mu{}^a\right)^2 . \tag{4.32}$$

If one engages in a simple analysis of this action, it results in the usual 3 PDF's of the massive spin-1 mode, times the number of spin-1 fields filling out an adjoint representation of the gauge group, $\mathcal{G}$. For SU(N), this gives (N^2-1) massive, interacting spin-1 fields, for a total of $3(N^2-1)$ PDF's. This compares nicely with the $3(N^2-1)$ PDF's that a 5-D SU(N) massless spin-1 theory has.

4.4 dRGT from Deconstruction

4.4.1 Deconstructing EC Gravity to dRGT Gravity

Next, I move to deconstruct Einstein-Cartan gravity, and see what kind of massive theory of gravity one obtains [124, 127, 128]. From the outset, however, there are choices which will generate different results. Firstly, should one deconstruct using the metric or vielbeins? Notice that we need to choose an offset in the definition of the $\partial_y e_\mu{}^a = m(e_\mu{}^a - \delta_\mu{}^a)$, since otherwise $\partial_y e_\mu{}^a = m(\delta^a_\mu + h_\mu{}^a) \neq m h_\mu{}^a$. But whichever is chosen, either an offset for the veilbein or the metric, it forces the other to be inconsistent, i.e. choosing the metric leads to

$$\begin{aligned} \partial_y g_{\mu\nu} &= m(g_{\mu\nu} - \eta_{\mu\nu}) \\ &= \partial_y (e_\mu{}^a \eta_{ab} e_\nu{}^b)\,, \end{aligned} \tag{4.33}$$

but contrarily

$$\begin{aligned} \partial_y g_{\mu\nu} &= \partial_y e_\mu{}^a \eta_{ab} e_\nu{}^b + e_\mu{}^a \eta_{ab} \partial_y e_\nu{}^b \\ &= m(e_\mu{}^a - \delta_\mu{}^a)\eta_{ab} e_\nu{}^b + e_\mu{}^a \eta_{ab} m(e_\nu{}^b - \delta_\nu{}^b) \\ &\neq m(g_{\mu\nu} - \eta_{\mu\nu})\,. \end{aligned} \tag{4.34}$$

The deconstructing using the metric ansatz was tried early on in the deconstruction program [108, 119–123, 129, 130]. Although much of the physics community is more familiar with GR in its metric formulation, the vielbein is in many senses a more natural object for gravity. On geometric grounds, it is the object that takes one to the locally inertial frame, and also it is the one that leads to natural generalisations that include torsion. On field-theoretic grounds, it is the formulation that allows gravity to couple to fermions and deals correctly with spin-couplings [131]. Stated differently, fermions are *ad hoc* if ones moves out of the Einstein-Cartan formalism, since a generic manifold does not furnish a representation for fermions [132]. Einstein-Cartan, however, has a globally well-defined local Lorentz group, from which representations for fermions can be drawn; additionally, the required interactions lead naturally to torsion. They are famously required in order to consistently write down self-interacting spin-$\frac{3}{2}$ fields, i.e. supergravity [131, 133]. This is because supergravity is the theory of local super-Poincaré symmetry, meaning that it has local SUSY, local Poincaré symmetry, amongst possible internal gauge symmetries. Therefore, I argue that this is the closest formulation of GR which relates back to representations of the Poincaré algebra, which was the starting point of Deconstruction. In either case, the Einstein-Cartan formulation certainly will deconstruct correctly into dRGT gravity. I review this argument now. One begins with the EC action in first-order form:

$$\mathcal{S}_{\text{EC}}[E^A, \Omega^A{}_B] = \frac{M^{D-2}}{4 \cdot (D-2)!} \int \mathcal{R}^{AB} E^{C_1} \cdots E^{C_{D-2}}\, \varepsilon_{ABC_1 \cdots C_{D-2}} \tag{4.35}$$

with vielbein $E^A := E_M{}^A dX^M$, curvature 2-form $\mathcal{R}^A{}_B := \mathrm{d}\Omega^A{}_B + \Omega^A{}_C \Omega^C{}_B$, and wedge products are implicit, $\omega\xi := \omega \wedge \xi$. Following the logic of the previous section, any mode which transforms non-trivially under the diffeomorphism or local Lorentz gauge symmetries needs to be removed, and one can see that again the axial gauge will work just fine:

$$\delta E_y{}^5 = \partial_y \xi^y E_y{}^5 \tag{4.36}$$
$$\delta E_\mu{}^5 = \lambda^5{}_a E_\mu{}^a \tag{4.37}$$
$$\delta E_y{}^a = \partial_y \xi^y E_y{}^a \tag{4.38}$$
$$\delta \Omega_y{}^{ab} = \Omega'_y{}^{ab} \supset \partial_y \lambda^{ab}\,, \tag{4.39}$$

where $E_y{}^A$ is the $y = X^D$ manifold coordinates, and $V^A = (V^a, V^{(y)})$, $a \in 0, \cdots, D-1$ for Lorentz indices. These represent $D + \frac{1}{2}D(D-1)$ distinct modes, but there are precisely that many gauge symmetries. Making the choices then[4] that removes these modes, the axial gauge

$$\begin{aligned} E_y{}^{(y)} &= 1 \\ E_\mu{}^{(y)} &= 0 \\ E_y{}^a &= 0 \\ \Omega_y{}^{ab} &= 0\,, \end{aligned} \tag{4.40}$$

is applied. Once the spin-connection is integrated out (which in first-order form is an auxiliary field), it generates the torsion-free conditions:

$$\mathrm{d}E^A + \Omega^A{}_B E^B = 0\,. \tag{4.41}$$

Upon substituting (5.17) into this equation, one derives the conditions

$$\Omega_y{}^{a(y)} = 0 \tag{4.42}$$
$$\Omega_\mu{}^{ay} = \partial_y E_\mu{}^a \tag{4.43}$$
$$\Omega_\mu{}^{ab} = \omega_\mu{}^{ab}[e_\mu{}^a] \tag{4.44}$$
$$E_{[\mu}{}^a \Omega_{\nu]}{}^{b(y)} \eta_{ab} = 0\,, \tag{4.45}$$

where $\omega_\mu{}^{ab}[E_\mu{}^a]$ is the D-dim spin connection $\mathrm{d}_D e$ for $e^a = E_\mu{}^a dx^\mu$. For notational convenience, I now define

$$\mathcal{K}^a = \Omega_\mu{}^{a\,(y)} \mathrm{d}x^\mu \tag{4.46}$$

[4]Actually, there is no obvious requirement from the outset that one should send $E_\mu{}^{(y)} = 0$, since $E_\mu{}^{(y)} \to \lambda^{(y)}{}_a E_\mu{}^a$ is not obviously a problem. This will be the topic of Chapter 5. For now, this may be taken as a simplifying ansatz since there is enough gauge symmetry to fix this mode to zero.

and so condition (4.45) simplifies as

$$e_a \mathcal{K}^a = 0\,. \tag{4.47}$$

Now, plugging in (5.17), (4.42-4.44), and (4.47) into the EC action (4.35), one will find the result

$$\mathcal{S}[e_\mu{}^a] = \frac{M^{D-2}}{4(D-3)!}\int \mathrm{d}y\,\left(\mathcal{R}^{ab} + \mathcal{K}^a\mathcal{K}^b\right) e^{c_1}\cdots e^{c_{D-3}}\,\varepsilon_{abc_1\cdots c_{D-3}} \tag{4.48}$$

with $\mathcal{R}^a{}_b := \mathrm{d}\omega^a{}_b + \omega^a{}_c\omega^c{}_b$. Next, apply the deconstruction prescription, $\partial_y e^a = m(e^a - I^a)$. Note that here one must make a choice about what the VEV of the vielbein will be; for simplicity, I choose the Poincaré invariant background vielbein, I^a which is the only choice of deconstruction that preserves Poincaré invariance. After the VEV has been chosen, it must be subtracted off the mass deformation, since the true mass eigenstate is $(e^a - I^a)$, not e^a. This is what breaks diffeomorphism invariance, a requirement of massive theories of gravity. (Note, however, that the theory retains *global* Poincaré invariance.) This then leads to an action of the form, $d = D-1$

$$\mathcal{S}[e_\mu{}^a] \;=\; \frac{M^{d-2}}{4(d-2)!}\int \left(\mathcal{R}^{ab} + \mathcal{K}^a\mathcal{K}^b\right) e^{c_1}\cdots e^{c_{d-2}}\,\varepsilon_{abc_1\cdots c_{d-2}}\,, \tag{4.49}$$

$$\begin{aligned} \text{given} \qquad & e^a\, I_a = 0\,, \\ & \mathcal{K}^a = m(e^a - I^a)\,, \end{aligned} \tag{4.50}$$

where the last equation is commonly known as the symmetric vielbein condition or the Deser-van Nieuwenhuizen condition (Henceforth, “DvN condition” or “DvN gauge”) whose importance was discussed in the previous subsection 3.2.3, and I have rescaled the Planck mass between the dimensions, $M_{D-1} = \left(\frac{1}{m}M_D^{D-2}\right)^{\frac{1}{D-3}}$, as usual. (For interacting theories like dRGT and PYM, my convention is that the coupling constant absorbs the dimensions obtained from the deconstructed integral, $\int \mathrm{d}y \;\to\; \frac{1}{m}\sum_I$, and then one may easily rescale the fields as desired to obtain the correct scaling dimension for their fluctuations.)

4.4.2 2-Site Deconstruction and Bigravity

It is worth noting that ghost-free bi-gravity, found in [45], in Einstein-Cartan formulation [90] can also be found following deconstruction procedure given by de

Rham, *et al*, in [124].

It is a straightforward generalisation of the previous section. Instead, one changes the deconstruction prescription to

$$\begin{aligned} E^a(x,y) &\rightarrow E^a_I(x) := \begin{pmatrix} e^a \\ f^a \end{pmatrix} && (4.51)\\ \partial_y e^a &\rightarrow m(e^a - f^a)\,, && (4.52)\\ \partial_y f^a &\rightarrow m(f^a - e^a)\,, && (4.53)\\ \int \mathrm{d}y &\rightarrow \sum_{I=1}^{2}\,, && (4.54) \end{aligned}$$

where in the first line the vector array indicates the site basis, so the first site has vielbein e^a and the second site has vielbein f^a.

This ansatz trivially generalises to the well known bigravity model

$$\begin{aligned} \mathcal{S}[e^a, f^b] &= \frac{M^{d-2}}{4(d-2)!}\int \left(\mathcal{R}^{ab} + \mathcal{K}^a\mathcal{K}^b\right) e^{c_1}\cdots e^{c_{d-2}}\,\varepsilon_{abc_1\cdots c_{d-2}} \\ &\quad + \left(\mathcal{Q}^{ab} + \mathcal{K}^a\mathcal{K}^b\right) f^{c_1}\cdots f^{c_{d-2}}\,\varepsilon_{abc_1\cdots c_{d-2}}\,, \qquad (4.55) \end{aligned}$$

$$\begin{aligned} \text{given}\qquad & e^a f_a = 0\,, && (4.56)\\ & \mathcal{K}^a = m(e^a - f^a)\,, && (4.57)\\ & \mathcal{R}^{ab} = \mathrm{d}\omega^{ab} + \omega^a{}_c\,\omega^{cb}\,, && (4.58)\\ & \mathcal{Q}^{ab} = \mathrm{d}\theta^{ab} + \theta^a{}_c\,\theta^{cb}\,, && (4.59) \end{aligned}$$

where $\omega^{ab}[e]$ is the spin connection for the first site, and $\theta^{ab}[f]$ is the connection for the second site, and again, $e^a = e_\mu{}^a \mathrm{d}x^\mu$ and $f_\mu{}^a \mathrm{d}x^\mu$ and the spin connection obeys the usual functional dependence on their associated vielbein, see Appendix C for further details.

Chapter 5

Obstructions to Interacting Charged Spin-2 Fields

5.1 Charged Spin-2 Fields?

Aside from considerations of cosmology and gravitational physics, massive spin-2 fields enter into particle physics and condensed matter physics in a natural manner. For instance, there are several known massive spin-2 resonances found in nuclear physics (e.g. spin-2 mesons such as the $\pi_2(1670)$, $\rho_3(1690)$, $\alpha_4(2040)$ [134]) and indeed some of the earliest work in massive spin-2 fields [2, 135, 136] dealt with how one could imagine resonances and related physics. Specifically in these instances, they are concerned with *charged* spin-2 fields, in other words complex massive spin-2 field, i.e. a complex 2-tensor field $H_{\pm,\,\mu\nu}$ that rotates under $\mathcal{U}(1)$ transformations, e.g. $H_{\pm,\,\mu\nu} \to e^{\pm i\theta} H_{\pm,\,\mu\nu}$; for obvious reasons, I will refer to these fields as being "charged massive spin-2 fields". In this context, one need not be interested in gravitational theory, although the close-knit relationship between spin-2 fields and gravitational theory is useful to exploit. A similar line was explored prior to the advent of dRGT theory in [137–140]. Note that it is not possible to have an interacting massless charged spin-2 fields, since it is impossible to have multiple interacting massless spin-2 fields as was shown by Weinberg [5, 6]; since the complex spin-2 fields are made up of two real spin-2 fields, an interacting charged spin-2 field would necessarily violate this condition. Therefore, charged spin-2 fields only make natural sense in the massive spin-2 context, which makes the advances in dRGT theory potentially quite interesting to shed new light on an old topic.

In a different arena, massive charged spin-2 fields have also arisen as an interesting and useful topic in AdS/CMT, namely for holographic descriptions of superconductors [141, 142]. The philosophy for AdS/CMT is to stick theory of superconductors theory on the boundary of AdS and use the known relations of AdS/CFT (for a review, see [31]) to attempt to use gravity in AdS to describe the strongly-coupled physics associated to superconductors (for a classic field-theoretic review of superconductors, see [62]). It is well-known in the literature that S-wave superconductivity can be modelled with a charged AdS scalar (in AdS, it creates a charged spin-0 hair on top of AdS black holes) that spontaneously breaks the $\mathcal{U}(1)$ symmetry in the holographic CMT system; however, in order to describe D-wave superconductivity one needs charged spin-2 hairs on the bulk AdS black hole, necessitating the use of a charged massive spin-2 field [142, 143]. Similarly, massive spin-2 fields have come about because of the ability to break translational invariance on the boundary theory (which corresponds to diffeomorphism invariance in the bulk) [144–147] that are naturally related to DC conductivity.

Therefore, it seems quite timely to understand if dRGT theory of ghost-free, massive spin-2 fields can be modified to give a new understanding of massive charged spin-2 fields. The purpose of this chapter shall be to explore this topic, and in the end, I shall develop a no-go theorem against dRGT being able to give new kinds of interactions, owing to structural incompatibilities between the ghost-free properties and the maintenance of a $\mathcal{U}(1)$ symmetry.

This Chapter is Outlined As Follows:

(1.) The first part of this chapter begins by reviewing the Federbush theory of a charged spin-2 field; this theory contains no self-interactions and instead only contains an interaction of a single photon field (abelian massless spin-1 field).

(2.) The second part deals with the generations of candidate theories from the insights of Dimensional Deconstruction outlined in Chapter 4. The deconstruction procedure can be modified to generate theories that contain interactions between complex massive spin-2 fields and a massless spin-1 field (i.e. a photon). I will show that this generates the most general kind of interactions, given a set of reasonable assumptions, for charged spin-2 theory based upon dRGT mass terms.

(3.) In the third part, I will show that this theory necessarily contains ghosts when one moves away from the Minkowski background. In doing so, I generate a novel, simplified method for determining the presence of PDF's beyond those contained for a complex ghost-free massive spin-2 theory, which acts as a means of checking for the existence of BD modes.

5.2 Federbush Theory of Massive Charged Spin-2 Fields

If one wishes to incorporate electromagnetic interactions for a (complex) massive spin-2 field $H_{\mu\nu} = \frac{1}{\sqrt{2}}(h_{\mu\nu} + i f_{\mu\nu})$, then one needs to add in local, Lorentz-invariant interactions between $H_{\mu\nu}$ and a photon A_μ. Typically, one would simply apply the usual "minimal coupling" inspired by a Nöther completion, i.e.

$$\partial_\mu \to D_\mu = \partial_\mu - iqA_\mu \,, \tag{5.1}$$

like one does for a scalar to create scalar QED or a fermion to create QED [63], to restore a local copy of the $\mathcal{U}(1)$ symmetry

$$\begin{aligned} H_{\mu\nu} &\to e^{i\theta} H_{\mu\nu} \,, \\ H^*_{\mu\nu} &\to e^{-i\theta} H^*_{\mu\nu} \,, \\ A_\mu &\to A_\mu - \frac{i}{g}\partial_\mu \theta \,. \end{aligned} \tag{5.2}$$

Notice here, however, that there is an ambiguity, since the following Lorentz-invariant, local interactions (up to the typical quartic order) are given by the most general action

$$\begin{aligned} \mathcal{S}[H, A] \;=\; & \int \mathrm{d}^4x \; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - H^{*\;\alpha}_{\mu} \left(\delta^{\mu\nu\rho}_{\alpha\beta\gamma} D_\mu D^\beta\right) H_\rho{}^\gamma \\ & - m^2 \left([H^*H] - [H^*][H]\right) + iq(2g_M - 1) H^{*\;\alpha}_{\mu} F_\alpha{}^\nu H_\nu{}^\mu \,. \end{aligned} \tag{5.3}$$

Notice that this leaves a full parameter, $g_M \in \mathbb{R}$, totally unfixed; this parameter can be interpreted as a gyromagnetic ratio since this interaction is Pauli-like [137]. This ambiguity comes about essentially because covariant derivatives do not commute, i.e. $[D_\mu, D_\nu]H_{\alpha\beta} = -qF_{\mu\nu}H_{\alpha\beta}$. Thus there is an ordering ambiguity, so for instance

$$D_\nu D^\mu H_{\alpha\beta} \neq D^\mu D_\nu H_{\alpha\beta} \tag{5.4}$$

$$D_\nu D^\mu H_{\alpha\beta} - D_\nu D^\mu H_{\alpha\beta} \;=\; -iqF_\nu{}^\mu H_{\alpha\beta} \,, \tag{5.5}$$

and thus there is a new, unspecific cubic term (a Pauli-like interaction) representing ordering ambiguity. This is, in general, a ghostly action for essentially the same non-Fierz-Pauli theories. However, In 1961, Federbush wrote down the first interacting, ghost-free theory of a massive spin-2 field interacting with a massless spin-1 photon.

The action takes on the specific form

$$\mathcal{S}_{\rm F} = \int \mathrm{d}^4x \, -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - H^{*\ \alpha}_{\mu}\left(\delta^{\mu\nu\rho}_{\alpha\beta\gamma}D_\mu D^\beta\right)H_\rho^{\ \gamma} \\ -m^2\left([H^*H]-[H^*][H]\right) . \tag{5.6}$$

Therefore, one has $g_M = \frac{1}{2}$ for Federbush. How does the Pauli interaction generate ghostly modes?

Perhaps unsurprisingly, a Stückelberg analysis will again elucidate the nature of the ghostly modes (analyses of this type were first done by Poratti, *et al*, in [138–140]). First, I introduce a $\mathcal{U}(1)$-covariant Stückelberg substitution

$$H_{\mu\nu} \to H_{\mu\nu} - \frac{1}{m}D_{(\mu}B_{\nu)} + \frac{1}{m^2}D_{(\mu}D_{\nu)}\pi\,, \tag{5.7}$$

where, again, I use weight-one conventions, $A_{(\mu}B_{\nu)} = \frac{1}{2}\left(A_\mu B_\nu - A_\nu B_\mu\right)$. Then, knowing how the canonical kinetic terms must arise, I apply the canonical kinetic diagonalising transformations

$$H_{\mu\nu} \to H_{\mu\nu} + \frac{1}{\sqrt{6}}\pi\eta_{\mu\nu}\,, \\ B_\mu \to +\frac{1}{\sqrt{2}}B_\nu \\ \pi \to \sqrt{\frac{2}{3}}\pi\,. \tag{5.8}$$

This leads to a canonically-normalised action, which at quadratic order is

$$S_{(2)} = \int \mathrm{d}^4x \, -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - H^{*\ \alpha}_{\mu}\left(\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\partial_\mu\partial^\beta\right)H_\rho^{\ \gamma} \\ -\partial_\mu\pi^*\partial^\mu\pi - \frac{1}{2}G^*_{\mu\nu}G^{\mu\nu} \tag{5.9}$$

with $G_{\mu\nu} = 2\partial_{[\mu}B_{\nu]}$. The lowest (clearly higher-order in derivative) interaction in the theory is

$$\mathcal{S} = \int \mathrm{d}^4x\,(2g_M-1)\frac{i}{\Lambda^4_{q,4}}\partial_\mu\partial^\alpha\pi^*F_\alpha^{\ \nu}\partial_\nu\partial^\mu\pi \tag{5.10}$$

suppressed by $\Lambda_{q,4}$.[1] As with the usual analyses that I have done up to now, employing a decoupling limit will allow me to remove the other operators, so long as I

[1]It is amusing to note that if a Kaluza-Klein relation $q = \frac{m}{M_{\rm Pl}}$ is applied, then $\Lambda_{q,n} = \Lambda_n$ of the ordinary decoupling limit scales of massive gravity; see chapter 3 for more details.

pick scalings

$$\begin{aligned} q &\to 0\,, \\ m &\to 0\,, \\ \Lambda_{q,4} = \frac{m}{q^{\frac{1}{4}}} &\to \text{const.} \end{aligned} \tag{5.11}$$

Therefore, in the classical contribution to the high-energy theory, I have a cut-off

$$\Lambda_C = \Lambda_{q,4}\,(2g_M - 1)^{-\frac{1}{4}} \tag{5.12}$$

for $g_M \neq \frac{1}{2}$. It is noteworthy that this scale goes to infinity when the Federbush g_M is chosen, owing to this interaction going to zero. This means that, much like the dRGT mass tunings, the scale for the lowest order interactions is raised to a higher scale! The new scale may be computed to be $\Lambda_{q,3}$, again in close analogy to dRGT. This may be computed, schematically, by using $H \sim H + \frac{1}{m}DB + \frac{1}{m^2}DD\pi$, I obtain[2]

$$\mathcal{S} \sim \delta_3 \left[\left(H^* + \frac{DB^*}{m} + \frac{DD\pi^*}{m^2} \right) DD \left(h + \frac{DB}{m} + \frac{DD\pi}{m^2} \right) \right]. \tag{5.13}$$

If one repeatedly makes use the of the schematic identity $\delta_3[DDDB] \sim \delta_3[qF\partial B]$, then one can then see that the interactions are now suppressed, at leading order, by

$$\mathcal{L}_{\lambda_{q,3}} = -\frac{i}{\sqrt{3}\Lambda_{q,3}^3} \delta^{\mu\nu\rho}_{\alpha\beta\gamma} \partial_\mu \partial^\nu \pi^* F_{\nu\rho} G_{\beta\gamma} + (\text{c.c.})\,. \tag{5.14}$$

Note, however, that this new interaction –similar to the Galileon interactions– leads to second-order equations of motion due to the critical anti-symmetry amongst the indices! Therefore, Federbush is ghost-free, at least up to $\Lambda_{q,3}$. Taking this argument to its full completion, the entire action is ghost-free to all orders. This shows that Federbush propagates 5 complex massive spin-2 PDF's and 2 massless spin-1 PDF's, as was proven in an alternate manner by Federbush in [135] and Poratti, *et al*, in [138–140].

5.2.1 Acausalities and Velo-Zwanziger

Ghost freedom is not sufficient to prove the consistency of a theory; even with a local, Lorentz-invariance, and ghost freedom, it was shown by Velo, *et al*, in

[2] In a schematic notation similar to the symmetric polynomials in Appendix B; this does not uniquely specify how to restore indices, but the reader need only follow the schematics since the calculated answer is provided.

[148–150] and Johnson, *et al*, [151] that these theories still exhibit superluminal velocities and acausality, similar to properties derived for massive gravity/Galileons five decades later [37, 38, 152]. In the case of the Velo-Zwanziger-type acauasalities (owing to the presence of a non-trivial background EM field, $\bar{F}_{\mu\nu}$) can be shown to arise for the Federbush theory. For instance, away from the decoupling limit, with a careful analysis, one can derive that the propagator for the helicity-0 modes are schematically deformed as

$$\mathcal{S}_{(2)} \supset \int \mathrm{d}^4x \, \frac{1}{\Lambda_{q,2}^2} \bar{F} |G_{\mu\nu}|^2 + \frac{1}{\Lambda_{q,2}^4} \bar{F}^2 |\partial\pi|^2 \,, \tag{5.15}$$

which leads to kinetic terms that will generically have new phase velocities like [150]

$$c_S = 1 + \epsilon^2 \,. \tag{5.16}$$

For $\epsilon > 0$, this implies the existence of superluminal modes. It has been argued that these issues are not insurmountable obstructions for a UV completion [153] and could potentially be resolved by quantum physics. Such ideas were further taken up in [55, 79, 114, 115] and use exotic UV physics to resolve the acausalities found here (effectively, the new physics would invalidate the analyses provided here). Although I remain agnostic to the existence of such UV-completion ideas, I have shown that this scenario could equally apply to fixing the superluminal velocities of the massive spin-2 Velo-Zwanziger.

5.3 Charged Deconstruction

5.3.1 Deconstruction Prescription

Given the existence of a non-self-interacting charged spin-2 fields theory, one can naturally ask if one could analyse the case for gravitationally interacting charged massive spin-2 fields. Given the discovery of dRGT and bigravity, this provides a new arena for taking on old questions, originally raised in the discussion of mesons [135, 136] is if the dRGT mass interactions will allow for a new perspective on this problem.

Actually, early work into Dimensional Deconstruction led me to a theory of a charged spin-2 field, which I explored in [128]. This is quite clearly a possibility from the standpoint of the “truncated Kaluza-Klein” deconstruction [124], where one obtains a spectrum of a massless spin-2 field and a complex massive spin-2 pair. For simplicity, I will deconstruct 4-D Einstein-Cartan to a 3-D theory with dRGT mass terms, since 3-D theories are considerably easier to work with; also, I will Wick rotate $t \to it$ so $g_{MN} = E_M{}^A \delta_{AB} E_N{}^B$, thus the height of local frame indices will be irrelevant. Recall that deconstruction proceeds by performing a (D+1)-split –here a (3+1)-split to a 3-D theory– and applying specific gauge fixings. Previously, I chose to set every field to radial gauge,

$$\begin{aligned} E_y{}^{(y)} &= 1\,, \\ E_\mu{}^{(y)} &= 0\,, \\ E_y{}^{a} &= 0\,, \\ \Omega_y{}^{ab} &= 0\,, \end{aligned} \tag{5.17}$$

however **I will now choose a less restrictive choice**, and allow for an extra vector mode to be present

$$\begin{aligned} E_\mu{}^{(y)} &= A_\mu \neq 0\,, \\ \Omega_y{}^{ab} &\neq 0\,, \end{aligned} \tag{5.18}$$

where $(y) = 3$ is the 4th Lorentz index, $A = 0, 1, 2, 3$. I have defined $a = 0, 1, 2$.

Prior to applying this gauge, I will split the spin connection up as

$$\Omega_M{}^{AB}\,\mathrm{d}x^M = \begin{pmatrix} \omega_\mu{}^{ab}\mathrm{d}x^\mu & \beta^{ab}\mathrm{d}y \\ K^a_\mu \mathrm{d}x^\mu & \lambda^a \mathrm{d}y \end{pmatrix}. \tag{5.19}$$

Using this convention, the (3+1)-split on the action leads to

$$\begin{aligned} \mathcal{S}_{\text{4-D EC}} = \frac{M_{\text{Pl}}^2}{4} \int \varepsilon_{abc} \Big[& \left(R^{ab} - K^a K^b\right) e^c \\ & + \left(\mathcal{D}\lambda^a - \partial_y K^a - \beta^{af} K^f\right) e^b e^c \\ & + \left(\mathcal{D}\beta^{ab} - \partial_y \omega^{ab} - \lambda^{[a} K^{b]}\right) A e^c \Big] \mathrm{d}y\,. \end{aligned} \tag{5.20}$$

Now, integrating out the spin connection, one obtains the well known torsion-free condition

$$\mathrm{d}E^A + \Omega^A{}_B E^B = 0\,, \tag{5.21}$$

which immediately splits up into the separate conditions; upon applying the gauge-fixing conditions, one finds

$$\begin{aligned} &K_\mu^a = \partial_y e_\mu{}^a + \beta^{ab} e_\mu{}^b + \lambda^a A_\mu \\ &F_{\mu\nu} = 2K^a_{[\mu} e_{\nu]}{}^a \\ &\lambda^a e^a_\mu = 0\,. \end{aligned} \tag{5.22}$$

Thus far, to wit, I have used up the following gauge fixings: 1 from y-diffs, 3 from local y-rotations (y-boosts if I Wick rotate back to Minkowski), I may use up the remaining local rotation symmetries to fix the condition

$$\beta^{ab} = -\frac{1}{2}F^{ab} \equiv -\frac{1}{2}F^{\mu\nu} e^a_\mu e^b_\nu\,. \tag{5.23}$$

This has the immediate consequences

$$K^a_\mu = \partial_y e^a_\mu - \frac{1}{2}F^{ab} e^b_\mu\,, \tag{5.24}$$

and

$$e_{[\mu}{}^a\, \partial_y e_{\nu]}{}^a = 0\,. \tag{5.25}$$

First, I would like to note that from chapter 3 that setting the Deser-van Nieuwenhuizen (i.e. DvN) condition is essential for the ghost-free properties of dRGT massive gravity. In Deconstruction, the above condition (5.25) leads to the DvN condition upon deconstructing e^a.

Applying these conditions to the action, I obtain the following

$$\begin{aligned} \mathcal{S}_{\text{4-D EC}} \;&=\; \frac{M_{\rm Pl}^2}{2}\int \varepsilon_{abc}\left(R^{ab}e^c + \partial_y e^a \partial_y e^b e^c + 2\partial_y \omega^{ab} A e^c\right) \mathrm{d}y \\ &\quad + \frac{M_{\rm Pl}^2}{2}\int \mathrm{d}^3x \mathrm{d}y\; e\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \end{aligned} \tag{5.26}$$

with $e = \mathrm{Det}(e_\mu{}^a)$; using the identity

$$F^{af} e^f \partial_y e^b e^c \propto F_{\mu\nu} e^\mu{}_a \partial_y e^\nu{}_a = 0\,, \tag{5.27}$$

and that it can be shown, using $De \sim Ae$, that the term is identically zero

$$\varepsilon_{abc} F^{ab} A \mathcal{D} e^c \propto A\,A = 0\,. \tag{5.28}$$

Thus there are some nice simplifications within the action.

5.3.2 Charged spin-2 Fields from Deconstruction

Now I turn to the actual application of the Deconstruction procedure. The immediate question is how to obtain a charged spin-2 mode in the spectrum of the deconstructed theory. The answer to this is straightforward, following [124], I may choose a "truncated" Kaluza-Klein tower,

$$\begin{aligned} \partial_y \phi(x^\mu, y) &\to m\alpha_{IJ}\phi_J(x^\mu)\,, \\ \int \mathrm{d}y\, f(y) &\to \frac{1}{m}\sum_{I=1}^{N} f_I\,. \end{aligned} \tag{5.29}$$

In other words, I can treat this like a Kaluza-Klein reduction (modulo the fact that I am freezing the dilaton ${E_y}^{(y)} = 1$), where I put in a cut-off to a finite largest mode. Here, I choose to only take three modes (which in deconstruction means 3 sites, $N = 3$ in my sums). The precise relationship between the α_{IJ} can be seen in the [124], but it was shown there that the spectrum of this theory is a single massless spin-2 mode and a complex massive spin-2 field. Applying this choice and deconstructing action (5.26) yields

$$\begin{aligned} S_{\text{3-D}} \;=\; & \frac{M_3}{4}\int \varepsilon_{abc}\sum_{I=1}^{N}\left(\mathcal{R}[\omega_I]^{ab} + q\sum_J A\alpha_{IJ}\omega_J^{ab} - m^2\sum_{J,K}\alpha_{IJ}\alpha_{IK}e_J^a e_K^b\right)e_I^c \\ & -\frac{1}{4}\int \mathrm{d}^3x\, e\, F_{\mu\nu}F^{\mu\nu}\,, \end{aligned} \tag{5.30}$$

again using the rescaling technique $M_3 \equiv M_{\rm Pl}^2/m$. Here the charge takes on a specific value

$$q_{\text{Deconstruction}} = \frac{m}{\sqrt{N}M_3} = \frac{m^2}{\sqrt{N}M_{\rm Pl}^2}\,, \tag{5.31}$$

although in principle one can consider generalising this action for arbitrary q. It is worth mentioning that the precise site for where one places the photon is ambiguous, so I make the choice to place it simply on the first site; this is inspired by the obstructions to multi-site matter found in [154–156]. Thus I pick for the first site, $E^a = {(e_0)_\mu}^a$ and let $e \equiv \mathrm{Det}(e_0)$.

5.3.3 Charged Deconstruction Spectrum has Complex Spin-2 Fields

The simplest way to see the appearance of the complex modes is to go to a presentation of the variables which manifests the complex properties. The most natural formulation which accomplishes this is the Fourier decomposition, which amounts to a field redefinition

$$\tilde{\Phi}_n = \frac{1}{\sqrt{N}} \sum_{I=1}^{2N+1} \Phi_I e^{2\pi i I n/N} \tag{5.32}$$

where I have chosen the parameterisation $\Phi_I = \{e^a_I, \omega^{ab}_I\}$. By construction, N is odd, and so one can readily see that the inverse redefinition is

$$\Phi_I = \frac{1}{\sqrt{N}} \sum_{n=-(N-1)/2}^{(N-1)/2} \tilde{\Phi}_n e^{-2\pi i I n/N} \,. \tag{5.33}$$

Crucially, notice that the Φ_I fields (in "site basis") are real, while the Fourier fields $\tilde{\Phi}_n$ are manifestly complex, and moreover are subject to the condition $\tilde{\Phi}^*_n = \tilde{\Phi}_{-n}$ in order for Φ_I to be a real variable. Physically, the site basis manifests certain geometric properties, but the Fourier basis manifests the complex structure. Indeed, if one linearised this theory following,

$$\kappa = \frac{1}{M_{\mathrm{Pl}}} \to 0,\, q \to 0 \,, \tag{5.34}$$

they would immediately discover a theory with one massless spin-2 field (LGR) and a massive complex spin-2 field with a $\mathcal{U}(1)$ symmetry, upon applying the redefinition (5.33). Just by noting the properties of a Fourier decomposition, (again with N is being odd) the mass/charge eigenbasis is always spanned by a tower of $(N-1)$ complex massive spin-2 fields $H_{+\,\mu}{}^a$ (with $H_+^\dagger = H_-$ and a single massless, neutral spin-2 mode H_0. A massless spin-2 field in 3-D has 0 PDF's (the constraints balance the phase space variables), and a massive spin-2 field has 2 PDF's. Therefore, a complex massive spin-2 field has 4 PDF's.[3]

Performing this limit demonstrates that charged deconstruction generates, at quadratic order, a spectrum containing (in principle $N-1$) complex massive spin-2 modes, plus a massless spin-2 mode. The natural question is whether the self-interactions respect this symmetry. If they do, then I will have found a consistent

[3] Again, this works exactly the same as a KK compactification over S^1, cf. [125, 126, 157].

(modulo causality concerns) action for a charged spin-2 field with both electromagnetic and gravitational interactions. If they do not, then I have shown that one is not permitted to deconstruct while keeping the vector mode present. As it turns out, these interactions will not preserve a $\mathcal{U}(1)$ invariance. I turn to this issue now.

5.4 Deconstructed Action in Fourier Variables

5.4.1 Interactions in Fourier Variables

For this section, I will be using an arbitrary number of sites (an N-site deconstruction), unless otherwise specified. I will now show that although the theory contains a charged spin-2 mode in its spectrum, its interactions necessarily violate the $\mathcal{U}(1)$ symmetry. This issue will be independent of the photonic interactions, and instead lie in the e_I self-interactions, so I will focus on the pure e_I terms (this is equivalent to setting $q = 0$).

To start with, I will explicitly make use of the Fourier variable formulation of the action, therefore if I expand out

$$\begin{aligned} \tilde{\Phi}_n &= \Phi_I e^{2\pi i I n/N} \,, \\ \implies \tilde{e}^a_n &= \frac{1}{\sqrt{N}} \sum_{I=-N}^{N} \tilde{e}^a_I \, e^{2\pi i I n/N} \,, \qquad (5.35) \\ \implies \tilde{\omega}^{ab}_n &= \frac{1}{\sqrt{N}} \sum_{I=-N}^{N} \tilde{\omega}^{ab}_I \, e^{2\pi i I n/N} \,. \qquad (5.36) \end{aligned}$$

Perhaps surprisingly, the new variables $\tilde{\omega}_+$ and $\tilde{\omega}_-$ are no longer connections. One can see that the effect of the field redefinition is that the $\tilde{\omega}_n$ now transform as tensors under the diagonal local Lorentz transformations, $e^a_I \to \Lambda^a{}_b e^b_I$. To see this, note for example that in the case $N = 3$ that the definition can be repackaged into

$$\tilde{\omega}^{ab}_1 = \frac{1}{\sqrt{3}} \left(\frac{1}{2}(\omega^{ab}_3 - \omega^{ab}_1) + \frac{1}{2}(\omega^{ab}_3 - \omega^{ab}_2) + i\frac{\sqrt{3}}{2}(\omega^{ab}_1 - \omega^{ab}_2) \right) . \qquad (5.37)$$

Remarkably, the connections all act to form difference combinations, ergo even though under the diagonal local Lorentz transformation in matrix notation acts on the site basis as

$$\omega_I \to \Lambda \omega_I \Lambda^{-1} - \mathrm{d}\Lambda \Lambda^{-1} \,, \qquad (5.38)$$

the variables $\tilde{\omega}^{ab}$ are no longer connections! Only $\tilde{\omega}_0$ transforms as a connection.

Additionally, I shall make use of a convenient notation allowed only in three dimensions. One uses a modified Hodge dual of the spin connection, namely

$$\omega^{ab} = \frac{1}{2}\varepsilon^{abc}\omega^{c}\,, \tag{5.39}$$

and thus $\omega^a = \varepsilon^{abc}\omega^{bc}$, where for this chapter a non-standard normalisation of the Hodge dual was chosen of $\frac{1}{(D-p)!}$ rather than $\frac{1}{p!}$. Covariant derivatives now take on the form $DV^a = \mathrm{d}V^a + \omega^{ab}V^b = \mathrm{d}V^a - \frac{1}{2}\varepsilon^{abc}\omega^b V^c$, and and using the dual curvature 2-form $\mathcal{R}^a = \varepsilon^{abc}\mathcal{R}^{bc}$ and the Einstein-Cartan action are re-expressed as

$$\begin{aligned} \mathcal{R}^a &= \varepsilon^{abc}\mathcal{R}^{bc} = \mathrm{d}\omega^a - \frac{1}{4}\varepsilon^{abc}\omega^b\omega^c\,, && (5.40)\\ \mathcal{S}_{\mathrm{EC}} &= M_3\int \mathcal{R}^a e^a\,, && (5.41) \end{aligned}$$

with $M_3 = \frac{1}{2\kappa^2}$. Now then, applying the field redefinition at the level of the action, one obtains a new action of the form

$$\begin{aligned} S_{\text{3-D}} = {} & M_3 \int \sum_I \Big[\frac{1}{N}\sum_{n_1,n_2}(\mathrm{d}\tilde{\omega}^a_{n_1}\tilde{e}^a_{n_2})e^{2\pi i I(n_1+n_2)/N} \\ & + \frac{1}{N^{3/2}}\sum_{n_1,n_2,n_3}\left(-\frac{1}{2}\varepsilon_{abc}\tilde{\omega}^a_{n_1}\tilde{\omega}^b_{n_2}\tilde{e}^c_{n_3}\right)e^{2\pi i I(n_1+n_2+n_3)/N} \\ & + \frac{m^2}{N^{3/2}}\sum_{n_1,n_2,n_3}\left(\varepsilon_{abc}\tilde{e}^a_{n_1}\tilde{e}^b_{n_2}\tilde{e}^c_{n_3}\right)\left(\sum_{J,K}\beta_{IJK}e^{\frac{2\pi i}{N}(In_1+Jn_2+Kn_3)}\right)\Big]\,, \end{aligned} \tag{5.42}$$

noting that I have denoted $\beta_{IJK} := \alpha_{IJ}\alpha_{IK}$ for expediency.

5.4.2 $\mathcal{U}(1)$ symmetry in the $N \to \infty$ limit

It is useful to start with a clear example of a theory with a $\mathcal{U}(1)$ invariance, which is the actual KK "deconstruction" (here we take the literal full KK prescription to "deconstruct"), but I add in the condition that $A_\mu = e_\mu{}^4 = 0$ (i.e. only keeping the tensor modes in the KK tower) and use the same gauge choices that have been applied in this and the previous section, e.g. one may confer to ([125])

for a 4-D to 3-D compactification. One then obtains an action

$$
\begin{aligned}
\mathcal{S}_{\mathrm{KK}} &= M_{\mathrm{Pl}}^2 \int_0^L \mathrm{d}y \int \Big[\frac{1}{L} \sum_{n_1,n_2} (\mathrm{d}\tilde{\omega}^a_{n_1} \tilde{e}^a_{n_2}) e^{2\pi i(n_1+n_2)y/L} \\
&+ \frac{1}{L^{3/2}} \sum_{n_1,n_2,n_3} \left(-\frac{1}{2}\varepsilon_{abc}\tilde{\omega}^a_{n_1}\tilde{\omega}^b_{n_2}\tilde{e}^c_{n_3}\right) e^{2\pi i(n_1+n_2+n_3)y/L} \\
&+ \frac{1}{L^{3/2}} \sum_{n_1,n_2,n_3} (-n_1 n_2) \left(\varepsilon_{abc}\tilde{e}^a_{n_1}\tilde{e}^b_{n_2}\tilde{e}^c_{n_3}\right) e^{2\pi i(n_1+n_2+n_3)y/L} \Big] . \qquad (5.43)
\end{aligned}
$$

I then can perform the integrals over y, which using the orthnormality relations,

$$
\int_0^L \mathrm{d}y e^{2\pi i n y/L} = L\delta_{n,0} \qquad (5.44)
$$

will put the action into the form

$$
\begin{aligned}
&= M_{\mathrm{Pl}}^2 \int \Big[\sum_{n_1,n_2} \delta_{n_1+n_2,0} (\mathrm{d}\tilde{\omega}^a_{n_1} \tilde{e}^a_{n_2}) \\
&+ \frac{1}{\sqrt{L}} \sum_{n_1,n_2,n_3} \delta_{n_1+n_2+n_3,0} \left(-\frac{1}{2}\varepsilon_{abc}\tilde{\omega}^a_{n_1}\tilde{\omega}^b_{n_2}\tilde{e}^c_{n_3}\right) \\
&+ \frac{1}{\sqrt{L}} \sum_{n_1,n_2,n_3} \delta_{n_1+n_2+n_3,0} (-n_1 n_2) \left(\varepsilon_{abc}\tilde{e}^a_{n_1}\tilde{e}^b_{n_2}\tilde{e}^c_{n_3}\right) \Big] . \qquad (5.45)
\end{aligned}
$$

This action, it can be seen with little effort is equivalent to $\lim_{N\to\infty} S_{3\text{-D}}$ of (5.42), in other words the infinite-site deconstruction. This elucidates two important facts. The first is that the $N \to \infty$ deconstruction theory has a $\mathcal{U}(1)$ symmetry. Secondly, the reason it obtains a $\mathcal{U}(1)$ symmetry is due to the appearance of the charge-conserving Kronecker delta's, $\delta_{n_1+\cdots n_M,0}$, i.e.

$$
\begin{aligned}
\delta_{\mathcal{U}(1)}\Phi_n &= in\Phi_n \qquad (5.46) \\
\Longrightarrow \int \delta_{\mathcal{U}(1)} \left(\Phi_{n_1}\cdots\Phi_{n_M}\delta_{n_1+\cdots n_M,0}\right) &\propto \int (n_1+\cdots+n_M)\delta_{n_1+\cdots n_M,0} \\
&= 0 . \qquad (5.47)
\end{aligned}
$$

Away from $N \to \infty$, one can see that the (finite) sums can no longer be written as integrals with orthonormality relations (5.44), and instead obey the finite orthonormality relations

$$
\sum_{I=1}^{N} e^{2\pi i I n/N} = N\delta_{n,kN} \neq \delta_{n,0} . \qquad (5.48)
$$

This orthonormality relation instead introduces an action of the form

$$
\begin{aligned}
\mathcal{S} &= M_3 \int \Big[\sum_{n_1,n_2} \delta_{n_1+n_2,0} (\mathrm{d}\tilde{\omega}^a_{n_1} \tilde{e}^a_{n_2}) \\
&+ \frac{1}{\sqrt{N}} \sum_{k=-1}^{1} \sum_{n_1,n_2,n_3} \delta_{n_1+n_2+n_3,kN} \left(-\frac{1}{2} \varepsilon_{abc} \tilde{\omega}^a_{n_1} \tilde{\omega}^b_{n_2} \tilde{e}^c_{n_3} \right) \\
&+ \frac{m^2}{N^{3/2}} \sum_{n_1,n_2,n_3} \left(\varepsilon_{abc} \tilde{e}^a_{n_1} \tilde{e}^b_{n_2} \tilde{e}^c_{n_3} \right) \left(\sum_{I,J,K} \beta_{IJK} e^{\frac{2\pi i}{N}(In_1+Jn_2+Kn_3)} \right) \Big] , \qquad (5.49)
\end{aligned}
$$

which manifestly fails to have the charge conserving Kronecker delta symbols! In fact, one can see that this only fails to affect the quadratic terms, owing to $n_1+n_2 = kN \implies k=0$ when $|n| \leq (N-1)/2$. For all of the interactions (cubic and higher), this property fails to hold and thus charge conservation disappears within the theory.

I close out this section with the explicit example of the $N=3$ action

$$
\begin{aligned}
\mathcal{S} &= M_3 \int \varepsilon_{abc} \Big(\mathcal{R}[\tilde{\omega}_0]^{ab} \, \tilde{e}^c_0 \\
&+ \left[(\mathrm{d}\tilde{\omega}^{ab}_1 + 2\tilde{\omega}^{ad}_0 \, \tilde{\omega}^{cb}_1) \, \tilde{e}^{*,c}_1 + c.c. \right] + \tilde{\omega}^{ad}_1 \tilde{\omega}^{*,db}_1 \tilde{e}^c_0 + m^2 \tilde{e}^a_1 \, \tilde{e}^{*,b}_1 \, \tilde{e}^c_0 \\
&+ \left[\tilde{\omega}^{ad}_1 \, \tilde{\omega}^{db}_1 \, \tilde{e}^c_1 + m^2 \tilde{e}^a_1 \, \tilde{e}^b_1 \tilde{e}^c_1 + c.c. \right] \Big) . \qquad (5.50)
\end{aligned}
$$

Here it is trivial to read off the charge conservation-violating interactions on the final line.

5.5 Obstructions to Restoring the $\mathcal{U}(1)$ Symmetry

In terms of a Deconstruction, I have shown that **one is not allowed to Deconstruct while keeping on vector modes unfrozen, thus** $A_\mu = 0$. This will have important consequences in Chapter 6, but for now I turn to natural question of whether or not one can be motivated by the observations in deconstruction to obtain a different theory of a massive, charged spin-2 field. The issue amounts to restoring the $\mathcal{U}(1)$ symmetry by hand, and see if this can consistently be maintained alongside diffeomorphism invariance, local Lorentz invariance, locality, and ghost freedom.

The obvious suggestion from deconstruction is to multiply the local, Lorentz-invariant, diff invariant terms by a projection operator that acts analogously to the Kronecker delta to maintain charge conservation; this is the most one can leverage at once, so this leaves ghost freedom incapable of being manifested. Starting with the mass terms, if I impose $\mathcal{U}(1)$ invariance, then this imposes constraints on what β_{IJK} can be. Note, however, that the mass term has a ghost-free form, so as long as the kinetic terms do not move substantially away from Einstein-Hilbert, then there is no ghost re-introduction. For the case of interest, with 3-sites this is restricted to the terms

$$\mathcal{S}_{\text{mass},\,\mathcal{U}(1)} = \int m^2 \left(c_1 \tilde{e}_0 \tilde{e}_0 \tilde{e}_0 + c_2 \tilde{e}_1 \tilde{e}_1^* \tilde{e}_0\right) . \tag{5.51}$$

Using the inverse field redefinition back into site basis, one obtains the mass terms with the coefficient tensor with non-zero entries

$$\begin{aligned} \beta_{111} &= c_1 + c_2 \\ \beta_{112} &= 3c_1 \\ \beta_{123} &= 6c_1 - 3c_2 \,. \end{aligned} \tag{5.52}$$

Also, imposing the tadpole cancellation condition, one has

$$\beta_{111} = \frac{36}{5}, \quad \beta_{112} = 3, \quad \beta_{123} = -\frac{63}{5} \,. \tag{5.53}$$

Kinetic term

Moving on to the kinetic interactions, the interactions from deconstruction that are cubic in e^a_I and ω^a_I follow the form

$$\mathcal{S}_{(3)} = \frac{M_3}{3^{1/2}} \int \sum_{k=-1}^{1} \sum_{n_1,n_2,n_3} \delta_{n_1+n_2+n_3,kN} \left(-\frac{1}{2}\varepsilon_{abc} \tilde{\omega}^a_{n_1} \tilde{\omega}^b_{n_2} \tilde{e}^c_{n_3}\right) . \tag{5.54}$$

Restricting further, the interactions containing purely charged states $\tilde{e}_{\pm 1}$, $\tilde{\omega}_{\pm 1}$, in other words terms which pick up a phase after a $\mathcal{U}(1)$ transformation, are given by

$$\mathcal{S}^{k=\pm 1}_{(3)} = -\frac{1}{2}\frac{M_3}{3^{1/2}} \int \varepsilon_{abc} \left(\tilde{\omega}^a_1 \tilde{\omega}^b_1 \tilde{e}^c_1 + \tilde{\omega}^a_{-1} \tilde{\omega}^b_{-1} \tilde{e}^c_{-1}\right) . \tag{5.55}$$

This can be off-loaded to site basis via (5.33), which takes on the form

$$\mathcal{S}^{k=\pm 1}_{\text{cubic}} = -M_3 \int \sum_{I,J,K} \gamma_{IJK} \varepsilon_{abc} \omega^a_I \omega^b_J e^c_K \,, \tag{5.56}$$

where

$$\gamma_{IJK} \equiv \frac{1}{3^2} \cos\left(\frac{2\pi}{3}(I+J+K)\right) . \tag{5.57}$$

Therefore, one simple way to restore $\mathcal{U}(1)$ invariance is just to subtract off the guilty terms, leaving only $\mathcal{U}(1)$-invariant interactions

$$\mathcal{S}_{\text{cubic}}^{k=0} = S_3 - S_3^{k=\pm 1} = M_3 \int \sum_I \varepsilon_{abc} \left[-\frac{1}{2} \omega_I^a \omega_I^b e_I^c + \sum_{J,K} \gamma_{IJK} \omega_I^a \omega_J^b e_K^c \right] . \tag{5.58}$$

However, this has the consequence of which can be reinterpreted as deforming the Einstein-Hilbert structure of the kinetic terms

$$\mathcal{S}_{\text{kin}}^{k=0} = S_{\text{GR}} + S_{\text{kin}}^{\text{new}} , \tag{5.59}$$

$$\Longrightarrow \mathcal{S}_{\text{kin, (3)}}^{k=0} = M_3 \int \sum_{IJK} \gamma_{IJK} \varepsilon_{abc} \omega_I^a \omega_J^b e_K^c . \tag{5.60}$$

I now turn to proving that, unsurprisingly, this deformation away from the Einstein-Hilbert kinetic structure is fatal to the underlying theory. Thus, the dRGT mass term does not buy any leeway with deforming the kinetic terms. Indeed, what is being uncovered here is similar to what was found earlier in [127] (where de Rham, *et al*, applied deconstruction to Gauss-Bonnet terms, and the results were shown to be ghostly), and will play a role in building up more powerful theorems regarding the dRGT theory of massive gravity [80] which I will discuss further in the conclusions. The precise identification between these interactions and those found in [127] may be seen through a field redefinition

$$\omega_1 \to \omega_1 + (\omega_2 - \omega_3) , \tag{5.61}$$

upon which

$$\mathcal{R}[\omega_1]e_1 \to \mathcal{R}[\omega_1]e_1 + \mathcal{R}[\omega_2]e_1 + \mathcal{R}[\omega_3]e_1 - 2(\omega_1 - \omega_3)(\omega_2 - \omega_3)e_1 . \tag{5.62}$$

The chief difference between these terms and [127] is that ω_I is still an auxiliary field in my formulation, thus one cannot immediately deduce the existence of ghosts just yet, and a more direct analysis is needed.

5.5.1 Projecting out the $\mathcal{U}(1)$-Violating Interactions

Taken all together, I have obtained a $\mathcal{U}(1)$ invariant, Lorentz invariant, and diff invariant action, which collecting all of the results of this section is

$$\begin{aligned}\mathcal{S} &= S_{kin}^{k=0} + S_{mass}^{k=0} \\ &= M_3 \int \varepsilon_{abc}\Big(R[\tilde{\omega}_0]^{ab}\tilde{e}_0^c \\ &\quad + \left[(\mathrm{d}\tilde{\omega}_1^{ab} + 2\tilde{\omega}_0^{ad}\tilde{\omega}_1^{cb})\tilde{e}_1^{*,c} + c.c.\right] + \tilde{\omega}_1^{ad}\tilde{\omega}_1^{*,db}\tilde{e}_0^c \\ &\quad + m^2 \tilde{e}_1^a\, \tilde{e}_1^{*,b}\, \tilde{e}_0^c\Big). \end{aligned} \tag{5.63}$$

A few general comments about the things that do succeed for this theory:

(1.) Supposing that the theory was consistent (or at least ghost free), then one would next want to incorporate $\mathcal{U}(1)$ interactions through the standard minimal coupling procedure, i.e. $\mathrm{d} \to D = \mathrm{d} - iqA$. After introducing the gauge field, one can easily show that it recovers the Federbush theory in the scaling limit $M_3 \to \infty$ where one loses gravitational interactions.

(2.) S_{kin}^{new} has diagonalized diff invariance, guaranteed by the form structure, as well as diagonalized local Lorentz invariance, which can be seen by expanding out the γ_{IJK} explicitly

$$\begin{aligned}\mathcal{S}_{kin}^{new} &= \frac{1}{9}M_3 \int \varepsilon_{abc}\left[2(\omega_1^a - \omega_2^a)(\omega_1^b - \omega_3^b) - (\omega_2^a - \omega_3^a)(\omega_2^b - \omega_3^b)\right] e_1^c \\ &\quad + (\mathbb{Z}_3 \text{ perms})\,, \end{aligned} \tag{5.64}$$

noting that the only scale in the kinetic terms is M_3.

(3.) Again, the spin connections are auxiliary fields, so they remain independent at the moment.

5.5.2 Generic non-linear completions

Unsurprisingly, the deconstruction-motivated charged spin-2 theory will contain spurious PDF's and have more than 5 complex PDF's (10 real PDF's). Rather than go through a detailed proof of this specific theory, I wish to make this no-go proof as strong as possible. To this end, I will write down the most general non-linear theory involving a linearly-realised $\mathcal{U}(1)$ symmetry acting on a complex spin-2 state, $\tilde{e}_\pm^a \to e^{\pm i\alpha}\tilde{e}_\pm^a$, and the standard gravitational properties with vielbein e^a, but give up entirely on any Einstein-Hilbert, but retain the wedge product structure. Although this analysis will only be done for the $3-D$ case (since this is where Hamiltonian analysis is easiest), if the theory existed in higher D-dimensions, it would have to come out via S^n dimensional reduction, where $D=3+n$. Taking all of this together, I have

* A dreibein and spin connection e^a, ω^{ab}, which are neutral and have 0 PDF's (DOF - diff/LLI constraints = 0), encoding the massless spin-2 field.

* A complex vielbein field $H^a_{\pm,\mu}, \Theta^a_{\pm,\mu}$, which carry the charged spin-2 DOF's. We take $H_- = H_+^*$. Under a $\mathcal{U}(1)$ transformation with parameter α, the spin-2 field $H_\pm$ transforms as $H_\pm \to e^{\pm iq\alpha}H_\pm$, and similarly for $\Theta_\pm$. They ought to have 4 (a complex massive spin-2 field, $2\times 2=4$).

* All fields transform under diffeormorphisms as forms, and as Lorentz vectors $X^a \to \Lambda^a{}_b X^b$.

* $\mathcal{U}(1)$ is easy to explore in this representation. The requisite diagonal Lorentz invariance enforces that the spin connection ω should appear only through the dual curvature $\mathcal{R}[\omega]^a = \frac{1}{2}\varepsilon^{abc}\mathcal{R}^{bc}[\omega]$ or the exterior covariant derivative $\mathcal{D} = \mathrm{d}+\omega$, modulo a Chern-Simons interaction which I shall not consider here.

* Again, I consider only theories that can be written explicitly with wedge products since this element is critical to all known ghost-freedom proofs.

Firstly, comparing to old notation, morally I have $\Theta^a_+ = \varepsilon^{abc}\tilde{\omega}_1^{bc}$ and similar for Θ^a_-, but I am only interested in writing down a Hamiltonian, thus $\Theta^a_\pm$ is only playing the role of the momenta conjugate to $H^a_{\pm,\mu}$.

Secondly, I can give some justification for the last line by noting that non-wedge product actions typically will have kinetic terms of the form

$$H^a_{+,\mu}H^b_{-,\nu}X^{\mu\nu}_{ab}\,, \tag{5.65}$$

given some tensor $X^{\mu\nu}_{ab}$ that is a function of fields. After introducing the Stückelberg fields by $H \sim H + \mathcal{D}\phi$, this will have the form

$$\mathcal{D}_\mu\phi^a\mathcal{D}_\nu\phi^b X^{\mu\nu}_{ab}\,, \tag{5.66}$$

which clearly leads to terms of the form

$$\dot{\phi}^a\dot{\phi}^b X^{00}_{ab}\,. \tag{5.67}$$

This manifestly gives all Stückelberg fields ϕ^a a kinetic term (owing to Lorentz invariance), clearly generating a BD ghost. Contrarily, the wedge/GKD structure can guarantee diffeomorphism invariance and the rough draft of the structure of ghost freedom.

With a bit of effort, one can derive the most generic action that follows the above criterion. It takes on the form

$$\begin{aligned} \mathcal{S} \;=\; & M_3 \int \varepsilon_{abc}\mathcal{R}[\omega]^{ab}e^c \\ & + \left[(c_1\mathcal{D}\Theta^a_+ H^a_- + c.c.) + c_2\mathcal{D}e^a_+ e^a_- + c_3\mathcal{D}\Theta^a_+\Theta^a_-\right] \\ & + \varepsilon_{abc}\left(c_4\Theta^a_+\Theta^b_- e^c + (c_5\Theta^a_+ H^b_- e^c + c.c.)\right) \\ & + \varepsilon_{abc}\left(m^2 H^a_+ H^b_- e^c + \Lambda\, e^a e^b e^c\right)\,, \end{aligned} \tag{5.68}$$

with a usual cosmological constant Λ.

It will be useful to use another set of field redefinitions away from this form. First, one can note that the kinetic terms can be put into the form

$$\begin{aligned} \mathcal{L}_{kin} &= \left(c_1\mathcal{D}\Theta^a_+ H^a_- + c.c.\right) + c_2\mathcal{D}e^a_+ e^a_- + c_3\mathcal{D}\Theta^a_+\Theta^a_- \\ &= c_2\mathcal{D}\left(\Theta^a_+ - C^{(+)}H^a_+\right)\left(\Theta^a_- - C^{(-),*}H^a_-\right) + c.c.\,, \end{aligned} \tag{5.69}$$

such that

$$C^{(\pm)} = \frac{c_1}{c_2}\left(-1 \pm \sqrt{1 - \frac{c_2 c_3}{|c_1|^2}}\right)\,, \tag{5.70}$$

and then performing the field redefinition

$$\begin{aligned} \Theta^a_+ - C^{(+)}H^a_+ &\;\to\; \Theta^a_+ \\ \Theta^a_+ - C^{(-)}H^a_+ &\;\to\; E^a_+\,. \end{aligned} \tag{5.71}$$

Using the condition that $\Theta_- = \Theta^*_+$, one is then allowed to set $c_2 = c_3 = 0$. Note that this field redefinition cannot be inverted when $|c_1|^2 = c_2c_3$; however, for this specific choice of parameters, the action is a perfect square and so after a field redefinition the action becomes $\mathcal{L} \sim \mathcal{D}E_+E_-$. Then $\Theta_\pm$ drops out of the kinetic term completely, and this will not linearise to Federbush. I will return to the issue of the correct linearisation properties momentarily.

This will change the factors in front of c_4, c_5, m^2, λ, so I rescale these parameters to compensate for this. Finally, I am able to rescale the fields to absorb c_1 and c_4, leaving the action

$$\begin{aligned}\mathcal{S} &= M_3 \int \varepsilon_{abc}\mathcal{R}[\omega]^{ab}e^c + \left(\mathcal{D}\Theta^a_+ H^a_- + cc\right) \\ &\quad +\varepsilon_{abc}\left(\Theta^a_+\Theta^b_- e^c + \left(c_5\Theta^a_+ H^b_- e^c + c.c.\right)\right) \\ &\quad +\varepsilon_{abc}\left(m^2 H^a_+ H^b_- e^c + \lambda e^a e^b e^c\right) \,. \end{aligned} \tag{5.72}$$

5.5.3 Reproducing Federbush

Although this is the most generic non-linear action with the desired properties (modulo the PDF's of the theory, which can only be determined after performing the Hamiltonian analysis), there is one implicit assumption I alluded to above: One needs to reproduce Federbush in the $M_3 \to \infty$ limit. A quick check of this leads one to the condition that $c_5 = 0$. This can be seen via

$$\begin{aligned} e^a &= \left(\delta^a_\mu + \frac{1}{2\sqrt{M_3}}h^a_\mu\right)\mathrm{d}x^\mu = I^a + \frac{1}{2\sqrt{M_3}}h^a \,, \\ \omega^{ab}_\mu &= \frac{1}{\sqrt{M_3}}\Theta^{ab}_\mu \,, \\ H^a_\pm &= \frac{1}{2\sqrt{M_3}}h^a_\pm \,, \\ \Theta^a_\pm &= \frac{1}{\sqrt{M_3}}\Theta^a_\pm \,. \end{aligned} \tag{5.73}$$

Neglecting the neutral, massless spin-2 sector, one has for the charged modes

$$\mathcal{S} = \int \left(\mathrm{d}\Theta^a_+ \wedge h^a_- + c.c.\right) + \varepsilon_{abc}\Theta^a_+ \wedge \Theta^b_- \wedge I^c + \left(c_5\varepsilon_{abc}h^a_+ \wedge \Theta^b_- \wedge I^c + c.c.\right) \,, \tag{5.74}$$

where I have restored wedge products to avoid confusion and compare more easily with linearised Einstein-Cartan. Upon integrating out the spin-connection $\Theta_\pm$ one finds the equation of motion

$$\Theta^a_{+,\mu} = \epsilon^{abc}\partial_b h^c_{+,\mu} + c_5 h^a_{+,\mu} \,, \tag{5.75}$$

which upon being plugged back into the action generates the usual, second-order acton plus a spurious term

$$\mathcal{S} = \mathcal{S}_{\text{Complex FP}} + c_5 \int \mathrm{d}^3x \epsilon^{\mu\nu\rho}\partial_\mu h^a_{+,\nu} h^a_{-,\rho} \,. \tag{5.76}$$

Therefore, I take $c_5 = 0$.

5.5.4 Deconstruction-Motivation is the Unique Ansatz

Collecting these results, I derive the final form of the action to be

$$\begin{aligned}\mathcal{S} &= M_3 \int \varepsilon_{abc}\mathcal{R}[\omega]^{ab}e^c + \left(\mathcal{D}\Theta^a_+ H^a_- + cc\right) \\ &\quad +\varepsilon_{abc}\left(\Theta^a_+\Theta^b_- e^c + m^2 H^a_+ H^b_- e^c + \lambda e^a e^b e^c\right) \,. \end{aligned} \tag{5.77}$$

If one relabels the variables so that $H^a_\pm \to \tilde{e}^a_{\pm 1}$ and $\Theta^a_\pm \to \varepsilon_{abc}\tilde{\omega}^{bc}_{\pm 1}$, then one can see that this is the unique ansatz *is* the deconstruction-motivated ansatz. Therefore, if one wants the properties listed in this section, one is uniquely led to this action! The lingering question is whether or not this action is free of ghosts; I now show this action contains spurious degrees of freedom.

5.6 Hamiltonian Analysis of the Ansatz

One could proceed with a Hamiltonian analysis á la ADM [81], however in practice this method will be quite cumbersome. Instead, I will make use of a new analysis in first-order formalism and represents an original formalism for this kind of Hamiltonian analysis.

This analysis will be predicated upon a few crucial details. First, I will incorporate all of the Stückelberg fields for the Lorentz and diffeomorphism invariance for the vector mode. Second, this will be done in first-order form where the momenta can easily be identified (at least in $3-D$). The advantage here is that aside from the would-be second-class constraints that remove the BD ghost, all constraints are first-class. Third, in the second-order form, one checks for the existence of a BD ghost based on whether or not the Hessian is invertible

$$\mathcal{H}^{ab} = \delta^2 S/\delta\dot{\phi}^a\delta\dot{\phi}^b \,, \tag{5.78}$$

cf. to [44, 158]. Typically, this is a very difficult condition to check. However, one can observe that there is an equivalent simpler condition. If and only if the theory has no Boulware-Deser ghost, then the Boulware-Deser ghost mode must not be found in the quadratic Lagrangian expanded about an arbitrary off-shell background. (This works for the usual reason, namely that the absence of the BD ghost is always related to whether or not the Stückelberg kinetic structures generated from mass terms fail to exist for ϕ^0 which can be seen at quadratic order so long as one considers all possible backgrounds.) By only concerning oneself with the quadratic action greatly simplifies the action, but also the manner in which the Stückelberg modes enter into the Lagrangian and the existence of spurious kinetic terms.[4]

5.6.1 Hamiltonian Setup

One begins by expanding the action to quadratic order. I use the variables

$$\begin{aligned} e^a_\mu &= \bar{e}^a_\mu + h^a_\mu \\ \omega^{ab}_\mu &= \bar{\omega}^{ab} + \Theta^{ab}_\mu \\ H^a_{\pm,\mu} &= \bar{H}^a_{\pm,\mu} + v^a_{\pm,\mu} \\ \Theta^a_{\pm,\mu} &= \bar{\Theta}^a_{\pm,\mu} + \mu^a_{\pm,\mu} \, . \end{aligned} \tag{5.79}$$

To reiterate, the backgrounds fields are off-shell, and as such there is no condition forcing them to obey any equations of motion. Next, one can introduce the Stückelberg fields at the level of the perturbations,

$$\begin{aligned} &v^a_\pm \to v^a_\pm + \bar{\mathcal{D}}\phi^a_\pm \\ &\mu^a_\pm \to \mu^a_\pm + \bar{\mathcal{D}}\lambda^a_\pm \, , \end{aligned} \tag{5.80}$$

choosing $\bar{\mathcal{D}}\phi^a = \mathrm{d}\phi^a + \bar{\omega}^{ab}\phi^b$ for the background covariant derivative. Note that the background diagonalised diffeomorphisms and local Lorentz symmetry are still realised in the usual manner.

Generic kinetic terms for arbitrary fluctuations, χ^a and ψ^b and a background field $\bar{\Phi}^a_\mu$, must have the form

$$\begin{aligned} \int \varepsilon_{abc}\, \bar{\mathcal{D}}\chi^a \bar{\mathcal{D}}\psi^b \bar{\Phi}^c &= \int \varepsilon_{abc} \left(-\chi^a \bar{\mathcal{D}}^2 \psi^b \bar{\Phi}^c + \chi^a \bar{\mathcal{D}}\psi^b \bar{\mathcal{D}}\bar{\Phi}^c \right) \\ &= \int \varepsilon_{abc} \left(-\chi^a \psi^d \bar{R}^{bd} \bar{\Phi}^c + \chi^a \bar{\mathcal{D}}\psi^b \bar{\mathcal{D}}\bar{\Phi}^c \right) \, . \end{aligned} \tag{5.81}$$

Crucially, one can make use of the formula $\bar{\mathcal{D}}^2\psi \sim \bar{R}\psi$ from the form structures in the theory, and the Bianchi identity $\bar{\mathcal{D}}\bar{R} = 0$. Another novel feature, again to demonstrate the upshot of forcing the action to have p-form notation, is that one can easily check that at quadratic-order the zero components $h^a_0\mathrm{d}t, \Theta^{ab}_0\mathrm{d}t, v^a_{\pm,0}\mathrm{d}t, \mu^a_{\pm,0}\mathrm{d}t$

[4] I will not review these here, but for the purposes of comparison: A straightforward analysis to cubic order also confirms the results given by our analysis. Likewise, this original analysis also reproduces the ghost-freedom of 3-D dRGT and bigravity in this streamlined method, providing explicit examples demonstrating that this method works precisely as advertised.

are all Lagrange multipliers because of the antisymmetry.

Degrees of freedom for healthy spin-2 fields in three-dimensions

In order to determine if the BD ghost persists, one must check the size of the physical phase space. Recall that my nomenclature is that the number DOF's is defined as (half of) the dimensionality of the phase space, whereas the number of PDF's is defined as (half of) the dimensionality of the constraint-reduced phase space (i.e. PDF's = DOF's − number of second-class constraints $-2\times$ number of first-class constraints). The DOF's are just an artefact of the choice of variables, but the PDF's are a physical, meaningful observable. Peeling off the component basis of the form, e.g. $e_i{}^a \mathrm{d}x^i \to e_i{}^a$, one can decompose the phase space DOF's into

- $\left(e_i{}^a, \, \omega_i{}^{ab}\right)$: 6 components $\times\, 2 = 12$ fields.
- $\{H^a_{\pm,i}, \, \Theta^a_{\pm,i}\}$: 6 components $\times\, 2 \times 2 = 24$ fields.
- $\{\phi^a_\pm, \, \lambda^a_\pm\}$: 3 components $\times\, 2 \ \times 2 = 12$ fields.

By construction, one also has several first-class constraints

- 3 diagonal diffeomorphism symmetries (with Lagrange multipliers e^a_0).
- 2×3 Stückelberg diffeomorphism symmetries (with Lagrange multipliers $H^a_{\pm,0}$).
- 3 local Lorentz symmetries (with Lagrange multipliers ω^{ab}_0).
- 2×3 Stückelberg local Lorentz symmetries (with Lagrange multipliers $\Theta^a_{\pm,0}$).

Therefore if one reduces the phase space using these constraints (or gauge-fixing the first-class constraints), one obtains an intermediary number of phase space DOF's:

$$\begin{aligned} &(12+24+12) \text{ dynamicalvariables} \\ &-2\times 18 \text{ first} - \text{classconstraints} \\ &= 2\times(2+2)+2\times(1+1) \text{ DOF's}\,. \end{aligned} \tag{5.82}$$

If there are no further constraints, this the physical phase space, so $2\times$ the number of PDF's. Notice, however, that if there are further PDF's, then the BD ghost survived. It is useful to split this down into the further Poincaré sup-representations. In 3-D, a massless spin-2 field has 0 PDF's, while a charged massive spin-2 field has $2\times 2 = 4$. Thus one should expect that these 4 PDF's (i.e. 8 phase space dimensions). The residual $+2\times(1+1)$ away from this number represent the potential BD ghosts.

If the theory has further constraints, this will uniquely kill the BD ghosts, but if there are no further constraints, this theory will possess ghostly modes. Alternatively, the absence of the constraints can been seen by the quadratic action fluctuating a full $2\times(2+2)+2\times(1+1)$ set of modes on an arbitrary background.

5.6.2 Reappearance of the Boulware-Deser Mode

There is little effort needed to see that $2\times(2+2)+2\times(1+1)$ modes are present in this theory. I will take a fixed gravitational background field $\bar{e}^a$, and assume a trivial background for the charged modes $h^a_\pm = \Theta^a_\pm = 0$, but it is assumed that the gravitational background follows the torsion-free condition $\bar{\mathcal{D}}\bar{e} = 0$.

Expanding (5.77) around this off-shell background, and introducing the Stückelberg fields as (5.80), one obtains

$$\begin{aligned} \mathcal{S} &= \int \bar{\mathcal{D}}\mu^a_+\left[v^a_- - \varepsilon^{abc}\lambda^b_-\bar{e}^c\right] + \bar{\mathcal{D}}\phi^a_+\left[\bar{R}^{ab}\lambda^b_- + m^2\varepsilon_{abc}v^b_-\bar{e}^c\right] + \text{c.c.} \\ &\quad + \mathcal{U}_{\text{N.D.}}\left(v^a_-,\, \mu^b_+,\, \phi^c,\, \lambda^d_-,\, e^a\right)\,, \end{aligned} \tag{5.83}$$

where $\mathcal{U}_{\text{N.D.}}$ are potential terms without any derivatives in the fluctuations.

In passing to the Hamiltonian analysis, one needs to extricate the terms with time derivatives on the fluctuation fields. Zooming in onto terms with only time derivatives, one finds that the momenta can be straightforwardly defined (again, another great advantage of working in first-order formalism),

$$\mathcal{S} \supset \int \mathrm{d}^3x\ \dot{\mu}^{+,a}_i P^{-,a}_i + \dot{\phi}^{+,a}\pi^{-,a} + \text{c.c.} \tag{5.84}$$

where I have denoted

$$\begin{aligned} P^{-,a}_i &= \varepsilon_{ij}v^{-,a}_i - \varepsilon^{abc}\lambda^{-,b}e^{0,c}_j \\ \pi^{-,a} &= \varepsilon^{abc}\varepsilon^{ij}\lambda^{-,b}R^c_{ij} + m^2\varepsilon^{abc}\varepsilon_{ij}v^{-,b}_i e^{0,c}_j\,. \end{aligned} \tag{5.85}$$

Curiously, but perhaps not surprisingly since the theory is in first-order form[5]

Now, the analysis proceeds by seeing if there is a linear combination of the conjugate momenta which drop out. To reiterate, if a combination of the conjugate

[5] When using the Hamiltonian for ordinary gravity in first-order, the Lorentz spin-connection (the generator of local Lorentz boosts) plays the role of the conjugate momenta to the vielbien, cf. [106, 107], which is the broken generator of diffeormopshisms/local translations [159]. Therefore, it is not entirely surprising the the Lorentz Stückelberg is the momenta conjugate to the diffeomorphism Stückelberg fields.

momenta drop out, then there is a second-class constraint. This will generate a secondary partner via the Dirac procedure [82, 84], yielding the desired pair of second-class constraints. If there is no such combination, then the BD ghost is not vanquished within this theory, and so the theory propagates a spurious PDF.

The first step is to solve for $\pi^{-,a}$ in terms of the field $P^{-,a}$. A simple substitution yields

$$\pi^{-,a} = \varepsilon^{abc}\varepsilon_{ij}\left[\left(\bar{R}^{b}_{ij} + m^2\varepsilon^{bpq}\bar{e}^{p}_{i}\bar{e}^{q}_{j}\right)\lambda^{-,c} + m^2 P^{-,b}_{i}\bar{e}^{c}_{j}\right] . \tag{5.86}$$

First, a sanity check. Since I am working to quadratic order and shuffling the interactions into an off-shell background, I ought to be able to see that when I set the background to Minkowski my argument results in ghost-free Federbush. Indeed, this is the case. In general, following the above formula, if the term proportional to $\lambda^{-,c}$ drops out, then π is just proportional to another conjugate momentum and so the momenta are not independent. This is another way of saying that a linear combination of momenta are zero, and thus the Dirac analysis argument is triggered and the pair of second-class constraints is generated.

When expanding off of Minkowski, i.e. $\bar{R} = 0$ and $\bar{e}^{a}_{i} = \delta^{a}_{i}$, it is easy to see that the generalised coordinate λ^{-} drops out of the formula for π^{-} and thus the action (which is the Federbush theory written in different variables) is ghost-free, as expected. Unfortunately, it is just as easy to see that for any deviation away from those choices, the generalised coordinate λ^{-} does **not** drop out of the formula, and **there is no linear combination of conjugate momenta that are trivial.** Thus, the Dirac analysis is not triggered, and a spurious PDF exists inside of the theory. It is amusing to note how this is essentially the same argument that is found in the Buchdahl condition [160, 161], the only difference is that it applies to the Stückelberg modes found inside of a massive theory.

5.7 Obstructions to Self-Interacting Charged Spin-2 from Group Theory

Given the difficulties in developing a theory of self-interacting massive charged spin-2 fields, one might wonder if there are related fundamental obstructions to the existence of these fields. A clear issue might be the impossibility of finding fields which transform in the prescribed manner actually cannot exist at the the level of groups.

5.7.1 Non-Existence of Specific $[\mathbf{ISO}(1,d) \times \mathbf{ISO}(1,d)] \rtimes \mathcal{U}(1)$

For self-interacting spin-2 fields, there are two broken copies of ISO$(1,d)$ symmetries; comparing to the previous sections, when the gravitational background is frozen.

For there to be charged spin-2 modes, then they have a specific variant of ISO$(1,d) \times$ ISO$(1,d)$ that is non-linearly realized, but still present. Specifically, in order to have the interpretation of being charged spin-2 fields, one must have the basic commutation relations

$$\begin{aligned} &[P_i^a,\, P_j^b] = 0 \\ &[Q,\, P_i^a] = \varepsilon_{ij} P_j^a \\ &[Q,\, M_i^{ab}] = \varepsilon_{ij} M_j^{ab} \,. \end{aligned} \tag{5.87}$$

in addition for each local Lorentz algebra $[P_i^a,\, M_i^{ab}]$ to be associated to each graviton, e.g. $\delta_{M_i^{cd}}(\lambda^{cd})e_j^a = \lambda^a{}_b e_i^a$, for $i = j$. Note that I have chosen to use $\mathcal{U}(1) \cong \mathrm{SO}(2)$, so i, j label the indices of 2-D real vectors of the charge multiplet, not the spatial components of spacetime. For concreteness, the specific group action is given by the usual $\mathcal{U}(1)$ rotations

$$\delta_{\mathcal{U}(1)}(\Theta)E^a = \Theta\varepsilon_{ij}E_j^a \Longrightarrow [Q,\, P_i^a] = \varepsilon_{ij}P_j^a \,. \tag{5.88}$$

Obviously then, $[Q, P_i^a] \neq 0$, so one must have an algebra from

$$G = \mathcal{U}(1) \ltimes [\mathrm{ISO}(1,d) \times \mathrm{ISO}(1,d)] \,, \tag{5.89}$$

where there is definitive non-commutation from (5.87).

The most natural question now is whether or not this is even consistent as a group. The unfixed commutation relations are given by

$$[P_i^a,\, M_j^{bc}] \;=\; ? \tag{5.90}$$

$$[M_i^{ab},\, M_j^{cd}] \;=\; ? \tag{5.91}$$

when $i \neq j$.

Firstly, we wish for $i = j$ to form the usual Poincaré algebra. Unfortunately, this theory seems to lack the requisite structure to finish off the algebra non-trivially, since a self-consistent algebra would need something analogous to an f^{ijk} structure, which cannot be furnished non-trivially for an abelian $\mathcal{U}(1)$ theory. Thus the only consistent choice is to set these to zero

$$\begin{aligned} [P_i^a,\, M_j^{bc}] &= 0 \\ [M_i^{ab},\, M_j^{cd}] &= 0\,, \end{aligned} \tag{5.92}$$

when $i \neq j$.

5.7.2 Checking the Jacobi identity

This leads directly to an inconsistency. Taking these generators $\{Q, M_1^{ab}, M_2^{ab}, P_1^c, P_2^c\}$ and the given commutation relations

$$\begin{aligned} &[Q,\, Q] = 0 \\ &[P_1^a,\, P_1^b] = [P_1^a,\, P_2^b] = 0 && \text{(Same for } 1 \leftrightarrow 2) \\ &[M_1^{ab},\, P_2^c] = [M_1^{ab},\, M_2^{cd}] = 0 && \text{(Same for } 1 \leftrightarrow 2) \\ &[M_1^{ab},\, P_1^c] = \eta^{bc} P_1^a - \eta^{ac} P_1^b && \text{(Same for 2)} \\ &[M_1^{ab},\, M_1^{cd}] = \eta^{ac} M_1^{bd} + \eta^{bd} M_1^{ac} - \eta^{ad} M_1^{bc} - \eta^{bc} M_1^{ad}. && \text{(Same for 2)}\,. \end{aligned} \tag{5.93}$$

From this, one can easily see that the Jacobi identity cannot be upheld

$$\begin{aligned} &\Big[Q, [M_1, M_2]\Big] + \Big[M_1, [M_2, Q]\Big] + \Big[M_2, [Q, M_1]\Big] \\ &= \Big[Q, 0\Big] + \Big[M_1, M_1\Big] + \Big[M_2, M_2\Big] \\ &= \eta^{ac}\left(M_1^{bd} + M_2^{bd}\right) + \eta^{bd}\left(M_1^{ac} + M_1^{ac}\right) - \eta^{ad}\left(M_1^{bc} - M_2^{bc}\right) - \eta^{bc}\left(M_1^{ad} + M_2^{ad}\right) \\ &\neq 0\,, \end{aligned} \tag{5.94}$$

therefore this cannot close under a Lie algebra structure, and thus one cannot exponentiate this structure to a consistent Lie group.

5.8 Comments about No-Go Theorem on Charged Spin-2 Fields

I have explored the case of charged spin-2 fields, due to the possible applications for dRGT theory in particle physics and condensed matter theory. Within the context of the Dimensional Deconstruction procedure, there is a natural method of obtaining a self-interacting theory of a charged spin-2 with both electromagnetic and gravitational backgrounds, namely that of 3-site Deconstruction where the vector modes are unfrozen ("charged deconstruction"). It is easy to show that this action fails to generate a complete $\mathcal{U}(1)$ due to a set of interactions that are only $\mathcal{U}(1)$ invariant when the number of sites is infinite; however, one can systematically subtract off of these guilty interactions. From there, I write down and analyse all possible generalisations of this structure, given a natural set of requirements, to include new types of interactions. I demonstrated that when one writes down the most general interactions, one is invariably lead back to the very same action generated by charged deconstruction with the $\mathcal{U}(1)$-violating interactions projected out. I then demonstrated, using a simple novel new method for checking the existence of ghosts, the existence of a spurious PDF in this charged spin-2 theory. It is worth reiterating that this must be interpreted as a ghostly mode appearing at a energy scale higher than Λ_3, thus it is possible for one to include new UV corrections or PDF's at a scale around or above Λ_3.

It is worth noting that this no-go theorem is complementary to Boulanger, *et al*, [162] where it was shown that one is forbidden from having spin-2 fields that are charged under nonabelian gauge groups, although they used very different methods to derive their theorem.

The techniques and arguments developed in this chapter also helped set up developments into two other important questions for massive spin-2 fields, thus it has greatly deepened and expanded theorems regarding massive spin-2 fields. Firstly, the techniques and concepts were expanded to deal with the case of gravity-matter couplings, supplementing earlier work [154–156]. This is a natural and important question in multi-gravity theories, since multiple metrics make matter couplings inherently ambiguous. For instance the techniques developed in this work have contributed to the ideas laid out in [163] and their applications for cosmology [164]. It was shown in [163] that the vielbein formulation cannot provide new ways to couple massive spin-2 fields to matter, and lead to the argument that one must pick a single site for a matter field to couple to.

Secondly, the techniques and the analyses of the chapter directly lead to the advancements showing that there cannot be any new kinetic terms for dRGT mass terms [165]. For instance, the complete uniqueness of the Einstein-Hilbert term even within the context of massive spin-2 fields was proven in [80] by my collaborator Matas. This result is particularly powerful since it explicitly links the dRGT ghost-free mass terms to Einstein-Hilbert kinetic terms for spin-2 fields, and the proof made extensive use of the techniques developed here for determining ghost-freedom of massive spin-2 theories.

Finally, this work will also be directly relevant to the following section, which deals with attempts to make dRGT more symmetric in a different fashion (i.e. enhancing the spacetime symmetry to supersymmetry, rather than incorporating internal symmetries). This is no-go theorem of particular importance in the context of supersymmetry; any no-go theorem on charged spin-2 fields immediately leads to a no-go theorem on BPS short supermultiplets, which necessitate massive spin-2 fields charged under a $\mathcal{U}(1)_{\mathrm{BPS}}$ symmetry.

Chapter 6

Deconstructing Supermultiplets

The outline of this chapter goes as follows:

(1.) I will discuss some motivations for studying supersymmetric (SUSY) theories of massive spin-2 fields. I will overview both the notion that SUSY theories have radically simplified quantum properties, which is appealing for potential future analyses of dRGT massive gravity, and the notion that SUSY theories are interesting in their own right for trying to understanding what types of SUSY QFT's are consistent.

(2.) Once one sets their sights on a supersymmetric theory containing massive spin-2 fields, one is then tasked with generating these theories. In general, it is very hard to find supersymmetric theories. To this end, I will derive a procedure for generating 4-D massive gauge theories from 5-D gauge theories, extending the Dimensional Deconstruction program from Chapter 4 to successfully incorporate supersymmetry. This is the first step in being able to systematically generating massive supersymmetric gauge theories with interactions; however, for the purposes of this chapter, I will be content with working out the procedure for free theories.

(3.) This will lead me to explore some 5-D $\mathcal{N}=2$ massless theories[1], specifically $\mathcal{N}=2$ super-Maxwell and linearised $\mathcal{N}=2$ supergravity. I will create a procedure that allows me to deconstruct these theories into 4-D massive $\mathcal{N}=1$ SUSY theories, where deconstruction breaks half of the supersymmetry. Thus, I fill in a gap in the literature where it has been speculated that there is a relationship between the 5-D massless and 4-D massive SUSY theories, e.g.

[1]The minimal amount of SUSY in 5-D is $\mathcal{N}=2$, owing to the Dirac fermions being the only kind of representation available for the supercharges.

[166, 167], by explicating the exact relationship and how the 4-D massive theories can be derived directly from the 5-D massless theories.

As for **(1.)**, it was discussed in Chapter 2 that massive gravity remains far from understood at the quantum level. Even a very simplified version of massive gravity, which cleanly illustrated consistency of a quantum massive spin-2 field, would be a major victory towards understanding and developing a fully consistent picture of massive gravity. Well beneath Λ_3, the theory appears to be largely understood, but the story surrounding reliable predictions around or above Λ_3 is considerably less clear (See [55] for a discussion on these issues). Therefore, it seems timely to develop modifications of massive gravity that might make analyses of the UV quantum physics more tractable. One will naturally be led to the study of supersymmetric theories.

On the other hand, one may be quite interested in SUSY in its own right. Supersymmetric field theories and supersymmetric gravitational theory (supergravity/SUGRA) have attracted much attention over the past four decades do to their alluring role in UV completions of gravity and gauge theories (See for instance [131, 168, 169]); therefore, for those interested in what types of supersymmetric theories are possible, it is intrinsically interesting if there is a supergravity theory with non-zero graviton mass. The earliest attempts at this are [129, 130], some of which were attempting to resolve unitarity issues in massive gravity (the BD ghost) via supersymmetry.

Points **(2.)** and **(3.)** will be taken up starting in section 6.3 of this chapter. I will continue discussion of these points there.

6.1 Review of Supersymmetry

Supersymmetry was originally developed in [170] for 4-D, and the first supergravity theory was developed in [171]. Stated simply, supersymmetry is an extension to the usual Poincaré spacetime symmetries, namely by adding in a fermionic charge (a Grassmann-valued spinor), Q. Traditionally, this is called a **supercharge**; these theories no longer obeys commutation relations from a Lie algebra, but now obey super-commutation relations of a super-Lie algebra [131, 168]. In terms of field content, the simplest SUSY theories have a single supercharge with the following new super-commutation relations

$$\{Q,\, \bar{Q}\} = 2i\gamma^{\mu}P_{\mu}\,, \tag{6.1}$$

$$[Q,\, P] = 0\,, \tag{6.2}$$

$$[Q,\, M_{\mu\nu}] = -\frac{1}{4}\gamma_{\mu\nu}Q\,. \tag{6.3}$$

In other words, anti-commutators of supercharges result in translations, supercharges commute with translations, and supercharges transform as fermions under boosts. There is also another charge that (chirally) rotates the supercharge, called the R charge, obeying

$$[Q,\, R] = i\gamma_5 Q\,. \tag{6.4}$$

As one can see, R follows an axial $\mathcal{U}(1)$ structure, which amounts to chiral rotations in the supercharge's spinor basis.

This may seem bizarre, since the Coleman-Mandula theorem [172] demonstrated that there are no further charges beyond internal symmetry algebras and the Poincaré algebra (or conformal algebra) that allow for interacting QFT's. This super-algebra is clearly an extension of the Poincaré algebra, but the crucial loophole is that the charge is fermionic. Shortly after the discovery of SUSY, it was found that there is an overarching theorem proving that the only extension to Poincaré representations are the super-Poincaré algebra (or superconformal algebra) with the R-symmetry algebra and internal symmetry algebras [173], called the Haag-Lopuszański-Sohnius theorem. Any other QFT containing fields with representations from larger symmetry group cannot interact with a consistent, non-trivial S-matrix.[2] Therefore, supersymmetry is literally the "most symmetric" type of interacting theory allowed within the whole framework of QFT. Since non-anomalous quantum corrections must respect the underlying symmetries of the theory, it should come as no surprise that SUSY greatly simplifies the underlying quantum version of these theories.

The spectrum of supersymmetric theories is set by the representations of the super-Poincaré algebra, some useful reviews go over these in more detail [131, 167, 168, 175]. Since the Poincaré algebra is a subgroup, they fall into collections of Poincaré reps, as one expects; however, unlike reps/multiplets coming from an internal symmetry algebra, supermultiplets necessarily contain collections of fields

[2] Outside of theories with an infinite tower of spins, e.g. [174] or string theory [29, 30]; I will not discuss these theories here. Also, since this only applies to representations, it follows that non-linearly realised symmetries have no restrictions from this theorem.

with mixed spin. In other words, representations of the super-Poincaré algebra contain fermions and bosons. The number of spins depends on the amount of supercharges and whether or not the supermultiplets have a mass. Much like Poincaré reps, supermultiplets are bifurcated into massive and massless reps, and there are non-trivial differences between massive and massless supermultiplets.

For $\mathcal{N}=1$ SUSY, a massless multiplet contains only a single Grassmann-valued operator that raises the spin of the base state, whereas a massive multiplet contains two Grassmann-valued operators that raise and fill out twice as many states (see [168] for details). The simpler argument comes from PDF's; it can be shown that for supermultiplets, bosons and fermions must have the same number of PDF's [168]. Therefore, supposing one starts off with a spin-$\frac{1}{2}$ fermion, one needs to give it a spin-1 partner. Explicitly, if they are massless, a spin-1 mode has 2 PDF's, so it can partner with a single Majorana fermion with 2 PDF's. Thus only one other field (the spin-1 mode) is needed to fill out the supermultiplet. If they all have a mass, then the spin-1 mode has 3 PDF's, while the massive spin-$\frac{1}{2}$ Majorana fermion only has 2 PDF's. The correct way to fill out the multiplet will be adding a single massive scalar and another Majorana fermion. Then the supermultiplet will be balanced by $2+2$ fermion PDF's $=3+1$ boson PDF's. The new states come from a new Grassmann-valued operator that acts on the base state (the super-spin state) to decrease the helicity by $\frac{1}{2}$. With some effort, it can be seen that the "superspin" state enters with $(+1)$ and (-1) R-charge, respectively (For a review of massive supermultiplets, see [167, 168]).

In other words, a massless $\mathcal{N}=1$ supermultiplet with superhelicity-Y falls into a collection of fields of spin

$$\begin{pmatrix} Y+\frac{1}{2} \\ Y \end{pmatrix}, \tag{6.5}$$

whereas massive $\mathcal{N}=1$ supermultiplets with superspin-Y fall into a collection of fields of spin

$$\begin{pmatrix} & Y+\frac{1}{2} & \\ Y & & \hat{Y} \\ & Y-\frac{1}{2} & \end{pmatrix}, \tag{6.6}$$

where the Y field has spin-Y with R-charge $+1$ and the $\hat{Y}$ field has spin-Y but R-charge -1.

6.1.1 Massless $\mathcal{N}=1$ Supermultiplets

Here I will take the simplest case of a massless spin-2 supermultiplet, with a superpartner spin-$\frac{3}{2}$. So, this gives a theory of a free massless spin-2 field $h_{\mu\nu}$ (the "graviton" field) and a massless Majorana spin-$\frac{3}{2}$ field ψ_μ (the "gravitino" field). The action follows the obvious choice of the linearised Einstein-Hilbert action combined with the Rarita-Schwinger action, i.e.

$$\mathcal{S}[h,\psi]=\int \mathrm{d}^4x \left[-\frac{1}{2}h_\mu{}^\alpha \delta^{\mu\nu\rho}_{\alpha\beta\gamma}\partial_\nu\partial^\beta h_\rho{}^\gamma-\frac{i}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}\,\partial_\nu\psi_\rho\right]. \tag{6.7}$$

This action has two types of gauge symmetries related to the massless gauge fields. The graviton obeys the obvious linearised diffeomorphisms, but the gravitino has a fermionic gauge symmetry,

$$\delta\psi_\mu=\partial_\mu\eta \tag{6.8}$$

where η is a Majorana fermionic gauge parameter. It is easy to check that the action is invariant under this symmetry, owing to integration by parts and the anti-symmetry of $\gamma^{\mu\nu\rho}$.

It also has global symmetries. The first kind are the usual Poincaré symmetries. The second kind of transformation that it is invariant under are the $\mathcal{N}=1$ supersymmetry transformations

$$\begin{aligned}\delta h_{\mu\nu}&=i\bar{\epsilon}\gamma_{(\mu}\psi_{\nu)}\,,\\ \delta\psi_\mu&=\gamma^{\alpha\beta}\partial_\alpha h_{\beta\mu}\epsilon\,.\end{aligned} \tag{6.9}$$

One can prove with some effort that these form a supersymmetry algebra,

$$\begin{aligned}[\delta_1,\,\delta_2]h_{\mu\nu}&=(2i\bar{\varepsilon}_2\gamma^\alpha\varepsilon_1)\,\partial_\alpha h_{\mu\nu}+\delta_{\text{Gauge}}h_{\mu\nu}\\ [\delta_1,\,\delta_2]\psi_\mu&=(2i\bar{\varepsilon}_2\gamma^\alpha\varepsilon_1)\,\partial_\alpha\psi_\mu+\delta_{\text{Gauge}}\psi_\mu+(\text{E.O.M.})\end{aligned} \tag{6.10}$$

where the first piece is the momentum shift induced by $\{Q,\,Q\}\sim\gamma^\mu P_\mu$, the second term are the δ_{Gauge}, coming from are the linearised diffeomorphism and supergauge transformations

$$\xi^\alpha = 2i\bar{\varepsilon}_2\gamma^\alpha\varepsilon_1 \tag{6.11}$$

$$\delta_{\text{G}}h_{\mu\nu} = \partial_{(\mu}\left[-\xi^\alpha h_{\nu)\alpha}\right] \tag{6.12}$$

$$\delta_{\text{G}}\psi_\mu = \partial_\mu[-\xi^\alpha\psi_\alpha]\,, \tag{6.13}$$

with the (E.O.M.) term finally being a zilch symmetry (see [159]), so this closes on the SUSY algebra (plus local symmetries) and keeps the action invariant.

6.1.2 Massless $\mathcal{N} = 2$ Supermultiplets

I will use information about $\mathcal{N} = 2$ massless supermultiplets, so I briefly discuss them here. For $\mathcal{N} = 2$ theories the superalgebra is extended over two supercharges, with the non-trivial commutation relations given by

$$\{Q^i,\, \bar{Q}^j\} = 2i\delta^{ij}\gamma^\mu P_\mu\,, \tag{6.14}$$
$$[Q^i,\, P_\mu] = 0\,, \tag{6.15}$$
$$[Q^i,\, M_{\mu\nu}] = -\frac{1}{4}\gamma_{\mu\nu}Q^i\,, \tag{6.16}$$
$$[Q^i,\, R] = i\gamma_5 Q^i\,, \tag{6.17}$$
$$[Q^i,\, R^j{}_k] = i\delta^j_k Q^k\,, \tag{6.18}$$

where the two supercharges work as usual, but now the R-symmetry is enhanced to $\mathcal{U}(2)_R = \mathrm{SU}(2)_R \times \mathcal{U}(1)_R$; the $\mathcal{U}(1)_R$ chirally rotates charges and is carried by R, and SU(2) does non-chiral, complex rotations of supercharges and is carried by $R^j{}_k$ in my notation.

Unsurprisingly, the introduction of new supercharges introduces new fields of differing spin, and thus enhances the size of the supermultiplet. For instance, the $\mathcal{N} = 2$ supergravity multiplet is spanned by a massless spin-2, two massless Majorana spin-$\frac{3}{2}$, and a massless spin-1, i.e.

$$\begin{pmatrix} & h_{\mu\nu} & \\ \psi_\mu{}^1 & & \psi_\mu{}^2 \\ & B_\mu & \end{pmatrix}\,, \tag{6.19}$$

where $h_{\mu\nu}$ is the graviton field, $\psi_\mu{}^i$ (for $i = 1,\, 2$) are the gravitini fields, and a real, non-axial B_μ, which I will call the graviphoton. Here one can easily see that the i indices act as the vector representation of the remaining R-symmetry after the Majorana condition is applied (i.e. only manifesting the real $\mathfrak{so}(2)_R$ subalgebra of $\mathfrak{su}(2)_R$). At linear level, it is given by an action of the form

$$\mathcal{S}[h,\psi,B] \;=\; \int \mathrm{d}^4x \left[-\frac{1}{2}h_\mu{}^\alpha\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\partial_\nu\partial^\beta h_\rho{}^\gamma - \frac{i}{2}\bar{\psi}_\mu{}^i\gamma^{\mu\nu\rho}\,\partial_\nu\psi_\rho{}^i - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}\right]\,, \tag{6.20}$$

with the usual $G_{\mu\nu} = 2\partial_{[\mu}B_{\nu]} = \partial_\mu B_\nu - \partial_\nu B_\mu$ and Einstein summation convention with the R-symmetry indices.

This action possesses all of the obvious symmetries, including the full gauge symmetries and the Poincaré symmetry. For the gravitini, they have an obvious doubled supergauge symmetry

$$\delta\psi_\mu{}^i = \partial_\mu \eta^i \,, \tag{6.21}$$

where η^i, $i = 1, 2$ are a pair of Majorana fermions that act as gauge parameters. In addition, this action is invariant under a more intricate set of $\mathcal{N} = 2$ SUSY transformations

$$\begin{aligned} \delta h_{\mu\nu} &= i\bar{\epsilon}^i \gamma_{(\mu}\psi_{\nu)}{}^i \\ \delta\psi_\mu{}^i &= \gamma^{\alpha\beta}\partial_\alpha h_{\beta\mu}\epsilon^i + \frac{i}{4\sqrt{6}}\gamma^{\alpha\beta}\gamma_\mu\, G_{\alpha\beta}\,\varepsilon^{ij}\epsilon^j \\ \delta B_\mu &= \frac{1}{\sqrt{2}}\epsilon^{ij}\epsilon^i\bar{\psi}_\mu{}^j \end{aligned} \tag{6.22}$$

where ε^i for $i = 1, 2$ are the two supersymmetry group parameters, which by construction are two Majorana fermions. Note that when this theory is promoted to having interactions, the two supersymmetries and the two supergauge transformations fuse into an $\mathcal{N} = 2$ local supersymmetry of an $\mathcal{N} = 2$ supergravity theory.

6.1.3 Massive $\mathcal{N} = 1$ Supermultiplets

Finally, I discuss massive $\mathcal{N} = 1$ supermultiplets containing a Fierz-Pauli mode. Owing to the previous discussion, one can check that this contains the same field content, roughly, of the $\mathcal{N} = 2$ supermultiplet. It has one massive spin-2 field, two massive Majorana spin-$\frac{3}{2}$ fields, and a massive spin-1 field,

$$\begin{pmatrix} & h_{\mu\nu} & \\ \psi_\mu{}^1 & & \psi_\mu{}^2 \\ & A_\mu & \end{pmatrix} , \tag{6.23}$$

where $h_{\mu\nu}$ is the graviton field, $\psi_\mu{}^i$ (for $i = 1, 2$) are the gravitini fields, and A_μ is an as-of-yet undetermined massive spin-1 mode. Again, this gives the PDF counting of $4 + 4 = 3 + 5$, where massive Majorana fermions have 4 PDF's. The only other thing one can prove is that each gravitino has the opposite R-charge assignment.

Thus, the $\mathcal{U}(1)_R$ symmetry transformation can be re-written as

$$\begin{pmatrix} \psi_\mu{}^1 \\ \psi_\mu{}^2 \end{pmatrix} \rightarrow \begin{pmatrix} e^{i\theta\gamma_5}\,\psi_\mu{}^1 \\ e^{-i\theta\gamma_5}\,\psi_\mu{}^2 \end{pmatrix}, \tag{6.24}$$

$$\Longrightarrow \psi_\mu{}^i \rightarrow \left(e^{i\theta\gamma_5\eta}\right)^i{}_j\,\psi_\mu{}^j \tag{6.25}$$

$$\Longrightarrow \delta\psi_\mu{}^i = \theta\, i\gamma_5\,\eta^{ij}\,\psi_\mu{}^j \tag{6.26}$$

$$\Longrightarrow \delta\bar{\psi}_\mu{}^i = -\theta\,\bar{\psi}_\mu{}^j\eta^{ji}\,i\gamma_5\,, \tag{6.27}$$

where I am denoting the 2×2 matrix diag$(1,-1)$ as η^{ij}.

It is worth noting already at this level that there are a great many choices that are now unconstrained. First, in principle, the parity of A_μ is totally unfixed; secondly, there are several choices of fermion masses one could take (Majorana, Dirac, mixed). Therefore, it is not immediately obvious what Lagrangian needs to be written down for the linear theory.

6.1.4 Towards Massive $\mathcal{N}=1$, $Y=\frac{3}{2}$ Supermultiplets

I will now argue that the consistent Lagrangian should be equivalent to one already known to the literature, namely the $\mathcal{N}=1$ Zinoviev theory [166]. I will now review the arguments that would lead one to choose the $\mathcal{N}=1$ Zinoviev theory.

One ideally wants to specify the aforementioned ambiguities through the superalgebra-motivated arguments. Specifically, one wishes to

(1.) Deduce the parity of A_μ.

(2.) Fix the fermion's mass terms (i.e. pick either Majorana, Dirac, or mixed).

One can see that **(1.)** can be cleared up with a clever use of the Stückelberg formalism + decoupling limit argument. If the IR is supersymmetric, then the decoupling limit (high momentum, UV theory) must be supersymmetric, too. In order to consistently flow to the UV, one needs to add in Stückelberg fields. If one applies the Stückelberg formalism and take $p \gg m$, then the massive graviton splits into $h_{\mu\nu}$, B_μ, π (helicity-2, helicity-1, and helicity-0 modes), the massive gravitini split into $\psi_\mu{}^i$, ξ^i (helicity-$\frac{3}{2}$ and helicity-$\frac{1}{2}$ modes), and the massive spin-1 mode turns into A_μ, φ (helicity-1 and helicity-0 modes). Naturally then, a portion of this system (in the decoupling limit) will need to form a massless Wess-Zumino supermultiplet, thus the two scalars and one of the fermions (e.g. some linear combination $\chi \sim a\chi^1 + b\chi^2$) will link up to form an $\mathcal{N} = 1$ supermultiplet

$$\begin{pmatrix} \chi \\ \pi, \varphi \end{pmatrix} . \tag{6.28}$$

As is well-known, this only works if one of the two scalar modes is parity-odd! This imposes a non-trivial constraint, since this scalar must either appear as a Stückelberg mode for the massive spin-2 or the massive spin-1 field. Since the Stückelberg shares the parity assignment of its progenitor, one is forced to pick the massive spin-1 mode to be parity-odd ($\mathcal{PT} : A_\mu \to A_\mu$), thus ensuring that its Stückelberg mode is parity-odd ($\mathcal{PT} : \varphi \to -\varphi$). This is because the graviton would have to have an unphysical $\mathcal{PT}$ transformation, i.e. $\mathcal{PT} : h_{\mu\nu} \to -h_{\mu\nu}$ in order for $\pi \to -\pi$. **Therefore A_μ is an axial vector.**

Returning to **(2.)**, one must figure out how to pick the correct mass term for the gravitini. I will start with describing the two types of mass terms that a pair of Majorana fermions are allowed to have. Temporarily, I will return to unitary gauge so there are no Stückelberg modes. Firstly, the general form of a mass term is given by

$$\mathcal{S}_{\text{Gravitini Mass}}[\psi] = \int \mathrm{d}^4x \, \frac{1}{2} m \bar{\psi}_\mu \left(A \gamma^{\mu\nu} \right) \psi_\nu \tag{6.29}$$

$$:= \int \mathrm{d}^4x \, \frac{1}{2} m A^{ij} \bar{\psi}_\mu{}^i \gamma^{\mu\nu} \psi_\nu{}^j \, . \tag{6.30}$$

Since A^{ij} is a 2×2 matrix, there is a simple Hermitian choice of basis

$$A^{ij} = \text{span}\left\{ \delta^{ij}, \Delta^{ij}, i\varepsilon^{ij}, \eta^{ij} \right\} \tag{6.31}$$

where I define

$$\delta^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad i\varepsilon^{ij} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \tag{6.32}$$

$$\Delta^{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \eta^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{6.33}$$

These are, of course, just a different presentation of the Pauli matrices along with the identity matrix. It turns out that this is too many matrices. Firstly, the choice of the mass matrix $A = i\varepsilon$ is identically zero in the action (6.29), therefore this cannot generate a mass term. Moreover, the two matrices in (6.33) actually generate the same mass term. One can prove this in matrix notation via noting that

$$\begin{aligned}\mathcal{S}_{\text{Mass}}[\psi] &= \int \mathrm{d}^4x\, \frac{1}{2} m\, \bar{\psi}_\mu \left(\Delta\gamma^{\mu\nu}\right) \psi_\nu \\ \mathcal{S}_{\text{Mass}}[\phi] &= \int \mathrm{d}^4x\, \frac{1}{2} m\, \bar{\phi}_\mu \left(\eta\gamma^{\mu\nu}\right) \phi_\nu \end{aligned} \tag{6.34}$$

are equivalent under the field redefinition

$$\begin{aligned}\psi_\mu &\to \frac{1}{\sqrt{2}} \left(\delta + \varepsilon\right) \phi_\mu \\ \bar{\psi}_\mu &\to \bar{\phi}_\mu \frac{1}{\sqrt{2}} \left(\delta - \varepsilon\right) ,\end{aligned} \tag{6.35}$$

where the second line follows from the first by trivial matrix manipulations of $\bar{\psi}_\mu{}^i$. Therefore, one is left with only two linearly independent choices for masses. I choose $A \in \{\delta, \eta\}$. The first is a Majorana mass term (δ) and the second is a Dirac mass term (Δ).

One can finally choose between these two using the $\mathcal{U}(1)_R$ symmetry (6.24). Clearly, only the $A = \eta$ Majorana mass terms are invariant under this symmetry. This is easier to see in the R-charge basis, so let me re-express the mass term in R-charge basis:

$$\mathcal{L}_{\text{mass}} \sim \frac{1}{2} \bar{\psi}_\mu \Delta\gamma^{\mu\nu}\psi = \bar{\psi}_\mu{}^1 \gamma^{\mu\nu} \psi_\mu{}^2 , \tag{6.36}$$

which is manifestly invariant under R-symmetry transformations given by (6.24). Therefore, I must choose this as my mass term for the gravitini. **The physical significance of these two formulations is that one is manifestly in mass eigenbasis ($\sim \bar{\psi}\eta\psi$) and the other is manifestly in R-charge eigenbasis ($\sim \bar{\psi}\Delta\psi$).** The first statement follows trivially by expanding the definition of η^{ij}, and the second I have just demonstrated.

Once one realises this, they are immediately led to the Zinoviev action [166]. I will now discuss the ramifications of these choices and the properties of the $\mathcal{N} = 1$ Zinoviev theory.

6.2 The $\mathcal{N}=1$ Zinoviev Theory

The $\mathcal{N}=1$ Zinoviev theory has an action of the form

$$\begin{aligned}\mathcal{S}[h,\psi,A] &= \int \mathrm{d}^4x \Big[-\frac{1}{2}h_\mu{}^\alpha \delta^{\mu\nu\rho}_{\alpha\beta\gamma}\partial_\nu\partial^\beta h_\rho{}^\gamma + \frac{1}{2}m^2 h_\mu{}^\alpha \delta^{\mu\nu}_{\alpha\beta} h_\nu{}^\beta \\ &\quad -\frac{i}{2}\bar{\psi}_\mu{}^i\gamma^{\mu\nu\rho}\,\partial_\nu\psi_\rho{}^i + \frac{1}{2}m\bar{\psi}_\mu{}^i\gamma^{\mu\nu}\Delta^{ij}\psi_\nu{}^j \\ &\quad -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu\Big]\,. \end{aligned} \tag{6.37}$$

Simply stated, this is a theory which combines the actions for a Fierz-Pauli graviton, two Rarita-Schwinger gravitini with a Dirac mass, and a Proca photon. While this action is a valid formulation of the $\mathcal{N}=1$ Zinoviev Lagrangian, I will promote this to the Stückelberg formulation and reintroduce all of the gauge symmetries. To do this I introduce the Stückelberg fields, B_μ, π,ϕ, and χ^i through the usual manual installation of the original gauge symmetry

$$h_\mu{}^\alpha \to h_\mu{}^\alpha - \frac{1}{2m}\left(\partial_\mu B^\alpha + \partial^\alpha B_\mu\right) + \frac{1}{m^2}\partial_\mu\partial^\alpha\pi\,, \tag{6.38}$$

$$\psi_\mu{}^i \to \psi_\mu{}^i - \frac{1}{m}\partial_\mu\chi^i\,, \tag{6.39}$$

$$\bar{\psi}_\mu{}^i \to \bar{\psi}_\mu{}^i - \frac{1}{m}\partial_\mu\bar{\chi}^i\,, \tag{6.40}$$

$$A_\mu \to A_\mu - \frac{1}{m}\partial_\mu\varphi\,. \tag{6.41}$$

This restores linearised diffeomorphism invariance, the $\mathcal{U}(1)$ invariance for the graviphoton B_μ Stückelberg field, both of the supergauge symmetries of the two Rarita-Schwinger fields, and finally the $\mathcal{U}(1)$ of the axial Proca photon. The terms added to the action come, as usual, from the mass terms. Substituting the above relations yields an action of the form

$$\begin{aligned}\mathcal{S}_{\text{mass}} &= \int \mathrm{d}^4x \Big[\frac{1}{2}m^2\delta^{\mu\nu}_{\alpha\beta}\Big(h_\mu{}^\alpha h_\nu{}^\beta - \frac{2}{m}\partial_\mu B^\alpha h_\nu{}^\beta + \frac{2}{m^2}\partial_\mu\partial^\alpha\pi h_\nu{}^\beta - \frac{1}{m^2}\partial_\mu B_\nu\partial^\alpha B^\beta\Big) \\ &\quad +\frac{1}{2}m\bar{\psi}_\mu{}^i\gamma^{\mu\nu}\Delta^{ij}\psi_\nu{}^j - \bar{\psi}_\mu{}^i\gamma^{\mu\nu}\Delta^{ij}\partial_\nu\chi^j \\ &\quad -\frac{1}{2}m^2 A_\mu A^\mu + mA_\mu\partial^\mu\varphi - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi\Big]\,. \end{aligned} \tag{6.42}$$

Following from the Stückelberg formalisms introduced in Chapter 2, the reader may confer to this for the spin-2 and spin-1 cases. To recap, the scalar Stückelberg π needs a diagonalising transformation in order to obtain a canonical kinetic term.

The Rarita-Schwinger gravitini also needs a diagonalisation transformation. First, I canonically scale the requisite fields

$$\pi \to \sqrt{\frac{2}{3}}\pi\,, \tag{6.43}$$

$$B_\mu \to \frac{1}{\sqrt{2}}B_\mu\,, \tag{6.44}$$

$$\chi^i \to \sqrt{\frac{2}{3}}\chi^i\,, \tag{6.45}$$

and then use the diagonalisation transformations

$$\begin{aligned} h_\mu{}^\alpha &\to h_\mu{}^\alpha - \frac{1}{\sqrt{6}}\pi\delta_\mu^\alpha\,, \\ \psi_\mu{}^i &\to \psi_\mu{}^i - \frac{i}{\sqrt{6}}\gamma_\mu\Delta^{ij}\chi^j\,, \\ \bar{\psi}_\mu{}^i &\to \bar{\psi}_\mu{}^i + \frac{i}{\sqrt{6}}\bar{\chi}^j\Delta^{ji}\gamma_\mu\,, \end{aligned} \tag{6.46}$$

to obtain an action that has all canonically normalised fields. For simplicity, I will expand the action into the form $\mathcal{S} = \mathcal{S}_0 + m\mathcal{S}_1 + m^2\mathcal{S}_2$. In this expansion, the canonically normalised Zinoviev action in Stückelberg formalism is

$$\begin{aligned} \mathcal{S}_0 \;=\; \int \mathrm{d}^4x\, \Bigg[& -\frac{1}{2}h_\mu{}^\alpha\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\partial_\nu\partial^\beta h_\rho{}^\gamma - \frac{i}{2}\bar{\psi}_\mu{}^i\gamma^{\mu\nu\rho}\,\partial_\nu\psi_\rho{}^i \\ & -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{i}{2}\bar{\chi}^i\gamma^\mu\partial_\mu\chi^i \\ & -\frac{1}{2}\partial_\mu\pi\partial^\mu\pi - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi \Bigg]\,, \end{aligned} \tag{6.47}$$

$$\begin{aligned} m\mathcal{S}_1 \;=\; \int \mathrm{d}^4x\, \Big[& -m\sqrt{2}\delta^{\mu\nu}_{\alpha\beta}h_\mu{}^\alpha\partial_\nu B^\beta + m\sqrt{3}\pi\partial_\mu B^\mu\,, \\ & +\frac{1}{2}m\bar{\psi}_\mu{}^i\gamma^{\mu\nu}\Delta^{ij}\psi_\nu{}^j + im\sqrt{\frac{3}{2}}\bar{\psi}_\mu{}^i\gamma^\mu\chi^i + m\bar{\chi}^i\Delta^{ij}\chi^j + mA_\mu\partial^\mu\varphi \Big] \end{aligned} \tag{6.48}$$

$$m^2\mathcal{S}_2 \;=\; \int \mathrm{d}^4x\, \left[-\frac{1}{2}m^2A_\mu A^\mu + \frac{1}{2}m^2h_\mu{}^\alpha\delta^{\mu\nu}_{\alpha\beta}h_\nu{}^\beta + m^2\left(\pi^2 - \sqrt{\frac{3}{2}}\pi h_\mu{}^\mu\right)\right]\,. \tag{6.49}$$

6.2.1 Gauge Symmetries of the Zinoviev Lagrangian

The Zinoviev action in Stückelberg formalism has many symmetries. First, I will list the Stückelberg symmetries. The graviton has four (abelian) linearised diffeomorphisms with bosonic vector gauge parameter ξ_μ,

$$\delta h_{\mu\nu} = \partial_{(\mu}\xi_{\nu)}\,, \quad \delta B_\mu = m\sqrt{2}\xi_\mu\,, \quad \delta\pi = 0\,. \tag{6.50}$$

From its vector Stückelberg B_μ, the spin-2 also has a $\mathcal{U}(1)$ gauge symmetry with bosonic scalar gauge parameter ξ. They are

$$\begin{aligned}
\delta h_{\mu\nu} &= \frac{m}{2}\eta_{\mu\nu}\xi\,,\\
\delta B_\mu &= \partial_\mu\xi\,,\\
\delta\pi &= m\sqrt{\frac{3}{2}}\xi\,.
\end{aligned} \tag{6.51}$$

The axial Proca field A_μ has the traditional Stückelberg $\mathcal{U}(1)$ symmetry with bosonic pseudo-scalar gauge parameter θ. They are

$$\delta A_\mu = \partial_\mu\theta\,, \quad \delta\phi = m\theta\,.$$

Finally, gravitini fields have two supergauge symmetries with fermionic Majorana group parameter η^i, given by

$$\begin{aligned}
\delta\psi_\mu{}^i &= \partial_\mu\eta^i + i\frac{m}{2}\gamma_\mu\Delta^{ij}\eta^j\,.\\
\delta\chi^i &= m\sqrt{\frac{3}{2}}\eta^i\,.
\end{aligned} \tag{6.52}$$

6.2.2 The Supersymmetry Transformations

Finally, there is the crucial global $\mathcal{N}=1$ supersymmetry. With a global, Majorana fermionic parameter ε, the SUSY variations are

$$\begin{aligned}
\delta h_{\mu\nu} &= \alpha^i\, i\bar{\epsilon}\gamma_{(\mu}\psi_{\nu)}{}^i\,, \\
\delta\psi^i &= \alpha^i\,\gamma^{\alpha\beta}\partial_\alpha h_{\beta\mu}\epsilon - \frac{m}{\sqrt{2}}\Big[\gamma_\mu\gamma^\alpha B_\alpha + i\sqrt{3}\gamma_5 A_\mu\Big]\alpha^i\epsilon \\
&\quad -\frac{i}{4\sqrt{2}}\gamma^{\alpha\beta}\gamma_\mu\Big[G_{\alpha\beta}-\sqrt{3}i\gamma_5\mathcal{F}_{\alpha\beta}\Big]\beta^i\epsilon + im\Big[\gamma^\alpha h_{\alpha\mu}+\gamma_\mu\pi\Big]\beta^i\epsilon\,, \\
\delta B_\mu &= \beta^i\,\frac{1}{\sqrt{2}}\bar{\epsilon}\psi^i + \alpha^i\, i\frac{\sqrt{3}}{2}\bar{\varepsilon}\gamma_\mu\chi^i\,, \\
\delta A_\mu &= \beta^i\,\sqrt{\frac{3}{2}}\bar{\epsilon}\gamma_5\psi_\mu{}^i + \alpha^i\frac{1}{2}\bar{\epsilon}\gamma_\mu\gamma_5\chi^i\,, \\
\delta\chi^i &= -\frac{1}{4}\gamma^{\alpha\beta}\Big[\sqrt{3}G_{\alpha\beta}+i\gamma_5\mathcal{F}_{\alpha\beta}\Big]\epsilon\alpha^i \\
&\quad -i\gamma^\alpha\Big[\partial_\alpha\pi+\gamma_5\partial_\alpha\varphi\Big]\beta^i\epsilon + im\gamma^\alpha\Big[\sqrt{3}B_\alpha - i\gamma_5 A_\alpha\Big]\beta^i\epsilon\,, \\
\delta\pi &= i\beta^i\bar{\epsilon}\chi^i\,, \\
\delta\varphi &= \beta^i\, i\bar{\epsilon}\gamma_5\chi^i\,.
\end{aligned} \tag{6.53}$$

To simplify the language, I have chosen to package the fermions consistently into vectors of $\mathfrak{so}(2)$, i.e. I write all of my fermions with i indices. Note, however, that there is no $\mathfrak{so}(2)$ symmetry (one can easily check that it is broken by the Dirac mass term), thus in the SUSY variations, there are special directions in the $\mathfrak{so}(2)_R$ basis that are picked out in order to preserve invariance of the mass terms

$$\begin{aligned}
\alpha^i &= \begin{pmatrix}0\\1\end{pmatrix} = -\eta^{ij}\alpha^j\,, \\
\beta^i &= \begin{pmatrix}-1\\0\end{pmatrix} = \varepsilon^{ij}\alpha^j = -\Delta^{ij}\alpha^j\,,
\end{aligned} \tag{6.54}$$

which preserves the $\mathcal{N}=1$ SUSY R-symmetry transformations (6.24).

To state this differently and more directly at the level of the action, the kinetic terms are invariant under arbitrary α^i (where β^i is still a shorthand for $\beta^i = -\varepsilon^{ij}\alpha^j$), since only the mass terms break the $\mathcal{N}=2$ structure; in fact, it was first pointed out in [166] that long massive multiplets generically have kinetic terms that repackage into $\mathcal{N}=2$ supermultiplets with $\mathcal{N}=2$ supersymmetry, specifically an $\mathcal{N}=2$ gravity supermultiplet and an $\mathcal{N}=2$ vector supermultiplet:

$$\begin{pmatrix} & h_{\mu\nu} & \\ {\psi_\mu}^1 & & {\psi_\mu}^2 \\ & B_\mu & \end{pmatrix} \oplus \begin{pmatrix} & A_\mu & \\ \chi^1 & & \chi^2 \\ & \pi,\ \varphi & \end{pmatrix}. \tag{6.55}$$

The presence of the mass terms (and Stückelberg interactions) breaks this symmetry down to a single copy of $\mathcal{N}=1$ SUSY.[3] One can see the broken $\mathcal{N}=2$ SUSY by noticing that portions of the SUSY variations without a factor of m in (6.53) are $\mathcal{N}=2$ SUSY variations with $\epsilon^i=\alpha^i\epsilon$, cf. to the $\mathcal{N}=2$ gravity supermultiplet's transformations (6.22). This is a useful way to interpret the α^i and corresponding β^i parameters. They determine how the $\mathcal{N}=1$ SUSY for massive representations can be embedded into broken $\mathcal{N}=2$ SUSY for massless representations.

6.3 Dimensional Deconstruction of Fermions

6.3.1 Incorporating Fermions

Knowing that there is a supersymmetric extension to Fierz-Pauli is useful, but in principle one would like to know what a supersymmetric extension of dRGT massive gravity might look like (if it is even possible, or if it must be a broken phase of another SUGRA theory, *etc*). Unfortunately, discovering supersymmetric theories is quite challenging, and it will therefore be quite helpful to develop a procedure for obtaining SUSY theories from higher dimensional ones. The most systematic way, tailored way of obtaining massive gauge theories is Dimensional Deconstruction (henceforth "deconstruction"), see Chapter 4 for my review of this technique. It would therefore be quite beneficial and powerful if one could generate a method for deconstructing 5-D massless SUSY theories (of which much is known) into 4-D massive SUSY theories (of which little is known). It has been speculated in the literature, e.g. [166, 167], that there appears to be a relationship between 5-D massless supermultiplets and 4-D massive supermultiplets.

[3]Since one knows that this is the dominant description in the decoupling limit, $E\gg m$, the UV theory actually uplifts to an $\mathcal{N}=2$ theory. This has been christened "supersymmetry uplifting". I will return to this point in the discussion section of this chapter.

The aim of this section is to extend the deconstruction prescription for bosons, which is

$$\Phi(x,y) \rightarrow \phi(x)\,, \tag{6.56}$$

$$\partial_y\Phi(x,y) \rightarrow m\phi(x)\,, \tag{6.57}$$

$$\int \mathrm{d}y\,\Phi(x,y)\Psi(x,y) \rightarrow \phi(x)\psi(x)\,, \tag{6.58}$$

to incorporate the case when Φ and Ψ are fermions. In other words, I will derive a procedure to deconstruct fermions, which is the natural first step towards deriving a deconstruction procedure for SUSY theories. In keeping with desire for clarity and conceptual simplicity, I will only consider free SUSY theories and I will only deconstruct over a single site. Thus I will deconstruct a theory of a single 5-D massless fermion field to a single 4-D massive fermion field, and I will not be concerned about defining cases with fermions with multiple sites. Before continuing, I will note earlier attempts at this idea, however none of them generate Zinoviev theory [129, 130, 176].

6.3.2 Engineering Fermion Mass

In 5-D, there are no Majorana (or Weyl) spinor representations for fermions. The only available representation is the ordinary Dirac fermion. Since I have an eventual interest in SUSY, I will remind the reader of an alternative description for 5-D Dirac fermions, called **symplectic-Majorana spinors** (henceforth "spM" fermion). This is a compromise between the elegance and beautiful mathematical properties of Majorana fermions and the requirement of containing a full Dirac representation. An spM fermion is constructed by first writing down two Dirac fermions, indexed by $i = 1, 2$,

$$\Psi \rightarrow \Psi^i\,. \tag{6.59}$$

From there, one imposes a condition on the spM fermion,

$$\bar{\Psi}^i = \left(\bar{\Psi}^j\right)^T \Omega^{ji}\mathcal{C}_5 \tag{6.60}$$

where $\bar{\Psi}^i := (\Psi^i)^\dagger\Gamma^0$, $\mathcal{C}_5$ is the 5-D charge matrix and Ω^{ij} is an symplectic form (i.e. a non-degenerate, invertible, anti-symmetric 2-tensor). For $i = 1, 2$, then $\Omega^{ij} = \varepsilon^{ij}$ is simply a Levi-Civita 2-tensor. This condition forces the 2 Dirac fermions to contain only a single Dirac fermion worth of information. However, spM fermions obey many novel identities and useful properties that are directly analogous to their 4-D Majorana fermion cousins. I review the exact definitions, useful identities, and the decomposition of 5-d spM fermions in section A.2.1. For later convenience, I define Γ^M, $M = 0, 1, 2, 3, 5$, as

$$\Gamma^M = \begin{pmatrix} \gamma^\mu \\ i\gamma_5 \end{pmatrix}, \tag{6.61}$$

which obey the Clifford-algebra relation

$$\{\Gamma^M, \Gamma^N\} = -2\eta^{MN}. \tag{6.62}$$

Taking all of this together, one can take the Dirac action for a spin-$\frac{1}{2}$ field Ψ in 5-D and re-write it as a spM action for a spin-$\frac{1}{2}$ field Ψ^i, i.e.

$$\begin{aligned} \mathcal{S}_{\mathrm{D}}[\Psi] &= \int \mathrm{d}^5x \left(\frac{i}{2} \bar{\Psi}\, \Gamma^M \partial_M \Psi \right), & (6.63) \\ \Leftrightarrow \mathcal{S}_{\mathrm{spM}}[\Psi^i] &= \int \mathrm{d}^5x \left(\frac{i}{2} \bar{\Psi}^i\, \Gamma^M \partial_M \Psi^i \right), & (6.64) \\ &= \int \mathrm{d}^4x \mathrm{d}y \left(\frac{i}{2} \bar{\Psi}^i\, \gamma^\mu \partial_\mu \Psi^i + \frac{i}{2} \bar{\Psi}^i\, (i\gamma_5) \partial_5\, \Psi^i \right). & (6.65) \end{aligned}$$

If I take this action and perform a (4+1)-split, $X^M = (x^\mu, y)$, one obtains

$$\mathcal{S}_{\mathrm{spM}}[\Psi^i] = \int \mathrm{d}^4x \mathrm{d}y \left(\frac{i}{2} \bar{\Psi}^i\, \gamma^\mu \partial_\mu \Psi^i + \frac{i}{2} \bar{\Psi}^i\, (i\gamma_5) \partial_y\, \Psi^i \right). \tag{6.66}$$

If I then decompose the 5-D symplectic-Majorana fermion into two 4-D Majorana fermions, i.e. plugging (A.28) into (6.65), the action takes on the form

$$= \int \mathrm{d}^4x \mathrm{d}y\, \frac{i}{2} \bar{\psi}^i\, \gamma^\mu \partial_\mu \psi^i + \frac{1}{2} \varepsilon^{ij} \bar{\psi}^i (\partial_y \psi^j). \tag{6.67}$$

If one wishes to generate fermion mass terms, then one can read off that they would need to deconstruct ∂_y derivatives on fermions following the rule

$$\begin{array}{rclclcl} \partial_y \psi^i & \to & m(-\varepsilon^{ij})\psi^j & \Longrightarrow & \mathcal{L}_{\mathrm{mass}} & = & \frac{1}{2} m \bar{\psi}^i\, \psi^i \\ \partial_y \psi^i & \to & m(\eta^{ij})\psi^j & \Longrightarrow & \mathcal{L}_{\mathrm{mass}} & = & \frac{1}{2} m \Delta^{ij} \bar{\psi}^i\, \psi^j \\ \partial_y \psi^i & \to & m(\Delta^{ij})\psi^j & \Longrightarrow & \mathcal{L}_{\mathrm{mass}} & = & \frac{1}{2} m \eta^{ij} \bar{\psi}^i\, \psi^j \end{array} \tag{6.68}$$

to get the Majorana mass, Dirac mass in R-charge eigenbasis, and Dirac mass in mass eigenbasis, respectively. The latter two are physically equivalent as Dirac mass terms, but the first is the Majorana mass. **This is not the typical scenario, since now one is forced to make the derivative mix the R-symmetry/symplectic indices.** To get a handle on this peculiar feature, I will give a group-theoretic description, analogous to the bosonic case in section 4.1.

6.3.3 Group-Theoretic Perspective

Fermions obey a dispersion relation that follows from the formula

$$\begin{aligned} \left(i\Gamma^M P_M\right)|F_{5-D}\rangle &= 0 \\ \Longrightarrow \left(i\partial\!\!\!/ + \epsilon\partial_5\right)|F_{4-D}\rangle &= 0, \end{aligned} \tag{6.69}$$

which I want to obtain the familiar bosonic Klein-Gordon relationship

$$\left(k^\mu k_\mu + m^2\right)|F_{4-D}\rangle = 0\,. \tag{6.70}$$

In order to preserve the massive dispersion relation for fermionic states this is actually less restrictive than for bosons. Here, I can see that I may insert a matrix, M s.t. $(\varepsilon M)^2 = m^2\delta$, into the deconstruction deformation procedure,

$$\partial_y|F\rangle \to M|F\rangle \text{ , so that } (\varepsilon P_y)^2 \to (\varepsilon M)^2 = m^2\,,$$

and I will maintain the desired dispersion relation! Thus if I have R-symmetry indices in 5-D, I have shown that they can be deformed in the deconstruction procedure to generate the different kinds of fermion masses in 4-D. For the case of obtaining the Dirac mass, one can see that the correct choice is

$$= (\varepsilon\eta)^2 = m^2(\Delta)^2 = m^2\delta\,. \tag{6.71}$$

Then one sees that this leads to a Dirac mass: $\frac{1}{2}\bar{\psi}^i\varepsilon^{ij}\partial_y\psi^j \to \frac{1}{2}\bar{\psi}^i\Delta^{ij}\psi^j$.

6.3.4 Prescription for Deconstructing Fermions

The other masses follow similar patterns as before, but since I am interested in SUSY, I will only consider the deconstruction procedure for the Dirac mass in the remainder of this chapter. Taken all of what I have developed above, if one wishes to obtain a Dirac mass, one must apply

$$\begin{aligned}\psi^i(x,y) &\rightarrow \psi^i(x)\,,\\ \partial_y\psi^i(x,y) &\rightarrow m\eta^{ij}\psi^j(x)\,,\\ \int \mathrm{d}y\,\bar{\psi}^i\phi^i &\rightarrow \bar{\psi}^i\phi^i\,.\end{aligned} \tag{6.72}$$

Alternatively, for 5-D spM fermions, one can check that this is equivalent to

$$\partial_y\Psi^i(x,y) \rightarrow m\Delta^{ij}\Psi^j(x)\,. \tag{6.73}$$

When I apply these rules to my action for a 5-D spM fermion, I obtain a 4-D action for two massive spin-$\frac{1}{2}$ Majorana fermions with a Dirac mass, i.e.

$$\mathcal{S}_{\mathrm{spM}} = \int \mathrm{d}^4x\mathrm{d}y\,\frac{i}{2}\bar{\psi}^i\,\gamma^\mu\partial_\mu\psi^i + \frac{1}{2}\varepsilon^{ij}\bar{\psi}^i(\partial_y\psi^j) \tag{6.74}$$

$$\rightarrow \mathcal{S}_{\mathrm{M}} = \int \mathrm{d}^4x\left(i\frac{1}{2}\bar{\psi}^i\gamma^\mu\partial_\mu\psi^i + \frac{1}{2}m\Delta^{ij}\bar{\psi}^i\psi^j\right)\,. \tag{6.75}$$

6.4 Super-Proca Theory a lá Deconstruction

6.4.1 5-D $\mathcal{N}=2$ Super-Maxwell Theory

I now turn to the question of whether or not this deconstruction procedure is consistent with SUSY, and whether I am actually able to derive 4-D massive SUSY theories from 5-D SUSY theories following this prescription. A useful 5-D gauge theory to use as a benchmark is 5-D super-Maxwell theory. 5-D super-Maxwell theory is given by the massless superhelicity-$\frac{1}{2}$ supermultiplet,

$$\begin{pmatrix} A_M \\ \Psi^i \\ \Phi \end{pmatrix}\,. \tag{6.76}$$

Which has one massless spin-1 A_μ field (photon), a massless spin-$\frac{1}{2}$ symplectic-Majorana field Ψ^i (photini), and one massless spin-0 Φ (axion[4]). Note that in 5-D, a massless spin-1 field has 3 PDF's, a spin-0 has 1 PDF, and a massless

[4]This contains no axion interactions since the theory is free; I chose this nomenclature because this field generates the pseudoscalar field in 4-D, and is considerably more elegant than, e.g., "sphotino".

spin-$\frac{1}{2}$ Dirac/symplectic-Majorana field has 4 PDF's; thus the counting goes 3 + 1 bosonic PDF's = 4 fermionic PDF's.

In my conventions, the 5-D $\mathcal{N} = 2$ super-Maxwell Lagrangian has the action

$$\mathcal{S} = \int \mathrm{d}^5x \left[-\frac{1}{4} F_{MN} F^{MN} + i\frac{1}{2} \bar{\Psi}^i \Gamma^M \partial_M \Psi^i - \frac{1}{2} (\partial_M \phi)^2 \right] , \tag{6.77}$$

with using the usual $F_{MN} = \partial_M A_N - \partial_N A_M$. Beyond the obvious $\mathcal{U}(1)$ gauge symmetry, $\delta A_M = \partial_M \xi$, this theory also obeys an $\mathcal{N} = 2$ 5-D SUSY symmetry. The variations for this symmetry are

$$\begin{aligned} \delta A_M &= i\bar{\varepsilon}^i \Gamma_M \Psi^i , \\ \delta \Psi^i &= -\frac{1}{2} \Gamma^{AB} F_{AB} \varepsilon^i - \Gamma^M \partial_M \phi \varepsilon^i , & (6.78) \\ \delta \phi &= i\bar{\varepsilon}^i \Psi^i . & (6.79) \end{aligned}$$

With a little effort, one can prove that they obey the 5-D SUSY algebra

$$\begin{aligned} [\delta_1, \delta_2] A_M &= \xi^R \partial_R A_M + \partial_M \theta , \\ [\delta_1, \delta_2] \Psi^i &= \xi^R \partial_R \Psi^i + (\text{E.O.M.}) , \\ [\delta_1, \delta_2] \phi &= \xi^R \partial_R \phi , & (6.80) \end{aligned}$$

with

$$\begin{aligned} \xi^M &= 2i\bar{\varepsilon}_2^i \Gamma^M \varepsilon_1^i & (6.81) \\ \theta &= -\xi^M A_M + 2i\bar{\varepsilon}_2^i \, \varepsilon_1^i . & (6.82) \end{aligned}$$

6.4.2 Deconstructing 5-D Super-Maxwell

The first step in the deconstruction procedure is to perform a $(4+1)$-split. For the vector mode, I split it as

$$\begin{aligned} A_M &= \begin{pmatrix} A_\mu \\ \pi \end{pmatrix} , & (6.83) \\ \Phi &= \phi , & (6.84) \end{aligned}$$

and I use the relations $\Psi^i = \mathcal{P}^{ij}\psi^j$ following section A.2.1. Then, one arrives at the action

$$\begin{aligned}
\mathcal{S}_{\text{5-D sMaxwell}} &= \int \mathrm{d}^4x\mathrm{d}y \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_y A_\mu - \partial_\mu \pi)^2 \right.\\
&\quad + i\frac{1}{2}\bar{\psi}^i\gamma^\mu\partial_\mu\psi^i + \frac{1}{2}\varepsilon^{ij}\bar{\psi}^i\partial_y\psi^j \\
&\quad \left. -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}(\partial_y\phi)^2\right] . \qquad (6.85)
\end{aligned}$$

I am explicitly using the identities (A.28) in section A.2.1. I will use the definition $\varepsilon^i = P^{ij}\epsilon^j$ for the SUSY parameter, which puts (6.78) into the (4+1)-split form

$$\begin{aligned}
\delta A_\mu &= i\bar{\epsilon}^i\gamma_\mu\psi^i ,\\
\delta\psi^i &= -\frac{1}{2}\gamma^{\alpha\beta}F_{\alpha\beta}\epsilon^i - i\gamma^\alpha(\partial_y A_\alpha - \partial_\alpha\pi)\varepsilon^{ij}\epsilon^j ,\\
&\quad +\gamma^\mu\gamma_5\partial_\mu\phi\,\varepsilon^{ij}\epsilon^j - i\gamma_5\partial_y\phi\epsilon^i ,\\
\delta\phi &= -i\varepsilon^{ij}\bar{\epsilon}^i\gamma_5\psi^j ,\\
\delta\pi &= \varepsilon^{ij}\bar{\epsilon}^i\psi^j . \qquad (6.86)
\end{aligned}$$

Finally, I will apply my deconstruction prescription. Collecting previous results, the prescription is given by

$$\begin{aligned}
&\partial_y A_\mu = mA_\mu ,\\
&\partial_y\psi^i = m\,\eta^{ij}\psi^j ,\\
&\partial_y\phi = m\phi . \qquad (6.87)
\end{aligned}$$

Together, they deform the action into

$$\begin{aligned}
\rightarrow \mathcal{S}_{\mathcal{N}=1\text{ Super-Proca}} &= \int \mathrm{d}^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2\left(A^\mu - \frac{1}{m}\partial_\mu\pi\right)^2 \right.\\
&\quad + i\frac{1}{2}\bar{\psi}^i\gamma^\mu\partial_\mu\psi^i + \frac{1}{2}m\Delta^{ij}\bar{\psi}^i\psi^j \\
&\quad \left. -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2\right] . \qquad (6.88)
\end{aligned}$$

A few comments are in order. Firstly, by not choosing a gauge prior to deconstructing, I obtained the action in the Stückelberg form where one can see that the Stückelberg mode is clearly given by $A_y = \pi$. If one applies the deconstruction rule on the gauge transformation $\delta A_y = \partial_y\theta$, then one manifestly obtains the Stückelberg symmetry transformation $\delta\pi = m\pi$! Secondly, one of the scalars must be a pseudoscalar for the same reason as the Zinoviev theory: There is a Wess-Zumino sub-supermultiplet sitting in this theory when $E \gg m$, which forces one of these scalars to be parity-odd/a pseudoscalar ($\mathcal{PT} : \phi \to -\phi$). Noting that the photon is not parity-odd/an axial vector, this imposes that ϕ must be a pseudoscalar as a consistency condition. In the next subsection, I prove this explicitly at the level of the transformations. Thirdly, I have not shown how one might take the 5-D $\mathcal{N} = 2$ transformations (6.86) and deconstruct these into the 4-D SUSY variations. I proceed to this task now.

6.4.3 4-D Supercharges from Deconstructed 5-D Supercharges

If I now apply the deconstruction prescription (6.87) onto the 5-D $\mathcal{N} = 2$ transformations (6.86), I obtain

$$\begin{aligned}
\delta A_\mu &= i\bar{\epsilon}^i \gamma_\mu \psi^i \,, \\
\delta \psi^i &= -\frac{1}{2}\gamma^{\alpha\beta} F_{\alpha\beta}\epsilon^i - i\gamma^\mu \partial_\mu \big(\pi - i\gamma_5 \phi\big)\varepsilon^{ij}\epsilon^j \,, \\
&\quad -m\big(i\gamma^\alpha A_\alpha \varepsilon^{ij} + i\gamma_5 \phi \delta^{ij}\big)\epsilon^j \,, \\
\delta\phi &= -i\varepsilon^{ij}\bar{\epsilon}^i \gamma_5 \psi^j \,, \\
\delta\pi &= \varepsilon^{ij}\bar{\epsilon}^i \psi^i \,.
\end{aligned} \tag{6.89}$$

This looks like an $\mathcal{N} = 2$ SUSY transformation, but from the fact that deconstruction has broken y-translations, so

$$\delta X = \xi \partial_y X \Longrightarrow \delta\mathcal{S} \neq 0 \tag{6.90}$$

means that there cannot be a full copy of $\mathcal{N} = 2$ supersymmetry (where X is any field in the supermultiplet). This follows from the 5-D $\mathcal{N} = 2$ SUSY algebra

$$\begin{aligned}
[\delta(\bar{\varepsilon}_1^i), \delta(\varepsilon_2^j)]X &= 2i\bar{\varepsilon}_2^i \Gamma^A \varepsilon_1^i \, \partial_A X \\
&= 2i\bar{\varepsilon}_2^i \Gamma^\mu \varepsilon_1^i \partial_\mu X + \underbrace{\bar{\varepsilon}_2^i \Gamma^5 \varepsilon_1^j \varepsilon^{ij} \partial_y X}_{A=0,\text{ must be imposed}} \,.
\end{aligned} \tag{6.91}$$

Obviously then, only one linear combination of $\varepsilon^i = \alpha^i \varepsilon$ survives. By direct calculation from the action,

$$\delta\mathcal{S} = \underbrace{\delta_0 \mathcal{S}_0}_{=0} + m\Big(\underbrace{\delta_0 \mathcal{S}_1 + \delta_1 \mathcal{S}_0}_{\propto(\Delta^{ij} - \varepsilon^{ij})\alpha^j}\Big) + m^2\Big(\underbrace{\delta_1 \mathcal{S}_1 + \delta_0 \mathcal{S}_2}_{\propto(\Delta^{ij} - \varepsilon^{ij})\alpha^j}\Big) + m^3 \underbrace{\delta_1 \mathcal{S}_2}_{=0} \tag{6.92}$$

leads to the condition

$$\varepsilon^{ij}\alpha^j = \Delta^{ij}\alpha^j \,, \tag{6.93}$$

which combined with $\beta^i = \varepsilon^{ij}\alpha^j$ leads to the previously known (6.54), i.e.

$$\begin{aligned} \alpha^i &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\eta^{ij}\alpha^j \,, \\ \beta^i &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \varepsilon^{ij}\alpha^j = -\Delta^{ij}\alpha^j \,. \end{aligned} \tag{6.94}$$

In other words, this picks out the same special direction in the $\mathfrak{so}(2)_R$ plane that the Zinoviev theory does! Likewise, this manifestly preserves the action.

Substituting these relations into (6.95) leads to the final $\mathcal{N} = 1$ SUSY variations for $\mathcal{N} = 1$ super-Proca theory

$$\begin{aligned} \delta A_\mu &= \alpha^i \left(i\bar{\epsilon}\gamma_\mu\psi^i\right) \,, \\ \delta\psi^i &= \alpha^i \left(-\frac{1}{2}\gamma^{\alpha\beta}F_{\alpha\beta}\epsilon\right) + \beta^i \left(-i\gamma^\mu\partial_\mu\big(\pi - i\gamma_5\phi\big)\epsilon\right) \,, \\ &\quad +\beta^i \left(-m\big(i\gamma^\alpha A_\alpha\epsilon\big) + \alpha^i \left(-mi\gamma_5\phi\epsilon\right) \,, \right. \\ \delta\phi &= \beta^i \left(-i\bar{\epsilon}\gamma_5\psi^i\right) \,, \\ \delta\pi &= \beta^i \left(\bar{\epsilon}\psi^i\right) \,. \end{aligned} \tag{6.95}$$

They obey the SUSY algebra

$$\begin{aligned} &[\delta_1,\, \delta_2]A_\mu = \xi^\nu\partial_\nu A_\mu + \partial_\mu\theta \,, \\ &[\delta_1,\, \delta_2]\psi^i = \xi^\nu\partial_\nu\psi^i + (\text{E.O.M.}) \,, \\ &[\delta_1,\, \delta_2]\pi = \xi^\nu\partial_\nu\pi + m\theta \\ &[\delta_1,\, \delta_2]\phi = \xi^\nu\partial_\nu\phi \end{aligned} \tag{6.96}$$

with

$$\xi^\alpha = 2i\bar{\varepsilon}_2\gamma^\alpha\varepsilon_1 \,, \tag{6.97}$$

$$\theta = -\xi^\nu A_\nu \,. \tag{6.98}$$

There are a few final comments for this theory. Firstly, notice that although the action does not give any indication for the parity of the two scalars, the parity can be read off immediately from the SUSY transformations, and even though the mode originally started off with even-parity assignment in 5-D, ϕ becomes parity odd (a pseudoscalar) under 4-D $\mathcal{PT}$ transformations. Secondly, the algebra closes on a Stückelberg gauge symmetry transformation, in direct analogy to the massless super-Maxwell case. Thirdly, this has a clean interpretation of deforming an $\mathcal{N} = 2$ 5-D supercharge Q^i for a massless supermultiplet into an $\mathcal{N} = 1$ 4-D supercharge for a massive supermultiplet.

6.5 Deconstructing $D = 5$ SUGRA to the Zinoviev Theory

6.5.1 Review of Linearised 5-D $\mathcal{N} = 2$ Supergravity

Now that I have demonstrated how to deconstruct fermions and how one can deconstruct SUSY theories into massive SUSY theories, I turn to the central issue of this chapter, namely if this procedure actually is capable of generating Zinoviev theory, with all of the information about SUSY variations and gauge symmetries, from 5-D $\mathcal{N} = 2$ linearised SUGRA. Note that this theory is the obvious choice to deconstruct, since it is the supermultiplet containing

$$\begin{pmatrix} H_{MN} \\ \Psi_M{}^i \\ A_M \end{pmatrix} . \tag{6.99}$$

Thus this multiplet contains a massless spin-2 field H_{MN} (graviton), one symplectic-Majorana spin-$\frac{3}{2}$ field $\Psi_M{}^i$ (gravitino), and one spin-1 field A_M (graviphoton) [177–180]. The action for linearised 5-D $\mathcal{N} = 2$ supergravity is given by

$$\mathcal{S} = \int \mathrm{d}^5 X \left[-\frac{1}{2} H_M{}^A \left(\delta^{MNR}_{ABC} \partial_N \partial^B \right) H_R{}^C - i\frac{1}{2} \bar{\Psi}_M{}^i \Gamma^{MNR} \partial_N \Psi_R{}^i - \frac{1}{4} F_{MN} F^{MN} \right] , \tag{6.100}$$

which is just a collection of the linearised Einstein-Hilbert action, the Rarita-Schwinger action for a symplectic-Majorana fermion, and a 5-D Maxwell action. In 5-D, the physical degrees of freedom counting goes as 5 PDF's for a massless spin-2 field, 8 PDF's for a massless symplectic-Majorana spin-$\frac{3}{2}$, and 3 PDF's for a massless spin-1 field. Other than the manifest Poincaré invariance and the obvious abelian gauge symmetries,

$$\begin{aligned}
\delta H_{MN} &= \partial_{(M}\xi_{N)} = \frac{1}{2}\left(\partial_M\xi_N + \partial_N\xi_M\right) \\
\delta\Psi_M{}^i &= \partial_M\Lambda^i \\
\delta A_M &= \partial_M\xi\,,
\end{aligned} \tag{6.101}$$

it enjoys an $\mathcal{N}=2$ global SUSY

$$\begin{aligned}
\delta H_M{}^A &= i\frac{1}{2}\bar{\varepsilon}^i\left(\Gamma_M\Psi^{A\,i} + \Gamma^A\Psi_M{}^i\right)\,, \\
\delta\Psi_M{}^i &= \Gamma^{AB}\partial_A H_{BM}\varepsilon^i + \frac{1}{2\sqrt{6}}\left(\Gamma^{AB}{}_M - 4\Gamma^A\delta^B_M\right)F_{AB}\varepsilon^i\,, \\
\delta A_M &= -i\sqrt{\frac{3}{2}}\bar{\varepsilon}^i\Psi_M{}^i
\end{aligned} \tag{6.102}$$

where the global $\mathcal{N}=2$ group parameter ε^i is a symplectic-Majorana fermion.

6.5.2 Deconstructing to the $\mathcal{N}=1$ Zinoviev Action

Now I will deconstruct the 5-D supergravity theory into the 4-D Zinoviev theory. With a little effort, it can be seen that the 5-D modes can be decomposed into the canonically-normalised 4-D modes in Stückelberg formalism upon making the decomposition

$$H_M{}^A = \begin{pmatrix} h_\mu{}^\alpha - \frac{1}{\sqrt{6}}\pi\delta^\alpha_\mu & \frac{1}{\sqrt{2}}B^\alpha \\ \frac{1}{\sqrt{2}}B_\mu & \sqrt{\frac{2}{3}}\pi \end{pmatrix}\,, \tag{6.103}$$

$$A_M = \begin{pmatrix} A_\mu \\ \varphi \end{pmatrix}\,. \tag{6.104}$$

After using this definition for the bosons, and then applying bosonic deconstruction, it straightforwardly leads to the actions for Fierz-Pauli and Proca 4-D theories with canonical kinetic terms. The major new piece here is deconstructing the Rarita-Schwinger action, so I will explicitly derive this piece of the action. Following the usual route of first applying the $(4+1)$-split, $X^M = (x^\mu,\, y)$, the 5-D Rarita-

Schwinger action decomposes into

$$\begin{aligned}
\mathcal{S}_{\text{RS}} &= \int \mathrm{d}^5x \left[-i\frac{1}{2}\bar{\Psi}_M{}^i\Gamma^{MNR}\partial_N\Psi_R{}^i\right], &(6.105)\\
&= \int \mathrm{d}^4x\mathrm{d}y \left[-i\frac{1}{2}\bar{\Psi}_\mu{}^i\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho{}^i - i\frac{1}{2}\bar{\Psi}_\mu{}^i\gamma^{\mu\nu}(i\gamma_5)\left[2\partial_\nu\Psi_y{}^i - \partial_y\Psi_\mu{}^i\right]\right], &(6.106)\\
&= \int \mathrm{d}^4x\mathrm{d}y \left[-i\frac{1}{2}\bar{\psi}_\mu{}^i\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho{}^i - \varepsilon^{ij}\bar{\psi}_\mu{}^i\gamma^{\mu\nu}\partial_\nu\psi_y{}^i + \frac{1}{2}\varepsilon^{ij}\bar{\psi}_\mu{}^i\gamma^{\mu\nu}\partial_y\psi_\mu{}^i\right]. &(6.107)
\end{aligned}$$

Next, I decompose the 5-D spM fermions into 4-D Majorana doublet, and deconstruct via $\partial_y\psi_\mu{}^i = m\eta^{ij}\psi_\mu{}^j$. This yields the following action

$$= \int \mathrm{d}^4x \left[-i\frac{1}{2}\bar{\psi}_\mu{}^i\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho{}^i + \frac{1}{2}m\bar{\psi}_\mu{}^i\gamma^{\mu\nu}\Delta^{ij}\psi_\nu{}^j - \bar{\psi}_\mu{}^i\gamma^{\mu\nu}\varepsilon^{ij}\partial_\nu\psi_y{}^j\right]. \quad (6.108)$$

Upon making the identification (and redefining ψ_μ)

$$\bar{P}^{ij}\Psi_M{}^j = \begin{pmatrix}\psi_\mu{}^i\\ \psi_y{}^i\end{pmatrix} \to \begin{pmatrix}\psi_\mu{}^i - \frac{i}{\sqrt{6}}\gamma_\mu\Delta^{ij}\chi^j\\ \sqrt{\frac{2}{3}}\eta^{ij}\chi^j\end{pmatrix}, \quad (6.109)$$

one arrives at fully diagonalised and canonically-normalised kinetic terms within the action. Explicitly, the Rarita-Schwinger action deconstructs to

$$\begin{aligned}
\mathcal{S}_{\text{RS-D}} = \int \mathrm{d}^4x \; &-\frac{i}{2}\bar{\psi}_\mu{}^i\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho{}^i + \frac{i}{2}\bar{\chi}^i\gamma^\mu\partial_\mu\chi^i\\
&+\frac{1}{2}m\bar{\psi}_\mu{}^i\gamma^{\mu\nu}\Delta^{ij}\psi_\nu{}^j + im\sqrt{\frac{3}{2}}\bar{\psi}_\mu{}^i\gamma^\mu\chi^i + m\bar{\chi}^i\Delta^{ij}\chi^j, \quad (6.110)
\end{aligned}$$

which when combined with the bosonic actions leads to the full action for $\mathcal{N} = 1$ Zinoviev theory. Therefore, I have shown that the linearised 5-D $\mathcal{N} = 2$ supergravity action can be deconstructed to the 4-D $\mathcal{N} = 1$ Zinoviev action (6.47).

Also, one can deconstruct the 5-D supergauge symmetries $\delta\Psi_M{}^i = \partial_M\Lambda^i$ using $\Lambda^i = \mathcal{P}^{ij}\eta^j$, $\partial_y\eta^i = m\eta^{ij}\eta^j$, and (6.109), obtaining the Stückelberg supergauge symmetries,

$$\begin{aligned}
\delta\psi_\mu{}^i &= \partial_\mu\eta^i + i\frac{m}{2}\gamma_\mu\Delta^{ij}\eta^j,\\
\delta\chi^i &= m\sqrt{\frac{3}{2}}\eta^i, \quad (6.111)
\end{aligned}$$

where Λ^i is a spM fermion and η^i is a Majorana doublet.

6.5.3 Deriving the Zinoviev SUSY Transformations

The next question is whether or not the Zinoviev transformations (6.53) can be obtained from deconstructing the 5-D SUSY variations (6.101), in the same manner as how the super-Proca $\mathcal{N} = 1$ transformations were obtained from the $\mathcal{N} = 2$ Maxwell SUSY transformations. First, I will collect the decomposition formulas for the 5-D modes into their 4-D constituents

$$H_M{}^A = \begin{pmatrix} h_\mu{}^\alpha - \frac{1}{\sqrt{6}}\pi\delta^\alpha_\mu & \frac{1}{\sqrt{2}}B^\alpha \\ \frac{1}{\sqrt{2}}B_\mu & \sqrt{\frac{2}{3}}\pi \end{pmatrix} , \tag{6.112}$$

$$\bar{P}^{ij}\Psi_M{}^j = \begin{pmatrix} \psi_\mu{}^i - \frac{i}{\sqrt{6}}\gamma_\mu\Delta^{ij}\chi^j \\ \sqrt{\frac{2}{3}}\eta^{ij}\chi^j \end{pmatrix} , \tag{6.113}$$

$$A_M = \begin{pmatrix} A_\mu \\ \varphi \end{pmatrix} . \tag{6.114}$$

From here, one simply needs to apply these definitions consistently to (6.101), following the deconstruction prescription

$$\begin{aligned} &\partial_y H_{MN} \to m H_{MN} \\ &\partial_y \psi_\mu{}^i \to m\eta^{ij}\psi_\mu{}^j \\ &\partial_y \chi^i = m\eta^{ij}\chi^j \\ &\partial_y A_M = mA_M \,. \end{aligned} \tag{6.115}$$

Performing a (4+1)-split on the transformations, substituting in the decompositions, and applying the deconstruction prescription is laborious. Here I will write down the major steps, starting with the spin-2 mode,

$$\begin{aligned}
\delta H_y{}^y &= -\bar{\varepsilon}^i\gamma_5\Psi_y{}^i = \varepsilon^{ij}\bar{\epsilon}^i\left(\sqrt{\frac{2}{3}}\eta^{jk}\chi^k\right),\\
&= \sqrt{\frac{2}{3}}\delta\pi\,,\\
\Longrightarrow \delta\pi &= i\beta^i\bar{\epsilon}\chi^i\,, \qquad (6.116)\\
&,\\
\delta H_\mu{}^y &= \frac{1}{2}\bar{\varepsilon}^i\left(\gamma_\mu\Psi_y{}^i + i\gamma_5\Psi_\mu{}^i\right),\\
&= \frac{1}{\sqrt{2}}\delta B_\mu\,,\\
\Longrightarrow \delta B_\mu &= \beta^i\frac{1}{\sqrt{2}}\bar{\epsilon}\psi^i + \alpha^i\, i\frac{\sqrt{3}}{2}\bar{\varepsilon}\gamma_\mu\chi^i\,, \qquad (6.117)\\
\delta H_\mu{}^\alpha &= i\bar{\varepsilon}^i\gamma_{(\mu}\Psi_{\nu)}{}^i = i\bar{\varepsilon}^i\gamma_{(\mu}\left(\psi_{\nu)}{}^i - \frac{i}{\sqrt{6}}\gamma_{\nu)}\Delta^{ij}\chi^j\right),\\
&= \delta h_\mu{}^\alpha - \frac{1}{\sqrt{6}}\delta\pi\delta_\mu^\alpha\,,\\
\Longrightarrow \delta h_\mu{}^\alpha &= \alpha^i\, i\bar{\epsilon}\gamma_{(\mu}\psi_{\nu)}{}^i\,. \qquad (6.118)
\end{aligned}$$

In the last equation, I have made use of the $\varepsilon^i = \alpha^i\varepsilon$ and related identities (6.54). The next easiest to work through are the spin-1 transformation rules. The major intermediary steps and the final results are given by

$$\begin{aligned}
\delta A_y &= -i\sqrt{\frac{3}{2}}\bar{\varepsilon}^i\Psi_y{}^i = -i\sqrt{\frac{3}{2}}\varepsilon^{ij}\bar{\epsilon}^i\gamma_5\left(\sqrt{\frac{2}{3}}\eta^{jk}\chi^k\right),\\
\Longrightarrow \delta\varphi &= \beta^i\, i\bar{\epsilon}\gamma_5\chi^i\,, \qquad (6.119)\\
& \qquad (6.120)\\
\delta A_\mu &= -i\sqrt{\frac{3}{2}}\bar{\varepsilon}^i\Psi_\mu{}^i = -i\sqrt{\frac{3}{2}}\varepsilon^{ij}\bar{\epsilon}^i\gamma_5\left(\psi_\mu{}^i - \frac{i}{\sqrt{6}}\gamma_\mu\Delta^{ij}\chi^j\right),\\
\Longrightarrow \delta A_\mu &= \beta^i\sqrt{\frac{3}{2}}\bar{\epsilon}\gamma_5\psi_\mu{}^i + \frac{1}{2}\bar{\epsilon}\gamma_\mu\gamma_5\chi^i\,. \qquad (6.121)
\end{aligned}$$

This completes the analysis of the bosonic transformations; a quick comparison to (6.53) shows that I have correctly obtained the bosonic transformation rules for the Zinoviev theory!

I now turn to the fermionic SUSY transformations, which are considerably more laborious than their bosonic counterparts. Once again, splitting the transformations, decomposing the modes, and deconstructing the y-derivatives yields

$$\begin{aligned}\bar{P}^{ij}\delta\Psi_5{}^j &= \frac{1}{\sqrt{2}}\gamma^{\alpha\beta}\partial_\alpha B_\beta - i\gamma^\alpha\left(\sqrt{\frac{2}{3}}\partial_\alpha\pi - \frac{1}{\sqrt{2}}\partial_y B_\alpha\right)\varepsilon^{ij}\epsilon^j\,,\\ &\quad -\frac{1}{2\sqrt{6}}\gamma^{\alpha\beta}\gamma_5 F_{\alpha\beta}\epsilon^i + \sqrt{\frac{2}{3}}\gamma^\alpha\gamma_5\left(\partial_y A_\alpha - \partial_\alpha\varphi\right)\varepsilon^{ij}\epsilon^j\,,\\ &= \sqrt{\frac{2}{3}}\eta^{ij}\delta\chi^j\,, \end{aligned} \tag{6.122}$$

for the gravitini. For the photino fields χ^i, I find

$$\begin{aligned}\bar{P}^{ij}\delta\Psi_\mu{}^j &= \gamma^{\alpha\beta}\partial_\alpha\left(h_{\beta\mu} - \frac{1}{\sqrt{6}}\pi\eta_{\beta\mu}\right)\epsilon^i + i\gamma^\alpha\left[\partial_y\left(h_{\mu\alpha} - \frac{1}{\sqrt{6}}\eta^{\mu\alpha}\right) - \frac{1}{\sqrt{2}}B_\alpha\right]\varepsilon^{ij}\epsilon^j\,,\\ &\quad +\frac{1}{2\sqrt{6}}\left[\gamma^{\alpha\beta}{}_\mu - 4\gamma^\alpha\delta^\beta_\mu\right]F_{\alpha\beta}\varepsilon^{ij}\epsilon^j - \frac{i}{\sqrt{6}}[\gamma^\alpha{}_\mu - 4\delta^\alpha_\mu]\gamma_5(\partial_y A_\alpha - \partial_\alpha\varphi)\epsilon^i\,,\\ &= \delta\psi_\mu{}^i - \frac{i}{\sqrt{6}}\gamma_\mu\Delta^{ij}\delta\chi^j\,. \end{aligned} \tag{6.123}$$

Next, if I compare these SUSY transformations to (6.53), it shows that this has failed to reproduce the immediate Zinoviev transformations. However, this is only a superficial difference. They may be put into the form originally written by Zinoviev in [166] by performing a field-dependent supergauge transformation, with supergauge parameter

$$\eta^i = \left(-\frac{1}{\sqrt{6}}\pi\alpha^i\epsilon + i\frac{1}{\sqrt{2}}\gamma^\alpha B_\alpha\beta^i\epsilon\right). \tag{6.124}$$

After which, I have manifestly obtained the Zinoviev transformations for the fermions, namely

$$\begin{aligned}\delta\psi_\mu{}^i &= \alpha^i\gamma^{\alpha\beta}\partial_\alpha h_{\beta\mu}\epsilon - \frac{m}{\sqrt{2}}\Big[\gamma_\mu\gamma^\alpha B_\alpha + i\sqrt{3}\gamma_5 A_\mu\Big]\alpha^i\epsilon\,,\\ &\quad -\frac{i}{4\sqrt{2}}\gamma^{\alpha\beta}\gamma_\mu\Big[G_{\alpha\beta} - \sqrt{3}i\gamma_5 F_{\alpha\beta}\Big]\beta^i\epsilon + im\Big[\gamma^\alpha h_{\alpha\mu} + \gamma_\mu\pi\Big]\beta^i\epsilon\,,\\ \delta\chi^i &= -\frac{1}{4}\gamma^{\alpha\beta}\Big[\sqrt{3}G_{\alpha\beta} + i\gamma_5 F_{\alpha\beta}\Big]\epsilon\alpha^i\,,\\ &\quad -i\gamma^\alpha\partial_\alpha\Big[\pi + \gamma_5\varphi\Big]\beta^i\epsilon + im\gamma^\alpha\Big[\sqrt{3}B_\alpha - i\gamma_5 A_\alpha\Big]\beta^i\epsilon\,. \end{aligned} \tag{6.125}$$

Again, a few final comments. First, one can easily read off that the lowest spin in the 5-D supermultiplet, here A_μ, has again flipped parity from what it was in the 5-D theory. A quick analysis of these SUSY variations clearly tells one that this is an axial vector and its Stückelberg φ mode is a pseudoscalar. Secondly, it is interesting that (given this interpretation) there is a simultaneous supergauge transformation alongside. Thirdly, I will not write down the commutators of the SUSY variations, but they do indeed reproduce the normal SUSY algebra, which closes on Stückelberg gauge transformations and EOM (ziltch symmetries), as one expects.

6.6 Discussion of Fermionic Deconstruction

I have explicitly demonstrated how one may deconstruct a symplectic-Majorana fermion from 5-D into a two different kinds of fermions in 4-D, namely an doublet of fermions with Majorana or Dirac masses. Following the work of Zinoviev and the shared criteria for a linear theory of massive supergravity, in order to keep an $\mathcal{N} = 1$ SUSY present, one must choose the Dirac mass in order to keep SUSY. If one does so, then I have shown that a 5-D $\mathcal{N} = 2$ super-Maxwell theory may be deconstructed into a 4-D $\mathcal{N} = 1$ super-Proca theory, and a 5-D $\mathcal{N} = 2$ linear SUGRA theory may be deconstructed to give a 4-D $\mathcal{N} = 1$ Zinoviev theory (i.e. an $\mathcal{N} = 1$ "super-Fierz-Pauli theory"). If the pattern holds, as seems likely, one may use this outlined procedure to extract seemingly unrelated theories of massive superspin-Y fields from higher dimensional theories of massless superhelicity-$(Y - \frac{1}{2})$ fields. Remarkably, I have shown how to do this entirely within the Stückelberg formalism and explicitly have extracted the 4-D transformations from the 5-D transformations, showing that the deconstruction procedure robustly and completely is generated from higher dimensions. The Zinoviev theory is not the theory obtained from any Kaluza-Klein compactification, making this more remarkable. Previously, Dimensional Deconstruction [118, 120, 121, 124] prescriptions have relied heavily upon a clear relationship to Kaluza-Klein and similar compactification arguments in order to keep the consistency of the resulting massive theory obvious. However, our procedure does seem naturally interpretable from the standpoint of extracting representations of D-dimensional representations from $(D + 1)$-dimensional representations.

Unfortunately, this says nothing about interacting theories, which I have reviewed in the purely bosonic case in section 4.3.1; even there it is quite a bit more complicated than these linear theories. There are several considerations here. For instance, it may be useful to see if one can export the results from this section to the superfield formalism. In 5-D, the $\mathcal{N} = 2$ SUSY (8 real supercharges) is too much for an ordinary superfield formalism, e.g. [181, 182], there are extensions to the superfield for $\mathcal{N} = 2$ theories (such as those discussed in [183], e.g. harmonic superspace). It may be possible, and indeed quite advantageous, before moving over to interacting massive SUSY theories to formulate the SUSY deconstruction procedure I have given into the a deformation directly upon an (e.g. harmonic) massless 5-D superfield action to 4-D massive superfield action. The presence of SUSY is manifest in this formalism, so the failure to uphold SUSY should be obvious under such a procedure. In the final chapter, I will review what happens in component formalism, but it would be quite beneficial.

Secondly, it is interesting to further map out how one might be able to find $\mathcal{N} = 1$ self-interacting massive supergravity theories (here it may or may not have a literal $\mathcal{N} = 1$ SUSY, but ideally it will have the same field content of Zinoviev theory with supergravity kinetic terms and dRGT/ghost-free mass terms). In the final chapter, I will go over the progress in building such a theory by Dimensionally Deconstructing 5-D $\mathcal{N} = 2$ supergravity.

Chapter 7

Decontructing Supergravity

I now move onto 1-site deconstructing the full, interacting theory of $\mathcal{N} = 2$ 5-D supergravity into a 4-D theory whose spectrum contains a massive graviton, two massive Majorana gravitini, and a massive pseudo-spin-1 field.[1] This theory may or may not possess supersymmetry; it cannot possess a local SUSY since this theory contains no massless spin-$\frac{3}{2}$ field, although it is not *a priori* impossible that it contains a global SUSY. In either case, a completely ghost-free theory of a massive gravitini field would be new to the literature (see for instance a discussion in [185]). At another level, it is interesting to see what qualities a deconstructed interacting theory has within the fermionic deconstruction procedure I developed in the previous chapter.

This chapter proceed as follows:

(1.) I will review the most important aspects of supergravity theory (SUGRA), including its tightly-knit relationship to local SUSY which forms the (super)gauge redundancy for interacting gravitino fields and briefly touch on the uniqueness theorems for supergravity theories. I will specifically outline 4-D and 5-D minimal supergravity theories.

(2.) I will then take the full 5-D SUGRA theory and perform the deconstruction procedure, including on fermionic interactions, at the level of the action. This will lead to a theory with many interesting and novel properties.

(3.) I will demonstrate that the presence of an $\mathcal{N} = 1$ global SUSY in the theory is greatly obstructed by the absence of a way to extract supercharges, in the manner performed in Chapter 6, when interactions are present. I will then conclude the chapter with some final remarks on what is clear about the de-

[1] This follows major results to appear in [184].

constructed theory thus far and possible ways to resolve the inadequacies of the current method.

7.1 Review of Supergravity

I will now quickly review the salient features of supergravity theories; stated most simply, supergravity theories are theories which contain interacting spin-$\frac{3}{2}$ field(s) and when the spin-$\frac{3}{2}$ field(s) are massless, the theory possess a local SUSY invariance (the gauge redundency for a massless spin-$\frac{3}{2}$ field). The supergauge symmetry $\psi_\mu \to \psi_\mu + \partial_\mu \epsilon$ of the linearised theory always combines with the global SUSY $\delta\psi \sim \partial h \epsilon$ into a combined local SUSY symmetry $\psi_\mu \sim D_\mu \epsilon$ (when linearised, the spin connection forms the ∂h contribution to the global symmetry). It is worth noting that the proofs for ordinary GR, its uniqueness, and its relation to diffeomorphism invariance in [3, 4, 6–8] may be extended into supergravity and local SUSY, e.g. see proofs contained in [133, 186, 187]. Namely, supergravity (SUGRA) is the unique theory of a massless, self-interacting spin-$\frac{3}{2}$ fields, it is the fundamental theory of supergauge/local SUSY invariance, and it has non-trivial soft-scattering implications (including containing GR as its spin-2 superpartner). To begin with a concrete example, I start with the first discovered theory, that of $\mathcal{N} = 1$ supergravity (SUGRA) in four spacetime dimensions.

7.1.1 $\mathcal{N} = 1$ and $\mathcal{N} = 2$ 4-D Supergravity

The simplest theory of supergravity in 4-D is $\mathcal{N} = 1$ supergravity, which contains a massless spin-2 field $e_\mu{}^a$ (graviton) and a massless spin-$\frac{3}{2}$ field (gravitino) [131, 133, 186]. The $\mathcal{N} = 1$ supergravity action is given by the combination of the Einstein-Cartan action and Rarita-Schwinger action, thus it is

$$\mathcal{S}_{\text{4-D SUGRA}} = \int \mathrm{d}^4x \, \frac{1}{2\kappa^2} e \, \mathcal{R}\,[\hat{\omega}] - i\frac{1}{2} e \bar{\psi}_\mu{}^i \gamma^{\mu\nu\rho} D_\nu \, [\hat{\omega}] \psi_\rho \,, \tag{7.1}$$

with $D_\mu \, [\hat{\omega}]\psi_\nu := \partial_\mu \psi_\nu - \frac{1}{4}\gamma_{ab}\hat{\omega}_\mu{}^{ab}\psi_\nu$. Upon integrating out the spin connection $\hat{\omega}$ and going into second-order form (see Appendix C for details), one will obtain 4-Fermi interactions from the contorsion squared $K^2 \sim (\bar{\psi}\psi)^2$, since for connection 1-forms $\hat{\omega} = \omega + K$, the curvature 2-form follows

$$\mathcal{R}[\hat{\omega}] = \mathrm{d}\hat{\omega} + \hat{\omega}\hat{\omega} = \mathcal{R}[\omega] + D[\omega]\, K + KK \,, \tag{7.2}$$

which in $\mathcal{L} = \mathcal{R}^{ab} e^c e^d \varepsilon_{abcd}$ gives rise to the Einstein-Hilbert term, a total derivative DK, and the aforementioned 4-Fermi interactions, respectively.

Aside from diffeomorphisms and local Lorentz transformations, this action is additionally invariant under a local $\mathcal{N} = 1$ SUSY

$$\delta_Q e_\mu{}^a = i\frac{1}{2}\kappa^2 \bar{\epsilon}\gamma^a \psi_\mu \tag{7.3}$$

$$\delta_Q \psi_\mu = D_\mu \epsilon\,, \tag{7.4}$$

where the superalgebra is filled out, using $X = e_\mu{}^a$ or ψ_μ, the closure rule

$$[\delta_Q(\epsilon_1),\, \delta_Q(\epsilon_2)]X = \delta_{\text{Diff}}\left(\xi^\nu\right) + \delta_{\text{LLT}}\left(\xi^\mu \omega_\nu{}^{ab}\right) + \delta_Q\left(\xi^\nu \psi_\nu\right)\,. \tag{7.5}$$

Here I have used the usual $\xi^\mu = i\frac{1}{2}\kappa^2 \bar{\epsilon}_2 \gamma^\mu \epsilon_1$ (note that my k value is now $\frac{1}{2}$ away from the previous chapter, which is a more convenient convention for SUGRA theories). These gauge transformations form the gauge redundancies for the gravitino field, and from this one can see that much like the spin-2 field couples to the conserved stress-energy current (implied by diff invariance) and the spin-1 field couples to the conserved color current (implied by $\mathcal{U}(1)$ or SU(N) invariance), the spin-$\frac{3}{2}$ field couples to the conserved supercurrent implied by SUSY invariance.

The $\mathcal{N} = 2$ theory contains an additional pair of fields, since $\mathcal{N} = 2$ doubles the size of the multiplet. Now it has content $(e_\mu{}^a,\, \psi_\mu{}^i,\, A_\mu)$, for $i = 1, 2$. It follows the same action as above, sending bispinors to $\bar{\psi}\psi \to \bar{\psi}^i \psi^i$ and includes a Pauli interaction, $\sim \varepsilon^{ij} \bar{\psi}_\mu{}^i (F^{\mu\nu} + \gamma_5 * F^{\mu\nu}) \psi_\nu{}^i$. The full theory can be derived via a dimensional reduction of 5-D SUGRA, e.g. as was done in [126, 177, 178].

7.1.2 $\mathcal{N} = 2$ 5-D Supergravity

The 5-D theory of $\mathcal{N} = 2$ 5-D SUGRA [177–179] contains the usual terms, i.e. the Einstein-Hilbert, Rarita-Schwinger kinetic terms, and the covariant Maxwell term. In order to keep the action invariant under the SUSY variations, they are supplemented with two interaction terms, a Pauli term and a Chern-Simons interaction term. The total action is therefore given by

$$\begin{aligned}\mathcal{S}_{\text{5-D SUGRA}} = \int \mathrm{d}^5 X\, & \frac{1}{2\kappa^2} E\,\mathcal{R}\left[\tilde{\Omega}\right] - i\frac{1}{2} E \bar{\Psi}_M{}^i \Gamma^{MNR} D_N \left[\tilde{\Omega} - \frac{1}{2}\Theta\right] \Psi_R{}^i - \frac{1}{4} E\, F_{MN} F^{MN} \\ & - \kappa \frac{i}{8}\sqrt{\frac{3}{2}} E\, \bar{\Psi}_P{}^i X^{MNPQ} \Psi_Q{}^i \left(F_{MN} + \hat{F}_{MN}\right) \\ & - \frac{1}{6\sqrt{6}} \kappa\, \varepsilon^{MNRSL} F_{MN} F_{RS} A_L\,. \end{aligned} \tag{7.6}$$

Note that local Minkowski fibre indices are labelled by $\{A,\, B, \dots\}$ and the tangent space indices are labelled by $\{M, N, R, P, Q, \dots\}$. For convenience, I will remind the reader that in my conventions,

$$
\begin{aligned}
G_{MN} &= E_M{}^A \eta_{AB} E_N{}^B && (7.7)\\
E &= \mathrm{Det}[E_M{}^A] && (7.8)\\
\mathcal{R}_{MN}{}^{AB}[\Omega] &= \partial_{[M}\Omega_{N]}{}^{AB} - \Omega_{[M}{}^{AC}\Omega_{N]C}{}^B && (7.9)\\
&= E_P{}^A E_Q{}^B\, \mathcal{R}_{MN}{}^{PQ}\,[G_{MN}]\ , && (7.10)
\end{aligned}
$$

the graviphoton-gravitini tensors are defined via

$$
\begin{aligned}
\hat{F}_{MN} &= (\partial_M A_N - \partial_N A_M) - i\frac{\sqrt{6}}{4}\kappa^2 \bar{\Psi}_M{}^i \Psi_N{}^i && (7.11)\\
X^{MNPQ} &= \Gamma^{MNPQ} + G^{MP}G^{NQ} - G^{NP}G^{MQ}\,, && (7.12)
\end{aligned}
$$

and the various spin connection constituents are defined via

$$
\begin{aligned}
O^{ABC} &= E^{M\,A}E^{N\,B}\left(\partial_{[M}E_{N]}{}^C\right) && (7.13)\\
\Omega_M{}^{AB}[E] &= E_{M\,C}\left(O^{ABC} - O^{BCA} - O^{CAB}\right) && (7.14)\\
\hat{K}_{M\,AB} &= i\frac{\kappa^2}{4}\left(\bar{\Psi}_A{}^i \Gamma_M \Psi_B{}^i + 2\bar{\Psi}_M{}^i \Gamma_{[A}\Psi_{B]}{}^i\right)\ , && (7.15)
\end{aligned}
$$

and

$$
\begin{aligned}
\Theta_{M\,AB} &= -i\frac{\kappa^2}{8}\left(\bar{\Psi}_P{}^i \Gamma^{PQ}{}_{MAB}\Psi_Q{}^i\right) && (7.16)\\
\tilde{K}_M{}^{AB} &= \hat{K}_M{}^{AB} + \Theta_M{}^{AB} && (7.17)\\
\hat{\Omega}_M{}^{AB} &= \Omega_M{}^{AB}[E] + \hat{K}_M{}^{AB} && (7.18)\\
\tilde{\Omega}_M{}^{AB} &= \Omega_M{}^{AB}[E] + \tilde{K}_M{}^{AB}\,. && (7.19)
\end{aligned}
$$

Noting that $\mathcal{R}[G_{MN}]$ is the usual Riemann tensor defined with Christoffel symbols $\sim(\partial\Gamma + \Gamma\Gamma)$, and that $\Psi_A{}^i := E_A{}^M \Psi_M{}^i$. It is also worth noting that the auxiliary field is $\tilde{\Omega}$, so just like the 4-D case, there are also four-Fermi interactions coming from the kinetic term of $\Psi_M{}^i$ after integrating out $\tilde{\Omega}$.

This action is invariant under the local SUSY transformations

$$\delta_Q E_M{}^A = i\frac{1}{2}\kappa^2\,\bar{\varepsilon}^i\Gamma^A\Psi_M{}^i \tag{7.20}$$

$$\delta_Q \Psi_M{}^i = \hat{D}_M\varepsilon^i + \frac{1}{4\sqrt{6}}\left[\Gamma_M{}^{PQ} - 4\Gamma^P\delta_M^Q\right]\hat{F}_{PQ}\varepsilon^i \tag{7.21}$$

$$\delta_Q A_M = -i\sqrt{\frac{3}{8}}\kappa\,\bar{\varepsilon}^i\Psi_M{}^i\,, \tag{7.22}$$

where, as one expects, ε^i is a symplectic-Majorana fermion.

7.2 Properties of the Conjectured Non-Linear Theory

The best case scenario for an interacting theory with a spectrum containing a massive graviton, massive gravitini, and a massive spin-1 field (graviphoton) is one that is ghost-free and contains an $\mathcal{N}=1$ SUSY (here the global Killing spinor would need to come from the reference vielbein). If such a theory existed, it would necessarily follow that it must obey the following diagram of scaling limits for its two fundamental parameters, M_{Pl} and m:

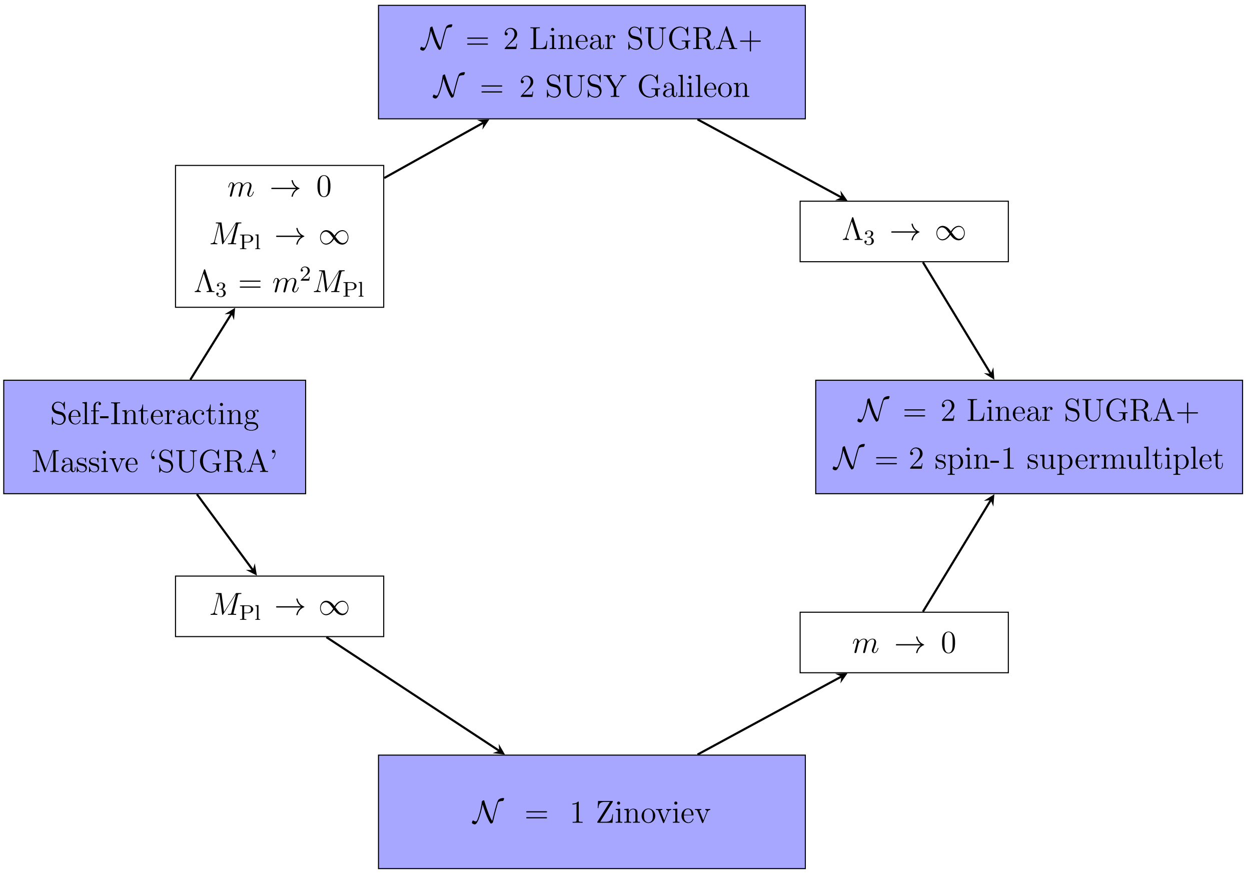

Figure 7.1: The scaling limits of the conjectured interacting theory of a massive supermultiplet. Notice that each limit of the theory ends with $\mathcal{N} = 2$ SUSY, even though the original massive theory only has $\mathcal{N} = 1$. The supersymmetry is enhanced in any $m \to 0$ limit, because the spin-$\frac{3}{2}$ are then massless and must have 2 local supersymmetries for their supergauge redundancies.

First, any limit which sends $m \to 0$ must uplift the SUSY from $\mathcal{N} = 1$ to $\mathcal{N} = 2$ (owing to the presence of two massless gravitini fields). Then the major question is what happens to the interactions when $M_{\rm Pl}$ is scaled. Quite simply, it should linearise to the $\mathcal{N} = 1$ Zinoviev theory (super-Fierz-Pauli) for the obvious reasons, so $M_{\rm Pl} \to \infty$ must result in Zinoviev theory. The decoupling limit,

$$\begin{aligned} &m \to 0\,, \\ &M_{\rm Pl} \to \infty\,, \\ &M_{\rm Pl} m^2 \to \Lambda_3^3\,, \end{aligned} \tag{7.23}$$

is another interesting question. Presuming that the fermionic interactions do not lower the scale of interactions away from $\left(\frac{1}{M_{\rm Pl} m^2}\right)^n \mathcal{O}^{4+3n}$, then the decoupling limit must be some kind of $\mathcal{N}=2$ supersymmetric Galileon theory (a vector multiplet) decoupled from a linearised $\mathcal{N}=2$ SUGRA theory. It is encouraging to note that some $\mathcal{N}=1$ super-Galileon theories were explored in [188, 189], so it is possible to have supersymmetric Galileon theories. This is a very interesting limit, although I will not explore it in detail in here.

I will instead focus on the fact that the theory should linearise to Zinoviev; this provides a good justification for applying Dimensional Deconstruction to 5-D $\mathcal{N}=2$ SUGRA, since whatever results from this will have the right field content and bosonic PDF's. I will reiterate here that even if the deconstructed SUGRA theory does not itself possess an $\mathcal{N}=1$ global SUSY, one should expect that this diagram approximately holds. One should expect that this theory ought to be a very good ansatz for ghost-freedom and having global SUSY, and by construction contains the same spectrum of fields and corresponding non-linear, gauge-invariant kinetic terms and, to linear order, the correct mass terms.

7.3 1-Site Deconstruction of $\mathcal{N}=2$ 5-D SUGRA

Now I turn to deconstructing the full 5-D SUGRA theory following the procedures in Chapter 4 and 6 following the 1-site prescription. To wit, this means that I use the gauge-fixing

$$E_M{}^A = \begin{pmatrix} e_\mu{}^a & 0 \\ 0 & 1 \end{pmatrix} \tag{7.24}$$

$$\Omega_y{}^{ab} = 0 \tag{7.25}$$

$$\Psi_M{}^i = \begin{pmatrix} P^{ij}\psi_\mu{}^j \\ 0 \end{pmatrix} \tag{7.26}$$

$$A_M = \begin{pmatrix} A_\mu \\ 0 \end{pmatrix}, \tag{7.27}$$

Again, this completely fixes all of the gauge symmetries for 5-D SUGRA –the local 5-D Diff($\mathcal{M}$) and $SO(1,4)$ for the fünfbein $E_M{}^A$, the local $U(1)$ for the graviphoton A_M, and both of the local $\mathcal{N}=2$ SUSY of the gravitini $\Psi_M{}^i$. This will cause the deconstructed $D=4$ action to sit in unitary gauge.

From here, I apply the deformation procedure

$$\begin{aligned} \partial_y e_\mu{}^a &\to m(e_\mu{}^a - \delta_\mu{}^a) && (7.28)\\ \partial_y \psi_\mu{}^i &\to m\Delta^{ij}\psi_\mu{}^j && (7.29)\\ \partial_y A_\mu &\to mA_\mu\,, && (7.30)\end{aligned}$$

where again $\Psi_\mu{}^i = P^{ij}\psi^j$ following section A.2.1. Due to the lengthiness of calculation, I will first apply this procedure to the purely bosonic actions, and then I will apply it to the covariant Rarita-Schwinger term and the other gravitini interaction terms.

7.3.1 Spin-2 Sector

I will repeat quickly, now taking into account the presence of torsion, the EC term.

$$\mathcal{S}_{\rm EC}[E,\Omega] = \int \frac{1}{4\cdot 3!\kappa^2}\mathcal{R}^{AB}\left[\tilde{\Omega}\right] E^C E^D E^F \varepsilon_{ABCDF} \tag{7.31}$$

Integrating out the auxiliary field $\tilde{\Omega}^{AB}$, leads to the algebraic torsion constraint

$$\mathrm{d}E^A + \tilde{\Omega}^A{}_B E^B = \tilde{T}^A\,. \tag{7.32}$$

Following Appendix C, I decouple this equation via the definition

$$\tilde{\Omega}^{AB} = \Omega^{AB}[E] + \tilde{K}^{AB}\,, \tag{7.33}$$

which upon substitution into into the original torsion-free condition satisfied by the usual definition of $\Omega[E] \sim E^{-1}\partial E$. This will deconstruct identically into the usual manner.

The second equation, $\tilde{K}^A{}_B E^B = \tilde{T}^A$ leads to 4-Fermi interactions which I shall address later in the next section on fermionic interactions. The pure graviton interactions, however, obviously then give rise to the exact same manner. Following the same results as those derived in Chapter 4, the deconstructed action for the pure spin-2 self-interactions gives rise to the ordinary dRGT theory of a massive spin-2 field

$$\begin{aligned} \mathcal{S}_{\rm dRGT} &= \frac{1}{4\kappa^2}\int \mathcal{R}^{ab}[\omega]e^c e^d\varepsilon_{abcd} + m^2\Big(e^a-\delta^a\Big)\Big(e^b-\delta^b\Big)e^c e^d\varepsilon_{abcd}\,,\\ & e^a_{[\mu}f_{\nu]\,a} = 0\,. \end{aligned} \tag{7.34}$$

7.3.2 Spin-1 Sector

Starting with the covariant graviphoton bosonic terms

$$\mathcal{S}_{\text{spin-1}} = \int \mathrm{d}^5X\, E\left(-\frac{1}{4}F_{MN}F^{MN} - \frac{1}{E\,6\sqrt{6}}\kappa\,\varepsilon^{MNRSL}F_{MN}F_{RS}A_L\right) \tag{7.35}$$

If one follows the deconstruction prescription, then one quickly sees that the Chern-Simons term vanishes identically under deconstruction, since $\mathcal{L}_{\text{CS}} \propto \varepsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}F_{\rho\sigma}A_y \propto A_y = 0$, leaving only the covariant Maxwell term. From here, one immediately derives a covariant Proca action from it, via

$$\mathcal{S}_{\text{5-D}} = \int \mathrm{d}^5X\, E\left(-\frac{1}{4}F_{MN}F^{MN}\right) = \int \mathrm{d}^4x\mathrm{d}y\, e\left(-\frac{1}{4}\left(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + 2\mathcal{F}_{y\mu}\mathcal{F}^{y\mu}\right)\right) \tag{7.36}$$

which under deconstruction naturally leads to

$$\rightarrow\ \mathcal{S}_{\text{4-D}} = \int \mathrm{d}^4x\, e\left(-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}m^2A_\mu A^\mu\right), \tag{7.37}$$

where I have used $E = 1\cdot e$, and the formula

$$G^{MN} = \begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.38}$$

These follow from my gauge-fixing conventions.

7.3.3 Deconstructing the Fermions

To begin the fermionic interactions, I will start with collecting the 4-Fermi interactions from its contributors; here the contributing terms are the squared contorsion terms from the Einstein-Cartan action (after integrating out $\tilde{\Omega}$) and from the contorsion piece in the covariant Rarita-Schwinger action. Starting with the EC action, one has

$$= \int \frac{1}{4\cdot 3!\kappa^2}\left(\mathrm{d}\tilde{\Omega}^{AB} + \tilde{\Omega}^{A}{}_{F}\tilde{\Omega}^{FB}\right)E^CE^DE^F\,\varepsilon_{ABCDF}\,. \tag{7.39}$$

The curvature 2-form decomposes under $\tilde{\Omega} = \Omega[E] + \tilde{K}$ into the Riemann tensor and contorsion squared pieces, i.e.

$$\begin{aligned}\tilde{\mathcal{R}}^{AB} &= \mathrm{d}\tilde{\Omega}^{AB} + \tilde{\Omega}^{A}{}_{F}\,\tilde{\Omega}^{FB} && (7.40)\\ &= \mathrm{d}\Omega^{AB}[E] + \Omega^{A}{}_{F}[E]\,\Omega^{FB}[E] + D_{\Omega[E]}\,\tilde{K}^{AB} + \tilde{K}^{A}{}_{F}\tilde{K}^{FB} && (7.41)\\ &= \mathcal{R}^{AB}[\Omega[E]] + D_{\Omega[E]}\,\tilde{K}^{AB} + \tilde{K}^{A}{}_{F}\tilde{K}^{FB}\,. && (7.42)\end{aligned}$$

Following the same as the 4-D $\mathcal{N}=1$ SUGRA case, the term proportional to $D\tilde{K}^{AB}$ is a total derivative, and leaving only the $\tilde{K}^2$ term in the action. The $\mathcal{R}EEE$ term obviously has already been accounted for in the dRGT action. Next, employing (7.17) which be derived from laborious but straightforward algebra, one finds finally finds the equation for $\tilde{K}^{AB}$ to be

$$\tilde{K}_{M}{}^{AB} = \frac{i\kappa^2}{4}\left[\bar{\Psi}_{A}{}^{i}\Gamma_{M}\Psi_{B}{}^{i} + 2\bar{\Psi}_{M}{}^{i}\Gamma_{[A}\Psi_{B]}{}^{i} - \frac{1}{2}\bar{\Psi}_{P}{}^{i}\,\Gamma^{PQ}{}_{M}{}^{AB}\Psi_{Q}{}^{i}\right]. \qquad (7.43)$$

Then, the Einstein-Cartan action's K^2 4-Fermi terms are given by

$$\mathcal{S}^{\mathrm{EC}}_{\text{5-D 4-Fermi}} = \int \frac{1}{4\cdot 3!\kappa^2}\left(\hat{K}^{A}{}_{F}\hat{K}^{FB} + 2\hat{K}^{A}{}_{F}\Theta^{FB} + \Theta^{A}{}_{F}\Theta^{FB}\right)E^{C}E^{D}E^{E}\,\varepsilon_{ABCDE} \qquad (7.44)$$

Then one may apply this split in the spin connection in the covariant Rarita-Schwinger action, $\bar{\Psi}(D+\tilde{K})\Psi$ which will split into a covariant kinetic term for a Rarita-Schwinger field and the desired contribution to the 4-Fermi interaction (Note that there is a shift in the connection for the gravitini, which also contributes outright to the 4-Fermi interaction terms). Honing in on the latter, one finds

$$\begin{aligned}\mathcal{S}_{\text{5-d RS}} &= \int \mathrm{d}^5X\Big(-i\frac{1}{2}E\bar{\Psi}_{M}{}^{i}\Gamma^{MNR}\underbrace{D_{N}\left[\tilde{\Omega}-\frac{1}{2}\Theta\right]}_{D[\Omega+\hat{K}-\frac{1}{2}\Theta]}\Psi_{R}{}^{i}\Big) && (7.45)\\ \Longrightarrow \mathcal{S}^{\mathrm{RS}}_{\text{4-Fermi}} &= \int \mathrm{d}^5X\left(-i\frac{1}{2}E\bar{\Psi}_{M}{}^{i}\Gamma^{MNR}\left[\left(-\frac{1}{4}\Gamma_{AB}\right)\left(\hat{K}_{N}{}^{AB} - \frac{1}{2}\Theta_{N}{}^{AB}\right)\right]\Psi_{R}{}^{i}\right) && (7.46)\end{aligned}$$

Combining all of this with those terms from the Pauli interactions $\bar{\Psi}\hat{F}\Psi \sim (\bar{\Psi}\Psi)^2$ with only 4-D derivatives, one must note that the Dimensional Deconstruction procedure, particularly since $\Psi_{y}{}^{i} = 0$, changes nothing about the overall 4-Fermi structure present in 4-D $\mathcal{N}=2$ SUGRA, since there are no derivatives are involved in the derivation of the 4-Fermi interactions. Thus, it follows deconstructing results identically in the same structure that one gets via straight dimensional reduction [126, 178, 180], and thus must be directly equivalent to the $\mathcal{N}=2$ 4-Fermi interaction, i.e. those found in [171, 190].

Rarita-Schwinger Kinetic Term

Next, I deconstruct the purely covariant (the portion of the spin connection containing only the vielbein contribution and no contorsion) Rarita-Schwinger action. This procedure results in

$$\mathcal{S}_{\text{5-d RS}} = \int \mathrm{d}^5X \left(-i\frac{1}{2}E\bar{\Psi}_M{}^i\Gamma^{MNR}\left(\partial_N - \frac{1}{4}\Gamma_{AB}\Omega_N{}^{AB}\right)\Psi_R{}^i\right) \quad (7.47)$$

The (4+1)-split on the action, employing the simplifications gauge-fixing applies on the spin connection in Chapter 4, results in

$$\begin{aligned}\mathcal{S}_{\text{5-d RS}} &= \int \mathrm{d}^4x\mathrm{d}y\, e\left[-i\frac{1}{2}\bar{\Psi}_\mu{}^i\Gamma^{\mu\nu\rho}\left(\partial_\nu\Psi_\rho{}^i - \frac{1}{4}\left[\Gamma_{ab}\Omega_\nu{}^{ab} + 2\Gamma_{a5}\Omega_\nu{}^{a5}\right]\Psi_\rho{}^i\right)\right] \\ &\quad + e\left[i\frac{1}{2}\bar{\Psi}_\mu{}^i\Gamma^{y\mu\nu}\left(\partial_y\Psi_\nu{}^i - \frac{1}{4}\Gamma_{AB}\Omega_y{}^{AB}\Psi_\nu{}^i\right)\right] \qquad (7.48) \\ &= \int \mathrm{d}^4x\mathrm{d}y\, e\left[-i\frac{1}{2}\bar{\Psi}_M{}^i\Gamma^{\mu\nu\rho}\left(\partial_\nu\Psi_\rho{}^i - \frac{1}{4}\left[\Gamma_{ab}\Omega_\nu{}^{ab} + 2\Gamma_{a5}\Omega_\nu{}^{a5}\right]\Psi_\rho{}^i\right)\right] \\ &\quad + e\left[i\frac{1}{2}\,\bar{\Psi}_\mu{}^i\Gamma^{y\mu\nu}\left(\partial_y\Psi_\nu{}^i - \frac{1}{4}\left[\Gamma_{ab}\Omega_y{}^{ab} + 2\Gamma_{a5}\Omega_y{}^{a5}\right]\Psi_\nu{}^i\right)\right]. \qquad (7.49)\end{aligned}$$

The deconstruction deformation on the gravitini yields an action of the form

$$\begin{aligned}\mathcal{S}_{\text{4-d RS}} &= \int \mathrm{d}^4x\, e\left[-i\frac{1}{2}\bar{\psi}_\mu{}^i\gamma^{\mu\nu\rho}\left(\partial_\nu\psi_\rho{}^i - \frac{1}{4}\gamma_{ab}\omega_\nu{}^{ab}\psi_\rho{}^i\right)\right] \\ &\quad + e\left[-i\frac{1}{4}\bar{\psi}_\mu{}^i\gamma^{\mu\nu\rho}(\gamma_a i\gamma_5)\left(\Omega_\nu{}^{a5}\right)\left(-\varepsilon^{ij}\gamma_5\psi_\rho{}^j\right)\right] \\ &\quad + e\left[i\frac{1}{2}\,\bar{\psi}_\mu{}^i\left(\gamma^{\mu\nu}i\gamma_5\right)\left(m\Delta^{ij}\right)\left(-\gamma_5\varepsilon^{jk}\psi_\nu{}^k\right)\right]. \qquad (7.50)\end{aligned}$$

After performing some simplifying algebra, one arrives at the usual action one expects, i.e. the covariant Rarita-Schwinger action, plus an odd interaction.
The Rarita-Schwinger action takes the obvious form

$$= \int \mathrm{d}^4x\, e\left[-i\frac{1}{2}\bar{\psi}_\mu{}^i\gamma^{\mu\nu\rho}D_\nu\psi_\rho{}^i + \frac{1}{2}\eta^{ij}m\,\bar{\psi}_\mu{}^i\gamma^{\mu\nu}\psi_\nu{}^j\right],$$

alongside this peculiar interaction that violates local Lorentz invariance

$$+e\left[\frac{1}{4}m\varepsilon^{ij}\bar{\psi}_{\mu}{}^{i}\gamma^{\mu\nu\rho}\gamma_{c}\psi_{\nu}{}^{j}\left(e_{\rho}{}^{c}-\delta_{\rho}{}^{c}\right)\right]. \tag{7.51}$$

Interestingly, the spin connection appears to contribute a new $\mathcal{O}(m)$ interaction between the vierbein and the Rarita-Schwinger field. However, this interaction drops out and can be seen to be re-expressible as

$$\gamma^{c}=\gamma^{\sigma}e_{\sigma}{}^{c}\Longrightarrow\bar{\psi}_{\mu}{}^{i}\gamma^{\mu\nu\rho}\gamma_{c}\psi_{\sigma}{}^{j}\left(e_{\rho}{}^{c}-\delta_{\rho}{}^{c}\right). \tag{7.52}$$

However, using the generalized Majorana identity and Clifford algebra properties one can derive the lemma that

$$\varepsilon^{ij}\bar{\psi}_{\mu}{}^{i}\gamma^{\mu\nu\rho}\gamma^{\sigma}\psi_{\nu}{}^{j} \quad = \quad \varepsilon^{ij}\bar{\psi}_{\mu}{}^{i}\gamma^{\mu\nu\rho\sigma}\psi_{\rho}{}^{j}. \tag{7.53}$$

Together, one can see that this causes this $\mathcal{O}(m)$ interaction to be identically zero

$$e\left[\frac{1}{4}m\varepsilon^{ij}\bar{\psi}_{\mu}{}^{i}\gamma^{\mu\nu\rho\sigma}\psi_{\nu}{}^{j}\left(g_{\rho\sigma}-e_{\sigma\,c}\delta_{\rho}{}^{c}\right)\right], \tag{7.54}$$

owing to the DvN condition $\delta_{[\nu}{}^{a}e_{\mu]\,a}=0$. This leaves only the covariant kinetic term and the Dirac mass,

$$\mathcal{S}_{\text{4-d RS}}=\int\mathrm{d}^{4}x\,e\left[-i\frac{1}{2}\bar{\psi}_{\mu}{}^{i}\gamma^{\mu\nu\rho}D_{\nu}\psi_{\rho}{}^{i}+\frac{1}{2}\eta^{ij}m\,\bar{\psi}_{\mu}{}^{i}\gamma^{\mu\nu}\psi_{\nu}{}^{j}\right]. \tag{7.55}$$

In other words, this leads to a purely covariant Rarita-Schwinger action with a Dirac mass!

Pauli Interactions

Finally, the Pauli interactions need to be deconstructed. After removing the already-accounted-for 4-Fermi interactions, one has

$$\begin{aligned}\mathcal{S}_{\text{5-D Pauli}} \quad = \quad & \int\mathrm{d}^{4}x\mathrm{d}y\left(-\kappa\frac{i}{4}\sqrt{\frac{3}{2}}E\,\bar{\Psi}_{\rho}{}^{i}X^{\mu\nu\rho\sigma}\Psi_{\sigma}{}^{i}\mathcal{F}_{\mu\nu}\right)\\ & +\left(-\kappa\frac{i}{4}\sqrt{\frac{3}{2}}e\,\bar{\Psi}_{\rho}{}^{i}X^{y\mu\rho\sigma}\Psi_{\sigma}{}^{i}F_{y\mu}\right),\end{aligned} \tag{7.56}$$

which after algebraically simplifying the Γ matrices becomes

$$
\begin{aligned}
= & \int \mathrm{d}^4x\mathrm{d}y \left(-\kappa \frac{-i}{2}\sqrt{\frac{3}{2}} e\, \bar{\psi}_\mu{}^i \left[\mathcal{F}^{\mu\nu} - i\gamma_5 (*\mathcal{F})^{\mu\nu}\right] (-\gamma_5 \varepsilon^{ij}\psi_\nu{}^j) \right) \\
& + \left(-\kappa \frac{i}{2}\sqrt{\frac{3}{2}} e\, \bar{\psi}_\rho{}^i \left[i\gamma_5 \gamma^{\mu\nu\rho}\right] \psi_\sigma{}^i \mathcal{F}_{y\mu} \right) . \qquad (7.57)
\end{aligned}
$$

The Deconstruction deformation leaves the term

$$
\mathcal{S}_{\text{4-D Pauli}} = \int \mathrm{d}^4x \left(\kappa i \sqrt{\frac{3}{8}} e\, \bar{\psi}_\mu{}^i \left[\mathcal{F}^{\mu\nu} - i\gamma_5 (*\mathcal{F})^{\mu\nu}\right] \gamma_5 \varepsilon^{ij} \psi_\nu{}^j \right) , \qquad (7.58)
$$

unchanged, but there is y-derivative on A_μ is converted to

$$
\int \mathrm{d}^4x\, m\kappa \left(\sqrt{\frac{3}{8}} e\, \bar{\psi}_\nu{}^i \left[\gamma_5 A_\mu \gamma^{\mu\nu\rho}\right] \psi_\rho{}^i \right) . \qquad (7.59)
$$

This leaves the original $D=4$ Pauli interaction between the gravitini fields and the axial vector field, plus a new order κm Velo-Zwanziger-type interaction [148, 149, 151].

7.3.4 The Action for a Massive Spin-$\frac{3}{2}$ on a Curved Space

Up to the $\mathcal{N}=2$ 4-Fermi interactions, the complete action is given by

$$
\begin{aligned}
\mathcal{S}_{\text{4-D mSUGRA}} = & \int \mathrm{d}^4x \frac{1}{2\kappa^2} \left(e\mathcal{R}[\omega] + \frac{2m^2}{2!} \delta^{\mu\nu\rho\sigma}_{abcd} (e-\delta)_\mu{}^a (e-\delta)_\nu{}^b e_\rho{}^c e_\sigma{}^d \right) \\
& + e \left(-i\frac{1}{2} \bar{\psi}_\mu{}^i \gamma^{\mu\nu\rho} D_\nu[\omega_e] \psi_\rho{}^i + \frac{1}{2} m\, \eta^{ij} \bar{\psi}_\mu{}^i \gamma^{\mu\nu} \psi_\nu{}^j \right) \\
& + e \left(-\frac{1}{4} \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - m^2 \frac{1}{2} A_\mu A^\mu \right) \\
& + e\,\kappa \left(i\sqrt{\frac{3}{8}}\, \bar{\psi}_\mu{}^i \left[\mathcal{F}^{\mu\nu} - i\gamma_5 (*\mathcal{F})^{\mu\nu}\right] \gamma_5 \varepsilon^{ij} \psi_\nu{}^j \right) \\
& + e\,\kappa m \left(\sqrt{\frac{3}{8}}\, \bar{\psi}_\nu{}^i \left[\gamma_5 A_\mu \gamma^{\mu\nu\rho}\right] \psi_\rho{}^i \right) \qquad (7.60)
\end{aligned}
$$

7.4 Difficulties Deconstructing the Non-Linear Superalgebra

In the Chapter 6, I demonstrated how one can readily read off the 4-D SUSY transformations by deconstructing the 5-D SUSY variations, as long as one forces the condition $\varepsilon^i = \alpha^i \varepsilon$ that breaks the $\mathfrak{so}(2)$-invariance. I will now show that there are issues to applying such a straightforward procedure to seeing the existence of a 4-D SUSY variation. This does not demonstrate the non-existence of an $\mathcal{N} = 1$ local SUSY, but it does make proving its existence considerably more difficult.

7.4.1 Superalgebra for $D = 5$ SUGRA

Since for interacting theories one needs to gauge-fix before applying the Deconstruction procedure, I will now note the obvious amendment to the procedure outlined in Chapter 6. First, one needs to fix the SUSY variations to preserve the gauge-fixing condition, which amounts to discovering and adding compensating gauge transformations. Thus one starts with

$$\delta_Q E_M{}^A = i\frac{1}{2}\kappa^2\, \bar{\varepsilon}^i \Gamma^A \Psi_M{}^i \tag{7.61}$$

$$\delta_Q \Psi_M{}^i = \hat{D}_M \varepsilon^i + \frac{1}{4\sqrt{6}} \left[\Gamma_M{}^{PQ} - 4\Gamma^P \delta_M^Q \right] \hat{F}_{PQ} \varepsilon^i \tag{7.62}$$

and

$$\delta_Q A_M = -i\sqrt{\frac{3}{8}}\kappa\, \bar{\varepsilon}^i \Psi_M{}^i \,. \tag{7.63}$$

Together, they obey the superalgebra

$$\left[\bar{\varepsilon}_1^i Q^i, \bar{\varepsilon}_2^j Q^j\right] X = \delta_{\text{Diff}}\left(\xi^M\right) X + \delta_{\text{LLT}}\left(\lambda^{AB}\right) X + \delta_Q\left(\eta^i\right) X + \delta_{\mathcal{U}(1)}\left(\zeta\right) X \,, \tag{7.64}$$

such that

$$\xi^M = i\frac{1}{2}\kappa^2\, \bar{\varepsilon}_2^i \Gamma^M \varepsilon_1^i \tag{7.65}$$

$$\lambda^{AB} = \xi^M \Omega_M{}^{AB} + \cdots \tag{7.66}$$

$$\eta^i = \xi^M \Psi_M{}^i \tag{7.67}$$

$$\zeta = \xi^M A_M + \cdots \,. \tag{7.68}$$

From here, I take a useful re-writing of the gauge-fixing conditions

$$
\begin{aligned}
E_{[MA]} &= 0 \qquad (7.69)\\
E_y{}^5 &= 1 \qquad (7.70)\\
E_y{}^A &= \delta_5^A\,, \qquad (7.71)
\end{aligned}
$$

noting that I have lowered A with respect to the 5-D η metric. This looks like more conditions than the original gauge conditions, but multiple equations are redundant. Together they can be explicitly checked to be equivalent to the previous gauge-fixing conditions (7.27). For the remaining fields, I keep them as

$$
\begin{aligned}
\Psi_y{}^i &= 0 \qquad (7.72)\\
A_y &= 0 \qquad (7.73)
\end{aligned}
$$

The compensated transformations $\delta_{Q'}X = \delta_Q X + \delta_G X$ are forced to obey the compensating equation

$$
\delta_{Q'}F \equiv 0\,. \qquad (7.74)
$$

Local Lorentz and diffeomorphism transformations require

$$
\begin{aligned}
\delta_{Q'}E_y{}^a &= \delta_Q E_y{}^a + \Lambda^a{}_B E_y{}^B + \partial_y \xi^N E_N{}^a = 0 \qquad (7.75)\\
\delta_{Q'}E_{MA} &= \delta_{Q'}E_{AM}\,. \qquad (7.76)
\end{aligned}
$$

Using the resummation techniques I developed in Chapter 3, I arrive at conditions

$$
\begin{aligned}
\xi^y &= -\int \mathrm{d}y\ \left(\delta_Q E_y{}^5\right) = 0 \qquad (7.77)\\
\xi^\mu &= -E^\mu{}_a \int \mathrm{d}y\ \left(\delta_Q E_y{}^a\right) = 0 \qquad (7.78)\\
\Lambda^a{}_5 &= -E^a{}_b \int \mathrm{d}y\, [\delta_Q + \delta_D(\xi)] E_y{}^a \qquad (7.79)\\
\Lambda_{ab} &= -\int_0^\infty \mathrm{d}\mu\ \exp(-\mu E)_a{}^c \left([\delta_Q + \delta_D(\xi)]E_{[cd]}\right)\exp(-\mu E)_b{}^d\,, \qquad (7.80)
\end{aligned}
$$

where I have made use of the gauge-fixings $\Psi_y{}^i = A_y = 0$. N.B. the term $E^a{}_b$ denotes the inverse vielbein; unfortunately, the Deser-van Nieuwenhuizen condition makes a difficult time of the indices, and thus vierbeins must explicitly be written down and cannot be inferred from indices being Latin or Greek.

7.4.2 Solving for the Residual Gauge Symmetries

Although the 5-D SUSY transformations need to obey the gauge transformations, it is worth noting that there is still a full set of 4-D gauge symmetries remaining in the gauge-fixed theory. This follows from noting that the gauge conditions, e.g. infinitesimal 4-D diff transformations on the 4-D vierbein, are still invariant subject to

$$\delta_D E_y{}^a = \partial_y \xi^N E_N{}^a = 0\,, \tag{7.81}$$

if one wishes to fix

$$\xi^N(x^\mu, y) = \xi^N(x^\mu)\,. \tag{7.82}$$

Now I check how this impacts the SUSY transformations, and if one can find a preserved $\mathcal{N} = 1$ copy of SUSY. In other words, I will only be concerned with the case where $\varepsilon^i(x, y) = \varepsilon^i(x)$ with some fixed y-dependence

$$\delta_Q \Psi_y{}^i \;=\; \hat{D}_y \varepsilon^i + \frac{\kappa}{4\sqrt{6}} \Gamma^{\mu\nu}\Gamma_5 \hat{F}_{\mu\nu}\varepsilon^i + \frac{\kappa}{\sqrt{6}}\Gamma^\mu \hat{F}_{y\mu}\varepsilon^i = 0\,, \tag{7.83}$$

which can be re-expressed as

$$= \;\partial_y \varepsilon^i - \frac{i\kappa^2}{16}(\Psi_\rho{}^j \Gamma_5 \Psi_\sigma{}^j)\Gamma^{\rho\sigma}\varepsilon^i + \frac{\kappa}{4\sqrt{6}}\Gamma^{\mu\nu}\Gamma_5 \hat{F}_{\mu\nu}\varepsilon^i + \frac{\kappa}{\sqrt{6}}\Gamma^\mu \partial_y A_\mu \varepsilon^i\,. \tag{7.84}$$

This local SUSY case differs from the bosonic case, where the simply forcing the restriction $\varepsilon^i(x, y) = \varepsilon^i(x)$ clearly will not function as desired. Thus, ε must vary in y in order to cancel off the affine contributions and solve this equation. Packaging those additional pieces (here I make the x dependence implicit) as

$$\partial_y \varepsilon^i(y) + M(y)\varepsilon^i(y) = 0\,, \tag{7.85}$$

where $M(y)$ is field-dependent, then I can use the ansatz

$$\varepsilon^i(y) = \varepsilon^i(0) - \int_0^y \mathrm{d}y'\, M(y')\varepsilon^i(y')\,. \tag{7.86}$$

A little algebra allows this to be written as an infinite series of integrals

$$\varepsilon^i(y) = \varepsilon^i(0) - \int_0^y \mathrm{d}y'\, M(y')\varepsilon^i(y') + \int_0^y \mathrm{d}y' \int_0^y \mathrm{d}y''\, M(y')M(y'')\varepsilon^i(y'') - \dots\,. \tag{7.87}$$

Although not a pithy equation, this does concretely demonstrate that a 4-D SUSY parameter $\varepsilon^i(0)$ with fixed y-dependence (but arbitrary x^μ dependence) can preserve the gauge conditions and by construction keep the action invariant. Now I turn to the case of deconstruction.

7.4.3 Obstructions to Deconstructing the SUSY Transformations

I now take the exact same logic of the previous section, but now in the presence of the deconstruction deformations. Thus, the equations for a local SUSY invariance (including the special case of global SUSY invariance) starts off as

$$\partial_y \varepsilon^i - \frac{i\kappa^2}{16}(\Psi_\rho{}^j \Gamma_5 \Psi_\sigma{}^j)\Gamma^{\rho\sigma}\varepsilon^i + \frac{\kappa}{4\sqrt{6}}\Gamma^{\mu\nu}\Gamma_5 \hat{F}_{\mu\nu}\varepsilon^i + \frac{\kappa}{\sqrt{6}}\Gamma^\mu \partial_y A_\mu \varepsilon^i = 0\,,$$

but now I must deform it under the very general form of Deconstruction via

$$\partial_y \varepsilon^i(y) \to -m\Delta^{ij} D_{IJ}\varepsilon^j_J\,. \tag{7.88}$$

Absorbing the non-y derivative terms into M again, I find the linear difference-like equation

$$-m\Delta^{ij} D_{IJ}\varepsilon^j_J + M_I \varepsilon^i_I = 0\,. \tag{7.89}$$

Now y-anti-derivatives are multi-linear matrix inversions, and not particularly elegant ones. One may write down a formal inverse operator for $\mathcal{D}_{IJ}$, which acts trivially on massless states (in order to define the pseudo inverse) but acts as an inverse on the massive states, i.e.

$$[m\mathcal{D}]^{-1}_{IJ} \equiv= \frac{1}{m}\mathcal{D}_{IJ}\,. \tag{7.90}$$

Unlike the continuum case, there is no recursion relation

$$\varepsilon^i_I + \frac{1}{m}\mathcal{D}_{IJ}(M_J\varepsilon_J) = 0 \tag{7.91}$$

Explicitly for the 1-site case, this equation immediately implies

$$\varepsilon_I = 0\,. \tag{7.92}$$

Therefore, this technique fails to reproduce a 4-D super-algebra and non-trivial SUSY variations. It is unclear if this is not the correct way of recovering the 4-D symmetry variations, although it does seem to be the most natural. So while this cannot easily be interpreted as a proof that of the absence of SUSY in either model, it does show that a natural method one would apply to deconstructing the symmetry fails.

One interesting way to prove SUSY invariance would be to take the 2-site model (freezing the lower $\mathcal{N}=1$ vector multiplet of the $\mathcal{N}=2$ SUGRA multiplet, leading a bimetric theory) which couples the $\mathcal{N}=1$ massive SUGRA multiplet with an ordinary $\mathcal{N}=1$ SUGRA. Critically, the ordinary $\mathcal{N}=1$ SUGRA multiplet contains a massless spin-$\frac{3}{2}$ field, and thus if this theory propagated the correct 10 bosonic PDF's and 10 fermionic PDF's, then the theory necessary contains a local SUSY gauge redundancy. This can be explicitly checked via a Dirac analysis. The methods I developed in Chapter 5 could also potentially be employed for counting the fermionic PDF's.[2]

7.5 Comments on the Non-Linear Theory

The limits of this theory seem to point to largely positive signs, despite the unclear nature of the SUSY charge. First, by construction, the linear limit of this theory is the $\mathcal{N}=1$ Zinoviev theory in the limit $M_{\rm Pl}\to\infty$. Second, one can look at what happens when modes of the theory are frozen. For instance, if one freezes the fermions, then the theory propagates $5+3$ massive bosonic PDF' since this merely is dRGT and covariant Proca. Another case is when the bosons are frozen. Supposing I freeze the vielbein to an Einstein vacuum $\bar{g}_{\mu\nu}$, such that $\bar{\mathcal{R}}_{\mu\nu}=k\bar{g}_{\mu\nu}$, and I freeze out the vector mode

$$\begin{aligned} e_\mu{}^a &= \bar{e}_\mu{}^a \\ A_\mu &= 0 \end{aligned} \tag{7.93}$$

Then the theory only has the massive Rarita-Schwinger fields (here with a Dirac mass) and 4-Fermi interactions. This theory is known to be ghost-free to all orders (See [185] for details and further references), so long as an Einstein background is chosen. Therefore it non-linearly propagates $4+4=8$ fermionic degrees of freedom. Of course, moving away from Einstein vacuum manifolds is precisely what the Rarita-Schwinger needs SUGRA in order to have a consistent gauge symmetry (e.g. see the proof of $\mathcal{N}=1$ invariance of SUGRA in [131]); thus one should not expect that away from the Einstein vacuum the theory should remain healthy.

[2] Although it may need to be slightly modified owing to the subtle differences in how fermionic PDF's are counted, see [84].

Explicitly, one must have in mind that in the high-momentum/massless limit the effectively massless Rarita-Schwinger fields will fail [160, 161]) and lose fermionic gauge invariance.

Finally, it was shown in [185] that massive, covariant Rarita-Schwinger actions with SUGRA 4-Fermi interactions have the energy scale of the irrelevant fermionic interactions raised to Λ_3 or higher. In addition to those terms, the deconstructed action also possesses some A_μ-${\psi_\mu}^i$ interactions, and it is easy to see that they only generate irrelevant terms suppressed by Λ_3 and higher. Therefore, this theory will be ghost-free up (contains no new higher-derivative interactions) to at least the scale Λ_3. Therefore, a consistent decoupling limit will exist. A further important issue is whether or not this decoupling limit or the LEEFT exemplifies an $\mathcal{N} = 1$ or 2 global SUSY.

Finally, I conclude this chapter by noting, similarly to Chapter 6, that it would also be beneficial in future work to incorporate the fermionic deconstruction procedure into the superspace formalism, by taking a 5-D superspace formalism (e.g. harmonic superspace) and deconstruct the superfields directly into 4-D ordinary superspace (see [183], which addresses methods that seem related to this goal). The results I have discovered here are robust enough to suggest a meaningful relationship between SUSY invariance and fermionic deconstruction. If one wishes to further explore these ideas, one could take a simpler interacting 5-D SUSY theory (for example an interacting matter supermultiplet like SYM or a supersymmetric NLSM) and see if it is able to be consistently deconstructed in a way that preserves the interactions. To reiterate the issues found in the superalgebra, they are circumvented the superspace formalism; this makes this approach all the more ideal for interacting massive gauge theories.

Chapter 8

Conclusion

8.1 Summary of Thesis Achievements

The question of consistent theories of massive spin-2 fields is raised in gravitational, cosmological, and even condensed matter physics. Given recent progress in developing ghost-free, self-interacting mass terms which linearise to Fierz-Pauli theory, it is timely to explore these interactions and the ensuing massive gauge theories of spin-2 fields. I have worked towards understanding these issues in variety of contexts.

The Λ_3-decoupling limit of ghost-free massive gravitational theories exhibits delicate and complex helicity-0 and helicity-1 interactions. I have shown that the most efficient manner of understanding these interactions is by making use of the Stückelberg formalism in the Einstein-Cartan variables, thus restoring a gauge theory picture for massive spin-2 fields. Here, one must use Stückelberg fields for both diffeomorphism and local Lorentz transformations, the gauge symmetries of ordinary Einstein-Cartan gravity (a massless spin-2 field). Additionally, I have developed a concise method for employing the Lorentz Stückelberg field and for solving their fully non-linear equations of motion. Using these techniques, one is able to directly apply the decoupling limit and write down a closed-form expression for the complete tower of interactions for dRGT massive gravity and related self-interacting spin-2 theories. This expression allows one to systematically express and analyse the vector modes present within spin-2 fields that have dRGT mass interactions, and directly enabled the research into the propagation of the helicity-1 modes [111], the existence of the Galileon duality [112–116], further no-go proofs on the kinds of possible matters interactions [163], extensions of my results to curved spacetimes [112] and multi-gravity [113], it is a necessary piece tot understand the Λ_2 non-Poincaré invariant decoupling limit [191], and this chapter developed necessary techniques for all of the following work in my thesis [127, 184, 192].

Dimensional Deconstruction program has shown how massive spin-2 theories can be generated from massless spin-2 fields in a higher dimension by de Rham, *et al*, in [124], following earlier work by [108, 118–124]. In general, this procedure makes a systematic (though not completely understood) connection between Poincaré-invariant massless fields in $(D+1)$-dimensions and massive Poincaré-invariant theories in D-dimensions. Following the successes of this program, I explored the possibility of modifying the deconstruction procedure to create different types of massive spin-2 theories, both to further the general understanding of Dimensional Deconstruction and gauge theories of massive spin-2 fields.

The first way that I did this was presented in Chapter 5 on a type of "Charged Deconstruction." I demonstrated that by not gauge-fixing the vector mode present prior in the deconstruction procedure leads to a theory of a charged spin-2 field, and gave a No-Go theorem on charged spin-2 fields with dRGT mass terms. Here I applied a 3-site deconstruction procedure and kept the vector mode unfrozen, with the interpretation of this mode being a photon mode that couples to the $\mathfrak{u}(1)$-currents generated by the theory. Thus, I demonstrated that this theory lacks a global $\mathcal{U}(1)$ invariance, and therefore Dimensional Deconstruction cannot form a consistent gauge theory of a charged spin-2 field. With a subtractive procedure, I cancelled off the $\mathcal{U}(1)$-violating terms to generate a new, $\mathcal{U}(1)$-invariant theory of a spin-2 field with dRGT mass terms. To make the argument more robust, I demonstrated that this in fact is the most general Lagrangian following several necessary criterion, i.e. Lorentz invariance, linearising to Federbush (the extension of Fierz-Pauli to include electromagnetic interactions), and so forth. Using this ansatz Lagrangian, I gave a proof using a novel new manner of counting PDF's in massive spin-2 theories that this theory necessarily contains spurious additional PDF's beyond those found in dRGT massive gravity. This signals the presence of Boulware-Deser ghosts, and thus rules out these theories, at least to energy scales. above Λ_3. Therefore, I demonstrated that the charged deconstruction procedure fails to maintain $\mathcal{U}(1)$ invariance, that subtracting off the $\mathcal{U}(1)$-violating interactions in charged deconstruction leads to a unique ansatz which has a spurious number of PDF's, and I developed a novel PDF-counting algorithm for massive spin-2 fields to demonstrate as such. The further development of this technique by a collaborator led to the proof of the uniqueness of the Einstein-Hilbert kinetic term [80] when the dRGT mass term is present, and similar theorems regarding the interplay between matter interactions and spin-2 fields with dRGT mass terms [163].

In Chapter 6, I explored Dimensional Deconstruction in the context of supersymmetric gauge theories. I created a novel method of extending the Dimensional Deconstruction program to theories containing fermions, including the case of supersymmetric gauge theories. Due to the highly non-trivial structural differences between spinors of differing dimensions, I focused on the case of 5-D massless (necessarily Dirac) fermions to 4-D Majorana doublets. Using this procedure, I explicitly demonstrated how one can obtain all 4-D mass terms for Majorana doublets from a simple deformation of the usual Dimensional Deconstruction procedure to break both the y-derivatives and a symplectic-Majorana indices with a mass doublet matrix. I showed how this procedure works explicitly for spin-$\frac{1}{2}$ fermions, where no gauge symmetry exists in order to obtain massive fermions, and spin-$\frac{3}{2}$ fermions, which realises a supergauge symmetry. I then demonstrate that this fermionic deconstruction procedure, choosing the R-symmetry preserving mass doublet matrix, a single $\mathcal{N}=1$ copy of the 5-D $\mathcal{N}=2$ SUSY is preserved. I explicitly demonstrate the existence of supersymmetric gauge theories being generated by this method; I deconstructed 5-D $\mathcal{N}=2$ Maxwell theory down to $\mathcal{N}=1$ super-Proca theory and 5-D $\mathcal{N}=2$ linear SUGRA to 4-D $\mathcal{N}=1$ Zinoviev theory. The $\mathcal{N}=1$ Zinoviev theory [166] is a supersymmetric completion of Fierz-Pauli theory, gaining two massive gravitini superpartners and a massive psuedo-spin-1 superpartner. I also explicitly demonstrated how in these linear theories, one is able to directly extract the $\mathcal{N}=1$ component SUSY transformations directly from a linear combination of their 5-D counterparts. This represents a substantial step forward in the ability to generate new theories of massive SUSY theories and view old massive SUSY gauge theories in a new light.

In Chapter 7, I begin the effort towards generating interaction SUSY gauge theories via the Dimensional Deconstruction approach I developed in the previous chapter. Specifically, I begin with the fully non-linear 5-D $\mathcal{N}=2$ SUGRA theory, and I performed 1-site deconstruction at the level of the action. The action has many interesting features. By construction, it linearises to $\mathcal{N}=1$ Zinoviev theory; since it descends from a SUGRA theory, it follows the properties necessary for having a Λ_3 decoupling limit outlined in earlier work by Rahman [185] where we have a doublet of Majorana fermions rather than a single massive spin-$\frac{3}{2}$ field. Although the existence of a global $\mathcal{N}=1$ SUSY, which would need to come from a borrowed global Killing spinor on the Minkowski reference spacetime, seems dubious, it is interesting that this theory should remain ghost-free at least to the scale of Λ_3, possibly including a full global $\mathcal{N}=1$ (or $\mathcal{N}=2$) SUSY, although the derivation or falsification of its existence is left to future work.

8.2 Outlook

Moving forward from this work, it seems that there are several potentially important new areas to explore. I shall focus largely on the case of SUSY theories with massive gauge fields, since I think this has the largest new frontier to explore, both in terms of old and new questions.

First, in terms of a concrete proposal, it seems like future progress for interacting SUSY theories lies in developing methods where the SUSY invariance is manifest. In the presence of interactions, checking the explicit SUSY-invariance of the resulting action in component form (and deriving said SUSY variations) is cumbersome and highly inefficient. Therefore, exporting my results on the fermionic Dimensional Deconstruction program into the arena of Superspace/Superfield program seems highly advantageous, pragmatic, and timely. In other words, developing an explicit Deconstruction of a massless gauge theory living in 5-D $\mathcal{N} = 2$ (e.g. harmonic) superspace to a massive gauge theory living in 4-D $\mathcal{N} = 1$ superspace. Since the results in component form are directly checked, the $\mathcal{N} = 1$ SUSY-preserving direction of $\mathcal{N} = 2$ R-symmetry space is detailed, and explicit examples are known, it should be relatively straightforward, although potentially tedious, to implement. Once this procedure exists, however, the systematic analysis of massive SUSY gauge theories will be straightforward and immediate owing to SUSY-invariance being trivial to check in superspace formalism. Coupled with a modification of the ghost-freedom analysis provided in Chapter 5, and further refined by my collaborator Matas in [80], one may even be able to come up with very efficient algorithms to test ghost-freedom and SUSY-invariance. If SUSY is present, then one could make repeated use of the theorem that fermionic PDF's are equal to the number of bosonic PDF's when SUSY is present (see [168]), noting that my PDF-counting algorithm gives a relatively simple analysis for counting the bosonic PDF's.

Second, although both qualitative and speculative, it may be interesting to return to the question of the vDVZ discontinuity and Vainshtein philosophy within the context of the work here and recent advances in field theory. It is worth noting that a great deal of the foundational work in massive gravity was conducted in the 1970's when the most of the modern perspectives on Quantum Field Theory were either nascent or non-existent. For example, a great deal of work was absent on interacting CFT's and non-Lagrangian field theories (see [193] and [31] for a modern review of some of these results) and also the complete Wilsonian RG paradigm had not yet been developed (including notions of asymptotic safety/non-trivial UV fixed points [78] and explicit examples of such [75]).

Since massive spin-2 fields naturally involve topics of quantum gravity, it may be convenient to return to the vDVZ discontinuity/UV-non-renormalisability of Proca-Yang-Mills theory (the whole of which resides purely in the QFT framework) and explore it from the light of the Vainshtein philosophy supplemented by modern QFT. It seems like many of the results derived in this thesis demonstrate the inadequacy of Lagrangian methods for massive gauge fields. If one views this in light of Vainshtein's philosophy –namely that irrelevant operators dominate and make the theory strongly-interacting (i.e. non-perturbative) at the decoupling-limit energy scale– with the recent understanding that interacting CFT's, who can play the role of UV-fixed points, are often non-Lagrangian. This is the core of the asymptotic safety scenario; from this standpoint, all of these results by myself and others should be neither worrying nor surprising. This is precisely what happens for $\mathcal{N}=2$ 5-D SYM with its dimensionful coupling constant $g_{5\text{-D}}$; although SYM is not perturbatively renormalisable, it is UV complete when viewed as a specific interacting CFT (whose existence is implied by String Theory) with relevant deformations; at scales $E \ll \frac{1}{g^2_{5\text{-D}}}$, this quantum theory flows in the IR to a theory dominated by classical SYM theory [75].

The Vainshtein philosophy may be dramatically updated in this framework for 4-D PYM; the idea would then be comprised of an asymptotic-safety setup plus two conditions. The first would be that one would need to obtain an interacting massive gauge theory (PYM, having $3 \times (N^2-1)$ PDF's, plus matter multiplets) in the IR, which would be viewed as an EFT with a tower of irrelevant operators; this is the obvious condition. The second[1] would be that the UV theory made up of two separate, decoupled (e.g. factorised) CFT's in the UV; both the ordinary YM UV-CFT ($2 \times (N^2-1)$ vector PDF's) and a new interacting, presumably non-Lagrangian CFT (with $1 \times (N^2-1)$ spin-0 PDF's), for a total UV CFT theory with $3 \times (N^2-1)$ at all energy scales larger than m. The RG flow would come from both the YM interactions and a new set of relevant (scaleful) operators coupling the two CFT sectors. Below the scale, the gluons would "feel" the scalar (here expressible as the Stückelberg mode), but above this scale, the theory would become ordinary YM and increasingly decoupled at higher energies $E \gg \Lambda_1$ from a strongly self-interacting, presumably non-Lagrangian CFT. In this scenario, the theory would be UV-complete and at the same time resolve the vDVZ discontinuity. Such a scenario can only hope to be explored in the presence of SUSY, since much of what is known

[1]To the author's knowledge, no one has setup a picture of massive gauge theories in this language before. Many consistency conditions would be rendered obvious if such a scenario could be constructed, and it would be provide a natural loophole to many –if not all– no-go results in the literature regarding massive, self-interacting gauge theories.

about interacting CFT's is only known in the SUSY versions of the theories, making my methods particularly relevant. Additionally, from a Dimensional Deconstruction perspective, it would be aesthetically pleasing if the quantum theory of a 4-D massive spin-1 field shared such a similar quantum structure to that of massless spin-1 field in 5-D. Although this discussion is quite qualitative and speculative, if such a scenario could be consistently concocted for a super-PYM –a theory potentially obtainable from my methods, since super-Proca theory was obtained from my methods– then it would represent the first UV-complete massive gauge theory that does not make use of a Higgs mechanism (require new UV PDF's), and would represent the first fundamental theory of a massive spin-1 field. I believe that this idea merits further inspection. This would be historically interesting in its own right, and could mark an important milestone in the derivation and creation of a fully consistent, fully quantum theory of a massive spin-2 field, should an analog scenario prove possible for its spin-2 cousin.

In conclusion, since dRGT was discovered in 2010 I believe that it fair to say that the physics community is undergoing a new renaissance in our understanding of massive gauge theories –one that would likely not have been guessed when Fierz and Pauli first wrote their equations down in 1939.

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Appendix A

Conventions and Nomenclature

A.1 Nomenclature

A complete glossary of acronyms:

- QFT = "Quantum Field Theory",
- EFT = "Effective Field Theory",
- IR = "Infra-red" (at low energies $E \sim 0$),
- UV = "Ultraviolet" (at high energies $E \sim \infty$),
- LEEFT = "Low-Energy Effective Field Theory",
- DL = "Decoupling limit",
- RG = "Renormalisation group",
- GR = "General Relativity",
- ADM = "Arnowitt-Deser-Misner",
- EC = "Einstein-Cartan",
- spM = "symplectic-Majorana",
- RS = "Rarita-Schwinger",
- LLT = "local Lorentz transformation/boost",
- SUSY = "supersymmetric",
- SUGRA = "supergravity",

- diff = "diffeomorphism",
- LGR = "Linearised General Relativity",
- FP = "Fierz-Pauli",
- mGR = "Massive Gravity",
- BD = "Boulware-Deser",
- dRGT = "de Rham-Gabadadze-Tolley gravity",
- DvN = "Deser-van Nieuwenhuizen",
- KK = "Kaluza-Klein",
- YM = "Yang-Mills theory",
- PYM = "Proca-Yang-Mills theory"
- DOF = "Degrees of freedom" (e.g. size of naïve configuration space)
- PDF = "Physical Degree of Freedom" (polarisation; half of the total initial conditions of time ODE's),
- GKD = "Generalised Kronecker Delta symbol",
- vDVZ = "van Dam-Veltman-Zakharov".

A.2 Conventions and Notations

My conventions are:

(1.) The same as Srednicki [63] (of specific interest for fermions are sections 33 through 43). Natural units are always employed, so by definition $\hbar = c = 1$ unless otherwise specified.

(2.) **Therefore, I use the** $(-, +, ..., +)$ **metric signature**, but my gamma matrices obey $\{\Gamma^A, \Gamma^A\} = -2\eta^{AB}$. Grassmann numbers obey $(ab)^\dagger = b^\dagger a^\dagger$.

(3.) Majorana spinors can easily be decomposed into Weyl spinors by following the recipe outlined in the above sections of Srednicki, or by directly decomposing γ^μ into σ^μ and $\bar{\sigma}^\mu$ in my formulas.

(4.) All 5-D fermions are given by capital Greek characters (e.g. Λ, Ψ), whereas all 4-D fermions are given by lower case characters (e.g. λ, ψ). This will also be true of my bosonic variables, the only exceptions being the vector fields A_μ and B_μ. 5-D symplectic-Majorana spinors (Ψ^i) will often have their indices referred to as "symplectic indices", but 4-D Majorana spinors ψ^i will be called "$\mathfrak{so}(2)_R$-symmetry" indices.

(5.) Gamma matrices in each dimension are also (un)capitalized following the previous convention, and obey $\Gamma^a = \gamma^a$ and $\Gamma^5 = i\gamma_5$. Note that $(\gamma_5)^2 = +1$.

(6.) I take weight-one objects $\Gamma^{A_1\cdots A_n} \equiv \frac{1}{n!}\big(\Gamma^{A_1}\cdots\Gamma^{A_n} + (\text{Perms})\big)$.

(7.) I make extensive use of the generalized Kronecker delta tensors, e.g. $\delta^{A_1\cdots A_D}_{B_1\cdots B_D} \equiv \varepsilon^{A_1\cdots A_D}\varepsilon_{B_1\cdots B_D}$ and $\delta^{AB}_{MN} = \delta^A_M\delta^B_N - \delta^A_N\delta^B_M$. Note that I always use weight 1 (anti)-symmetrisation, e.g. $\Gamma^{ABC} = \Gamma^{[A}\Gamma^B\Gamma^{C]} = \frac{1}{3!}\delta^{ABC}_{MNR}\Gamma^M\Gamma^M\Gamma^R$.

(8.) In my definition of the SUSY algebra, one has an ambiguity,

$$[\bar{\varepsilon}_1 Q,\, \bar{Q}\varepsilon_2] = k\bar{\varepsilon}_2\Gamma^M\varepsilon_1 P_M + (\text{Gauge Symmetries}) \tag{A.1}$$

I make the choice that $k := 2$ for SUSY theories, but $k := \frac{1}{2}$ for SUGRA theories. This is just because these are the most natural coefficients to use in those theories, but they are related by rescalings of the SUSY parameter $\varepsilon_{\text{SUSY}} \to 2\varepsilon_{\text{SUGRA}}$. Otherwise, there is no difference.

(9.) **Note that I also follow the variational convention of Srednicki**. In this variational convention, functional differentiation by a fermion ψ is defined with an extra minus sign

$$\delta\mathcal{S}[\psi] := \int d^Dx \left(-\frac{\delta_R\mathcal{S}}{\delta\psi(x)}\right)\delta\psi(x) \tag{A.2}$$

Therefore, $\delta\psi \to -\delta\psi$ away from usual variational conventions. (This causes my SUSY algebra to have positive k parameters rather than negative.)

(9.) Supplemental to Srednicki's QFT conventions, I use gravitational conventions

$$\frac{1}{2}M_{\text{Pl}}^2 = \frac{1}{2\kappa^2} = \frac{1}{8\pi G} \tag{A.3}$$

$$\mathcal{S}_{EH} = \frac{1}{2\kappa^2}\int \mathrm{d}^Dx\,\sqrt{-g}\mathcal{R}[g] \tag{A.4}$$

$$h_{\mu\nu} \text{ s.t. } e_\mu{}^a = \delta_\mu{}^a + \kappa h_\mu{}^\alpha\,. \tag{A.5}$$

I will make a few notes about other common conventions that I do **not** follow in this thesis. In many texts that use fermions in 4-D and 5-D, their $\bar{\psi}$ will always refer to the Majorana conjugate (e.g. [126, 180]); secondly for those texts the placement of symplectic indices i on symplectic-Majorana fermions Ψ^i vs $\Psi_i = \Omega_{ij}\Psi^j$ and the height of the index refers to chirality , e.g. $\psi^i := L\psi^i$. Contrariwise, I will **not** indicate chirality in 4-D fermions with this index, so $\psi^i :\neq R\psi^i$ or $L\psi^i$, and my $\bar{\psi}^i$ always indicates the Dirac conjugate,

$$\bar{\psi} \equiv (\psi)^\dagger \gamma^0 \tag{A.6}$$
$$\bar{\psi}^i \equiv (\psi^i)^\dagger \gamma^0 \tag{A.7}$$
$$\bar{\Psi}^i \equiv (\Psi^i)^\dagger \Gamma^0 \tag{A.8}$$

As such, the height of the symplectic index does not signify anything, and thus I will always be written upstairs to prevent clutter in my notation.

Explicitly, one can check my identities and formulas by using the 4x4 spinor matrices:

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{A.9}$$
$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \tag{A.10}$$
$$\mathcal{C}_4 = -i\gamma^0\gamma^2 = \begin{pmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, \tag{A.11}$$
$$\mathcal{C}_5 = \mathcal{C}_4\gamma_5 = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, \tag{A.12}$$
$$L = \frac{1}{2}(1-\gamma_5) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \tag{A.13}$$
$$R = \frac{1}{2}(1+\gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \tag{A.14}$$

and these 2x2 matrices

$$I = \delta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{A.15}$$

$$\sigma^1 = \Delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{A.16}$$

$$\sigma^2 = i\varepsilon = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{A.17}$$

$$\sigma^3 = \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{A.18}$$

$$\tag{A.19}$$

I choose to keep my Weyl spinor-based tensors (i.e. $(\sigma^i)^\alpha{}_\beta$, $I^\alpha{}_\beta$) and the symplectic/$\mathfrak{so}(2)_R$ symmetry-based matrices (i.e. δ^{ij}, η^{ij}, ε^{ij}, Δ^{ij}) separated with distinct symbols, even though they are numerically equivalent (up to factors) to the Pauli matrices. Unquestionably, the existence of these matrices within the SUSY/R-symmetry structure, which I prove in Chapter 6, would straightforwardly indicate a higher dimensional spinor origin to all of these R-symmetry objects (almost certainly within a 6-D theory), I choose to hide this fact for conceptual clarity and keep a clean 4-D notation.

A.2.1 Treatise on Spinors

Fermions in 5-D are generally less well known than their 2, 4, 6, and 10 dimensional counterparts, and I have my own convention to decomposing them into 4-D fermions. Therefore, I will give a quick sermon on 4-D and 5-D spinors/fermions.

For 4-D spinors, there are 3 distinct representations to choose from: Weyl spinors, Dirac spinors, and Majorana spinors. The irreducible unitary representation (rep) is Weyl, thus the other two can always be recast as Weyl fermions. A Dirac fermion is composed of a L-handed and a R-handed Weyl spinor, which when the spinors as in Weyl basis gives the simple decomposition

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} , \tag{A.20}$$

where ψ_L, ψ_R are 2-dimensional complex, Grassmann-valued vectors (Weyl spinors).

Majorana spinors then are Dirac spinors subjected to the Majorana constraint:

$$\begin{aligned}\bar{\psi} &= \psi^T \mathcal{C}_4\,, \\ \Longrightarrow \psi^\dagger \gamma^0 &= \psi^T \mathcal{C}_4\,,\end{aligned} \tag{A.21}$$

where C_4 is the 4-D charge conjugation matrix. This kills half of the degrees of freedom of the Dirac fermion, and acts as an effective "reality condition".

By contrast, fermions in 5-D can only come from one spinor representation, namely the Dirac representation. This means it is 4-d vector of complex Grassmann numbers and, rather unpleasantly, they have no nice Majorana properties that are often power the identities underpinning SUSY invariance. For this reason, it has become popular in 5-D (and 6-D) SUSY theories to make use of an equivalent, but far more elegant, fermion structure called a **symplectic-Majorana fermion**.[1] Symplectic-Majorana fermions are defined by taking two Dirac fermions $\Psi \to \Psi^i$, $i = 1, 2$, and then recovering a single Dirac fermion worth of information by imposing the relation

$$\bar{\Psi}^i \equiv (\Psi^i)^\dagger \Gamma^0 = (\Psi^j)^T \varepsilon^{ji} \mathcal{C}_5\,, \tag{A.22}$$

which returns one back to a single Dirac fermion worth of information. A few comments are in order. First, the odd appearance[2] of the symplectic matrix ε^{ij} is due to the fact that in 5-D, the absence of a Majorana representation is because the natural charge conjugation operator, $\mathcal{C}_5$ fails to be a star operator; it obeys $\bar{\bar{\bar{\Psi}}}^M = -\Psi$, rather than $\bar{\bar{\bar{\Psi}}}^M = \Psi$. But the introduction of ε resolves the sign problem, $\bar{\bar{\bar{\Psi}}}^{\text{spM}} = -\varepsilon^2\Psi = \Psi$, at the price of having an indexed object. Thus, in 5-D, one is forced to make use of a charge conjugation that mixes the spinor basis and a new symplectic index in order to talk about Majorana-like fermions. This leads to a nice set of identities I will list in section A.2.2.

A natural question now is what relationship, if any, exists between 4-D Majorana fermions and a 5-D symplectic-Majorana fermion. At the level of DOF, two 4-D Majorana fermions have 8 real grassmann numbers of information, which matches the 8 from a symplectic-Majorana (i.e. Dirac) rep. This leads one to conjecturing a conversion formula between them (which I often call a "descending relation" or a "decomposition" in this text), which I take to have the form

$$\Psi^i = P^{ij}\psi^j\,, \tag{A.23}$$

[1] Further details can be found in [131, 159].

[2] If one has more fermions, they can mix them up with a more complicated symplectic form, Ω^{ij}, but I will not use this fact.

where P^{ij} operates on both the symplectic/$\mathfrak{so}(2)_R$-symmetry indices *and* on the spinor basis. This operator must simultaneously satisfy conditions (A.22) and (A.21). I use a solution of the form

$$P^{ij} = \frac{1}{\sqrt{2}}\Big[I\delta^{ij} - \gamma_5\varepsilon^{ij}\Big]\,, \tag{A.24}$$

where, explicitly, the matrices I and γ_5 operate on the spinor basis, but δ and ε operate on the symplectic/$\mathfrak{so}(2)_R$ indices. Then the inverse is given by

$$\psi^i = \bar{P}^{ij}\Psi^j = \frac{1}{\sqrt{2}}\Big[I\delta^{ij} + \gamma_5\varepsilon^{ij}\Big]\Psi^j\,. \tag{A.25}$$

From these relations, I have derived the following formulas:

$$\bar{\Psi}^i = \bar{\psi}^j P^{ji}\,, \tag{A.26}$$

$$\bar{\Psi}^i N\Psi^i = \begin{cases} \bar{\psi}^i N\psi^i & \text{if } \{N,\, \gamma_5\} = 0\,, \\ -\varepsilon^{ij}\bar{\psi}^i N\gamma_5\psi^j & \text{if } [N,\, \gamma_5] = 0\,, \end{cases} \tag{A.27}$$

$$P^{ij}\Psi^j = -\gamma_5\varepsilon^{ij}\psi^j\,, \tag{A.28}$$

$$\bar{P}^{ij}\Psi^j = \psi^i\,, \tag{A.29}$$

which I frequently use; note that I am using N in (A.28) to represent an arbitrary spinor operator, e.g. $N = I$, γ^μ, $\gamma^{\mu\nu}$, $\gamma^\mu\gamma^{\nu\rho}$, etc. It is interesting to note that that pure bispinors ($N = I$) are $\mathcal{PT}$-odd in 4-D but even in 5-D.

A.2.2 Useful Formulas for Fermions

Here I use $X^M = (x^\mu, y)$ when I need to compare 5-D to 4-D identities, but many of these identities are dimension independent. It will be explicitly stated when this is the case.

Firstly, I will list some useful R-symmetry identities

$$\Delta^{ij}\eta^{jk} = \varepsilon^{ik} = -\eta^{ij}\Delta^{jk}\,, \tag{A.30}$$

$$\varepsilon^{ij}\eta^{jk} = \Delta^{ik} = -\eta^{ij}\varepsilon^{jk}\,, \tag{A.31}$$

$$\Delta^{ij}\varepsilon^{jk} = \eta^{ik} = -\varepsilon^{ij}\Delta^{jk}\,. \tag{A.32}$$

Secondly, I will write down some Γ matrix "recursion relations". I will start by noting the definitions of Γ matrices

$$\begin{aligned}
\Gamma^A &= \Gamma^A && \text{(A.33)}\\
\Gamma^{AB} &= \frac{1}{2!}\delta^{AB}_{MN}\Gamma^M\Gamma^N && \text{(A.34)}\\
\Gamma^{ABC} &= \frac{1}{3!}\delta^{ABC}_{MNR}\Gamma^M\Gamma^M\Gamma^R \\
&= \frac{1}{3}\left(\Gamma^A\Gamma^{BC}+\Gamma^B\Gamma^{CA}+\Gamma^C\Gamma^{AB}\right) && \text{(A.35)}\\
\Gamma^{ABCD} &= \frac{1}{4!}\delta^{ABCD}_{MNRS}\Gamma^M\Gamma^N\Gamma^R\Gamma^S && \text{(A.36)}\\
\Gamma^{ABCDE} &= \frac{1}{5!}\delta^{ABCDE}_{MNRSL}\Gamma^M\Gamma^N\Gamma^R\Gamma^S\Gamma^L\,. && \text{(A.37)}
\end{aligned}$$

From this, one may use $[\Gamma^M, \Gamma^N] = -2\eta^{MN}$, the GKD recursion relations (given in totality by (B.8), and shown as an example for gamma in (A.35) to obtain the dimension-independent Γ **recursion formulas**

$$\Gamma^A = \Gamma^A \tag{A.38}$$

$$\Gamma^{AB} = \Gamma^A\Gamma^B + \eta^{AB} \tag{A.39}$$

$$\Gamma^{ABC} = \Gamma^A\Gamma^B\Gamma^C + \Gamma^A\eta^{BC} - \Gamma^B\eta^{AC} + \Gamma^C\eta^{AB} \tag{A.40}$$

$$\begin{aligned}
\Gamma^{ABCD} &= \Gamma^A\Gamma^B\Gamma^C\Gamma^D \\
&+ \Gamma^A\Gamma^B\eta^{CD} - \Gamma^A\Gamma^C\eta^{BD} + \Gamma^A\Gamma^D\eta^{BC} \\
&+ \Gamma^C\Gamma^D\eta^{AB} - \Gamma^B\Gamma^D\eta^{AC} + \Gamma^B\Gamma^C\eta^{AD} \\
&+ \eta^{AB}\eta^{CD} - \eta^{AC}\eta^{BD} + \eta^{AD}\eta^{BC} && \text{(A.41)}
\end{aligned}$$

$$\begin{aligned}
\Gamma^{ABCDE} &= \Gamma^A\Gamma^B\Gamma^C\Gamma^D\Gamma^E \\
&+ \Gamma^A\Gamma^B\Gamma^E\eta^{CD} - \Gamma^A\Gamma^C\Gamma^E\eta^{BD} + \Gamma^A\Gamma^D\Gamma^E\eta^{BC} - \Gamma^B\Gamma^C\Gamma^D\eta^{AE} + \Gamma^B\Gamma^C\Gamma^E\eta^{AD} \\
&+ \Gamma^A\Gamma^C\Gamma^D\eta^{BE} + \Gamma^A\Gamma^B\Gamma^D\eta^{CE} - \Gamma^A\Gamma^B\Gamma^C\eta^{DE} + \Gamma^B\Gamma^D\Gamma^E\eta^{AC} - \Gamma^C\Gamma^D\Gamma^E\eta^{AB} \\
&+ \Gamma^A\eta^{BC}\eta^{DE} - \Gamma^A\eta^{BD}\eta^{CE} + \Gamma^A\eta^{BE}\eta^{CD} \\
&+ \Gamma^B\eta^{AC}\eta^{DE} - \Gamma^B\eta^{AD}\eta^{CE} + \Gamma^B\eta^{AE}\eta^{CD} \\
&+ \Gamma^C\eta^{AB}\eta^{DE} - \Gamma^C\eta^{AD}\eta^{BE} + \Gamma^C\eta^{AE}\eta^{BD} \\
&+ \Gamma^D\eta^{AB}\eta^{CE} - \Gamma^D\eta^{AC}\eta^{BE} + \Gamma^D\eta^{AE}\eta^{BC} \\
&+ \Gamma^E\eta^{AB}\eta^{CD} - \Gamma^E\eta^{AC}\eta^{BD} + \Gamma^E\eta^{AD}\eta^{BC}\,. && \text{(A.42)}
\end{aligned}$$

Fortunately, supergravity never requires a Γ with more than 5 indices, so I stop here. However, with deepest regret, I inform that reader that supergravity makes use of Γ^{MNRPQ} (in dimensions 5 and higher) for the derivation of torsion and contorsion generated from Rarita-Schwinger kinetic terms (i.e. gravitini) on curved spacetimes.

Thirdly, I will list the dimension-dependent **contraction identities** in 4-D and 5-D, which are as follows:

$$\begin{array}{ccccccc} & \text{5-D} & & & & \text{4-D} & \\ \Gamma^{ABCDE}\Gamma_E & = & (-1)\Gamma^{ABCD} & & & & \\ \Gamma^{ABCD}\Gamma_D & = & (-2)\Gamma^{AB} & & \gamma^{abcd}\gamma_d & = & (-1)\gamma^{abc} \\ \Gamma^{ABC}\Gamma_C & = & (-3)\Gamma^{AB} & & \gamma^{abc}\gamma_c & = & (-2)\gamma^{ab} \\ \Gamma^{AB}\Gamma_B & = & (-4)\Gamma^{A} & & \gamma^{ab}\gamma_b & = & (-3)\gamma^{a} \\ \Gamma^{A}\Gamma_A & = & (-5) & & \gamma^{a}\gamma_a & = & (-4) \end{array}$$

which holds true for contractions applied on any side so long as it is the index 'closest' to it, e.g. $\gamma^{a\bullet}\gamma_\bullet = \gamma_\bullet\gamma^{\bullet a}$).

Fourthly, I will write down the dimension-dependent Majorana "exchanging identities". Starting in 5-D, for any two 5-D symplectic-Majorana spinors, the **symplectic-Majorana exchanging identities** [159] are given by

$$\bar{\Lambda}^i_1\Lambda^i_2 = -(+1)\bar{\Lambda}^i_2\Lambda^i_1 \tag{A.43}$$

$$\bar{\Lambda}^i_1\Gamma^A\Lambda^i_2 = -(+1)\bar{\Lambda}^i_2\Gamma^A\Lambda^i_1 \tag{A.44}$$

$$\bar{\Lambda}^i_1\Gamma^{AB}\Lambda^i_2 = -(-1)\bar{\Lambda}^i_2\Gamma^{AB}\Lambda^i_1 \tag{A.45}$$

$$\bar{\Lambda}^i_1\Gamma^{ABC}\Lambda^i_2 = -(-1)\bar{\Lambda}^i_2\Gamma^{ABC}\Lambda^i_1 \tag{A.46}$$

The overall minus sign is due to the symplectic form from the definition of charge conjugation, with the crucial sign sign factors are (± 1); these coefficients are often labelled as t_N for a gamma matrix, $\Gamma^{(N)} \equiv \Gamma^{M_1\cdots M_N}$. The **generalized symplectic-Majorana exchanging identity**, which handles all cases, is

$$\bar{\Lambda}^i_1\Gamma^{(N_1)}\cdots\Gamma^{(N_p)}\Lambda^i_2 = (-1)\left(t_{N_1}\cdots t_{N_p}\right)\bar{\Lambda}^i_2\Gamma^{(N_p)}\cdots\Gamma^{(N_1)}\Lambda^i_1\,, \tag{A.47}$$

or more simply, to reverse bispinors, one simply reverses the multiplication order of the $\Gamma^{(N)}$'s, and multiplies by minus one times all of the t_N's of each $\Gamma^{(N)}$. It has been shown that t_N has mod-4 periodicity (See [159]), $t_{N+4} = t_N$, so this gives a formula for any $\Gamma^{(N)}$; this periodicity holds in all dimensions.

The four dimensional **Majorana identities** are given by

$$\bar{\lambda}_1\lambda_2 = (+1)\bar{\lambda}_2\lambda_1 \tag{A.48}$$
$$\bar{\lambda}_1\gamma^a\Lambda_2 = (-1)\bar{\lambda}_2\gamma^a\lambda_1 \tag{A.49}$$
$$\bar{\lambda}_1\gamma^{ab}\lambda_2 = (-1)\bar{\lambda}_2\gamma^{ab}\lambda_1 \tag{A.50}$$
$$\bar{\lambda}_1\gamma^{abc}\lambda_2 = (+1)\bar{\lambda}_2\gamma^{abc}\lambda_1\,, \tag{A.51}$$

with the **generalized Majorana identity** given by

$$\bar{\lambda}_1\gamma^{(N_1)}\cdots\gamma^{(N_p)}\Lambda_2^i = \left(t_{N_1}\cdots t_{N_p}\right)\,\bar{\lambda}_2\gamma^{(N_p)}\cdots\gamma^{(N_1)}\lambda_1\,. \tag{A.52}$$

where the t_N's are read off the same way as 5-D (i.e. the coefficient in parenthesis).

A.2.3 A Useful Formalism for γ Matrices

Famously Γ matrices form a complex Clifford algebra $\mathcal{C}\ell(D-1,\,1)$, that is defined by

$$\{\gamma^\mu,\,\gamma^\nu\} = -2\eta^{\mu\nu}\,, \tag{A.53}$$

so $a = 1,\,\cdots,\,D$. The higher rank $\gamma^{\mu_1\cdots\mu_N}$ objects obey even more complex (anti)-commutation relationships, which are critical to the structure of SUSY and SUGRA. Frequently, one winds up with permutations and re-orderings of γ matrices, e.g. expressions like

$$\gamma^{\mu\nu}\gamma^\rho - \gamma^\rho\gamma^{\mu\nu}\,. \tag{A.54}$$

I have found a useful formalism for dealing with the large combinatorics associated to γ identities, which eases proofs and cumbersome notation.

The conversion to my gamma formalism proceeds by assigning each gamma matrix with a unique (i.e. free) index a number. Firstly, in the above expression one could make a dictionary, e.g.

$$\begin{aligned} \mu &\rightarrow 1\,, \\ \nu &\rightarrow 2\,, \\ \rho &\rightarrow 3\,. \end{aligned} \tag{A.55}$$

Secondly, if one has gamma matrices that are contracted (e.g. an expression like

$\gamma^\alpha\gamma^\mu\gamma_\alpha$), I give that repeated label a Greek character, e.g.

$$\gamma^\alpha \to \alpha \,. \tag{A.56}$$

Thirdly, I turn the γ matrices into these (now exotic) numbers; additionally, I have it that brackets represent the higher rank γ matrices, so e.g.

$$\begin{aligned}
&\gamma^\mu \to 1\\
&\gamma^\nu \to 2\\
&\gamma^\rho \to 3\\
&\gamma^\alpha \to \alpha\\
&\gamma^{\mu\nu} \to [12]\\
&\gamma^{\mu\nu\rho} \to [123]\,,
\end{aligned} \tag{A.57}$$

where **these 'numbers' are actually stand-ins for matrices**; alternatively, one can simply think of them as being exotic Clifford numbers. So for example, the expression $\gamma^\alpha\gamma^\mu\gamma_\alpha \to \alpha 1 \alpha$.

Fourthly, all numerical coefficients are put in factor form (e.g. $(2 \to \frac{2}{1})$ to differentiate them from the matrices being represented by numbers (In practice, on paper one can draw a circle around, or underline, numerical factors to differentiate them).

Fifthly, and finally, the only rules I impose on my numbers are the following,

$$ab \;=\; -ba - \frac{2}{1}(ab)\,, \quad \forall\, a, b \in \{1, 2, 3, \alpha\} \tag{A.58}$$

$$\alpha\alpha \;=\; \frac{-D}{1}\,. \tag{A.59}$$

In other words, these exotic numbers anti-commute up to a Clifford term, and all of the Greek numbers contract to create a pure number (equal to minus the spacetime dimension). The final critical rule is that when two contracted indices touch, one ends up with the expected $\gamma^\alpha\gamma_\alpha = -D \to \alpha\alpha := \frac{-D}{1}$. (A.59)

These two rules force an isomorphism between this simplified formalism and the original γ matrices; the dictionary forms a 1-to-1 correspondence between all expressions. From here, one can start writing down old equations in a new form, so for example

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\eta^{\mu\nu} \quad \to \quad 12 + 21 = \frac{-2}{1}(12) \tag{A.60}$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\gamma^{\mu\nu} \quad \to \quad 12 - 21 = \frac{2}{1}[12]\,. \tag{A.61}$$

Adding both of those equations, one ends up with a nice, frequently used relation,

$$[12] = 12 + (12)\,. \tag{A.62}$$

Returning to the original example expression $\gamma^{\mu\nu}\gamma^\rho - \gamma^\rho\gamma^{\mu\nu}$, one has

$$\begin{aligned} \to [12]3 - 3[12] &= \Big(123 + 3(12)\Big) - \Big(312 + 3(12)\Big) = 123 - 312 \\ &= \begin{array}{ll} 123 & -312 \\ & +\tfrac{2}{1}(13)2 \\ & -\tfrac{2}{1}1(23) \end{array} && \text{(A.63)} \\ &= \frac{2}{1}\Big(2(13) - 1(23)\Big) && \text{(A.64)} \\ &\to \; 2\left(\gamma^\nu\eta^{\mu\rho} - \gamma^\mu\eta^{\nu\rho}\right)\,. && \text{(A.65)} \end{aligned}$$

The middle lines look convoluted, however they obey an extremely trivial pattern. The parenthesis tracks which two numbers are being permuted (as well as being the Minkowski metric), and thus all cubic terms $123, 231, 132, 312$, etc, are all equivalent at cubic order, up to $\pm$ and subleading linear terms of the form $a(bc)$. In other words, every time one permutes the numbers passed one another, a term is picked up (from the Clifford anti-commutator) and the sign of the cubic term alternates. The subleading terms (those underneath the term -312) are generated by anti-commutator terms, and one continues doing the permutations until one has the "normal ordering" of 123. The ± 1 is determined by the sign of the last item in the stack, so here -123. This cancels the other normal-ordered terms.

This procedure systematically simplifies any sequence of unordered, anti-symmetrised γ expression as well as provides expedited ways of proving all of the γ matrix formulas generated in subsection A.2.3. For instance, converting (A.35) (these recursion formulas are extraordinarily powerful in this context) into this formalism, one finds

$$\frac{3}{1}[123] = 1[23] + 2[31] + 3[12]\,. \tag{A.66}$$

It is instructive to prove then that, using $[12] = 12 + (12)$, one has

$$[123] = 123 + 1(23) - 2(31) + 3(12)\,, \tag{A.67}$$

and to showcase a formula with contracted indices, that one can derive

$$\alpha 1 \alpha = \frac{D-2}{1} 1 \tag{A.68}$$

These formulas in component notation are the well-known relations

$$\begin{aligned} \gamma^{\mu\nu\rho} &= \gamma^\mu\gamma^\nu\gamma^\rho + \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu} + \gamma^\rho\eta^{\mu\nu} \\ \gamma^\alpha\gamma^\mu\gamma_\alpha &= (D-2)\gamma^\mu\,. \end{aligned}$$

Proving SUSY invariance and closure of the algebra for component fields is rapidly sped up in this formalism, as are the usual gamma identities used in all fermion calculations.

Appendix B

GKD Symbols and Symmetric Polynomials

I make copious use of the **generalized Kronecker delta** (GKD) tensors, which are the tensor index formulation of weighted, anti-symmetric permutations of indices, the basis of symmetric polynomials, the basis of p-forms and exterior calculus, and so on. They are highly powerful objects, so I now review them carefully. To begin with the most basic object, one has the definition

$$\delta^{M_1 \cdots M_D}_{N_1 \cdots N_D} := \varepsilon^{M_1 \cdots M_D} \varepsilon_{N_1 \cdots N_D} \, . \tag{B.1}$$

One can also define GKD tensors with indices lower than the the number of spacetime dimensions, which I define as[1]

$$\delta^{M_1 \cdots M_n}_{N_1 \cdots N_n} := n! \delta^{[M_1}_{N_1} \cdots \delta^{M_n]}_{N_n} \tag{B.2}$$

First, this is an inelegant way to write down the GKD tensors, but it is a valid definition. So for instance, they then trivially work for relations of the form

$$T_{[\mu_1 \cdots \mu_n]} = \frac{1}{n!} \delta^{\mu'_1 \cdots \mu'_n}_{\mu_1 \cdots \mu_n} T_{\mu'_1 \cdots \mu'_n} \, . \tag{B.3}$$

Second, notice the well known relation

$$n! \delta^{[M_1}_{N_1} \cdots \delta^{M_D]}_{N_D} := \varepsilon^{M_1 \cdots M_D} \varepsilon_{N_1 \cdots N_D} \tag{B.4}$$

[1]The factor of $n!$ exists because I am using weight-one conventions, $A_{[ab]} = \frac{1}{2}[A_{ab} - A_{ba}]$. So the GKD tensor will always expand out into sums of ordinary Kronecker delta tensors times unity, e.g. $\delta^{ab}_{cd} = \delta^a_c \delta^b_d - \delta^a_d \delta^b_c$.

shows how the 2 definitions agree for the case $n = D$. Third, note that there are only as many GKD tensors as there are spacetime dimensions, owing to the Levi-Civita tensor being the largest anti-symmetric form that a spacetime can hold without implying triviality. Fourth and finally, notice that once one moves to curved spacetimes, one has

$$\varepsilon^{\mu_1\cdots\mu_D} = \frac{1}{\sqrt{-g}}\tilde{\varepsilon}^{\mu_1\cdots\mu_D} \tag{B.5}$$

$$\varepsilon_{\mu_1\cdots\mu_D} = \sqrt{-g}\,\tilde{\varepsilon}_{\mu_1\cdots\mu_D} \tag{B.6}$$

with $\tilde{\varepsilon}$ being the totally anti-symmetric Levi-Civita symbol. I also use the convention that $\tilde{\varepsilon}^{\mu_1\cdots\mu_D} := (-1)\tilde{\varepsilon}_{\nu_1\cdots\nu_D} g^{\mu_1\nu_1}\cdots g^{\mu_D\nu_D}$. These together imply that the GKD tensor

$$\delta^{\mu_1\cdots\mu_D}_{\nu_1\cdots\nu_D} = \varepsilon^{\mu_1\cdots\mu_D}\varepsilon_{\nu_1\cdots\nu_D} = \tilde{\varepsilon}^{\mu_1\cdots\mu_D}\tilde{\varepsilon}_{\nu_1\cdots\nu_D} \tag{B.7}$$

remains a tensor even when promoted to curved spacetimes, since the determinant factors identically cancel!

B.1 Some Useful GKD Formulas

The power of GKD tensors comes from their beautiful, dimension-independent recursion relations and their contraction identities. The **recursion relations** for fourth-rank and lower GKD tensors is given by

$$\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} = \delta^{\mu}_{\alpha}\delta^{\nu\rho\sigma}_{\beta\gamma\delta} - \delta^{\mu}_{\beta}\delta^{\nu\rho\sigma}_{\gamma\delta\alpha} + \delta^{\mu}_{\gamma}\delta^{\nu\rho\sigma}_{\delta\alpha\beta} - \delta^{\mu}_{\delta}\delta^{\nu\rho\sigma}_{\alpha\beta\gamma}\,, \tag{B.8}$$

$$\delta^{\mu\nu\rho}_{\alpha\beta\gamma} = \delta^{\mu}_{\alpha}\delta^{\nu\rho}_{\beta\gamma} + \delta^{\mu}_{\beta}\delta^{\nu\rho}_{\gamma\alpha} + \delta^{\mu}_{\gamma}\delta^{\nu\rho}_{\alpha\beta}\,, \tag{B.9}$$

$$\delta^{\mu\nu}_{\alpha\beta} = \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\,, \tag{B.10}$$

$$\delta^{\mu}_{\alpha} = \delta^{\mu}_{\alpha}\,. \tag{B.11}$$

In other words, one can expand them as $\delta_p = \sum_\sigma \delta_1\delta_{p-1}$, where even-numbered tensors get alternative signs as one sums over the permutations, but odd-numbered have no sign as instead one sums over permutations of the bottom indices.

Their very useful (but dimension-dependent) **contraction identities** generate, in 4 and 5 dimensions, formulas such as

$$\begin{array}{cc} \text{5-D} & \text{4-D} \\ \delta^{MNRS\bullet}_{ABCD\bullet} = (1)\delta^{MNRS}_{ABCD} & \delta^{\mu\nu\rho\bullet}_{\alpha\beta\gamma\bullet} = (1)\delta^{\mu\nu\rho}_{\alpha\beta\gamma} \\ \delta^{MNR\bullet}_{ABC\bullet} = (2)\delta^{MNR}_{ABC} & \delta^{\mu\nu\bullet}_{\alpha\beta\bullet} = (2)\delta^{\mu\nu}_{\alpha\beta} \\ \vdots & \\ \delta^{M\bullet}_{A\bullet} = (4)\delta^{M}_{A} & \delta^{\mu\bullet}_{\alpha\bullet} = (3)\delta^{\mu}_{\alpha} \\ \delta^{M}_{M} = 5 & \delta^{\mu}_{\mu} = 4\,, \end{array}$$

following the general D-dimensional rule of $\delta^{A_1\cdots A_{D-k}C_1\cdots C_k}_{B_1\cdots B_{D-k}C_1\cdots C_k} = k!\,\delta^{A_1\cdots A_{D-k}}_{B_1\cdots B_{D-k}}$.

B.2 p-Forms and Exterior Calculus via GKD Tensors

All totally anti-symmetric tensors with p indices may be recast as objects called a p-form, which fall into an exterior algebra. These objects are defined via contractions into "basis vectors"

$$\frac{1}{0!}1, \quad \frac{1}{1!}\mathrm{d}x^{\mu}, \quad \frac{1}{2!}x^{\mu_1}\wedge \mathrm{d}x^{\mu_2}\,, \dots\,, \frac{1}{D!}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_D}\,. \tag{B.12}$$

The wedge product over coordinate basis vectors is defined via GKD tensors as

$$\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p} \;=\; \frac{1}{p!}\delta^{\mu_1\cdots\mu_p}_{\nu_!\cdots\nu_p}\mathrm{d}x^{\nu_1}\cdots\mathrm{d}x^{\nu_p}\,. \tag{B.13}$$

So in particular, for a $p = D$ form in D-dimensions, one has

$$\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_D} \;=\; \delta^{\mu_1\mu_2\cdots\mu_D}_{1\,2\,\cdots\,D}\mathrm{d}x^0\,\mathrm{d}x^1\,\cdots\,\mathrm{d}x^D \tag{B.14}$$

$$\;=\; \varepsilon^{\mu_1\cdots\mu_D}\,\mathrm{d}x^0\,\mathrm{d}x^1\,\cdots\,\mathrm{d}x^D\,. \tag{B.15}$$

The basis can now be seen as the volume measure of a D-dimensional integral times the Levi-Civita symbol. In fact, p-forms also have integral measures over p-dimensional hypersurfaces. More details on this can be found in [132]. A p-form is created from contracting a antisymmetric rank-p tensor $A_{\mu_1\cdots\mu_p}$ into the basis vectors

$$\begin{aligned} A &= A_{\mu_1\cdots\mu_p}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p} & \text{(B.16)}\\ &= \delta^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_p}T_{\mu_1\cdots\mu_p}\left(\frac{1}{p!}\mathrm{d}x^{\nu_1}\,\cdots\,\mathrm{d}x^{\nu_p}\right). & \text{(B.17)}\end{aligned}$$

This object is manifestly diffeomorphism invariant.

The **wedge product** between a p-form A and a q-form B

$$A\wedge B=\delta^{\mu_1\mu_2\cdots\mu_p\nu_1\cdots\nu_q}_{\rho_1\cdots\rho_{(p+q)}}A_{\mu_1\cdots\mu_p}B_{\nu_1\cdots\nu_q}\left(\frac{1}{(p+q)!}\mathrm{d}x^{\rho_1}\,\cdots\mathrm{d}x^{\rho_{(p+q)}}\right), \tag{B.18}$$

there is a conversion factor of $\frac{p!q!}{(p+q)!}$ exists simply to keep the wedged coordinate basis weight-one; however, this always refactors (by design) into a $\frac{1}{(p+q)!}$ so I neglect these factors and only keep them around in the basis vectors, i.e. $\frac{1}{(p+q)!}\mathrm{d}x^{\mu_1}\cdots\mathrm{d}x^{\mu_{(p+1)}}$. This object is also manifestly diffeomorphism invariant. This product obeys important anti-symmetry identity

$$A\wedge B=(-1)^{pq}B\wedge A\,, \tag{B.19}$$

thus an even-rank form always commutes, but two odd-rank forms anti-commute. $A\wedge B$ is manifestly a $(p+q)$-form.

The **exterior derivative** of an arbitrary p-form A can be defined simply via

$$\mathrm{d}A=\delta^{\nu\mu_1\cdots\mu_p}_{\rho_1\cdots\rho_{(p+1)}}\partial_\nu A_{\mu_1\cdots\mu_p}\left(\frac{1}{(p+1)!}\mathrm{d}x^{\rho_1}\,\cdots\mathrm{d}x^{\rho_{(p+1)}}\right). \tag{B.20}$$

One can prove the useful relations for p-form A and q-form B from the previous two formulas, for instance

$$\mathrm{d}(A\wedge B)=\mathrm{d}A\wedge B+(-1)^pA\wedge\mathrm{d}B\,. \tag{B.21}$$

Note that the $(-1)^p$ comes from permuting the index for the partial derivative past A's p indices.

One can **integrate over volume forms over** $\mathcal{M}$ with a neat conversion formula between form notation and component notation (and back again) via

$$\begin{aligned}
\mathcal{I} &= \int A && \text{(B.22)}\\
&= \int \delta^{\mu_1\cdots\mu_D}_{\nu_1\cdots\nu_D} A_{\mu_1\cdots\mu_D}\left(\frac{1}{D!}\mathrm{d}x^{\nu_1}\wedge\cdots\wedge\mathrm{d}x^{\nu_D}\right) && \text{(B.23)}\\
&= \int \tilde{\varepsilon}^{\mu_1\cdots\mu_D} A_{\mu_1\cdots\mu_D}\left(\mathrm{d}x^1\,\cdots\,\mathrm{d}x^D\right) && \text{(B.24)}\\
&= \int \mathrm{d}^D x\,\tilde{\varepsilon}^{\mu_1\cdots\mu_D} A_{\mu_1\cdots\mu_D} && \text{(B.25)}\\
&= \int \mathrm{d}^D x\,\sqrt{-g}\varepsilon^{\mu_1\cdots\mu_p} A_{\mu_1\cdots\mu_D} && \text{(B.26)}
\end{aligned}$$

where in the second to last line I have used $\mathrm{d}^D x = \mathrm{d}x^1\cdots\mathrm{d}x^D$ and to get to the last line I have made use of $\varepsilon^{\mu_1\cdots\mu_D} = \frac{1}{\sqrt{-g}}\tilde{\varepsilon}^{\mu_1\cdots\mu_D}$; again for more details, I refer the reader to [132]. Passive diffeomorphism invariance is guaranteed by the form structure. I list one last formula that applies to a manifold without boundary (or whose p-forms die at the boundary), known as the integration by parts formula for p-form A and q-form B

$$\int_{\mathcal{M}} \mathrm{d}A\wedge B = \int_{\mathcal{M}} (-1)^{(p+1)} A\wedge \mathrm{d}B \tag{B.27}$$

It may be useful to a reader not familiar with p-form (or component notation) to convert each line their preferred notation and see how each step is generated in their preferred language.

B.2.1 Hodge Duality

Hodge duals are an operator that relates p-forms to $(D-p)$-forms, namely that a p-form A can always be exchanged for a $(D-p)$-form $*A$ with no loss of content. This is done via contracting into a Levi-Civita tensor, i.e.

$$*A = \delta^{\mu_1\cdots\mu_{(D-p)}}_{\nu_1\cdots\nu_{(D-p)}}\left(\frac{1}{p!}\varepsilon_{\mu_1\cdots\mu_{(d-p)}}{}^{\rho_1\cdots\rho_p}A_{\rho_1\cdots\rho_p}\right)\left(\frac{1}{(D-p)!}\mathrm{d}x^{\nu_1}\cdots\mathrm{d}x^{\nu_{D-p}}\right). \tag{B.28}$$

Note that this can been seen at the level of independent basis; for instance in 3 dimensions, a 2-form has 3 independent components, the same as a $(3-2)$-form $=$ 1-form.

Under my definitions, there are two potentially useful formulas

$$*(*A) = (-1)^{p(D-p)}A \tag{B.29}$$

$$\int A \wedge *B = \int \mathrm{d}^D x\, \sqrt{-g} A_{\mu_1 \cdots \mu_p} B^{\mu_1 \cdots \mu_p} \tag{B.30}$$

for when **both** A and B are p-forms, and thus $A \wedge *B$ is necessarily a volume form.

N.B. In this text, I will use as a convention that I suppress wedge products, i.e. $AB := A \wedge B$**. In this section, I have used explicit wedge product symbols for clarity. Elsewhere, I will simply write** AB **and I will convert between these formulas following the rules listed in this section.**

B.3 Symmetric Polynomials via GKD Tensors

Symmetric polynomials are merely the natural 2-tensor "polynomials" created by the GKD tensors. In some meaningful sense, they are like "double p-forms" (only the second indices do not contract into a coordinate basis $\mathrm{d}x^\mu$). For instance, given a group of tensors $(X_i)_\mu{}^\alpha$, where $i = 1, \ldots, P$, one can define rank-P symmetric polynomial

$$\delta_P[X_1 \cdots X_P] := \delta^{\mu_1 \cdots \nu_P}_{\alpha_1 \cdots \alpha_P} (X_1)_{\mu_1}{}^{\alpha_1} \cdots (X_P)_{\mu_P}{}^{\alpha_P} \,. \tag{B.31}$$

These are called symmetric polynomials because the dual anti-symmetry in the top and bottom indices of the GKD tensor means that all objects inside the polynomial commute, just like as though they were real variables, e.g.

$$\delta_2[X_1 X_2] = \delta_2[X_2 X_1] \,. \tag{B.32}$$

So for instance,

$$\delta_P[X_1 X_2 \cdots X_P] = \delta_P[X_2 X_1 \cdots X_P] \tag{B.33}$$

$$= \delta_P[X_P X_1 \cdots X_{P-1}] \,, \tag{B.34}$$

and so forth. This sets up a very useful homomorphism between tensors in the symmetric polynomial and real polynomials. So for instance, it follows the simple

Pascal triangle/Newton multi-nomial expansions

$$\delta_P[(X+Y)^p] = \delta_P[X^p + pX^{p-1}Y + \cdots + pXY^{p-1} + Y^p] \tag{B.35}$$

$$\begin{aligned}\delta_P[(X+Y+Z)^p] &= \delta_P[X^p + Y^p + Z^p \\ &\quad + p\left(X^{p-1}(Y+Z) + Y^{p-1}(X+Z) + Z^{p-1}(X+Y)\right) \\ &\quad + \cdots]\end{aligned} \tag{B.36}$$

with $X_\mu{}^\alpha$, $Y_\mu{}^\alpha$, and $Z_\mu{}^\alpha$. Calculus for this reason is also greatly simplified so long as one keeps track of whether or not the derivative is upstairs or downstairs.

B.4 Bosons with Spin $s \leq 2$

There is a unified way of writing down all kinetic terms with GKD tensor for bosons with spin less than or equal to 2 in all dimensions:

$$\mathcal{S}_{s=0} = \int \mathrm{d}^D x \, \frac{1}{2}\varphi \left(\delta^\mu_\alpha \partial_\mu \partial^\alpha\right) \varphi \tag{B.37}$$

$$\mathcal{S}_{s=1} = \int \mathrm{d}^D x \, -\frac{1}{2} A_\mu \left(\delta^{\mu\nu}_{\alpha\beta} \partial_\nu \partial^\beta\right) A^\alpha \tag{B.38}$$

$$\mathcal{S}_{s=2} = \int \mathrm{d}^D x \, -\frac{1}{2} h_\mu{}^\alpha \left(\delta^{\mu\nu\rho}_{\alpha\beta\gamma} \, \partial_\nu \partial^\beta\right) \, h_\rho{}^\gamma \tag{B.39}$$

The sign is picked out by unitarity/positive-definite Hamiltonian. The last equation for the spin-2 field is nothing more than the action for linearised GR. This is a massive simplification over the typical way of writing it via,

$$\mathcal{S}_{s=2} = \int \mathrm{d}^D x \, -\frac{1}{2} h_\mu{}^\mu \Box \, h_\nu{}^\nu + \cdots . \tag{B.40}$$

Appendix C

Review of Einstein-Cartan Formalism

C.1 The Vielbein $e_\mu{}^a$

Physicists are typically taught General Relativity using the original formulation involving metric tensors and their associated Christoffel symbols. There is, however, an alternative formulation of General Relativity that uses a different set of variables, called a “vielbein” and its associated “spin connection”. In this section, I focus on the vielbein, using all of the usual language of differential geometry [132].

At its core level, a vielbein contains precisely the same information as the metric tensor, but is directly monitoring the difference between the global coordinates and the local rest frame. In other words, the **vielbein** starts is defined as the map between the local inertial coordinates at a point $x_0 \in \mathcal{M}$ amd the global coordinate atlas. Suppose the local coordinates, which for simplicity I denote as $X^a(x, x_0)$ and is essentially the Riemann-normal coordinates at the point x_0, one has

$$e_\mu{}^a(x_0) = \frac{\partial X^a(x, x_0)}{\partial x^\mu}\Big|_{x=x_0}. \tag{C.1}$$

It is best to use a new coordinate label, since at this point there is a surviving Lorentz symmetry not shared in the global coordinate system. This may be viewed as a spray of coordinate functions, up to local Lorentz boosts, which produce the local inertial coordinate at x_0, i.e. $g_{ab}(x_0) = \eta_{ab}$; at a physical level, one should expect that this must contain all of the information of the metric since this is also the information a metric contains. Indeed, this is so. Instead of working with a spray of local coordinate fields X^a, one can simply choose to work with the vielbien itself as defined above. Notice that the defining relation, and indeed the pragmatic definition of the vielbein, is that it follows the change-of-coordinates rule

$$\begin{aligned} g_{\mu\nu}(x_0) &= \left[\frac{\partial X^a}{\partial x^\mu}\eta_{ab}\frac{\partial X^a}{\partial x^\mu}\right]_{x=x_0} && \text{(C.2)} \\ &= e_\mu{}^a(x_0)\eta_{ab}e_\nu{}^b(x_0) && \text{(C.3)} \\ \Longrightarrow g_{\mu\nu} &= e_\mu{}^a\eta_{ab}e_\nu{}^b\,. && \text{(C.4)} \end{aligned}$$

In other words the vielbein can be thought of as the map which takes a global metric $g_{\mu\nu}$ to the locally inertial metric η_{ab} . Likewise, the vielbein drags vectors from the tangent space to the local inertial frame's space, $e_\mu{}^a : V^\mu \to V^a = e_\mu{}^a V^\mu$. Because $e_\mu{}^a$ is a linear map that represents a change of coordinates, it necessarily contains an inverse mapping that is defined naturally through a matrix inverse. Let E be the $D \times D$ matrix formulation

$$E = e_\mu{}^a \tag{C.5}$$

then there is a related object called the **inverse vielbein** defined in the obvious manner

$$e_a{}^\mu := [E^{-1}]_a{}^\mu \tag{C.6}$$

It is left to the reader to demonstrate to themselves that formulas like $g^{\mu\nu}e_\nu{}^b\eta_{ab} = e_a{}^\mu$ and $\mathrm{Det}(E) = \sqrt{-g}$ hold true, so one may always raise and lower with the respective metrics in the usual manner. Note that it is conventional to define e (without indices) to be $e := \mathrm{Det}(E)$.

Now, of course, it is worth pointing out gauge redundancies/symmetries. Famously, there is no unique global coordinate system, thus the metric is unique only up to diffeormorphisms. Likewise, there is no unique local inertial frame, there is only a unique one up to local Lorentz boosts $X^a(x) \to \Lambda^a{}_b(x)X^b(x)$. Since the metric can be reinterpreted as a Lorentz-invariant contraction of indices, it is trivial to see that local Lorentz boosts have no affect on the metric

$$\begin{aligned} e_\mu{}^a(x) &\to \Lambda^a{}_b(x)e_\mu{}^b(x) && \text{(C.7)} \\ \Longrightarrow g_{\mu\nu}(x) &\to g'_{\mu\nu} = g_{\mu\nu}(x)\,, && \text{(C.8)} \end{aligned}$$

using $g_{\mu\nu} = e_\mu{}^a\eta_{ab}e_\nu{}^b$ and the well-known identity $\Lambda^a{}_c\eta_{ab}\Lambda^b{}_d = \eta_{cd}$. Likewise, the vielbein transforms as a covector under diffeormorphisms for the expected reasons

$$e_\mu{}^a \to e'_\mu{}^a(x') = \frac{\partial x'^\nu}{\partial x^\mu}e_\nu{}^a(x(x'))\,. \tag{C.9}$$

Thus, the vielbein at first sight appears to be a bizarre re-encoding of the metric to accommodate two coordinate systems. As one will discover, however, there are many advantages to this variable. But first, I need to introduce the spin connection.

C.2 The Spin Connection $\omega_\mu{}^a{}_b$

The spin connection is the second crucial variable in the Einstein-Cartan formulation of General Relativity. This object is noticeably similar to that of the gauge connections of Yang-Mills. Indeed, this is a connection that knows about the curvature of the spacetime as one moves about the manifold, much like the Yang-Mills gauge connection $A_\mu{}^a$ knows about the "curvature" induced on objects which are charged under color. The principle of equivalence requires all objects in the manifold to be charged under diffs (i.e. have a contribution to the stress-energy). Before getting into this physics, I will introduce the spin connection as the connection needed to preserve local Lorentz symmetry.

Suppose one is given a vector field in the local inertial space, $\Phi^a(x)$. This vector will transform under a local Lorentz boost as[1]

$$\Phi^a(x) \to \Lambda^a{}_b(x)\Phi^b(x)\,. \tag{C.10}$$

Notice, however, if one wishes to build up an action or write down field equations for this object, then one needs to define how derivatives act on this object. In order to build locally Lorentz-invariant equations, one needs to apply the covariant derivative trick used in Yang-Mills, namely

$$\partial_\mu\Phi^a \to D_\mu\Phi^a = \partial_\mu\Phi^a + \omega_\mu{}^a{}_b\Phi^b\,, \tag{C.11}$$

and then give the spin connection the transformation properties necessary to keep this derivative gauge covariant

$$\begin{aligned}\Omega_\mu{}^a{}_b &\to \Lambda^a{}_c\Omega_\mu{}^c{}_d\Lambda^d{}_b - \partial_\mu\Lambda^a{}_c\Lambda_b{}^c & \text{(C.12)}\\ D_\mu\Phi^a &\to \Lambda^a{}_b D_\mu\Phi^b & \text{(C.13)}\end{aligned}$$

[1]N.B. From now on, I shall drop the explicit functional dependence on the manifold, i.e. $\Phi^a(x) = \Phi^a$.

C.2.1 In Matrix Notation

If one puts this into matrix notation, so that the Lorentz indices are suppressed, i.e.

$$\Omega_\mu{}^a{}_b \to \Omega_\mu, \quad \Lambda^a{}_b \to \Lambda\,, \tag{C.14}$$

then this has the usual elegance of the Yang-Mills formalism

$$\begin{aligned} D'_\mu \Phi' &= \partial_\mu \Phi' + \Omega'_\mu \Phi' &\text{(C.15)}\\ &= \partial_\mu(\Lambda\Phi) + (\Lambda\omega_\mu\Lambda^{-1} - \partial_\mu\Lambda\Lambda^{-1})\Lambda\Phi &\text{(C.16)}\\ &= \Lambda(\partial_\mu\Phi + \omega_\mu\Phi)\,. &\text{(C.17)} \end{aligned}$$

C.2.2 In p-Form Notation

If one puts this into p-form notation, i.e. as a matrix-valued 1-form $\omega = \omega_\mu \mathrm{d}x^\mu$, then one gets another boost in the simplicity of the formalism and a deeper connection to the geometry.

$$\begin{aligned} D\Phi &= \mathrm{d}\Phi + \omega\Phi\,, &\text{(C.18)}\\ \omega \to \omega' &= \Lambda\omega\Lambda^{-1} - \mathrm{d}\Lambda\Lambda^{-1}\,, &\text{(C.19)}\\ \Phi \to \Phi' &= \Lambda\Phi &\text{(C.20)} \end{aligned}$$

implies

$$\begin{aligned} D\Phi \to D'\Phi' &= \mathrm{d}\Phi' + \Omega'\Phi' &\text{(C.21)}\\ &= \mathrm{d}(\Lambda\Phi) + (\Lambda\omega\Lambda^{-1} - \mathrm{d}\Lambda\Lambda^{-1})\Lambda\Phi &\text{(C.22)}\\ &= \mathrm{d}\Lambda\Phi + \Lambda\mathrm{d}\Phi + \Lambda\omega\Phi - \mathrm{d}\Lambda\Phi &\text{(C.23)}\\ &= \Lambda(\mathrm{d}\Phi + \omega\Phi) &\text{(C.24)}\\ &= \Lambda D\Phi\,. &\text{(C.25)} \end{aligned}$$

This is as simple of a notation as one can muster. Note one can also use p-forms without the matrix notation, e.g. $D\Phi^a = \mathrm{d}\Phi^a + \omega^a{}_b\Phi^b$. The notation that is most practical and convenient depends strongly on the context.

C.3 The Curvature 2-Form $\mathcal{R}^a{}_b$

Using the exterior calculus and re-writing the connection as a matrix-valued 1-form $\omega = \omega_\mu{}^a{}_b \mathrm{d}x^\mu$, one can make further crucial analogies with Yang-Mills theory. Changing gears, one knows from calculations in Yang-Mills (or else one can compute by hand), that there is a formula

$$\begin{aligned} D^2\Phi &= \mathcal{R}\Phi && \text{(C.26)} \\ \text{s.t. } \mathcal{R} &= \mathrm{d}\omega + \omega\,\omega && \text{(C.27)} \\ &= \left(\partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^a{}_c\, \omega_\mu{}^{cb}\right)\left(\frac{1}{2}\mathrm{d}x^\mu \mathrm{d}x^\nu\right), && \text{(C.28)} \end{aligned}$$

where in the last line I have expressed all of the suppressed indices. This 2-form object, the **curvature 2-form** $\mathcal{R}$ is the only gauge covariant object,

$$\mathcal{R} \to \Lambda \mathcal{R} \Lambda^{-1} \tag{C.29}$$

$$\text{i.e. } \mathcal{R}^a{}_b \to \Lambda^a{}_c\, \mathcal{R}^c{}_d\, \Lambda_b{}^d \tag{C.30}$$

that can be constructed from the spin connection ω. This object also obeys a **Bianchi identity**

$$D\mathcal{R} = 0 \tag{C.31}$$

It should come as no surprise to the reader that this object is merely the Riemann tensor moved from spacetime coordinates μ, ν, $\rho, \ldots$ to local Lorentz (i.e. Minkowski fibre) indices a, b, $\ldots$. Before this can be seen, it is necessary to discuss how this variable relates to the vielbein.

C.3.1 The Vielbein Postulate

There is a simple relation between the vielbein $e_\mu{}^a$ and the spin connection $\omega_\mu{}^{ab}$. In essence, it forces the vielbein to be covariantly constant with a connection on each index. In other words,

$$D_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - \Gamma^\rho_{\mu\nu}\, e_\rho{}^a + \omega_\mu{}^a{}_b\, e_\nu{}^b = 0\,. \tag{C.32}$$

This formula is called the **vielbein postulate**. Later on, I will prove that this follows from a variational principle, but for now I merely take it as a fact. It is quite obviously closely related to the Levi-Civita condition

$$\nabla_\rho g_{\mu\nu} = 0\,, \tag{C.33}$$

where ∇ only acts with Christoffel symbols, but D acts with both Christoffel for spacetime indices and spin connections on Lorentz indices. This relation can be explicated by noting that

$$\begin{aligned}
\nabla_\rho g_{\mu\nu} &= D_\rho g_{\mu\nu} && \text{(C.34)}\\
&= D_\rho \left(e_\mu{}^a \eta_{ab} e_\nu{}^b\right) && \text{(C.35)}\\
&= \left(D_\rho e_\mu{}^a\right) \eta_{ab} e_\nu{}^b + e_\mu{}^a \eta_{ab} \left(D_\rho e_\nu{}^b\right) && \text{(C.36)}\\
&= 0 && \text{(C.37)}\\
\Leftrightarrow\; & D_\rho e_\mu{}^a = 0\,. && \text{(C.38)}
\end{aligned}$$

(It is trivial to prove that $D_\rho \eta_{ab} = 0$.) There are two important facts now to mention.

C.3.2 Curvature 2-form is the Riemann Tensor

The first important fact to mention is that **the previously defined curvature 2-form is the Riemann tensor in disguise.**[2] To start, note that the vielbein postulate guarantees a nice covariant conversion between tangent space indices and Lorentz indices, so covariant derivatives play well together

$$\begin{aligned}
\partial_\rho \Phi^a + \omega_\rho{}^a{}_b \Phi^b &= D_\rho \Phi^a && \text{(C.39)}\\
&= D_\rho (e_\nu{}^a \Phi^\nu) && \text{(C.40)}\\
&= e_\nu{}^a D_\rho \Phi^\nu && \text{(C.41)}\\
&= e_\nu{}^a \nabla_\rho \Phi^\nu && \text{(C.42)}\\
&= e_\nu{}^a \left(\partial_\rho \Phi^\nu - \Gamma^\nu_{\rho\alpha} \Phi^\alpha\right)\,. && \text{(C.43)}
\end{aligned}$$

Critically, the line (C.42) follows by making use of the fact that Φ^ν is now a tangent vector, so there is no contribution to it from the spin connection and only the

[2] So long as there is no torsion, a complication which I will address shortly in section C.3.4.

Christoffel symbol. From here, the equation $D^2\Phi^a = \mathcal{R}^a{}_b\Phi^b$ can be re-written as

$$\begin{aligned}
\mathcal{R}_{\mu\nu}{}^a{}_b\Phi^b &= [D_\mu, D_\nu]\Phi^a && \text{(C.44)}\\
&= D_{[\mu}D_{\nu]}\Phi^a = D_{[\mu}D_{\nu]}(e_\rho{}^a\Phi^\rho) && \text{(C.45)}\\
&= e_\rho{}^a D_{[\mu}D_{\nu]}(\Phi^\rho) && \text{(C.46)}\\
&= e_\rho{}^a \nabla_{[\mu}\nabla_{\nu]}\Phi^\rho && \text{(C.47)}\\
&= e_\rho{}^a(\mathcal{R}_{\mu\nu}{}^\rho{}_\sigma\Phi^\sigma) && \text{(C.48)}\\
&= e_\rho{}^a\mathcal{R}_{\mu\nu}{}^\rho{}_\sigma(e_b{}^\sigma\Phi^b) && \text{(C.49)}\\
\Longrightarrow \mathcal{R}_{\mu\nu}{}^a{}_b &:= \mathcal{R}_{\mu\nu}{}^\rho{}_\sigma e_\rho{}^a e_b{}^\sigma && \text{(C.50)}
\end{aligned}$$

where in the third line I have used the vielbein postulate, the fourth makes use of $D_\mu\Phi^\rho = \nabla_\mu\Phi^\rho$ since Φ no longer has Lorentz indices, and the fifth I have made use of the defining formula for the Riemann tensor. Thus, this proves the equivalence of the two, up to conversations of tangent space indices to flat space indices with vielbein.[3]

C.3.3 Torsion-Free Condition

Secondly, if one looks at the form of the vielbein postulate

$$\partial_\mu e_\nu{}^a - \Gamma^\rho_{\mu\nu}\, e_\rho{}^a + \omega_\mu{}^a{}_b\, e_\nu{}^b = 0\,,$$

it suggests that one has the ability to trade in the spin connection in terms of Christoffel symbol, and *vice versa*, up to a term with a partial derivative of the vielbein. This actually is stronger, this equations forces a definition for both the Christoffel symbol and for the spin connection in terms of the vielbein; in other words, the spin connection is also a function of the vielbein. I now give an elegant proof of this without appealing to any knowledge of the Christoffel symbols.

The vielbein postulate can actually can be rephrased as an elegant 2-form equation by anti-symmetrising the spacetime indices and contracting them into the 2-form basis $\mathrm{d}x^\mu\mathrm{d}x^\nu$. One finds that anti-symmetrising kills the Christoffel symbol, since without torsion $\Gamma^\alpha_{[\mu\nu]} = 0$, leaving only spin connection equation

$$\partial_{[\mu}e_{\nu]}{}^a + \omega_{[\mu}{}^a{}_b\, e_{\nu]}{}^b = 0 \Longrightarrow De^a = 0 \tag{C.51}$$

[3]Note that I use an non-standard definition of the Riemann tensor that is weight-one, or half the usual value.

Solution for the Spin Connection

The spin connection can be solved in this form. Taking this equation

$$\partial_{[\mu}e_{\nu]}{}^c + \omega_{[\mu}{}^c{}_d\, e_{\nu]}{}^d = 0\,, \tag{C.52}$$

one may sequester the spin connection to the RHS, and then convert all of the indices to Lorentz indices and raise them.

$$\begin{aligned}
e^{\mu\,a}e^{\nu\,b}\left(\partial_\mu e_\nu{}^c - \partial_\nu e_\mu{}^c\right) &= -e^{\mu\,a}e^{\nu\,b}\left(\omega_\mu{}^c{}_d\, e_\nu{}^d - \omega_\nu{}^c{}_d\, e_\mu{}^d\right) && \text{(C.53)}\\
&= -e^{\mu\,a}\omega_\mu{}^{cb} + e^{\nu b}\omega_\nu{}^{ca}\,, && \text{(C.54)}\\
O^{abc} &:= e^{\mu\,a}e^{\nu\,b}\left(\partial_\mu e_\nu{}^c - \partial_\nu e_\mu{}^c\right) && \text{(C.55)}\\
&= e^{\mu\,a}\omega_\mu{}^{bc} + e^{\nu\,b}\omega_\nu{}^{ca} && \text{(C.56)}
\end{aligned}$$

where on the second to last line, I have defined the **objects of anholonomity** O_{abc}, a sort of geometric precursor to the spin connection; on the last line, I have simply used the above equation of motion exchanging $b \leftrightarrow c$ on the first term. Using the form of the object of anholonomity in form (C.56), one may note that one can add permutations of the form $abc - bca - cab$ to isolate a single factor of $\omega_\mu{}^a{}_b e_\mu{}^b$. Specifically, one finds

$$\begin{aligned}
O^{abc} - O^{bca} - O^{cab} &= -2e^{\mu\,c}\omega_\mu{}^{ab}\,, && \text{(C.57)}\\
\therefore\quad \omega_\mu{}^{ab} &= -\frac{1}{2}e_{\mu\,c}\left(O^{abc} - O^{bca} - O^{cab}\right) && \text{(C.58)}
\end{aligned}$$

This solves for $\omega_\mu{}^{ab}$ in terms of the vielbein exclusively once one substitutes in the definition of the object of anholonomity (C.56) which only depends on derivatives of the vielbein.

C.3.4 Torsion and Contorsion

Famously, the Einstein-Cartan formalism allows for torsion. The **torsion** is typically defined by modifying the Christoffel symbol to contain an antisymmetric component $\Gamma^\mu_{\alpha\beta} \to \Gamma^\mu_{\alpha\beta} - T^\mu{}_{[\alpha\beta]}$, so $\Gamma^\mu_{[\alpha\beta]} = T^\mu{}_{\alpha\beta}$. This causes a form equation $T^a = e_\mu{}^a T^{mu}{}_{\alpha\beta}\frac{1}{2}\mathrm{d}x^\alpha \mathrm{d}x^\beta$, the torsion-free condition now becomes the **torsion equation** of the form

$$\hat{D}e^a = T^a \tag{C.59}$$

where $\hat{D} = \mathrm{d} + \hat{\omega}$ is the spin connection that has torsion. This equation is not much harder to solve than the torsion-free condition, since one can choose to shift the equation into two equations. So if I split the connection as

$$\hat{\omega}^{ab} = \omega^{ab}[e] + K^{ab}\,, \tag{C.60}$$

then

$$\underbrace{(de^a + \omega^a{}_b[e]e^b)}_{=0} + K^a{}_b e^b = T^a\,. \tag{C.61}$$

The new tensor $K_\mu{}^{ab}$ is called the **contorsion**. Using the exact same equation as (C.57), one can see that

$$K_\mu{}^{ab} = \frac{1}{2} e_{\mu\, c}(T^{abc} - T^{bca} - T^{cab}) \tag{C.62}$$

where $T^{abc} := e^{\mu\, a} e^{\nu\, b} T^c{}_{\mu\nu}$. Geometrically, the torsion causes the geodesics to twist as they move through the spacetime; physically, torsion appears only in the presence of fermions. To see this, however, I will need to introduce the action and the equations of motion. I now move to this task right after writing down the Bianchi identities.

C.3.5 Bianchi Identities

One may take the original Bianchi and incorporate torsion. The generalised **Bianchi identities** for the torsion and curvature 2-forms are given by

$$D\mathcal{R}^a{}_b = 0 \tag{C.63}$$

$$DT^a = R^a{}_b e^b\,. \tag{C.64}$$

C.4 Action Principle and Field Equations

C.4.1 Einstein-Cartan Action in First-Order Formalism

The action for Einstein-Cartan gravity is given elegantly in p-form notation as

$$\mathcal{S}_{\mathrm{EC}}[e, \omega] = \frac{1}{4(D-2)!\kappa^2} \int \mathcal{R}^{ab}[\omega] e^{c_1} \cdots e^{c_{D-2}} \tilde{\varepsilon}_{abc_1 \ldots c_{D-2}} \tag{C.65}$$

again with the vielbein 1-form $e^c = e_\mu{}^c \mathrm{d}x^\mu$ and curvature 2-form defined as $\mathcal{R}^{ab}[\omega] = \mathrm{d}\omega^{ab} + \omega^a{}_d \omega^{db}$. At this level, the spin connection 1-form ω^{ab} is merely a field that one varies with, in addition to the vielbein.

C.4.2 Varying the Spin Connection: Torsion-Free Condition

Integrating out the spin connection leads to the torsion free condition. To see this, note that

$$\delta\mathcal{R}^{ab}[\omega] = \mathrm{d}(\delta\omega^{ab}) + \delta\omega^a{}_d\,\omega^{db} + \omega^a{}_d\,\delta\omega^{db} = D(\delta\omega^{ab})\,. \tag{C.66}$$

Therefore, varying the action WRT ω^{ab} leads to

$$\begin{aligned} \delta\mathcal{S}_{\mathrm{EC}}[e,\omega] &= \frac{1}{4(D-2)!\kappa^2}\int (D\delta\omega^{ab})e^{c_1}\cdots e^{c_{D-2}}\tilde{\varepsilon}_{abc_1\ldots c_{D-2}} = 0 & \text{(C.67)}\\ &= \frac{(D-2)}{2(D-2)!\kappa^2}\int \delta\omega^{ab} De^c e^d e^{d_1}\cdots e^{d_{D-3}}\tilde{\varepsilon}_{abcd_1\cdots d_{D-3}} & \text{(C.68)}\\ & & \text{(C.69)} \end{aligned}$$

using integration by parts for forms and $D(e^{c_1}\cdots e^{D-2}) = +(D-2)(De^c e^{d_1})\cdots e^{d_{D-3}}$ (explicitly relabeling indices). This leads us naturally to the condition

$$\propto (\delta\omega^{ab}De^c) = 0 \Longrightarrow De^c = 0\,. \tag{C.70}$$

Therefore, one obtains the same constraints as the previous section, so the connection is an auxiliary variable that must be solved for algebraically in terms the vielbein e^a.

N.B. This can also be confirmed, with much more effort, in component notation by integrating out the spin connection in component form. Doing this yields the equation

$$T^\alpha{}_{\mu\nu} = -\kappa^2\left(S^\alpha{}_{\mu\nu} + \frac{1}{D-2}S^\beta{}_{\beta\mu}\delta^\alpha_\nu - \frac{1}{D-2}S^\beta{}_{\beta\nu}\delta^\alpha_\mu\right) \tag{C.71}$$

where for convenience I have defined the **spin density** and its associated torsion tensor as

$$\begin{aligned} S^\alpha{}_{\mu\nu} &= \frac{1}{e}\frac{\delta\mathcal{S}_{\mathrm{EC}}}{\delta\omega_\alpha{}^{ab}}e^a{}_\mu e^b{}_\nu & \text{(C.72)}\\ T^a{}_{\mu\nu} &= e_\alpha{}^a T^\alpha{}_{\mu\nu}\,. & \text{(C.73)} \end{aligned}$$

Then equation (C.71) will generate the torsion-free condition $De^a = 0$.

In the presence of fermions, for instance there is a covariant matter action for spin-$\frac{1}{2}$ fields [159] is

$$\mathcal{S}_{\mathrm{m}}[\psi, e] = \int d^D x\, i\frac{1}{2}\bar{\psi}\gamma^\mu \left(\partial_\mu - \frac{1}{4}\gamma_{ab}\omega_\mu{}^{ab}\right)\psi \tag{C.74}$$

or the covariant Rarita-Schwinger action for gravitino fields

$$\mathcal{S}_{\mathrm{m}}[\psi, e] = \int d^D x\, -i\frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho} \left(\partial_\nu - \frac{1}{4}\gamma_{ab}\omega_\nu{}^{ab}\right)\psi_\rho\,, \tag{C.75}$$

one clearly has a non-zero contribution to the spin density $\frac{\delta \mathcal{S}_{\mathrm{m}}}{\delta\omega^{ab}} \neq 0$ from the matter Lagrangians, which induces torsion terms upon integrating out the spin connection! Using (C.71) calculations the generated torsion from these terms, but do remember to keep track of the minus sign from moving the torsion over to the RHS.

From which, one ends up with

$$De^a = T^a\,, \tag{C.76}$$

a true torsion equation. Note, also, that this technically implies the entire vielbein postulate once one uses the canonical definition of the Christoffel symbol.

C.4.3 Varying Vielbeins: Einstein Equations

One may re-write the Einstein-Cartan action in component form, employing the formula

$$\varepsilon_{a_1\cdots a_D} = e^{a_1}{}_{\mu_1}\cdots e^{a_D}{}_{\mu_D}\varepsilon_{\mu_1\cdots\mu_D} \tag{C.77}$$

then

$$\begin{aligned}
\mathcal{S}_{\mathrm{EC}}[e,\omega] &= \frac{1}{4(D-2)!\kappa^2}\int \mathcal{R}^{ab}[\omega]\left(e^{c_1}\cdots e^{c_{D-2}}\tilde{\varepsilon}_{abc_1\ldots c_{D-2}}\right)\\
&= \frac{1}{2(D-2)!\kappa^2}\int d^Dx\, e\,\mathcal{R}_{\mu\nu}{}^{ab}[\omega]\left(e^{c_1}_{\rho_1}\cdots e^{c_{D-2}}_{\rho_{D-2}}\tilde{\varepsilon}_{abc_1\ldots c_{D-2}}\tilde{\varepsilon}^{\mu\nu\rho_1\cdots\rho_{D-2}}\right)\\
&= \frac{1}{2(D-2)!\kappa^2}\int d^Dx\, e\,\mathcal{R}_{\mu\nu}{}^{ab}[\omega]\left(e_a{}^{\mu'}e_b^{\nu'}\delta^{\mu\nu\rho_1\cdots\rho_{D-2}}_{\mu'\nu'\rho_{\ldots}\rho_{D-2}}\right)\\
&= \frac{1}{2\kappa^2}\int d^Dx\, e\,\mathcal{R}_{\mu\nu}{}^{ab}[\omega]\left(e_a{}^{\mu'}e_b{}^{\nu'}\delta^{\mu\nu}_{\mu'\nu'}\right) && \text{(C.78)}\\
&= \frac{1}{2\kappa^2}\int d^Dx\, e\, e_a{}^\mu e_b{}^\nu\mathcal{R}_{\mu\nu}{}^{ab} && \text{(C.79)}
\end{aligned}$$

where in the second line I rewritten the volume form into a component action following (B.26). On the third line I have used the (C.77), and then recombined the tangent space index Levi-Civita's into a GKD tensor; then I replaced all of the $e_\rho{}^a e_a{}^{\rho'} = \delta_\rho^{\rho'}$ into contractions on the GKD tensor. In the fourth line I have used the GKD contraction identities (B.12).

If one recalls $e = \sqrt{-g}$ and $\mathcal{R}_{\mu\nu}{}^a{}_b := \mathcal{R}_{\mu\nu}{}^\rho{}_\sigma e_\rho{}^a e_b{}^\sigma$, then one sees that this is already the Einstein-Hilbert action. A very straightforward variation of the action WRT the $e_\mu{}^a$ yields the Einstein equations. This is particularly simple since one is free to hold the spin connection fixed under variations, since

$$\frac{\delta \mathcal{S}_{\text{EC}}}{\delta e_\rho{}^c} = \frac{\delta \mathcal{S}_{\text{EC}}}{\delta e_\rho{}^c}\bigg|_{\omega_\mu{}^{ab}} + \underbrace{\frac{\delta \mathcal{S}_{\text{EC}}}{\delta \omega_\mu{}^{ab}}}_{=0} \frac{\delta \omega}{\delta e_\rho{}^c} = \frac{\delta \mathcal{S}_{\text{EC}}}{\delta e_\rho{}^c}\bigg|_{\omega_\mu{}^{ab}}, \tag{C.80}$$

and therefore because $\omega_\mu{}^{ab}$ is an auxiliary variable, its implicit variations WRT $\delta e_\rho{}^c$ drop out explicitly.

Once this vielbein is varied, it yields the Einstein equation

$$G^a = \mathcal{R}^a - \frac{1}{2}\mathcal{R} e^a = \kappa^2 \mathcal{T}^a, \tag{C.81}$$

given the Ricci 1-form $\mathcal{R}^a = R_{\mu\nu}{}^{ab} e_b{}^\nu \mathrm{d}x^\mu$ and Ricci scalar $\mathcal{R} = \mathcal{R}_{\mu\nu}{}^{ab} e_a{}^\mu e_b{}^\nu$, and $\mathcal{T}^a$ is the stress-energy 1-form $T_{\mu\nu} e^{a\,\nu} \mathrm{d}x^\mu$. (Note that in the presence of torsion, the tensor equations are not symmetric.)

These are the essential features of General Relativity in the Einstein-Cartan formalism. Note that although General Relativity often refers to the torsion-free, Levi-Civita connection variant of gravity, I will weaken the definition of General Relativity to include torsion to allow for couplings to fermion fields (necessary for SUSY).