

DOCTORAL THESIS

STRONG INTERACTIONS FOR COSMOLOGY

A thesis on models of Inflation with non-minimal coupling



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A thesis on models of Inflation with non-minimal coupling

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Dissertation for the degree of Doctor of Philosophy

CP³-Origins

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Submitted to the University of Southern Denmark

April 15, 2015

ACKNOWLEDGEMENTS

My studies and research for the degree of PhD were mainly performed at CP3-Origins. During this time I have been privileged to discuss and work with many outstanding researchers. I owe them all a big thanks for their friendliness, enthusiasm in physics, and great collaboration.

During my PhD I spend time in the Theoretical Cosmology Group at McGill university. I thank my host Robert Brandenberger for taking me in and for patience in our physics discussions. I am grateful to all the people in the group for making my stay a fruitful experience.

I am truly thankful to my advisor, Francesco Sannino, for having given me all these opportunities, for his immense support at every step, and for all the scientific advice.

PREFACE AND PUBLICATIONS

During my phd.-studies I contributed with work in the following publications

1. P. Channuie, J. J. Joergensen and F. Sannino, “Minimal Composite Inflation,” *JCAP* **1105** (2011) 007 [arXiv:1102.2898 [hep-ph]].
2. F. Bezrukov, P. Channuie, J. J. Joergensen and F. Sannino, “Composite Inflation Setup and Glueball Inflation,” *Phys. Rev. D* **86** (2012) 063513 [arXiv:1112.4054 [hep-ph]].
3. P. Channuie, J. J. Joergensen and F. Sannino, “Composite Inflation from Super Yang-Mills, Orientifold and One-Flavor QCD,” *Phys. Rev. D* **86** (2012) 125035 [arXiv:1209.6362 [hep-ph]].
4. J. Joergensen, F. Sannino and O. Svendsen, “Primordial tensor modes from quantum corrected inflation,” *Phys. Rev. D* **90** (2014) 4, 043509 [arXiv:1403.3289 [hep-ph]].
5. A. Codello, J. Joergensen, F. Sannino and O. Svendsen, “Marginally Deformed Starobinsky Gravity,” *JHEP* **1502** (2015) 050 [arXiv:1404.3558 [hep-ph]].

This thesis will focus on results and discussions in [1, 2, 4, 5]

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ENGLISH ABSTRACT

In this work we analyze single field slow-roll models of inflation with an explicit non-minimal coupling to gravity. We will introduce the non-minimal coupling by doing large field inflation on a quartic potential and find that it leads to a lowering of the tensor-to-scalar ratio as compared to the minimally coupled case. It may also alleviate the problem of tiny values of the inflaton self-coupling. However, this comes at the cost of a very large non-minimal coupling $\xi \sim 10^4$. We will use this example as template for models of composite inflation. We consider models where the inflaton emerges as a composite scalar field in a low-energy effective field theory description of an underlying gauge dynamics, which is free from fundamental scalars. We will find that inflation may be realized in a pure Yang-Mills theory, where the inflaton emerges as a glueball, as well as in technicolor-like models, where the inflaton emerges in a manner similar to the composite Higgs of Minimal Walking Technicolor. Also, we will describe corrections on top of a quartic potential with non-minimal coupling, and find that even small quantum correction may shift the tensor-to-scalar ratio significantly towards higher values. We compare this discussion with $f(R)$ -theories of inflation, in particular the Starobinsky model. We argue that corrections stemming from integrating out matter fields embedded in the gravitational theory, may be probed in the (r, n_s) -plane if inflation is driven by a $f(R)$ -theory of gravity.

We start the thesis by providing a careful review of the inflationary paradigm. First we consider the perfectly homogeneous and isotropic universe and introduce inflation as a resolution to concerns about the initial conditions necessary for Big Bang cosmology. Next we consider perturbations on top of the homogeneous background and explain that inflation may also serve as a theory for the origin of structure in the universe.

DANSK SAMMENFATNING

I denne afhandling analyseres modeller for slow-roll inflation, som er beskrevet af et enkelt skalarfelt med ikke minimal kobling til tyngdegraften. Vi introducerer den ikke minimale kobling ved at se på inflation beskrevet ved et fjerde ordens potentiale og høje værdier af inflationsfeltet. Vi finder at den ikke minimale kobling leder til mindskelse af tensor-til-skalar forholdet, sammenlignet med det minimalt koblede tilfælde. Den ikke minimale kobling kan også afhjælpe problemer vedrørende små værdier af inflationsfeltets selvkobling. Dette kræver dog at den ikke minimale kobling er meget stor $\xi \sim 10^4$. Vi bruger dette som skabelon for modeller for sammensat inflation. Vi ser på modeller hvor inflatonen opstår som et sammensat skalarfelt i en lav-energi effektiv beskrivelse af underlæggende stærkt vekselvirkende gauge dynamik, som ikke har fundamentale skalarfelter. Vi finder at inflation kan realiseres i ren Yang-Mills teori, hvor inflationen fremkommer som en glueball, og også i technicolor-lignende modeller, hvor inflatonen fremkommer på en lignende måde som den sammensatte Higgs partikel i Minimal Walking Technicolor. Vi beskriver også korrektioner til fjerdeordens potentialet med ikke minimal kobling og finder at selv små korrektioner kan øge tensor-til-skalar forholdet betydeligt. Vi sammenligner denne diskussion med $f(R)$ -teorier for inflation, især Starobinsky modellen. Vi argumenterer for at korrektioner, der fremkommer ved at integrere stoffelter der er indlejret i den gravitationelle teori ud, kan testes i (r, n_s) -planen, hvis inflation er drevet af en $f(R)$ -teori for tyngdekraft.

Vi indleder afhandlingen med et resume af inflationsparadigmet. Først behandler vi det perfekt homogene og isotropiske univers og introducerer inflation som en løsning til problemer vedrørende startbetingelserne i Big Bang kosmologi. Derefter ser vi på perturbationer af den homogene baggrund og forklarer at inflation også kan betragtes som en teori for strukturdannelse i universet.

INTRODUCTION

The theory of inflation was developed in the early 1980's to solve a number of puzzles in cosmology, in particular the flatness, horizon and homogeneity problems of the standard hot Big Bang model. The idea is that an early phase of accelerated expansion prior to radiation domination can serve to stretch out any primordial inhomogeneities and create an almost flat universe, thereby providing initial conditions which are in agreement with observations. In addition, inflation allows regions of the universe which are not in causal contact at late times, to originate from within a causally connected patch at the earliest times. This provides a causal explanation for the long range correlations observed in the universe, in particular the almost perfect isotropy of the cosmic microwave background (CMB). A crucial element of the inflationary scenario is, that besides explaining the initial conditions, it also serves as a theory for the origin of structure in the universe. Small scale quantum vacuum fluctuations of the energy density are thought to be stretched to enormous scales by the inflationary expansion, thereby creating a spectrum of perturbations on top of the homogeneous background. These perturbations serve as seeds for structure formation which proceeds by means of gravitational instability and are imprinted as temperature anisotropies in the CMB. This provides a remarkable link between small scale quantum physics in the early universe and structure on the largest cosmological scales in the present day observable universe.

The thesis is structured as follows. In chapter 1 we introduce the inflationary paradigm as a solution to the flatness and horizon problem of standard (pre-inflationary) Big Bang cosmology. We take the cosmological principle as starting point and from there on derive the conditions needed for inflation. We then describe the simple scenario in which inflation is modeled by means of a single scalar field which rolls slowly on its potential.

In chapter 2 we consider fluctuations on top of the homogeneous background solution derived in chapter 1. We will present the famous calculation of the primordial spectrum of perturbations generated by quantum vacuum fluctuations during inflation. These perturbations provide an important link between the underlying physics model

describing inflation and observables. The power spectrum of the perturbations will be used to confront inflationary model building with data throughout the remainder of the thesis.

From chapter 3 and onwards we begin investigating specific models of inflation. In chapter 3 we will introduce a non-minimal coupling between the inflaton field and gravity. This leads to several interesting consequences which we explore. In particular it leads to a lowering of the tensor-to-scalar ratio r , as compared to minimally coupled models in general. This feature is favored by current experiments. It also alleviates the problem of tiny values for the inflaton self-coupling. The models we present throughout the remainder of the thesis all feature a non-minimal coupling term and the results derived in chapter 3 will be used throughout.

In chapter 4 we presents two models of composite inflation. First we consider a model where the inflaton emerges as the lightest glueball field associated with a pure Yang-Mills theory. We will see that it is possible to achieve inflation with a glueball inflaton in agreement with current data. In this chapter we also consider the issue of the unitarity cut-off related to the introduction of a non-minimal coupling term. Next we consider a model in which the inflaton emerges as a composite field of a strongly interacting and nonsupersymmetric gauge theory, featuring purely fermionic matter. As templates for the discussion, we use models of dynamical electroweak symmetry breaking, in particular Minimal Walking Technicolor. We then investigate whether it is possible for the lightest composite scalar to serve both as a composite Inflaton and a composite Higgs and find that within our framework it is not.

In chapter 5 we consider small corrections on top of the quartic potential of an inflaton with non-minimal coupling to gravity. The corrections may be thought of as quantum corrections, which typically lead to a potential which carries a non-integer power of the field. We will find that even small corrections shift the predicted tensor-to-scalar ratio r significantly towards higher values, and hence that quantum corrected potentials may account for a sizable amplitude of primordial tensor modes.

In chapter 6 we consider inflation within $f(R)$ -theories of gravity. In particular we consider the Starobinsky model of inflation, and find that it is connected to matter scalar field models with a non-minimal coupling to gravity. We then consider quantum-induced marginal deformations of the Starobinsky action, and find that such deformations significantly shift the predicted tensor-to-scalar towards higher values. At last we discuss sources for these corrections. In chapter 7 we conclude.

INFLATION IN HOMOGENEOUS AND ISOTROPIC SPACE-TIME

In this chapter we introduce the inflationary scenario of the early universe. We will take the cosmological principle and the Einstein equations as our starting point, and from there on derive the conditions needed for inflation. We will then see that inflation may solve the flatness and horizon problem of standard Big Bang cosmology, and see how the physics may be described by a single scalar field.

1.1 The cosmological principle and FRW spacetime

Our starting point will be the cosmological principle, which states that on large enough scales the universe is spatially homogenous and isotropic. It has been validated by a variety of observations, for example by red shift surveys which suggest that the universe is homogenous and isotropic when coarse grained on 100 Mpc scales and perhaps most spectacularly by the isotropy of the cosmological microwave background (CMB). In general relativity this may be translated to the statement that space-time can be foliated into a series of maximally symmetric space-like slices, which each gives a snapshot of the universe at an instant of time. We therefore consider space-time to be $R \times \Sigma$, where R represents the time direction and Σ is a maximally symmetric three-manifold. The most general metric consistent with the cosmological principle is the Friedmann-Robertson-Walker (FRW) metric. Up to normalizations $a \rightarrow \lambda a, r \rightarrow \lambda^{-1} r, k \rightarrow \lambda^2 k$, with λ a distance

scale it reads

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right). \quad (1.1)$$

In the present normalization $a(t)$ is the dimensionless scale factor. It determines the size of the spacial slice Σ at time t . The line element is expressed via comoving coordinates (r, θ, ϕ) and cosmic time t . This is the conventional coordinate choice in which the symmetries of the universe are clearly manifest. Only comoving observers, i.e. observers with constant comoving coordinates will see the universe and hence the CMB as isotropic. Physical coordinates are related to comoving coordinates in the following way: The physical separation Δx_{phys} between two comoving observers with separation Δx_{com} scales as

$$\Delta x_{\text{phys}} = a(t) \Delta x_{\text{com}}. \quad (1.2)$$

By taking a time derivative this may be turned into the Hubble law $v_{\text{phys}} = \frac{\dot{a}}{a} \Delta x_{\text{phys}}$ which relates the recessional velocity of two comoving points, for example two comoving galaxies to the physical separation. The constant of proportionality is the Hubble constant H , which measures the fractional rate of expansion (we will only consider expanding space-times)

$$H = \frac{\dot{a}}{a}. \quad (1.3)$$

The metric provides two important scales which characterize FRW space-time. One is the Hubble scale H^{-1} which represents the characteristic time scale of evolution of the scale factor and by multiplying with c , the distance light can travel during that time. If the expansion is decelerating $\ddot{a} < 0$, it is a good estimate of the age and size of the observable universe. The other is the curvature scale R_{curv} , which sets the distance scale at which curvature effects become significant

$$R_{\text{curv}} = a|k|^{-1/2}. \quad (1.4)$$

In the present normalization the curvature parameter has dimensions of $(\text{length})^{-2}$ and signature $k = +1, -1$ for positively curved Σ and negatively curved Σ respectively. Note that the Hubble law is only valid at small distance scales compared to H^{-1} and R_{curv} .

1.1.1 Friedmann equations

The dynamics of FRW space-time is characterized by the evolution of the scale factor $a(t)$, which is related to the energy-momentum density of the universe by the Einstein

equations. Without a cosmological constant term $\Lambda g_{\mu\nu}$ they read

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}. \quad (1.5)$$

G is Newtons constant which we shall express in terms of the reduced Plank mass M_P which is defined by $M_P^{-2} = 8\pi G$ in units where $\hbar = c = 1$. We shall often work in units where also $M_P = 1$. The symmetries of FRW space-time reduce the Einstein equations to just two coupled ordinary differential equations called the Friedmann equations. To see this consider first the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. The Ricci tensor $R_{\mu\nu}$ and the scalar curvature R are given by contractions of the Riemannian curvature tensor $R_{\mu\sigma\nu}^{\rho}$

$$R_{\mu\nu} = R_{\mu\rho\nu}^{\rho} = \partial_{\rho}\Gamma_{\mu\nu}^{\rho} - \partial_{\nu}\Gamma_{\mu\rho}^{\rho} + \Gamma_{\mu\nu}^{\rho}\Gamma_{\rho\sigma}^{\sigma} - \Gamma_{\mu\rho}^{\sigma}\Gamma_{\nu\sigma}^{\rho}, \quad R = g^{\mu\nu}R_{\mu\nu}. \quad (1.6)$$

$\Gamma_{\mu\nu}^{\rho}$ is the Christoffel connection which is related to the metric by

$$\Gamma_{\mu\nu}^{\rho} = \frac{1}{2}g^{\rho\sigma}(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\sigma\mu} - \partial_{\sigma}g_{\mu\nu}). \quad (1.7)$$

Inserting the FRW metric (1.1) the Ricci tensor and the scalar curvature become

$$R_{00} = -3\frac{\ddot{a}}{a}, \quad R_{0i} = 0, \quad R_{ij} = \left[\frac{\ddot{a}}{a} + 2H^2 + \frac{2k}{a^2}\right]g_{ij}, \quad R = 6\left[\frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2}\right], \quad (1.8)$$

Where g_{ij} is the spatial part of the FRW metric. We shall model the energy-momentum of the universe by a perfect fluid

$$T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + pg_{\mu\nu}. \quad (1.9)$$

$\rho(t)$ and $p(t)$ are the energy density and pressure respectively and u^{μ} is the four velocity of the fluid. The fluid is at rest in comoving coordinates such that the cosmological principle is respected $u^{\mu} = (1, 0, 0, 0)$, hence the energy momentum tensor takes the form

$$T_{\nu}^{\mu} = \text{diag}[-\rho(t), p(t), p(t), p(t)]. \quad (1.10)$$

Inserting this in the Einstein equation we obtain the Friedmann equations which are the two promised differential equations

$$H^2 = \frac{1}{3M_P^2}\rho - \frac{k}{a^2}, \quad (1.11)$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{1}{6M_P^2}(\rho + 3p). \quad (1.12)$$

Oftentimes the first equation will be called *the* Friedmann equation while the second equation will be called the acceleration equation.

We considered the Einstein equations without an explicit cosmological constant term $\Lambda g_{\mu\nu}$. This term may be included by redefining/decomposing the energy density and pressure

$$\rho \rightarrow \tilde{\rho} + M_p^2 \Lambda, \quad p \rightarrow \tilde{p} - M_p^2 \Lambda. \quad (1.13)$$

The tilde has temporarily been introduced to denote the contributions from matter and radiation. This leads to the notion of a vacuum energy $\rho_{\text{vac}} = M_p^2 \Lambda$ with negative pressure $p_{\text{vac}} = -\rho_{\text{vac}}$. The Friedmann equations then read

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_p^2} \rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \quad (1.14)$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_p^2} (\rho + 3p) + \frac{\Lambda}{3}. \quad (1.15)$$

A universe dominated by a cosmological constant provides the simplest example of Inflation. We will return to this point shortly.

1.1.2 Standard Big Bang model

In this section we briefly review the basics of the Standard hot Big Bang model, in which the universe is in a thermal radiation dominated state at the earliest times. We start by solving the Friedmann equation for the simple cases where the universe is dominated by either matter, radiation, curvature or a cosmological constant. To do this, we first consider the time component of energy momentum conservation $\nabla_\mu T_0^\mu = 0$

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (1.16)$$

We also define the equation of state parameter $w = \frac{p}{\rho}$ and consider it to be constant for simplicity. The values of w for the different types of stress-energy are listed in table 1.1. By integrating the continuity equation one finds scaling laws for the energy density

$$\rho = \rho_0 a^{-3(1+w)}, \quad (1.17)$$

Where at present time t_0 , the scale factor has been normalized to unity $a(t_0) = 1$. Inserting this in the Friedmann equation 1.14 yields

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(w+1)}}, \quad w \neq -1 \quad (1.18)$$

$$a(t) = e^{H(t-t_0)}, \quad w = -1. \quad (1.19)$$

	w	$\rho(a)$	$a(t)$	$a(\tau)$
Radiation	$\frac{1}{3}$	a^{-4}	$t^{1/2}$	τ
Matter	0	a^{-3}	$t^{2/3}$	τ^2
Curvature	$-\frac{1}{3}$	a^{-2}	t^1	
Λ	-1	a^0	e^{Ht}	$-\frac{1}{H\tau}$

Table 1.1: FRW solutions for a universe dominated by radiation, matter, curvature and a cosmological constant. Solutions in terms of conformal time $d\tau = \frac{dt}{a}$ are included.

From the Friedmann equation we also find that the curvature contribution may be treated as a fictitious energy with $\rho_k = -\frac{3M_p^2 k}{a^2}$ and $w = -\frac{1}{3}$. The solutions for the different types of stress-energy may then be listed as in table 1.1.

If more than one species contribute to the energy density, ρ and p denote the sum of all components

$$\rho \equiv \sum \rho_i, \quad p \equiv \sum p_i, \quad w_i = \frac{p_i}{\rho_i}. \quad (1.20)$$

If the species are non-interacting the scaling laws applies throughout the expansion such that a flat universe $k = 0$ initially will be dominated by radiation. The energy density of radiation scales both with a volume factor a^{-3} and redshift of wavelength a^{-1} which combines to give a^{-4} . Hence matter which only scales with volume a^{-3} will eventually become the dominant constituent. At later times the evolution will be dominated by vacuum energy which does not scale at all. Note also that the scaling laws for matter and radiation implies infinite energy density and temperature at an initial singularity $a \rightarrow 0$ for $t \rightarrow 0$. This leads to the notion of a hot Big Bang at some finite time $t = 0$ in the past. We have arrived at the hot Big Bang picture of the universe: A cosmological singularity at finite time in the past, followed by a hot radiation dominated phase, which gradually cools as the universe expands. At later times matter will be the dominant constituent and eventually vacuum energy.

The Friedmann equation (1.11) provides a time dependent critical energy density ρ_c for which the universe is spatially flat

$$\rho_c = 3M_p^2 H^2. \quad (1.21)$$

It is convenient to express the actual energy density ρ as a fraction of the critical value by defining the density parameter $\Omega \equiv \rho/\rho_c$. The Friedmann equation then takes the form

$$\left(\frac{H}{H_0}\right)^2 = \sum_i \Omega_{i,0} a^{-3(1+w_i)} + \Omega_{k,0} a^{-2}, \quad (1.22)$$

Which implies a consistency relation at present time

$$\sum_i \Omega_{i,0} + \Omega_{k,0} = 1. \quad (1.23)$$

According to observations of the CMB and large-scale structure [14, 16], the present day universe is flat, dominated by dark energy, has a considerable amount of dark matter and only traces of baryonic matter and radiation

$$\Omega_b = 0.0499(22), \quad \Omega_{\text{DM}} = 0.265(11), \quad \Omega_\Lambda = 0.685^{+0.017}_{-0.016}, \quad \Omega_k \simeq 0. \quad (1.24)$$

The universe went from being radiation dominated to matter dominated at $a_0/a_{\text{eq}} \sim 3 \times 10^3$, the CMB was emitted at $a_0/a_{\text{rec}} \sim 1100$ and dark energy became the dominant constituent at $a_0/a_\Lambda \sim \frac{1}{2}$, where the scale factor at present time a_0 , have been included explicitly.

This concludes our brief review of the standard Hot Big Bang model. We have left out almost all details concerning the different phases of evolution, however, the description should be sufficient for the purpose of this chapter which is to introduce the concept of Inflation. In the next sections we will follow Guth [6] and introduce inflation as a solution to the flatness and horizon problems of standard hot Big Bang cosmology.

1.1.3 Flatness problem

The flatness problem comes from considering the Friedmann equations in a universe with matter and radiation, but no vacuum energy. To state and quantify the problem we rewrite the Friedmann equations in terms of the critical density $\Omega = \rho/\rho_c$

$$\Omega - 1 = \frac{k}{a^2 H^2} \quad (1.25)$$

$$\frac{\ddot{a}}{a} = -\frac{1}{2} H^2 \Omega (1 + 3w). \quad (1.26)$$

Combining these equations with the derivative of the first we obtain

$$\frac{d\Omega}{d \ln a} = (1 + 3w) \Omega (\Omega - 1). \quad (1.27)$$

A flat universe $\Omega = 1$ therefore remains flat at all times. This is an unstable fixed point if the strong energy condition $1 + 3w > 0$ is satisfied (valid for radiation $w = \frac{1}{3}$ and matter $w = 0$).

$$1 + 3w > 0 \quad \Rightarrow \quad \frac{d|\Omega - 1|}{d \ln a} > 0.$$

Any deviation from flatness is amplified by the subsequent expansion, hence the flatness of the universe at present time $\Omega_0 \simeq 1$ represents an initial fine tuning problem. This is referred to as the flatness problem of standard Big Bang cosmology in which the universe is initially dominated by radiation and later matter. On the other hand if $1 + 3\omega < 0$ (valid for example for a cosmological constant $w = -1$), the universe evolves towards flatness:

$$1 + 3\omega < 0 \quad \Rightarrow \quad \frac{d|\Omega - 1|}{d \ln a} < 0. \quad (1.28)$$

From (1.26) we see that this leads to accelerated expansion. The flatness problem may therefore be solved by introducing a period of accelerated expansion prior to radiation domination. The inflationary paradigm does exactly that. We may also state the flatness problem and its solution in terms of the comoving Hubble scale $(aH)^{-1}$. From the Friedmann equation we infer the following behavior

$$\frac{d}{dt} (aH)^{-1} < 0 \quad \rightarrow \quad \text{Expansion towards flatness} \quad (1.29)$$

$$\frac{d}{dt} (aH)^{-1} > 0 \quad \rightarrow \quad \text{Expansion away from flatness.} \quad (1.30)$$

The first condition applies to matter and radiation while the second applies to a cosmological constant. A shrinking comoving Hubble scale may be taken as the defining feature of inflation, it implies accelerated expansion since $\frac{d}{dt} (aH)^{-1} = \frac{-\ddot{a}}{(aH)^2}$.

1.1.4 Horizon problem

The isotropy of the CMB pose another problem in standard Big Bang cosmology called the horizon problem. The problem arise since the surface of last scattering consists of many $\sim 10^4$ causally disconnected patches as illustrated in Fig. 1.1. It is highly unlikely that each patch, independently of the others should produce the same spectrum of black body radiation to make the CMB appear isotropic today. To be a bit more precise we consider particle horizons $R_H(t)$ which are the distance light can travel between the initial singularity and time t . Photons travel along null paths which for radial trajectories in a flat universe are characterized by $dr = dt/a$. The comoving distance light can travel between times t_1 and t_2 is then

$$\Delta r = \int_{t_1}^{t_2} \frac{dt}{a(t)} = \frac{n}{1-n} H_0^{-1} \left(a_2^{\frac{1-n}{n}} - a_1^{\frac{1-n}{n}} \right) \quad \text{for } a(t) \propto t^n, \quad n < 1. \quad (1.31)$$

Thus the comoving horizon size at time t is $R_H(t) \sim H_0^{-1} a(t)^{\frac{1-n}{n}}$. At present time $R_H(t_0) \sim H_0^{-1}$ and we see explicitly that the Hubble scale H^{-1} provides a good estimate

for the size and age of the observable universe if its constituents are matter and radiation. When we look at the CMB we are observing the universe at scale factor $a_{\text{rec}} \simeq 1/1100$. Today the comoving distance to a point on the surface of last scattering is then well approximated by the horizon size $\Delta r \sim H_0^{-1}$. At recombination the comoving horizon size of such a point is $R_H(t_{\text{rec}}) \sim H_0^{-1} \sqrt{a_{\text{rec}}} \sim 10^{-2} H_0^{-1}$, were we assumed that the universe is matter dominated from t_{rec} until present time. Hence widely separated points on the surface of last scattering have non overlapping horizons at the time of recombination. So far we have compared the radius of two spheres. By including area and volume factors we find that the surface of last scattering consist of $\sim 10^4$ disconnected patches and $\sim 10^6$ disconnected volumes at the time of recombination.

The horizon problem may be solved by introducing an early period of inflation prior to radiation domination. To see this and for later convenience we switch to conformal time τ defined by

$$d\tau = \frac{dt}{a} . \quad (1.32)$$

The FRW metric is then conformally related to a static Minkowsky metric

$$ds^2 = a(\tau)^2 [-d\tau^2 + dr^2] , \quad (1.33)$$

Where we again restricted ourselves to radial propagation in a flat universe for the sake of simplicity (Generalization to curved spatial slides is straightforward). Conformal time allows us to draw light cones and infer causal relationships in a mannar similar to that of special relativity. With these coordinates the particle horizon is conveniently given by the age of the universe in conformal time:

$$\tau = \int_0^t \frac{dt'}{a(t')} = \int_0^a d \ln a \left(\frac{1}{aH} \right) . \quad (1.34)$$

The size is the width of the past light cone projected onto the surface $\tau = 0$ defined by the initial singularity, see Fig. 1.1. The integral has been written in terms of the comoving hubble scale $(aH)^{-1}$ which is a more useful scale in inflationary cosmology than the particle horizon. We shall follow standard conventions and call $(aH)^{-1}$ *the* horizon. As we have seen it is about the size of the particle horizon during matter and radiation domination, but this does not hold in general. We classify comoving length scales λ with associated wave number k according to their size relative to the horizon

$$\frac{k}{aH} \ll 1 \quad \Rightarrow \quad \text{scale } \lambda \text{ inside the horizon} \quad (1.35)$$

$$\frac{k}{aH} \gg 1 \quad \Rightarrow \quad \text{scale } \lambda \text{ outside the horizon.} \quad (1.36)$$

If a scale is larger than the horizon size causal physics cannot affect it. In standard Big Bang cosmology $\frac{d}{dt}(aH)^{-1} > 0$ such that scales which are outside the horizon at earlier times, such as the CMB scale cf. the horizon problem, may enter the horizon at later times. It is now clear that the horizon problem may be solved by an early period of inflation in which $\frac{d}{dt}(aH)^{-1} < 0$. In this scenario the CMB scale may initially be inside the comoving horizon such that causal physics can equilibrate it. However during Inflation the scale exit the horizon. When inflation ends the standard hot Big Bang commences and the comoving horizon size starts growing such that the CMB scale eventually reenters the horizon, see Fig. 1.2. In this scenario τ will get most of its contribution from early times cf. (1.34) and will be much larger than the estimate aH^{-1} provided by standard Big Bang cosmology.

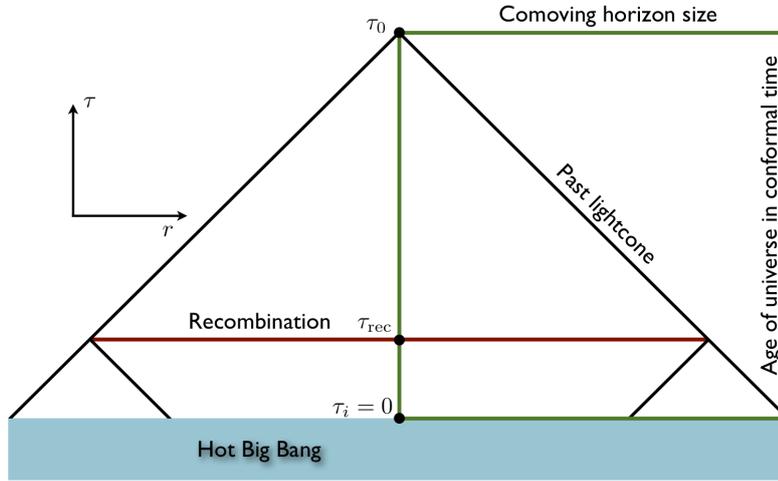


Figure 1.1: Conformal diagram of standard Big Bang cosmology. The past light cones at the surface of last scattering does not overlap. This is the source of the horizon problem in standard Big Bang cosmology. In the text we estimated the surface of last scattering to consist of $\sim 10^4$ causally disconnected patches. The figure is inspired by [24]

The inflationary paradigm may be visualized by the conformal diagram in Fig. 1.3, as we now explain. In standard hot Big Bang cosmology the universe is dominated by radiation early on such that there is an initial singularity $a(\tau_i \equiv 0) = 0$. However in the inflationary paradigm we assume that prior to radiation domination, there is a period of inflation $\frac{d}{dt}(aH)^{-1} < 0$. For the purpose of this discussion we assume that the universe is dominated by a cosmological constant in this period. This is the simplest case of inflation. Then $a \propto e^{Ht}$ and in conformal time the scale factor evolves as

$$a(\tau) = -\frac{1}{H\tau}. \quad (1.37)$$

Hence the initial singularity is pushed to the infinity past in conformal time, $a \rightarrow 0$

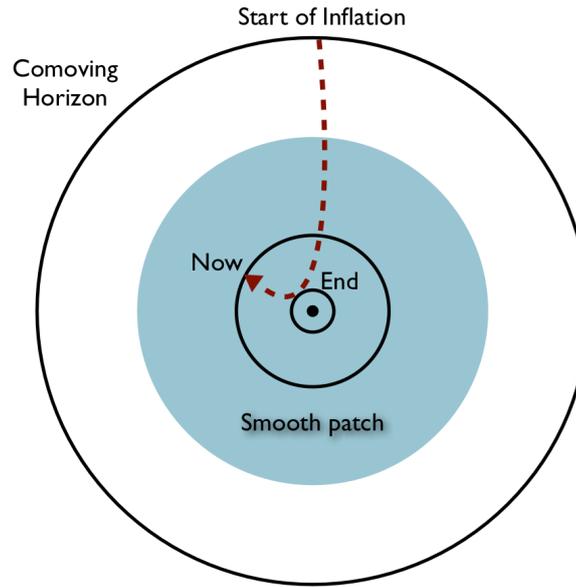


Figure 1.2: Evolution of the comoving Hubble radius $\frac{1}{aH}$ in a universe which undergoes a period of inflation prior to radiation domination. The comoving Hubble radius shrinks dramatically during inflation. This allows the present day horizon to lie within a "smooth patch" that was well inside the horizon at the start of inflation. This solves the flatness and horizon problems. The figure is inspired by [17]

for $\tau \rightarrow -\infty$ thereby allowing past light cones to overlap. Note that the scale factor becomes infinite at $\tau = 0$. This is because we have assumed pure de Sitter space with $H = \text{constant}$. In this case inflation lasts forever, with $\tau = 0$ corresponding to the infinite future $t \rightarrow \infty$. In more realistic models, inflation ends at some finite time which is characterized by the breakdown of (1.37) as an approximation valid during inflation. In these models $\tau = 0$ does not correspond to the initial singularity but a transition from inflation to radiation dominated expansion called reheating.

1.2 Inflation from a scalar field

In the preceding sections we introduced the inflationary paradigm as a solution to the flatness and horizon problems of the standard hot Big Bang model. We considered a simple model in which inflation is driven by a cosmological constant. This is not a realistic model since the universe stays dominated by the cosmological constant at all times such that inflation never ends. In order to transition from inflation to radiation domination the vacuum-like energy during inflation must be time dependent. This is traditionally modeled by introducing a single scalar field ϕ , the inflaton. We start by

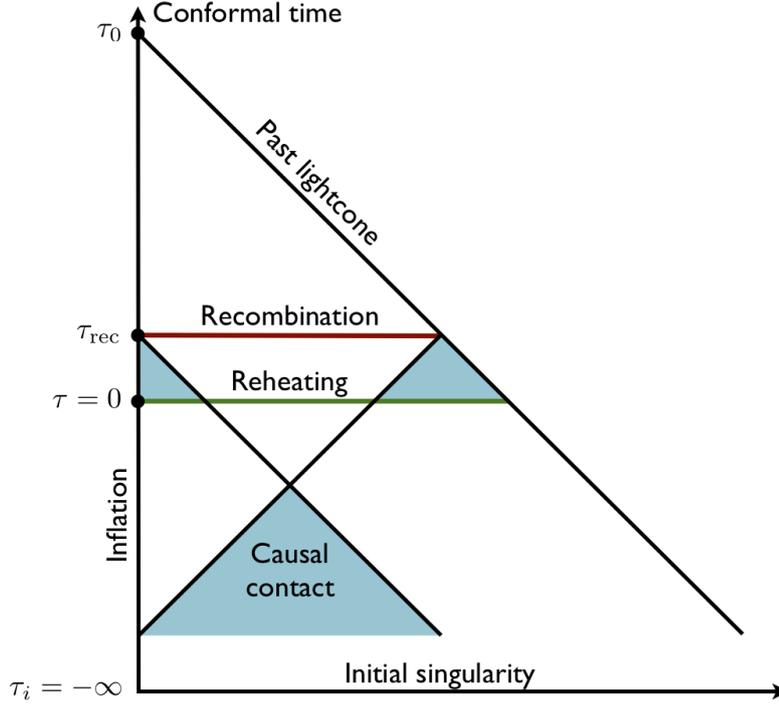


Figure 1.3: Conformal diagram of inflationary cosmology. Inflation pushes the initial singularity to the infinity past in conformal time, thereby allowing past light cones at recombination to overlap. Inflation ends in a reheating phase at $\tau \sim 0$. During reheating the vacuum like energy of the inflationary sector is converted to other sectors. The figure is inspired by [25]

considering the action of a scalar field with a minimal coupling to gravity

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_P^2 R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right], \quad (1.38)$$

Where $V(\phi)$ is the potential energy associated with the field. Later we will consider the more general case of non-minimally coupled theories in which we add the term $\frac{1}{2} R \xi \phi^2$ to the action. The field is split into a classical homogeneous background $\phi(t)$ and fluctuations $\delta\phi(t, \mathbf{x})$

$$\phi(t, \mathbf{x}) = \phi(t) + \delta\phi(t, \mathbf{x}). \quad (1.39)$$

The near isotropy of the CMB suggest that we may treat $\delta\phi(t, \mathbf{x})$ as small perturbations which evolve on a classical homogenous background solution given by $\phi(t)$ and the FRW metric (1.1). In this chapter we are only concerned with the evolution of the homogeneous background while fluctuations are considered later. The energy momentum tensor is

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\mu\rho} \phi \partial_\rho \phi + V(\phi) \right). \quad (1.40)$$

For the homogeneous background it is of the perfect fluid form (1.10) with

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad (1.41)$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) . \quad (1.42)$$

If the potential energy of the field is dominant $V(\phi) \gg \dot{\phi}^2$ we recover the vacuum like behavior of inflation characterized by negative pressure $\omega = \frac{p}{\rho} < 0$, accelerated expansion $\omega < -\frac{1}{3}$ and hence shrinking horizon.

1.2.1 Slow-roll Inflation

To quantify better under which conditions inflation may arise we consider the equation of motion of the inflaton and the Friedmann equations. We neglect spatial curvature since the curvature term in the Friedmann equation become less and less important as inflation gets under way. Then

$$\begin{aligned} \ddot{\phi} + 3H\dot{\phi} + V'(\phi) &= 0 \quad (1.43) \\ H^2 &= \frac{1}{3M_p^2} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) \\ \frac{\ddot{a}}{a} &= H^2 (1 - \epsilon_H), \quad \epsilon_H \equiv \frac{3}{2} (\omega + 1) = \frac{\dot{\phi}^2}{2M_p^2 H^2} = -\frac{\dot{H}}{H^2} . \end{aligned}$$

We have defined the first Hubble slow roll parameter ϵ_H . Accelerated expansion occurs if $\epsilon_H < 1$ and the de sitter limit $p \rightarrow -\rho$ is equivalent to $\epsilon_H \rightarrow 0$. Slow-roll inflation is characterized by the conditions

$$\begin{aligned} \dot{\phi}^2 &\ll V(\phi) \quad (1.44) \\ |\ddot{\phi}| &\ll |3H\dot{\phi}|, |V'(\phi)| . \end{aligned}$$

The smallness of $\ddot{\phi}$, as compared to the Hubble friction term and the slope of the potential, ensures that accelerated expansion is sustained for a sufficient period. This may be parametrized by smallness a second Hubble slow-roll parameter η_H

$$\eta_H = -\frac{\ddot{\phi}}{H\dot{\phi}} = -\frac{1}{2} \frac{\ddot{H}}{H\dot{H}} . \quad (1.45)$$

Higher derivatives of the Hubble parameter can be used to quantify slow-roll, however, the two slow-roll parameters defined here are sufficient for our purposes. The amount of expansion is parametrized by the number of e-foldings N

$$dN \equiv H dt = d \ln a . \quad (1.46)$$

A period of inflation is then characterized by

$$N \equiv \ln \frac{a_{\text{end}}}{a_{\text{start}}} . \quad (1.47)$$

Using N as time variable the Hubble slow-roll parameters take on a convenient form which illustrate that they parametrize deviations from de Sitter expansion by measuring the (non-)constancy of H

$$\epsilon_H = -\frac{d \ln H}{dN} , \quad \eta_H = \epsilon_H - \frac{1}{2\epsilon_H} \frac{d\epsilon_H}{dN} . \quad (1.48)$$

If the slow-roll conditions (1.44) are satisfied we may approximate the equation of motion of the inflaton field and the Friedmann equation as

$$\begin{aligned} 3H\dot{\phi} + V'(\phi) &\simeq 0 \\ H^2 &\simeq \frac{V}{3M_P^2} . \end{aligned} \quad (1.49)$$

It follows that necessary conditions for slow-roll inflation also may be parametrized by smallness of two *potential* slow roll parameters $\epsilon_V \ll 1$ and $|\eta_V| \ll 1$, which are defined as

$$\epsilon_V = \frac{M_P^2}{2} \left(\frac{V'(\phi)}{V} \right)^2 , \quad \eta_V = M_P^2 \frac{V''(\phi)}{V} . \quad (1.50)$$

The slow-roll conditions are then expressed as conditions on the shape of the potential and leads to the notion of a classical field which experiences large Hubble friction and hence rolls slowly on a nearly flat potential, as illustrated in Fig. 1.4. During slow-roll inflation $\epsilon_H \simeq \epsilon_V$ and $\eta_H \simeq \eta_V - \epsilon_V$. Most often we shall express our results in terms of the potential slow-roll parameters.

1.2.2 Attractor behavior

There is an important subtlety related to the fact that the order of differentiation is reduced by one when going from (1.43) to the slow-roll approximation (1.49). The conditions $\epsilon_V \ll 1$ and $|\eta_V| \ll 1$ only restrict the shape of the potential and as such are necessary but not sufficient conditions for slow-roll. No matter what the potential looks like, (1.43) being second order, we are free to choose initial conditions $\dot{\phi}_i$ that violates slow-roll. It is therefore an additional assumption that the solution for a given potential satisfies (1.49). However, as we demonstrate below the solutions exhibit attractor behavior. Solutions with different initial conditions in general converge on a slow roll trajectory in phase space (if the potential allows it to exist), hence further

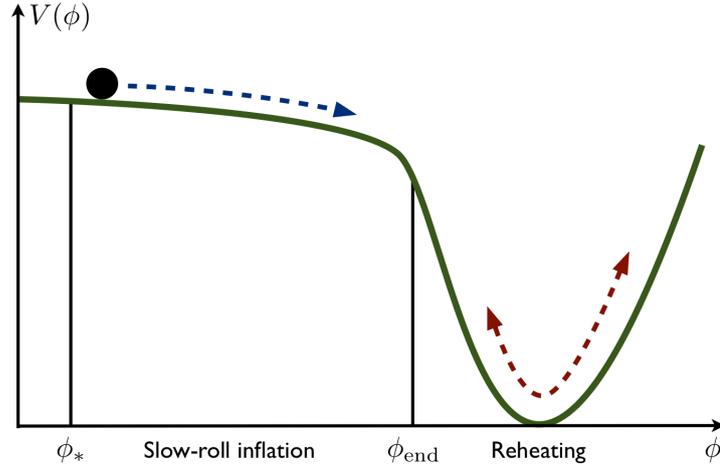


Figure 1.4: A classical inflaton field experience large Hubble friction and rolls slowly on a nearly flat potential. Scales relevant for the CMB anisotropies exit the horizon at ϕ_* about $N_* = 60$ e-folds before the end of inflation. The universe reheats via coherent oscillations of the inflaton field after the end of inflation.

evolution is independent of the initial conditions. This is vital for the predictive power of slow-roll inflation. To demonstrate the attractor behavior we follow [17] and consider the Hamilton-Jacobi approach which treats $H(\phi)$ as the fundamental quantity and ϕ as the time variable. By substituting the time derivative of the second equation in (1.43) into the first, we find

$$\dot{\phi} = -2M_P^2 H'(\phi). \quad (1.51)$$

This may be integrated to specify the relation between ϕ and t . We will assume that $\dot{\phi} > 0$ in the following. Insertion in the Friedmann equation yield the Hamilton-Jacobi equation¹

$$\left(H'(\phi)\right)^2 - \frac{3H^2(\phi)}{2M_P^2} = -\frac{V(\phi)}{2M_P^4}. \quad (1.52)$$

Now consider $H(\phi) = H_0(\phi) + \delta H(\phi)$ where H_0 is a solution of the Hamilton-Jacobi equation and δH a linear perturbation. We linearize to find

$$H_0'(\phi) \delta H'(\phi) \simeq \frac{3}{2M_P^2} H_0 \delta H. \quad (1.53)$$

¹This approach allows us to consider $H(\phi)$ rather than $V(\phi)$ as the fundamental quantity to be specified. For any specified function $H(\phi)$, it produces a potential $V(\phi)$ which admits the given $H(\phi)$ as an exact (inflationary) solution. It also provides yet another nice representation of the Hubble slow-roll parameters: $\epsilon_H = 2M_P^2 (H'(\phi)/H)^2$ and $\eta_H = 2M_P^2 H''(\phi)/H$. In the slow-roll approximation $\epsilon_H \simeq \epsilon_V$ and $\eta_H \simeq \eta_V - \epsilon_V$.

This is readily integrated to give the solution

$$\delta H(\phi) = \delta H(\phi_i) \exp \left[\frac{3}{2M_P^2} \int_{\phi_i}^{\phi} \frac{H_0(\phi)}{H_0'(\phi)} d\phi \right]. \quad (1.54)$$

We assume that $\dot{\phi} > 0$ hence the integrand is negative and initial perturbations die away as ϕ increases, thereby demonstrating the attractor behavior. Inflationary solutions are particularly attractive. To see this clearly we insert (1.51) in the definition for the number of e-folds (1.46) to obtain $dN = -\frac{1}{2M_P^2} \frac{H}{H'(\phi)} d\phi = \frac{d\phi}{M_P \sqrt{2\epsilon_H}}$. (1.54) then reads

$$\delta H(\phi) = \delta H(\phi_i) \exp [-3(N - N_i)]. \quad (1.55)$$

During inflation $\epsilon_H < 1$, the number of e-folds rapidly become large and perturbations to the inflationary solution are diluted exponentially.

1.2.3 End of inflation

Let us now estimate how much inflation is needed. To solve the horizon problem we require that the present day horizon fits inside the horizon at the beginning of inflation, as illustrated in Fig. 1.2.

$$(a_0 H_0)^{-1} < (a_{\text{start}} H_{\text{start}})^{-1}. \quad (1.56)$$

We assume for simplicity that the universe has been radiation dominated since the end of inflation and until present time, $H \propto a^{-2}$. Then

$$\frac{a_0 H_0}{a_{\text{end}} H_{\text{end}}} \sim \frac{a_{\text{end}}}{a_0} \sim \frac{T_0}{T_{\text{RH}}} \sim 10^{-28}, \quad (1.57)$$

Where we utilized that the temperature of a black body spectrum redshifts as a^{-1} and $T_0 = 10^{-3} eV \sim 2.7$ K is the temperature at present time. T_{RH} is the reheating temperature which is the temperature at the beginning of the radiation-dominated phase after inflation where the universe has thermalized. If we assume that inflation takes place around the GUT scale 10^{16} GeV (later we will find this to be a reasonable estimate), we may estimate $T_0/T_{\text{RH}} \sim 10^{-28}$. Hence we require that $(aH)^{-1}$ at least shrink by a factor 10^{28} during inflation. The Hubble constant as approximately constant during inflation such that $H_{\text{start}} \sim H_{\text{end}}$, then

$$\frac{a_{\text{end}}}{a_{\text{start}}} > 10^{28} \quad \Rightarrow \quad N = \ln \frac{a_{\text{end}}}{a_{\text{start}}} > 64. \quad (1.58)$$

This is the famous statement that the horizon problem may be solved by about 60 e-folds of inflation prior to radiation domination. One may show that this solves also

the Flatness problem by a related procedure. It is a very rough estimate though. More accurate results may be obtained by taking into account model dependent dynamics, particularly during the reheating phase which determines the reheating temperature T_{RH} .

Inflation ends when the slow-roll conditions are violated

$$\epsilon_H(\phi_{\text{end}}) = 1, \quad \epsilon_V(\phi_{\text{end}}) \simeq 1. \quad (1.59)$$

At this point the field has ‘picked up speed’ rolling down its potential such that the kinetic energy is comparable to the potential energy $\dot{\phi}^2 \sim V(\phi)$ and accelerated expansion ceases. The field begins to oscillate coherently about the minimum of its potential, see figure 1.4 and this dominates the energy density of the universe as any pre-inflationary entropy would have been redshifted away. This marks the beginning of reheating which transform the low entropy cold universe into a high entropy hot universe dominated by radiation. We will not describe reheating in this thesis, but just note that it typically consist of a phase of preheating followed by a phase of thermalization as reviews for example in [19]. The start of the preheating phase may be taken to be at $\eta_H \sim 1$ where $\ddot{\phi}$ becomes significant and the field starts oscillating. The energy associated with the coherent oscillations of the inflaton field can then be transferred to other sectors by means of parametric resonance. Afterwards the particles created in this process scatter and thermalize to form a thermal background.

In the next chapter we describe (the generation of) perturbations on top of the homogeneous inflationary background. These provide the seeds for anisotropies in the CMB and structure formation and hence impose observational constraints on inflationary model building. As a prelude, we end this chapter by noting that we may associate a certain number of e-folds N_* to a fluctuation in the following way: Fluctuations may be characterized by their co-moving wavenumber k . During inflation the horizon shrinks such that fluctuations which are initially smaller than the horizon eventually cross outside the horizon, see Fig. 2.1. We mark horizon crossing by $k \sim a_* H_*$, the associated background value of the inflaton field by ϕ_* and the number of e-folds to go before the end of inflation by N_* . Then

$$N_* = \ln \frac{a_{\text{end}}}{a_*} \simeq \frac{1}{M_P^2} \int_{\phi_{\text{end}}}^{\phi_*} \frac{V}{V'} d\phi, \quad \Leftrightarrow \quad k = a_* H_* . \quad (1.60)$$

We have inserted the Hamilton-Jacobi results $dN = -\frac{1}{2M_P^2} \frac{H}{H'(\phi)} d\phi = \frac{d\phi}{M_P \sqrt{2\epsilon_H}} \simeq \frac{d\phi}{M_P \sqrt{2\epsilon_V}}$ to express N_* in terms of the potential. Two important observables for inflation are the (dimensionless) power spectra of scalar perturbations $\Delta_{\mathcal{R}}^2(k)$ and tensor perturbations

$\Delta_h^2(k)$. They are typically expanded about a pivot scale k_*

$$\begin{aligned}\Delta_{\mathcal{R}}^2(k) &= A_s \left(\frac{k}{k_*} \right)^{n_s - 1 + \frac{1}{2} dn_s / d \ln k \ln(k/k_*) + \dots} \\ \Delta_h^2(k) &= A_t \left(\frac{k}{k_*} \right)^{n_t + \frac{1}{2} dn_t / d \ln k \ln(k/k_*) + \dots},\end{aligned}\quad (1.61)$$

Where $A_{s/t}$ is the amplitude of scalar/tensor perturbations and $n_{s/t}$ is the spectral index. We will consider this in more detail in the next chapter where we derive the predictions from inflation on these quantities. When we confront models of inflation with data in later sections we use the Planck results expressed at the pivot scale $k_* = 0.05 \text{ Mpc}^{-1}$. The time at which this scale crossed the horizon depends both on the specific inflationary potential and on the model dependent reheating mechanism. It may be estimated as [12]

$$N_* \simeq 71.21 - \ln \frac{k_*}{a_0 H_0} + \frac{1}{4} \ln \frac{V_{\text{hor}}}{M_{\text{P}}^4} + \frac{1}{4} \ln \frac{V_{\text{hor}}}{\rho_{\text{end}}} + \frac{1 - 3w_{\text{int}}}{12(1 + w_{\text{int}})} \ln \frac{\rho_{\text{th}}}{\rho_{\text{end}}}, \quad (1.62)$$

Where only the first two terms are model independent. V_{hor} is the potential energy when the present horizon scale left the horizon during inflation, ρ_{end} is the energy density at the end of inflation, ρ_{th} is an energy scale at which the universe has thermalized and w_{int} characterizes the effective equation of state between these times. For $k_* = 0.05 \text{ Mpc}^{-1}$ the first term is ~ 3 and for a large class of inflationary models the rest adds up to give the commonly assumed range $50 < N_* < 60$ and most often we will just assume that the pivot scale k_* crossed the horizon $N_* = 60$ e-foldings before the end of inflation.

Their main equations of this section are the slow-roll results (1.50) and (1.60)

$$\epsilon_V = \frac{M_{\text{P}}^2}{2} \left(\frac{V'(\phi)}{V} \right)^2, \quad \eta_V = M_{\text{P}}^2 \frac{V''(\phi)}{V}, \quad N_* \simeq \frac{1}{M_{\text{P}}^2} \int_{\phi_{\text{end}}}^{\phi_*} \frac{V}{V'} d\phi. \quad (1.63)$$

They allow us to probe whether a potential is suitable for slow-roll inflation. The procedure goes as follows: We set $\epsilon_V(\phi_{\text{end}}) = 1$ to find the field value at which inflation ends. Then we test whether the potential allows for $N_* = 60$ e-folds of slow roll inflation with $\epsilon_H < 1$ and $\eta_H < 1$ prior to the end of inflation. In the next chapter we will see that the values of $V(\phi_*)$, $\epsilon(\phi_*)$ and $\eta(\phi_*)$ determines the shape of the primordial power spectra of perturbations (1.61) (to first order in slow-roll approximation) and may be used to confront models with data.

PRIMORDIAL PERTURBATIONS FROM INFLATION

In the previous chapter we considered the homogenous and isotropic universe and introduced the inflationary paradigm as a solution to the flatness and horizon problems of the standard Hot Big Bang model. However, a crucial element of the inflationary scenario is, that besides explaining the initial conditions, it also serves as a theory for the origin of structure in the universe. As we shall see in this chapter, inflation can both explain the origin of primordial inhomogeneities and predict their spectrum. This provides a link between the inflationary theory of the early universe and observations, as the primordial inhomogeneities are imprinted as temperature anisotropies in the CMB and provide the seeds for large-scale structure formation.

In this chapter we present the famous calculation of the primordial spectrum of inhomogeneities generated by quantum fluctuations during inflation [10, 11]. We will see that the primordial spectra of density perturbations may be obtained in excellent agreement with data, by considering vacuum quantum fluctuations of the inflaton field about the homogeneous background solution we described in the previous chapter. During inflation these quantum fluctuations are stretched to super horizon scales, where they freeze in and become classical. When inflation ends, the horizon starts growing, and eventually the perturbations will reenter the horizon as classical density perturbations and provide the seeds for structure formation which proceeds by means of gravitational instability, see Fig. 2.1. This provides a remarkable link between small scale quantum physics of the early universe and structure on the largest cosmological scales in the

present day universe.

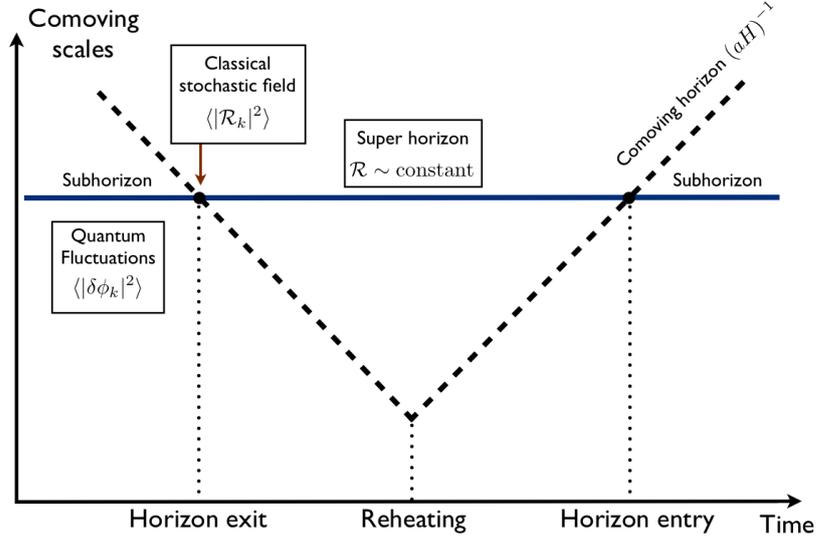


Figure 2.1: Scalar vacuum quantum fluctuations of the system inflaton+gravity freeze in at horizon crossing during inflation. The fluctuations are quantified by perturbations of the comoving curvature perturbation \mathcal{R} . The perturbations become classical at super horizon scales $k < aH$, and may be treated as a classical stochastic field upon horizon entry. Also, the perturbations stay nearly constant while they are super horizon thereby easing the comparison of the conditions at horizon crossing with late time observables. The figure is inspired by [25]

The primordial spectra of perturbations may be calculated via linear perturbation theory in general relativity. Observations of the CMB reveal that the universe was nearly homogenous at the time of decoupling. Small inhomogeneities only existed at the 10^{-5} level which suggests that the conditions of the early universe may be accurately described by means of small perturbations on top of the homogeneous background solution. It is therefore natural to split quantities such as the metric, matter fields and the stress tensor into a homogenous background $\bar{X}(t)$, that depends only on cosmic time, and a small spatially dependent perturbation $\delta X(x, t)$

$$X(x, t) = \bar{X}(t) + \delta X(x, t) . \quad (2.1)$$

The smallness of the inhomogeneities ensures that a linear expansion of the Einstein equations approximates the full non-linear solution to very high accuracy

$$\delta G_{\mu\nu} = 8\pi G\delta T_{\mu\nu} . \quad (2.2)$$

2.1 Background equations

Before we consider perturbations, we briefly list the background equations which will be used over and over in this chapter. We will use both the cosmic time t and conformal time τ formalisms. The Friedmann equations read

$$H^2 = \frac{\bar{\rho}}{3}, \quad \dot{H} + H^2 = -\frac{1}{6}(\bar{\rho} + 3\bar{p}), \quad (2.3)$$

Where an overbar indicate that we are considering the background value and have set $M_p = 1$, which will be the case most often in the following. Using conformal time the equations read

$$\mathcal{H}^2 = \frac{\bar{\rho}}{3}a^2, \quad \mathcal{H}' = -\frac{1}{6}(\bar{\rho} + 3\bar{p})a^2, \quad \mathcal{H} \equiv \frac{a'}{a}, \quad (2.4)$$

Where a prime denotes a derivative with respect to conformal time τ . The continuity equation is

$$\dot{\rho} = -3H\bar{\rho}(1+w), \quad \bar{\rho}' = -3\mathcal{H}\bar{\rho}(1+w), \quad w \equiv \frac{\bar{p}}{\bar{\rho}}. \quad (2.5)$$

Defining the speed of sound $c_s^2 \equiv \dot{p}/\dot{\rho} = \bar{p}'/\bar{\rho}'$ this may be turned into

$$\mathcal{H}' = -\frac{1}{2}(1+3w)\mathcal{H}^2, \quad \frac{w'}{1+w} = 3\mathcal{H}(w - c_s^2). \quad (2.6)$$

Specializing to a single matter scalar field yields

$$\begin{aligned} \ddot{\phi} + 3H\dot{\phi} + V'(\phi) &= 0 \\ H^2 &= \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \\ \dot{H} + H^2 &= H^2(1 - \epsilon_H), \quad \epsilon_H \equiv \frac{3}{2}(\omega + 1) = \frac{\dot{\phi}^2}{2H^2} = -\frac{\dot{H}}{H^2}. \end{aligned} \quad (2.7)$$

During slow-roll we have $\ddot{\phi} \simeq 0$, $\dot{\phi}^2 \simeq 0$ and $H^2 \simeq V/3$. This may be parametrized by $\epsilon_H \ll 1$ and $\eta_H \ll 1$.

2.2 Perturbations of the metric

We start our discussion by reviewing some general aspects of perturbation theory at the level of the linearized Einstein equations. In later sections, where we derive the power spectra of primordial perturbations by quantizing the scalar and tensor degrees of freedom. Then we will work directly with the action of the system inflation+gravity.

We are interested in perturbations of the inflaton field and the metric around the homogenous background solution which we considered in the previous chapter

$$\phi(\tau, \mathbf{x}) = \phi(\tau) + \delta\phi(\tau, \mathbf{x}), \quad g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}, \quad (2.8)$$

Where $g_{\mu\nu}^{(0)}$ is the FRW background metric (1.1) with flat spacial slices $k = 0$. Using the conformal time formalism, the line element may be expressed as

$$ds^2 = a^2(\tau) \left[-(1 + 2\Phi) d\tau^2 - 2B_i dx^i d\tau + (\delta_{ij} + \delta g_{ij}) dx^i dx^j \right], \quad (2.9)$$

Defining the metric perturbations Φ , B_i and δg_{ij} , which are functions of both space and time. It is convenient to decompose the metric perturbations into scalar, vector and tensor parts (SVT-decomposition). This classification is based upon the symmetries of the homogenous background, which at a given point of time is invariant under spatial translations and rotations. The SVT-decomposition is powerful since the Einstein equations for scalars, vectors and tensors do not mix at linear order, and therefore can be treated separately [11]. The vector B_i is split into the gradient of a scalar B and a divergenceless vector \hat{B}_i

$$B_i = -\partial_i B + \hat{B}_i, \quad \partial^i \hat{B}_i = 0, \quad (2.10)$$

Where spatial indices are raised and lowered by the unit metric δ_{ij} , since contracting a perturbation with the full spatial metric gives the contraction with the unit metric plus a second order term which we neglect in the first half of this chapter. The perturbation δg_{ij} is also decomposed in the standard manner

$$\delta g_{ij} = \underbrace{-2\Psi\delta_{ij} + 2\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial^2\right)E}_{\text{Scalar}} - \underbrace{\partial_{(i}\hat{E}_{j)}}_{\text{Vector}} + \underbrace{\gamma_{ij}}_{\text{Tensor}}, \quad \partial^i \hat{E}_i = 0, \quad \partial^i \gamma_{ij} = \gamma^i_i = 0. \quad (2.11)$$

Symmetrization is defined by $\partial_{(i}\hat{E}_{j)} = \frac{1}{2}(\partial_i\hat{E}_j + \partial_j\hat{E}_i)$. The ten degrees of freedom of the metric fluctuation have thus been decomposed into 4 scalar + 4 vector + 2 tensor degrees of freedom

- Scalars Φ, B, E, Ψ
- Vectors \hat{B}_i, \hat{E}_i
- Tensors γ_{ij}

Our focus will be on scalar and tensor perturbations, since vector perturbations decay in an expanding universe and are usually not important. Scalar perturbations are

considered to be the most important in cosmology. They are induced by matter inhomogeneities and exhibit gravitational instability which lead to the formation of structure in the universe. In the coming sections we will describe how quantum vacuum fluctuations of the inflaton field during inflation produce a primordial spectrum of scalar fluctuations which provides the seeds for structure formation. Tensor perturbations does not exhibit gravitational instability and does not couple to matter at linear order. However, they are predicted to be generated during inflation. It seems that we have not yet observed primordial tensor fluctuations [14], but if we do at some point in the future, it may constrain inflationary model building as it is related to the energy scale of inflation and can constrain the underlying particle physics model. We will return to these points in later sections.

2.3 Gauge transformations

The theory of cosmological perturbations is complicated by the freedom in the choice of coordinates. Contrary to the homogeneous and isotropic universe where a preferred coordinate system exists (1.1), there are no obvious preferred coordinate systems for analyzing perturbations. If one is not careful this freedom may lead to the appearance of fictitious perturbations which are artifacts of the coordinate choice and do not describe real physics. On the other hand it allows us to pick convenient coordinate systems which can simplify our computations significantly.

We approach the issue by considering small amplitude transformations of the coordinates

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \xi^\mu . \quad (2.12)$$

We will follow the passive approach to gauge transformations, which goes as follows [11]. We pick coordinates $x^\mu(p)$ on our physical perturbed spacetime manifold \mathcal{M} . Then to any function Q on \mathcal{M} we assign a background function $Q^{(0)}(x^\mu)$, which is a fixed function of the coordinates. What will be relevant in the following is that we assign the FRW metric $g_{\mu\nu}^{(0)}$ to the full perturbed metric $g_{\mu\nu}$ on \mathcal{M} . $g_{\mu\nu}^{(0)}$ is not a geometrical object on \mathcal{M} , it is a fixed function ie. it has the same functional form when acting on the old and new coordinates in (2.12). The perturbation δQ of Q in the coordinates system x^μ is then defined as

$$\delta Q(p) = Q(x^\mu(p)) - Q^{(0)}(x^\mu(p)) \quad (2.13)$$

Changing the coordinates we get

$$\delta\tilde{Q}(p) = \tilde{Q}(\tilde{x}^\mu(p)) - Q^{(0)}(\tilde{x}^\mu(p)). \quad (2.14)$$

Thus the way we split Q into background and perturbation parts depends on the coordinate choice. The transformation of the perturbation induced by the coordinate transformation (2.12) is called a gauge transformation

$$\delta Q(p) \rightarrow \delta\tilde{Q}(p). \quad (2.15)$$

The gauge transformation laws of the metric perturbations may easily be found via the tensor transformation law

$$\tilde{g}_{\mu\nu}(\tilde{x}^\rho) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x^\rho). \quad (2.16)$$

We expand to first order in ξ^ρ using (2.8) and (2.12)

$$\tilde{g}_{\mu\nu} = g_{\mu\nu}^{(0)}(\tilde{x}^\rho) + \delta\tilde{g}_{\mu\nu} \simeq g_{\mu\nu}^{(0)}(x^\rho) + \delta g_{\mu\nu} - g_{\mu\beta}^{(0)}\partial_\nu\xi^\beta - g_{\alpha\nu}^{(0)}\partial_\mu\xi^\alpha. \quad (2.17)$$

The left hand side depends on \tilde{x}^μ and the right hand side on x^μ . Note that the difference between $\frac{\partial\xi^\alpha}{\partial\tilde{x}^\mu}$ and $\frac{\partial\xi^\alpha}{\partial x^\mu}$ is second order small thus there is no need to distinguish between them. Next we expand terms on the right hand side to first order using again (2.12), for example $g_{\mu\nu}^{(0)}(x^\rho) \simeq g_{\mu\nu}^{(0)}(\tilde{x}^\rho) - \partial_\sigma g_{\mu\nu}^{(0)}(\tilde{x}^\rho)\xi^\sigma$. We may then read off the gauge transformation law

$$\begin{aligned} \delta g_{\mu\nu} &\rightarrow \delta\tilde{g}_{\mu\nu} = \delta g_{\mu\nu} - \partial_\sigma g_{\mu\nu}^{(0)}\xi^\sigma - g_{\mu\beta}^{(0)}\partial_\nu\xi^\beta - g_{\alpha\nu}^{(0)}\partial_\mu\xi^\alpha \\ &= \delta g_{\mu\nu} + a^2 \left[-\partial_\mu\xi^\alpha\eta_{\alpha\nu} - \partial_\nu\xi^\beta\eta_{\mu\beta} - 2\frac{a'}{a}\xi^0\eta_{\mu\nu} \right], \end{aligned} \quad (2.18)$$

Where we have inserted the flat background metric $g_{\mu\nu}^{(0)} = a^2\eta_{\mu\nu}$. The inflaton field is a scalar under (2.12) and the perturbation is easily found to transform as

$$\delta\phi \rightarrow \delta\tilde{\phi} = \delta\phi - \partial_\sigma\phi^{(0)}\xi^\sigma = \delta\phi - \phi'\xi^0. \quad (2.19)$$

Where $\phi^{(0)}$ is the background value $\phi(\tau)$ from (2.8). From (2.18) we may read off the transformation laws for the components of the SVT-decomposition. We shall only consider the scalar metric perturbations, since vector perturbations will not be relevant for us and since tensor perturbations are gauge invariant.

2.4 Scalar perturbations

The metric for the background and scalar metric perturbations is

$$ds^2 = a^2(\tau) \left[-(1 + 2\Phi) d\tau^2 + 2\partial_i B dx^i d\tau + \left((1 - 2\Psi) \delta_{ij} + 2\partial_i \partial_j E \right) dx^i dx^j \right]. \quad (2.20)$$

The term $-\frac{1}{3}\partial^2 E \delta_{ij}$ has been absorbed in Ψ which is then called the curvature perturbation since the intrinsic scalar curvature of constant conformal time hyper surfaces, which may be derived from (1.6), is given by

$$R^{(3)} = \frac{4}{a^2} \partial^2 \Psi. \quad (2.21)$$

The curvature perturbation will be an important quantity in coming sections. We may read off the gauge transformation laws for scalar metric perturbations by comparing (2.18) and (2.20). Before we do that it is convenient to split the spatial part of ξ^μ into scalar and vector parts

$$\xi^\mu = (\xi^0, \xi^i), \quad \xi^i = -\partial^i \xi + \hat{\xi}^i, \quad \partial_i \hat{\xi}^i = 0. \quad (2.22)$$

If we start with a pure scalar perturbation and perform a coordinate transformation we may in general introduce fictitious vector perturbations. The part $\hat{\xi}^i$ is responsible for the fictitious vector perturbations, whereas ξ^0 and $\partial_i \xi$ change only the scalar part of the perturbation. We shall therefore drop the vector term $\hat{\xi}^i$. Hence the transformation laws are

$$\begin{aligned} \tilde{\Phi} &= \Phi - \xi^{0'} - \frac{a'}{a} \xi^0 \\ \tilde{B} &= B + \xi' + \xi^0 \\ \tilde{\Psi} &= \Psi + \frac{a'}{a} \xi^0 \\ \tilde{E} &= E + \xi. \end{aligned} \quad (2.23)$$

In the argument made above we did not include fictitious tensor perturbations, the reason is that tensor perturbations are gauge invariant. This means that if we start with a pure tensor perturbation and perform a coordinate transformation we *will* in general introduce fictitious scalar and vector perturbations, but the tensor part of the perturbation stays the same.

2.4.1 Perturbed energy momentum tensor

Having described scalar perturbations of the metric tensor in the previous section we now turn to perturbations of the stress energy tensor $T_{\mu\nu} = T_{\mu\nu}^{(0)} + \delta T_{\mu\nu}$, where the background $T_{\mu\nu}^{(0)}$ is of the perfect fluid form (1.9). The perturbation $\delta T_{\mu\nu}$ can be divided into 5 + 5 degrees of freedom of which 5 are of the perfect fluid form and 5 describe the anisotropic stress. The perfect fluid degrees of freedom may be taken to be the density perturbation, pressure perturbation and velocity perturbation

$$\rho(\tau, \mathbf{x}) = \bar{\rho}(\tau) + \delta\rho(\tau, \mathbf{x}), \quad p(\tau, \mathbf{x}) = \bar{p}(\tau) + \delta p(\tau, \mathbf{x}), \quad u^i = \delta u^i = \frac{1}{a} v^i(\tau, \mathbf{x}), \quad (2.24)$$

Where an overbar indicates the background value and v_i is the coordinate velocity which equal the fluid velocity observed by a comoving observer. These degrees of freedom keep the full tensor $T_{\mu\nu}$ in the perfect fluid form

$$T_{\nu}^{\mu} = (\rho + p) u^{\mu} u_{\nu} + p \delta_{\mu\nu}. \quad (2.25)$$

If we write $u^{\mu} = \bar{u}^{\mu} + \delta u^{\mu} = (a^{-1} + \delta u^0, a^{-1} v_i)$ and use the metric (2.9) to obtain u_{μ} and the contraction $u^{\mu} u_{\mu} = -1$ we obtain to first order

$$\begin{aligned} T_{\nu}^{\mu} &= T_{\nu}^{(0)\mu} + \delta T_{\nu}^{\mu} \\ &= \begin{pmatrix} -\bar{\rho} & 0 \\ 0 & \bar{p} \delta_{ij} \end{pmatrix} + \begin{pmatrix} -\delta\rho & (\bar{\rho} + \bar{p})(v_i + B_i) \\ -(\bar{\rho} + \bar{p})v_i & \delta p \delta_{ij} \end{pmatrix}. \end{aligned} \quad (2.26)$$

The 5 remaining degrees of freedom describe perturbations away from the perfect fluid form. We parametrize them by Π_{ij} which is conventionally defined as

$$\delta T_j^i \equiv \bar{p} \left(\frac{\delta p}{\bar{p}} \delta_{ij} + \Pi_{ij} \right). \quad (2.27)$$

Π_{ij} is symmetric and traceless and parametrize the anisotropic stress. In the simplest cosmological models where matter is describe by one or a set of canonically normalized scalar fields the anisotropic stress vanishes $\Pi_{ij} = 0$.

The energy momentum tensor may be decomposed into scalar, vector and tensor parts using the same procedure as in the previous section. The coordinate velocity and the anisotropic stress are decomposed as

$$v_i = -\partial_i v + \hat{v}_i, \quad \partial_i \hat{v}_i = 0, \quad (2.28)$$

$$\Pi_{ij} = \Pi_{ij}^S + \Pi_{ij}^V + \Pi_{ij}^T, \quad \Pi_{ij}^S = \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \right) \Pi. \quad (2.29)$$

As in the previous section we are only interested in the scalar part of the perturbation. Using the tensor transformation law for the stress energy tensor as we did for the metric

tensor (2.16) and the decomposition of coordinate transformation (2.22) we obtain the gauge transformation laws

$$\begin{aligned}\tilde{\delta\rho} &= \delta\rho - \bar{\rho}'\xi^0 \\ \tilde{\delta p} &= \delta p - \bar{p}'\xi^0 \\ \tilde{v} &= v + \xi' \\ \tilde{\Pi} &= \Pi.\end{aligned}\tag{2.30}$$

2.4.2 Linearized Einstein equations

In the previous sections we introduced first order scalar perturbations of the metric tensor and the energy momentum tensor. Their dynamics couple through the perturbed Einstein equation such that a perturbation in the energy momentum density will induce a perturbation in the curvature of space time

$$\delta R_{\mu\nu} - \frac{1}{2}\delta R g_{\mu\nu} = \delta G_{\mu\nu} = 8\pi G\delta T_{\mu\nu},\tag{2.31}$$

Where the zeroth order terms have been dropped by means of the background Einstein equations. As we have seen the perturbations depend on the coordinate choice and therefore one has to be careful when extracting physics from the Einstein equations. One may proceed either by doing gauge invariant calculations where the perturbed Einstein tensor and stress energy tensor are expressed in a gauge invariant manner, or one may proceed by computing gauge invariant quantities in a specific gauge. In the following we will proceed by the last approach by pick the Newtonian gauge.

2.4.3 Newtonian gauge

First recall that we decomposed the coordinate transformation into two scalar modes ξ^0, ξ and two vector modes ξ^i (2.22). In general it is therefore possible to perform a coordinate transformation which eliminate two of the scalar and two of the vector metric perturbations, leaving 2 + 2 + 2 degrees of freedom in the SVT-decomposition.

The Newtonian gauge is defined by

$$E = B = 0.\tag{2.32}$$

Starting with an arbitrary scalar perturbation we see from (2.23), that this may be accomplished by $\xi = -E$ and $\xi^0 = -B + E'$. The metric for scalar degrees of freedom then reads

$$ds^2 = a^2(\tau) \left[-(1 + 2\Phi) d\tau^2 + (1 - 2\Psi) \delta_{ij} dx^i dx^j \right]. \quad (2.33)$$

The full Christoffel symbols $\Gamma = \Gamma^{(0)} + \delta\Gamma$ may be found from (1.7)

$$\begin{aligned} \Gamma_{00}^0 &= \mathcal{H} + \Phi', & \Gamma_{0j}^0 &= \partial_j \Phi, & \Gamma_{ij}^0 &= \mathcal{H} \delta_{ij} - [2\mathcal{H}(\Phi + \Psi) + \Psi'] \delta_{ij}, \\ \Gamma_{00}^k &= \partial_k \Phi, & \Gamma_{0j}^k &= \mathcal{H} \delta_{kj} - \Psi' \delta_{kj}, & \Gamma_{ij}^k &= -(\partial_j \delta_{ki} + \partial_i \Psi \delta_{kj}) + \partial_k \Psi \delta_{ij}. \end{aligned} \quad (2.34)$$

The perturbation to the Einstein tensor then read to first order

$$\begin{aligned} \delta G_0^0 &= a^{-2} \left(-2\partial^2 \Psi + 6\mathcal{H}\Psi' + 6\mathcal{H}^2 \Phi \right) \\ \delta G_i^0 &= -2a^{-2} \partial_i (\Psi' + \mathcal{H}\Phi) \\ \delta G_j^i &= a^{-2} \left[2\Psi'' + \partial^2 (\Phi - \Psi) + \mathcal{H} (2\Phi' + 4\Psi') + (4\mathcal{H}' + 2\mathcal{H}^2) \Phi \right] \delta_{ij} \\ &\quad + a^{-2} \partial_i \partial_j (\Psi - \Phi). \end{aligned} \quad (2.35)$$

From the $i \neq j$ components of the last equation and from (2.26) we see that if no anisotropic stress is present $\Pi = 0$, then $\Psi = \Phi$. Hence the scalar metric fluctuations are described by just one degree of freedom. The perturbed energy momentum tensor reads

$$\delta T_\nu^\mu = \begin{pmatrix} -\delta\rho & -(\bar{\rho} + \bar{p}) \partial_i v \\ (\bar{\rho} + \bar{p}) \partial_i v & \delta p \delta_{ij} \end{pmatrix}. \quad (2.36)$$

And the field equations are then

$$\begin{aligned} \partial^2 \Phi &= \frac{3}{2} \mathcal{H}^2 \left[\frac{\delta\rho}{\bar{\rho}} + 3\mathcal{H} (1 + w) v \right] \\ \Phi' + \mathcal{H}\Phi &= \frac{3}{2} \mathcal{H}^2 (1 + w) v \\ \Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2) \Phi &= \frac{3}{2} \mathcal{H}^2 \frac{\delta p}{\bar{\rho}}, \end{aligned} \quad (2.37)$$

Where we have inserted the equation of state parameter $w = \bar{p}/\bar{\rho}$ since it makes the coming derivation cleaner. One may study a variety of interesting phenomena by means of the linearized Einstein equations in different gauges. In particular growth of inhomogeneities in the various phases of the early universe and also large scale inhomogeneities at later times where perturbations are still small [11]. However, this is out of the scope of this chapter, were we seek to derive the primordial power spectrum of perturbations generated by inflation. In the following we will use the linearized Einstein equations to

prove that the an important gauge invariant quantity \mathcal{R} called the comoving curvature perturbation, is constant on super horizon scales.

2.4.4 Comoving curvature perturbation

The comoving curvature perturbation is defined as

$$\mathcal{R} = -\Psi - \mathcal{H}(v - B). \quad (2.38)$$

This quantity is gauge invariant by construction. The comoving gauge is defined by $v = B = 0$. In this gauge the threading is comoving, ie. given by the world lines of comoving observers and the slicing is comoving, ie. orthogonal to the worldliness of comoving observers. Hence $\mathcal{R} = -\Psi$ such that \mathcal{R} gives the curvature perturbation on constant time slices in the comoving gauge (2.21).

2.4.5 A conservation law

We now consider the comoving curvature perturbation in the Newtonian gauge. Then $B = 0$ such that

$$\mathcal{R} = -\Phi - \mathcal{H}v = -\Phi - \frac{2(\mathcal{H}^{-1}\Phi' + \Phi)}{3(1+w)}, \quad (2.39)$$

Where we have inserted a linearized Einstein equation. We do one time differentiation and insert the background equations (2.6) to obtain

$$-\frac{3}{2}(1+w)\mathcal{H}^{-1}\mathcal{R}' = \mathcal{H}^{-2}\Phi'' + 3\mathcal{H}^{-1}\Phi' + 3c_s^2(\mathcal{H}^{-1}\Phi' + \Phi) - 3w\Phi. \quad (2.40)$$

The total entropy is another important gauge invariant quantity

$$\mathcal{S} = \mathcal{H}\left(\frac{\delta p}{\bar{p}'} - \frac{\delta \rho}{\bar{\rho}'}\right) \Rightarrow \delta p = c_s^2[\delta \rho - 3(\bar{\rho} + \bar{p})\mathcal{S}]. \quad (2.41)$$

If $\mathcal{S} = 0$ the perturbation is adiabatic. This will be the case for example if the primordial curvature perturbations are generated by single field inflation. However, if inflation is driven by more than one field, isocurvature perturbations are expected to be generated. For a fluid with different components with energy density ρ_i these may be quantified by the gauge invariant quantities \mathcal{S}_{ij}

$$\mathcal{S}_{ij} = \mathcal{H}\left(\frac{\delta \rho_i}{\bar{\rho}'_i} - \frac{\delta \rho_j}{\bar{\rho}'_j}\right). \quad (2.42)$$

Inserting (2.41) into (2.40) and using the perturbed Einstein equations we obtain

$$-\frac{3}{2}(1+w)\mathcal{H}^{-1}\mathcal{R}' = \mathcal{H}^{-2}c_s^2\partial^2\Phi + \frac{9}{2}(1+w)\mathcal{S}. \quad (2.43)$$

Hence for adiabatic curvature perturbations

$$-\frac{3}{2}(1+w)\mathcal{H}^{-1}\mathcal{R}'_{\mathbf{k}} = \left(\frac{k}{\mathcal{H}}\right)^2 c_s^2\Phi_{\mathbf{k}}, \quad (2.44)$$

Where we have switched to fourier space. For super horizon perturbations, i.e for fourier modes with $k \ll \mathcal{H}$ the right hand side is negligible and we may conclude that for adiabatic perturbations, the comoving curvature perturbation stays constant outside the horizon. This is an important result which allow us to relate initial perturbations generated during inflation to late time observables. The big picture is as follows: For sub horizon modes the curvature perturbation oscillates during inflation. This will be obvious in the following sections where we treat the system of gravity plus matter field perturbations at the level of the action during inflation. We shall see that the sub horizon modes at linear order are described by independent harmonic oscillators with time dependent frequencies. The modes will be quantized and when each mode crosses the horizon during inflation its quantum vacuum fluctuation will 'freeze out' and stay constant as described above. We assume that each mode is in its quantum vacuum state at horizon crossing since any initial excitation would have been redshifted away by the prior expansion. When inflation ends the horizon starts to shrink. Hence the perturbation modes which froze during inflation eventually reenters the horizon. Outside the horizon, the quantum nature of the field disappears as the canonical commutators quickly goes to zero. Hence the perturbations reenter the horizon as classical perturbations and the quantum expectation value at horizon crossing may be identified with the ensemble average of a classical stochastic field.

2.4.6 Restriction to scalar field theory

Having described some general aspects of linear perturbation theory we now restrict our attention to the case where matter is described by a single scalar field which will be the inflaton. We consider the action

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \right]. \quad (2.45)$$

The energy momentum tensor reads

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right). \quad (2.46)$$

We insert the ansatz $\phi \equiv \phi + \delta\phi$ and the metric for background and scalar perturbations

$$g_{\mu\nu} = \begin{pmatrix} -1 - 2\Phi & \partial_i B \\ \partial_i B & (1 - 2\Psi)\delta_{ij} + \partial_i \partial_j E \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -1 + 2\Phi & \partial_i B \\ \partial_i B & (1 + 2\Psi)\delta_{ij} - \partial_i \partial_j E \end{pmatrix}. \quad (2.47)$$

The perturbation to the stress energy tensor then reads at first order

$$\begin{aligned} \delta T_0^0 &= a^{-2} (\phi'^2 \Phi - \phi' \delta\phi') - \frac{dV}{d\phi} \delta\phi \\ \delta T_i^0 &= -a^{-2} \phi' \partial_i \delta\phi \\ \delta T_0^i &= a^{-2} (\phi'^2 \partial_i B + \phi' \partial_i \delta\phi) \\ \delta T_j^i &= \left(-a^{-2} \phi'^2 \Phi + a^{-2} \phi' \delta\phi' - \frac{dV}{d\phi} \delta\phi \right) \delta_{ij}. \end{aligned} \quad (2.48)$$

There is no anisotropic stress since the off diagonal components of δT_j^i vanishes. Hence the Einstein equations force $\Phi = \Psi$. We have only seen this statement to hold in the Newtonian gauge (2.35), however it is correct in general which can be seen by expressing the Einstein equations in a gauge invariant manner.

By comparing (2.48) to (2.36) we see that $v+B = -\frac{\delta\phi}{\phi'}$ such that the comoving curvature perturbation reads

$$\mathcal{R} = -\Psi + \mathcal{H} \frac{\delta\phi}{\phi'}. \quad (2.49)$$

Let us now consider a comoving slicing which is the slicing orthogonal to the worldlines of comoving observers. There is no flux of energy measured by these observers, that is $T_{0i} = 0$. Thus

$$\delta\phi = 0 \quad \Rightarrow \quad \mathcal{R} = -\Psi. \quad (2.50)$$

Hence on a slicing where the inflation field is unperturbed, the comoving curvature perturbation equals the curvature perturbation. We shall use this gauge in the following.

2.5 Action for scalar perturbations at linear order

Having described some general aspects of linear perturbations theory, we now turn to quantization of the scalar and tensor degrees of freedom during inflation. We require an action for linearized cosmological perturbations, hence we expand the action of the coupled system inflaton+gravity to second order in the fluctuating degrees of freedom. The starting point is the action for gravity and a canonical scalar field ϕ , minimally coupled to gravity.

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[R - (\nabla\phi)^2 - 2V(\phi) \right] \quad (2.51)$$

Where $M_{pl}^{-2} = 8\pi G = 1$. It is conventional to treat fluctuations in the ADM formalism, in which space-time is sliced into three-dimensional hypersurfaces with metric

$$ds^2 = -N^2 dt^2 + h_{ij} (dx^i + N^i dt) (dx^j + N^j dt), \quad (2.52)$$

Where $N(\mathbf{x})$ is called the lapse function and $N_i(\mathbf{x})$ the shift function. h_{ij} is the three-dimensional metric on the constant time hyper surface. In matrix form the metric reads

$$g_{\mu\nu} = \begin{pmatrix} [1.2]g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} [1.5] - N^2 + N_i N^i & N_j \\ N_i & h_{ij} \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} [1.5] - \frac{1}{N^2} & \frac{N^j}{N^2} \\ \frac{N^i}{N^2} & h^{ij} - \frac{N^i N^j}{N^2} \end{pmatrix}. \quad (2.53)$$

h^{ij} is the inverse of the 3-metric. Note that $g_{ij} = h_{ij}$, but $g^{ij} \neq h^{ij}$ in general. N , N_i and h_{ij} contain the same information as the metric perturbations in (2.20) and we could have stuck with the old parametrization. However, the ADM formalism provides a much cleaner notation, since N and N_i will appear as Lagrange multipliers in the action. Hence their equations of motion are algebraic constraint equations which can be solved perturbatively and reinserted in the action. This gives a procedure for obtaining the quadratic action, which describes perturbations at linear order.

In the ADM formalism the action (2.51) becomes [11, 20, 21]

$$S = \frac{1}{2} \int d^4x \sqrt{h} \left[NR^{(3)} + N^{-1} (E_{ij} E^{ij} - E^2) + N^{-1} (\dot{\phi} - N^i \partial_i \phi)^2 - N h^{ij} \partial_i \phi \partial_j \phi - 2NV \right]. \quad (2.54)$$

$R^{(3)}$ is the intrinsic scalar curvature of the spacial slices, and E_{ij} is related to the extrinsic curvature of the spacial slices,

$$E_{ij} \equiv \frac{1}{2} (\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i), \quad E = h^{ij} E_{ij}, \quad (2.55)$$

Where ∇_i is the Levi-Civita connection associated with the metric h_{ij} . As promised, N and N^i appear as Lagrange multipliers in the action (2.54). Before we solve the constraint equations it is convenient to fix the gauge. We choose the comoving gauge in which the inflation field is unperturbed

$$\delta\phi = 0, \quad h_{ij} = a^2 \left[(1 + 2\mathcal{R}) \delta_{ij} + \gamma_{ij} \right] + \mathcal{O}(2), \quad \partial_i \gamma_{ij} = \gamma_{ii} = 0. \quad (2.56)$$

In this gauge \mathcal{R} parametrize all scalar degrees of freedom and γ_{ij} the tensor degrees of freedom. It is the canonical action of these quantities we seek.

Variation of the action with respect to N and N_i yields two constraint equations called the Hamiltonian and momentum constraint respectively

$$R^{(3)} - N^{-2} (E_{ij} E^{ij} - E^2) - N^{-2} \dot{\phi}^2 - 2V = 0 \quad (2.57)$$

$$\nabla_i \left[N^{-1} (E_j^i - E \delta_{ij}) \right] = 0. \quad (2.58)$$

We will solve the constraint equations perturbatively by expanding the Lagrange multipliers in powers of \mathcal{R}

$$N \equiv N^0 + N^1 + N^2 + \dots \quad (2.59)$$

$$\psi \equiv \psi^0 + \psi^1 + \psi^2 + \dots \quad (2.60)$$

$$\beta_i \equiv \beta_i^0 + \beta_i^1 + \beta_i^2 + \dots, \quad (2.61)$$

Where for example $\psi^1 = \mathcal{O}(\mathcal{R})$. As we did in previous sections, we have decomposed the shift vector into scalar and vector parts

$$N_i = \partial_i \psi + \beta_i, \quad \text{where} \quad \partial_i \beta_i = 0. \quad (2.62)$$

Then we can set $\beta_i = 0$ when we consider scalar degrees of freedom separately.

2.5.1 Solution of constraint equations

We now set out to solve the constraint equations perturbatively. We will first consider the scalar mode \mathcal{R} and therefore drop the tensor modes for the time being $\gamma_{ij} = 0$. The starting point for the calculation is simply to plug the 3-metric along with the gauge choice into the action (2.54). It is convenient to write the 3-metric as

$$h_{ij} = a^2 e^{2\mathcal{R}} \delta_{ij}, \quad h^{ij} = a^{-2} e^{-2\mathcal{R}} \delta_{ij}, \quad h \equiv \det(h) = a^6 e^{6\mathcal{R}}, \quad (2.63)$$

$$\dot{h}_{ij} = 2a^2 e^{2\mathcal{R}} (H + \dot{\mathcal{R}}) \delta_{ij}. \quad (2.64)$$

The Christoffel symbols associated with the metric are (1.7)

$$\Gamma_{ij}^k = \partial_j \mathcal{R} \delta_{ik} + \partial_i \mathcal{R} \delta_{kj} - \partial_k \mathcal{R} \delta_{ij}. \quad (2.65)$$

The Ricci tensor of the spacial slices is then (1.6)

$$R_{ij} = -\partial_i \partial_j \mathcal{R} - \partial^2 \mathcal{R} \delta_{ij} + \partial_i \mathcal{R} \partial_j \mathcal{R} - (\partial \mathcal{R})^2 \delta_{ij}. \quad (2.66)$$

Where $\partial^2 \equiv \delta_{ij} \partial_i \partial_j$ and $(\partial \mathcal{R})^2 \equiv \delta_{ij} \partial_i \mathcal{R} \partial_j \mathcal{R}$. That is, the indexes are summed over but not contracted. The intrinsic scalar curvature follows by contraction

$$R^{(3)} = h^{ij} R_{ij} = -2a^{-2} e^{-2\mathcal{R}} (2\partial^2 \mathcal{R} + (\partial \mathcal{R})^2). \quad (2.67)$$

Inserting (2.64) and (2.65) into (2.55) the extrinsic curvature become

$$E_{ij} = a^2 e^{2\mathcal{R}} (H + \dot{\mathcal{R}}) \delta_{ij} - \partial_{(i} N_{j)} + 2N_{(i} \partial_{j)} \mathcal{R} - N_k \partial_k \mathcal{R} \delta_{ij}, \quad (2.68)$$

$$E = h^{ij} E_{ij} = 3(H + \dot{\mathcal{R}}) - a^{-2} e^{-2\mathcal{R}} (\partial_i N_i + N_i \partial_i \mathcal{R}), \quad (2.69)$$

Where $\partial_{(i}N_{j)} = \frac{1}{2}(\partial_i N_j + \partial_j N_i)$. The term which is relevant for the action is

$$E_{ij}E^{ij} - E^2 = -6(H + \dot{\mathcal{R}})^2 + 4a^{-2}e^{-2\mathcal{R}}(H + \dot{\mathcal{R}})(\partial_i N_i + N_i \partial_i \mathcal{R}) - a^{-4}e^{-4\mathcal{R}} \left[(\partial_i N_i)^2 + 2(N_i \partial_i \mathcal{R})^2 - (\partial_{(i} N_{j)} - 2N_{(i} \partial_{j)} \mathcal{R})^2 \right]. \quad (2.70)$$

Let us first consider solutions to the constraint equations at zeroth order in \mathcal{R} . This is just unperturbed FRW space-time and we expect the background equations to emerge. Indeed, the Hamiltonian constraint equation (2.57) becomes

$$-6H^2 - \dot{\phi}^2 - 2V(N^0)^2 + 4a^{-2}H\partial_i N_i^0 = 0. \quad (2.71)$$

Setting $N^0 = 1$ and $N_i^0 = 0$, which reduces (2.52) to the FRW metric, we obtain the Friedman equation

$$H^2 = \frac{1}{3} \left(\frac{1}{2} \dot{\phi}^2 + V \right). \quad (2.72)$$

The momentum constraint equation (2.58) vanishes identically.

To solve the constraint equations to first order we set $N = 1 + N^1$, $N_i = \partial_i \psi^1$. We set $\beta_i^1 = 0$ since we are dealing with scalar d.o.f. The curvature terms are

$$R^{(3)} = -4a^{-2}\partial^2 \mathcal{R}, \quad (2.73)$$

$$E_{ij} = a^2 e^{2\mathcal{R}} (H + \dot{\mathcal{R}}) \delta_{ij} - \partial_i \partial_j \psi^1, \quad (2.74)$$

$$E_{ij}E^{ij} - E^2 = -6H^2 - 12H\dot{\mathcal{R}} + 4a^{-2}H\partial^2 \psi^1, \quad (2.75)$$

Where we have left an exponential unexpanded which is often convenient. The momentum constraint equation (2.58) becomes

$$\nabla_i \left[2(1 - N^1)(H + \dot{\mathcal{R}}) \delta_{ij} + a^{-2}(\partial_i \partial_j \psi^1 - \partial^2 \psi^1 \delta_{ij}) \right] = 0. \quad (2.76)$$

Since $\Gamma_{ij}^k \sim \mathcal{O}(\mathcal{R})$ the covariant derivative reduce to $\nabla_i = \partial_i$ for the terms involving ψ^1 at first order and are then easily seen to vanish. We are left with

$$\nabla_i [HN^1 - \dot{\mathcal{R}}] = 0. \quad (2.77)$$

The Hamiltonian constraint equation (2.57) becomes

$$a^{-2}(\partial^2 \mathcal{R} + H\partial^2 \psi^1) - \frac{1}{2}N^1\dot{\phi}^2 + 3H(HN^1 - \dot{\mathcal{R}}) = 0, \quad (2.78)$$

Where the Friedman equation has been inserted to eliminate zero order terms and to express the potential V in terms of $\dot{\phi}^2$ and H . To first order the constraint equations are then solved by

$$N^1 = \frac{\dot{\mathcal{R}}}{H}, \quad \psi^1 = -\frac{\mathcal{R}}{H} + \varphi, \quad \partial^2 \varphi = \frac{a^2 \dot{\phi}^2}{2H^2} \dot{\mathcal{R}}. \quad (2.79)$$

2.5.2 The Quadratic action

The next step is to plug the solutions (2.79) back into the action (2.54) and expand the action to second order in perturbations. We need a second order expansion since the *linearized* equations of motion for the coupled system gravity + scalar field are obtained by a *quadratic* action. For this purpose it is not necessary to solve the constraint equations to second order. The reason being that second order terms of the Lagrange multipliers will be multiplied by zero order constraint equations, which vanish by the background equations. The action becomes

$$\begin{aligned}
S = \frac{1}{2} \int d^4x \left[& ae^{\mathcal{R}} \left(1 + \frac{\dot{\mathcal{R}}}{H} \right) \left(-4\partial^2 \mathcal{R} - 2(\partial \mathcal{R})^2 - 2a^2 e^{2\mathcal{R}} V \right) \right. \\
& + a^3 e^{3\mathcal{R}} \left(1 + \frac{\dot{\mathcal{R}}}{H} \right)^{-1} \left(-6(H + \dot{\mathcal{R}})^2 + \dot{\phi}^2 \right) \\
& \left. + 4a^{-2} e^{-2\mathcal{R}} (H + \dot{\mathcal{R}}) \left(\partial^2 \psi^1 + \partial_i \psi^1 \partial_i \mathcal{R} \right) - a^{-4} \left((\partial^2 \psi^1)^2 - (\partial_i \partial_j \psi^1)^2 \right) \right]. \quad (2.80)
\end{aligned}$$

Some, but not all third and higher order terms have been eliminated. At this point it takes some massaging to turn the action into the desired form. The guiding principle is to force out the target form below and show that the remaining parts are total derivatives. This is done by using the background equations and lots of integration by parts. For example the last term (which is the last line of (2.70)) is a total derivative $a^{-4} \partial_i (\partial_i \psi^1 \partial^2 \psi^1 - \partial_j \psi^1 \partial_i \partial_j \psi^1)$ which is omitted. The result is

$$S = \frac{1}{2} \int dt d^3x \frac{\dot{\phi}^2}{H^2} \left[a^3 \dot{\mathcal{R}}^2 - a (\partial \mathcal{R})^2 \right]. \quad (2.81)$$

Next we transition to conformal time and the Mukhanov variable v in terms of which the action is canonical

$$v \equiv z\mathcal{R}, \quad z \equiv \frac{a\dot{\phi}}{H} = \frac{a\phi'}{\mathcal{H}}, \quad dt = a d\tau. \quad (2.82)$$

After some additional integration by parts we arrive at the Mukhanov-Sasaki action, which describes a (canonically normalized) scalar field with time dependent mass $\frac{z''}{z}$.

$$S = \frac{1}{2} \int d\tau d^3x \left[(v')^2 - (\partial v)^2 + \frac{z''}{z} v^2 \right]. \quad (2.83)$$

The equation of motion is

$$v'' - \partial^2 v - \frac{z''}{z} v = 0. \quad (2.84)$$

In momentum space it reads

$$v_k'' + \left(k^2 - \frac{z''}{z} \right) v_k = 0, \quad v(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} v_{\mathbf{k}}(\tau) e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (2.85)$$

The mode equation only depends on the magnitude of \mathbf{k} and we may therefore drop the vector subscript on the mode functions $v_{\mathbf{k}}$.

2.5.3 Quantization

Given the action (2.83) quantization of cosmological scalar perturbations may be performed by the standard canonical quantization procedure in the Heisenberg picture. The field v and its conjugate momentum are promoted to operators \hat{v} and $\hat{\pi}$ and equal time commutation relations are imposed

$$[\hat{v}(\tau, \mathbf{x}), \hat{\pi}(\tau, \mathbf{x}')] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}') \quad (2.86)$$

$$[\hat{v}(\tau, \mathbf{x}), \hat{v}(\tau, \mathbf{x}')] = [\hat{\pi}(\tau, \mathbf{x}), \hat{\pi}(\tau, \mathbf{x}')] = 0. \quad (2.87)$$

Then each fourier mode (2.85), which represents an independent harmonic oscillator with time dependent frequency is expressed via creation and annihilation operators

$$\hat{v}_{\mathbf{k}} = v_{\mathbf{k}}(\tau) \hat{a}_{\mathbf{k}} + v_{\mathbf{k}}^*(\tau) \hat{a}_{-\mathbf{k}}^\dagger, \quad \hat{v}(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} [v_{\mathbf{k}} \hat{a}_{\mathbf{k}} + v_{\mathbf{k}}^* \hat{a}_{-\mathbf{k}}^\dagger] e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (2.88)$$

The ladder operators are independent of time while the mode functions $v_{\mathbf{k}}(\tau)$ which satisfy the equation of motion carry the time dependency. They are the classical solutions of the problem. Let us now consider vacuum fluctuations of the field. One needs to proceed with care since there is in general no unique choice of vacuum state for quantum fields in a curved spacetime. In the case at hand this general statement has settled in the fact that there is no unique choice of vacuum state for a collection of harmonic oscillators with time dependent frequencies. We will set this problem aside for the moment and just introduce a vacuum state $|0\rangle$ which we assume behaves in the standard harmonic oscillator fashion when acted upon by the ladder operators. We may then consider the variance of the field

$$\langle 0 | |v(\tau, \mathbf{x})|^2 | 0 \rangle = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} v_{\mathbf{k}} v_{\mathbf{k}'}^* \langle 0 | \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}'}^\dagger | 0 \rangle \quad (2.89)$$

$$= \int d \ln k \frac{k^3}{2\pi^2} |v_{\mathbf{k}}(\tau)|^2, \quad (2.90)$$

Where we used the standard commutation relation (2.93) and $\hat{a}_{\mathbf{k}}|0\rangle = 0$. This defines the power spectrum

$$\Delta_v^2(k, \tau) \equiv \frac{k^3}{2\pi^2} |v_{\mathbf{k}}(\tau)|^2. \quad (2.91)$$

A related quantity which is often used is $P_v(k, \tau)$. This is defined as the fourier transform of the real space two point correlation function $\langle 0 | \hat{\varphi}(\tau, 0) \hat{\varphi}(\tau, \mathbf{x}) | 0 \rangle$. Hence

$$\langle 0 | \hat{\varphi}_{\mathbf{k}} \hat{\varphi}_{\mathbf{k}'} | 0 \rangle = (2\pi)^3 P_v(k, \tau) \delta^{(3)}(\mathbf{k} + \mathbf{k}'). \quad (2.92)$$

The definitions are related by $\Delta_v^2(k, \tau) = \frac{k^3}{2\pi^2} P_v(k, \tau)$.

The power spectra are important quantities which are used to confront inflationary model building with real data. It is evident that they are determined by the mode functions $v_{\mathbf{k}}$. These are solutions of a second order differential equation (2.85) and are therefore fixed by imposing two boundary conditions. One boundary condition may be obtained by demanding that the operators a and a^\dagger obey the standard commutation relations for creation and annihilation operators, which we used above. This may be done without loss of generality. Substituting (2.88) into (2.86) one finds that this leads to a normalization condition in terms of the Wronskian of the mode functions

$$i(v_{\mathbf{k}}^* v_{\mathbf{k}'}' - v_{\mathbf{k}'}^* v_{\mathbf{k}}) = 1 \quad \Rightarrow \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}'). \quad (2.93)$$

The second boundary conditions may be set by choosing a vacuum state for the fluctuations $\hat{a}_{\mathbf{k}}|0\rangle = 0$. The standard choice is the Minkowski vacuum of a comoving observer in the far past, which may be justified by the following reasoning: In the far past $\tau \rightarrow -\infty$ all comoving scales of observational interest were far inside the horizon $|k\tau| \gg 1, k \gg aH$. Thus the mode equation (2.85) reduce to that of a free field in Minkowski space

$$v_{\mathbf{k}}'' + k^2 v_{\mathbf{k}} = 0. \quad (2.94)$$

There are two independent solutions $v_{\mathbf{k}} \propto e^{\pm ik\tau}$. Demanding that $|0\rangle$ is the ground state of the Hamiltonian \hat{H} a unique positive frequency solution exists

$$\lim_{\tau \rightarrow -\infty} v_{\mathbf{k}} = \frac{e^{-ik\tau}}{\sqrt{2k}}. \quad (2.95)$$

The boundary conditions (2.93) and (2.95) then fix the mode functions $v_{\mathbf{k}}$ and hence the power spectrum $\Delta_v^2(k, \tau)$ on all scales.

2.5.4 Power spectrum of scalar perturbations

So far we have only set the boundary conditions for the solutions. To obtain the power spectrum we still need to solve the mode equation (2.85). This is in general rather complicated since z depends on the background dynamics. However approximate analytic solutions may be found in the de Sitter limit and in the slow-roll approximation.

The solution in the de Sitter limit gives the correct result for the power spectrum during slow-roll inflation at horizon crossing. Since the derivation is very fast we shall consider this limit in the following. A slightly more formal approach is to take the slowly varying background during inflation into account, which to first order in slow-roll parameters evolves as $a(\tau) = -\frac{1}{H\tau(1-\epsilon)}$. Then to first order one can show that $\frac{z''}{z} = \mathcal{H}^2(2 + 5\epsilon - 3\eta)$. A solution may then be found in terms of Hankel functions.

In the de Sitter limit $\epsilon = \eta = 0$ and the last formula reduce to $\frac{z''}{z} = \frac{2}{\tau^2}$. The equation of motion (2.85) reads

$$v_k'' + \left(k^2 - \frac{2}{\tau^2}\right)v_k = 0. \quad (2.96)$$

The general solution is

$$v_k = \alpha \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) + \beta \frac{e^{ik\tau}}{\sqrt{2k}} \left(1 + \frac{i}{k\tau}\right). \quad (2.97)$$

The boundary conditions (2.93) and (2.95) fix $\alpha = 1, \beta = 0$. This leads to the unique Bunch-Davies mode functions

$$v_k = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right). \quad (2.98)$$

The power spectrum becomes

$$\Delta_v^2(k, \tau) \equiv \frac{k^3}{2\pi^2} |v_k(\tau)|^2 = \frac{a^2 H^2}{4\pi^2} (1 + k^2 \tau^2), \quad (2.99)$$

Where we inserted the de Sitter evolution $a(\tau) = -\frac{1}{H\tau}$. On super horizon scales $\frac{k}{aH} = |k\tau| \ll 1$ this approaches a constant $\Delta_v^2(k, \tau) = \frac{a^2 H^2}{4\pi^2}$. Recalling the definition of v in terms of the comoving curvature perturbation \mathcal{R} from (2.82), $\mathcal{R}^2 = \frac{H^2}{a^2 \dot{\phi}^2} v^2$ we may write down the dimensional power spectrum of \mathcal{R} at the time of horizon crossing $k = a_* H_*$

$$\Delta_{\mathcal{R}}^2(k) = \frac{H_*^2}{4\pi^2} \frac{H_*^2}{\dot{\phi}_*^2}. \quad (2.100)$$

We saw earlier that \mathcal{R} approaches a constant on super horizon scales. The spectrum at horizon crossing therefore approximately determines the future spectrum until a given mode re-enters the horizon. During slow-roll inflation the Hubble parameter evolves as

$$H^2 \simeq \frac{1}{3M_p^2} V(\phi), \quad \epsilon_H = \frac{1}{2M_p^2} \frac{\dot{\phi}^2}{H^2}, \quad (2.101)$$

Where we have reinserted the Planck mass M_p . Inserting this we obtain the following quasi de Sitter result which relates the power spectrum to the shape of the potential

$$\Delta_{\mathcal{R}}^2(k) = \frac{1}{24\pi^2 M_p^4} \frac{V_*}{\epsilon_{V_*}}. \quad (2.102)$$

(2.100) and (2.102) are the famous primordial power spectrum of scalar perturbations generated by inflation. During inflation $V \simeq \text{const}$ and $\epsilon_V \simeq \text{const} \ll 1$ such that the power spectrum is nearly scale invariant, i.e. independent of $k = aH$. The scale (in)dependence may be characterized by the spectral index n_s of scalar perturbations

$$n_s - 1 \equiv \frac{d \ln \Delta_{\mathcal{R}}^2}{d \ln k} = 2\eta_{V^*} - 6\epsilon_{V^*}. \quad (2.103)$$

This result may be derived by using the slow roll conditions to show that $\frac{d}{d \ln k} = -M_P^2 \frac{V'}{V} \frac{d}{d\phi}$ and $\frac{d\epsilon}{d \ln k} = 4\epsilon_V^2 - 2\epsilon_V \eta_V$. Exact scale invariance corresponds to $n_s = 1$ and is obtained in the de Sitter limit $\epsilon_H = \eta_H = 0$. The power spectrum is often expanded about the pivot scale which we take as $k_* = 0.05 \text{Mpc}^{-1}$ to make contact with the Planck results [13]

$$\Delta_{\mathcal{R}}^2(k) = A_s \left(\frac{k}{k_*} \right)^{n_s - 1 + \frac{1}{2} \frac{dn_s}{d \ln k} \ln(k/k_*) + \dots}. \quad (2.104)$$

The experimental value of the amplitude of scalar perturbations is about $A_s = 2.2 \cdot 10^{-9}$ such that

$$A_s = \frac{1}{24\pi^2 M_P^4} \frac{V_*}{\epsilon_{V^*}} \Leftrightarrow \frac{V_*}{\epsilon_{V^*}} \simeq (0.0269 M_P)^4. \quad (2.105)$$

As we stated earlier we will assume that the pivot scale k_* left the horizon $N_* = 60$ e-folds before the end of inflation.

Another important result which stems from the calculations in this chapter, is that the primordial perturbations are predicted to be Gaussian. Here we are referring to the statistical properties of the classical stochastic field, which describes the curvature perturbations at super horizon scales and at horizon re-entry. In this context 'Gaussian' means that the coefficients of the fourier modes of the field, are drawn from a gaussian distribution. The variance and power spectrum of the distribution are determined by the underlying inflationary theory, and are derived using the quantum theory, as presented in this section. Gaussianity arise since each scalar mode is described by a quantum harmonic oscillator and by construction started out in its ground state. The linear 'free field' expansion of the action then forces it to stay in its ground state.

To probe non-gaussianity, one needs to expand the action to third order in perturbations, such that interactions are taken into account. This enables one to determine the bispectrum, which is the fourier equivalent of the three-point function

$$\langle \mathcal{R}_{k_1} \mathcal{R}_{k_2} \mathcal{R}_{k_3} \rangle = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B_{\mathcal{R}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3). \quad (2.106)$$

If the statistical properties are truly described by a gaussian distribution, the bispectrum and all higher order correlations will be determined by the power spectrum. We will not

describe non-Gaussianity in this thesis, but just note that standard single field slow-roll inflation, which demands that the interactions of the inflaton are weak, predicts deviations from Gaussianity to be vanishingly small [20]. This statement may be quantified by the Creminelli and Zaldarriaga result [22]

$$\lim_{k_1 \rightarrow 0} \langle \mathcal{R}_{k_1} \mathcal{R}_{k_2} \mathcal{R}_{k_3} \rangle = -(2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) (n_s - 1) P_{\mathcal{R}}(k_1) P_{\mathcal{R}}(k_3) \quad (2.107)$$

Which states that the bispectrum in the squeezed limit $k_1 \rightarrow 0$, is suppressed by the slow-roll parameters. A detection of non-Gaussianity in the squeezed limit can therefore rule out single field inflation. However, the single field scenario is favored by current experiments [15].

2.6 Action of tensor perturbations at linear order

An important feature of inflation is that it predicts a spectrum of primordial gravitational waves. Similarly to scalar perturbations the tensor spectrum is generated by quantum vacuum fluctuations which freeze out at horizon crossing. The analysis of tensor modes parallels the analysis of scalar modes and we shall just sketch it briefly. In fact it is simpler since there are no gauge ambiguities and since there is no source for tensor perturbations at linear order if the anisotropic stress vanishes. Hence we may obtain a quadratic action for tensor modes by expanding the Einstein-Hilber action to second order in tensor perturbations

$$S_{EH} = \frac{1}{2} M_P^2 \int d^4x \sqrt{-g} R. \quad (2.108)$$

For this purpose we consider the metric for a flat FRW background plus tensor perturbations

$$ds^2 = a^2 \left[-d\tau^2 + (\delta_{ij} + 2h_{ij}) dx^i dx^j \right], \quad \partial_i h_{ij} = h_{ii} = 0. \quad (2.109)$$

We now use the symbol h for tensor perturbations which is conventional. The quadratic action becomes

$$S = \frac{M_P^2}{8} \int dx^3 d\tau a^2 \left[(h'_{ij})^2 - (\partial_k h_{ij})^2 \right]. \quad (2.110)$$

The symmetric, transverse and traceless conditions on h_{ij} leave two physical degrees of freedom which may be parametrized by two fixed polarization tensors e_{ij}^+ and e_{ij}^\times . We follow [25] and define the fourier expansion as

$$h_{ij} = \int \frac{d^3k}{(2\pi)^3} \sum_{s=+, \times} e_{ij}^s(k) h_{\mathbf{k}}^s(\tau) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (2.111)$$

With $k^i \epsilon_{ij} = \epsilon_{ii} = 0$ and $\epsilon_{ij}^s \epsilon_{ij}^{s'} = 2\delta_{ss'}$. We also define the variable $\mu_{\mathbf{k}}^s$ in terms of which the action is of canonical form

$$\mu_{\mathbf{k}}^s \equiv \frac{a}{2} M_P h_{\mathbf{k}}^s. \quad (2.112)$$

Then

$$S = \sum_{s=+,X} \frac{1}{2} \int d^3\mathbf{k} d\tau \left[(\mu_{\mathbf{k}}^{s'})^2 - \left(k^2 - \frac{a''}{a} \right) (\mu_{\mathbf{k}}^s)^2 \right]. \quad (2.113)$$

The mode equation is

$$\mu_{\mathbf{k}}^{s''} + \left(k^2 - \frac{a''}{a} \right) \mu_{\mathbf{k}}^s. \quad (2.114)$$

We have essentially two copies of the quadratic action for scalar perturbations considered in section 2.5.2, one for each polarization state. The only difference is that the term $\frac{z''}{z}$ in (2.83) is replaced by $\frac{a''}{a}$ in (2.113) reflecting that tensor perturbations are massless. In fact (2.114) is the same as the equation of motion of a massless scalar field during a de Sitter expansion. The modes $\mu_{\mathbf{k}}^s$ are quantized in the same way as we quantized the scalar mode in section 2.5.3. If we specialize to de Sitter space we have $\frac{z''}{z} = \frac{a''}{a} = \frac{2}{\tau^2}$ which reproduce (2.96). Hence by comparing to (2.99) we see that the power spectrum approaches a constant on super horizon scales $\frac{k}{aH} = |k\tau| \ll 1$. At horizon crossing where modes freeze in it reads

$$\Delta_h^2(k) = \frac{2}{\pi^2} \frac{H_*^2}{M_P^2} \simeq \frac{2}{3\pi^2} \frac{V_*}{M_P^4}, \quad (2.115)$$

Where both polarization states have been included. We may also define a spectral index of tensor perturbations n_t to parametrize the scale (in)dependence

$$n_t \equiv \frac{d \ln \Delta_h^2}{d \ln k} = -2\epsilon_{V_*}. \quad (2.116)$$

As for scalar modes, the power spectrum may be expanded about the pivot scale k_*

$$\Delta_h^2(k) = A_t \left(\frac{k}{k_*} \right)^{n_t + \frac{1}{2} d n_t / d \ln k \ln(k/k_*) + \dots}. \quad (2.117)$$

Let us briefly return to the topic of freeze in. In section 2.4.5 we used the linearized Einstein equations in the Newtonian gauge to show that the comoving curvature perturbation \mathcal{R} freeze in on super horizon scales, in the absence of entropy perturbations. This conservation law applies to a general energy momentum tensor and therefore also after reheating where the universe is no longer dominated by the (single) inflaton field. This allows one to treat the primordial perturbation as approximately constant until a

given mode reenters the horizon and participate in the dynamics on sub horizon scales. However, during inflation the freeze in behavior may be deduced in a more straightforward way by means of the mode equations (2.85) and (2.114). For slow-roll inflation we may, as explained earlier, write $\frac{z''}{z} \simeq \frac{a''}{a} \simeq 2\mathcal{H}^2 = 2(aH)^2$, with equalities in de Sitter space. Then for sub horizon modes $k \gg aH$ one obtains the oscillatory behavior which we considered when defining the Minkowsky vacuum

$$\mu_k^{s''} + k^2 \mu_k^s = 0 \quad \Rightarrow \quad \mu_k^s = \frac{e^{-ik\tau}}{\sqrt{2k}}, \quad k \gg aH. \quad (2.118)$$

On super horizon scales the modes scale with the scale factor

$$\mu_k^{s''} - \frac{a''}{a} \mu_k^s = 0 \quad \Rightarrow \quad \mu_k^s \propto a, \quad k \ll aH. \quad (2.119)$$

From the mode equation it is also clear that the solution changes from oscillatory to growing at horizon crossing $k \sim aH$. Recalling that the variables we are interested in are obtained by dividing with the scale factor, for example $h_k^s \propto \frac{\mu_k^s}{a}$ for tensor modes, we clearly see the freeze in behavior at horizon crossing.

2.6.1 Energy scale of Inflation

It is conventional to normalize the amplitude of tensor perturbations by the amplitude of scalar perturbations, thereby defining the 'tensor to scalar ratio' r

$$r = \frac{\Delta_h^2}{\Delta_{\mathcal{R}}^2} \simeq 16\epsilon_{V_*}. \quad (2.120)$$

From (2.116) this yields the consistency relation of single field inflation

$$r = -8n_t, \quad (2.121)$$

Which states that the shape of the tensor spectrum is not independent from the other parameters. If this relation turns out to be falsified by experiments, the single field scenario is not valid.

By plugging the power spectra and the observed value of of the amplitude of scalar perturbations A_s , as measured by Planck (2.105), into the definition of r we obtain

$$V_* = \frac{3\pi^2 A_s}{2} r M_p^4 = \left(1.88 \times 10^{16} \text{GeV}\right)^4 \frac{r}{0.10}. \quad (2.122)$$

Hence the tensor-to-scalar ratio is a direct measure of the energy scale of inflation. Large values $r > 0.10$ corresponds to inflation occurring at GUT scale energies. It seems that

primordial tensor modes have not yet been observed. The Planck mission [13] sets bounds at about $r < 0.12$ at 95% CL (this bound depends on specifics such as running of spectral index, choice of pivot scale and which experiments are included). Hence the expression above yields an upper bound for the energy scale of inflation.

Using the result $dN \simeq \frac{d\phi}{M_p \sqrt{2\epsilon_V}}$ from section 1.2.2 the tensor-to-scalar ratio may be expressed by the Lyth bound

$$\frac{\Delta\phi}{M_p} \simeq \frac{1}{\sqrt{8}} \int_{N_{\text{end}}}^{N_*} \sqrt{r} . \quad (2.123)$$

This gives the field evolution between the times when the CMB fluctuations left the horizon and the end of inflation. During slow-roll $r(N)$ does not evolve much. The bound then implies that ‘large’ values of the tensor to scalar ratio $r > 0.01$ is connected to *large field inflation* $\Delta\phi > M_p$.

INFLATION IN NON-MINIMALLY COUPLED THEORIES

So far we have considered a scalar field inflaton which is minimally coupled to gravity. In this chapter we consider the more general case where an explicit non-minimal coupling term $\xi\phi^2 R$ is added to the action. This leads to several interesting consequences which we explore. In particular it leads to lowering of the tensor-to-scalar ratio r , a feature which is favored by current experiments. It also alleviates the problem of tiny values for the inflaton self-coupling. A coupling of this type is in general allowed by all symmetries of the scalar field sector and gravity. In fact the coupling is inevitable, as renormalization of a scalar field in curved space-time requires introduction of divergent counter terms of this type [44].

We consider the following action

$$S_I = \int d^4x \sqrt{-g} \left[\frac{M^2 + \xi\phi^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]. \quad (3.1)$$

The dimensionless non-minimal coupling ξ turns the effective Planck mass into a dynamical quantity. The present day value M_P is related to the vacuum expectation value v of the inflationary potential by

$$M_P^2 = M^2 + \xi v^2. \quad (3.2)$$

Most often we will consider models where the contribution ξv^2 is negligible. In that case the present day value of the planck mass is independent of the model parameters and we may safely identify M with the Planck mass $M \simeq M_P$.

It is in general rather complicated to analyze inflation with this action. The easiest way to proceed is by performing a conformal transformation which brings the gravitational part of the action into the standard Einstein-Hilbert form [32, 33]. This is achieved by

$$g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \Omega^2 = \frac{M_P^2 + \xi\phi^2}{M_P^2}. \quad (3.3)$$

The transformation law for the Ricci scalar may be found from (1.6) and (1.7)

$$R = \Omega^2 \left[\hat{R} + 6\hat{\square} \ln \Omega - 6\hat{g}^{\mu\nu} (\partial_\mu \ln \Omega) (\partial_\nu \ln \Omega) \right]. \quad (3.4)$$

The transformed action then reads

$$S_E = \int d^4x \sqrt{-\hat{g}} \left[\frac{1}{2} M_P^2 R - \frac{1}{2} \Omega^{-2} (1 + 6M_P^2 (\Omega')^2) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \Omega^{-4} V(\phi) \right]. \quad (3.5)$$

The d'Alembertian term has been removed by an integration by parts and hats are omitted for convenience. The original action (3.1) is said to be formulated in the *Jordan frame* whereas the transformed action is formulated in the *Einstein frame*. We are treating the same physics in both frames, but using different time and length scales to do so. In general this may give rise to apparent differences between observables in the two frames. However, the inflationary observables we consider are the same in both frames. See for example [34] which finds that comoving curvature perturbations in the Jordan frame \mathcal{R} coincide with the transformed values $\hat{\mathcal{R}}$ in the Einstein frame. Tensor perturbations are also invariant under conformal transformations, hence the inflationary observables (A_s , n_s , r and n_t) are the same in both frames. As we shall see, this allows us to use the conformal transformation as a trick to obtain the slow-roll results in an easy way.

The next step is to bring the Einstein frame action into canonical form by performing a field redefinition $\phi \rightarrow \chi$. The canonically normalized field $\chi(\phi)$ is defined by

$$\frac{1}{2} g^{\mu\nu} \partial_\mu \chi(\phi) \partial_\nu \chi(\phi) = \frac{1}{2} \left(\frac{d\chi}{d\phi} \right)^2 g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \quad (3.6)$$

Where the chain rule factor $(d\chi/d\phi)^2$ is read off as the coefficient of the the non-canonical kinetic term in the action above. It is convenient to write this factor as

$$\frac{d\chi}{d\phi} = \sqrt{\Omega^{-2} + \frac{3}{2} M_P^2 \left(\frac{d}{d\phi} \ln \Omega^2 \right)^2} = \sqrt{\frac{1 + (\xi + 6\xi^2)\phi^2/M_P^2}{(1 + \xi\phi^2/M_P^2)^2}}. \quad (3.7)$$

The action then takes the form

$$S_E = \int d^4x \sqrt{-\hat{g}} \left[\frac{1}{2} M_P^2 R - \frac{1}{2} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - U(\chi) \right]. \quad (3.8)$$

Having brought the action in canonical Einstein-Hilbert form, we may analyze inflation within the standard slow-roll paradigm presented in the preceding chapters. The slow-roll parameters are given by the Einstein frame potential

$$\epsilon_V = \frac{1}{2}M_P^2 \left(\frac{U'(\chi)}{U(\chi)} \right)^2, \quad \eta_V = M_P^2 \frac{U''(\chi)}{U(\chi)}, \quad N_* = \frac{1}{M_P^2} \int_{\chi_{\text{end}}}^{\chi_*} \frac{U(\chi)}{U'(\chi)} d\chi, \quad (3.9)$$

Where the Einstein frame potential $U(\chi)$ is just a rescaling of the Jordan frame potential

$$U(\chi) = \Omega^{-4} V(\phi). \quad (3.10)$$

From this it is clear that the non-minimal coupling term may help facilitate slow-roll inflation by flattening the potential at large field values. In particular, if the potential is of the form $\lambda\phi^4$, the Einstein frame potential approaches a constant in the large field regime $\phi \gg M_P/\sqrt{\xi}$, since $\Omega^{-4} \sim M_P^4/(\xi^2\phi^4)$.

In the following we will consider the case where the non-minimal coupling is large $\xi \gg 1$. As we shall see this is consistent with the observed amplitude of density perturbations for all the models we consider. In this case the field redefinition (3.7) provides two important scales $\frac{M_P}{\xi}$ and $\frac{M_P}{\sqrt{\xi}}$, such that models in general splits into three distinct regimes.

- *Small field regime* $\phi \ll M_P/\xi$. For large ξ it is clear from the field redefinition (3.7) that the solution $\chi(\phi)$ separates into two distinct behaviors separated by the scale M_P/ξ . For small field values $\phi \ll M_P/\xi$ the expression in (3.7) approaches unity such that

$$\chi \sim \phi, \quad \text{for } \phi \ll \frac{M_P}{\xi}. \quad (3.11)$$

Also, the conformal factor Ω^2 in (3.3) approaches unity such that $U(\chi) = V(\phi)$. Therefore the Einstein frame action is the same as the Jordan frame action, but with a *minimal* coupling to gravity. Hence the presence of the non-minimal coupling term is negligible. Of course, this behavior is already clear from the Jordan frame action (3.1). This ensures that after inflation where ϕ settles into v , the universe takes on the standard Einstein-Hilbert gravitational form.

If on the other hand $\phi \gg \frac{M_P}{\xi}$ the conformal factor approaches $\Omega^2 \simeq \xi\phi^2/M_P^2$ and the Ω^{-2} term may be neglected from the second expression in (3.7). The solution then approaches

$$\chi \simeq \sqrt{\frac{3}{2}}M_P \ln \Omega^2(\phi) \quad \text{for } \phi \gg \frac{M_P}{\xi}. \quad (3.12)$$

This form is relevant for the inflationary stage and (pre)heating.

- *Large field, inflationary regime* $\phi \gg M_P/\sqrt{\xi}$. In this regime the non-minimal coupling term flattens the potential to an extent where slow-roll inflation is viable. The field redefinition approaches the solution

$$\chi \simeq \sqrt{6}M_P \ln \frac{\sqrt{\xi}\phi}{M_P} \quad \text{for} \quad \phi \gg \frac{M_P}{\sqrt{\xi}}. \quad (3.13)$$

As we shall see, the typical scale for the end of inflation is $\phi_{\text{end}} \sim M_P/\sqrt{\xi}$ and $N_* = 60$ gives approximately $\phi_* \sim 10M_P/\sqrt{\xi}$.

- *Intermediate regime relevant for preheating* $M_P/\xi < \phi < M_P/\sqrt{\xi}$. The slow-roll approximation breaks down at $\phi_{\text{end}} \sim M_P/\sqrt{\xi}$, this marks the beginning of the reheating phase. In this regime the logarithm in (3.13) may be expanded as

$$\chi \simeq \sqrt{\frac{3}{2}} \frac{\xi\phi^2}{M_P}. \quad (3.14)$$

3.1 Inflation from a quartic potential

In this section we analyze inflation from a quartic potential

$$V(\phi) = \frac{\lambda}{4}\phi^4. \quad (3.15)$$

This potential is suitable for the chaotic inflation scenario proposed by Linde [35], where the field starts at a large value and rolls slowly towards the origin. It has also been analyzed with a non-minimal coupling term, see for example [36, 37, 38]. In particular, the quartic potential gives the large field behavior of the non-minimally coupled Higgs-inflation model [39, 40, 41, 42], which has served as great inspiration for much of the work in this thesis. In this model the Higgs serves both as the standard model Higgs at low energies and as a single component inflaton field at high energies. The models we consider in the following sections reproduce the quartic potential in certain limits and we will often express our results in terms of deviations from “ ϕ^4 -inflation”.

3.1.1 Minimal coupling

We are mainly interested in models with a non-minimal coupling to gravity. However, to appreciate what the non-minimal coupling term does, we first consider the case where

the inflaton is *minimally* coupled to gravity

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_P^2 R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\lambda}{4} \phi^4 \right]. \quad (3.16)$$

Note that this potential does not resemble the flat schematic potential in figure 1.4 and the discussion related to this figure. However, if we apply the slow-roll conditions (1.49) we find

$$\dot{\phi} \simeq -\frac{V'(\phi)}{3H} \propto \phi. \quad (3.17)$$

Hence the potential energy $V(\phi) \propto \phi^4$ grows much faster than the kinetic energy $\dot{\phi}^2 \propto \phi^2$ and as long as the field is far enough out on the potential, the slow-roll approximations are self-consistent. We apply the slow-roll approximation and obtain from (1.50)

$$\epsilon_V = 8 \left(\frac{M_P}{\phi} \right)^2, \quad \eta_H \simeq \eta_V - \epsilon_V = 4 \left(\frac{M_P}{\phi} \right)^2. \quad (3.18)$$

Inflation ends when

$$\epsilon_V(\phi_{\text{end}}) \simeq 1 \quad \Rightarrow \quad \phi_{\text{end}} \simeq \sqrt{8} M_P. \quad (3.19)$$

Assuming that the pivot scale k_* crossed the horizon $N_* = 60$ e-folds before the end of inflation yields (1.60)

$$\phi_* \simeq \sqrt{8N_* + 1} M_P \simeq 22 M_P. \quad (3.20)$$

Hence we obtain inflation at super planckian field values. To stay out of the domain of quantum gravity the self-coupling λ needs to be small such that the energy density can be much less than the Planck density. In fact, matching the potential to the observed value of scalar perturbations A_s in (2.105) requires that λ is extremely small

$$\frac{V_*}{\epsilon_{V_*}} = (0.0269 M_P)^4 \quad \Leftrightarrow \quad \lambda \sim 10^{-13}. \quad (3.21)$$

The presence of an extremely small parameter is generic to minimally coupled models, and represents a fine tuning problem. The values of r and n_s are estimated from (2.120) and (2.103)

$$r \simeq 0.26, \quad n_s \simeq 0.95. \quad (3.22)$$

The high value of r places the model well outside the 99.7% CL region in the (n_s, r) plane as measured by Planck [13], and is effectively ruled out, see Fig. 3.1.

3.1.2 Non-minimal coupling

Adding a non-minimal coupling term changes the picture. The action now reads

$$S_J = \int d^4x \sqrt{-g} \left[\frac{M_P^2 + \xi\phi^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\lambda}{4} \phi^4 \right]. \quad (3.23)$$

We use our previous findings (3.10) and (3.13) to find the Einstein frame potential in the large field inflationary regime $\phi \gg M_P / \sqrt{\xi}$. It is useful to express the potential in terms of both the original field ϕ , defined in the Jordan frame, and the canonical Einstein frame field χ

$$\begin{aligned} U(\chi(\phi)) &= \Omega^{-4} V(\phi) \simeq \frac{M_P^4 \lambda}{4\xi^2} \left(1 - \frac{M_P^2}{\xi\phi^2} \right)^2 \\ U(\chi) &\simeq \frac{\lambda M_P^4}{4\xi^2} \left(1 - \exp \left[\frac{-2\chi}{\sqrt{6}M_P} \right] \right)^2. \end{aligned} \quad (3.24)$$

The slow-roll parameters (3.9) are

$$\epsilon_V = \frac{4}{3} \left(e^{\frac{2\chi}{\sqrt{6}M_P}} - 1 \right)^{-2} \simeq \frac{4}{3} e^{\frac{-4\chi}{\sqrt{6}M_P}} \quad (3.25)$$

$$\eta_V \simeq -\frac{4}{3} e^{\frac{-2\chi}{\sqrt{6}M_P}}. \quad (3.26)$$

At this point the conformal transformation to the Einstein frame and the subsequent field-redefinition has served its purpose. It allowed us to do the standard slow-roll approximation outlined in section 1.2.1, and find the slow-roll conditions in a simple way. We now reinsert the field redefinition (3.13) to express the results in terms of ϕ

$$\epsilon_V \simeq \frac{4}{3} \frac{M_P^4}{\xi^2 \phi^4}, \quad \eta_V \simeq -\frac{4}{3} \frac{M_P^2}{\xi \phi^2}. \quad (3.27)$$

The field value at the end of inflation is

$$\epsilon_V(\phi_{\text{end}}) \sim 1 \quad \Rightarrow \quad \phi_{\text{end}} \simeq 1.07 \frac{M_P}{\sqrt{\xi}}. \quad (3.28)$$

Expressing also the number of e-folds (3.9) in terms of the Jordan frame field ϕ we get

$$\begin{aligned} N_* &= \frac{1}{M_P^2} \int_{\phi_{\text{end}}}^{\phi_*} \frac{U}{dU/d\phi} \left(\frac{d\chi}{d\phi} \right)^2 d\phi \simeq \frac{3\xi}{4M_P^2} (\phi_*^2 - \phi_{\text{end}}^2). \\ \Rightarrow \phi_* &\simeq 9 \frac{M_P}{\sqrt{\xi}}, \quad \text{for } N_* = 60. \end{aligned} \quad (3.29)$$

At first sight the large field approximation $\phi \gg M_P / \sqrt{\xi}$ seems inconsistent since ϕ_* is only about one order of magnitude larger than $M_P / \sqrt{\xi}$. Exact analytical solutions

to (3.8) do exist [43], however, they approach the large field approximation rapidly and the approximation is in fact quite good. To generate the proper value of scalar perturbations A_s we match the potential to (2.105). In the previous section we found that this condition require the self-coupling to be extremely small $\lambda \sim 10^{-13}$. However, with the non-minimal coupling as an additional parameter the condition instead yields a relation between ξ and λ

$$\frac{U_*}{\epsilon_{V_*}} = (0.0269M_P)^4 \quad \Rightarrow \quad \xi \sim 48000 \sqrt{\lambda}. \quad (3.30)$$

Hence the problem of the tiny inflaton self-coupling is alleviated, at the price however, of a large non-minimal coupling to gravity, which begs for a fundamental explanation. Note also that the initial assumption $\xi \gg 1$ is self-consistent for sensible values of λ . The values of r and n_s may be estimated from (2.120) and (2.103)

$$r \simeq 0.0033, \quad n_s = 0.966. \quad (3.31)$$

This lies well inside the 95% CL region, as determined by Planck, in the (r, n_s) -plane [13], see Fig. 3.1.

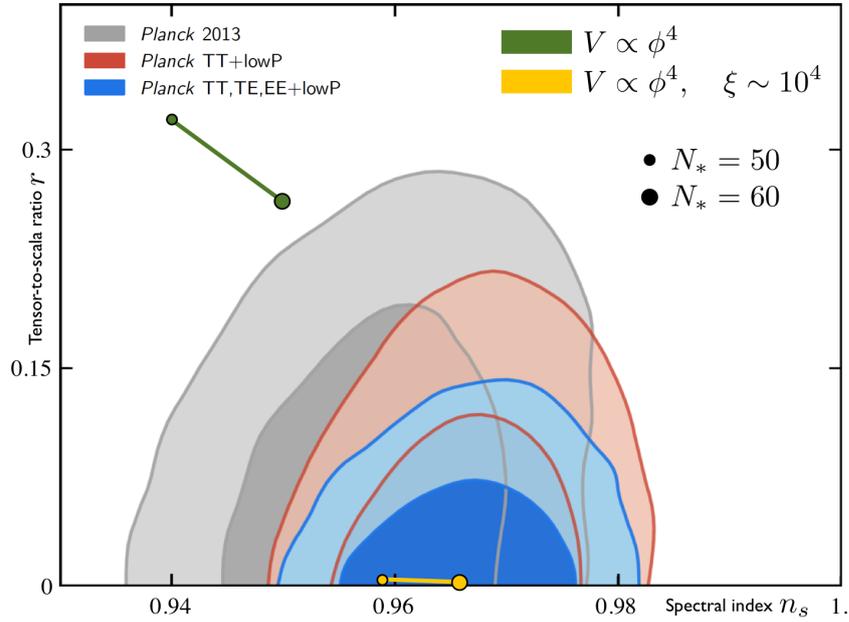


Figure 3.1: Comparison with Planck results in the (r, n_s) -plane [13]. The quartic potential with minimal coupling to gravity lies well outside the 99.7% CL region. The presence of a large non-minimal coupling ξ flattens the potential and push the model well inside the region favored by Planck results. In particular this model predicts an almost vanishing amplitude of primordial tensor modes. In this figure and all other figures representing the (r, n_s) -plane, dark colors represent $1\sigma \simeq 68\%$ CL regions and light colors represent $2\sigma \simeq 95\%$ CL regions. $3\sigma \simeq 99.7\%$ CL regions are not shown.

COMPOSITE INFLATION

This chapter is based on work published in [1] and [2]. We present two models of inflation in which the inflaton emerges as a composite state in a low energy effective field theory description of a strongly interacting gauge theory, free from fundamental scalars. First we consider a model where the inflaton emerges as the lightest glueball field associated with a pure Yang-Mills theory. In this section we also consider the issue of the unitarity cut-off related to the introduction of a non-minimal coupling term. Next we consider a model in which the inflaton emerges as a composite field of a strongly interacting and nonsupersymmetric gauge theory, featuring purely fermionic matter. As templates for this discussion, we use models of dynamical electroweak symmetry breaking, in particular Minimal Walking Technicolor. We then investigate whether it is possible for the lightest composite scalar to serve both as a composite Inflaton and a composite Higgs.

4.1 Glueball inflation

In this section we present a model where the inflaton emerges as the lightest glueball field associated with, in absence of gravity, a pure Yang-Mills theory. We will see that it is possible to achieve inflation with a glueball inflaton. Furthermore the natural scale of compositeness associated with the underlying Yang-Mills gauge theory, for the consistency of the underlying model, turns out to be of the order of the grand unification scale. We will consider both the metric and Palatini formulations of general relativity and briefly introduce the latter. We will find that the metric case behaves much better than the Palatini one. We will also investigate constraints set by tree-level unitarity of in-

flaton scattering at the effective action level in the Einstein frame for both formulations. We discover that the unitarity cutoff, i.e. the scale above which the model ceases to be valid and gravitational corrections must be taken into account, is the Planck scale for the metric formulation while it is the strongly coupled Yang-Mills scale for the Palatini one. The metric formulation therefore provides a consistent picture for a successful glueball inflation model.

4.1.1 Glueball action and inflation

Pure Yang-Mills theories featuring only gluonic-type fields are one of the simplest examples of strongly coupled theories. It is therefore natural to investigate composite inflation using these theories. The candidate for the inflaton is the interpolating field describing the lightest glueball.

$$\Phi(x) = \frac{\beta}{g} \text{Tr} [\mathcal{G}^{\mu\nu} \mathcal{G}_{\mu\nu}] , \quad (4.1)$$

where $\mathcal{G}^{\mu\nu}$ is the standard non-abelian field strength and β is the full beta function of the theory in any renormalization scheme. Φ is written in a renormalization scheme-independent way and therefore associated to a physical quantity. The Yang-Mills trace anomaly constrains the low energy effective Lagrangian for the lightest glueball state [45, 46, 47] to be:

$$L_{\text{GI}} = -\Phi^{-\frac{3}{2}} \partial_\mu \Phi \partial^\mu \Phi - V_{\text{GI}}, \quad V_{\text{GI}} = \frac{\Phi}{2} \ln \left(\frac{\Phi}{\Lambda^4} \right) , \quad (4.2)$$

Where Λ is the confinement scale. The generalization of this action, at the effective Lagrangian level, allowing also for a description of the topological properties of the theory can be found here [48, 49]. This generalization, and associated operators, by construction cannot affect the potential above nor the following analysis involving gravity. The reason being that the resulting action *must saturate* the underlying trace-anomaly only via the effective potential above. We discuss, however, the naive effects of higher order operators on graviton-scattering in section (4.1.3).

We introduce a non-minimal coupling to gravity such that the action reads in the Jordan frame

$$S_J = \int d^4x \sqrt{-g} \left[\frac{M^2 + \xi \Phi^{\frac{1}{2}}}{2} R - \Phi^{-\frac{3}{2}} \partial_\mu \Phi \partial^\mu \Phi - \frac{\Phi}{2} \ln \left(\frac{\Phi}{\Lambda^4} \right) \right] . \quad (4.3)$$

In this model we will keep the explicit dependence on M , which is not the present day Planck mass M_P . It is convenient to introduce the field ϕ possessing unity canonical

dimension $\Phi = \phi^4$, then

$$S_J = \int d^4x \sqrt{-g} \left[\frac{M + \xi \phi^2}{2} R - 16 \partial_\mu \phi \partial^\mu \phi - 2\phi^4 \ln \left(\frac{\phi}{\Lambda} \right) \right]. \quad (4.4)$$

We follow the procedure outlined in the previous chapter and impose the conformal transformation (3.3). We then land in the Einstein frame

$$S_E = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} g^{\mu\nu} R_{\mu\nu} - 16\Omega^{-2} \left(1 + \frac{3f\Omega^{-2}\xi^2\phi^2}{16M_P^2} \right) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \Omega^{-4} V_{\text{GI}} \right]. \quad (4.5)$$

The term which is proportional to the parameter f which has just been introduced, stems from the transformation of the Ricci scalar. Thus far we have set $f = 1$ which corresponds to the standard metric formalism in which the field equations are derived by varying the action with respect to the metric tensor $g_{\mu\nu}$. In this formalism the connection $\Gamma_{\mu\nu}^\alpha$ depends on the metric in the standard way, as it is given by the Christoffel connection (1.7). However, one may also consider the Palatini formalism which corresponds to $f = 0$. In this formalism $g_{\mu\nu}$ and $\Gamma_{\mu\nu}^\alpha$ are treated as independent variables. Hence the Ricci scalar does not transform under the conformal transformation, and the term proportional to f in the action above vanish.

The reason for considering the Palatini formalism is that [87] pointed out that it may alleviate the unitarity problem of Higgs inflation. This problem has been debated much in the literature. It concerns unitarity of the scattering amplitude of inflaton field fluctuations $\delta\phi$ at tree-level during inflation. In [51, 52, 53] it is argued that the presence of the non-minimal coupling ξ places the cut-off at $E \sim M_P/\xi$. This scale is much smaller than the energy density scale at inflation and only slightly larger than the Hubble scale, which is the relevant momentum scale for processes during inflation

$$V(\phi_{\text{end}})^{1/4} \sim \lambda^{1/4} \frac{M_P}{\sqrt{\xi}}, \quad H \sim \lambda^{1/2} \frac{M_P}{\xi}, \quad (4.6)$$

Where we are using results from section 3.1.2. This suggest that the inflationary analysis of the theory is false, since inflation happens at energies where an UV completion, which is not provided seems to be needed. However, in [42, 43] it is argued that tree-level unitarity is not violated, if the classical background during inflation is taken into account instead of the vacuum one. We will return to these points in the context of the Glueball model of inflation in conning sections.

Using the Einstein frame action (4.23), we are now able to determine the slow-roll parameters and constraints relevant for inflation. First we introduce the canonically normalized field $\chi(\phi)$ by the field redefinition (3.6). The chain rule factor $\frac{1}{2} \left(\frac{d\chi}{d\phi} \right)^2$ is read of as the coefficient of the kinetic term in the action above. Since this contains f , the

solution $\chi(\phi)$ depends on whether $f = 0$ or $f = 1$, which suggest that we consider the metric and Palatini formalisms separately. However, by applying the chain rule to (3.9) it is straightforward to obtain expressions which are valid for both $f = 0$ and $f = 1$ and which do not require an explicit solution of the field redefinition

$$\epsilon_V = \frac{M_P^2}{2} \left(-4\Omega^{-1}\Omega' + \frac{V'}{V} \right)^2 \left(\frac{1}{\chi'} \right)^2, \quad (4.7)$$

Where primes denote derivatives with respect to ϕ and we have omitted the subscript GI on the potential. In the large field regime $\phi^2 \gg M/\sqrt{\xi}$ this approaches

$$\epsilon \simeq \frac{1}{64 \ln\left(\frac{\phi}{\Lambda}\right)^2 \left(\xi^{-1} + f \cdot \frac{3}{16}\right)}. \quad (4.8)$$

Inflation ends when $\epsilon = 1$ such that:

$$\frac{\phi_{\text{end}}}{\Lambda} = \exp\left(\frac{1}{8\sqrt{\left(\xi^{-1} + f \cdot \frac{3}{16}\right)}}\right). \quad (4.9)$$

At this point it is already clear that the Palatini formalism is not feasible. We require that $\xi \gg 1$ to generate the proper amplitude of scalar perturbations, hence ϕ_{end} is many orders of magnitude larger than the scale Λ , for $f = 0$, and the effective field theory description is inconsistent. However, we will keep the explicit dependence on the f -term a little while longer to see that it introduces other inconsistencies. In the large field limit the number of e-folds is:

$$N_* = \frac{1}{M_P^2} \int_{\phi_{\text{end}}}^{\phi_*} \frac{V}{-4\Omega^{-1}\Omega'V + V'} \chi'^2 d\phi \simeq \left[16 \left(\xi^{-1} + f \cdot \frac{3}{16} \right) \ln\left(\frac{\phi}{\Lambda}\right)^2 \right]_{\phi_{\text{end}}}^{\phi_*}. \quad (4.10)$$

Assuming $N_* = 60$ e-folds yields

$$\frac{\phi_*}{\Lambda} \simeq \exp\left(\sqrt{\frac{60}{16\left(\xi^{-1} + f \cdot \frac{3}{16}\right)}}\right). \quad (4.11)$$

We match the model to the Planck normalization condition (2.105)

$$\frac{U_*}{\epsilon_*} = (0.0269M_P)^4. \quad (4.12)$$

This condition helps estimating the magnitude of the non-minimal coupling. We deduce

$$U_* \simeq \frac{2M_P^4}{\xi^2} \ln\left(\frac{\phi_*}{\Lambda}\right) \simeq \frac{2M_P^4}{\xi^2} \sqrt{\frac{3.75}{\xi^{-1} + f \cdot 0.1875}}. \quad (4.13)$$

while

$$\epsilon_* \simeq \frac{1}{64 \ln\left(\frac{\phi_{ini}}{\Lambda}\right)^2 \left(\xi^{-1} + f \cdot \frac{3}{16}\right)} = 0.0042. \quad (4.14)$$

We can therefore determine the magnitude of the non-minimal coupling which, depending whether we used the Palatini or the metric formulation, assumes the following value:

$$\xi \simeq 1.4 \cdot 10^6 \quad \text{Palatini}, \quad \text{and} \quad \xi \simeq 6.1 \cdot 10^4 \quad \text{Metric} \quad (4.15)$$

As for the case of Higgs inflation, and other earlier approaches [31, 32, 34, 36, 37, 38, 54] we see that a phenomenologically large value of ξ is needed for generating the correct size of the observed amplitude of density fluctuations. A more complete treatment for all these models would require, in the future, a mechanism for generating such a large coupling.

The value of the non-minimal coupling allows us to estimate the initial and final value of the composite glueball field Φ . We have in units of the strong scale Λ :

$$\frac{\phi_{\text{end}}}{\Lambda} \sim 10^{63.5}, \quad \frac{\phi_*}{\Lambda} \sim 10^{986} \quad \text{Palatini.} \quad (4.16)$$

$$\frac{\phi_{\text{end}}}{\Lambda} \sim 1.3, \quad \frac{\phi_*}{\Lambda} \sim 88 \quad \text{Metric.} \quad (4.17)$$

From these results it is clear that the metric formulation provides a more natural range of values for ϕ . The effective action built here is a generating functional for the trace anomaly and therefore the associated potential V cannot be quantum modified. This may protect the inflationary scenario even for large values of the scalar field.

We may relate the strongly coupled scale Λ with M by recalling that we are working in the large field regime $\phi \gg M/\sqrt{\xi}$. To be consistent with the results above we therefore require in the metric formalism

$$\Lambda > \frac{M}{\sqrt{\xi}}. \quad (4.18)$$

If we plug in the reduced Planck mass $M_p = 2.44 \cdot 10^{18}$ GeV we obtain a characteristic value:

$$\Lambda > 0.9 \cdot 10^{16} \text{ GeV}, \quad (4.19)$$

Which is the typical scale for grand unification.

4.1.2 Tree-level unitarity cut-off

In this section we consider the constraints set by tree-level unitarity of the inflaton field. For the present purpose it is convenient first to shift the overall glueball potential, before coupling it non-minimally to gravity, in such a way that the potential evaluated on the ground state has zero energy:

$$V_{\text{GI}} \rightarrow 2\phi^4 \ln\left(\frac{\phi}{\Lambda}\right) + \frac{\Lambda^4}{2e}, \quad (4.20)$$

Where e is the Euler number. With this shift the ground state of the theory assumes the same value in the Jordan and Einstein frame and reads:

$$\langle\phi\rangle = e^{-\frac{1}{4}}\Lambda = v. \quad (4.21)$$

The previous inflationary analysis remains unmodified by this shift. Furthermore we are interested in the large field regime $\phi \gg M/\sqrt{\xi}$ which may be approximated by setting $M = 0$ in the action. The following relation is then natural:

$$M_{\text{P}}^2 \simeq \xi v^2, \quad \Rightarrow \quad \Omega = \frac{\phi}{v}. \quad (4.22)$$

In the Einstein frame we then have:

$$S_E = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_{\text{P}}^2 R - 16 \frac{v^2}{\phi^2} \left(1 + \frac{3}{16} f\xi \right) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{v^4}{\phi^4} \left[2\phi^4 \ln\left(\frac{\phi}{e^{\frac{1}{4}}v}\right) + \frac{v^4}{2} \right] \right]. \quad (4.23)$$

We are now equipped with the needed ingredients to tackle the issue of tree-level unitarity at the effective Lagrangian level during the inflationary period. More specifically we are concerned with violation of tree-level unitarity of the scattering amplitude concerning the inflaton field fluctuations $\delta\phi$ around its classical time dependent background $\phi_c(t)$ during the inflationary period. Following the analysis performed in [42] we can, in first approximation, neglect the time dependence of the classical field and write:

$$\phi = \phi_c + \delta\phi, \quad (4.24)$$

since the fluctuations are expected to encapsulate the high frequency modes of the inflaton. To estimate the actual cutoff of the tree-level scattering amplitude we analyze independently the kinetic and potential term for the inflaton in the Einstein frame. Starting from the kinetic term it is straightforward to show that around the classical background it may be written as:

$$T = \frac{v^2}{2\phi_c^2} (32 + 6f\xi) (\partial\delta\phi)^2 \sum_{n=0}^{\infty} (n+1) \frac{(-\delta\phi)^n}{\phi_c^n}, \quad (4.25)$$

Where T indicates that we are considering the kinetic term. It is possible to canonically normalize the first term of the series, i.e. the kinetic term for a free field by rescaling the fluctuations as follows:

$$\frac{\delta\phi}{\phi_c} = \frac{\delta\tilde{\phi}}{v\sqrt{32+6f\xi}}. \quad (4.26)$$

Under this field redefinition (4.25) becomes:

$$T = \frac{(\partial\delta\tilde{\phi})^2}{2} \sum_{n=0}^{\infty} (n+1) \frac{(-\delta\tilde{\phi})^n}{(32+6f\xi)^{\frac{n}{2}} v^n}. \quad (4.27)$$

For the potential term the higher order operators are also of the form:

$$C \frac{(\delta\tilde{\phi})^n}{(32+6f\xi)^{\frac{n}{2}} v^n}, \quad (4.28)$$

Where C denote an arbitrary constant. In the metric formalism $f = 1$ this implies that the tree-level cutoff for unitarity is:

$$\sqrt{\xi}v \simeq M_{\text{P}}, \quad (4.29)$$

Where we reinserted (4.22). In the Palatini formalism the f -term which involve the non-minimal coupling ξ drop out from the denominators above, and the cut-off is simply v . This result shows that the cut-off, in both formulations, is background independent. Quite nicely the unitarity cutoff in the metric formulation corresponds to the Planck scale and therefore tree-level unitarity is safe in this approach, however this is not the case for the Palatini formulation. These results are in complete agreement with the findings for successful inflation in the previous section.

4.1.3 Graviton exchange for Composite Inflation

Similar to the case of Higgs inflation we have introduced a non-minimal coupling to gravity of the type $\xi\phi^2 R$ which is allowed by all known symmetries of the underlying strongly coupled theory and gravity. In [51, 52, 53, 58] it is argued that, although this term superficially appears to be a dimension four operator, expanding it around flat space, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}/M_{\text{P}}$, leads to a dimension five operator plus an infinite tower of higher dimensional operators:

$$\xi\phi^2 R \sim \xi\phi^2 \frac{\square h}{M_{\text{P}}} + \dots \quad (4.30)$$

This indicates that generic non-minimally coupled theories become strongly interacting at scales $\Lambda_{\text{NRG}} \sim M_{\text{P}}/\xi$. The new scale Λ_{NRG} emerges because gravity in four dimensions

is non renormalizable and NRG stands for Non Renormalizable Gravity. In the case of minimally coupled theories, this scale is simply M_P . Therefore, without any protecting mechanism, the interaction with gravity can lead to a series of corrections to the low energy effective Lagrangian. Using the canonically normalized field ϕ , one naively expects the following corrections to any potential, and in our specific case to V_{GI} :

$$V = V_{GI}(\Lambda) + \phi^4 \sum_{n>0} a_n \left(\frac{\phi}{\Lambda_{NRG}} \right)^n + \xi \phi^2 R \sum_n b_n \left(\frac{\phi}{\Lambda_{NRG}} \right)^n. \quad (4.31)$$

The new interactions are suppressed by $\Lambda_{NRG} \sim M_P/\xi$ while the new strongly coupled dynamics has a scale $\Lambda \sim M_P/\sqrt{\xi}$. The coefficients a_n and b_n , due to graviton exchange, depend on the behavior of gravity above the scale Λ_{NRG} . Unless a protecting mechanism exists, and taking all the coefficients a_n and b_n to be of order unity, the flatness of the inflationary potential in the Einstein frame can be questioned. This is not only the case in Higgs inflation, but also of many minimal models of inflation, such as $m^2\phi^2$ chaotic inflation, since in these cases $\phi > \Lambda_{NRG}$ during inflation.

Although no actual resolution to this potential issue was presented in [51, 52, 53, 58], it was, however, pointed out that currently we have no experimental evidence that a_n and b_n must be of order unity and that there is still the logical possibility that graviton exchange is softer than naive estimate suggested in [59] leaving our potential unaltered. We could therefore work in the same spirit of Higgs or chaotic inflation with the further benefit that, as we showed above, the inflaton-inflaton scattering is better behaved than in models of Higgs inflation.

In composite inflation, there is already a symmetry principle partially constraining the effective potential V_{GI} . This constraint requires the action for ϕ to be such that, at zero external momentum, the matter trace-anomaly, in the Jordan frame, has to reproduce the Yang-Mills trace anomaly and therefore automatically requires $a_n = 0$ for any $n > 0$. The situation for the b_n coefficients is more delicate since they involve derivatives vanishing at zero momentum, however, it would seem natural that also these coefficients have to vanish.

4.1.4 Summary of the different energy scales

In this section we summarize the various scales and associated operators involved in the model, before and after coupling our underlying gauge theory to gravity.

We started our analysis by introducing the *simplest* non-abelian gauge theory known, i.e. the pure $SU(N)$ Yang-Mills gauge theory. The fundamental Lagrangian for this gauge

theory, in absence of the θ -angle operator, is constituted by *only one* renormalizable conformal operator¹:

$$L_{\text{Fund}} = -\frac{1}{4} \sum_{a=1}^{N^2} \mathcal{G}_a^{\mu\nu} \mathcal{G}_{\mu\nu,a}. \quad (4.32)$$

First principle lattice simulations have shown that this theory confines and via dimensional transmutation a renormalization invariant physical scale is generated. This scale is identifiable with the scale Λ of the glueball theory introduced in the previous sections. Using the renormalization group equations, lattice simulations, as well as our experience from ordinary quantum chromodynamics² the fundamental theory can be used in the perturbative regime to describe the dynamics of the theory at energy scales of the order of 100Λ and above. For energies below this scale and to describe the vacuum properties of the theory the effective potential given in (4.2) works and it has been used recently in [89] also to determine cosmological properties.

When coupling our theory to gravity we can, of course, use directly the unique operator constituting the fundamental gauge theory (4.32), and use, for example first principle lattice simulations. However, because we were interested in slow roll conditions near the ground state of the underlying gauge theory we used the simplest and most appropriate analytic description, i.e. the one in terms of the glueball effective theory. As an important consistency check we showed that inflation starts at energy scales just below or near the energy scales above which the underlying gauge dynamics is perturbative and described by a single renormalizable operator. We have also showed that the natural scale for Λ is the grand-unified scale which is orders of magnitude smaller than the Planck scale. Therefore we expect the perturbative dynamics of the gauge theory to set-in before we arrive at the Planck scale. We showed, furthermore, that inflaton-inflaton scattering would only be affected by Planck scale physics making our analysis, from this point of view, more solid than Higgs inflation.

The grand-unified scale here is defined as the energy at which the standard model gauge couplings, in a given renormalisation scheme, unify. Given that the standard model alone does not unify, an extension perhaps also including dark matter is needed. The standard model couplings are weak at the unification point. However the inflationary model is still strongly coupled at this scale (now identified with Λ). Therefore, a potential unification of the standard model and the new inflationary gauge dynamics can only take place at or around the Planck scale which is not accessible with our current understanding of the gravitational corrections.

¹If we add also the θ -angle operator we have one more renormalizable conformal operator which does not affect the classical equations of motion.

²Which is Yang-Mills with quarks.

There is, however, another scale to worry about, i.e. the one associated to graviton scattering. In the last section we have shown that, like in Higgs inflation and several other scenarios, this problem arises at a new scale $\Lambda_{\text{NRG}} < \Lambda$. The fact that this scale Λ_{NRG} is smaller than Λ , i.e. where inflation takes place, might spoil the inflationary scenario unless a mechanism for softening this behavior emerges. Due to the fact that this mechanism, as stressed above [58], must be active above the scale Λ_{NRG} this implies the following scenarios for composite inflation. If the scale where this mechanism emerges is below 10 to 100 Λ then the effective description given in (4.2) is valid and we can use the further constraint $a_n = 0$ needed to correctly saturate the trace anomaly of the underlying gauge theory. If the mechanism is introduced at scales between 100 Λ and M_{P} the underlying Lagrangian, before coupling to gravity, reduces to (4.32). In this energy range the underlying gauge theory is perturbative and therefore one can use any mechanism that works for Higgs inflation. Finally, if the scale at which this mechanism takes place is above M_{P} a more complete theory of gravity is needed. This shows that our model has, in the worst case scenario, the same limitations of Higgs inflation for graviton scattering but works better for inflaton scattering.

4.1.5 Comparison with Planck results

In this section we plot the model in the (r, n_s) -plane and compare with Planck results. We consider just standard metric formalism with $f = 1$ since, as we have seen, the Palatini one with $f = 0$ is inconsistent. The easiest way to proceed is to calculate directly from the Einstein frame potential, which is obtained by inserting the field-redefinition (3.13) into $\Omega^{-4}V_{\text{GI}}$

$$U(\chi) = \underbrace{\frac{2M_{\text{P}}^4}{\xi^2} \left(1 + \exp \left[\frac{-2\chi}{\sqrt{6}M_{\text{P}}} \right] \right)^{-2}}_{\phi^4\text{-Inflation}} \left(\ln \frac{M_{\text{P}}}{\sqrt{\xi}\Lambda} + \frac{\chi}{\sqrt{6}M_{\text{P}}} \right). \quad (4.33)$$

The underbraced term has the same form as the quartic potential we investigated in the previous chapter (3.24), however, the presence of the linear term will alter the results. The slow-roll parameters read

$$\epsilon_V \simeq \frac{1}{12} \left(\frac{4M_{\text{P}}^2}{\xi\phi^2} + \left(\ln \frac{\phi}{\Lambda} \right)^{-1} \right)^2, \quad (4.34)$$

$$\eta_V \simeq -\frac{4}{3} \frac{M_{\text{P}}^2}{\xi\phi^2} + \frac{4}{3} \frac{M_{\text{P}}^2}{\xi\phi^2} \ln \frac{\phi}{\Lambda}. \quad (4.35)$$

This may be compared to (3.28) which are the slow-roll parameters for the quartic potential. As we discussed earlier, consistency of our approach sets a bound on the confinement scale $\Lambda > M_p/\sqrt{\xi}$ (4.18). By inserting the lower bound $\Lambda \sim M_p/\sqrt{\xi}$ we clearly see that the first term in both ϵ_V and η_V , which are the slow-roll parameters of the quartic potential, are subdominant. Hence

$$\epsilon_V \simeq \frac{1}{12} \left(\ln \frac{\phi}{\Lambda} \right)^{-2}, \quad \eta_V \simeq -\frac{4}{3} \left(\frac{\phi}{\Lambda} \right)^{-2} \left(1 - \ln \frac{\phi}{\Lambda} \right), \quad (4.36)$$

Where we have reproduced our previous result for ϵ_V (4.14), which were obtained by a slightly different procedure. Assuming $N_* = 60$ e-folds and inserting our previous findings (4.17), we obtain from (2.120) and (2.103)

$$r = 0.067, \quad n_s = 0.975. \quad (4.37)$$

This lies inside the 95% CL as determined by Planck [13] in the (r, n_s) -plane, see Fig. (4.1). The results for $N_* = 50$ have also been plotted to illustrate the uncertainty in the reheating temperature. The presence of the logarithmic term in the potential raise the r and n_s values as compared to the pure quartic potential.

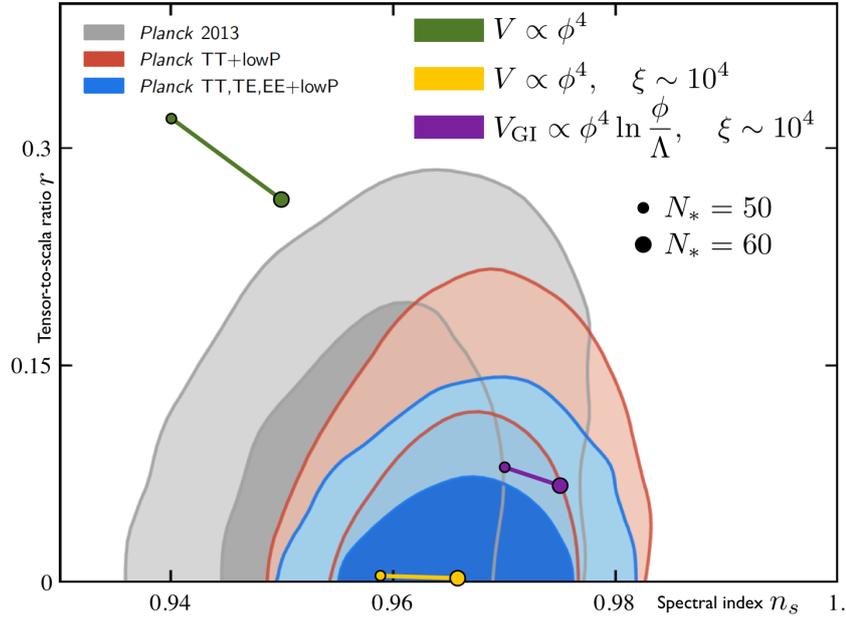


Figure 4.1: Comparison with Planck results in the (r, n_s) -plane [13]. The Glueball model lies inside the 95% CL region. Results for $N_* = 50$ and $N_* = 60$ are plotted to illustrate the uncertainty in the reheating temperature. The presence of the logarithmic term in the potential raise the r and n_s values as compared to the pure quartic potential

4.2 Minimal composite inflation

In this section we consider models in which the inflaton emerges as a composite field of a four dimensional strongly interacting and nonsupersymmetric gauge theory, featuring purely fermionic matter. The presentation is based on work published in [1]. As templates for this discussion, we use models of dynamical electroweak symmetry breaking, also known as Technicolor [76, 77]. In these models the Higgs sector of the SM is replaced by a new underlying four-dimensional gauge dynamics free from fundamental scalars, and the Higgs itself emerges as a composite ‘techni-hadron’, thereby breaking the electroweak symmetry. The simplest models of Technicolor passing precision tests are known as Minimal Walking Technicolor models (MWT) [78, 79, 80, 81]. We will use these specifically to construct the Minimal Composite Inflation scenario, such that the Inflaton emerges in a manner similar to the composite Higgs of MWT. The spectrum of the theory features a light composite scalar as compared to the compositeness scale of the theory $\Lambda_{MCI} \simeq 4\pi v$, where v is the scale of the fermion condensate. This composite scalar will be the Inflaton and we will use a low energy effective description of the underlying technicolor dynamics to describe its potential. This description is valid up to the compositeness scale $\Lambda_{MCI} \simeq 4\pi v$. We then investigate whether it is possible for this scalar to serve both as a composite Inflaton and a composite Higgs.

4.2.1 Underlying Minimal Conformal Gauge Theory for Inflation

We will only provide a very brief description of the underlying gauge theory, since the effective description of inflation in the end will reduce to that of a simple quartic potential which we considered in section 3.1.2. As we shall see, by matching this potential with the observed amplitude of density perturbations, we straightforwardly find that the composite scenario cannot serve to explain both inflation and dynamical electroweak symmetry breaking. The reason is that the effective description cannot be valid at both the electroweak scale and the inflationary scale. Therefore, we proceed by just quoting how a composite inflationary scenario may be set up by using the MWT model as template.

We consider as underlying gauge theory the $SU(N)$ gauge group with $N_f = 2$ Dirac massless fermions transforming according to the adjoint representation of $SU(N)$. This theory has a quantum global symmetry $SU(4)$ expected to break spontaneously to $SO(4)$ when the fermion condensate forms. The associated effective Lagrangian has been

constructed explicitly in [75]³. For $N = 2$ we recover the MWT model, however for the composite inflation purpose any N can be considered. To discuss the symmetry properties of the theory it is convenient to use the Weyl basis for the underlying fermions and arrange them in the following vector transforming according to the fundamental representation of $SU(4)$

$$Q = \begin{pmatrix} U_L \\ D_L \\ -i\sigma^2 U_R^* \\ -i\sigma^2 D_R^* \end{pmatrix}, \quad (4.38)$$

where U_L and D_L are the left handed techniup and technidown respectively, and U_R and D_R are the corresponding right handed particles. We are using a Technicolor friendly notation to allow for a straightforward identification of these states with the ones relevant at the electroweak scale. Assuming the standard breaking to the maximal diagonal subgroup, the $SU(4)$ symmetry spontaneously breaks to $SO(4)$. Such a breaking is driven by the following condensate

$$\langle Q_i^\alpha Q_j^\beta \epsilon_{\alpha\beta} E^{ij} \rangle = -2 \langle \bar{U}_R U_L + \bar{D}_R D_L \rangle, \quad (4.39)$$

where the indices $i, j = 1, \dots, 4$ denote the components of the tetraplet of Q , and the Greek indices indicate the ordinary spin. The matrix E is a 4×4 matrix defined in terms of the 2-dimensional unit matrix as

$$E = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \quad (4.40)$$

Here $\epsilon_{\alpha\beta} = -i\sigma_{\alpha\beta}^2$ and $\langle U_L^\alpha U_R^{*\beta} \epsilon_{\alpha\beta} \rangle = -\langle \bar{U}_R U_L \rangle$. A similar expression holds for the D techniquark. The above condensate is invariant under an $SO(4)$ symmetry. This leaves us with nine broken generators with associated Goldstone bosons. The fundamental Lagrangian replacing the inflaton one is:

$$\mathcal{L}_{\text{Inflation}} \rightarrow -\frac{1}{4} \mathcal{F}_{\mu\nu}^a \mathcal{F}^{a\mu\nu} + i\bar{Q}_L \gamma^\mu D_\mu Q_L + i\bar{U}_R \gamma^\mu D_\mu U_R + i\bar{D}_R \gamma^\mu D_\mu D_R$$

with the techni-inflation field strength $\mathcal{F}_{\mu\nu}^a = \partial_\mu \mathcal{A}_\nu^a - \partial_\nu \mathcal{A}_\mu^a + g_{TC} \epsilon^{abc} \mathcal{A}_\mu^b \mathcal{A}_\nu^c$, $a, b, c = 1, \dots, N^2 - 1$. For the left handed technifermions the covariant derivative might or might not include the SM fields. This model becomes MWT if $N = 2$. In that case we gauge the

³An equally interesting possibility is the use of pseudoreal representations of the underlying gauge group for which the expected pattern of symmetry breaking is $SU(2N_f) \rightarrow Sp(2N_f)$ which has been investigated in [27, 28] however our main physical results are general.

left and right symmetries appropriately and further identify the techni-inflation with the composite Higgs. The MWT covariant derivative reads:

$$D_\mu Q_L^a = \left(\delta^{ac} \partial_\mu + g_{TC} \mathcal{A}_\mu^b \epsilon^{abc} - i \frac{g}{2} \vec{W}_\mu \cdot \vec{\tau} \delta^{ac} - i g' \frac{y}{2} B_\mu \delta^{ac} \right) Q_L^c. \quad (4.41)$$

\mathcal{A}_μ are the techni gauge bosons, W_μ are the gauge bosons associated to $SU(2)_L$ and B_μ is the gauge boson associated to the hypercharge. τ^a are the Pauli matrices and ϵ^{abc} is the fully antisymmetric symbol. In the case of right handed techniquarks the third term containing the weak interactions disappears and the hypercharge $y/2$ has to be opportunely modified according to whether it is an up or down techniquark to avoid gauge anomalies.

4.2.2 Effective theory for minimal composite inflation

The effective theory consists of a composite inflaton σ and its pseudoscalar partner, as well as nine pseudoscalar Goldstone bosons and their scalar partners. These can be assembled in the matrix

$$M = \left[\frac{\sigma + i\Theta}{2} + \sqrt{2}(i\Pi^a + \tilde{\Pi}^a) X^a \right] E, \quad E = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \quad (4.42)$$

which transforms under the full $SU(4)$ group according to

$$M \rightarrow u M u^T, \quad \text{with } u \in SU(4). \quad (4.43)$$

The X^a 's, $a = 1, \dots, 9$ are the generators of the $SU(4)$ group which do not leave the vacuum expectation value (VEV) of M invariant

$$\langle M \rangle = \frac{v}{2} E. \quad (4.44)$$

The connection between the composite scalars and the underlying technifermions can be derived from the transformation properties under $SU(4)$, by observing that the elements of the matrix M transform like technifermion bilinears

$$M_{ij} \sim Q_i^\alpha Q_j^\beta \epsilon_{\alpha\beta} \quad \text{with } i, j = 1 \dots 4. \quad (4.45)$$

The effective Lagrangian is

$$\mathcal{L}_{\text{MWT}} = -\frac{1}{2} \text{Tr} \left[D_\mu M D^\mu M^\dagger \right] - \mathcal{V}(M), \quad (4.46)$$

where the potential reads

$$\begin{aligned} \mathcal{V}(M) &= -\frac{m^2}{2} \text{Tr} [M M^\dagger] + \frac{\lambda}{4} \text{Tr} [M M^\dagger]^2 + \lambda' \text{Tr} [M M^\dagger M M^\dagger] \\ &\quad - 2\lambda'' [\text{Det}(M) + \text{Det}(M^\dagger)], \end{aligned} \quad (4.47)$$

The potential $\mathcal{V}(M)$ is $SU(4)$ invariant. It produces a VEV which parameterizes the techniquark condensate, and spontaneously breaks $SU(4)$ to $SO(4)$. In terms of the model parameters the VEV is

$$v^2 = \langle \sigma \rangle^2 = \frac{m^2}{\lambda + \lambda' - \lambda''} . \quad (4.48)$$

while the inflaton mass is:

$$M_I^2 = 2 m^2 . \quad (4.49)$$

The linear combination $\lambda + \lambda' - \lambda''$ corresponds to the composite inflaton self coupling. We have nine Goldstones which might or not acquire any mass or, be absorbed by gauging some of the global symmetries of the theory as it happens for some of the Goldstones when the MCI is identified with the MWT model. The remaining scalar and pseudoscalar masses are

$$\begin{aligned} M_{\Theta}^2 &= 4v^2 \lambda'' \\ M_{A^\pm}^2 = M_{A^0}^2 &= 2v^2 (\lambda' + \lambda'') \\ M_{\Pi_{UU}}^2 &= M_{\Pi_{UD}}^2 = M_{\Pi_{DD}}^2 = 2v^2 (\lambda' + \lambda'') . \end{aligned} \quad (4.50)$$

Besides the techni-scalar sector we expect other higher spin bound states to appear in the low energy effective theory. However, we focus on the scalar sector here and note that it would be interesting to investigate the effects of the spin one states in the future.

4.2.3 Coupling to gravity

Having introduced the effective theory for inflation above, we investigate now whether it is possible to identify it with a composite Higgs. It is therefore natural to use the framework of Higgs-inflation developed in [39]. Here it was proposed that the inflationary expansion of the early Universe can be linked to the SM by identifying the SM Higgs boson with the inflaton. The salient feature of the Higgs-inflation model is the non-minimal coupling of the Higgs doublet field (H) to gravity. This is introduced by adding a non-minimal coupling term $\xi H^\dagger H R$ to the standard gravity-matter action. The potential relevant for inflation is a quartic one as described in section 3. It was found in [39] that with ξ of the order 10^4 the model leads to successful inflation, provides the graceful exit from it, and produces a spectrum of primordial fluctuations in good agreement with observations. Here we would like to use this framework in order to test the hypothesis that a composite model of inflation can serve as a natural model for

a rapid expansion of the Universe. We therefore consider the following action in the Jordan frame

$$S_{J,MCI} = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} R + \frac{1}{2} \xi \text{Tr} [MM^\dagger] R + \mathcal{L}_{MCI} \right]. \quad (4.51)$$

The non-minimally coupled term in the Lagrangian corresponds at the fundamental level to a four-fermion interaction term coupled to the Ricci scalar in the following way:

$$\frac{\xi}{2} \frac{\langle QQ \rangle^\dagger QQ}{\Lambda_{ECI}^4} R, \quad (4.52)$$

with $\Lambda_{ECI} \geq 4\pi v$ a new high energy scale where this operator is supposed to be generated via some new dynamics, lets call it "Extended Conformal Inflation" (ECI) dynamics. This may be thought of as the equivalent of "Extended Technicolor" (ETC) which explains the origin of fermion masses in Technicolor models. In this framework the SM fermion mass terms appear, in a low energy effective description, as four-fermion operators, suppressed by the scale Λ_{ETC}

$$m_{\text{SM fermion}} \sim g_{ECT}^2 \frac{\langle QQ \rangle_{\Lambda_{ECI}}}{\Lambda_{ECT}^2}, \quad (4.53)$$

Where the subscript indicates that the operator is evaluated at the ECT scale. Now back to Inflation. We do not know the details of this new ECI sector, however, they are not relevant for the present discussion. Using the renormalization group equation for the chiral condensate we expect

$$\langle QQ \rangle_{\Lambda_{ECI}} \sim \left(\frac{\Lambda_{ECI}}{\Lambda_{MCI}} \right)^\gamma \langle QQ \rangle_{\Lambda_{MCI}}, \quad (4.54)$$

Where $\Lambda_{MCI} = 4\pi v$ and γ is the anomalous scaling dimension of the operator. We assumed the underlying theory to be near conformal in the energy range $\Lambda_{MCI} \leq \mu \leq \Lambda_{ECI}$ and therefore γ is almost constant. If the fixed value is $\gamma \sim 2$ the explicit dependence on the Λ_{ECI} disappears since $M \sim \langle QQ \rangle_{\Lambda_{MCI}} / \Lambda_{MCI}^2$. In other words for γ around two the ECI dynamics decouples from the lower energy inflationary physics.

The inflaton is identified with the field σ . The other scalars are the nine goldstone bosons, (Π^a) with $a = 1 \dots 9$, which we assume to become the longitudinal degrees of freedom of the conveniently gauged $SU(4)$ flavor symmetry. This is expected to break spontaneously to $SO(4)$ via the dynamically generated techni-fermionic condensate. The remaining composite scalars Θ and $\tilde{\Pi}^a$ are massive, and for (near) conformal field theories, expected to be heavier than σ . Therefore it is sensible to consider the σ in isolation. Moreover, at large number of colors we are guaranteed that the inflaton has

a narrow width and therefore decoupled from the rest of the strongly coupled states making its effective description robust. However, we will not limit our analysis only to the large N limit.

The relevant composite inflaton effective action reads:

$$\mathcal{S}_{\text{J,MCI}} = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} \Omega^2 R - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma + \frac{m^2}{2} \sigma^2 - \frac{\kappa}{4} \sigma^4 \right], \quad (4.55)$$

where

$$\Omega^2 = \left(\frac{M_P^2 + \xi \sigma^2}{M_P^2} \right), \quad \text{and} \quad \kappa = (\lambda + \lambda' - \lambda''). \quad (4.56)$$

We proceed as in section 3 by applying the conformal transformation $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ and analyze Inflation in the Einstein frame using the large field approximation $\sigma \gg M_P \sqrt{\xi}$. The results are exactly the same as in section 3.1.2. In particular we find that

$$\epsilon_V \simeq \frac{4}{3} \frac{M_P^4}{\xi^2 \sigma_{\text{end}}^4} \Rightarrow \sigma_{\text{end}} \simeq \left(\frac{4}{3} \right)^{1/4} \frac{M_P}{\sqrt{\xi}} \simeq 1.07 \frac{M_P}{\sqrt{\xi}}. \quad (4.57)$$

The number of e-foldings during inflation is

$$N_* \simeq \frac{3\xi}{4M_P^2} (\sigma_*^2 - \sigma_{\text{end}}^2), \quad (4.58)$$

which combined with (4.57) allows us to write:

$$\sigma_* = \sqrt{\left(\frac{4N_*}{3} + (1.07)^2 \right)} \frac{M_P}{\sqrt{\xi}}. \quad (4.59)$$

Setting $N_* = 60$ we get

$$\sigma_* \sim 9 \frac{M_P}{\sqrt{\xi}}. \quad (4.60)$$

To generate the proper amplitude of the density perturbations the potential must satisfy the normalization condition (2.105)

$$\frac{U_*}{\epsilon_{V_*}} \simeq (0.0269 M_P)^4, \quad (4.61)$$

We therefore deduce

$$\xi = \frac{N_*}{(0.0269)^2} \sqrt{\frac{\kappa}{3}} \sim 48000 \sqrt{\kappa}. \quad (4.62)$$

For a strongly coupled theory we expect κ to be of the order unity and therefore $\xi \sim 48000$. This analysis resembles very closely the one for the SM Higgs inflation, except

that our effective theory for the composite inflaton cannot be utilized for arbitrary large value of scalar field. The effective theory is valid for:

$$\sigma < 4\pi v , \quad (4.63)$$

implying

$$v > \frac{9 M_P}{4\pi \sqrt{\xi}} \sim 0.81 \times 10^{16} \text{ GeV} . \quad (4.64)$$

Where we inserted the reduced Planck mass of 2.44×10^{18} GeV. This phenomenological constraint on v forbids the identification of the composite inflaton with the composite Higgs. This lower bound on the scale of composite inflation arises from having assumed the effective theory to be valid during the inflationary period. This bound may be weakened if we consider directly the underlying strongly coupled gauge theory, however, this is beyond the scope of this initial investigation.

TENSOR MODES FROM QUANTUM CORRECTED POTENTIALS

This chapter is based on work published in [4].

In this chapter we consider small corrections on top of the quartic potential. These may be thought of as arising from quantum corrections, which typically lead to a potential which carries a non-integer power of the field. On general grounds any renormalizable field theory will receive quantum corrections to the potential. One can think of the E. Weinberg and Coleman perturbative quantum corrections to the classical scalar potential of any field theory as a simple example of these type of corrections [137, 138]. We phenomenologically characterize these corrections to the ϕ^4 theory by introducing a real parameter γ as follows:

$$V_{eff} = \lambda \phi^4 \left(\frac{\phi}{\Lambda} \right)^{4\gamma}, \quad (5.1)$$

with Λ a given energy scale. Of course, model by model, one can compute the specific potential as in [139]. Nevertheless we will see that it is possible to provide useful information on a large class of models corresponding to different values of γ using this simple approach. We delay further discussion about the origin of these corrections until section 6.3, where we consider how they may be thought to arise in the context of Starobinsky Inflation.

For completeness we analyze the cases in which ϕ couples both minimally and non-minimally to gravity. In the previous sections we have seen that the presence of the non-minimal coupling term $\xi\phi^2 R$ flattens the quartic potential in the Einstein frame

and gives very small values of the tensor-to-scalar ratio r . However, in this chapter we find that even small corrections to the quartic potential significantly shift r towards higher values and that this result is largely independent of the number of e-foldings and hence the reheating temperature. Originally we compared our findings with the BICEP2 results, which indicated the presence of primordial tensor modes [124]. In this section we compare only with the Planck2015 results and note that, independently on the validity of the BICEP2 results, it is interesting to know whether quantum corrected potentials can account for nonzero tensor modes.

5.1 Coupling to gravity and slow-roll inflation

We consider the action of a scalar field non-minimally coupled to gravity:

$$S_J = \int d^4x \sqrt{-g} \left[\frac{M_P^2 + \xi \phi^2}{2} R - g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V_{eff}(\phi) \right]. \quad (5.2)$$

We follow the procedure outlined chapter 3 and assume that inflation takes place in the large field regime $\phi \gg \frac{M_P}{\sqrt{\xi}}$. In this regime the canonical Einstein frame field is (3.13)

$$\chi \simeq \sqrt{6} M_P \ln \frac{\sqrt{\xi} \phi}{M_P}, \quad \phi \gg \frac{M_P}{\sqrt{\xi}}, \quad (5.3)$$

And the Einstein frame potential takes the form

$$\begin{aligned} U(\chi) = \Omega^{-4} V(\phi(\chi)) &= \frac{M_P^4}{(M_P^2 + \xi \phi^2)^2} \lambda \phi^4 \left(\frac{\phi}{\Lambda} \right)^{4\gamma} \\ &= \underbrace{\frac{\lambda M_P^4}{\xi^2} \left(1 + \exp \left[\frac{-2\chi}{\sqrt{6} M_P} \right] \right)^{-2}}_{\phi^4\text{-Inflation}} \underbrace{\left(\frac{M_P}{\sqrt{\xi} \Lambda} \right)^{4\gamma} \exp \left[\frac{4\gamma \chi}{\sqrt{6} M_P} \right]}_{\text{Corrections from } \gamma}. \end{aligned} \quad (5.4)$$

The underbraced ‘ ϕ^4 -Inflation’-term refers to the potential one would obtain by setting $\gamma = 0$, that is, non-minimally coupled ϕ^4 -Inflation. As we have seen large field asymptotic flatness of this term makes non-minimally coupled ‘ ϕ^4 -Inflation’ viable. However, quantum corrections which we parametrize by γ , may spoil this feature of the potential.

As outlined in chapter 3 it is straightforward to analyze inflation in the Einstein frame. We proceed by the standard slow-roll approach and compute the slow-roll parameters in the large field limit using the field χ and its potential $U(\chi)$. These may be

expressed in terms of the Jordan frame field ϕ by reinserting (5.3):

$$\epsilon_V = \frac{M_p^2}{2} \left(\frac{dU/d\chi}{U} \right)^2 \sim \underbrace{\frac{4M_p^4}{3\xi^2\phi^4}}_{\phi^4\text{-Inflation}} + \frac{8M_p^2}{3\xi\phi^2}\gamma + \frac{4}{3}\gamma^2 \quad (5.5)$$

$$\eta_V = M_p^2 \left(\frac{d^2U/d\chi^2}{U} \right) \sim \underbrace{-\frac{4M_p^2}{3\xi\phi^2} + \frac{12M_p^4}{3\xi^2\phi^4}}_{\phi^4\text{-Inflation}} + \frac{16M_p^2}{3\xi\phi^2}\gamma + \frac{8}{3}\gamma^2. \quad (5.6)$$

So far γ can assume any value and the only approximation made is the one in (5.3).

Setting $\epsilon_V(\phi_{\text{end}}) \simeq 1$ we find

$$\phi_{\text{end}} = \frac{2M_p}{\sqrt{\xi}} \frac{1}{\sqrt{\sqrt{12} - 4\gamma}} = (1.07 + 0.32\gamma) \frac{M_p}{\sqrt{\xi}} + \mathcal{O}(\gamma^2) \quad \text{for } \xi \gg 1. \quad (5.7)$$

From the first identity we derive the universal bound:

$$\gamma < \frac{\sqrt{3}}{2}. \quad (5.8)$$

Assuming the quantum corrections to be perturbative, in the underlying inflaton theory, we can expand for small values of γ and obtain the right-hand side of (5.7). We set $\xi \gg 1$ since $\xi \sim 10^4$ is required to generate the proper amplitude of density perturbations. As we have seen, this is a general feature of non-minimally coupled theories of single-field inflation [3, 2, 1, 39, 143, 144]. A relatively small ξ can be realized but it requires an extremely small λ as noted in [145]. We will quantify this relation between ξ and λ later, see equation (5.18).

The number of e-folds N_* before the end of inflation is

$$\begin{aligned} N_* &= \frac{1}{M_p^2} \int_{\chi_{\text{end}}}^{\chi_*} \frac{U}{dU/d\chi} d\chi = \frac{3}{2} \int_{\phi_{\text{end}}}^{\phi_*} \frac{1 + \frac{\xi\phi^2}{M_p^2}}{1 + \gamma \left(1 + \frac{\xi\phi^2}{M_p^2}\right)} \frac{1}{\phi} d\phi \\ &\sim \frac{3}{4\gamma} \ln \left[1 + \gamma \frac{\xi\phi^2}{M_p^2} \right]_{\phi_{\text{end}}}^{\phi_*}. \end{aligned}$$

Combining the previous equation with (5.7) we deduce

$$\begin{aligned} \phi_* &\sim \sqrt{\frac{1}{\gamma} \left(\exp \frac{4\gamma N_*}{3} - 1 \right)} \frac{M_p}{\sqrt{\xi}} \quad (5.9) \\ &= (1.16 + 0.385(\gamma N) + 0.107(\gamma N)^2 + \mathcal{O}(\gamma N)^3) \sqrt{N_*} \frac{M_p}{\sqrt{\xi}} \\ &= \left(\underbrace{8.94}_{\phi^4\text{-Inflation}} + 179\gamma + 2980\gamma^2 + \mathcal{O}(\gamma^3) \right) \frac{M_p}{\sqrt{\xi}} \quad \text{for } \xi \gg 1, N_* = 60. \end{aligned}$$

We expanded in γ to clarify how the result deviates from ϕ^4 -inflation. It is evident that the γ -correction push inflation to higher field values. An expansion is justified only for tiny values of γ .

5.1.1 Unitarity test via Inflaton-Inflaton scattering

Next, we turn to the constraints set by tree-level unitarity of inflaton-inflaton scattering. We follow the approach outlined in section 4.1.2 and consider the Einstein frame action in the large field regime:

$$\mathcal{S}_E = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_p^2 g^{\mu\nu} R_{\mu\nu} - 6 \frac{M_p^2}{\phi^2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{M_p^4}{\xi^2} \lambda \left(\frac{\phi}{\Lambda} \right)^{4\gamma} \right]. \quad (5.10)$$

Violation of tree-level unitarity of the scattering amplitude, concerns fluctuations of the inflaton around its classical homogeneous background:

$$\phi(\vec{x}, t) = \phi_c(\vec{x}, t) + \delta\phi(\vec{x}, t). \quad (5.11)$$

In first approximation we neglect the time dependence of the background during the inflationary period and write $\phi_c(t) = \phi_c$. To estimate the cutoff we expand the kinetic and potential term around the background. The kinetic term for the fluctuations then takes the form

$$\frac{36M_p^2}{\phi_c^2 \left(1 + \frac{\delta\phi}{\phi_c}\right)^2} (\partial\delta\phi)^2 = \frac{6M_p^2}{\phi_c^2} (\partial\delta\phi)^2 \sum_{n=0}^{\infty} (n+1) \left(\frac{-\delta\phi}{\phi_c} \right)^n. \quad (5.12)$$

The first term of the series, i.e. the kinetic term for a free field, may be canonically normalized by a field redefinition

$$\frac{\delta\phi}{\phi_c} = \frac{\delta\tilde{\phi}}{\sqrt{12}M_p}. \quad (5.13)$$

The kinetic term then takes the form

$$T = \frac{1}{2} (\partial\delta\tilde{\phi})^2 \sum_{n=0}^{\infty} (n+1) \left(\frac{-\delta\tilde{\phi}}{\sqrt{12}M_p} \right)^n. \quad (5.14)$$

Expanding the potential, the leading higher order operators take on the same form

$$\frac{\gamma\lambda M_p^4}{\xi^2} \left(\frac{\phi_c}{\Lambda} \right)^{4\gamma} \left(\frac{\delta\tilde{\phi}}{\sqrt{12}M_p} \right)^n. \quad (5.15)$$

From these expression, we determine the cutoff of the theory which controls the physical suppression of higher order operators:

$$\Lambda_{UC} \sim \sqrt{12}M_P. \quad (5.16)$$

This implies that the theory is valid, from the unitarity point of view, till the Planck scale.

5.1.2 Phenomenological constraints

We are now equipped to confront the inflationary potential with experiments. We start by considering the constraints set by the observed amplitude of density perturbation A_s ,

$$A_s = \frac{1}{24\pi^2 M_P^4} \left| \frac{U^*}{\epsilon^*} \right| = 2.2 \cdot 10^{-9} \Leftrightarrow \left| \frac{U^*}{\epsilon^*} \right| = (0.0269 M_P)^4. \quad (5.17)$$

For a minimally coupled quartic potential this imposes a constraint on the self coupling, which must be unnaturally small: $\lambda \sim 10^{-13}$ [147]. In the present case (5.17) yields a relation between ξ , λ and γ . We can self-consistently solve for $\xi \gg 1$

$$\xi = \left(\frac{3\lambda}{4 \cdot 0.0269^4} \left(\frac{M_P}{\Lambda} \right)^{4\gamma} \frac{\left(\exp \frac{4\gamma N}{3} - 1 \right)^2 \left(\frac{1}{\gamma} \exp \frac{4\gamma N}{3} - \frac{1}{\gamma} \right)^{2\gamma}}{\gamma^2 \left(\exp \frac{4\gamma N}{3} + \gamma \right)^2} \right)^{\frac{1}{2+2\gamma}}. \quad (5.18)$$

The resulting constraint is plotted in Fig. 5.1. The magnitude of ξ needed to produce the observed amplitude of scalar perturbations decreases for increasing γ to a certain point from which it increases monotonically. Expanding in γ and setting $N_* = 60$ and $\lambda = \frac{1}{4}$ the relation takes on a more readable form:

$$\begin{aligned} \xi = & \underbrace{48000}_{\phi^4\text{-Inflation}} + (-2.27 \cdot 10^6 + 9.57 \cdot 10^4 \ln \frac{M_P}{\Lambda})\gamma \\ & + \left(7.46 \cdot 10^7 - 4.63 \cdot 10^6 \ln \frac{M_P}{\Lambda} + 9.57 \cdot 10^4 \ln \left(\frac{M_P}{\Lambda} \right)^2 \right) \gamma^2 + \mathcal{O}(\gamma^3). \end{aligned}$$

Next we consider the scalar spectral index n_s and the tensor-to-scalar power ratio r

$$r = 16\epsilon_* = \underbrace{\frac{64M_P^4}{3\xi^2\phi_*^4}}_{\phi^4\text{-Inflation}} + \frac{128M_P^2}{3\xi\phi_*^2}\gamma + 8\gamma^2, \quad (5.19)$$

$$n_s = 2\eta_* - 6\epsilon_* + 1 = 1 - \underbrace{\frac{8M_P^2}{3\xi\phi_*^2}}_{\phi^4\text{-Inflation}} - \frac{16M_P^2}{3\xi\phi_*^2}\gamma - \frac{8\gamma^2}{3}. \quad (5.20)$$

Using (5.9) and expanding in γ we obtain

$$r = \underbrace{\frac{11.8}{N_*^2}}_{\phi^4\text{-Inflation}} + \frac{16.3\gamma}{N_*} + 8.73\gamma^2 + \mathcal{O}(\gamma^3), \quad \text{for } \xi \gg 1 \quad (5.21)$$

$$= \underbrace{0.0033}_{\phi^4\text{-Inflation}} + 0.27\gamma + 8.73\gamma^2 + \mathcal{O}(\gamma^3) \quad \text{for } \xi \gg 1, N_* = 60. \quad (5.22)$$

$$n_s = \underbrace{1 - \frac{1.98}{N_*}}_{\phi^4\text{-Inflation}} + \left(1.30 - \frac{3.96}{N_*}\right)\gamma + (-0.0699 - 0.262N_*)\gamma^2 + \mathcal{O}(\gamma^3) \quad \text{for } \xi \gg 1$$

$$= \underbrace{0.967}_{\phi^4\text{-Inflation}} + 1.23\gamma - 15.8\gamma^2 + \mathcal{O}(\gamma^3) \quad \text{for } \xi \gg 1, N_* = 60. \quad (5.23)$$

The expansions show that the (r, n_s) -values are sensitive to even small corrections. This sensitivity arises since the asymptotic flatness of the potential in the large field regime $\phi \gg M_P \sqrt{\xi}$ is lost when the γ -term is present. See also figure 5.2 where we compare with Planck2015 results [13]. We find that the spectral index n_s is sensitive to corrections at small γ . For example, the model cross outside the 95% CL region at $\gamma \simeq 0.016$ and $N_* = 60$. For larger, but still small values of γ , the tensor-to-scalar ratio is sensitive to corrections. For example we find that r values in the interval $r = [0.1, 0.2]$ are achieved for γ -values in the interval $\gamma \simeq [0.068, 0.097]$, $N_* = 60$. Note also that for $r > 0.1$ which

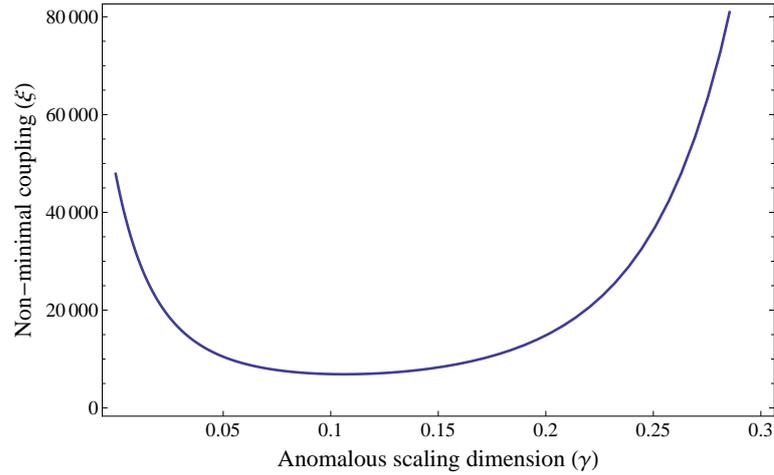


Figure 5.1: Here we plot (5.18) as a function of γ for $\xi \gg 1$, $N = 60$, $\frac{M_P}{\Lambda} = 1$ and $\lambda = \frac{1}{4}$. As γ increases from zero the magnitude of ξ needed to produce the correct amount of scalar perturbations decreases. The minimum is obtained at $\gamma \sim 0.1$. For this value of γ , the model produce a significant amount of tensor modes.

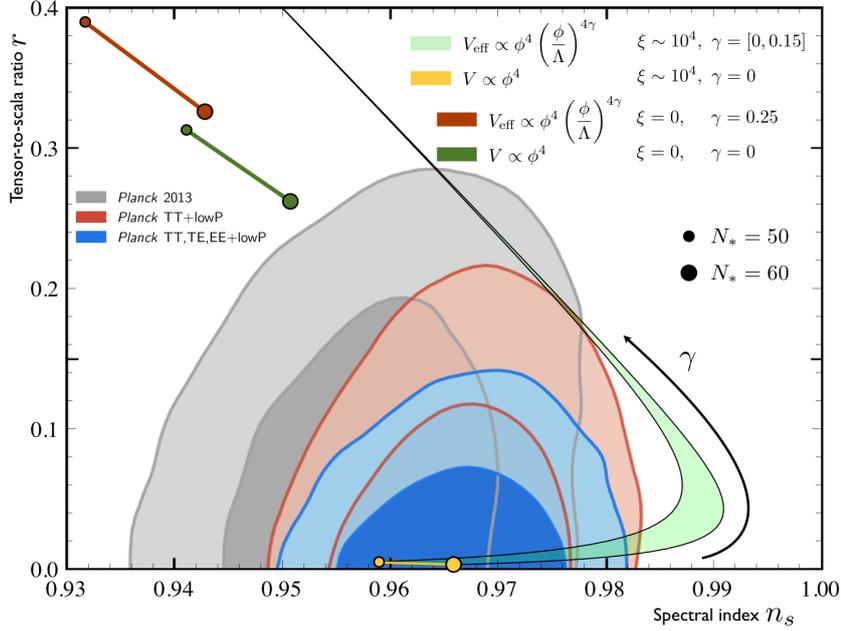


Figure 5.2: Comparison with Planck results in the (r, n_s) -plane [13]. The corrected potential with non-minimal coupling gives the light green region. This region is obtained by letting N_* and γ span the intervals $N_* = [50, 60]$ and $\gamma = [0, 0.15]$. Note that we used the full dependence on γ , derived in (5.9), (5.19), (5.20), to generate this region. For small values of γ the spectral index is sensitive to corrections. For example we find that the model cross outside the 95% CL region at $\gamma \simeq 0.016$ and $N_* = 60$. For larger, but still small values of γ , the tensor-to-scalar ratio is sensitive to corrections. For example we find that r values in the interval $r = [0.1, 0.2]$ are achieved for γ -values in the interval $\gamma \simeq [0.068, 0.097]$, $N_* = 60$. Note also that for $r > 0.1$ which corresponds to $\gamma > 0.068$, the predictions are largely independent of the number of e-folds N_* .

corresponds to $\gamma > 0.068$, the predictions are largely independent of the number of e-folds N_* and hence the reheating temperature.

For reference we summarize the results one would obtain if the model were minimally coupled to gravity. V_{eff} then produce standard minimally coupled power-law inflation. Within the slow-roll approximation, the scalar spectral index and the tensor-to-scalar ratio are:

$$r = 16\epsilon_* = \frac{128M_P^2(1+\gamma)}{\phi_i^2} = \frac{16(1+\gamma)}{N+1+\gamma}. \quad (5.24)$$

$$n_s = 2\eta_* - 6\epsilon_* + 1 = 1 - \frac{8M_P^2(1+\gamma)(3+2\gamma)}{\phi_i^2} = \frac{N-2-\gamma}{N+1+\gamma}. \quad (5.25)$$

These lines are plotted in figure 5.2 for $\gamma = 0$ and $\gamma = 0.25$. Contrary to the non-minimally coupled case, we find a strong dependence on N_* for any value of γ . We also find that the results are not as dependent on γ as compared to the non-minimally

coupled model. To summarize, we have shown that the inflationary parameters r and n_s are sensitive to small corrections of the potential, which may be thought of as arising from quantum corrections. In particular, quantum corrected potentials may produce a significant amplitude of primordial tensor modes. Our analysis is sufficiently simple and general to provide useful constraints for a general class of quantum field theories that can be used to drive inflation.

INFLATION VIA MODIFIED GRAVITY

In the previous chapters we have modelled inflation by modifying the energy-momentum tensor in standard Einstein gravity. In particular we have seen that inflation occurs if the energy-momentum tensor is dominated by a vacuum like component with negative equation of state parameter $w < -1/3$. Such conditions, as well as a graceful exit from the inflationary phase, were achieved by introducing a single matter scalar field, the Inflaton ϕ . In this chapter we consider another approach where inflation is driven directly by the gravitational part of the action. This requires one to go beyond standard Einstein gravity and consider modified versions, for example in the context of $f(R)$ -theories [149]. In these theories the action is

$$S = \int d^4x \sqrt{-g} \frac{M_P^2}{2} f(R) + \int d^4x \mathcal{L}_M(g_{\mu\nu}, \psi_M), \quad (6.1)$$

Where $f(R)$ is an arbitrary function of the Ricci scalar R and \mathcal{L}_M is a matter Lagrangian which is minimally coupled to gravity. This includes the Starobinsky model of inflation [7], which is one of the earliest models of inflation. The Starobinsky model features an R^2 -term added to the Einstein-Hilbert action

$$f(R) = R + \frac{R^2}{6M^2}, \quad (6.2)$$

Where M is a new mass scale. We consider the Starobinsky model of Inflation in detail below. We begin our discussion by considering the field equations associated to the general action (6.1). These may be found by varying the action with respect to $g_{\mu\nu}$

$$F(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \square F(R) = M_P^{-2} T_{\mu\nu}^M, \quad (6.3)$$

Where $F(R) \equiv \partial f / \partial R$ and $T_{\mu\nu}^M$ is the energy-momentum tensor of the matter fields. One obtains the standard Einstein equations (1.5) by setting $f(R) = R$ and $F(R) = 1$. By taking the trace of the field equations we get

$$3\Box F(R) + F(R)R - 2f(R) = M_p^{-2} g^{\mu\nu} T_{\mu\nu}^M. \quad (6.4)$$

This reveals an extra propagating scalar degree of freedom $\psi \equiv F(R)$ as compared to standard Einstein gravity. We will soon see that this extra scalar degree of freedom may be used to drive inflation. In Einstein gravity the term $\Box F(R)$ vanishes and $R = -M_p^2 g^{\mu\nu} T_{\mu\nu}^M$ such that the Ricci scalar is determined by the matter content in the standard manner.

In the following we consider vacuum solutions with $T_{\mu\nu}^M = 0$. In section 6.3 we will consider the effects of integrating out matter fields. Also we consider flat FRW space-time with metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \quad (6.5)$$

From (1.8) the Ricci scalar is

$$R = 6(2H^2 + \dot{H}), \quad (6.6)$$

With H the Hubble constant. Since we are studying inflation we are interested in (quasi) de Sitter solutions with H and R constant. In this case the term $\Box F(R)$ vanished from the trace equation which then reads

$$F(R)R - 2f(R) = 0. \quad (6.7)$$

The model $f(R) \propto R^2$ solves this condition and gives rise to an exact de Sitter solution. We may consider this as a correction to Einstein gravity and write

$$f(R) = R + \frac{R^2}{6M^2}, \quad \Rightarrow \quad F(R) = 1 + \frac{R}{3M^2}, \quad (6.8)$$

Where M is a mass scale. Then at high R -values where the R^2 -term dominates we obtain quasi de Sitter expansion $F(R)R - 2f(R) \simeq 0$. This is the famous Starobinsky model of inflation [7, 8]. During inflation R decreases such that Inflation ends when the quadratic term becomes smaller than the linear term $R \sim M^2$. We will see this explicitly below.

To be a bit more precise we analyze Inflation via the slow-roll approximation outlined in section 1.2.1. We first insert the Starobinsky model and the FRW-metric (6.5) in the field equations (6.3) to obtain

$$\begin{aligned} \ddot{H} - \frac{\dot{H}^2}{2H} + \frac{1}{2}M^2 H &= -3H\dot{H} \\ \ddot{R} + 3H\dot{R} + M^2 R &= 0. \end{aligned} \quad (6.9)$$

The first equation is the (0,0)-component which have been inserted in the (i,i)-component to obtain the second equation. When deriving these equations it is useful to know that the FRW-metric yields

$$\square F = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu F \right) = - \left(\frac{d^2}{dt^2} + 3H \frac{d}{dt} \right) F, \quad (6.10)$$

$$\nabla_\mu \nabla_\nu F = \partial_\mu \partial_\nu F - \Gamma_{\mu\nu}^0 \dot{F}, \quad \Gamma_{00}^0 = 0, \quad \Gamma_{ij}^0 = a\dot{a}\delta_{ij}. \quad (6.11)$$

As we did earlier, we quantify slow-roll by smallness of the Hubble slow-roll parameters

$$\epsilon_H = \left| \frac{\dot{H}}{H^2} \right| \ll 1, \quad \eta_H = \left| \frac{\ddot{H}}{H\dot{H}} \right| \ll 1. \quad (6.12)$$

The first two terms in (6.9) may then be neglected. From (6.8) we find that $R \simeq 12H^2$, hence \ddot{R} can also be neglected. The slow-roll approximation then becomes

$$\dot{H} \simeq -\frac{1}{6}M^2 \quad (6.13)$$

$$3H\dot{R} + M^2R \simeq 0. \quad (6.14)$$

The first term may readily be integrated to obtain the slow-roll solution

$$H \simeq H_i - \frac{1}{6}M^2(t - t_i) \quad (6.15)$$

$$a \simeq a_i \exp \left[H_i(t - t_i) - \frac{1}{12}M^2(t - t_i)^2 \right] \quad (6.16)$$

$$R \simeq 12H^2 - M^2, \quad (6.17)$$

Where i denotes the initial conditions. It can be shown that the slow-roll trajectory is an attractor in phase space [148] and hence the further evolution is largely independent on the Initial conditions, as we discussed in section 1.2.2. Accelerated expansion occurs as long as the slow-roll parameter ϵ_H is smaller than unity

$$\epsilon_H = -\frac{\dot{H}}{H^2} \simeq \frac{M^2}{6H^2}. \quad (6.18)$$

Hence inflation occurs for $H^2 > M^2$. Inflation ends when $\epsilon_H = 1$, i.e $H_{\text{end}} \simeq M/\sqrt{6}$. It follows that this corresponds to the time at which the Ricci scalar decreases to $R \sim M^2$.

6.1 Starobinsky Inflation in the Einstein frame

The $f(R)$ -theory (6.4) may be cast in a form that features a potential for the extra scalar degree of freedom which appeared above [149]. This can be done by considering the

following linear representation in terms of a new field y

$$S = \int d^4x \sqrt{-g} \frac{M_P^2}{2} [f(y) + f'(y)(R - y)] . \quad (6.19)$$

We set $T_{\mu\nu}^M = 0$ since we will insert the Starobinsky model shortly. The equation of motion for y is

$$f''(y)(R - y) = 0 . \quad (6.20)$$

If $f''(y) \neq 0$ it follows that $y = R$ and we recover the original action (6.4). By inserting the scalar degree of freedom $\psi \equiv f'(y) = F(y)$ in (6.19) the action may be expressed as

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_P^2 \psi R - V(\psi) \right], \quad V(\psi) = \frac{1}{2} M_P^2 (y(\psi)\psi - f(y(\psi))) . \quad (6.21)$$

Hence we have obtained an action for the scalar degree of freedom ψ with potential $V(\psi)$ which is equivalent to the $f(R)$ -theory. It appears to have the same form as the non-minimally coupled models we considered earlier (3.1), except that there is no kinetic term. We will discuss similarities and differences within the framework of Starobinsky inflation shortly. First we proceed by performing a conformal transformation, in the same manner as in chapter 3. To do this it is convenient to reinsert $F(R)$ and write the action in the form

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_P^2 F R - V \right]. \quad (6.22)$$

Let us briefly repeat the steps of the conformal transformation. The metric and Ricci scalar transform as

$$g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \quad R = \Omega^2 \left[\hat{R} + 6\hat{\square} \ln \Omega - 6\hat{g}^{\mu\nu} (\partial_\mu \ln \Omega)(\partial_\nu \ln \Omega) \right]. \quad (6.23)$$

The transformed action then reads

$$S = \int d^4x \sqrt{-\hat{g}} \left[\frac{1}{2} M_P^2 F \Omega^{-2} \left(\hat{R} + 6\hat{\square} \ln \Omega - 6\hat{g}^{\mu\nu} (\partial_\mu \ln \Omega)(\partial_\nu \ln \Omega) \right) - \Omega^{-4} V \right]. \quad (6.24)$$

We land in the Einstein frame where the action is linear in \hat{R} if we choose

$$\Omega^2 = F . \quad (6.25)$$

We also see that the action may be canonically normalized by the field redefinition

$$\chi = \sqrt{\frac{3}{2}} M_P \ln F . \quad (6.26)$$

Defining the Einstein frame potential $U(\chi)$ as

$$U(\chi) = \Omega^{-4}V = \frac{V}{F^2} = \frac{M_P(FR - f)}{2F^2}, \quad (6.27)$$

The action finally takes the form

$$S_E = \int d^4x \sqrt{-\hat{g}} \left[\frac{1}{2} M_P^2 R - \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - U(\chi) \right], \quad (6.28)$$

We may now follow the same steps as earlier, and analyze inflation using the Einstein frame potential within the standard slow-roll paradigm. We proceed by inserting the Starobinsky model

$$f(R) = R + \frac{R^2}{6M^2} \quad \Rightarrow \quad F(R) = 1 + \frac{R}{3M^2}. \quad (6.29)$$

The field redefinition then reads

$$\chi = \sqrt{\frac{3}{2}} M_P \ln \left(1 + \frac{R}{3M^2} \right). \quad (6.30)$$

Using this relation, the Einstein frame potential (6.27) becomes

$$U(\chi) = \frac{3M^2 M_P^2}{4} \left(1 - \exp \left[\frac{-2\chi}{\sqrt{6} M_P} \right] \right). \quad (6.31)$$

Except for the overall coefficient, this is the same as the large field limit of the quartic potential with non-minimal coupling (3.24). The two potentials coincide if we make the identification

$$M^2 = \frac{\lambda}{3\xi^2} M_P^2. \quad (6.32)$$

Hence, by using our earlier results we find that the Planck constraint on the amplitude of scalar perturbations (3.30) constrains the mass parameter M to be $M \sim 10^{-5} M_P$. The slow-roll parameters are the same as for the quartic potential since the overall coefficient of the potential drop out in the derivation

$$\epsilon_V = \frac{4}{3} \left(e^{\frac{2\chi}{\sqrt{6} M_P}} - 1 \right)^{-2} \simeq \frac{4}{3} e^{-\frac{4\chi}{\sqrt{6} M_P}} \quad (6.33)$$

$$\eta_V \simeq -\frac{4}{3} e^{-\frac{2\chi}{\sqrt{6} M_P}}. \quad (6.34)$$

Note that the similarity only holds in the large field approximation of the quartic potential. Setting $N_* = 60$, the Starobinsky model then gives the same values of r and n_s which we obtained earlier (3.31)

$$r \simeq 0.0033, \quad n_s = 0.966. \quad (6.35)$$

This is in excellent agreement with results from Planck [13].

6.2 Comparison with the quartic potential

Let us briefly touch upon the similarities of the Starobinsky model and the quartic potential with a non-minimal coupling. We follow [150] which provides a nice comparison between Higgs inflation [39] and the Starobinsky model. We begin by noting that in the linear representation which we considered earlier, the action may explicitly be written as [23]

$$S = \int d^4x \sqrt{-g} \frac{M_P^2}{2} \left[R + \frac{2R\psi}{M_P M} - \frac{6\psi^2}{M_P^2} \right] \quad (6.36)$$

$$\rightarrow \int d^4x \sqrt{-g} \frac{M_P^2}{2} \left[R + \frac{R^2}{6M^2} \right]. \quad (6.37)$$

Note that ψ now has mass dimension 2. The arrow indicates that we may obtain the Starobinsky action in the pure $f(R)$ form by integrating out ψ using its equation of motion. Consider now the action for the non-minimally coupled quartic potential

$$S_J = \int d^4x \sqrt{-g} \left[\frac{M_P^2 + \xi\phi^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\lambda}{4} \phi^4 \right]. \quad (6.38)$$

During slow-roll inflation the kinetic term is by definition negligible. The action then reads

$$S_J = \int d^4x \sqrt{-g} \frac{M_P^2}{2} \left[R + \frac{\xi\phi^2}{M_P^2} R - \frac{\lambda}{2M_P^2} \phi^4 \right]. \quad (6.39)$$

Hence the inflaton is an auxiliary field in this regime, and may be integrated out by means of its equation of motion

$$\phi^2 = \frac{\xi R}{\lambda}. \quad (6.40)$$

Inserting this in the Jordan frame action we obtain

$$S_J = \int d^4x \sqrt{-g} \frac{M_P^2}{2} \left[R + \frac{\xi^2 R^2}{2\lambda M_P^2} \right]. \quad (6.41)$$

Therefore, the non-minimally coupled quartic potential is equivalent to the Starobinsky model during inflation. If we make the identification $M^2 = \frac{\lambda}{3\xi^2} M_P^2$ it exactly coincides with the $f(R)$ -representation in (6.37). Of course, this is the same conclusion as the one we drew earlier using the Einstein frame actions. However, this representation clarifies that the equivalence arises since the kinetic term in the model with the quartic potential is negligible during inflation. We implicitly made the same approximation when we derived the Einstein frame potential for the quartic potential in section 3.1.2. The approximation was made by going to the large field regime $\phi \gg M_P \sqrt{\xi}$ in the field

redefinition (3.7) \rightarrow (3.12). It is also important to note that the non-minimal coupling ξ naturally appears in the model with quartic potential (6.38), whereas it can be absorbed in the auxiliary field in the linear representation of the Starobinsky model (6.39), (6.37). Hence ξ is redundant in the Starobinsky model. This difference arise since there is no kinetic term for the auxiliary field in the linear representation of the Starobinsky model.

Let us now consider the observables of the two models and at what level they differ. In particular we consider differences in the (r, n_s) -plane by comparing the slow-roll parameter ϵ_V of the two models. To do so, we compare the action of the quartic potential with non-minimal coupling and kinetic term (6.38) with the linear representation (6.39) of the Starobinsky model. We compute the slow-roll parameter ϵ_V for both models in the Einstein frame. The procedure is as described in chapter 3

$$\epsilon_V = \frac{1}{2} M_P \left(\frac{1}{U} \frac{dU}{d\phi} \right)^2 \left(\frac{d\phi}{d\chi} \right)^2. \quad (6.42)$$

The Einstein frame potential U is the same for the two models whereas the field redefinition $\chi(\phi)$, which is related to the kinetic term, differ. Recall from (3.7) that the field redefinition for the model with quartic potential and kinetic term is

$$\frac{d\chi}{d\phi} = \sqrt{\Omega^{-2} + \frac{3}{2} M_P^2 \left(\frac{d}{d\phi} \ln \Omega^2 \right)^2}. \quad (6.43)$$

In the Starobinsky model the term Ω^{-2} vanish. This is exactly equivalent to the large field approximation for the model with quartic potential and kinetic term. Of course, we do not perform the large field approximation here, since the slow-roll parameters would then coincide. Using (3.29) as the number of e-foldings one may find the following relation between the slow-roll parameters [150]

$$\frac{\epsilon_{V, \phi^4\text{-Inflation}}}{\epsilon_{V, \text{Starobinsky}}} = \frac{8N\xi}{1 + \frac{4}{3}N + 8N\xi} \simeq 1 - \frac{10^{-5}}{6\lambda}. \quad (6.44)$$

The difference is extremely small and we do not expect observable differences in the (r, n_s) -plane unless there is a strong dependence on model dependent post inflationary physics.

6.3 Marginally deformed Starobinsky Gravity

This section is based on work published in [5].

In the previous section we have seen that gravity itself may be responsible for inflation. This requires one to go beyond standard Einstein gravity, for example by modifying

the gravitational action via $f(R)$ -theories. In particular we have seen that Inflation occurs in the Starobinsky model. We now consider quantum-induced marginal deformations of the Starobinsky action. We parametrize the deformations by $R^{2(1-\alpha)}$, where α is a positive parameter smaller than one half. The Starobinsky model is recovered for $\alpha = 0$. As we shall see, deformations of the Starobinsky action may lead to a sizeable amplitude of primordial tensor modes, even for small α . Originally we compared the deformed model with the BICEP2 results [124] which indicated the presence of primordial tensor modes [124]. In this section we compare only with the Planck results and note that independently on the validity of the BICEP2 results, it is interesting to know how deformations of the Starobinsky model alter the inflationary observables. In particular we argue that deformations may arise if a matter theory of particle physics is embedded in the gravitational theory.

6.3.1 Motivation

According to [126], cosmology can be used qualitatively to establish the quantization of gravity. In fact, by combining cosmological observations with an effective field theory (EFT) treatment of gravity [127, 128] one can start estimating the parameters entering gravity's effective action. An actual discovery of primordial tensor modes can therefore be used to determine these parameters at the inflationary scale, which may turn out to be close to the grand unification energy scale.

To lowest order, the effective action for gravity can be parametrized as

$$S = \int d^4x \sqrt{-g} \left[-\frac{M_p^2}{2} R + a_0 R^2 + a_1 R^3 + c_0 C^2 + e_0 E + \dots \right].$$

Beyond an expansion in the Ricci scalar R , we formally included the Weyl conformal tensor C^2 and the Euler four dimensional topological term E . However we can drop E since it is a total derivative. Furthermore when gravity is quantized around the Friedmann Lemaitre Robertson Walker metric the Weyl terms are sub-leading since the geometry is conformally flat [130]. We are left with an $f(R)$ form of the EFT. In particular the first two terms reproduce the Starobinsky model. Higher powers of R , C^2 and E are naturally suppressed by the Planck mass scale. If inflation occurs at energy scales much below the Planck scale the EFT is accurate. We must, however, take into account also marginal deformations including, for example, logarithmic corrections to the action above. Because of the similarity between the EFT description of gravity and the chiral Lagrangian for Quantum Chromo Dynamics we expect the quantum-induced logarithmic corrections

to play a fundamental role for a coherent understanding of low energy gravitational dynamics at the inflationary scale. This is exactly what happens in hadronic processes involving pions at low energies.

6.3.2 Inflation In the modified Starobinsky model

We encode these ideas as deformations of the Starobinsky action

$$S = \int d^4x \sqrt{-g} \left[\frac{M_p^2}{2} R + a M_p^{4\alpha} R^{2(1-\alpha)} \right], \quad (6.45)$$

Where a is now a dimensionless parameter. This may be turned into the form (6.37) by replacing a with the dimensionfull parameter $a \rightarrow \frac{1}{12M^2 M_p^2}$ and $\alpha = 0$, however we will follow [5] and use the notation in the action above. The equivalence between the Starobinsky model and non-minimally coupled large field ϕ^4 -inflation, allows us to map the deformed Starobinsky action into the model with potential $\lambda \left(\frac{\phi}{\Lambda}\right)^{4\gamma}$, which we considered in section 5.1

$$S_I = \int d^4x \sqrt{-g} \left[\frac{M_p^2 + \xi \phi^2}{2} R - g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \lambda \phi^4 \left(\frac{\phi}{\Lambda}\right)^{4\gamma} \right]. \quad (6.46)$$

During Inflation the kinetic term is negligible, which as we have seen, corresponds to the large field regime $\phi \gg M_p \sqrt{\xi}$ with large non-minimal coupling ξ . The action then reads

$$S_I = \int d^4x \sqrt{-g} \left[\frac{M_p^2 + \xi \phi^2}{2} R - \lambda \phi^4 \left(\frac{\phi}{\Lambda}\right)^{4\gamma} \right]. \quad (6.47)$$

This is equivalent to the linear representation of the deformed Starobinsky action (6.45) if we make the following identifications

$$\alpha = \gamma/(1 + 2\gamma), \quad a^{1+2\gamma} = \left(\frac{\xi}{4} \frac{1 + 2\gamma}{1 + \gamma}\right)^{2(1+\gamma)} \frac{1}{\lambda(1 + 2\gamma)}. \quad (6.48)$$

These results are obtained straightforwardly by following the steps outlined in section 6.2. As we have seen ξ is redundant in the linear representation of the Starobinsky model, however we will retain the explicit dependence on ξ to ease the comparison between the two models. The slow-roll analysis is the same as in section 5.1. It leads to the Einstein frame potential

$$U(\chi) = \underbrace{\frac{\lambda M_p^4}{\xi^2} \left(1 + \exp\left[\frac{-2\chi}{\sqrt{6}M_p}\right]\right)^{-2}}_{\phi^4\text{-Inflation}} \underbrace{\xi^{-2\gamma} \exp\left[\frac{4\gamma\chi}{\sqrt{6}M_p}\right]}_{\text{Corrections from } \gamma}, \quad (6.49)$$

As well as the tensor-to-scalar ratio and scalar spectral index

$$r = \underbrace{0.0033}_{\phi^4\text{-Inflation}} + 0.27\gamma + 8.73\gamma^2 + \mathcal{O}(\gamma^3), \quad N_* = 60. \quad (6.50)$$

$$n_s = \underbrace{0.967}_{\phi^4\text{-Inflation}} + 1.23\gamma - 15.8\gamma^2 + \mathcal{O}(\gamma^3), \quad N_* = 60. \quad (6.51)$$

The underbraced ϕ^4 -terms refers to the potential one would obtain by setting $\gamma = 0$ and the Starobinsky model. The expansions show that the (r, n_s) -values of the Starobinsky model are sensitive to even small corrections in γ (or equivalently α). In particular we find that deformations of the Starobinsky action may lead to primordial tensor modes.

We argued in section section 6.2 that the Starobinsky model and the non-minimally coupled quartic potential with kinetic term are probably indistinguishable in the (r, n_s) -plane. Note that the same argument holds for the deformed Starobinsky model and the deformed quartic potential with a kinetic term, since the models have the same Einstein frame potential in the large field limit.

In Fig. 6.1 generic modifications of the Starobinsky model are confronted with planck data (The plot is of course the same as Fig. 5.2). We observe that cosmology may constrain the deformation parameter α , and as we will show shortly, α holds information regarding the generic particle content embedded in this gravity model of inflation.

6.3.3 Field theoretical approach to quantum gravity

We now argue that these marginal deformations, expected from a purely phenomenological standpoint, arise naturally within a field-theoretical approach to quantum gravity. To gain insight we start by expanding (6.45) in powers of α and write

$$S_J \simeq \int d^4x \sqrt{-g} \left\{ \frac{M_p^2}{2} R + aR^2 \left[1 - 2\alpha \log \left(\frac{R}{M_p^2} \right) \right] + \mathcal{O}(\alpha^2) \right\}. \quad (6.52)$$

The logarithmic term is reminiscent of what one would obtain via trace-log evaluations of quantum corrections. There are several possible sources for these corrections. They may arise for example by integrating out matter fields, or they can arise directly from gravity loops. To sum-up the entire series of logarithmic corrections, and hence recover the $R^{2(1-\alpha)}$, we expect that a renormalization group improved computation is needed. This suggests that we would be able to determine α if a more fundamental theory was at our disposal. In the absence of a full theory of quantum gravity we start here by comparing different predictions for the coefficient of the logarithmic term in (6.52) stemming out from

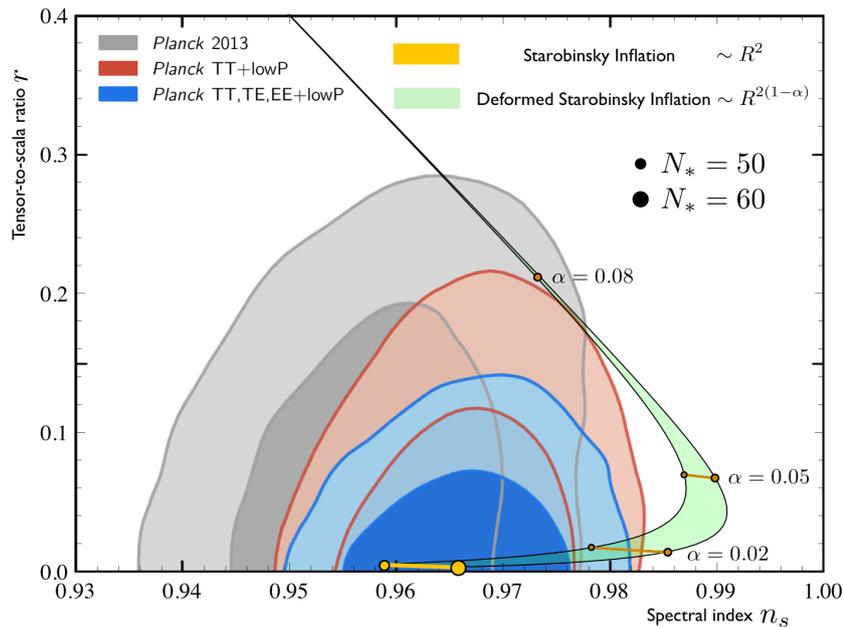


Figure 6.1: Comparison with Planck results in the (r, n_s) -plane [13]. The marginally deformed Starobinsky model gives the light green region. This region is obtained by letting N_* and α span the intervals $N_* = [50, 60]$ and $\alpha = [0, 0.15]$. We find that the r and n_s depends sensitively on the value of α , which is related to the microscopic theory dictating the trace-log quantum corrections.

1. Integrating out minimally coupled non-interacting N_S real scalar fields [133] (only non-conformal invariant matter contributes).
2. Gravity corrections via the effective field theory (EFT) approach [127, 128, 129].
3. gravity corrections within higher derivative gravity (HDG) [134].

For dimensional reasons these corrections can be parametrized by an $a(R/\mu^2)R^2$ term, where a is now a function of R/μ^2 , with μ the renormalization scale. Explicit computations via heat kernel methods show [135] that leading order quantum fluctuations will induce a logarithmic form for a as in (6.52). This fact alone immediately shows the link between the exponent α and the coefficient of the beta function related to the coupling of the R^2 term, as a scale derivative with respect to the mass scale in (6.52) shows. But we can give a better argument noticing that, because a depends on the ratio R/μ^2 , we have $2R\partial_{Ra} = -\mu\partial_{\mu}a$ and one can determine the R dependence once the beta function, with respect to μ , of a is known. Non-local $R^2 \log(-\square/\mu^2)$ quantum corrections can also be derived in a similar way [127]. To the lowest order the beta function is $\mu\partial_{\mu}a = \frac{C}{(4\pi)^2}$ with C a constant depending on the source of quantum corrections considered. After an

RG improvement, the equation for a reads

$$R \partial_R a = -\frac{C}{2(4\pi)^2} a. \quad (6.53)$$

The improvement is related to the appearance of a factor $a(R/\mu^2)$ on the right hand-side of the equation above. If one sets $a(R/\mu^2) = 1$ on the right-hand side, we only obtain the first logarithmic correction of (6.52). Using (6.53) we construct the log-resummed solution

$$a(R) = a(R_0) \left(\frac{R}{R_0} \right)^{-\frac{C}{2(4\pi)^2}}. \quad (6.54)$$

Here $R_0 = \mu_0^2$ is a given renormalization scale. We therefore have $\alpha = \frac{C}{4(4\pi)^2}$ and the constant a in (6.45) is $a(R_0)$. If $C > 0$ this would naturally lead to a positive α . An explicit evaluation of C gives [134, 133]:

$$\begin{aligned} C &= \frac{N_S}{72} && \text{minimally coupled scalars} \\ C &= \frac{1}{4} && \text{EFT gravity} \\ C &= \frac{5}{36} && \text{HDG.} \end{aligned} \quad (6.55)$$

Remarkably we deduce a *positive* exponent regardless of the underlying theory used to determine the associated quantum corrections to the gravitational action. Massive particles (we consider scalars of mass m for simplicity) lead to the beta function $\mu \partial_\mu a = \frac{C}{(4\pi)^2} (1 + m^2/\mu^2)^{-1}$ [134]. When the renormalisation scale is taken to be the Planck mass the effect of the mass term is negligible. Smaller renormalisation scales generally tend to reduce the value of C and thus of α , but in particular they do not affect its sign.

From (6.55) we deduce that quantum gravitational contributions can account, at most, for a 3% increase in r as compared to the original Starobinsky model. Therefore any larger value of r can only be generated by adding matter corrections. This in turn can be used to constrain particle physics models minimally coupled to $f(R)$ gravity. Furthermore, as it is evident from Fig 6.1, for small r the spectral index (n_s) depends strongly on the particular value of α . For example we find that if $N_s \sim 90$ or higher, the contour cross outside the one sigma confidence level provided by Planck. This corresponds to $\alpha \sim 0.02$. To exemplify our results further, we may compare this with popular models of grand unification (GUT) such as minimal SU(5) that features 34 scalars and (non)minimal SO(10) featuring (297) 109 scalars. It is clear that only models with a low content of scalars are preferred by current experiments. Values of r around and above 0.2 can be achieved only by allowing for the presence of thousands of scalars. This corresponds to the upper part of Fig. 6.1.

To conclude, we have found that if inflation is driven by an $f(R)$ theory of gravity, a natural form for this function is the marginally deformed Starobinsky action provided in (6.45) with a positive α . The size of α is related to the microscopic theory dictating the trace-log quantum corrections. This form can be tested by current and future experimental results and constitutes a natural generalization of the original Starobinsky action.

CONCLUSIONS

In this work we have reviewed the theory of inflation by first introducing inflation as a solution to the flatness and horizon problems of pre-inflationary cosmology. We have derived the conditions needed for inflation, and modeled inflation by means of a single scalar field within the slow-roll paradigm. Secondly we have considered perturbations on top of the homogeneous background solution. We have seen that the primordial power spectra of scalar and tensor perturbations, as predicted by inflation are nearly scale invariant. In addition the statistical properties are very well described by a gaussian distribution, if inflation is driven by a single canonical scalar field. These properties are in excellent agreement with observations.

We considered models of inflation which feature a non-minimal coupling to gravity. We introduced the non-minimal coupling by doing large field inflation on a quartic potential and we outlined the procedure for obtaining slow-roll results in the Einstein frame. We found that the presence of a large non-minimal coupling lowers the tensor-to-scalar ratio as compared to the minimally coupled case, a feature which is favored by current experiments. We also found that it alleviates the problem of tiny values of the inflaton self-coupling if the non-minimal coupling is very large $\xi \sim 10^4$.

Next we presented two models of composite inflation, in which the inflaton emerges as a composite scalar in a low energy effective theory description of a strongly interacting gauge theory, free from fundamental scalars. We considered inflation from Glueballs of a pure Yang-Mills theory and inflation from a Minimal Walking Technicolor - like theory. Both models were able to produce a successful inflationary phase in agreement

with data. Also we found the compositeness scale of the underlying theory to be of the order of the grand unification scale in both cases. For the technicolor - like scenario, we investigated whether it was possible for the composite scalar to serve both as a composite inflaton and a composite Higgs. It turned out not to be possible within our framework.

We further analyzed inflation from a quartic potential with also a non-minimal coupling to gravity, by considering small corrections on top of the potential. We found that even small corrections shift the tensor-to-scalar ratio r significantly towards higher values, and hence that quantum corrected potentials may account for a sizable amplitude of primordial tensor modes. Next we considered the Starobinsky model of inflation and described how it is connected to matter scalar field models with non-minimal coupling, and at what level they differ. We considered quantum-induced marginal deformations of the Starobinsky action, and found that such deformations significantly shift the predicted tensor-to-scalar ratio towards higher values. At last we discussed sources for these corrections and argued that if inflation is driven by an $f(R)$ -theory of gravity, the inflationary observables r and n_s may constrain matter theories embedded in the gravitational theory.

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