New and Topologically Massive Gravity, from the Outside In

By

COLIN CUNLIFF
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M.S. (University of California, Davis) 2007

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Approved:

Professor Steve Carlip, Chair

Professor Andreas Albrecht

Professor Andrew Waldron

Committee in Charge

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Abstract

This thesis examines the asymptotically anti-de Sitter solutions of higher-derivative gravity in 2+1 dimensions, using a Fefferman-Graham-like approach that expands solutions from the boundary (at infinity) into the interior. First, solutions of topologically massive gravity (TMG) are analyzed for values of the mass parameter in the range $\mu \geq 1$. The traditional Fefferman-Graham expansion fails to capture the dynamics of TMG, and new terms in the asymptotic expansion are needed to include the massive graviton modes. The linearized modes of Carlip, Deser, Waldron and Wise [1] map onto the non-Einstein solutions for all μ, with nonlinear corrections appearing at higher order in the expansion. A similar result is found for new massive gravity (NMG), where the asymptotic behavior of massive gravitons is found to depend on the coupling parameter m^2. Additionally, new boundary conditions are discovered for a range of values $-1 < 2m^2l^2 < 1$ at which non-Einstein modes decay more slowly than the rate required for Brown-Henneaux boundary conditions. The holographically renormalized stress tensor is computed for these modes, and the relevant counterterms are identified up to unphysical ambiguities.

Dedicated to Babak and Xerxes, two Persians who keep me grounded.

Contents

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Chapter 1

Two Paths to Higher-Derivative, (2+1)-Dimensional Gravity

Recent years have seen a surge of interest in higher-derivative, (2+1)-dimensional models of gravity. With the conjecture that topologically massive gravity becomes a chiral theory at a critical value of the coupling constants, the work of Li, Song and Strominger [2] at the beginning of 2008 motivated more than 200 follow-up studies addressing stability of the theory. Similarly, the discovery in early 2009 of new massive gravity [3, 4] as the first higher-derivative, non-linear extension of the Fierz-Pauli massive spin-2 model [5] has generated more than 250 citations at last count. Since we live in a (3+1)-dimensional world with gravitational interactions well-explained by general relativity, it is reasonable to question the reasons for all this interest in higher-derivative, three-dimensional models.

This heightened attention is motivated from two different directions. From the three-dimensional side, the study of general relativity and quantum gravity has been remarkably productive; yet the theory lacks the dynamics of (3+1)-dimensional general relativity. This lack of dynamics motivates consideration of higher-derivative models, which add gravitational degrees of freedom that, it is hoped, make the theory more like its four-dimensional counterpart. The second path finds motivation in 3+1 dimensions to explore massive gravity models, a subset of which include the higher-derivative models explored here. Then a reduction in dimensions provides a simpler setting for studying these models. These two

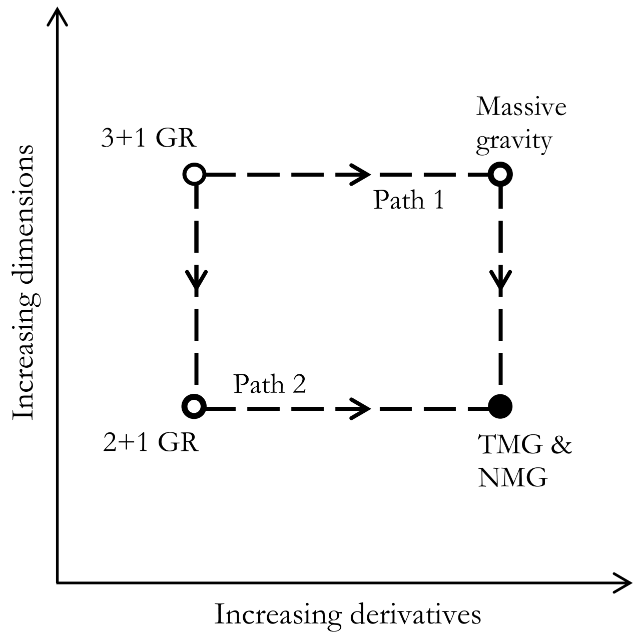

Figure 1.1: Graphical depiction of the two paths to higher-derivative, (2+1)-dimensional gravity. Both paths begin with general relativity in 3+1 dimensions. More complex theories are obtained by moving up (increasing dimensions) or to the right (increasing derivatives). These are not the only methods of complicating general relativity—in particular, other massive gravity theories can be obtained by adding non-derivative potential terms to the action.

paths are represented pictorially in Figure 1.1.

Let us begin by exploring the first path. The development of gravity tracks unsolved problems, and we can look to Newtonian gravity for guidance on how a particular model can resolve these problems. Shortly after Newton's theory was developed, his force-law was applied to planetary orbits, and a discrepancy between the observed and predicted orbit of Uranus was noticed as early as 1788 [6, 7]. At the time, it was not known whether this discrepancy indicated a flaw in the theory or in observations, and in fact it was neither. In 1846, nearly 60 years after this error was discovered, Adams and le Verrier independently postulated the existence of "dark matter"—in this case, the planet Neptune—to explain the observed deviations, and Neptune was instantly discovered [6, 7]. However, when applied to Mercury, Newtonian gravity predicts a perihelion precession 43 arc sec/century *less* than the observed value. Again a new planet, Vulcan, was expected to explain the deviation from the predicted value [7, 8]. However, Vulcan was never observed, and resolution in this case did not occur until a major "modification" to gravity, with the introduction of Einstein's theory of general relativity (GR).

General relativity is a geometric theory of gravity in which the gravitational field is the spacetime metric, and the curvature of the metric is determined by the matter and energy content of other fields according to the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{1.1}$$

The Ricci tensor $R_{\mu\nu}$ and scalar R are constructed from the metric and its derivatives and measure the curvature of spacetime. The stress-energy tensor $T_{\mu\nu}$ provides the contribution from the matter and radiation sectors, and G_N is Newton's gravitational constant. Here, Λ is the cosmological constant. The geometric nature of the theory is summed up succinctly in John Archibald Wheeler's famous quote, "Spacetime tells matter how to move; matter tells spacetime how to curve" [9]. Shortly after its introduction, Einstein applied general relativity to the orbit of Mercury, and the theory was found to give a value for the perihelion precession consistent with observation [10]. Additionally, general relativity makes other predictions that have been confirmed by experiment or observation, including the bending of light in gravitational fields, gravitational red-shifting of light, and time-dilation.

The lesson from the Newtonian examples is that deviations between observations and predictions may result from insufficient data—the existence of Neptune, in the case of the orbit of Uranus—or they may indicate a flaw of with the underlying theory. We are now at a similar juncture in cosmology. Evidence from Type Ia supernovae [11] indicate the accelerating expansion of the universe, and general relativity together with the observed matter content is insufficient to explain this expansion. If general relativity is the correct theory of gravity, then theory demands the existence of "dark energy," with an energy density accounting for 72% of the total energy density of universe. Another possibility is that general relativity requires modification in the infrared (IR), at ultra large distances, such that the accelerating expansion can be explained without the need for dark energy. One possible mechanism that achieves this is to give a small mass to the graviton. This possibility has generated a resurgence of interest in massive gravity models (see [13] and [7] for reviews of recent developments).

Additional motivations exist for modifying general relativity in the ultraviolet (UV) regime. In particular, general relativity is perturbatively non-renormalizable, due to the fact that the perturbation parameter G_N has dimensions of length-squared [14]. It has long been known that the addition of curvature-squared terms to the Einstein-Hilbert action improve renormalizability in 3+1 dimensions [18]; however, these terms generically introduce ghosts that spoil unitarity [19].

Still, it appears that general relativity requires modification at both the infrared (IR) and ultraviolet (UV) scales. Higher-curvature terms may be able to accomplish both goals by improving the renormalizability of the theory and by giving the graviton a small mass. As with many other models, it is helpful to consider such theories in a lower-dimensional setting. Thus concludes the first path toward (2+1)-dimensional, higher derivative gravity.

The second path begins with the final step of the first path: dimensional reduction of general relativity to 2+1 dimensions. Lower-dimensional models have proven productive in virtually every branch of physics. For example, one-dimensional potentials in quantum mechanics are sufficiently complex to provide examples of tunneling and quantization of bound states. In quantum field theory, two-dimensional potentials of scalar fields demonstrate spontaneous symmetry breaking and Nambu-Goldstone bosons. Thus an obvious approach to the study of gravity in 3+1 dimensions is exploration of its lower-dimensional counterpart.

Research into (2+1)-dimensional gravity has already proven remarkably productive, and I mention three particular milestones here. First, the description of point particles as conical defects [20] makes these systems realistic models of cosmic strings in 3+1 dimensions [24][1].

[1] For a review of early work on point particles in 2+1 dimensions, as well as the corresponding Aharonov-Bohm-like effect, see the introduction of [25], as well as Chapter 3 of [26].

The discovery by Gott in 1991 that spacetimes with a pair of cosmic strings admit closed timelike curves (CTCs) [27] makes the lower-dimensional model an ideal testing ground for causality violation. Second, the formulation of (2+1)-dimensional general relativity as a Chern-Simons theory [28, 29] permits computation of topology-changing amplitudes and provides new tools for exploring the role of topology and topology change in the quantum theory [29, 30]. Finally, the black hole of Bañados, Teitelboim and Zanelli [31, 32] has become the focus of much work, since it is the simplest purely gravitational black hole that can be constructed, and is therefore ideal for exploring microscopic state counting and questions about the statistical mechanics of black holes.[2]

Gravity in 2+1 dimensions has also been useful for exploring many conceptual questions relating to the classical and quantum theories. In particular, a large number of approaches to quantizing the theory have been developed, and it has been possible to explore the relationships between these various approaches. For a thorough review of the subject, see [26].

General relativity in 2+1 dimensions shares the same conceptual underpinnings as the (3+1)-dimensional theory, so it should come as no surprise that this research has been valuable. What is surprising is that research in lower-dimensional gravity did not begin in earnest until the mid-1980s. No doubt this lack of interest was due to the apparent triviality of the theory: (2+1)-dimensional general relativity possesses no local degrees of freedom (no gravitational waves) and has no Newtonian limit [34, 36]. This can be seen most easily by noting that the symmetries of the Riemann tensor in n dimensions reduce the number of independent components to $n^2(n^2-1)/12$. In three dimensions, the Riemann tensor possesses six independent components, the same number as the Ricci tensor, and can be written entirely in terms of the Ricci tensor

$$R_{\mu\nu\lambda\rho} = g_{\mu\lambda}R_{\nu\rho} + g_{\nu\rho}R_{\mu\lambda} - g_{\mu\rho}R_{\nu\lambda} - g_{\nu\lambda}R_{\mu\rho} - \frac{1}{2}R(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}). \tag{1.2}$$

[2] For a recent review, see [33].

Then for metrics satisfying the vacuum Einstein equations (1.1), the curvature tensor is just that of constant-curvature spacetime,

$$R_{\mu\nu\lambda\rho} = \Lambda(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) \tag{1.3}$$

For regions free of matter, spacetime is locally flat ($\Lambda = 0$), de Sitter ($\Lambda > 0$) or anti-de Sitter ($\Lambda < 0$). In other words, a nonzero source (stress-energy tensor) affects the curvature only locally, and its effects do not propagate out.

This lack of dynamics can also be seen by counting degrees of freedom. In a foliation into surfaces of constant time, the two-dimensional spatial metric has three independent components, and its conjugate momentum has three more independent components. Three degrees of freedom can be removed by the choice of coordinates, and the other three are removed by initial value constraints, leaving no remaining local degrees of freedom. Additionally, a simple look at the geodesic equation reveals that two point particles do not experience a gravitational attraction [36].

Thus, general relativity in 2+1 dimensions is not a good model for dynamics in (3+1)-dimensional spacetimes. Then an interesting question is whether the theory can be deformed to give it dynamical degrees of freedom similar to those possessed by the higher-dimensional theories. Perhaps the simplest way is to couple gravity to a scalar field (the dilaton). In certain limits, these dilaton models even possess Newtonian-like behavior [37]. However, this goes beyond the scope of this paper, and these models will not be explored here. Another way to add degrees of freedom is to add higher-derivative terms to the action. Generically, such terms introduce ghosts or tachyons, rendering the theory unstable. However, two notable examples exist in three dimensions: the topologically massive gravity of Deser, Jackiw and 't Hooft [38, 39]; and the new massive gravity of Bergshoeff, Hohm and Townsend [3, 4]. These models are introduced in later sections.

And so both paths lead to the study of higher-derivative, lower-dimensional models of gravity. I am motivated, in particular, to exploration of the asymptotically anti-de Sitter solutions of topologically massive and new massive gravity. This thesis is organized as follows. The remainder of this chapter briefly introduces the two theories, and also provides a review of anti-de Sitter spacetime and the Fefferman-Graham expansion, which will be used in later chapters. Chapter 2 finds that the original Fefferman-Graham expansion fails to capture the dynamics of the TMG and proposes a new term in the expansion which captures the asymptotic behavior of the propagating massive graviton. Chapter 3 carries out the same analysis for new massive gravity. In addition, new boundary conditions—relaxations from the Brown-Henneaux boundary conditions—are identified, and the renormalized boundary stress-energy tensor is obtained with these new asymptotics. These two chapters are based on the papers "Topologically Massive Gravity from the Outside In" [40] and "Non-Fefferman-Graham Asymptotics and Holographic Renormalization in New Massive Gravity" [41]. The final chapter discusses possible extensions of the methods presented here to higher-dimensional and higher-derivative extensions of new massive gravity.

1.1 Asymptotically Anti-de Sitter Spacetimes

Because (2+1)-dimensional black holes only exist in spacetimes with a negative cosmological constant, and because (2+1)-dimensional gravity provides a perfect setting for exploring the AdS_3/CFT_2 correspondence, I will focus solely on asymptotically AdS_3 solutions. Section 1.1.1 begins with a review of pure anti-de Sitter solution, while Section 1.1.2 develops the notion of "asymptotically" anti-de Sitter spacetimes.

1.1.1 Anti-de Sitter Space

General relativity with a negative cosmological constant permits anti-de Sitter space as an exact solution. Anti-de Sitter space is a conformally flat, constant-curvature, maximally symmetric solution to the Einstein equations, with curvature given by

$$R_{\mu\nu\rho\sigma} = \frac{1}{l^2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \tag{1.4}$$

where l^2 is the AdS radius ($\Lambda = -1/l^2$). Anti-de Sitter space can be obtained by restricting a flat four-dimensional metric of signature $(- - ++)$:

$$\mathrm{d}s^2 = -\mathrm{d}u^2 - \mathrm{d}v^2 + \mathrm{d}x^2 + \mathrm{d}y^2 \tag{1.5}$$

to the hypersurface defined by

$$-v^2 - u^2 + x^2 + y^2 = -l^2. \tag{1.6}$$

A system of coordinates covering the whole of the manifold may be introduced by setting

$$\begin{aligned} u &= l\cosh\rho\sin\tau, \qquad & v &= l\cosh\rho\cos\tau \\ x &= l\sinh\rho\cos\theta, \qquad & y &= l\sinh\rho\sin\theta \end{aligned} \tag{1.7}$$

with $0 \leq \rho < \infty$, and $0 \leq \tau, \theta < 2\pi$. Inserting (1.7) into (1.5) gives

$$\mathrm{d}s^2 = l^2\left[-\cosh^2\rho\,\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2\rho\,\mathrm{d}\theta^2\right] \tag{1.8}$$

for the metric of anti-de Sitter space. Note that both τ and θ are angles, and this permits closed-timelike curves. To avoid this, we can "unwrap" the τ coordinate and remove the identification $\tau = \tau + 2\pi$ to obtain the universal covering of anti-de Sitter space. This metric can be written in Schwarzschild-like coordinates with the identification $\tau = t/l$ and $r = l\sinh\rho$:

$$\mathrm{d}s^2 = -[(r/l)^2 + 1]\mathrm{d}t^2 + [(r/l)^2 + 1]^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2, \tag{1.9}$$

which is the metric (2.7) with $M = -1$, $J = 0$ (and ϕ replaced by θ).

Another coordinate system that will be useful is the Poincaré system, defined in terms of the embedding space by

$$z = \frac{u+x}{l}, \qquad \beta = \frac{y}{u+x}, \qquad \gamma = \frac{-v}{u+x} \tag{1.10}$$

These coordinates cover only one of the regions where $u + x$ has a definite sign. The metric in these coordinates reads

$$\mathrm{d}s^2 = \frac{l^2 \mathrm{d}z^2}{z^2} + \frac{z^2}{l^2}(-\mathrm{d}\gamma^2 + \mathrm{d}\beta^2). \tag{1.11}$$

While Poincaré AdS_3 and global AdS_3 are locally equivalent, they have different global properties. In particular, Poincaré AdS has zero mass, whereas global AdS_3 has mass $-1/8G$ [42]. Thus Poincaré coordinates are not well-suited to studying global properties.

1.1.2 Conformally Compact Manifolds

In order to accommodate more general solutions, we need to define what it means for a metric to "asymptotically approach" anti-de Sitter space. This discussion begins with the definition of conformally compact manifolds. Consider a $(d + 1)$-dimensional manifold with metric (M, g), where M is the interior of a manifold-with-boundary $\bar{M}$, and the bulk metric g becomes singular on the boundary, denoted ∂M. Suppose the existence of a smooth, non-negative defining function z on $\bar{M}$ such that $z(\partial M) = 0$, $\mathrm{d}z(\partial M) > 0$, and $z(M) > 0$. This can be used to define a non-degenerate metric on $\bar{M}$,

$$\bar{g} = z^2 g. \tag{1.12}$$

Then, in the language of Penrose [43], the pair (M, g) is labeled conformally compact, and the choice of defining function determines a particular conformal compactification of (M, g),

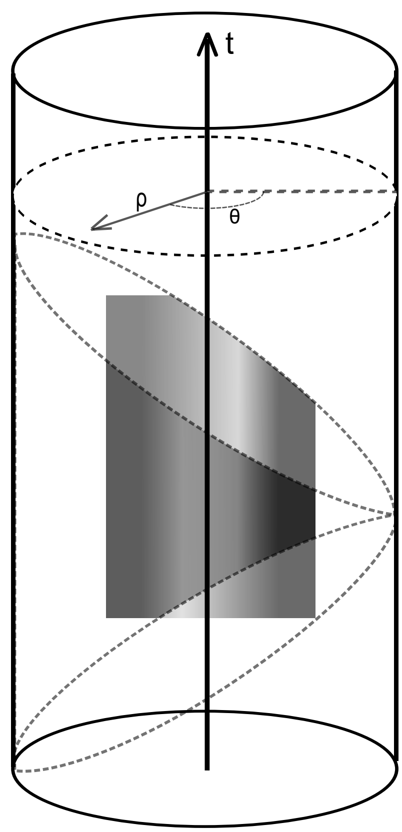

Figure 1.2: The universal covering of AdS_3 in global coordinates. The shaded region corresponds to the portion of anti-de Sitter space covered by a single Poincaré patch.

with boundary located at $z = 0$.

The metric $\bar{g}$ induces a metric $g_{(0)}$ on the boundary ∂M. However, this metric is not unique, as a different defining function conformally rescales the boundary metric. The bulk metric (M, g) thus induces a conformal structure $(\partial M, [g_{(0)}])$ on the boundary, where $[g_{(0)}]$ denotes a conformal class of metrics.

The connection to anti-de Sitter space becomes apparent in the expansion of the curvature tensor (of the bulk metric) in powers of z, yielding

$$R_{\mu\nu\rho\sigma} = -\bar{g}^{\alpha\beta} \nabla_\alpha z \nabla_\beta z \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right) + \mathcal{O}\left(z^{-3} \right) \tag{1.13}$$

Note that $\bar{g}^{-1}$ is of order z^{-2}, so the leading term is of order z^{-4}. If, in addition, there exists a defining function that, to leading order, satisfies $\bar{g}^{\alpha\beta} \nabla_\alpha z \nabla_\beta z = \frac{1}{\ell^2}$, then the manifold M is called asymptotically locally anti-de Sitter (AlAdS), in the sense that the curvature tensor approaches that of AdS with radius ℓ near the boundary. Note that no restriction has been placed on the boundary topology. This notion is more general than what is often defined as "asymptotically anti-de Sitter" (AAdS), which usually indicates metrics which asymptote to the exact AdS metric at infinity (see, for instance [44]). In a slight abuse of notation, I will often use the term "asymptotically anti-de Sitter" interchangeably for both cases.

1.1.3 Fefferman-Graham Expansion

On the manifold-with-boundary $\bar{M}$, the defining function z can be used as one of the coordinates to bring the metric $\bar{g}$ to Gaussian normal form near the boundary,

$$\bar{g}_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}\bar{s}^2 = \mathrm{d}z^2 + g_{ij} \mathrm{d}x^i \mathrm{d}x^j. \tag{1.14}$$

where Greek indices run over $d+1$ dimensions, and Latin indices run over the d non-radial coordinates. Then the bulk (physical) metric becomes

$$\mathrm{d}s^2 = z^{-2}\left(\mathrm{d}z^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j\right). \tag{1.15}$$

The d- dimensional metric induced on a hypersurface of constant z can be expanded in powers of z

$$g_{ij}(z, x^k) = g^{(0)}_{ij}(x^k) + \cdots, \tag{1.16}$$

where the subleading terms vanish on the boundary as $z \to 0$.

The specific form the expansion depends on the bulk theory. Fefferman and Graham [45] first derived the asymptotic expansion for general relativity in $d+1$ dimensions

$$g_{ij}(r, x^k) = g^{(0)}_{ij} + z^2 g^{(2)}_{ij} + \cdots + z^d\left(g^{(d)}_{ij} + \ln z\, h^{(d)}_{ij}\right) + \cdots \tag{1.17}$$

They found that coefficients of odd powers of z vanish, and subleading terms $g^{(2k)}$, $2k < d$, are fixed by the boundary metric $g^{(0)}$. However, only part of $g^{(d)}$ is determined by the boundary. Specifically, the trace and covariant divergence of $g^{(d)}$ are solved in terms of the boundary metric, leaving other components free. The "log" term $h^{(d)}$ is present only in even dimensions $d > 2$ and is given by the metric variation of the conformal anomaly. In three dimensions, the expansion truncates at fourth order, $g = g^{(0)} + z^2 g^{(2)} + z^4 g^{(4)}$, and the metric can be found exactly [46].

However, these results are all consequences of the equations of motion and therefore are specific to Einstein gravity. In general, different bulk theories lead to different asymptotic expansions. Even with a gravitational action given by the Einstein-Hilbert action, gravity coupled to other fields can yield different expansions. For example, three dimensional Einstein gravity coupled to a free massless scalar field is of the form (1.17) with a non-zero log term $h^{(2)}$ [47]. In this case, $h^{(2)}$ is determined by the boundary values of the fields and does not indicate a new degree of freedom of the metric. Other potentials for a scalar field can yield odd powers of z [48] and even log-squared terms [49] in the asymptotic expansion.

Higher derivative gravitational theories can have even more exotic behavior. For example, topologically massive gravity (TMG) at the critical point $\mu\ell = 1$ can also have the log term $h^{(2)}$ in the asymptotic expansion [50, 51, 52, 53]. However, several points distinguish this case from that of general relativity coupled to a free massless scalar. In that example, the log term is fixed in terms of the boundary fields. But in critical TMG, one component of the log term is unconstrained by the equations of motion. Its inclusion the asymptotic expansion is allowed but not required, and this phenomena is related to the presence of new bulk degrees of freedom in the metric. This apparent freedom in the asymptotic expansion initiated a vigorous discussion in the literature about appropriate boundary conditions, with some arguing that different boundary conditions yield different theories [54]. However, other researchers, motivated by AdS/CFT, argued that the bulk theory determines the correct asymptotic expansion [55].

Another unusual feature of critical TMG is that the log term in TMG is only available at the critical point $\mu\ell = 1$. At all other values of the mass parameter, the equations of motion force the log term to vanish. So, unlike the case of GR coupled to a free massless scalar, the asymptotic expansion of TMG depends not just on the form of the bulk action but also *on the parameters*. This is reminiscent of Einstein gravity coupled to a massive scalar with higher-than-quadratic polynomial potential [56]. In that case, the asymptotic behavior of both the metric and scalar is dependent on the value of the mass, with the most relaxed asymptotics occurring when the scalar saturates the Breitenlohner-Freedman bound [57]. So the appearance of parameter-dependent asymptotic expansion in higher-derivative gravity, though surprising, is not unique. The asymptotic expansion of TMG at non-critical values of the mass parameter will be examined in Chapter 2.

New massive gravity exhibits some features similar to topologically massive gravity. As in TMG, NMG possesses a critical point $2m^2 = -\sigma$ at which the log term $h^{(2)}$ is allowed but not required, and either choice of boundary conditions seems acceptable. Also, as in TMG, this term vanishes by the equations of motion at non-critical values. NMG also possesses another critical point $2m^2 = +\sigma$ at which interesting new solutions have been found. In particular, new type black holes [4, 58] have been discovered with relaxed asymptotics containing odd powers of the radius

$$g_{ij} = g_{ij}^{(0)} + z g_{ij}^{(1)} + z^2 g_{ij}^{(2)} + z^3 g_{ij}^{(3)} + \cdots \tag{1.18}$$

Here, too, this expansion only works at this particular point in the space of parameters. At generic values of the mass parameter, the coefficients of the odd-powered terms vanish. Also, similar to the log term at $2m^2 = -\sigma$, the subleading term $g^{(1)}$ is *allowed* but not required, i.e. it is not determined by the boundary metric but results from the increased bulk degrees of freedom of the metric. Additionally, the theory at $2m^2 = +\sigma$ allows a new log term [59, 60] $g = g^{(0)} + z \ln z h^{(1)} + z g^{(1)} + \cdots$.

To date, the asymptotic behavior of NMG at generic values of m^2 remains unknown. That odd-powered and log asymptotics are allowed only at critical values of m^2 hints at the prospect that the correct asymptotic expansion depends on the value of the mass parameter. This question will be explored in Chapter 3.

1.2 Topologically Massive Gravity

One of the early examples of a higher derivative theory of gravity is the topologically massive gravity (TMG) model developed by Deser, Jackiw and Templeton [38, 39]. The theory is obtained by adding the gravitational Chern-Simons term to the Einstein-Hilbert action

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^3 x \sqrt{-g} \left[\sigma \left(R - 2\Lambda \right) + \frac{1}{2\mu} \epsilon^{\mu\nu\rho} \left(\Gamma^{\alpha}_{\mu\beta} \partial_\nu \Gamma^{\beta}_{\rho\alpha} + \frac{2}{3} \Gamma^{\alpha}_{\mu\gamma} \Gamma^{\gamma}_{\nu\beta} \Gamma^{\beta}_{\rho\alpha} \right) \right] \tag{1.19}$$

with or without the cosmological constant Λ, and with $2\kappa^2 = 16\pi G$. Here σ is used to control the sign of the Einstein-Hilbert piece of the action; the sign of the coupling μ is not fixed;

and $\epsilon^{\mu\nu\rho}$ is the three-dimensional anti-symmetric tensor, related to the Levi-Civita symbol by $\epsilon^{\mu\nu\rho} = \tilde{\epsilon}^{\mu\nu\rho}/\sqrt{-g}$.

Though the Einstein-Hilbert action (in $2+1$ dimensions) and the Chern-Simons action separately are topological, the inclusion of both terms in the action (1.19) makes the theory *not* topological. The name comes "topologically massive" comes from the massive nature of the field excitations and from the importance of the Chern-Simons action on topological field theories (see, for example, [61]), and it was later realized that this term arises naturally in the effective action for massless fermions coupled to Einstein gravity in three dimensions [62, 63, 64].

Variation of the Chern-Simons action with respect to the metric yields the Cotton tensor

$$C_{\mu\nu} = \epsilon_{\mu}{}^{\alpha\beta}\nabla_{\alpha}\left(R_{\beta\nu} - \frac{1}{4}Rg_{\beta\nu}\right), \tag{1.20}$$

which is also sometimes referred to as the York tensor [65]. In three dimensions, the Cotton tensor plays the role of the Weyl tensor, since it is invariant under conformal transformations and vanishes only if the three-dimensional space is conformally flat. Additionally, the Cotton tensor is symmetric, traceless, and covariantly conserved by virtue of the Bianchi identities. The resulting equations of motion

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0 \tag{1.21}$$

are now third-order in derivatives of the metric. Note that, because the Cotton tensor is traceless, all solutions have constant scalar curvature. Additionally, the Cotton tensor vanishes identically for solutions to the Einstein equations of motion, so all Einstein solutions are also solutions of topologically massive gravity. Despite the additional derivatives, the theory is unitary, with no ghosts or tachyons, and describes a single massive spin-2 degree of freedom. To obtain the particle content of the linear theory, the authors of [38] were able to write the linearized action (without cosmological constant) as

$$-\frac{\sigma}{4\kappa^2}\int d^3x\phi\left(\Box-\mu^2\right)\phi \tag{1.22}$$

where ϕ is the transverse-trace part of the perturbation

$$\phi=\left(\delta_i\delta_j+\hat{\partial}_i\hat{\partial}_j\right)h^{ij}, \qquad \text{with } \hat{\partial}_i\equiv\partial_i/\sqrt{-\nabla^2} \tag{1.23}$$

The field ϕ is the only dynamical part of the metric. In this form, it becomes apparent that (1.22) describes a massive degree of freedom. Also, note that, independent of the sign of the mass parameter μ, the mass-squared is always positive, which guarantees the absence of tachyons from the theory. However, the sign of the kinetic term is controlled by the sign of the Einstein-Hilbert piece of the action, and the absence of ghosts demands the "wrong sign" Einstein-Hilbert action $\sigma=-1$. This fact will become important in later discussions. While (1.22) appears to be the action of massive scalar, its Lorentz transformation properties confirm that it is a spin-2 field with helicity determined by the sign of μ.

After its introduction in the early 1980s, research into topologically massive gravity continued sporadically. It was expected that the higher-derivative nature of the model would improve renormalizability, and this was confirmed in [66, 67, 68], although see [69] for a more recent study. Additionally, many new classical non-Einstein solutions have been found [70, 71, 72, 73]. Quantization of the theory was explored early on in [74, 75, 76, 77], and more recently in [78] and [79].

Cosmological completion of the theory—the addition of a cosmological constant—was not examined until recently, with most attention paid to the case $\Lambda<0$. Despite early recognition that cosmological topologically massive gravity (CTMG) also contains the BTZ black hole solutions of Einstein gravity [80, 71], exploration of their properties in CTMG came much later [81, 82, 83, 84]. It is amusing to note that, as interest in cosmological topologically massive gravity has waxed and waned, many new solutions have been discovered and rediscovered independently. For example, exact AdS pp-waves have been discovered on at least five separate occasions [71, 82, 84, 85, 86], although the solutions of [82] were not originally recognized as wave solutions.

Cosmological topologically massive gravity was approached from the perspective of the AdS/CFT correspondence in several recent papers [87, 88, 2]. Kraus and Larsen [87] realized that the Chern-Simons action is not completely diffeomorphism-invariant. However, the non-invariance is entirely located on the boundary and thus does not spoil the consistency of the bulk gravitational theory. In the boundary theory, this gravitational anomaly shows up as the non-conservation of the boundary stress energy tensor. Additionally, the left- and right-central charges of the Virasoro algebras split,

$$c_{R/L} = \frac{3\ell}{2G}\left(1 \pm \frac{1}{\mu\ell}\right) \tag{1.24}$$

where ℓ is the AdS radius. Solodukhin [88] reproduced their results by integrating the bulk equations to find the boundary stress energy tensor.

Research in the field exploded after the chiral gravity conjecture of Li, Song and Strominger [2], which was motivated in part by Witten's conjecture [89] that the CFT dual to AdS_3 is a chiral theory. Because cosmological topologically massive gravity is a maximal parity-violating theory, this offered an ideal testing ground for Witten's proposal. The authors of [2] noted that the theory possesses a critical point $\mu\ell = \pm 1$ at which one of the central charges vanishes, and the asymptotic symmetry group is generated by a single Virasoro algebra [90]. This theory will be explored in more detail in the next chapter.

1.3 New Massive Gravity

More recently, a *parity-preserving* three-dimensional model of massive gravity was discovered by Bergshoeff, Hohm and Townsend [3, 4] in early 2009. This "new" massive gravity, also referred to as BHT gravity[3] is obtained by adding a specific combination of curvature-squared terms to the action

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^3x\sqrt{-g}\left[\sigma R - 2\lambda + \frac{1}{m^2}K\right], \tag{1.25}$$

where the new term in the action is given by

$$K \equiv R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2. \tag{1.26}$$

This model describes a unitary theory of massive, spin-2 gravitons with both polarization states of helicity ± 2. Variation of the action with respect to the metric yields the fourth-order equations of motion

$$\sigma\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \lambda g_{\mu\nu} + \frac{1}{2m^2}K_{\mu\nu} = 0\,, \tag{1.27}$$

where the curvature-squared term is given by

$$K_{\mu\nu} = 2\Box R_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu R - \frac{1}{2}\Box R g_{\mu\nu} + 4R_{\mu\alpha\nu\beta}R^{\alpha\beta} - \frac{3}{2}RR_{\mu\nu} - R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu} + \frac{3}{8}R^2 g_{\mu\nu}. \tag{1.28}$$

This new tensor enjoys the interesting property that its trace is just the Lagrangrian from which it came, $g^{\mu\nu}K_{\mu\nu} = K$. Unlike the Cotton tensor considered previously, this new tensor is not, in general, traceless, and the solutions of (1.27) are not restricted to constant-scalar-curvature solutions. This can be seen by taking the trace of (1.27)

$$\sigma R = 6\lambda + \frac{1}{m^2}K. \tag{1.29}$$

By definition, solutions of maximal symmetry are those for which the Ricci tensor is proportional to the metric, $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$. Plugging this condition into (1.27) produces an equation

[3]BHT is often used as a preservative to keep foods fresh, and I humbly suggest that the founders add "preserves freshness" to the list of positive attributes of the theory.

$$\Lambda^2 + 4\sigma m^2 \Lambda - 4\lambda m^2 = 0 \tag{1.30}$$

relating the *effective* cosmological constant Λ to the cosmological parameter λ appearing in the action. For most purposes, it is more convenient to work in terms of the effective cosmological constant.

Unitarity can be demonstrated by showing equivalence of the linear theory to the Fierz-Pauli action [5]. The relevant part of the action is

$$S = \frac{1}{m^2(\Lambda - 2m^2\sigma)} \int \mathrm{d}^3x \sqrt{-\bar{g}} \left[-\frac{1}{2} k^{\mu\nu} \mathcal{G}_{\mu\nu}(k) - \frac{1}{4} M_{eff}^2 (k^{\mu\nu} k_{\mu\nu} - k^2) \right], \tag{1.31}$$

where $k_{\mu\nu}$ carries the physical content of the theory and is (roughly) the linearized Schouten tensor. The kinetic term is just the linearized Einstein tensor,

$$\mathcal{G}_{\mu\nu}(k) \equiv R^{(1)}_{\mu\nu} - \frac{1}{2} R^{(1)} \bar{g}_{\mu\nu} - 2\Lambda k_{\mu\nu} + \Lambda k \bar{g}_{\mu\nu}, \tag{1.32}$$

and the effective massive is given by

$$M_{eff}^2 = -\sigma m^2 + \frac{1}{2}\Lambda. \tag{1.33}$$

Thus the physical content of the linearized theory is defined by a *second-order* action (1.31), which is nothing more than the Fierz-Pauli action for a massive spin-2 particle. The reverse route is also possible—for example, Dalmazi and Mendonça [91] were able to *derive* new massive gravity as a non-linear extension of Fierz-Pauli via Noether gauge embedment, and this has also been achieved through other methods in [92]. This rather remarkable result shows that new massive gravity is the first curvature-squared non-linear extension of Fierz-Pauli theory.

Around an anti-de Sitter background, negative values of the mass are allowed (i.e., tachyons are avoided) as long as the mass satisfies the Breitenlohner-Freedman bound [93, 57]

$$M_{eff}^2 \geq \Lambda \tag{1.34}$$

It should be mentioned that this bound was derived for scalar fields, though several authors have argued that it applies equally to spin 2 fields [86, 94]. Additionally, ghosts (with negative kinetic energy) are avoided as long as

$$m^2(\Lambda - 2m^2\sigma) > 0 \tag{1.35}$$

Both conditions are required for unitarity of bulk gravitons. Several regions of parameter space lead to unitarity (of the bulk theory), and these regions are categorized and explored in [4]. Bulk unitarity has also been confirmed in several other studies [95, 96, 97, 98, 99].

Canonical analysis of the full theory is slightly more difficult than in topologically massive gravity, and the NMG equivalent of (1.22) will not be given here. Instead, I will simply note some of the results. The pure-K theory, given by the action (1.25) without the Einstein or cosmological terms, was first examined by Deser in [96]. The linearized "K-theory" propagates massless spin-2 modes, and the linearized action is Weyl-invariant and power-counting renormalizable. Introduction of the Einstein-Hilbert term breaks Weyl symmetry, thus giving mass to the graviton [100, 101]. This result offers an obvious interpretation of the theory: the K-term provides the dynamics of the graviton, and the Einstein-Hilbert term provides the mass. This perspective also explains the role of the sign σ of the Einstein-Hilbert part of the action: instead of determining positivity of kinetic energy, as in 3+1 dimensions, σ enters into the expression for the mass in a nontrivial way. Unfortunately, inclusion of the Einstein-Hilbert term also seems to destroy the power-counting renormalizability of the theory [96], although Oda [102] uses a different approach and concludes that the theory *is* renormalizable. Other studies of the Hamiltonian structure of new massive gravity have also been performed [103, 104, 105]

Since the initial publication, research has quickly spread in many directions. Methods for finding and categorizing solutions have been explored in [106, 107, 108, 109]. Many new classical solutions have been found, including AdS waves and their generalizations [59, 110, 111, 112], warped AdS and warped black holes [113, 114], new 'hairy' black holes [4, 58, 115], and even asymptotically Lifshitz black holes [116]. The supergravity extension was given in [117, 118]. Asymptotically AdS_3 boundary conditions were investigated in [119, 120]. Newtonian limits were investigated in [98]. Bañados and Thiesen have also explored the similarity between new massive gravity and certain bi-metric theories [121], and this relationship was further explored in [122]. The theory with negative cosmological correspondence has also been a productive setting for exploring the AdS/CFT correspondence [123, 119, 124, 125, 126, 127, 128, 129, 130, 60, 131, 132] and black hole solutions [133, 134, 135, 136].

A full accounting of all the research on new massive gravity is beyond the scope of this introduction. A more targeted introduction will be given in Chapter 3.

1.4 Conventions

The metric has the "mostly positive" signature $(-,+,+)$, and the Christoffel connection is defined as

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right) \tag{1.36}$$

and gives the unique metric-compatible covariant derivative $\nabla g = 0$. The Riemann tensor is defined by the commutator of covariant derivatives acting on a vector

$$[\nabla_\mu, \nabla_\nu] V^\sigma = R^{\sigma}{}_{\rho\mu\nu} V^\rho \tag{1.37}$$

where

$$R^{\sigma}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\sigma}_{\rho\nu} - \partial_\nu \Gamma^{\sigma}_{\rho\mu} + \Gamma^{\sigma}_{\mu\lambda} \Gamma^{\lambda}_{\rho\nu} - \Gamma^{\sigma}_{\nu\lambda} \Gamma^{\lambda}_{\rho\mu} \tag{1.38}$$

is the Riemann tensor. The Ricci tensor and Ricci scalar are defined by

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} \qquad R = g^{\mu\nu} R_{\mu\nu} \tag{1.39}$$

The d'Alembertian is $\Box = \nabla^{\mu}\nabla_{\mu}$. Greek indices run over all D dimensions. When one coordinate is singled out in an ADM-like foliation (usually the radial coordinate in this paper), Latin indices run over the remaining $D-1$ dimensions.

Chapter 2

Topologically Massive Gravity from the Outside In

The asymptotically anti-de Sitter solutions of cosmological topologically massive gravity (TMG) are analyzed for values of the mass parameter in the range $\mu \geq 1$. At non-chiral values, a new term in the Fefferman-Graham expansion is needed to capture the bulk degree of freedom. The CDWW modes provide a basis for the pure non-Einstein solutions at all μ, with nonlinear corrections appearing at higher order in the expansion.

2.1 Introduction

Topologically massive gravity (TMG) [38, 39] with a negative cosmological constant appears to be the simplest theory that contains both black holes and local gravitational degrees of freedom, making it a potentially useful toy model to explore various questions in quantum gravity. However, at arbitrary values of the coupling constants, the theory appears to be unstable, containing either positive mass black holes and negative energy gravitons, or the reverse, depending on the choice of sign of the Einstein-Hilbert piece of the action. Recently, Li et al. [2] proposed that with suitable boundary conditions, the local bulk degree of freedom disappears at the "chiral" coupling $\mu\ell = 1$, allowing the possibility of choosing the sign of the action such that only positive mass black holes are included. With these boundary conditions, the theory becomes chiral, in that the asymptotic symmetry consists of a single copy of the Virasoro algebra [90, 137, 50], and the theory was dubbed chiral gravity [2, 54].

The claim that chiral gravity admits only non-negative energy modes has been the subject of much debate in the literature. Several non-perturbative studies found a single local propagating degree of freedom at all values of μ [137, 138, 139]; however, the boundary conditions satisfied by these modes was not investigated. At the linearized level, other authors found negative-energy modes at $\mu\ell = 1$ [86, 1, 52, 140], but these modes either were not chiral or required different boundary conditions than those used in [2]. After some confusion in the literature, Maloney et al. [54] proposed that TMG at the critical value could be divided into two theories depending on the choice of boundary conditions: chiral gravity with Brown-Henneaux [44] boundary conditions and log gravity with relaxed boundary conditions that include a logarithmic term in the asymptotic expansion of the metric. For related work on boundary conditions, see [50, 51]. Additionally it was shown in [54] that all stationary, axially symmetric solutions of chiral gravity are the familiar BTZ black holes [32] and have non-negative energy. The authors found that the proposed counterexamples either required the relaxed boundary conditions, and thus were solutions to log gravity, or developed linearization instabilities at second order, and they speculated that all asymptotically anti-de Sitter non-Einstein solutions at the critical point are in fact solutions to log gravity.

Recently, Compère et al. [141] discovered a new class of non-Einstein solutions of chiral gravity using the Fefferman-Graham expansion [45] with Brown-Henneaux boundary conditions. They further examined a subset of the general solution that included linear perturbations from AdS_3 and BTZ backgrounds. Of the solutions examined, all contained either naked singularities or closed timelike curves and, unless they can be excluded as unphysical, may render chiral gravity unstable.

In an effort to understand the asymptotic behavior of non-Einstein solutions, this paper extends the work of Compère et al. to non-chiral values $\mu\ell > 1$. New terms in the Fefferman-Graham expansion are needed to capture the bulk degrees of freedom at all values of the mass parameter, and for each value of μ, a similar phenomenon occurs: one of the equations of motion disappears, leaving one piece of the metric unconstrained by the equations of motion. Section 2.4 gives the solution to second order in this new term. The division between Einstein and non-Einstein solutions becomes explicit in this formalism: one set of terms in the expansion captures all Einstein solutions, and the second set captures the non-Einstein solutions. Section 2.4.1 examines the special case of chiral gravity, where the solution of [141] is given in light-cone coordinates. In this formalism, we see the chiral point is just the point at which the two sets of terms overlap. Section 2.5 compares the general asymptotic expansion to two known asymptotically anti-de Sitter solutions that also exhibit modified Fefferman-Graham asymptotics: the AdS waves of [85, 142], which are exact solutions, and the linearized CDWW modes of [86, 1]. The CDWW modes provide a complete basis of the non-Einstein solutions only, and additional ingredients are needed to include the Einstein solutions. The solution given in Section 2.5.2 agrees with CDWW to second order; however, nonlinear deviations from CDWW are found for several integral values of μ at higher order and are likely to exist for generic μ. I conclude with a discussion on the significance of these solutions on the stability of chiral gravity.

2.2 Setup and Equations of Motion for Topologically Massive Gravity

The equation of motion for topologically massive gravity (TMG) [38, 39] with a cosmological constant is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0, \tag{2.1}$$

where $C_{\mu\nu}$ is the Cotton tensor

$$C_{\mu\nu} = \epsilon_{\mu}{}^{\alpha\beta}\nabla_{\alpha}\left(R_{\beta\nu} - \frac{1}{4}Rg_{\beta\nu}\right). \tag{2.2}$$

In the discussion below, it will be convenient to work in units where $\Lambda = -1$. By the Bianchi identity, the Cotton tensor is symmetric, traceless, and covariantly conserved. Taking the trace of (2.1) reveals that solutions are spacetimes of constant scalar curvature

$$R = -6, \tag{2.3}$$

and the equation becomes

$$R_{\mu\nu} + 2g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0. \tag{2.4}$$

Pure Einstein solutions are those for which $R_{\mu\nu} = -2g_{\mu\nu}$, and it is apparent from (2.2) that the Cotton tensor is identically zero for all Einstein metrics. Thus topologically massive gravity contains all of the ordinary Einstein solutions plus the massive propagating modes for which the Cotton tensor is non-zero.

2.3 Failure of Traditional Fefferman-Graham Expansion

In particular, topologically massive gravity admits asymptotically anti-de Sitter (AdS) solutions which can be written in Gaussian normal coordinates as

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{2.5}$$

where

$$g_{ij}^{(0)} = \lim_{\rho\to\infty} \mathrm{e}^{-2\rho/\ell} g_{ij}(x,\rho) \tag{2.6}$$

is the metric on the boundary, and ℓ is the AdS radius given by $\Lambda = -\ell^{-2}$. Recall that Greek indices run over all coordinates, and Latin indices run over the non-radial coordinates. Fefferman and Graham showed that metrics satisfying the Einstein equations can always be expanded near the boundary in powers of e^{ρ} [45]:

$$g_{ij} = \mathrm{e}^{2\rho/\ell}\left(g_{ij}^{(0)} + \mathrm{e}^{-2\rho/\ell}g_{ij}^{(2)} + \mathrm{e}^{-4\rho/\ell}g_{ij}^{(4)} + \cdots\right). \tag{2.7}$$

It will be convenient to fix the AdS radius at $\ell = 1$, since this can easily be reinstated through dimensional analysis. In three-dimensional Einstein gravity, the expansion terminates at $g_{(4)}$ (all higher order terms are zero), and the first three terms are sufficient to capture all AdS_3 and BTZ solutions to pure Einstein gravity [46]. It is interesting to note that this result also holds in higher-dimensional general relativity with a conformally flat boundary metric.

The perception exists in the literature (see, e.g. [88]) that the presence of higher derivatives in the equations of motion turns on higher order terms in (2.7), and that the expansion continues indefinitely. This belief is due to the existence of a propagating local degree of freedom in the metric in TMG. After all, in higher- dimensional general relativity, which possesses local degrees of freedom, the metric generically has an infinite expansion. However, the Cotton tensor modifies the equations of motion in a subtle way that does not guarantee that (2.7) is the correct expansion to use in TMG.

This section addresses the assumption that (2.7) is the correct asymptotic expansion for the metric in TMG by applying the expansion (2.7) to the equations of motion (2.1). As in Einstein gravity, the "second-order" $\{\rho,\rho\}$ and $\{i,j\}$ equations vanish identically, leaving some components of the metric unconstrained. Since the Ricci scalar is constant, the subleading terms in the expansion should vanish, and this is used to fix the trace of $g^{(2)}$ in terms of the boundary metric,

$$\mathrm{Tr}g^{(2)} = -\frac{1}{2}R(g^{(0)}), \tag{2.8}$$

where the inverse boundary metric is used to take the trace, and $R(g^{(0)})$ is the Ricci scalar constructed from the two-dimensional boundary metric. The divergence of $g^{(2)}$ is also determined by the boundary fields according to the $\{\rho, i\}$ equations:

$$\left(\delta_i^k + \frac{1}{\mu}\epsilon_i{}^k\right)\left(\nabla^j g_{ij}^{(2)} - \partial_i \mathrm{Tr} g^{(2)}\right) = 0 \tag{2.9}$$

Note that these are just the Einstein equations, with the divergence equation (2.9) multiplied by a factor depending on the mass parameter μ. At generic values, they have the same solution as Einstein gravity. However, part of the divergence constraint vanishes at the critical point $\mu = 1$, and this case is addressed separately in Section 2.4.1.

At "fourth order," all three sets of equations are present, and these equations determine the metric $g^{(4)}$ in terms of $g^{(2)}$ and the boundary metric. The $\{\rho, \rho\}$ equation fixes the trace,

$$\mathrm{Tr} g_{(4)} = \frac{1}{4}\mathrm{Tr} g_{(2)}^2 - \frac{1}{4\mu}\epsilon^{ij}\nabla_i \nabla^k g_{kj}^{(2)} \tag{2.10}$$

while the $\{i, j\}$ equations fix the other components

$$\begin{aligned}\left(\delta_i{}^k + \tfrac{3}{\mu}\epsilon_i{}^k\right)\left[-4g_{kj}^{(4)} + 2\left(g_{(2)}^2\right)_{kj} + g_{kj}^{(0)}\mathrm{Tr}\left(4g_{(4)} - \tfrac{3}{2}g_{(2)}^2\right) + \tfrac{1}{2}g_{kj}^{(0)}\left(\mathrm{Tr} g_{(2)}\right)^2 - g_{kj}^{(2)}\mathrm{Tr} g_{(2)}\right] \\ = \tfrac{1}{\mu}\epsilon_{ij}\mathrm{Tr}\left(4g_{(4)} - g_{(2)}^2\right) + \tfrac{1}{\mu}\epsilon_i{}^k\nabla_k\left[\nabla^l g_{lj}^{(2)} - \partial_j \mathrm{Tr} g_{(2)}\right].\end{aligned} \tag{2.11}$$

The $\{\rho, i\}$ equations are simply derivatives of combinations of the above equations and place no new constraints on $g^{(4)}$. These equations, and the Fefferman-Graham expansion of the Ricci and Cotton tensors, are included in the appendix. Note that (2.11) also has a critical point $\mu = 3$ at which one of the constraints on $g^{(4)}$ is modified.

Analysis of these equations is greatly simplified when considering solutions which asymptote to the exact anti-de Sitter metric at the boundary. In light-cone coordinates $u, v = t \pm \phi$, this is equivalent to the choice $g_{uv}^{(0)} = -1$. An orientation is chosen so that $\sqrt{-g}\epsilon^{\rho v u} = +1$.

Then the equations (2.8) and (2.9) become

$$\begin{aligned} \mathrm{Tr} g_{(2)} = 2g^{(2)}_{uv} &= 0 \qquad (2.12) \\ \left(1-\frac{1}{\mu}\right)\partial_v g^{(2)}_{uu} &= 0 \\ \left(1+\frac{1}{\mu}\right)\partial_u g^{(2)}_{vv} &= 0 \qquad (2.13) \end{aligned}$$

For all positive μ, $g^{(2)}_{vv} = \bar{L}(v)$, a function of v only; however, the equation for $g^{(2)}_{uu}$ disappears at $\mu = 1$, leaving this component of the metric unconstrained by the equations of motion. The critical point $\mu = 1$ is considered in the next section, and for now only $\mu \neq 1$ is considered. At $\mu \neq 1$, the solution reduces to the Einstein solution $g^{(2)}_{uu} = L(u)$.

The fourth order equations simplify considerably from (2.10) and (2.11) to

$$\begin{aligned} g^{(4)}_{uv} &= -\frac{1}{4}g^{(2)}_{uu}g^{(2)}_{vv} - \frac{1}{8\mu}\partial_v^2 g^{(2)}_{uu} \\ \left(1-\frac{3}{\mu}\right)g^{(4)}_{uu} &= \frac{-1}{4\mu}\partial_u\partial_v g^{(2)}_{uu} \\ \left(1+\frac{3}{\mu}\right)g^{(4)}_{vv} &= 0 \qquad (2.14) \end{aligned}$$

These equations possess their own critical point $\mu = 3$. When $\mu \neq 3$, the equations completely determine the components of $g_{(4)}$ in terms of the $g_{(2)}$, and the solution is either the chiral solution (2.36) when $\mu = 1$ or the Einstein solution (2.21) for $\mu \neq 1, 3$. However, at $\mu = 3$, one of the equations disappears, leaving $g^{(4)}_{uu}$ unconstrained, and the full solution depends on the three functions $g^{(2)}_{uu} = L(u)$, $g^{(2)}_{vv} = \bar{L}(v)$, and $g^{(4)}_{uu} = F(u,v)$.

This feature is repeated in the sixth order equations:

$$
\begin{aligned}
g^{(6)}_{uv} &= \left(\frac{-1}{3}-\frac{1}{6\mu}\right) g^{(2)}_{vv} g^{(4)}_{uu} - \frac{1}{12\mu}\partial^2_- g^{(4)}_{uu} \\
\left(1-\frac{5}{\mu}\right) g^{(6)}_{uu} &= \left(1-\frac{6}{\mu}\right)\left[\frac{-2}{3} g^{(2)}_{uu} g^{(4)}_{uv} - \frac{1}{6}\left(g^{(2)}_{uu}\right)^2 g^{(2)}_{vv}\right] + \frac{1}{24}\left(1-\frac{3}{\mu}\right) g^{(2)}_{uu}\partial^2_v g^{(2)}_{uu} \\
&\quad -\frac{1}{6\mu}\partial_u\partial_v g^{(4)}_{uu} + \frac{1}{6\mu}\partial^2_u g^{(4)}_{uv} + \frac{1}{24\mu} g^{(2)}_{vv}\partial^2_u g^{(2)}_{uu} + \frac{1}{24\mu}\left(\partial_v g^{(2)}_{uu}\right)^2 \\
\left(1+\frac{5}{\mu}\right) g^{(6)}_{vv} &= \left(1+\frac{6}{\mu}\right)\left[\frac{-2}{3} g^{(2)}_{vv} g^{(4)}_{uv} - \frac{1}{6}\left(g^{(2)}_{vv}\right)^2 g^{(2)}_{uu}\right] + \frac{1}{24} g^{(2)}_{vv}\partial^2_v g^{(2)}_{uu} - \frac{1}{6\mu}\partial^2_v g^{(4)}_{uv} \\
&\quad -\frac{1}{24\mu} g^{(2)}_{uu}\partial^2_v g^{(2)}_{vv} + \frac{1}{6\mu}\partial_v\partial_u g^{(4)}_{vv} - \frac{1}{24\mu}\left(\partial_v g^{(2)}_{vv}\right)\left(\partial_v g^{(2)}_{uu}\right) \qquad (2.15)
\end{aligned}
$$

At $\mu \neq 5$, $g_{(6)}$ is determined entirely in terms of the lower order terms. At the critical value $\mu = 5$, the $g^{(6)}_{uu}$ component is unconstrained. The equations are easily solved in each of the cases $\mu = 3$, $\mu = 5$ and $\mu \neq 1, 3, 5$ and displayed in Table (2.1).

	$\mu = 3$	$\mu = 5$	$\mu \neq 1,3,5$
$g^{(4)}_{uu}$	$F(u,v)$	0	0
$g^{(4)}_{uv}$	$-\frac{1}{4}L(u)\bar{L}(v)$	$-\frac{1}{4}L(u)\bar{L}(v)$	$-\frac{1}{4}L(u)\bar{L}(v)$
$g^{(6)}_{uu}$	$\frac{1}{12}\partial_u\partial_v F$	$F(u,v)$	0
$g^{(6)}_{uv}$	$\frac{-7}{18}F\bar{L} - \frac{1}{36}\partial^2_v F$	0	0

Table 2.1: The non-critical $\mu \neq 1$ solutions to sixth order. In all cases, $g^{(4)}_{vv} = g^{(6)}_{vv} = 0$. Together with (2.36), these are the complete set of solutions with the un-modified Fefferman-Graham expansion (2.7) that asymptote to the exact AdS metric at the boundary.

When $\mu \neq 1, 3, 5$, $g_{(6)}$ is identically zero, and the full solution (with the un-modified Fefferman-Graham expansion (2.7)) is equivalent to the Einstein solution. For generic values of μ, I have expanded the equations out to tenth order using Maple and have confirmed that $g_{(8)} = g_{(10)} = 0$. Additionally,the Cotton tensor $C_{\mu\nu} = 0$ to tenth order, indicating that the "traditional" Fefferman-Graham expansion (2.7) contains only solutions of Einstein gravity at generic values of the mass parameter.

The solutions at $\mu = 3, 5$ share several important features. First, the full solution is characterized by three functions $g^{(2)}_{uu} = L(u)$, $g^{(2)}_{vv} = L(v)$ and the unconstrained term $g^{(4)}_{uu} =$

$F(u,v)$ in the case $\mu = 3$ or $g^{(6)}_{uu} = F(u,v)$ in the case $\mu = 5$. The first nonzero component of the Cotton tensor is

$$C_{uu} = \begin{cases} \left(\frac{1}{2}\partial_u\partial_v F\right) \mathrm{e}^{-2\rho} & \mu = 1 \\ 12F\mathrm{e}^{-2\rho} & \mu = 3 \\ 60F\mathrm{e}^{-4\rho} & \mu = 5 \end{cases} \tag{2.16}$$

Thus at $\mu = 3, 5$, the requirement for a non-Einstein solution is simply that $F \neq 0$. In contrast, at the chiral point $\mu = 1$, non-Einstein solutions require the more stringent requirement that $\partial_u\partial_v F \neq 0$.

These results point to two possible explanation: either i) (2.7) is not the correct expansion for asymptotically anti-de Sitter solutions of TMG, or ii) the non-Einstein propagating modes exist only at $\mu = 1, 3, 5, \ldots$.[1] However, the second explanation conflicts with earlier research. Perturbative methods have found massive propagating modes for a wide range of parameter space [1, 86, 2, 143, 144, 145]. Similarly, non-perturbative degree-of-freedom counting techniques have found a single propagating (non-Einstein) degree of freedom at all values of the mass parameter [138, 139, 137, 143]. Thus we are lead to the conclusion that (2.7) is not the correct expansion for topologically massive gravity.

2.4 A Modified Fefferman-Graham Expansion for TMG

The failure of the traditional Fefferman-Graham expansion in TMG is not surprising, since it was derived for metrics satisfying the Einstein equations. As noted in [55], the bulk equations of motion determine the correct expansion, and we should expect modifications of general relativity to exhibit different asymptotic behavior. The goal of this section is to determine the appropriate expansion of TMG at all values of the mass parameter.

Because all solutions of Einstein gravity are also solutions of TMG, at least the terms in (2.7) are needed in the expansion for TMG. However, these terms alone do not capture the

[1]Note that inclusion of a log term $\rho h^{(2)}$ in the expansion (2.7) does not resolve this problem, since the log term is only non-zero at the critical point $\mu = 1$.

propagating degree of freedom of TMG: at $\mu \neq 1, 3, 5, \ldots$, the series (2.7) still terminates at $g_{(4)}$, and the Cotton tensor vanishes to all orders. Thus to include all Einstein and non-Einstein solutions, at all values of the mass parameter, I propose adding a new term in the expansion

$$g_{ij} = \mathrm{e}^{2\rho}\left(g_{ij}^{(0)} + \mathrm{e}^{-2\rho}g_{ij}^{(2)} + \mathrm{e}^{-4\rho}g_{ij}^{(4)}\right) \qquad \text{Einstein} \tag{2.17}$$

$$+\, \mathrm{e}^{2\rho}\left(\mathrm{e}^{-n\rho}g_{ij}^{(n)} + \mathrm{e}^{-(n+2)\rho}g_{ij}^{(n+2)} + \cdots\right) \qquad \text{non-Einstein} \tag{2.18}$$

with the exponent n to be determined by the equations of motion. However, no assumptions are made a priori about n.

Note that the presence of the $g^{(n)}$ term generically turns on an entire series of terms of order $mn + 2k$, where m, k are positive integers. Thus, if $n > 2$, the next term in the expansion occurs at order $n + 2$ and includes terms linear in $g^{(2)}$ and $g^{(n)}$. However, when $n < 2$, the next term in the series occurs at order $2n$ and contains terms quadratic in $g^{(n)}$. To keep things simple, I only address the case $n > 2$.

The technique now is to expand the equations of motion to "first order" in $g^{(2)}$ and "first order" in $g^{(n)}$. As long as $n \neq 1, 2$, the "first order" equations for $g^{(2)}$ contain no $g^{(n)}$ terms and are unchanged from (2.8) and (2.9). Similarly, the first order equations for $g^{(n)}$ will not include $g^{(2)}$ for generic n. Note that this is true for both cases $n < 2$ and $n > 2$ discussed above, and so the first order equations for $g^{(n)}$ are the same for both cases. The first contraint comes from the expansion of the Ricci scalar,

$$^{3}R = -6 + \mathrm{e}^{-n\rho}(3n - n^2)\mathrm{Tr}g^{(n)} + \cdots \tag{2.19}$$

and the fact that subleading terms must vanish. Then the trace of $g^{(n)}$ must vanish for all $n \neq 3$. The trace constraint from the Ricci scalar vanishes for $n = 3$. However, it is picked up by the $\{\rho\rho\}$ equation of motion at precisely this point (see appendix), and so the trace of $g^{(n)}$ vanishes at all values of n.

Other equations are derived in the appendix. Of particular interest is the $\{ij\}$ equation of motion:

$$\frac{n}{2}(n-2)\left[\delta_i^k + \frac{n-1}{\mu}\epsilon_i{}^k\right] g_{kj}^{(n)} = 0 \tag{2.20}$$

The constraints on $g^{(n)}$ from the $\{ij\}$ equation vanish when $n = 0, 2$, and these correspond to the Einstein solutions. Additionally, a "chiral half" of the equation vanishes when $n = \mu + 1$, and part of $g^{(n)}$ is unconstrained by the equations of motion. The $\{\rho i\}$ equation is essentially just the derivative of (2.20) and places no new constraint on $g^{(n)}$. Thus the asymptotic behavior of the non-Einstein solutions, i.e. the propagating massive graviton modes, is *determined by the value of the mass parameter*.

Again, the situation becomes transparent in the light-cone coordinates of the previous section. The solution for $g_{(2)}$ and $g_{(4)}$ at non-critical values is

$$\begin{aligned} g_{uu}^{(2)} &= L(u) \qquad \text{and} \qquad g_{vv}^{(2)} = \bar{L}(v) \\ g_{uv}^{(4)} &= -\frac{1}{4}L(u)\bar{L}(v). \end{aligned} \tag{2.21}$$

As noted previously, the solutions encompassed in these terms contain *only* the ordinary Einstein solutions [46], including the BTZ black hole [32]. Equation (2.20) becomes

$$\left(1 - \frac{n-1}{\mu}\right) g_{uu}^{(n)} = 0 \tag{2.22}$$

$$\left(1 + \frac{n-1}{\mu}\right) g_{vv}^{(n)} = 0 \tag{2.23}$$

Now the solution for the non-Einstein branch becomes clear. For all positive μ, $g_{vv}^{(n)} = 0$, and $g_{uu}^{(n)}$ becomes unconstrained at $n = \mu + 1$. Also, note that the equations are invariant under a chirality flip, $\mu \to -\mu$ and $u \leftrightarrow v$.

To "second order", the equations of motion contain terms linear in both $g^{(n)}$ and $g^{(2)}$, and the generic solution for the non-Einstein branch is

$$\begin{aligned} g^{(n)}_{uu} &= F(u,v) \\ g^{(n+2)}_{uu} &= \frac{1}{2\mu+6}\partial_u\partial_v F \\ g^{(n+2)}_{uv} &= -\frac{(\mu+1)^2-2}{2\mu(\mu+3)}\bar{L}F - \frac{1}{2\mu(\mu+3)}\partial_v^2 F \\ g^{(n+4)}_{uu} &= \cdots . \end{aligned} \tag{2.24}$$

For the full solution (2.21) and (2.24), we find a non-zero Cotton tensor (and therefore non-Einstein solutions) *only* when $F(u,v) \neq 0$. The first non-zero component of the Cotton tensor is $C_{uu} = \left(\frac{1}{2}n^3 - \frac{3}{2}n^2 + n\right) F\mathrm{e}^{(2-n)\rho}$. Thus, the function F contains the massive, propagating modes unique to TMG.

This explains the behavior observed in the previous section. There it was found that the Fefferman-Graham expansion with only even powers of e^{ρ} yields only Einstein solutions *except* at the critical points $\mu = 1, 3, 5, \ldots$. At these values, some of the equations vanish, and only at these critical points does the Fefferman-Graham expansion contain non-Einstein solutions. From (2.20), it is clear that these are precisely the values of the mass parameter at which the non-Einstein branch of solutions labelled by $n = \mu + 1$ overlaps with the Einstein branch, i.e., these are the values at which n is a positive even integer. As with the solutions at $\mu = 3, 5$ in Table (2.1), the general solution is characterized by the three functions $g^{(2)}_{uu} = L(u)$, $g^{(2)}_{vv} = \bar{L}(v)$, and $g^{(n)}_{uu} = F(u,v)$, and all higher-order terms in the expansion (2.18) are fixed in terms of these three functions.

The non-Einstein branch of solutions satisfies Brown-Henneaux boundary conditions in the parameter range $\mu \geq 1$. The solution (2.24) indicates that larger values of the mass parameter correspond to steeper decay, and this behavior has an obvious interpretation. The coupling μ^{-1} gives the relative weight between the Einstein and Chern-Simons contributions to the action (1.19). As μ increases, a TMG perturbation from an Einstein background decreases in strength and the massive modes are more localized in the interior.

2.4.1 Critical Gravity

At the critical point $\mu = 1$, the non-Einstein branch of solutions "overlaps" the $n = 2$ Einstein solutions. When two linearly independent solutions degenerate, a log branch appears, with solution given by

$$h^{\text{new}}_{\mu\nu} = \lim_{\mu \to 1} \frac{g^{(n)}_{\mu\nu} - g^{(2)}_{\mu\nu}}{\mu - 1} \tag{2.25}$$

Then the asymptotic expansion acquires a log term [52, 55]

$$g_{ij} = \mathrm{e}^{2\rho} \left(g^{(0)}_{ij} + \rho \mathrm{e}^{-2\rho} h^{(2)}_{ij} + \mathrm{e}^{-2\rho} g^{(2)}_{ij} + \cdots \right). \tag{2.26}$$

The theory at $\mu = 1$ with this expansion has been referred to as "critical topologically massive gravity" (CTMG).

This approach was first used by Grumiller and Johansson [52] to construct new log modes to the linearized equations of motion, and their discovery generated much discussion regarding the consistency of these boundary conditions and their relationship to the chirality and positivity conjectures of [2, 54]. While the log term in the expansion (2.26) breaks the Brown-Henneaux boundary conditions [44] for Einstein gravity, this does not necessarily spoil the asymptotic symmetries of the theory. Even pure Einstein gravity, coupled to massive scalar fields, can acquire a different asymptotic expansion with relaxed fall-off [47, 146, 48, 147, 49, 56].

The requirements for consistency were explicitly spelled out in [50], following early work on boundary conditions by Henneaux and Teitelboim [148]:

1. They are invariant under the Anti-de Sitter group.
2. They decay sufficiently slowly to allow interesting solutions.
3. They decay sufficiently fast to yield finite charges.

Several papers explored this question [51, 50, 53] and confirmed that the asymptotic symmetry of CTMG with expansion (2.26) consists of two copies of the Virasoro algebra. Additionally, the variational principle was explored, and a finite stress-energy tensor for CTMG was constructed in [52, 55]. So with conditions (1) and (3) met, I proceed to construct interesting asymptotic solutions for CTMG.

The procedure follows that of the generic case: plug the expansion (2.26) into the equations of motion; collect terms of the same order; and solve order-by-order. The equations to first order in $h^{(2)}$ are

$$\begin{aligned} \mathrm{Tr}\, h^{(2)} &= 0 \qquad (2.27) \\ \left(\delta_i^n + \frac{1}{\mu}\epsilon_i{}^n\right) h_{nj}^{(2)} &= 0 \qquad (2.28) \\ \left(\delta_i^n + \frac{1}{\mu}\epsilon_i{}^n\right) \nabla^j h_{nj}^{(2)} &= 0 \qquad (2.29) \end{aligned}$$

Thus it appears as if the equations for $h^{(2)}$ follow the same pattern as those for $g^{(n)}$, namely, that one component of $h^{(2)}$ is completely unconstrained. In light-cone coordinates, $h_{uv}^{(2)}$ and $h_{vv}^{(2)}$ vanish, and the above equations for $h_{uu}^{(2)}$ become

$$\begin{aligned} \left(1 - \frac{1}{\mu}\right) h_{uu}^{(2)} &= 0 \qquad (2.30) \\ \left(1 - \frac{1}{\mu}\right) \partial_v h_{uu}^{(2)} &= 0 \qquad (2.31) \end{aligned}$$

Thus part of the divergence equation (2.29) vanishes at first order. However, precisely this part that vanishes is recovered at *second order*

$$\left(\delta_i^n + \frac{1}{\mu}\epsilon_i{}^n\right)\left[\left(\nabla^j g_{jk}^{(2)} - \partial_k \mathrm{Tr} g^{(2)}\right) - \frac{1}{2}\left(\nabla^j h_{jk}^{(2)} - \partial_k \mathrm{Tr} h^{(2)}\right)\right] + \frac{1}{\mu}\epsilon^{nk}\nabla_k h_{ni}^{(2)} = 0 \qquad (2.32)$$

In light-cone coordinates, this reduces to

$$\left(1 - \frac{1}{\mu}\right)\left[\partial_v g_{uu}^{(2)} - \partial_u g_{uv}^{(2)} - \frac{1}{2}\left(\partial_v h_{uu}^{(2)} - \partial_u h_{uv}^{(2)}\right)\right] + \frac{1}{\mu}\left(\partial_v h_{uu}^{(2)} - \partial_u h_{uv}^{(2)}\right) = 0 \qquad (2.33)$$

Thus, equations (2.32) and (2.29) together force the entire divergence of $h^{(2)}$ to vanish.

These equations have a couple of interesting features. First, note that the equations of motion force $h^{(2)}$ to vanish at $\mu \neq 1$, and thus the log term is not present in the general expansion (2.18). Second, the trace *and divergence* of $h^{(2)}$ vanish, unlike the non-Einstein term in the general expansion, where only a chiral half of the divergence of $g^{(n)}$ vanishes. Oddly, now the divergence of $g^{(2)}$ is not completely constrained. The solution to subleading order in light-cone coordinates is

$$h^{(2)}_{uu} = L(u) \tag{2.34}$$

$$g^{(2)}_{uu} = F(u,v), \qquad g^{(2)}_{vv} = \bar{L}(v) \tag{2.35}$$

Similar to the solution at generic μ, the critical solution is characterized by three functions. Now, however, the general solution possesses *two* features not available in Einstein gravity: the log term is allowed, and the constraints on $g^{(2)}$ are relaxed.

After the existence of log modes with finite energy were discovered [52], Maloney, Song and Strominger [54] modified their definition of chiral gravity to be the critical theory $\mu = 1$ with Brown-Henneaux boundary conditions imposed. Log gravity was defined as the full theory with relaxed boundary conditions, and chiral gravity was (thought to be) the subset with the log branch turned off. They conjectured that solutions of chiral gravity have non-negative energy, and they further conjectured that all solutions of chiral gravity are locally AdS_3, i.e. are Einstein solutions. They proved that all stationary, axially symmetric solutions are just the ordinary Einstein solutions; however, they were not able to completely rule out the existence of non-Einstein solutions, and indeed we can see from the general solution (2.35) that non-Einstein solutions persist even without the log branch of solutions. In light cone coordinates, the solution to third order ($g_{(6)}$) is

$$
\begin{aligned}
g^{(2)}_{uu} &= F(u,v) \\
g^{(2)}_{vv} &= \bar{L}(v) \\
g^{(4)}_{uu} &= \frac{1}{8}\partial_u\partial_v F \\
g^{(4)}_{uv} &= -\frac{1}{4}\bar{L}F - \frac{1}{8}\partial_v^2 F \\
g^{(6)}_{uu} &= \frac{1}{8}F\partial_v^2 F + \frac{1}{96}\partial_u^2\partial_v^2 F - \frac{1}{96}\left(\partial_v F\right)^2 \\
g^{(6)}_{uv} &= -\frac{1}{16}\bar{L}\partial_u\partial_v F - \frac{1}{96}\partial_u\partial_v^3 F \\
g^{(6)}_{vv} &= \frac{1}{9}\bar{L}\partial_v^2 F + \frac{1}{144}\left(\partial_v\bar{L}\right)\left(\partial_v F\right) + \frac{1}{288}\partial_v^4 F
\end{aligned}
\tag{2.36}
$$

In agreement with [141], I find that Einstein solutions are the subset for which the function F depends only on u. When $\partial_v F = 0$, the Cotton tensor vanishes to all orders, and the expansion (2.26) terminates at $g_{(4)}$. Note that the requirement for vanishing Cotton tensor at the chiral value is more stringent that at non-chiral values, for which $F = 0 \leftrightarrow C_{\mu\nu} = 0$. As noted in [141], the existence of non-Einstein solutions to chiral gravity does not necessarily refute the positivity conjecture, since any solutions will have to satisfy regularity conditions in the interior.

2.5 Comparison to Known AAdS Solutions

Partial confirmation for the validity of the general expansion (2.18) can be obtained by a comparison with known asymptotically anti-de Sitter solutions. Many such solutions exist, and I have chosen two to explore in more detail: AdS pp-waves, which are exact solutions with finite Fefferman-Graham expansion; and the CDWW modes [86], which are solutions to the linearized equations and do not have a terminating expansion. Both solutions exist over a wide range of values for the mass parameter.

The surprising thing about the expansion (2.18) is not that it contains non-integral powers of e^ρ, but that the general expansion was not discovered sooner. Indeed, to my knowledge the earliest example of an exact solution with non-integral powers of the radius was discovered nearly twenty years ago by Nutku [71]. Additionally, AdS waves have been discovered and rediscovered several times over the years [71, 82, 84, 85, 142, 86], and these solutions are discussed next.

2.5.1 AdS Waves

AdS waves are a kind of exact gravitational wave propagating on an AdS background, and are conformally related to the *p*lane-fronted, *p*arallel-ray (pp) waves on a Minkowski background. As in the case of pp-waves, the wave fronts of AdS waves are null surfaces defined by u, v =const., and the rays follow a null Killing field k^μ. However, the wave fronts of AdS waves are not *p*lanes but hyperboloids with constant curvature proportional to $-\ell^{-2}$. Additionally, the null Killing field k^μ is a geodesic but is not a closed one-form and is therefore not covariantly constant, or, *p*arallel. Thus AdS and pp waves have different geometric properties. For a review of the properties of AdS waves, see [149, 150]

The metric of AdS waves is given by [151, 152]

$$g_{\mu\nu} = g_{\mu\nu}^{\text{AdS}} - F k_\mu k_\nu \tag{2.37}$$

where F is a smooth function independent of the integral parameter along k^μ. Choosing a null vector field $k^\mu \partial_\mu = \partial_v$, the AdS wave metric in Gaussian normal coordinates

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 - 2e^{2\rho}\mathrm{d}u\mathrm{d}v - F(\rho, u)\mathrm{d}u^2 \tag{2.38}$$

AdS wave solutions of topologically massive gravity were found in [82, 85] and further developed in [84, 142, 153]. The solution can be written in terms of a single unconstrained function of u,

$$F(\rho, u) = F_1(u)\mathrm{e}^{(1-\mu)\rho} \tag{2.39}$$

Note that this solution is a more restricted version of the general ansatz (2.18) that nonetheless has the same radial behavior. The solution (2.38) is exact, with finite Fefferman-Graham expansion.

The AdS waves also exhibit the same critical points as the general solution. At the critical points $\mu = 1$, the solution is [85, 142, 154, 155]

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 - 2e^{2\rho}\mathrm{d}u\mathrm{d}v - F_1(u)\rho\mathrm{d}u^2 \tag{2.40}$$

Also of interest is the fact that the profile function F satisfies the Klein-Gordon equation

$$\Box F = \mu_{\text{eff}}^2 F \tag{2.41}$$

where effective mass μ_{eff} is defined by $\mu_{\text{eff}}^2 = \mu^2 - 1$, or $\mu_{\text{eff}}^2 = \mu^2 - \ell^{-2}$ with the AdS radius reinstated. Thus the presence of a cosmological constant reduces the physical topological mass. However, at the critical point $\mu = 1$, the log mode (2.40) does not satisfy a Klein-Gordon equation.

2.5.2 Revisiting CDWW

Previously, Carlip et al. [86, 1] found a complete set of solutions – the CDWW modes – to the linearized equations of motion at all values of μ. These solutions share some important features with (2.24), namely

- the solutions are invariant under $\mu \to -\mu$ and a chirality flip ($u \leftrightarrow v$), and
- they exhibit a μ-dependent asymptotic behavior, with different fall-off conditions for each component of the metric.

Given the similarities, it's natural to ask if (2.24) contains the CDWW modes.

However, CDWW solved for the linearized Einstein tensor, and these modes must first be converted into perturbations of the metric in Gaussian normal coordinates before a direct comparison can be made. This can always be done in three dimensions, since a perturbation of the Einstein tensor uniquely determines a perturbation in the metric. The solutions for each component of the linearized Einstein tensor are the Bessel functions given in eqn. (A.28) of [1], for example

$$\mathcal{H}_{uu} = \frac{\omega_+^2}{\omega} \exp[\mathrm{i}(\omega_+ u + \omega_- v)] z J_{\mu-2}(\omega z) + h.c. \tag{2.42}$$

is shown here for comparison. In this expression, z is the radial coordinate related to the ρ of the previous section by $z = \mathrm{e}^{-\rho}$; ω_+ and ω_- are eigenvalues of the $SL(2,\mathbb{R})$ generators $\mathrm{i}\partial_u$ and $\mathrm{i}\partial_v$, and $\omega^2 = -2\omega_+\omega_-$. Eqn (5.5) of [1] relates the Einstein tensor to the metric perturbations via differential equations such as

$$\mathcal{H}_{uu} = -\frac{1}{2} z\partial_z (z\partial_z + 2)\hat{h}_{uu}, \tag{2.43}$$

where the metric in this coordinate system is given by

$$\mathrm{d}s^2 = \frac{\mathrm{d}z^2}{z^2} + \frac{2}{z^2}\mathrm{d}u\mathrm{d}v + \left(\hat{h}_{uu}\mathrm{d}u^2 + 2\hat{h}_{uv}\mathrm{d}u\mathrm{d}v + \hat{h}_{vv}\mathrm{d}v^2\right). \tag{2.44}$$

The final step is to expand the modes (2.42) in powers of z and solve (2.43) for the metric perturbations order by order. The CDWW modes as *metric peturbations* are

$$\begin{aligned}
\hat{h}_{uu} &= \frac{-\omega_+^2 \omega^{\mu-3} \exp[\mathrm{i}(\omega_+ u + \omega_- v)]}{2^{\mu-3}(\mu+1)\Gamma(\mu)} \left[z^{\mu-1} + \frac{\omega^2 z^{\mu+1}}{2^2(\mu+3)} + \frac{\omega^4 z^{\mu+3}}{2^5(\mu+5)(\mu+3)\mu}\right] + \cdots \\
\hat{h}_{uv} &= \frac{\omega^{\mu+1} \exp[\mathrm{i}(\omega_+ u + \omega_- v)]}{2^{\mu}(\mu+3)\Gamma(\mu+2)} \left[z^{\mu+1} - \frac{\omega^2}{2^2(\mu+5)} z^{\mu+3}\right] + \cdots \\
\hat{h}_{vv} &= \frac{-\omega_-^2 \omega^{\mu+1} \exp[\mathrm{i}(\omega_+ u + \omega_- v)]}{2^{\mu+1}(\mu+5)\Gamma(\mu+4)} z^{\mu+3} + \cdots
\end{aligned} \tag{2.45}$$

With the identification

$$F(u,v) = -\frac{\omega_+^2 \omega^{\mu-3}}{(\mu+1)2^{\mu-3}\Gamma(\mu)} \exp[\mathrm{i}(\omega_+ u + \omega_- v)] \tag{2.46}$$

the CDWW modes can be written as

$$\begin{aligned}
\hat{h}_{uu} &= F(u,v)z^{\mu-1} + \frac{z^{\mu+1}}{2(\mu+3)}\partial_u\partial_v F + \frac{(\mu+1)z^{\mu+3}}{8(\mu+5)(\mu+3)\mu}\partial_u^2\partial_v^2 F + \mathcal{O}(z^{\mu+5}) \\
\hat{h}_{uv} &= \frac{-z^{\mu+1}}{2\mu(\mu+3)}\partial_v^2 F - \frac{z^{\mu+3}}{4(\mu+5)(\mu+3)\mu}\partial_u\partial_v^3 F + \mathcal{O}(z^{\mu+5}) \\
\hat{h}_{vv} &= \frac{z^{\mu+3}}{4(\mu+5)(\mu+3)(\mu+2)\mu}\partial_v^4 F + \mathcal{O}(z^{\mu+5})
\end{aligned} \tag{2.47}$$

Note that the linear perturbation $\hat{h}_{ij}$ is related to the coefficients in the modified Fefferman-Graham expansion (2.18) by $g_{ij} = z^{-2}\hat{h}_{ij}$, i.e. the expansion used in (2.18) has an extra z^{-2} (or $\mathrm{e}^{2\rho}$) factored out. Comparison with the asymptotic solutions (2.24) and (2.36) reveals that CDWW correctly gives all numerical coefficients for pure derivatives of $F(u,v)$. However, the full solution is more than just a background Einstein solution plus the CDWW modes. It also contains interactions between the background and CDWW, e.g. the piece proportional to $\bar{L}F$ in (2.24). Additionally, nonlinear deviations from CDWW appear at higher orders. In the chiral solution (2.36), for example, CDWW fails to include the non-linear terms $F\partial_v^2 F$ and $(\partial_v F)^2$ in the $g_{uu}^{(6)}$ component, which is unsurprising given that the CDWW modes solve the linearized equations of motion.

While the $(\mu+5)$ and higher order solution at generic μ has not been found, solutions at integral values of μ have been explored using Maple, and several patterns emerge. For all μ examined, the first nonlinear deviation from CDWW appears in the g_{uu} component at order $2\mu+4$ and contains terms proportional to $F\partial_v^2 F$ and $(\partial_v F)^2$. In the case of odd μ, these deviations only appear at higher orders in the expansion (2.18). In the case of even μ, this nonlinearity turns on a new series of terms in the expansion (2.18), although these new terms do not represent new degrees of freedom. Note that, even at these higher orders, CDWW still gives the correct numerical coefficients of the pure derivative terms. I expect these features to hold for generic μ. Table 2.2 shows the first departure from CDWW for $\mu = 1$, 2 and 3.

	first deviation from CDWW
$\mu = 1$	$g_{uu}^{(6)} = \frac{1}{96}\partial_u^2\partial_v^2 F + \frac{1}{8}F\partial_v^2 F - \frac{1}{96}\left(\partial_v F\right)^2$
$\mu = 2$	$g_{uu}^{(8)} = \frac{17}{320}F\partial_v^2 F - \frac{1}{160}\left(\partial_v F\right)^2$
$\mu = 3$	$g_{uu}^{(10)} = \frac{1}{11520}\partial_u^3\partial_v^3 F - \frac{1}{240}\left(\partial_v F\right)^2 + \frac{11}{360}F\partial_v^2 F$
$\mu = 4$	$g_{uu}^{(12)} = \frac{9}{448}F\partial_v^2 F - \frac{1}{336}\left(\partial_v F\right)^2$

Table 2.2: Non-linearities in the g_{uu} component.

The CDWW modes at $\mu = 1$ were originally proposed as a counter-example to the positivity theorem – while the modes blow up in the interior, Carlip et. al [86] created finite-energy superpositions of the modes with negative energy. However, the original CDWW modes (2.45), as well as several other proposed counterexamples to positivity, were shown to develop a linearization instability at second order [54, 156]. For example, the GKP modes [140] at second order require the relaxed logarithmic boundary conditions and thus are not a linear approximation to an exact solution of chiral gravity. This is not the case with CDWW – the nonlinear completion of CDWW, including a background Einstein metric, interaction terms, and the nonlinear terms of Table 2.2, *is* a solution of chiral gravity satisfying strict Brown-Henneaux boundary conditions. This extended CDWW should be reconsidered as a potential candidate for violating the positivity theorem. The next step, left for the interested reader, is to construct finite-energy superpositions of this extended CDWW.

2.6 Discussion

It remains an open question whether physical non-Einstein solutions of chiral gravity exist. The technique used here - working from the boundary in and solving the equations of motion order by order - offers an alternative approach to the perturbative techniques used in most of the papers on the subject. Using this approach, I have constructed the general solution of TMG with strict Brown-Henneaux boundary conditions at all values of the mass parameter $\mu \geq 1$. The solutions at each μ share the same basic structure and can be written as the sum of an Einstein metric, a purely non-Einstein metric, and interactions between the two.

The non-Einstein solution is characterized by a single function F which can be expanded on to the CDWW modes of [86], with nonlinear corrections to CDWW appearing at higher order. In particular, the general solution at the critical value $\mu = 1$ contains these extended CDWW modes, and these modes do not require the relaxed boundary conditions of log gravity. Since chiral gravity shares these features with non-chiral TMG, which is generally thought to be unstable,[2] this result raises questions about the classical stability of chiral gravity. However, the task of constructing physically significant non-Einstein solutions to chiral TMG remains incomplete. Of the solutions (2.36) examined in [141] all contained either naked singularities or naked closed timelike curves which violate causality, rendering the solutions unphysical. However, they considered only a subset of functions F with finite Fefferman-Graham expansion, and some superposition of CDWW modes offers an intriguing possibility.

Additionally, this work offers some insight on the proposals of Maloney, Song and Strominger [54]. They originally envisioned chiral gravity as critical cosmological topologically massive gravity with Brown-Henneaux boundary conditions, and conjectured that chiral gravity may exist as a consistent theory by itself or as a superselection sector of log gravity. They further claimed that the CDWW modes do not provide a counterexample to positivity because the theory suffers a linearization instability at the critical point, and the CDWW modes are not chiral.

However, this work does not support that claim. While the nonlinear completion of the GKP modes *does* require the log boundary conditions, the nonlinear completion of the CDWW modes at the critical point does not require a relaxation from Brown-Henneaux. Instead, these results can be interpreted as supporting Carlip's proposition [156] that the theory contains *three* sectors: the log theory with relaxed boundary conditions, associated with non-zero $h^{(2)}$; a sector with bulk modes, which requires $g^{(2)}_{uu}(u, v)$; and a chiral sector with no bulk modes, for which $g^{(2)}_{uu} = L(u)$.

[2] See [157] for an argument that the BTZ black hole is stable to perturbations at all values of μ.

Chapter 3

New Massive Gravity

The asymptotic behavior of new massive gravity (NMG) is analyzed for all values of the mass parameter satisfying the Breitenlohner-Freedman bound. The traditional Fefferman-Graham expansion fails to capture the dynamics of NMG, and new terms in the asymptotic expansion are needed to include the massive graviton modes. New boundary conditions are discovered for a range of values $-1 < 2m^2l^2 < 1$ at which non-Einstein modes decay more slowly than the Brown-Henneaux boundary conditions. The holographically renormalized stress tensor is computed for these modes, and the relevant counterterms are identified up to unphysical ambiguities.

3.1 Introduction

Higher derivative extensions of general relativity have recently been the focus of much attention. String theory and other quantum gravity models generically predict the existence of such terms, and they generally improve the renormalizability of the theory. This increased focus is also partly motivated by recent models of low-energy modifications of general relativity that could provide an alternative to dark energy [13]. However, analysis of the dynamics of such theories is in general a difficult task, complicated by the nonlinearity of the equations of motion. The situation is improved in lower-dimensional models, where the reduction in degrees of freedom simplifies the dynamics while retaining many of the properties of higher-dimensional models.

New massive gravity (NMG) is a particular three-dimensional model with a specific combination of curvature-squared terms in the action [3, 4]. Generically, theories with curvature-squared terms contain massive spin-2 and massless ghost-like scalar modes; in NMG, however, the coefficients in the action are chosen in such a way that the scalar modes are excised from the theory [96]. Unlike its cousin topologically massive gravity (TMG) [39, 38], the theory is parity preserving, and was originally investigated as the non-linear completion of Fierz-Pauli theory. NMG shares some features with TMG. In particular, it admits anti-de Sitter (AdS_3) spacetime as a vacuum solution and permits a larger class of asymptotically anti-de Sitter solutions than Einstein gravity alone [59, 58, 110, 112]. Thus NMG formulated around an AdS background has proven a fruitful toy model for exploring the AdS/CFT correspondence [123, 119, 133, 115, 125, 130, 124, 136, 131, 129, 132].

NMG also shares with TMG the undesirable feature that massive gravitons and BTZ black holes appear with opposite sign energy, reigniting discussions about the consistency of 3D (topologically or new) massive gravity about an AdS_3 vacuum. Discussions in TMG centered on the consistency of strong boundary conditions that (it was hoped) could truncate massive graviton ghosts and render the background stable [2, 54]. To be specific, TMG possesses a critical point in parameter space at which both Brown-Henneaux [44] and relaxed log boundary conditions [50, 51] seem acceptable, in the sense that they yield finite charges at infinity and preserve the asymptotic symmetries. The theory with log boundary conditions is conjectured to be dual to a logarithmic conformal field theory, which is known to be non-unitary, while the theory with Brown-Henneaux boundary conditions is dual to a chiral CFT and is conjectured to be stable (see [54] for a review). NMG similarly possesses critical points in the space of parameters at which multiple boundary conditions are possible. This discussion of boundary conditions in NMG is thus essential to conclude the stability of the theory and to establish which geometries contribute to the partition function of the quantum theory.

Early approaches to the question of appropriate boundary conditions used the linearized theory to identify and determine the consistency of possible boundary conditions [123, 119]. Attention has primarily focused on two critical points at which novel solutions appear. NMG possesses a chiral point, analogous to that of TMG, at which log deformations from BTZ are allowed [59, 110]. At another critical point, new type black holes [58], characterized by a kind of gravitational 'hair,' have been found that require a different relaxation of the Brown-Henneaux boundary conditions. However, beyond these two critical points, only the Brown-Henneaux boundary conditions have been investigated and shown to be consistent at all points in parameter space [119]. The possible relaxation from Brown-Henneaux at non-critical values has not been explored.

The Fefferman-Graham expansion [45] provides a natural tool for determining the asymptotic behavior of the metric and for addressing the question of appropriate boundary conditions for asymptotically (locally) anti-de Sitter spacetimes. In Einstein gravity, it has already proven to be an important tool in holographic renormalization (see [158] for a review), and in computation of correlation functions in the boundary CFT. Recently, it has been applied to NMG at a critical value of the coupling for the purpose of constructing the renormalized boundary stress tensor [132]. However, the expansion used in [132] applies only at that particular point in parameter space, and the authors point out that the generic asymptotic expansion remains unknown.

This paper explores the asymptotic expansion of the metric at all values of the mass parameter in NMG. After covering the basics of NMG, I review the definition of asymptotically anti-de Sitter spacetimes and the derivation of the Fefferman-Graham expansion, drawing particular attention to those steps that rely on the bulk equations of motion. These are the steps at which the derivation of the asymptotic expansion in NMG diverges from that of Einstein gravity. The traditional Fefferman-Graham expansion is then applied to NMG and shown to be insufficient for recovering the non-Einstein solutions, except at a few special points in the parameter space. The next section introduces a modified asymptotic expansion that captures both Einstein and non-Einstein solutions. All known exact asymptotically AdS solutions are shown to have asymptotic behavior given by this modified expansion. This approach correctly identifies the weakened asymptotics at the critical points found in previous studies and also finds new regions of parameter space at which the massive non-Einstein modes obey weaker-than-Brown-Henneaux fall off. The Brown-York boundary tensor is constructed for these modes in the parameter range $-1 < 2m^2 < 1$. Several possible counterterms are considered, and the renormalized boundary tensor is obtained up to unphysical ambiguities. The central charge of the dual CFT is determined by the trace of the renormalized stress tensor. The final section summarizes the results, with some comments on implications and future steps.

3.2 Setup and Equations of Motion for New Massive Gravity

The bulk action of new massive gravity (NMG) is given by [3, 4, 92]

$$S = \frac{\xi}{2\kappa^2}\int \mathrm{d}^3x\sqrt{-g}\left[\sigma R + 2\lambda + \frac{1}{m^2}K\right]\,, \tag{3.1}$$

where $2\kappa^2 = 16\pi G$, and the constants ξ and σ are introduced to control the overall sign of the action and the Einstein-Hilbert piece and take the values ±1. In addition to the gravitational constant G and the cosmological parameter λ, NMG contains the mass parameter m^2 of mass dimension two.

The new tensor K is a specific combination of curvature squared terms defined by

$$K = R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\,. \tag{3.2}$$

The equations of motion given by variation of the action with respect to the metric are

$$\xi\left[\sigma G_{\mu\nu} - \lambda g_{\mu\nu} + \frac{1}{2m^2}K_{\mu\nu}\right] = 0\,, \tag{3.3}$$

where

$$K_{\mu\nu} = 2\Box R_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu R - \frac{1}{2}\Box R g_{\mu\nu} + 4R_{\mu\alpha\nu\beta}R^{\alpha\beta} - \frac{3}{2}RR_{\mu\nu} - R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu} + \frac{3}{8}R^2 g_{\mu\nu}. \tag{3.4}$$

I will allow both positive and negative values of m^2 and consider both signs of the Einstein-Hilbert action; however, ξ will be set to unity from this point forward.

NMG admits an AdS_3 vacuum with effective cosmological constant Λ related to the bare cosmological parameter λ by

$$\Lambda = 2m^2\left[\sigma \pm \sqrt{1 - \frac{\lambda}{m^2}}\right]\,. \tag{3.5}$$

The AdS radius is given by $\Lambda = -\frac{1}{L^2}$. It will turn out convenient to use the effective cosmological constant and AdS radius through the remainder of the paper. Note that many other papers express their results in terms of the cosmological parameter λ appearing in the action, and the parameter $\lambda = -\ell^{-2}$, and care must be taken when comparing this paper to other work.

3.3 Failure of the "Traditional" Fefferman-Graham Expansion

Before investigating the possibility of a parameter-dependent expansion, I first demonstrate the need for a modified expansion by explicitly showing the failure of the "traditional" Fefferman-Graham expansion to capture the dynamics of NMG at all but a few critical values of the mass parameter. As noted in the previous section, this failure is unsurprising, since the Fefferman-Graham expansion was originally derived for metrics satisfying the Einstein equations.

First note that the coordinate system employed in (1.15) covers only part of the boundary, and it is more convenient to work in a global coordinate system. The radial coordinate transformation $z = \mathrm{e}^{-\rho/L}$ moves the boundary to infinity, and the expansion for three dimensional general relativity becomes

$$\begin{aligned} \mathrm{d}s^2 &= \frac{\mathrm{d}\rho^2}{L^2} + g_{ij}\mathrm{d}x^i\mathrm{d}x^j \\ g_{ij} &= e^{2(\rho/L)}\left(g_{ij}^{(0)} + e^{-2(\rho/L)}g_{ij}^{(2)} + e^{-4(\rho/L)}g_{ij}^{(4)} + \cdots\right). \end{aligned} \tag{3.6}$$

For simplicity, the AdS radius is fixed at $L = 1$, since it can always be reinstated later through dimensional analysis.

The technique now is to plug this expansion into the equations of motion and solve order by order. For the moment, consider the expansion (3.6) with the log term $\rho\mathrm{e}^{-2\rho}h^{(2)}$ excluded. This should not affect the results, as previous work has shown the log term to be consistent only at the chiral point $2m^2 = +\sigma$, and the goal here is the solution at generic values of m^2. In general, the boundary metric $g^{(0)}$ is a free field and is not fixed by the equations of motion. However, to simplify the expansion, attention is restricted to solutions which asymptote to exact AdS_3 with light-cone coordinates on the boundary, i.e. the boundary metric is chosen so $g_{uv}^{(0)} = -1$ with diagonal components vanishing.[1] Gauge-independent equations are given in the appendix.

In general relativity, the second order equations of motion fix the trace and divergence of $g^{(2)}$, and the other components are undetermined. This shows up in the vanishing of the $\{ij\}$ equations of motion. In new massive gravity, it is also true that the $\{ij\}$ equations

[1] Note the distinction between "asymptotically locally anti-de Sitter" (AlAdS) spacetimes, in which the boundary metric is treated as a free field and the curvature tensor approaches that of anti-de Sitter space near the boundary, and "asymptotically anti-de Sitter" (AAdS) spacetimes, in which the metric asymptotes to the exact AdS metric at the boundary. In a slight abuse of notation, I use AAdS to refer to both cases, though the context should make clear which notion is appropriate.

vanish identically, and only the $\{\rho\rho\}$ and $\{\rho i\}$ equations restrict the metric. In new massive gravity, these equations become

$$\left(2\sigma - \frac{1}{m^2}\right) g^{(2)}_{uv} = 0 \tag{3.7}$$

$$\left(\sigma + \frac{1}{2m^2}\right) \partial_v g^{(2)}_{uu} = \left(\sigma - \frac{1}{2m^2}\right) \partial_u g^{(2)}_{uv} \tag{3.8}$$

$$\left(\sigma + \frac{1}{2m^2}\right) \partial_u g^{(2)}_{vv} = \left(\sigma - \frac{1}{2m^2}\right) \partial_v g^{(2)}_{uv} \tag{3.9}$$

These are the same restrictions on $g^{(2)}$ that appear in Einstein gravity, multiplied by a prefactor dependent on σ and m^2. These equations exhibit the two critical points $2m^2 = \pm\sigma$ that have previously been explored in the literature. At the chiral point $2m^2 = +\sigma$, the constraint on $g^{(2)}_{uv}$ vanishes, while the off-diagonal components have the same constraints as in Einstein gravity. In a gauge-independent language, the constraints on the divergence of $g^{(2)}$ are maintained, while the constraint on the trace disappears. Conversely, at the critical point $2m^2 = -\sigma$, the constraint on the trace of $g^{(2)}$ holds but the constraints on the divergence of $g^{(2)}$ vanish. These solutions will be explored in more detail in the next section; however, the solution at generic values of the mass parameter is just the Einstein solution

$$g^{(2)}_{uu} = L(u) \qquad\qquad g^{(2)}_{vv} = \bar{L}(v). \tag{3.10}$$

The program continues by plugging in the generic solution for $g^{(2)}$ and solving for $g^{(4)}$. The fourth order equations generically constrain $g^{(4)}$ in terms of $g^{(2)}$ and $g^{(0)}$, but of course there are now three sets of equations: one each for the critical points $2m^2 = \pm\sigma$, and the generic set using the second-order solution above. The equations for generic m^2 are

$$\{\rho\rho\}: \qquad \left(\sigma - \tfrac{1}{2m^2}\right)\left(4g^{(4)}_{uv} + g^{(2)}_{uu}g^{(2)}_{vv}\right) = 0 \tag{3.11}$$

$$\{uu\}: \qquad \left(\sigma + \tfrac{17}{2m^2}\right) g^{(4)}_{uu} = 0 \tag{3.12}$$

$$\{vv\}: \qquad \left(\sigma + \tfrac{17}{2m^2}\right) g^{(4)}_{vv} = 0 \tag{3.13}$$

$$\{uv\}: \qquad \text{same as } \rho\rho\text{-eqn} \tag{3.14}$$

$$\{\rho u\}: \quad \left(\sigma + \tfrac{17}{2m^2}\right) \partial_v g^{(4)}_{uu} = \left(\sigma - \tfrac{1}{2m^2}\right) \partial_u \left(g^{(4)}_{uv} + \tfrac{1}{4}g^{(2)}_{uu}g^{(2)}_{vv}\right) \tag{3.15}$$

$$\{\rho v\}: \quad \left(\sigma + \tfrac{17}{2m^2}\right) \partial_u g^{(4)}_{vv} = \left(\sigma - \tfrac{1}{2m^2}\right) \partial_v \left(g^{(4)}_{uv} + \tfrac{1}{4}g^{(2)}_{uu}g^{(2)}_{vv}\right) \tag{3.16}$$

Only the first three equations are necessary to solve for $g^{(4)}$, with the $\{uv\}$ duplicating the $\{\rho\rho\}$ equation, and the $\{\rho x^i\}$ equations being derivatives of combinations of the other equations. This system has a new critical value $2m^2 = -17\sigma$ at which some of the constraints on $g^{(4)}$ vanish. The vanishing of constraints is related to the additional degrees of freedom of the metric in NMG. However, at generic values of the mass parameter, the only non-zero component is

$$2m^2 \neq \pm\sigma, -17\sigma: \qquad g^{(4)}_{uv} = -\frac{1}{4}g^{(2)}_{uu}g^{(2)}_{vv} = -\frac{1}{4}L(u)\bar{L}(v) \tag{3.17}$$

At sixth order, the same pattern emerges. There are now four sets of equations: one each for the critical values $2m^2 = \pm\sigma, -17\sigma$ and a generic set. Again, the generic equations are just the Einstein equations multiplied by some pre-factor:

$$\{\rho\rho\}: \quad \left(\sigma - \tfrac{1}{2m^2}\right) g^{(6)}_{uv} = 0 \tag{3.18}$$

$$\{uu\}: \quad \left(\sigma + \tfrac{49}{2m^2}\right) g^{(6)}_{uu} = 0 \tag{3.19}$$

$$\{vv\}: \quad \left(\sigma + \tfrac{49}{2m^2}\right) g^{(6)}_{vv} = 0 \tag{3.20}$$

As is the case for the fourth-order equations, the $\{uv\}$ equation duplicates the $\{\rho\rho\}$

equation, and the $\{\rho x^i\}$ equations are just derivatives of the other equations. These equations exhibit a new critical value $2m^2 = -49\sigma$ at which some of the constraints on $g^{(6)}$ vanish. However, at generic values of the mass parameter, all components of $g^{(6)}$ vanish. It seems reasonable to assume that (at generic values of the mass parameter) all higher-order terms vanish as in Einstein gravity, and this has been confirmed to tenth order. Then at generic values of the mass parameter, the traditional Fefferman-Graham expansion (3.6) truncates at fourth order and contains only the same solutions allowed in Einstein gravity.

Clearly, the expansion (3.6) fails to capture the dynamics of new massive gravity. If (3.6) is the correct asymptotic expansion for NMG, these results suggest that the theory at generic values of m^2 contains only the ordinary Einstein solutions, but special values yield more degrees of freedom. However, this conflicts with earlier work. Perturbative methods have shown that new massive gravitons of NMG exist at a wide range of values of m^2. Non-perturbative degree of freedom counting techniques find the same number of degrees of freedom at all values of the mass parameter [104, 103, 105]. Additional counter-examples exist in the literature. For example, the AdS pp-waves of Ayon-Beato [59], and their generalization to the Type N solutions [112, 108], are exact solutions that exist at all values of the mass parameter. For the range $2m^2 \leq -\sigma$, the AdS waves are asymptotically anti-de Sitter and satisfy the strict Brown-Henneaux boundary conditions. However, these solutions can not be put into the form (3.6), except at the critical values mentioned earlier. These solutions will be discussed in greater depth in the next section.

Taken together, these results indicate that (3.6) is not the correct asymptotic expansion for new massive gravity at generic values of m^2. Inclusion of the log term $h^{(2)}$in (3.6) will not help, since that term is only non-zero at $2m^2 = -\sigma$. Similarly, inclusion of odd terms $g^{(1)}, \ldots$ will not remedy this problem, as $g^{(1)}$ is allowed only at the critical point $2m^2 = \sigma$.

Note again that this failure of the Fefferman-Graham expansion at generic values has been obscured in the literature, since previous studies exploring the asymptotic expansion of NMG have focused on the critical values $2m^2 = \pm\sigma$ at which the FG expansion *does* capture the new degrees of freedom of the theory.

3.4 A Modified Asymptotic Expansion for NMG

3.4.1 Modified Asymptotics and Solution at Generic m^2

The goal of this section is to remedy the failure of the Fefferman-Graham expansion noted in the previous section, which is accomplished by adding new terms to the asymptotic expansion. Note that, regardless of the value of m^2, solutions of Einstein gravity are also solutions of new massive gravity. Therefore, the generic expansion must still include the original terms of the FG expansion. Another way of seeing this is that the constraints that fix the trace and divergences of $g^{(2)}$ in terms of the boundary metric hold at all values of $2m^2 \neq \pm\sigma$, and so $g^{(2)}$ is required in the generic expansion. Similarly, the equations of motion fix $g^{(4)}$ in terms of $g^{(2)}$ except at $2m^2 \neq -17\sigma$, so the fourth-order term is also present in the generic expansion.

But the Einstein terms are insufficient to capture the dynamics at generic values of the mass parameter. To that end, I propose a new term in the expansion

$$g_{ij}(\rho, x^i) = \mathrm{e}^{2\rho} \left(g^{(0)}_{ij} + \mathrm{e}^{-2\rho} g^{(2)}_{ij} + \mathrm{e}^{-4\rho} g^{(4)}_{ij} + \mathrm{e}^{-n\rho} g^{(n)}_{ij} + \cdots \right) \tag{3.21}$$

for some exponent n, to be determined later. For the moment, the expansion will be carried through in a generic gauge, without any particular choice of coordinate system or boundary metric. Next, the equations of motion are expanded to first order in n:

$$\{\rho\rho\}: \qquad \left(\sigma - \tfrac{1}{2m^2}\right) \tfrac{-n}{2} \mathrm{Tr} g^{(n)} = 0 \tag{3.22}$$

$$\{ij\}: \quad \tfrac{n(2-n)}{2} \left[\sigma + \tfrac{1}{m^2}\left(n^2 - 2n + \tfrac{1}{2}\right)\right] g^{(n)}_{ij} + \tfrac{n(n-2)}{2} \left[\sigma + \tfrac{n(n-2)}{2m^2}\right] g^{(0)}_{ij} \mathrm{Tr} g^{(n)} = 0 \tag{3.23}$$

$$\{\rho i\}: \quad -\tfrac{n}{2} \left[\sigma + \tfrac{1}{m^2}\left(n^2 - 2n + \tfrac{1}{2}\right)\right] \nabla^k g^{(n)}_{ki} + \tfrac{n}{2} \left[\sigma + \tfrac{n(n-2)}{2m^2}\right] \partial_i \mathrm{Tr} g^{(n)} = 0 \tag{3.24}$$

The derivation is given in the appendix.

Away from the critical point $2m^2 = \sigma$, the $\{\rho\rho\}$ equation imposes the vanishing of the trace of $g^{(n)}$, regardless of the value of n. The $\{ij\}$ equations are more interesting. The constraints on the non-trace part of $g^{(n)}$ vanish whenever the pre-factor, a quartic polynomial in n, vanishes. This pre-factor has four roots:

$$n = 0, 2, 1 \pm \sqrt{\frac{1}{2} - \sigma m^2} \tag{3.25}$$

The four roots correspond to four branches of solutions, which should be expected since the equations of motion are fourth order in derivatives of the metric. The first two roots $n = 0, 2$ are also present in the Einstein limit $m^2 \to \infty$ and match up with the terms $g^{(0)}$ and $g^{(2)}$ which are already present in the ordinary Fefferman-Graham expansion. The next two roots $n_\pm = 1 \pm \sqrt{\frac{1}{2} - \sigma m^2}$ correspond to the non-Einstein solutions. These are the pieces of the expansion, new to NMG, which capture the dynamics of the theory. Now the connection between the asymptotic expansion and the linearized approach becomes apparent. The exponential behavior of the non-Einstein solutions can also be written as $n = 1 \pm m_{eff}$, where $m_{eff}^2 = \frac{1}{2} - \sigma m^2$ is the effective mass of the massive graviton modes found in [123, 120]. Additionally, the AdS pp-waves found in [59] satisfy the Klein-Gordon equation with the same effective mass. Note that imposing Brown-Henneaux boundary conditions $n \geq 2$ amounts to choosing only the positive branch and restricting to the parameter range $2m^2 \leq -\sigma$.

This explains the behavior observed in the previous section. There it was found that the Fefferman-Graham expansion, containing only even powers of e^ρ in the expansion, yields only Einstein solutions *except* at the special values $2m^2 = -\sigma, -17\sigma, -49\sigma, \cdots$. At these values, some of the equations vanish, and only at these critical points does the Fefferman-Graham expansion contain non-Einstein solutions. From (3.25), it becomes clear that these are precisely the values of the mass parameter at which the non-Einstein branch of solutions

n_+ overlaps with the Einstein branch, i.e., these are the values at which n_+ is a positive even integer.

Restricting attention only to those solutions satisfying Brown-Henneaux boundary conditions, we see that larger values of the mass parameter correspond to steeper asymptotics. This phenomena has an obvious interpretation. The coupling m^{-2} gives the relative weight between the Einstein and NMG contributions to the action. As m^2 increases, an NMG perturbation from an Einstein background decreases in strength, and the new degrees of freedom are more localized in the interior. Conversely, when $2m^2 = \sigma$, the NMG and Einstein contributions in the action have the same relative weight, and NMG perturbations from an Einstein background show up at the same order in the asymptotic expansion as the Einstein degrees of freedom.

Additionally, the general theory possesses two points at which two branches of solutions degenerate. When this occurs, the set of solutions labeled by (3.25) fails to span the space of linearly independent solutions, and new log solutions appear. At the chiral point $2m^2 = -\sigma$, the negative branch degenerates with the boundary at $n = 0$, and the positive branch degenerates with the $n = 2$ Einstein term. Here the generic solution is

$$g_{ij} = \mathrm{e}^{2\rho}\left(\rho h_{ij}^{(0)} + g_{ij}^{(0)} + \rho\mathrm{e}^{-2\rho}h_{ij}^{(2)} + \mathrm{e}^{-2\rho}g_{ij}^{(2)} + \cdots\right) \tag{3.26}$$

This is similar to what happens in topologically massive gravity at the chiral point, at which the new log term $h^{(2)}$ is allowed.

NMG also possesses another critical point at $2m^2 = +\sigma$, which has no analogue in topologically massive gravity. Here, the two non-Einstein branches degenerate with each other at $n_+ = n_- = 1$, and the generic solution is given by

$$g_{ij} = \mathrm{e}^{2\rho}\left(g_{ij}^{(0)} + \rho\mathrm{e}^{-\rho}h_{ij}^{(1)} + \mathrm{e}^{-\rho}g_{ij}^{(1)} + \cdots\right). \tag{3.27}$$

The next section compares known exact solutions from the literature to the asymptotic behavior of the modified asymptotic expansion (3.21).

3.4.2 Comparison to Known Solutions in Fefferman-Graham Coordinates

Only the sub-leading behavior of non-Einstein perturbations has been established. Finding exact solutions is more difficult. For generic values of the exponent $n > 2$, the next term in the expansion occurs at order $n+2$ and contains mixing between the Einstein background and the non-Einstein perturbation. However, self-interactions quadratic in $g^{(n)}$ occur at order $2n$, and a non-zero $g^{(n)}$ turns on an infinite series with exponential behavior $ln+2k$, where k and l are positive integers. Thus a generic perturbation from an Einstein background seems to generate an expansion that continues indefinitely, making this a poor tool for finding exact solutions.

Still, exact asymptotically AdS solutions are known, and a test of the modified expansion is whether it can accommodate all known solutions. BTZ black holes [31, 32] are solutions of NMG at all values of the mass parameter. In Gaussian normal coordinates, the BTZ metric can be written as

$$
\begin{aligned}
\mathrm{d}s^2 \;&=\; \mathrm{d}\rho^2 - \left(r_+^2 \sinh^2 r - r_-^2 \cosh^2 r\right) \mathrm{d}t^2 \\
&\quad + \left(r_+^2 \cosh^2 r - r_-^2 \sinh^2 r\right) \mathrm{d}\phi^2 - 2r_- r_+ \mathrm{d}t\mathrm{d}\phi
\end{aligned}
\tag{3.28}
$$

where r_+ and r_- are the inner and outer horizons. Properties of the BTZ metric are explored in a later section, but for now we note that this solution corresponds to the Einstein branch of solutions, with a nonzero traceless $g^{(2)}$, and a second-order piece given by Tr $g^{(4)} = \frac{1}{4}$Tr $g^2_{(2)}$. These solutions result from turning off the non-Einstein branches $g^{(n_\pm)}$.

Log deformations of extremal BTZ black holes have also been found at the chiral point $2m^2 = -\sigma$ [110, 59]. These solutions are the NMG analogues of the log black holes of topologically massive gravity, with metric

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \mathrm{e}^{2\rho}(-\mathrm{d}t^2 + \mathrm{d}\phi^2) + (2k\rho + r_+^2)(\mathrm{d}\phi - \mathrm{d}t)^2 \tag{3.29}$$

These solutions require turning on the log term $h^{(2)}$ in the general expansion (3.26), and reduce to the extremal BTZ metric as $k \to 0$.

New "hairy" black holes, referred to in the literature as "new type black holes," have also been discovered at the critical point $2m^2 = +\sigma$ [4, 58]. These solutions have no counterpart in TMG. The static solution is given by

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 - a\sinh^2\rho \mathrm{d}t^2 + (a\cosh\rho + c)^2\,\mathrm{d}\phi^2 \tag{3.30}$$

and results from turning on the $g^{(1)}$ term in the general expansion at the critical point (3.27). This solution can also be boosted to a rotating black hole without spoiling the asymptotics [58].

No exact solutions have been found that make use of the new log term $h^{(1)}$ that becomes available at the critical point (3.27). In particular, no one has yet found a log deformation of a new type black hole, although I know of no reason forbidding such a solution. Linearized modes *have* been constructed which require a nonzero $h^{(1)}$ [4, 60], although these modes do not appear to have a terminating power series expansion. The linear theory at this point has interesting properties, and massive excitations become partially massless in the sense of Deser and Waldron [159], although see [105] for evidence that the "partially massless" nature of the theory does not extend beyond the linear level.

While it is gratifying to see that the general solutions accommodates the asymptotic behavior of the solutions given above, these solutions are not good tests of the generic solution because they occur only at the critical points $2m^2 = \pm\sigma$, which have already been well-examined in the literature. A better test is to compare the expansion at generic m^2 (3.21) with non-Einstein AAdS solutions at non-critical points. The only known solutions that fit the bill are the AdS waves [59] and Type N solutions [112, 108], both of which exist at all values of the mass parameter. AdS waves are a kind of exact gravitational wave, conformally related to pp-waves, propagating on an AdS background with a null Killing vector. In Fefferman-Graham coordinates, the AdS wave solutions can be written in light cone gauge as

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 - 2\mathrm{e}^{2\rho}\mathrm{d}u\mathrm{d}v + F_\pm(u)\mathrm{e}^{(2-n_\pm)\rho}\mathrm{d}u^2 \tag{3.31}$$

where F is unconstrained by the equations of motion. The exponent n is exactly the function of m^2 found for the generic perturbation, $n_\pm = 1 \pm \sqrt{1/2 - \sigma m^2}$, and thus these solutions represent a subset of the general asymptotic expansion given in (3.21) with an expansion that terminates at first order in $g^{(n)}$. Interestingly, the profile function F satisfies a Klein-Gordon equation

$$\Box F = m^2_{\text{eff}} F, \tag{3.32}$$

with effective mass give by $m^2_{\text{eff}} = m^2 - \frac{1}{2}$. This shifting of the bare mass is similar to what happens for the AdS waves of topologically massive gravity. At the critical point $2m^2 = +\sigma$, this value for the effective mass saturates the Breitenlohner-Freedman bound [57, 93].

Surprisingly, the expansion (3.21) also accommodates exact solutions that are not asymptotically anti-de Sitter. The Type N solutions of Ahmedov and Aliev [108, 112] are exact solutions with constant scalar curvature that extend "beyond the boundary." These are generalizations of the AdS waves (3.31) in which the traceless part of the Ricci tensor can be written as the exterior product of a null vector with itself. The metric for these solutions is given by

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + 2\cosh^2\rho\mathrm{d}u\mathrm{d}v + \left[Z(u,\rho) - (v)^2\cosh^2\rho\right]\mathrm{d}u^2 \tag{3.33}$$

where

$$Z(u,\rho) = F_1(u)\left(\mathrm{e}^{(2-n_-)\rho} + \mathrm{e}^{-n_-\rho}\right) + F_2(u)\left(\mathrm{e}^{(2-n_+)\rho} + \mathrm{e}^{-n_+\rho}\right) + F_3(u)\left(\mathrm{e}^{2\rho} - \mathrm{e}^{-2\rho}\right) \tag{3.34}$$

and the exponent is again given by $n_\pm = 1 \pm \sqrt{1/2 - \sigma m^2}$. These solutions were shown to reduce to the AdS waves in an appropriate limit. While they have a more complicated expansion than the AdS waves, the first sub-leading terms are still of order $e^{(2-n_\pm)\rho}$.

Thus, the generic expansion (3.21) with non-Einstein asymptotics determined by (3.25) gives the correct asymptotic behavior of all known exact AAdS solutions.

3.4.3 Boundary Conditions

The near-boundary asymptotic expansion employed in the previous section provides a natural setting to address the question of appropriate boundary conditions. Early on, the Brown-Henneaux boundary conditions were shown to be consistent at all values of the mass parameter . The log solutions (at the chiral point $2m^2 = -\sigma$) and the new type black holes (at the critical point $2m^2 = +\sigma$) spurred investigations into these relaxed boundary conditions, and they were also found to be consistent at these points in parameter space. Consistency, in this case, means that asymptotic symmetries of AdS preserve the boundary conditions, and conserved charges, expressed as surface integrals at infinity, remain finite.

However, the possibility of relaxing the Brown-Henneaux boundary conditions has been addressed only at the critical points $2m^2 = \pm\sigma$. The solutions associated with the non-Einstein branches indicate several other possibilities for relaxing the boundary conditions at generic couplings. The asymptotic behavior of these solutions can be divided into four categories,[2] to be addressed separately.

1. The negative branch in the parameter range $2m^2 < -1$, corresponding to exponent $n_- < 0$ and the new term $e^{-n\rho} g_{ij}^{(n)}$ in the asymptotic expansion;

2. The negative branch in the range $-1 < 2m^2 < 1$, corresponding to exponent $0 < n_- < 1$;

3. The positive branch in the range $-1 < 2m^2 < 1$, with exponent $1 < n_+ < 2$;

[2] For the moment, I restrict discussion to the case $\sigma = +1$. The negative case can be easily identified from (3.25).

4. The positive branch in the range $2m^2 < -1$, with expansion exponent $n > 2$.

The first category consists of solutions that extend "beyond the boundary" and violate AdS asymptotics. While some known exact solutions make use of this branch, these solutions cannot properly be called "asymptotically AdS". The second category consists of solutions which asymptote to the AdS metric at the boundary but have slower-than-Brown-Henneaux fall-off. Note that these solutions also decay more slowly than the fall-off for the new type black holes examined in [132].

The positive branch of solutions in the parameter range $-1 < 2m^2 < 1$ also asymptote to the AdS metric at the boundary but break the Brown-Henneaux boundary conditions. In this range, the metric has asymptotic expansion

$$g_{ij} = \mathrm{e}^{2\rho}\left(g_{ij}^{(0)} + \mathrm{e}^{-n_+\rho} g_{ij}^{(n_+)} + \mathrm{e}^{-2\rho} g_{ij}^{(2)} + \cdots\right) \tag{3.35}$$

with exponent $1 < n_+ < 2$. Though weaker than Brown-Henneaux, these solutions decay faster than the $n = 1$ boundary conditions at the critical point $2m^2 = +1$, which have already been shown to be consistent. This behavior offers the intriguing possibility of new boundary conditions in a previously unexplored region of parameter space. A next step to determine the consistency of these boundary conditions is to tackle the subject of holographic renormalization in this parameter range, which is addressed in the next section.

The fourth category contains solutions which decay faster than Brown-Henneaux. Because they appear deeper in the interior, they are unlikely to affect either the counter-terms necessary for holographic renormalization or the conserved charges.

3.5 Holographic Renormalization with Relaxed Boundary Conditions

Previously, investigation of appropriate boundary conditions has been limited to i) Brown-Henneaux boundary conditions at all values of the mass parameter [119], and ii) relaxations of Brown-Henneaux at the critical points $2m^2 = \pm\sigma$, and holographic renormalization of the boundary stress tensor has only been achieved under these conditions [131, 129, 125, 132]. However, results from the previous section indicate other possibilities for relaxing the boundary conditions over a range of parameters, and I explore one of these possibilities here. In particular, the n_+ branch of solutions in the range $-1 < 2m^2 < 1$ falls off slower than Brown-Henneaux but faster than the $\mathrm{e}^{-\rho}$ fall-off that was previously found to be consistent [129, 132]. In this section I obtain the Brown-York stress tensor [160] with these relaxed asymptotics and determine the appropriate counterterms necessary for renormalization.

Note that there are two further possibilities that will not be explored. The n_- branch in the range $-1 < 2m^2 < 1$ asymptotes to the AdS metric but obeys weaker-than Brown-Henneaux boundary conditions. However, the exponent falls in the range $0 < n_- < 1$, and the Brown-York stress tensor must be expanded at least to second order in $g^{(n_-)}$ to include all divergent terms,

$$T_{ij}^{BY} = \mathrm{e}^{2\rho} T_{ij}^{(0)} + \mathrm{e}^{(2-n_-)\rho} T_{ij}^{(n_-)} + \mathrm{e}^{(2-2n_-)\rho} T_{ij}^{(2n_-)} + \cdots + \mathrm{e}^{0} T_{ij}^{(2)} + \cdots \tag{3.36}$$

This higher-order expansion is required for the negative branch because terms quadratic in $g^{(n_-)}$ are found at order $\mathrm{e}^{(2-2n_-)\rho}$, which is also divergent. This more difficult problem is postponed for future research. Also, in the parameter range $2m^2 < -1$, the n_- branch extends "beyond the boundary" and breaks the asymptotic symmetries of anti-de Sitter space. While a renormalized stress tensor can be obtained for some non-AAdS spacetimes (see [131]), holographic renormalization for these cases will not be explored here.

3.5.1 Brown-York Stress Tensor with Relaxed Asymptotics

I begin by reviewing construction of the holographic stress tensor, with special attention paid to the variational principle. For a general $(d+1)$-dimensional gravitational action S, variation with respect to the metric takes the following form:

$$
\begin{aligned}
\delta S \;=\; & \frac{1}{16\pi G_N}\int_M \mathrm{d}^{d+1}x\sqrt{-g}\,[(\cdots)\delta g_{\mu\nu}] + \frac{1}{16\pi G_N}\int_{\partial M}\mathrm{d}^d x\sqrt{-\gamma}\,[(\cdots)\delta\gamma_{ij}] \\
& + \frac{1}{16\pi G_N}\int_{\partial M}\mathrm{d}^d x\sqrt{-\gamma}\left[\sum_k (\cdots)(\partial_\eta)^k\delta\gamma_{ij}\right]
\end{aligned}
\tag{3.37}
$$

where $g_{\mu\nu}$ is the bulk metric, and γ_{ij} is the induced metric on the boundary ∂M. The bulk term vanishes when the equations of motion are satisfied. The second term vanishes by imposing Dirichlet boundary conditions $\delta\gamma_{ij}|_{\partial M}=0$, and it is this term that identifies the Brown-York stress tensor [160]

$$
\delta S = \frac{1}{2}\int_{\partial M}\mathrm{d}^d x\sqrt{-\gamma}T^{ij}\delta\gamma_{ij}. \tag{3.38}
$$

However, for an action containing up to n derivatives, the boundary term will also include terms of the form $(\partial_\eta)^k\delta\gamma_{ij}$, with $k=1,\ldots,n-1$. Then solutions to the bulk equations of motion satisfying Dirichlet boundary conditions fail to extremize the action, thus undermining the standard variational principle.

A well-posed variational principle can be restored in one of three ways. The first possibility exploits the falloff conditions on the metric, and makes use of the fact that the boundary term is evaluated at infinity. If the coefficients of the terms $(\partial_\eta)^k\gamma_{ij}$ vanish at infinity, then the variational principle does not need to modified, and (3.38) yields a well-defined stress tensor. This property was used, for example, to define the boundary stress tensor in cosmological topologically massive gravity [87, 52, 55]. More recently, the stress tensor for new massive gravity at the chiral point was found by exploiting the fact that the coefficients in (3.37) decay at infinity [125]. However, this approach merely sidesteps the problem and does not provide a method for removing the unwanted terms (3.37) in the event that their coefficients do not decay. Thus the results of [125] are not expected to hold true at generic values of the mass parameter.

A second possibility is to impose additional boundary conditions on derivatives of the metric, $(\partial_\eta)^k \gamma_{ij} = 0$, forcing the boundary terms (3.37) to vanish. However, this approach implies the existence of $n-1$ new "stress tensors" on the boundary, which clashes with the AdS/CFT conjecture.

The third approach is to add boundary terms to the action that, while leaving the bulk equations of motion unchanged, precisely cancel the unwanted boundary terms (3.37). For the second-order Einstein-Hilbert action, this is accomplished by adding the Gibbons-Hawking term [161]. However, a corresponding boundary term for generic higher-derivative theories may not exist except in a few special cases, for example [162]. In $f(R)$ theories, the problem is solved by using an auxiliary scalar ϕ, and reformulating the action as second-order in derivatives of the metric [163]. This process of reducing the number of derivatives in the action greatly simplifies the task of constructing a generalized Gibbons-Hawking term and restoring the variational principle.

Recently, Hohm and Tonni have extended the auxiliary field approach to new massive gravity [131]. In this approach, the NMG action (3.1) is written in terms of an auxiliary field $f_{\mu\nu}$:

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^3 x \sqrt{-g} \left[\sigma R + 2\lambda + f^{\mu\nu} G_{\mu\nu} - \frac{m^2}{4} \left(f^{\mu\nu} f_{\mu\nu} - f^2 \right) \right]. \tag{3.39}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor. On-shell, the auxiliary tensor is proportional to the Schouten tensor

$$f_{\mu\nu} = \frac{2}{m^2} \left(R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \right) \tag{3.40}$$

and the equation of motion is

$$\begin{aligned}\sigma G_{\mu\nu} + \lambda g_{\mu\nu} - \frac{m^2}{2}\left[f^\alpha_\mu f_{\nu\alpha} - f f_{\mu\nu} - \frac{1}{4} g_{\mu\nu}\left(f^{\alpha\beta} f_{\alpha\beta} - f^2\right)\right] \\ +2 f_{\alpha(\mu} G^\alpha_{\nu)} + \frac{1}{2} R f_{\mu\nu} - \frac{1}{2} f R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f^{\alpha\beta} G_{\alpha\beta} \\ +\frac{1}{2}\left[D^2 f_{\mu\nu} - 2 D^\alpha D_{(\mu} f_{\nu)\alpha} + D_\mu D_\nu f + \left(D^\alpha D^\beta f_{\alpha\beta} - D^2 f\right) g_{\mu\nu}\right] &= 0\end{aligned} \tag{3.41}$$

In this form, the action is only second-order in derivatives of the metric, which allows for construction of the generalized Gibbons-Hawking term. The original work [131] used a generic ADM-like decomposition of the metric

$$\mathrm{d}s^2 = N^2 \mathrm{d}\eta^2 + \gamma_{ij}\left(\mathrm{d}x^i + N^i \mathrm{d}r\right)\left(\mathrm{d}x^j + N^j \mathrm{d}r\right) \tag{3.42}$$

to derive the generalized Gibbons-Hawking term and the boundary stress tensor. For our purposes, it will be convenient to continue in the Gaussian normal coordinates of (3.6), corresponding to the gauge-fixing $N^i = 0$ and $N = 1$. Then in the discussion above, the boundary metric $\gamma_{ij} \to g_{ij}$, and the radial coordinate $\eta \to \rho$. For Einstein-Hilbert gravity, the Gibbons-Hawking term is just the trace of the extrinsic curvature,

$$K_{ij} = -\frac{1}{2}\partial_\rho g_{ij} \qquad \text{and} \qquad K = g^{ij} K_{ij}. \tag{3.43}$$

It is also useful to decompose the auxiliary tensor in radial and non-radial components:

$$f^{\mu\nu} = \begin{pmatrix} s & h^i \\ h^i & f^{ij} \end{pmatrix} \tag{3.44}$$

Then the generalized Gibbons-Hawking term for new massive gravity is just

$$S_{GH} = \frac{1}{2\kappa^2}\int_{\partial M_3}\left(-2\sigma K - \hat{f}^{ij} K_{ij} + \hat{f} K\right). \tag{3.45}$$

With this action, the Brown-York stress energy tensor for NMG was obtained in [131, 129],

$$
\begin{aligned}
8\pi G T^{ij}_{BY} &= \sigma\left(K^{ij} - K g^{ij}\right) + \frac{1}{2}(\hat{s} - \hat{f})(K^{ij} - K g^{ij}) - \nabla^{(i}\hat{h}^{j)} + \frac{1}{2}\mathcal{D}_\rho \hat{f}^{ij} \\
&\quad + K^{(i}_k \hat{f}^{j)k} + g^{ij}\left(\nabla_k \hat{h}^k - \frac{1}{2}\mathcal{D}_\rho \hat{f}\right), \qquad (3.46)
\end{aligned}
$$

where the first term is just the Brown-York tensor for Einstein gravity. Expressions for the "covariant ρ-derivative" $\mathcal{D}_\rho$ and hatted quantities are given in [131]. Note that in Gaussian normal coordinates, $\mathcal{D}_\rho$ becomes the ordinary ρ derivative ∂_ρ, and $\hat{}$ has no effect: $\hat{f} = \gamma_{ij} f^{ij}$, $\hat{h}^i = h^i$, $\hat{s} = s$.

We are now in a position to expand the Brown-York stress tensor using the modified asymptotic expansion given in (3.35). To simplify notation, I am dropping the subscript from n_+. Then

$$
\begin{aligned}
8\pi G T^{ij}_{BY} &= \mathrm{e}^{-2\rho}\left(\sigma + \frac{1}{2m^2}\right) g^{ij}_{(0)} \\
&\quad + \mathrm{e}^{-(n+2)\rho}\left[\left(\frac{n-2}{2}\sigma + \frac{2n^3 - 4n^2 + n - 2}{4m^2}\right) g^{ij}_{(n)}\right. \\
&\quad + \left.\left(\frac{-n}{2}\sigma + \frac{-n^3 + 2n^2 - 2n}{4m^2}\right) g^{ij}_{(0)} \mathrm{Tr} g^{(n)}\right] \\
&\quad + \mathrm{e}^{-4\rho}\left[\frac{-1}{4m^2} R^{(0)} g^{ij}_{(0)} - \left(\sigma + \frac{1}{m^2}\right) g^{ij}_{(0)} \mathrm{Tr} g^{(2)}\right] + \mathcal{O}(\mathrm{e}^{-(n+4)\rho}) \qquad (3.47)
\end{aligned}
$$

There are now two divergences coming from the first two terms in the stress-energy tensor.[3]

3.5.2 Counter-terms and Renormalized Stress Tensor

In the AdS/CFT dictionary, the expectation value of the stress-energy tensor of the dual CFT is just the Brown-York stress-energy tensor evaluated at the boundary,

$$
\langle T^{ij}_{\mathrm{CFT}} \rangle = \lim_{\rho\to\infty} T^{ij}_{\mathrm{BY}} \qquad (3.48)
$$

[3]Note that the stress-tensor with indices lowered (given in Appendix B) picks up two factors of $\mathrm{e}^{2\rho}$, and this is the form of the stress tensor that will be used in Section 3.6 to compute conserved charges.

However, the expression (3.47) diverges at the boundary. These divergences can be removed through the process known as holographic renormalization, in which counterterms constructed from quantities intrinsic to the boundary are added to the boundary action to cancel the near-boundary divergences. The new renormalized stress-energy tensor can be written

$$T^{ij}_{\text{ren}} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{ij}} \left(S_{\text{bulk}} + S_{GH} + S_{c.t.} \right). \tag{3.49}$$

In this section I construct the relevant counter-terms and obtain the renormalized stress tensor. This stress tensor gives the correct central charge of the dual CFT, as well as the mass and angular momentum of BTZ black holes, and is consistent with previous results obtained by other methods. Previous work has explored holographic renormalization at the chiral point $2m^2 = -\sigma$ [131, 125] and the critical point $2m^2 = +\sigma$ [129, 132], and their results are briefly reviewed next.

Holographic renormalization at the chiral point $2m^2 = -\sigma$ was first explored in [131, 125]. At this point, the correct expansion (3.26) corresponds to turning on the log branch $h^{(2)}$. Then log divergences appear in the Brown-York stress tensor (3.47). The counter-term required to remove both the leading order and sub-leading log divergences is just proportional to the boundary cosmological constant, as in Einstein gravity:

$$S_{c.t.} = -\left(\sigma + \frac{1}{2m^2} \right) \frac{1}{8\pi G} \int \mathrm{d}^2x \sqrt{-g} \tag{3.50}$$

This counter-term is sufficient for solutions obeying Brown-Henneaux boundary conditions at all values of the mass parameter. Interestingly, it is also sufficient to cancel the log divergence at the chiral point – no new counter-terms are needed. This is reminiscent of what happens in topologically massive gravity at the chiral point. The authors of [131] explored the asymptotic symmetry algebra of the renormalized stress tensor and confirmed that it reproduces the correct central charge of the dual CFT.

At the critical point $2m^2 = +\sigma$, the counter-term (3.50) is insufficient to cancel divergences in the boundary stress tensor. The full asymptotic expansion is given by (3.27); however, previous work studying holographic renormalization at the critical point has considered only the solutions with the $g^{(1)}$ branch of solutions turned on [129, 132]. The log branch associated with non-zero $h^{(1)}$ has not been considered in this context and remains an open question.[4] With a non-zero $g^{(1)}$, the Brown-York tensor (3.47) has a sub-leading divergence of order e^{ρ}. The counter-term necessary for removing both leading and sub-leading divergences is

$$S_{c.t.} = \frac{m^2}{2}\left(\sigma + \frac{1}{2m^2}\right) 8\pi G \int \mathrm{d}^2x\sqrt{-g}\hat{f}. \tag{3.51}$$

Note that NMG brings with it an expanded set of possible counter-terms, with the only criterion being that they be constructed from purely local objects that are invariant under boundary-preserving diffeomorphisms. Here I consider the expanded set of counter-terms given in [132]:

$$S_{c.t.} = \frac{1}{8\pi G}\int \mathrm{d}^2x\sqrt{-g}\left(A + B\hat{f} + C\hat{f}^2 + Df_{kl}f^{kl}\right), \tag{3.52}$$

with coefficients fixed by the requirement that the renormalized stress tensor remain finite. The expansion of the counter-terms is also given in the appendix. The leading-order divergence is removed when

$$\left(\sigma + \frac{1}{2m^2}\right) + A - \frac{2}{m^2}B + \frac{4}{m^4}C + \frac{2}{m^4}D = 0 \tag{3.53}$$

At sub-leading order, there appear to be two divergences: one proportional to $g^{(n)}$ and the other proportional to $\mathrm{Tr}g^{(n)}$, which can be removed by the counter-terms when

$$\frac{n+2}{2}\sigma + \frac{2n^3-4n^2+n+2}{4m^2} + A - \frac{2}{m^2}B + \frac{4}{m^4}C + \frac{2}{m^4}D \;=\; 0 \tag{3.54}$$

$$-\frac{n}{2}\sigma + \frac{-n^3+2n^2-2n}{4m^2} + \frac{n}{m^2}B - \frac{4n}{m^4}C - \frac{2n}{m^4}D \;=\; 0 \tag{3.55}$$

[4]The linearized theory with non-zero $h^{(1)}$ has been referred to as "partially massless NMG", and its properties have been explored from the perspective of the dual CFT in [60].

However, recall that the on-shell equations of motion (3.22) fix $\mathrm{Tr}g^{(n)} = 0$, and so the constraint coming from (3.55) is not needed. Also, recall that the exponent n is not an independent parameter. The term $g^{(n)}$ is only non-zero when $n = 1 + \sqrt{\frac{1}{2} - \sigma m^2}$. With this value of n, the equation (3.54) reduces to (3.53), and removal of the $g^{(n)}$ divergence places no new restrictions on the counter-terms. Oddly, *either* counter-term (3.50) (with to $B = C = D = 0$) or the relaxed asymptotics at the critical point (3.51) (with to $A = C = D = 0$) is sufficient for removing divergences from the stress energy tensor. Thus the requirement that counterterms remove divergences in the boundary stress tensor is not enough to fix the counterterm. However, these ambiguities are unphysical, and any choice of A, B, C, D satisfying (3.53) leads to the same renormalized stress tensor.

With divergences removed, the renormalized stress tensor is

$$\begin{aligned} 8\pi G T_{ij}^{\mathrm{ren}} &= \left(\sigma + \frac{1}{2m^2}\right) g_{ij}^{(2)} + g_{ij}^{(0)} \mathrm{Tr}g^{(2)} \left(-\sigma - \frac{1}{m^2} + \frac{2}{m^2}B - \frac{8}{m^2}C - \frac{4}{m^2}D\right) \\ &\quad + g_{ij}^{(0)} R^{(0)} \left(-\frac{1}{4m^2} + \frac{1}{m^2}B - \frac{4}{m^2}C - \frac{2}{m^2}D\right) \end{aligned} \tag{3.56}$$

Recall the on-shell $\{\rho\rho\}$ equation of motion fixes

$$R^{(0)} = -2\mathrm{Tr}g^{(2)}, \tag{3.57}$$

and so the on-shell renormalized stress tensor becomes

$$8\pi G T_{ij}^{ren} = \left(\sigma + \frac{1}{2m^2}\right)\left(g_{ij}^{(2)} - g_{ij}^{(0)} \mathrm{Tr}g^{(2)}\right). \tag{3.58}$$

The trace anomaly can be written in terms of the boundary Ricci scalar

$$8\pi G T = \frac{1}{2}\left(\sigma + \frac{1}{2m^2}\right) R^{(0)} \tag{3.59}$$

which is consistent with the central charge

$$c = \left(\sigma + \frac{1}{2m^2}\right)\frac{3}{2G} \tag{3.60}$$

and reproduces results from [123, 131].

Several features differentiate these results from the case $2m^2 = +\sigma$, and the results above do not hold smoothly in the limit $n \to 1$. The divergent terms in the Brown-York tensor (3.47) give the correct expressions in this limit [132]; however, the equation of motion forcing the trace of $g^{(n)}$ to vanish (3.22) disappears in the limit $n \to 1$, and thus the Brown-York stress tensor has two independent constraints on the counter-terms needed to renormalize the stress tensor. This explains why the counter-term found at $2m^2 = +\sigma$ (3.51) differs from the counter-term found at the chiral point $2m^2 = -\sigma$ (3.50). Second, the $g^{(1)}$ terms must be expanded to second-order and mix with the $g^{(2)}$ terms. Thus, they modify expressions for the conserved charges. However, the expression for the renormalized stress tensor (3.58) makes it clear that, for generic $1 < n < 2$, the non-Einstein mode $g^{(n)}$ does not contribute to the boundary stress tensor or to the conserved charges.

This result is remarkably uninteresting. It is surprising that the renormalized stress tensor, even with these relaxed asymptotics, is just proportional to the stress tensor for AdS space [42], and that the conserved charges of the non-Einstein modes vanish in this parameter range. The results from this paper and others on holographic renormalization indicate that non-Einstein modes *never* contribute to conserved charges *except* at the critical points $2m^2 = \pm\sigma$. At present, I do not have any explanation for why this should be the case.

3.6 Conserved Charges and Black Holes

The machinery of the boundary stress tensor developed in the previous section allows one to compute conserved charges, in particular the mass and angular momentum of specific solutions. Unfortunately, this is not a very interesting question for the parameter range $-1 < 2m^2 < 1$ examined in Sec. 3.5, since the non-Einstein solutions do not contribute to conserved charges. Instead, I take this opportunity to examine previously explored black hole solutions and verify their results using the holographic approach.

The holographic approach to conserved charges was first developed for Einstein gravity by Balasubramanian and Kraus [42] and allows us to define a conserved charge associated to a given asymptotic Killing vector ξ^i. The first step involves an ADM-like decomposition of the constant-ρ two-dimensional metric

$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = -\hat{N}^2\mathrm{d}t^2 + R^2(\mathrm{d}\phi + \hat{N}_\phi \mathrm{d}t)^2. \tag{3.61}$$

Then the charge associated to ξ is given by

$$Q(\xi) = \int_\Sigma \mathrm{d}s u^i \xi^j T^{\text{ren}}_{ij} \tag{3.62}$$

where Σ denotes a space-like surface with volume element (line element) $\mathrm{d}s$, and u^i is a time-like unit vector normal to the constant-t surface. In Gaussian normal coordinates, the integral over the spatial boundary becomes $\int_\Sigma \mathrm{d}s \to \int_0^{2\pi} \mathrm{d}\phi$. Then the mass and angular momentum are given by

$$M = Q[\partial_t] = \int_0^{2\pi} \mathrm{d}\phi T_{00} \qquad J = Q[\partial_\phi] = -\int_0^{2\pi} \mathrm{d}\phi T_{10}. \tag{3.63}$$

Now we are in a position to compute conserved charges of the asymptotically anti-de Sitter black hole solutions of new massive gravity. Note that, in most cases, these results have also been confirmed in the Abbott-Deser-Tekin method for conserved charges in higher-derivative, asymptotically anti-de Sitter spacetimes [164, 165, 166].

3.6.1 BTZ Black Holes

The BTZ black hole [31, 32] is a solution of pure three-dimensional Einstein gravity with a negative cosmological constant, and is locally equivalent to AdS3. Despite the triviality of Einstein gravity, the BTZ black hole has an event horizon, and nonvanishing entropy and mass. The Schwarzschild form of the BTZ metric is

$$\mathrm{d}s^2 = \frac{r^2}{(r^2 - r_+^2)(r^2 - r_-^2)}\mathrm{d}r^2 - \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}\mathrm{d}t^2 + r^2 \left(\mathrm{d}\phi - \frac{r_+ r_-}{r^2}\mathrm{d}t\right)^2 \tag{3.64}$$

These solutions exist for all values of m^2. They can be converted to Gaussian normal coordinates by the radial coordinate transformation

$$r^2 = \mathrm{e}^{2\rho} + \frac{1}{2}(r_+^2 + r_-^2) + \frac{1}{16}\mathrm{e}^{-2\rho}(r_+^2 - r_-^2)^2, \tag{3.65}$$

and the Fefferman-Graham expansion of the metric now reads

$$\mathrm{e}^{-2\rho} g_{ij}\mathrm{d}x^i \mathrm{d}x^j = -\mathrm{d}t^2 + \mathrm{d}\phi^2 + \mathrm{e}^{-2\rho}\left[\frac{1}{2}(r_+^2 + r_-^2)(\mathrm{d}t^2 + \mathrm{d}\phi^2) - 2r_+ r_- \mathrm{d}t\mathrm{d}\phi\right] + \mathcal{O}(\mathrm{e}^{-4\rho}). \tag{3.66}$$

From this metric, it is fairly straight forward to plug in to the stress tensor (3.58). The conserved charges are found to be proportional to those for the BTZ black hole in general relativity [42],

$$M = \frac{r_+^2 + r_-^2}{8G}\left(\sigma + \frac{1}{2m^2}\right), \qquad J = \frac{r_+ r_-}{4G}\left(\sigma + \frac{1}{2m^2}\right). \tag{3.67}$$

This result was first derived using the holographic stress tensor by Hohm and Tonni [131] and was later confirmed in [132]. The mass and angular momentum also match those found in [113, 167][5] using the Abbott-Deser-Tekin approach to conserved charges.

[5]The sign choice for the m^2 coupling is inconsistent in the literature, with no single convention dominating. Clemént, for instance, uses a negative sign in [110], while I adopt the positive sign choice of Nam et. al in [167], and simply allow m^2 to take positive and negative values.

Using Wald's method [168] to compute the entropy, Clemént [113] also found the entropy to be renormalized by a factor proportional to the central charge,

$$S_{BTZ} = \frac{A_H}{4G}\left(\sigma + \frac{1}{2m^2L^2}\right) \qquad A_H \equiv 2\pi L r_+ \tag{3.68}$$

The Hawking temperature and horizon angular velocity are derived in the usual way from the metric in ADM form

$$T_H = \frac{1}{4\pi} r\partial_r(N^2)\bigg|_{r_+} \qquad \Omega_h = -N^\phi\big|_{r_+} \tag{3.69}$$

and give the same values as that for a BTZ black hole in general relativity:

$$T_H = \frac{r_+^2 - r_-^2}{2\pi L r_+} \qquad \Omega_h = \frac{r_-}{L r_+} \tag{3.70}$$

Since the mass, angular momentum, and entropy are all renormalized by the same factor, the first law of black hole thermodynamics is easily verified

$$\mathrm{d}M = T_H \mathrm{d}S + \Omega_h \mathrm{d}J \tag{3.71}$$

From (3.67), physical black holes $M > 0$ have positive mass whenever the central charges are positive. However, the main results of [4, 123] show that unitarity of bulk gravitons requires *negative* central charges. Thus new massive gravity presents the same bulk/boundary conflict that appeared in cosmological topologically massive gravity, and we cannot choose the parameters in such way that both BTZ black holes and bulk gravitons have non-negative energy. Both theories also exhibit a chiral point that offers a potential way out of the dilemma. At the chiral point of topologically massive gravity, it was argued that the massive graviton modes could be excised by imposing Brown-Henneaux boundary conditions [2, 54], and then it would be possible to choose the 'right' sign of the Einstein-Hilbert action to give the BTZ black hole positive mass. The reverse may be true in new massive gravity: at the chiral point $2m^2 = -\sigma$, the mass of the BTZ black hole vanishes for *both* signs of the Einstein-Hilbert action, allowing us to choose $\sigma = -1$ to allow for positive energy bulk graviton modes, without the need to impose boundary conditions a priori. However, new 'log' black holes also exist at the chiral point, and their mass is checked next.

3.6.2 Log Black Holes

The general solution at the chiral point $2m^2 = -\sigma$ features a new 'log' branch of non-Einstein solutions (3.26), and log deformations of the extremal BTZ black hole were discovered shortly after the introduction of new massive gravity [110]. These solutions are comparable to the log black holes of topologically massive gravity, and are also locally equivalent to the log AdS wave found in [59]. The metric in Schwarzschild coordinates can be obtained from (3.64) by taking the extremal limit $r_+ = r_-$ and adding a log piece to metric

$$\mathrm{d}s^2 = \frac{r^2 \mathrm{d}r^2}{(r^2 - r_+^2)^2} - \frac{(r^2 - r_+^2)^2}{r^2}\mathrm{d}t^2 + r^2 \left(\mathrm{d}\phi - \frac{r_+^2}{r^2}\mathrm{d}t\right)^2 + k \ln |r^2 - r_+^2| \left(\mathrm{d}\phi - \mathrm{d}t\right)^2 . \tag{3.72}$$

The same radial transformation

$$r^2 = \mathrm{e}^{2\rho} + r_+^2 \tag{3.73}$$

converts the metric to Fefferman-Graham coordinates

$$g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{e}^{2\rho}(-\mathrm{d}t^2 + \mathrm{d}\phi^2) + (2k\rho + r_+^2)(\mathrm{d}\phi - \mathrm{d}t)^2. \tag{3.74}$$

A couple points about this case are noteworthy. First, the divergent terms in the Brown-York stress energy tensor disappear at this point in parameter space, and no counterterm is required for renormalization. This can also be seen in the fact that the counterterm derived in [131] (see (3.50)) vanishes at the chiral point. Second, the stress-energy tensor must be derived using the full Fefferman-Graham expansion at this point (3.26), and in general will contain terms involving $h^{(2)}$. Thus, we cannot simply plug the metric (3.74) into the renormalized stress-energy tensor derived in the previous section (3.58). The correct stress tensor was previously examined in [131, 129], and the conserved charges were found to be

$$M = \frac{2k\sigma}{G} \qquad J = \frac{2k\sigma}{G} \tag{3.75}$$

Again, it is gratifying to note that these results match conserved charges found using the ADT approach to conserved charges [119, 110]. Clemént [110] showed this metric is devoid of naked singularities when $k < 0$, so the mass is only positive with $\sigma = -1$. Thus, at the chiral point, positivity both for bulk graviton modes and black holes require the "wrong-sign" Einstein-Hilbert term.

3.6.3 Static New Type Black Holes

First realized in [4, 58], new massive gravity at the critical point $2m^2 = +\sigma$ admits static new type black holes, in Schwarzschild coordinates

$$\mathrm{d}s^2 = -(r^2 + br + c)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2 + br + c} + r^2\mathrm{d}\phi^2 \tag{3.76}$$

These solutions have scalar curvature

$$R = -6 - \frac{2b}{r} \tag{3.77}$$

and thus have no equivalent in topologically massive gravity. For a certain range of parameters, these solutions have inner and outer horizons given by $r_\pm = \frac{1}{2}(-b \pm \sqrt{b^2 - 4c})$. BTZ black holes are recovered in the limit $b \to 0$, with mass given by $c = -4GM$.

The metric (3.76) can be cast in Fefferman-Graham coordinates via the radial transformation

$$r = \mathrm{e}^\rho - \frac{b}{2} + \frac{1}{4}\mathrm{e}^{-\rho}\left(\frac{b^2}{4} - c\right) \tag{3.78}$$

Then to leading order, the constant-ρ two-dimensional metric becomes

$$g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{e}^{2\rho}(-\mathrm{d}t^2+\mathrm{d}\phi^2)-b\mathrm{e}^{\rho}\mathrm{d}\phi^2+\frac{1}{2}\left[\left(\frac{1}{4}b^2-c\right)\mathrm{d}t^2+\left(\frac{3}{4}b^2-c\right)\mathrm{d}\phi^2\right]+\mathcal{O}(\mathrm{e}^{-\rho}) \quad (3.79)$$

This case with exponential fall-off $n = 1$ must be treated separately from the generic case $1 < n < 2$ treated in the previous section, and the conserved charges cannot be obtained by plugging the above metric into the general renormalized stress tensor (3.58). While the expansion of the Brown-York stress tensor (3.47) produces the correct divergent pieces in the limit $n \to 1$, the constant term in (3.47) will now include expressions quadratic in $g^{(1)}$. Additionally, the trace of $g^{(1)}$ is no longer forced to vanish by the equations of motion, and the second constraint on the counterterms (3.55) is required to remove this divergence. Finally, the condition that the counterterms remove divergences from the stress tensor is not sufficient to completely determine the renormalized stress tensor, and instead the counterterms themselves contribute to the renormalized stress tensor. The renormalized stress tensor was found in [132]

$$\begin{aligned}8\pi G T_{ij} \;&=\; 2g^{(2)}_{ij} - g^{(1)}_{ik}\,g_{(1)j}^{\ \ \ \ k} + \frac{1}{2}g^{(1)}_{ij}\mathrm{Tr}g^{(1)} \\ &\quad +g^{(0)}_{ij}\left[\frac{1}{2}R^{(0)} - \mathrm{Tr}g^{(2)} + \left(4(C+D)+\frac{1}{4}\right)\left(\mathrm{Tr}g^{(2)}\right)^2\right]\end{aligned} \quad (3.80)$$

where C and D are the coefficients appearing in (3.52). Thus, another condition is needed to unambiguously determine the renormalized stress tensor. Matching with conserved charges found through other methods requires the additional condition $C + D = 0$ [129]. With these conditions, it becomes possible to use the renormalized stress tensor to compute the mass, and we find

$$M = \frac{b^2 - 4c}{4G}. \quad (3.81)$$

This result matches the total energy found through other methods. In particular, the Abbott-Deser-Tekin approach was originally used in [58] to compute the mass. Later [167] applied the super angular momentum method in the SO(2,1) reduction approach developed by Clemént, supplement by the ADT formalism, to arrive at the same results.

The absence of a global charge associated to b implies b is a sort of 'gravitational hair'. This trait is reflected in the first law of thermodynamics by the fact that there is no chemical potential associated to it, and a variation of b can be consistently reabsorbed by a shift of the global charges. It is reassuring to find that the mass M is consistent with the first law of black hole thermodynamics. The thermodynamics of these solutions have been investigated in [58, 133, 136]. The temperature is given by the inverse of the Euclidean time period to be

$$T = \frac{r_+ - r_-}{4\pi} \tag{3.82}$$

The entropy has been computed using both the Wald formula [58, 135] and the Cardy formula [133]

$$S = \frac{1}{4G}(A_+ - A_-), \qquad A_\pm = 2\pi r_\pm \tag{3.83}$$

From here, it is straightforward to verify that

$$\mathrm{d}M = T\mathrm{d}S. \tag{3.84}$$

3.6.4 Rotating New Type Black Holes

The solution (3.76) can be generalized to a rotating black hole by means of a boost in $\phi - t$ plane. The new metric takes the more complicated form

$$\mathrm{d}s^2 = -N^2(r)F^2(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{F^2(r)} + r^2\left(\mathrm{d}\phi + N^\phi(r)\mathrm{d}t\right)^2 \tag{3.85}$$

where

$$N(r) \equiv 1 + \frac{b\alpha}{2H(r)} \qquad N^{\phi}(r) \equiv \frac{\sqrt{\alpha(1-\alpha)}}{r^2}(c + bH(r)) \tag{3.86}$$

$$F(r) \equiv \frac{H(r)}{r}\left[H^2(r) + b(1-\alpha)H(r) + \frac{b^2}{4}\alpha^2 + c(1-2\alpha)\right]^{1/2} \tag{3.87}$$

$$H(r) \equiv \left[r^2 - \frac{b^2}{4}\alpha^2 + c\alpha\right]^{1/2} \tag{3.88}$$

and α is the rotation parameter $0 \leq \alpha \leq \frac{1}{2}$, which vanishes in the static case.

This metric can be put into Gaussian normal form, up to relevant order, by the radial transformation

$$r = \mathrm{e}^{\rho} + \beta_1 + \beta_2\mathrm{e}^{-\rho} + \mathcal{O}(\mathrm{e}^{-2\rho}) \tag{3.89}$$

where

$$\beta_1 = -\frac{b}{2}(1-\alpha) \qquad \beta_2 = \frac{1}{4}\left[\frac{b^2}{4}(1 - 2\alpha + 2\alpha^2) - c\right] \tag{3.90}$$

Then the metric components have the expansion

$$g_{tt} = -\mathrm{e}^{2\rho} - b\alpha\mathrm{e}^{\rho} - \left[b\beta_1 + \beta_1^2 + 2\beta_2 + c\right] + \mathcal{O}(\mathrm{e}^{-\rho}) \tag{3.91}$$

$$g_{\phi\phi} = \mathrm{e}^{2\rho} - b(1-\alpha)\mathrm{e}^{\rho} + (\beta_1^2 + 2\beta_2) + \mathcal{O}(\mathrm{e}^{-\rho}) \tag{3.92}$$

$$g_{t\phi} = b\sqrt{\alpha(1-\alpha)}\mathrm{e}^{\rho} + \sqrt{\alpha(1-\alpha)}(b\beta_1 + c) + \mathcal{O}(\mathrm{e}^{-\rho}) \tag{3.93}$$

The metric in this form can be plugged in to the stress tensor given in the previous section (3.80) to yield the conserved mass and angular momentum [58, 133]

$$M = \frac{b^2 - 4c}{16G} \qquad \frac{J}{M} = 2\sqrt{\alpha(1-\alpha)} \tag{3.94}$$

Note that this solution still reduces to the rotating BTZ black hole in the limit $b \to 0$.

3.7 Conclusion

In this chapter, I have adopted the Fefferman-Graham approach to examine the asymptotic behavior of New Massive Gravity at generic couplings. At generic values of the mass parameter, the traditional Fefferman-Graham expansion fails to capture the dynamics of the theory, demonstrating the need for a more general expansion. The expansion at all values of the mass parameter is derived and used to find the asymptotic behavior of non-Einstein solutions to first order. At the critical points $2m^2 = \pm\sigma$, some of the branches of solutions degenerate, and new logarithmic solutions becomes possible at these values. The validity of the general asymptotic expansion is confirmed by comparing to known exact solutions in the literature, and all known asymptotically AdS solutions are shown to match the asymptotic behavior found in this paper.

The solutions indicate a range of the mass parameter $-1 < 2m^2 < 1$ in which the non-Einstein solutions asymptote to the AdS metric with slower fall-off than Brown-Henneaux. In particular, the positive branch solutions, though weaker than Brown-Henneaux, decay faster than the relaxed asymptotics at the critical point $2m^2 = \sigma$, which have previously been shown to be consistent. Using the auxiliary tensor formulation of Hohm and Tonni [131], I have computed the Brown-York stress energy tensor with these asymptotics, and the appropriate counterterms required to remove divergences are determined up to unphysical ambiguities. The holographically renormalized stress tensor is found and gives the correct central charge of the dual CFT.

However, the results are intriguingly vague. First, the new divergences in the unrenormalized stress tensor stemming from the relaxed asymptotics cancel on-shell, with no new constraints on appropriate counterterms beyond those already needed for Brown-Henneaux boundary conditions. Second, the renormalized stress tensor is exactly that of the theory with Brown-Henneaux boundary conditions, i.e., without the non-Einstein modes. In other words, the non-Einstein solutions, despite the weaker fall-off, do not contribute to the boundary stress tensor or to the conserved charges. More work is needed to interpret these results, especially in the context of the boundary CFT, and it would be interesting to confirm these results using the Abbott-Deser-Tekin method for conserved charges.

Another intriguing question is the possibility of holographic renormalization of non-AdS asymptotics. Hohm and Tonni's original approach was applied successfully to asymptotically Lifshitz solutions [131], despite the fact that such solutions break the asymptotic symmetries of AdS spacetime. The modified Fefferman-Graham expansion (3.21) found here contains one branch of solutions – the negative branch associated with n_- in the parameter range $2m^2 < -1$ – that extend "beyond the boundary" and also break the asymptotic symmetries, and it would be interesting to see if the approach of [131] can be used to define a finite stress tensor for these asymptotics. The existence of exact solutions with this behavior, namely a subset of the AdS pp-waves and the Type N solutions, makes this question a potentially interesting one.

Chapter 4

Conclusion and Next Steps

Understanding the asymptotic structure of a theory has always been important for addressing questions related to the variational principle, global charges, and especially for understanding the relationship between bulk and boundary degrees of freedom. In particular, the Fefferman-Graham expansion in asymptotically anti-de Sitter spacetimes has been an important tool in the exploration of the proposed AdS/CFT correspondence. It has proven useful (in Einstein gravity) in constructing the boundary stress tensor, finding conserved charges, deriving correlation functions and obtaining the central charge of the dual CFT, and identifying degrees of freedom. Thus it is reasonable to expect the Fefferman-Graham expansion also to play a role in extensions to Einstein gravity. We have long known that the expansion is modified for Einstein gravity coupled to other fields; however, the idea that the expansion might also be modified for purely gravitational, higher-derivative extensions to general relativity has only recently gained footing.

With minimal assumptions, I have determined the general asymptotic expansion for both topologically massive gravity and new massive gravity in asymptotically (locally) anti-de Sitter spacetimes. The general solution can be decomposed into Einstein and non-Einstein terms, and the Einstein solutions obey the same fall-off conditions as in general relativity. The asymptotic behavior of the non-Einstein branches depends on the coupling parameters μ and m^2, requiring a modification of the usual Fefferman-Graham expansion.

For new massive gravity, I have continued the analysis a step further. The modified

Fefferman-Graham expansion identifies a range of parameters for which the non-Einstein branch decays more weakly than Brown-Henneaux, but still asymptotes to the AdS metric at infinity. I have constructed the Brown-York boundary stress-energy tensor for this relaxed fall-off and determined the central charge of the dual CFT.

These techniques can be easily extended to other theories with asymptotically AdS sectors. The research initiated by Bergshoeff, Hohm and Townsend [3] has taken off in many new and exciting directions, and a modified Fefferman-Graham approach offers important tools for exploring these new models. The next obvious extension is the addition of the gravitational Chern-Simons terms to the action of new massive gravity. This "generalized massive gravity" (GMG) has many interesting properties, including mass-splitting of the different helicities, and has been the subject of much investigation since the introduction of new massive gravity.

More recent extensions include two different approaches to higher-derivative extensions of new massive gravity: the sixth- and higher- order extended new massive gravity (ENMG), and Born-Infeld new massive gravity (BI-NMG). Additional research has looked at extensions of new massive gravity at the critical point $2m^2 = +\sigma$ to higher dimensions. Despite the fact that unitarity is not preserved in higher dimensions, there is some indication that critical NMG with appropriate boundary conditions is classically equivalent to Einstein gravity. I will take a few moments to explore both extensions—higher derivative and higher dimension—in more detail.

4.1 Critical Gravity in Arbitrary Dimensions

Critical gravity [169] is the extension of the action (3.1) to higher dimensions,

$$S = \frac{\sigma}{\kappa^2}\int \mathrm{d}^D x\sqrt{-g}\left[R + \frac{(D-1)(D-2)}{\ell^2} + \frac{1}{m^2(D-2)}\left(R^{\mu\nu}R_{\mu\nu} - \frac{D}{4(D-1)}R^2\right)\right], \quad (4.1)$$

formulated around the critical point $2L^2m^2 = \sigma(D-2)$, where D is the (bulk) spacetime dimension and L the AdS radius. The curvature-squared part of the action is just a particular combination of the Weyl-squared and Gauss-Bonnet tensors. As with new massive gravity in three dimensions, this particular combination of curvature-squared terms contains no scalar modes, and the action can also be written in terms of an auxiliary tensor as a second-order action. Additionally, the theory is consistent with the requirements of the holographic c-theorem [170].

It was observed that ghost-like massive modes have asymptotic fall-off slower than massless modes, and the authors of [171] speculated that if these massive modes could be truncated by appropriate boundary conditions, the theory would be classically equivalent to Einstein gravity. This work was more recently extended to non-critical gravity [172, 173], where the slower fall-off of massive modes was observed for a range of couplings

$$\frac{D^2-6D+7}{4(D-2)L^2} < m^2 \leq \frac{D-2}{2L^2}. \tag{4.2}$$

Most of these studies are based on the linearized equations of motion; however, the modified Fefferman-Graham approach employed here offers a natural way to formulate questions concerning boundary conditions and could complement the linearization studies. For $D = 3$, (4.2) is precisely the range at which both the positive and negative branches $n_\pm$ of non-Einstein solutions i) asymptote to the AdS metric at infinity, and ii) have weaker than Brown-Henneaux asymptotics. Imposing Brown-Henneaux boundary conditions would excise both branches, leaving only the Einstein solutions. Thus the approach taken in this paper may provide additional evidence of the equivalence of higher-dimensional (non-)critical gravity and Einstein gravity.

4.2 Extended- and Born-Infeld new massive gravities

Two higher-derivative extensions of new massive gravity were discovered concurrently less than two years after the initial formulation of NMG. The extension to cubic curvature terms was first derived by Sinha [124] with only the constraint that the theory satisfy a holographic c-theorem. The action obtained is

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^3x \sqrt{-g} \left[\sigma R + \frac{1}{\ell^2} + \frac{1}{m^2} K + \frac{1}{12\mu^4} K' \right], \tag{4.3}$$

where the first three terms comprise the familiar new massive gravity action, and the new term is

$$K' = 17R^3 - 72 R_{\mu\nu} R^{\mu\nu} R + 64 R^{\nu}_{\mu} R^{\rho}_{\nu} R^{\mu}_{\rho}. \tag{4.4}$$

The corresponding equations of motion are now sixth order in derivatives of the metric. The action was also generalized to eighth order, and an algorithm was devised to extend the action to arbitrary order in [174].

In the same month, a Born-Infeld extension was introduced [175], with the action

$$S = -\frac{2m^2}{\kappa^2} \int \mathrm{d}^3x \sqrt{-g} \left[\sqrt{\det \left(\delta^{\mu}_{\nu} + \frac{\sigma}{m^2} G^{\mu}_{\nu} \right)} - 1 - \frac{1}{2m^2 \ell^2} \right], \tag{4.5}$$

where G denotes the Einstein tensor. In some sense, this can be thought of as an infinite-order extension of new massive gravity. A small-curvature expansion of the action (4.5) precisely gives the action of ENMG (4.3), in particular the coefficients of the cubic terms (4.4). This action is also consistent with a holographic c-theorem. Moreover, equivalence of the two actions has been now been verified to fourth order in the curvature.

Despite this equivalence, some important differences arise at the non-linear level. In particular, ENMG has degenerate vacua, while BI-NMG offers a unique vacuum [176]. Additionally, ENMG possesses a critical point, similar to the critical point $2m^2 = +\sigma$, at which new type black holes are allowed. These new type black holes have the same metric (3.76) as the NMG solution. However, these black holes are *not allowed* in BI-NMG, and it is not entirely clear why this should be the case [177]. An asymptotic approach may provide some insight.

Appendix A

Appendices: Gaussian Normal Coordinates

It may be useful to write recurring quantities in terms of the Gaussian normal coordinates used throughout most of this paper. In these coordinates, the metric takes the form:

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{A.1}$$

Note that this is just a particular gauge of the ADM-like metric (see Hohm & Tonni [131]).

A.1 Ingredients for Einstein Gravity

Christoffel Symbol components

$$\Gamma^{\rho}_{ij} = -\frac{1}{2}g_{ij,\rho} \text{ or } -\frac{1}{2}g' \tag{A.2}$$

$$\Gamma^{i}_{\rho j} = \frac{1}{2}g^{ik}g_{kj,\rho} \text{ or } \frac{1}{2}g^{-1}g' \tag{A.3}$$

$$\Gamma^{i}_{jk} = \frac{1}{2}g^{il}\left(g_{kl,j} + g_{lj,k} - g_{jk,l}\right) \text{ or } {}^{2}\Gamma^{i}_{jk} \tag{A.4}$$

where $'$ denotes differentiation with respect to ρ. In these coordinates, the extrinsic curvature of a fixed ρ surface is reduces to $K_{ij} = -\Gamma^{\rho}_{ij}$.

Riemann Tensor components

$$R^i_{\rho j\rho} = \left[-\frac{1}{2}g^{-1}g'' + \frac{1}{4}g^{-1}g'g^{-1}g'\right]^i_j \quad (A.5)$$

$$R^i_{\rho jk} = \frac{1}{2}{}^2\nabla_j\left(g^{-1}g'\right)^i_k - \frac{1}{2}{}^2\nabla_k\left(g^{-1}g'\right)^i_j \quad (A.6)$$

$$R^i_{jkl} = {}^2R^i{}_{jkl} - \frac{1}{4}g^{im}g'_{mk}g'_{jl} + \frac{1}{4}g^{im}g'_{ml}g'_{jk} \quad (A.7)$$

where ${}^2R^i{}_{jkl}$ represents the curvature tensor for the two-dimensional surface at constant ρ. Similarly, ${}^2\nabla_j$ is the covariant derivative on the surface at constant ρ. From this point forward, the pre-index on all covariant derivatives will be suppressed and will be written as ∇_j.

Ricci Tensor components

$$R_{\rho\rho} = -\frac{1}{2}\mathrm{Tr}\left(g^{-1}g''\right) + \frac{1}{4}\mathrm{Tr}\left(g^{-1}g'g^{-1}g'\right) \quad (A.8)$$

$$R_{\rho i} = \frac{1}{2}\nabla_j\left(g^{-1}g'\right)^j_i - \frac{1}{2}\partial_i\mathrm{Tr}\left(g^{-1}g'\right) \quad (A.9)$$

$$R_{ij} = {}^2R_{ij} - \frac{1}{2}g''_{ij} - \frac{1}{4}\mathrm{Tr}\left(g^{-1}g'\right)g'_{ij} + \frac{1}{2}\left(g'g^{-1}g'\right)_{ij} \quad (A.10)$$

Ricci Scalar

$$R = {}^2R - \mathrm{Tr}\left(g^{-1}g''\right) + \frac{3}{4}\mathrm{Tr}\left(g^{-1}g'g^{-1}g'\right) - \frac{1}{4}\left[\mathrm{Tr}\left(g^{-1}g'\right)\right]^2 \quad (A.11)$$

Note that in these coordinates, the 3D Einstein-Hilbert action is

$$\begin{aligned} S_{EH} = &\frac{1}{16\pi G}\int_M \mathrm{d}^2x\,\mathrm{d}\rho\sqrt{-g}\left({}^2R + (\mathrm{Tr}K)^2 - \mathrm{Tr}K^2 - 2\Lambda\right) \\ &-\frac{1}{8\pi G}\int_{\partial M}\mathrm{d}^2x\sqrt{-g}\mathrm{Tr}K, \end{aligned} \quad (A.12)$$

and so the Gibbons-Hawking term is just the negative of the boundary term

$$S_{GH} = \frac{1}{8\pi G} \int_{\partial M} \mathrm{d}^2 x \sqrt{-g} \mathrm{Tr} K. \tag{A.13}$$

A.2 Cotton Tensor

Levi-Civita Tensor

Let $\tilde{\epsilon}$ be the completely antisymmetric Levi-Civita *symbol*. Then the Levi-Civita tensor is defined as

$$\epsilon_{ij} = \tilde{\epsilon}_{ij} \sqrt{-\mathrm{det} g} \tag{A.14}$$

Cotton Tensor components

$$C_{\rho\rho} = \epsilon^{ij} \left[\nabla_i R_{\rho j} - \frac{1}{2} \left(g^{-1} g' \right)_i^k R_{kj} \right] \tag{A.15}$$

$$C_{\rho i} = -\epsilon_n{}^k \nabla_k R_i^n + \frac{1}{2} \epsilon^{kn} g'_{ki} R_{\rho n} \tag{A.16}$$

$$C_{ij} = -\epsilon_i{}^k \left[\partial_\rho R_{kj} - \frac{1}{2} \left(g^{-1} g' \right)_j^n R_{kn} - \nabla_k R_{\rho j} - \frac{1}{2} g'_{kj} R_{\rho\rho} \right] \tag{A.17}$$

It may also be useful to construct higher-curvature terms, such as those appearing in the new massive gravity, in the Gaussian normal coordinates of (A.1). However, this step is not necessary for all purposes. For example, Hohm and Tonni [131] avoid this step when deriving the generalized Gibbons-Hawking term by using the auxiliary tensor formulation of new massive gravity. Similarly, I have sidestepped this question in the Fefferman-Graham expansion of new massive gravity by first expanding the components of the Ricci tensor, and then plugging in to the tensors in the NMG equations of motion. This process is used in the next section to derive the Fefferman-Graham expansion for NMG.

Appendix B

Fefferman-Graham Expansion

Expansion of same tensors as in first appendix.

B.1 "Traditional" Fefferman-Graham Expansion

I start by using the "traditional" Fefferman-Graham expansion for Einstein gravity, and expand out to "sixth" order:

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\rho^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j \\ g &= e^{2\rho}\left(g^{(0)} + e^{-2\rho}g^{(2)} + e^{-4\rho}g^{(4)} + e^{-6\rho}g^{(6)} + \cdots\right) \end{aligned} \tag{B.1}$$

where indices have been suppressed. The inverse metric is given by

$$g^{-1} = e^{-2\rho}\left[g^{-1}_{(0)} - e^{-2\rho}g^{-1}_{(2)} - e^{-4\rho}\left(g^{-1}_{(4)} - g^{2}_{(2)}\right) + \cdots\right] \tag{B.2}$$

where indices are raised and lowered with the boundary metric $g^{(0)}$. The two-dimensional Christoffel symbols can also be expanded

$$\Gamma^i_{jk} = \Gamma^{i(0)}_{jk} + \mathrm{e}^{-2\rho}\Gamma^{i(2)}_{jk} + \mathrm{e}^{-4\rho}\Gamma^{i(4)}_{jk} + \cdots \tag{B.3}$$

where

$$\begin{aligned}
\Gamma^{i(0)}_{jk} &= \frac{1}{2} g^{il}_{(0)} \left(g^{(0)}_{kl,j} + g^{(0)}_{lj,k} - g^{(0)}_{jk,l} \right) \\
\Gamma^{i(2)}_{jk} &= -\frac{1}{2} g^{il}_{(2)} \left(g^{(0)}_{kl,j} + g^{(0)}_{lj,k} - g^{(0)}_{jk,l} \right) + \frac{1}{2} g^{il}_{(0)} \left(g^{(2)}_{kl,j} + g^{(2)}_{lj,k} - g^{(2)}_{jk,l} \right) \\
\Gamma^{i(4)}_{jk} &= \cdots
\end{aligned}$$

However, it's more useful to write the higher-order Christoffel symbols in terms of covariant derivatives using equations derived from metric compatibility $\nabla g = 0$. The Christoffel symbols in the covariant derivative and the metric can both be expanded to give

$$\nabla^{(0)}_k g^{(0)}_{ij} = 0 \tag{B.4}$$

$$\nabla^{(0)}_k g^{(2)}_{ij} = \Gamma^{l(2)}_{ki} g^{(0)}_{lj} + \Gamma^{l(2)}_{kj} g^{(0)}_{il} \tag{B.5}$$

Using index permutation tricks, the higher-order Christoffel symbols can be written in terms of covariant derivatives:

$$\Gamma^{k(2)}_{ij} = \frac{1}{2} g^{kl}_{(0)} \left(\nabla_i g^{(2)}_{jl} + \nabla_j g^{(2)}_{li} - \nabla_l g^{(2)}_{ij} \right) \tag{B.6}$$

From this point forward, all covariant derivatives and Christoffel symbols have been expanded using the rules above, and the superscript in $\nabla^{(0)}$ is dropped to simplify notation. Now, the Ricci tensor components expanded to "fourth" order are

$$R_{\rho\rho} = -2 + \mathrm{e}^{-4\rho} \mathrm{Tr} \left(-4 g_{(4)} + g^2_{(2)} \right) \tag{B.7}$$

$$\begin{aligned}
R_{ij} = {} & -2\mathrm{e}^{2\rho} g^{(0)}_{ij} + \mathrm{e}^{0} \left[R^{(0)}_{ij} - 2 g^{(2)}_{ij} + g^{(0)}_{ij} \mathrm{Tr} g_{(2)} \right] \\
& + \mathrm{e}^{-2\rho} \left[R^{(2)}_{ij} - 6 g^{(4)}_{ij} + 2 \left(g^2_{(2)} \right)_{ij} + g^{(0)}_{ij} \mathrm{Tr} \left(2 g_{(4)} - g^2_{(2)} \right) \right]
\end{aligned} \tag{B.8}$$

$$\begin{aligned}
R_{\rho i} = {} & \mathrm{e}^{-2\rho} \left[-\nabla_j {g_{(2)}}^j{}_i + \partial_i \mathrm{Tr} g_{(2)} \right] \\
& + \mathrm{e}^{-4\rho} \left[-\nabla_j \left(2 g_{(4)} - g^2_{(2)} \right)^j_i + \partial_i \mathrm{Tr} \left(2 g_{(4)} - \frac{3}{4} g^2_{(2)} \right) - \frac{1}{2} {g_{(2)}}^j{}_i \partial_j \mathrm{Tr} g_{(2)} \right]
\end{aligned} \tag{B.9}$$

where the left-hand side are components of the full three-dimensional Ricci tensor, and the Ricci tensor/scalars appearing on the right-hand side are the two-dimensional Ricci tensor/scalar components constructed from g_{ij}. The Ricci scalar is

$$\begin{aligned} R &= -6 + \mathrm{e}^{-2\rho}\left(R^{(0)} + 2\mathrm{Tr}g^{(2)}\right) \\ &\quad + \mathrm{e}^{-2\rho}\left[g^{ij}_{(0)}R^{(2)}_{ij} - g^{ij}_{(2)}R^{(0)}_{ij} + \mathrm{Tr}\left(-4g^{(4)} + g^2_{(2)}\right) - \left(\mathrm{Tr}g^{(2)}\right)^2\right] \end{aligned} \tag{B.10}$$

The fact that ${}^3R = -6$ (for TMG only) and ${}^2R_{ij} \propto g_{ij}$ can also be used to simplify the expressions above and give the following:

$${}^2R^{(0)}_{ij} = -g^{(0)}_{ij}\mathrm{Tr}g_{(2)} \tag{B.11}$$

$${}^2R^{(2)}_{ij} = -g^{(2)}_{ij}\mathrm{Tr}g_{(2)} + \frac{1}{2}g^{(0)}_{ij}\mathrm{Tr}\left(4g_{(4)} - g^{-1}_{(2)}g_{(2)}\right) + \frac{1}{2}g^{(0)}_{ij}\left(\mathrm{Tr}g_{(2)}\right)^2 \tag{B.12}$$

The Cotton tensor contains the Levi-Civita *tensor*, which must also be expanded before the full Cotton tensor can be obtained. Let $\tilde{\epsilon}$ be the Levi-Civita symbol (a tensor density). Then

$$\epsilon^{ij} = \tilde{\epsilon}^{ij}/\sqrt{-g} \tag{B.13}$$

$$= \mathrm{e}^{-2\rho}\epsilon^{ij}_{(0)} - \frac{1}{2}\mathrm{e}^{-4\rho}\epsilon^{ij}_{(0)}\mathrm{Tr}g_{(2)} - \mathcal{O}(\mathrm{e}^{-6\rho}) \tag{B.14}$$

Now the Cotton tensor components are given by

$$C_{\rho\rho} = -\mathrm{e}^{-4\rho}\epsilon_{(0)}^{ij}\nabla_i\nabla_k g_{(2)}{}^k{}_j \tag{B.15}$$

$$\begin{aligned} C_{ij} &= \mathrm{e}^{-2\rho}\epsilon_i^{(0)k}\left[-12g_{kj}^{(4)} + 6\left(g_{(2)}^2\right)_{kj} - g_{kj}^{(0)}\mathrm{Tr}\left(-8g_{(4)} + \frac{7}{2}g_{(2)}^2\right) + \frac{3}{2}g_{kj}^{(0)}\left(\mathrm{Tr}g_{(2)}\right)^2\right. \\ &\quad \left. -3g_{kj}^{(2)}\mathrm{Tr}g_{(2)} - \nabla_k\nabla_l g_{(2)}{}^l{}_j + \nabla_k\partial_j\mathrm{Tr}g_{(2)}\right] \end{aligned} \tag{B.16}$$

$$\begin{aligned} C_{\rho i} &= \mathrm{e}^{-2\rho}\epsilon_i{}^n\left[-\nabla_j g_{(2)}{}^j{}_n + \partial_n\mathrm{Tr}g_{(2)}\right] \\ &\quad +\mathrm{e}^{-4\rho}\epsilon_n{}^k\nabla_k\left[4g_{(4)}{}^n{}_i - 2\left(g_{(2)}^2\right)_i^n - \delta_i^n\mathrm{Tr}\left(4g_{(4)} - \frac{3}{2}g_{(2)}^2\right) - \frac{1}{2}\delta_i^n\left(\mathrm{Tr}g_{(2)}\right)^2 + g_{(2)}{}^n{}_i\mathrm{Tr}g_{(2)}\right] \\ &\quad +\mathrm{e}^{-4\rho}\epsilon_i{}^n\left[-\nabla_j\left(2g_{(4)} - g_{(2)}^2\right)_n^j + \partial_n\mathrm{Tr}\left(2g_{(4)} - \frac{3}{4}g_{(2)}^2\right) - \frac{1}{2}g_{(2)}{}^j{}_n\partial_j\mathrm{Tr}g_{(2)}\right. \\ &\quad \left. -\frac{1}{2}\mathrm{Tr}g_{(2)}\left(-\nabla_j g_{(2)}{}^j{}_n + \partial_n\mathrm{Tr}g_{(2)}\right)\right] \end{aligned} \tag{B.17}$$

it is now fairly straightforward to construct the equations of motion. The "second" and "fourth" order equations of TMG are quoted in the text (2.8), (2.9), (2.10) and (2.11). However, the fourth order $\{\rho, i\}$ equations provide no new constraints on $g^{(4)}$ and are included here for reference:

$$\begin{aligned} &\nabla_j\left(2g_{(4)} - g_{(2)}^2\right)_i^j - \partial_i\mathrm{Tr}\left(2g_{(4)} - \frac{3}{4}g_{(2)}^2\right) + \frac{1}{2}g_{(2)}{}_i^l\partial_l\mathrm{Tr}g_{(2)} \\ &+\frac{1}{\mu}\epsilon_i{}^k\left[\nabla_j\left(2g_{(4)} - g_{(2)}^2\right)_k^j - \partial_k\mathrm{Tr}\left(2g_{(4)} - \frac{3}{4}g_{(2)}^2\right) + \frac{1}{2}g_{(2)}{}_k^l\partial_l\mathrm{Tr}g_{(2)}\right] \\ &+\frac{1}{\mu}\epsilon_n{}^k\nabla_k\left[\left(R_{(2)} - 4g_{(4)} + 2g_{(2)}^2\right)_i^n + \delta_i^n\mathrm{Tr}\left(2g_{(4)} - g_{(2)}^2\right)\right] \\ &+\frac{1}{2\mu}\mathrm{Tr}g_{(2)}\epsilon_i{}^n\left[-\nabla_j g_{(2)}{}_n^j + \partial_n\mathrm{Tr}g_{(2)}\right] = 0 \end{aligned} \tag{B.18}$$

In the light-cone coordinates with AdS boundary metric, these equations become

$$\left(1 + \frac{3}{\mu}\right)\partial_v g_{uu}^{(4)} = \left(1 - \frac{1}{\mu}\right)\partial_u\left(g_{uv}^{(4)} + 2g_{uu}^{(2)}g_{vv}^{(2)}\right) \tag{B.19}$$

$$\left(1 - \frac{3}{\mu}\right)\partial_u g_{vv}^{(4)} = \left(1 + \frac{1}{\mu}\right)\partial_v\left(g_{uv}^{(4)} + 2g_{uu}^{(2)}g_{vv}^{(2)}\right) \tag{B.20}$$

B.2 Critical TMG Expansion

At the critical point $\mu = 1$, the full asymptotic expansion is given by

$$g_{ij} = \mathrm{e}^{2\rho} g^{(0)}_{ij} + \rho h^{(2)}_{ij} + g^{(2)}_{ij} + \cdots \tag{B.21}$$

Plugging this expansion into the Ricci tensor gives

$$\begin{aligned}
R_{\rho\rho} &= -2 + \mathrm{e}^{-2\rho}\mathrm{Tr}h^{(2)} + \cdots && \text{(B.22)}\\
R_{\rho i} &= \rho\mathrm{e}^{-2\rho}\left(-\nabla^j h^{(2)}_{ji} + \partial_i \mathrm{Tr}h^{(2)}\right) && \text{(B.23)}\\
&\quad +\mathrm{e}^{-2\rho}\left[-\nabla^j\left(g^{(2)}_{ij} - \frac{1}{2}h^{(2)}_{ij}\right) + \partial_i \mathrm{Tr}\left(g^{(2)} - \frac{1}{2}h^{(2)}\right)\right] + \cdots \\
R_{ij} &= -2\mathrm{e}^{2\rho} g^{(0)}_{ij} + \rho\left(g^{(0)}_{ij}\mathrm{Tr}h^{(2)} - 2h^{(2)}_{ij}\right) && \text{(B.24)}\\
&\quad + \left[R^{(0)}_{ij} + g^{(0)}_{ij}\mathrm{Tr}\left(g^{(2)} - \frac{1}{2}h^{(2)}\right) + h^{(2)}_{ij} - 2g^{(2)}_{ij}\right] + \cdots
\end{aligned}$$

where again the boundary metric is used to raise and lower indices, and all covariant derivatives are with respect to the boundary metric. The Ricci scalar is given by

$$R = -6 + 2\rho\mathrm{e}^{-2\rho}\mathrm{Tr}h^{(2)} + \mathrm{e}^{-2\rho}\left[R^{(0)} + 2\mathrm{Tr}g^{(2)} + \mathrm{Tr}h^{(2)}\right] + \cdots \tag{B.25}$$

The Cotton tensor has components

$$\begin{aligned}
C_{\rho\rho} &= \mathcal{O}(\rho\mathrm{e}^{-4\rho}) && \text{(B.26)}\\
C_{\rho i} &= -\rho\mathrm{e}^{-2\rho}\epsilon_i{}^n\nabla^j h^{(2)}_{jn} && \text{(B.27)}\\
&\quad +\mathrm{e}^{-2\rho}\left\{-\epsilon^n{}_k\nabla^k h^{(2)}_{ni} + \epsilon_i{}^n\left[-\nabla^j\left(g^{(2)}_{jn} - \frac{1}{2}h^{(2)}_{jn}\right) + \partial_n \mathrm{Tr}\left(g^{(2)} - \frac{1}{2}h^{(2)}\right)\right]\right\} + \cdots \\
C_{ij} &= \epsilon_i{}^k h^{(2)}_{kj} + \cdots && \text{(B.28)}
\end{aligned}$$

Then the EOMs at first order are

$$\begin{aligned} \mathrm{Tr}\, h^{(2)} &= 0 && \text{(B.29)} \\ \left(\delta_i^n + \frac{1}{\mu}\epsilon_i{}^n\right) h^{(2)}_{nj} &= 0 && \text{(B.30)} \\ \left(\delta_i^n + \frac{1}{\mu}\epsilon_i{}^n\right) \nabla^j h^{(2)}_{nj} &= 0 && \text{(B.31)} \end{aligned}$$

Thus it appears as if one component of $h^{(2)}$ is completely unconstrained. In light-cone coordinates, $h^{(2)}_{uv}$ and $h^{(2)}_{vv}$ vanish, and the above equations for $h^{(2)}_{uu}$ become

$$\begin{aligned} \left(1 - \frac{1}{\mu}\right) h^{(2)}_{uu} &= 0 && \text{(B.32)} \\ \left(1 - \frac{1}{\mu}\right) \partial_v h^{(2)}_{uu} &= 0 && \text{(B.33)} \end{aligned}$$

Thus at $\mu = 1$, the divergence equation vanishes at first order. However, precisely this piece that is missing from the divergences equation is recovered at *second order*:

$$\left(\delta_i^k + \frac{1}{\mu}\epsilon_i{}^k\right) \left[\left(\nabla^j g^{(2)}_{jk} - \partial_k \mathrm{Tr} g^{(2)}\right) - \frac{1}{2}\left(\nabla^j h^{(2)}_{jk} - \partial_k \mathrm{Tr} h^{(2)}\right)\right] + \frac{1}{\mu}\epsilon^{nk}\nabla_k h^{(2)}_{ni} = 0 \qquad \text{(B.34)}$$

In light-cone coordinates, this becomes

$$-\left(1 - \frac{1}{\mu}\right) \partial_v \left(g^{(2)}_{uu} - \frac{1}{2} h^{(2)}_{uu}\right) - \frac{1}{\mu}\partial_v h^{(2)}_{uu} = 0, \qquad \text{(B.35)}$$

where I have used the vanishing of the trace of $g^{(2)}$ and $h^{(2)}$ to simplify the equation. It is clear that, even at $\mu = 1$, the divergence constraint $\partial_v h^{(2)}_{uu}$ remains.

B.3 CDWW Modes

The modes obtained in [1, 86] were first written as perturbations of the Einstein tensor. In three dimensional gravity, perturbations of the Einstein tensor correspond to perturbations of the metric, and it is possible to map one onto the other.

The authors of [86] start in the Poincaré patch

$$\mathrm{d}s^2 = \frac{\mathrm{d}z^2}{z^2} + \frac{1}{z^2}\left[2\mathrm{d}u\mathrm{d}v + z^2\left(\hat{h}_{uu}\mathrm{d}u^2 + 2\hat{h}_{uv}\mathrm{d}u\mathrm{d}v + \hat{h}_{vv}\mathrm{d}v^2\right)\right] \tag{B.36}$$

Fluctuations of the Einstein tensor are given by

$$\mathcal{H}_{vv} = \frac{\omega_-^2}{\omega}\mathrm{e}^{\mathrm{i}[\omega_+ u + \omega_- v]} z J_{\mu+2}(\omega z) + \text{h.c.} \tag{B.37}$$

$$\mathcal{H}_{vz} = -\mathrm{i}\omega_- \mathrm{e}^{\mathrm{i}[\omega_+ u + \omega_- v]} z J_{\mu+1}(\omega z) + \text{h.c.} \tag{B.38}$$

$$\mathcal{H}_{zz} = -\omega \mathrm{e}^{\mathrm{i}[\omega_+ u + \omega_- v]} z J_{\mu}(\omega z) + \text{h.c.} = -2\mathcal{H}_{uv} \tag{B.39}$$

$$\mathcal{H}_{uz} = \mathrm{i}\omega_+ \mathrm{e}^{\mathrm{i}[\omega_+ u + \omega_- v]} z J_{\mu-1}(\omega z) + \text{h.c.} \tag{B.40}$$

$$\mathcal{H}_{uu} = \frac{\omega_+^2}{\omega}\mathrm{e}^{\mathrm{i}[\omega_+ u + \omega_- v]} z J_{\mu-2}(\omega z) + \text{h.c.} \tag{B.41}$$

where the J are Bessel functions of the first kind.

The relationship between the Einstein tensor fluctuations and the metric fluctuations is given by

$$\mathcal{H}_{uu} = -\frac{1}{2} z\partial_z\left(z\partial_z + 2\right)\hat{h}_{uu} \tag{B.42}$$

$$\mathcal{H}_{uv} = \frac{1}{2} z\partial_z\left(z\partial_z + 2\right)\hat{h}_{uv} \tag{B.43}$$

$$\mathcal{H}_{vv} = -\frac{1}{2} z\partial_z\left(z\partial_z + 2\right)\hat{h}_{uu} \tag{B.44}$$

It may not be possible to invert (B.44) and write the metric perturbations in terms of Bessel functions, although [86] show this is the case for solutions at the chiral point. However, it is always possible to expand the Bessel functions in (B.41) in powers of z, and then invert (B.44) to get the metric perturbations to arbitrary order.

B.4 Modified FG Expansion

Here I present various formulae useful in solving the equations of motion. The modified asymptotic expansion

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\rho^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \\ \gamma_{ij} &= e^{2\rho}g_{ij} = e^{2\rho}\left(g^{(0)}_{ij} + e^{-n\rho}g^{(n)}_{ij} + e^{-2\rho}g^{(2)}_{ij} + \mathcal{O}(e^{-(n+2)\rho})\right) \end{aligned} \tag{B.45}$$

is used to expand the equations of motion, and all terms are expanded to first order in $g^{(n)}$ and first order in $g^{(2)}$. The generic expansion holds for $n \neq 1$ and n not a positive even integer. Note that when $n = 1$, expressions quadratic in $g^{(1)}$ appear at the same order as terms linear in $g^{(2)}$, and this case must be treated separately. For the Fefferman-Graham expansion when $n = 1$, see [132]. The same is true for, say, $n = 4$, where terms quadratic in $g^{(2)}$ appear at the same order as linear $g^{(4)}$ terms.

For generic n, the Ricci tensor components are

$$R_{\rho\rho} = -2 + \frac{-n^2+2n}{2}e^{-n\rho}\mathrm{Tr}g^{(n)} + \mathcal{O}^{-(n+2)\rho} \tag{B.46}$$

$$R_{\rho i} = -\frac{n}{2}e^{-n\rho}\left(\nabla^k g^{(n)}_{ki} - \partial_i \mathrm{Tr}g^{(n)}\right) - e^{-2\rho}\left(\nabla^k g^{(2)}_{ki} - \partial_i \mathrm{Tr}g^{(2)}\right) + \mathcal{O}^{-(n+2)\rho} \tag{B.47}$$

$$\begin{aligned} R_{ij} &= -2e^{2\rho}g^{(0)}_{ij} + e^{(2-n)\rho}\left[\left(-\frac{1}{2}n^2 + n - 2\right)g^{(n)}_{ij} + \frac{n}{2}g^{(0)}_{ij}\mathrm{Tr}g^{(n)}\right] \\ &\quad + e^0\left[R^{(0)}_{ij} - 2g^{(2)}_{ij} + g^{(0)}_{ij}\mathrm{Tr}g^{(2)}\right] + \mathcal{O}^{-n\rho} \end{aligned} \tag{B.48}$$

The Ricci scalar is given by

$$R = -6 + \left(3n - n^2\right)e^{-n\rho}\mathrm{Tr}g^{(n)} + e^{-2\rho}\left[R^{(0)} + 2\mathrm{Tr}g^{(2)}\right] + \mathcal{O}^{-(n+2)\rho} \tag{B.49}$$

The Cotton tensor to first order in n is given by

$$C_{\rho\rho} = \mathcal{O}(\mathrm{e}^{-(n+2)\rho}) \tag{B.50}$$

$$C_{ij} = \mathrm{e}^{(2-n)\rho}\epsilon_i{}^k\left[\frac{-n}{2}(n-2)(n-1)g_{ij}^{(n)} + \frac{n}{2}\mathrm{Tr}g^{(n)}\right] \tag{B.51}$$

$$C_{\rho i} = \mathrm{e}^{-n\rho}\left[\frac{-n}{2}\epsilon_i{}^k\nabla^l g_{lk}^{(n)} + \frac{n^2-2n}{2}\epsilon^{nk}\nabla_k g_{in}^{(n)}\right] \tag{B.52}$$

The new tensor in the equations of motion consist of several pieces, and it is useful to expand each piece separately.

$R_{\mu\alpha\nu\beta}R^{\alpha\beta}$

$$R_{\rho\alpha\rho\beta}R^{\alpha\beta} = 4 + \left(\frac{3n^2}{2} - 4n\right)e^{-n\rho}\mathrm{Tr}g_{(n)} \tag{B.53}$$

$$R_{\rho\alpha i\beta}R^{\alpha\beta} = \frac{n}{2}e^{-n\rho}\left[\nabla^k g_{ki}^{(n)} - \partial_i \mathrm{Tr}g_{(n)}\right] \tag{B.54}$$

$$R_{i\alpha j\beta}R^{\alpha\beta} = 4e^{2\rho}g_{ij}^{(0)} + e^{(2-n)\rho}\left[\left(\frac{n^2}{2} - n + 4\right)g_{ij}^{(n)} + \left(n^2 - \frac{7n}{2}\right)g_{ij}^{(0)}\mathrm{Tr}g_{(n)}\right] \tag{B.55}$$

$RR_{\mu\nu}$

$$RR_{\rho\rho} = 12 + \left(5n^2 - 12n\right)e^{-n\rho}\mathrm{Tr}g_{(n)} \tag{B.56}$$

$$RR_{\rho i} = 3ne^{-n\rho}\left[\nabla^k g_{ki}^{(n)} - \partial_i \mathrm{Tr}g_{(n)}\right] \tag{B.57}$$

$$RR_{ij} = 12e^{2\rho}g_{ij}^{(0)} + e^{(2-n)\rho}\left[\left(3n^2 - 6n + 12\right)g_{ij}^{(n)} + \left(2n^2 - 9n\right)g_{ij}^{(0)}\mathrm{Tr}g_{(n)}\right] \tag{B.58}$$

$R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu}$

$$R_{\alpha\beta}R^{\alpha\beta}g_{\rho\rho} = 12 + \left(4n^2 - 12n\right)e^{-n\rho}\mathrm{Tr}g_{(n)} \tag{B.59}$$

$$R_{\alpha\beta}R^{\alpha\beta}g_{\rho i} = 0 \tag{B.60}$$

$$R_{\alpha\beta}R^{\alpha\beta}g_{ij} = 12e^{2\rho}g_{ij}^{(0)} + e^{(2-n)\rho}\left[12g_{ij}^{(n)} + \left(4n^2 - 12n\right)g_{ij}^{(0)}\mathrm{Tr}g_{(n)}\right] \tag{B.61}$$

$R^2 g_{\mu\nu}$

$$R^2 g_{\rho\rho} = 36 + \left(12n^2 - 36n\right) e^{-n\rho} \mathrm{Tr} g_{(n)} \tag{B.62}$$

$$R^2 g_{\rho i} = 0 \tag{B.63}$$

$$R^2 g_{ij} = 36 e^{2\rho} g_{ij}^{(0)} + e^{(2-n)\rho} \left[36 g_{ij}^{(n)} + \left(12n^2 - 36n\right) g_{ij}^{(0)} \mathrm{Tr} g_{(n)}\right] \tag{B.64}$$

$\Box R g_{\mu\nu}$

$$\Box R g_{\rho\rho} = -n^2(n-2)(n-3) e^{-n\rho} \mathrm{Tr} g_{(n)} \tag{B.65}$$

$$\Box R g_{\rho i} = 0 \tag{B.66}$$

$$\Box R g_{ij} = -n^2(n-2)(n-3) e^{(2-n)\rho} g_{ij}^{(0)} \mathrm{Tr} g_{(n)} \tag{B.67}$$

$\nabla_\mu \nabla_\nu R$

$$\nabla_\rho \nabla_\rho R = n^3(3-n) e^{-n\rho} \mathrm{Tr} g_{(n)} \tag{B.68}$$

$$\nabla_\rho \nabla_i R = n(n-3)(n+1) e^{-n\rho} \partial_i \mathrm{Tr} g_{(n)} \tag{B.69}$$

$$\nabla_i \nabla_j R = n^2(n-3) e^{(2-n)\rho} g_{ij}^{(0)} \mathrm{Tr} g_{(n)} \tag{B.70}$$

$\Box R_{\mu\nu}$

$$\Box R_{\rho\rho} = e^{-n\rho} \left(-\frac{n^4}{2} + 2n^3 - n^2\right) \mathrm{Tr} g_{(n)} \tag{B.71}$$

$$\Box R_{\rho i} = e^{-n\rho} \left[\left(-\frac{1}{2}n^3 + n^2 + n\right) \nabla^k g_{ki}^{(n)} + \left(\frac{1}{2}n^3 - n^2 - 2n\right) \partial_i \mathrm{Tr} g_{(n)}\right] \tag{B.72}$$

$$\Box R_{ij} = e^{(2-n)\rho} \left[\left(-\frac{n^4}{2} + 2n^3 - n^2 - 2n\right) g_{ij}^{(n)} + \left(\frac{n^3}{2} - 2n^2 + n\right) g_{ij}^{(0)} \mathrm{Tr} g_{(n)}\right] \tag{B.73}$$

The gauge independent equations for $g^{(n)}$ are given in the text. For $g^{(2)}$, the equations of motion are

$$\{\rho\rho\}: \quad \tfrac{1}{2}\left(\sigma - \tfrac{1}{2m^2}\right)\left(R^{(0)} + 2\mathrm{Tr}g^{(2)}\right) = 0 \tag{B.74}$$

$$\{\rho i\}: \quad \left(\sigma + \tfrac{1}{2m^2}\right)\left(\partial_i \mathrm{Tr}g^{(2)} - \bar{\nabla}^j g^{(2)}_{ji}\right) = 0 \tag{B.75}$$

B.5 Brown-York Stress Energy Tensor in Fefferman-Graham Coordinates

Here I present various formulae useful in computation of the Brown-York stress energy tensor. The exponent n is taken to be in the range $1 < n < 2$. In this range, all quantities need be expanded only to "first order" in $g^{(n)}$ and $g^{(2)}$ in order to find all finite and divergent pieces of the boundary stress energy tensor.

Using the metric (B.45), the inverse metric is given by

$$\gamma^{ij} = e^{-2\rho}\left(g^{ij}_{(0)} - e^{-n\rho}g^{ij}_{(n)} - e^{-2\rho}g^{ij}_{(2)} + \cdots\right) \tag{B.76}$$

where indices are raised and lowered with the inverse metric. The extrinsic curvature and trace are just

$$\begin{aligned} K_{ij} \equiv -\frac{1}{2}\partial_\rho\gamma_{ij} &= -e^{2\rho}g^{(0)}_{ij} + \frac{n-2}{2}e^{(2-n)\rho}g^{(n)}_{ij} + \mathcal{O}(e^{-n\rho}) & \text{(B.77)} \\ K = K_{ij}\gamma^{ij} &= -2 + \frac{n}{2}e^{-n\rho}\mathrm{Tr}g^{(n)} + e^{-2\rho}\mathrm{Tr}g^{(2)} + \cdots & \text{(B.78)} \end{aligned}$$

The auxiliary tensor is proportional to the Schouten tensor, $m^2 f^{\mu\nu} = 2\left(R^{\mu\nu} - \frac{1}{4}Rg^{\mu\nu}\right)$

and is expanded as

$$
\begin{aligned}
m^2 f^{\rho\rho} &= -1 + \frac{-n^2+n}{2} e^{-n\rho} \mathrm{Tr} g^{(n)} - \frac{1}{2} e^{-2\rho} \left(R^{(0)} + 2\mathrm{Tr} g^{(2)} \right) + \cdots && \text{(B.79)} \\
m^2 f^{\rho i} &= -n e^{-(n+2)\rho} \left(\nabla_k g^{ki}_{(n)} - \partial^i \mathrm{Tr} g^{(n)} \right) - 2 e^{-4\rho} \left(\nabla_k g^{ki}_{(2)} - \partial^i \mathrm{Tr} g^{(2)} \right) && \text{(B.80)} \\
m^2 \hat{f}^{ij} &= -e^{-2\rho} g^{ij}_{(0)} + e^{-(n+2)\rho} \left[\left(-n^2 + 2n + 1 \right) g^{ij}_{(n)} + \frac{n^2-n}{2} g^{ij}_{(0)} \mathrm{Tr} g^{(n)} \right] && \text{(B.81)} \\
& \quad + e^{-4\rho} \left[R^{ij}_{(0)} - \frac{1}{4} R^{(0)} g^{ij}_{(0)} + \frac{1}{2} g^{ij}_{(2)} + \frac{1}{2} g^{ij}_{(0)} \mathrm{Tr} g^{(2)} \right] && \text{(B.82)} \\
m^2 \hat{f} &= \gamma_{ij} f^{ij} = -2 + n e^{-n\rho} \mathrm{Tr} g^{(n)} + e^{-2\rho} \left(R^{(0)} + 2\mathrm{Tr} g^{(2)} \right) + \cdots && \text{(B.83)}
\end{aligned}
$$

Some final pieces necessary for computation of the boundary stress tensor include

$$
\begin{aligned}
-\nabla^{(i} \hat{h}^{j)} &= \mathcal{O}(e^{-(n+4)\rho}) && \text{(B.84)} \\
\frac{m^2}{2} \mathcal{D}_\rho \hat{f}^{ij} &= e^{-2\rho} g^{ij}_{(0)} + e^{-(n+2)\rho} \left[\frac{n+2}{2} (n^2 - 2n - 1) g^{ij}_{(n)} + \frac{n+2}{4} (-n^2 + n) g^{ij}_{(0)} \mathrm{Tr} g^{(n)} \right] \\
& \quad + e^{-4\rho} \left[-4 R^{ij}_{(0)} + R^{(0)} g^{ij}_{(0)} - 2 g^{ij}_{(2)} - 2 g^{ij}_{(0)} \mathrm{Tr} g^{(2)} \right] && \text{(B.85)} \\
m^2 K^{(i}_k f^{j)k} &= e^{-2\rho} g^{ij}_{(0)} - e^{-(n+2)\rho} \left[\frac{-2n^2 + 5n + 2}{2} g^{ij}_{(n)} + \frac{n^2-n}{2} g^{ij}_{(0)} \mathrm{Tr} g^{(n)} \right] && \text{(B.86)} \\
& \quad - e^{-4\rho} \left[2 R^{ij}_{(0)} - \frac{1}{2} R^{(0)} g^{ij}_{(0)} + 2 g^{ij}_{(2)} + g^{ij}_{(0)} \mathrm{Tr} g^{(2)} \right] \\
\nabla_k \hat{h}^k &= \mathcal{O}(e^{-(n+2)\rho}) && \text{(B.87)} \\
-\frac{m^2}{2} \mathcal{D}_\rho \hat{f} &= \frac{1}{2} n^2 g^{ij}_{(0)} \mathrm{Tr} g^{(n)} + e^{-2\rho} \left[R^{(0)} + 2\mathrm{Tr} g^{(2)} \right] && \text{(B.88)}
\end{aligned}
$$

In addition to some of the pieces above, possible counterterms (3.52) include

$$
\begin{aligned}
m^4 \hat{f}^2 &= 4 - 4n e^{-n\rho} \mathrm{Tr} g^{(n)} - 4 e^{-2\rho} \left(R^{(0)} + 2\mathrm{Tr} g^{(2)} \right) + \mathcal{O}(e^{-(n+2)\rho}) && \text{(B.89)} \\
m^4 f_{ij} f^{ij} &= 2 - 2n e^{-n\rho} \mathrm{Tr} g^{(n)} + e^{-2\rho} \left(-2R^{(0)} - 4\mathrm{Tr} g^{(2)} \right) && \text{(B.90)}
\end{aligned}
$$

All together, the Brown-York stress energy tensor with indices lowered is

$$
\begin{aligned}
8\pi G T_{ij}^{\mathrm{BY}} &= e^{2\rho}\left(\sigma + \frac{1}{2m^2}\right) g_{ij}^{(0)} \\
&+ e^{(2-n)\rho}\left[\frac{4\sigma m^2 + 2n\sigma m^2 + 2n^3 - 4n^2 + n + 2}{4m^2} g_{ij}^{(n)}\right. \\
&\qquad \left. + \frac{-2nm^2\sigma - n^3 + 2n^2 - 2n}{4m^2} g_{ij}^{(0)} \mathrm{Tr} g^{(n)}\right] \\
&+ e^{0}\left[2\left(\sigma + \frac{1}{2m^2}\right) g_{ij}^{(2)} - \frac{1}{4m^2} R^{(0)} g_{ij}^{(0)} - \left(\sigma + \frac{1}{m^2}\right) g_{ij}^{(0)} \mathrm{Tr} g^{(2)}\right]
\end{aligned}
\tag{B.91}
$$

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