HARMONIC REPRESENTATIVES OF INSTANTONS AND SELF-DUAL MONOPOLES

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§1.Introduction

Local solutions to self-dual Yang-Mills and monopole equations correspond to holomorphic bundles on certain auxiliary manifolds [1,2,3]. These bundles have to obey additional conditions (such as triviality on some lines, reality, etc) which place not-easy-to-work-with constraints on their patching functions. However, there is an unconstrained quantity V^{++} which incorporates all the information about the bundle quite naturally. It was introduced in studying the N=2 supersymmetry [4] and turned out to be relevant in self-dual problems also [5]. In §2 we recall briefly the construction for self- dual Yang-Mills equations [5]. Global solutions are given by ADHM construction [6] and in the rest of §2 we give expressions for V^{++} corresponding to them. This problem was discussed also in [7] where V^{++} for instantons of t'Hooft type were found. §3 is devoted to the analogous interpretation of self-dual monopole equation and Nahm general construction [8,9] of solutions.

§2.Self-dual Yang-Mills equations

We will use only spinor indices $\alpha,\dot\alpha=1,2$, which are raised and lowered with the help of ε-tensor. For example, $x^{\alpha\dot\alpha}$ are coordinates. Let $\nabla_{\alpha\dot\alpha}$ be the covariant derivatives. Then the curvature is

$$F_{\alpha\dot\alpha\beta\dot\beta}=[\nabla_{\alpha\dot\alpha},\nabla_{\beta\dot\beta}]=\varepsilon_{\alpha\beta}F_{\dot\alpha\dot\beta}+\varepsilon_{\dot\alpha\dot\beta}F_{\alpha\beta} \tag{1}$$

where $F_{\alpha\beta}=F_{\beta\alpha}$, $F_{\dot\alpha\dot\beta}=F_{\dot\beta\dot\alpha}$. The self-duality equations are equivalent to $F_{\dot\alpha\dot\beta}=0$. So in this case we have

$$[\nabla_{\alpha\dot\alpha},\nabla_{\beta\dot\beta}]=\varepsilon_{\dot\alpha\dot\beta}F_{\alpha\beta} \tag{2}$$

Let us introduce harmonics $u^{+\dot\alpha}$: they lie in the fundamental representation of SU(2) and are subjected to SU(2)-invariant condition

$$u^{+\dot\alpha}u^{-}_{\dot\alpha}=1 \tag{3}$$

where $u^-_{\dot\alpha}=\overline{u^+_{\dot\alpha}}$. This is the three-dimensional sphere S^3. The action of SU(2) on S^3 commutes with the action of $U(1):u^+_{\dot\alpha}\Rightarrow e^{i\phi}u^+_{\dot\alpha}$ and so we have the action of SU(2) on $CP^1=S^3/U(1)$. We will work with quantities with definite U(1)-charge (a quantity $f^{(+k)}$ with the U(1)-charge k transforms as $f^{(+k)}\Rightarrow e^{i\phi}f^{(+k)}$ under the U(1)-transformations). There are three important operators preserving the condition $u^{+\dot\alpha}u^-_{\dot\alpha}=1$:

$$
\begin{aligned}
D^{++}&=u^{+\dot\alpha}\,\partial/\partial u^{-\dot\alpha}\\
D^{(0)}&=u^{+\dot\alpha}\partial/\partial u^{+\dot\alpha}-u^{-\dot\alpha}\partial/\partial u^{-\dot\alpha}\\
D^{--}&=u^{-\dot\alpha}\partial/\partial u^{+\dot\alpha}
\end{aligned}
\tag{4}
$$

Multiplying equation (2) by $u^{+\dot\alpha}u^{+\dot\beta}$ and denoting $u^{+\dot\alpha}\nabla_{\alpha\dot\alpha}$ by ∇^+_α we get

$$[\nabla^+_\alpha,\nabla^+_\beta]=0 \tag{5}$$

To reconstruct the self-duality equation from (5) we must only remember that ∇^+_α is linear in $u^{+\dot\alpha}$. This means that

$$[D^{++},\nabla^+_\alpha]=0 \tag{6}$$

Conversely, one can easily deduce that if (6) holds then $\nabla^{+\alpha}$ is linear in $u^{+\dot\alpha}$. Thus we see that the set of equations (5),(6) is equivalent to the original self-duality equation. But now we can solve (5). Indeed, (5) means that $\nabla^+_\alpha = h\partial^+_\alpha h^{-1}$, where $\partial^+_\alpha = u^{+\dot\alpha}\partial/\partial x^{\alpha\dot\alpha}$. The gauge transformation h depends not only on x but on u also. This means that the derivative D^{++} becomes nontrivial:

$$D^{++}\Rightarrow \mathcal{D}^{++} = h^{-1}D^{++}h = D^{++}+V^{+} \tag{7}$$

where $V^{++} = h^{-1}D^{++}(h)$. Now the equation (6) becomes

$$\partial^+_\alpha V = 0 \tag{8}$$

which simply means that V^{++} does not depend on $x^-_\alpha=u^{-\dot\alpha}x_{\alpha\dot\alpha}$. So we found the general local solution to the self-duality equation. It is described by the quantity V^{++} depending only on $x^+_\alpha=u^{+\dot\alpha}x_{\alpha\dot\alpha},u^{+\dot\alpha},u^{-\dot\alpha}$. Conversely, for almost arbitrary V^{++} (V^{++} must satisfy only some conditions of inequality type) we can find a gauge transformation h such that $h^{-1}D^{++}h = D^{++}+V^{++}$ and then $A_{\alpha\dot\alpha}(x)$ from the equation $u^{+\dot\alpha}(\partial/\partial x^{\alpha\dot\alpha}+A_{\alpha\dot\alpha})=h\partial^+_\alpha h^{-1}$.

This is just the interpretation of Ward construction in terms of V^{++}. We followed closely the treatment in [5].

Now let us proceed to the global solutions of self-duality equation describing instantons. They are given by the ADHM construction. We restrict ourselves to the case of SU(2) group. ADHM construction for k-instantons involves matrices $\lambda_{i\alpha\dot\alpha}, B_{ij\alpha\dot\alpha}$, $i,j=1,\dots,k$, subjected to the following constraints [6,10]:

(a) $B_{ij\alpha\dot\alpha}=B_{ji\alpha\dot\alpha}$,

(b) $\lambda_i{}^{\alpha}{}_{\dot\beta}\lambda_{j\alpha\dot\gamma}+B_{is}{}^{\alpha}{}_{\dot\beta}B_{sj\alpha\dot\gamma}=\rho_{ij}\epsilon_{\dot\beta\dot\gamma}$, $\rho_{ij}\in R$ (we sum over repeated indices),

(c) The equations $v_o{}^{\alpha}{}_{\dot\beta}(x)\lambda_{j\alpha\dot\gamma} + v_s{}^{\alpha}{}_{\dot\beta}(x)(B_{sj\alpha\dot\gamma}-\delta_{sj}x_{\alpha\dot\gamma})=0$ for $v_a=(v_o,v_i)$ has only one everywhere nonzero solution up to multiplication of $v^{\alpha}{}_{\dot\beta}(x)$ by $q_{\dot\gamma}{}^{\dot\beta}(x)$. (Multiplication by q corresponds to gauge transformations).

Now, if v is a normalized solution of the equations (c), $v_a{}^{\alpha\dot\beta}v_{a\alpha\dot\gamma}=\delta^{\dot\beta}{}_{\dot\gamma}$ and $(A_{\alpha\dot\alpha})_{\dot\beta}{}^{\dot\gamma}=v_a{}^{\sigma}{}_{\dot\beta}\partial/\partial x^{\alpha\dot\alpha}(v_{a\sigma}{}^{\dot\gamma})$ then $\nabla_{\alpha\dot\alpha}=\partial/\partial x^{\alpha\dot\alpha}+A_{\alpha\dot\alpha}$ is a self-dual connection.

Actually, the gauge transformation h for this connection $\nabla_{\alpha\dot\alpha}$ was found in [11] (though in different notations). Adjusting the results of [11] to our notations, put $(\xi_1)_{\alpha\dot\beta}=x_{\alpha\dot\beta}+F^-{}_{\alpha}u^+{}_{\dot\beta}$, $(\xi_2)_{\alpha\dot\beta}=x_{\alpha\dot\beta}+G^-{}_{\alpha}u^+{}_{\dot\beta}$ with arbitrary $F^-{}_{\alpha},G^-{}_{\alpha}$. Then the following identity holds [11]:

$$v_a{}^{\alpha}{}_{\dot\beta}(\xi_1)v_{a\alpha}{}^{\dot\gamma}(x)v_b{}^{\beta}{}_{\dot\gamma}(x)v_{b\beta\dot\tau}(\xi_2)=v_a{}^{\alpha}{}_{\dot\beta}(\xi_1)v_{a\alpha\dot\tau}(\xi_2). \tag{9}$$

Introduce $\xi(c)_{\alpha\dot\beta}=x^+{}_{\alpha}u^-{}_{\dot\beta}+c^-{}_{\alpha}u^+{}_{\dot\beta}$, where c is an arbitrary functions depending on harmonic variables u only. Put

$$h(x,u,c)_{\dot\beta}{}^{\dot\gamma}=v_a{}^{\alpha}{}_{\dot\beta}(x)v_{a\alpha}{}^{\dot\gamma}(\xi(c)) \tag{10}$$

Using (9) one can show that $h^{-1}(x,u,c)_{\dot\beta}{}^{\dot\gamma}=v_a{}^{\alpha}{}_{\dot\beta}(\xi(c))v_{a\alpha}{}^{\dot\gamma}(x)$ and $\nabla^+{}_{\alpha}=h\partial^+{}_{\alpha}h^{-1}$. So we conclude that h is the needed gauge transformation.

Now we have to compute $V^{++}=h^{-1}D^{++}(h)$. Again using (9) we can rewrite it in the form

$$V^{++}(x^+,u,c)_{\dot\beta}{}^{\dot\gamma}=v_a{}^{\alpha}{}_{\dot\beta}(\xi(c))D^{++}v_{a\alpha}{}^{\dot\gamma}(\xi(c)) \tag{11}$$

Using the arbitrariness of c we can further simplify this expression. Namely, let $c^-{}_{\alpha}=d\delta_{\alpha}{}^{\dot\beta}u^-{}_{\dot\beta}$, where d is a constant. Surprisingly, it turns out that there exists the limit

$$v_{lim}(x_H,u)=\lim_{d\to\infty}v(\xi(c)) \tag{12}$$

where $(x_H)_{\alpha\dot\beta}=x^+{}_{\alpha}u^-{}_{\dot\beta}$. Inserting v_{lim} in the expression for V^{++} we come (after some computations) to the following statement:

Theorem.
$$V^{++}{}_{\alpha}{}^{\beta} = -\lambda_i{}^{+}{}_{\alpha}(C^{-2})_{ij}\lambda_j{}^{+\beta}$$
where $\lambda_i{}^{+}{}_{\alpha}=u^{+\dot\alpha}\lambda_{i\alpha\dot\alpha}$, C_{ij} is the matrix inverse to $B_{ij}{}^{+-}-x^{+-}\delta_{ij}$, $B_{ij}{}^{+-}=u^{+\dot\beta}u^{+\dot\alpha}B_{ij\dot\alpha\dot\beta}$, $x^{+-}=u^{+\dot\alpha}u^{-\dot\beta}x_{\dot\beta\dot\alpha}$. (Note that we replaced the undotted index of $x_{\alpha\dot\alpha}$ with the dotted one.)

§3. Self-dual monopoles

Self-dual monopoles also have an interpretation in terms of holomorphic bundles subjected to some constraints [2,3]. In this section we find, as in the case of instantons, the general local solution in terms of the unconstrained quantity V^{++} (which now depends on variables specific to three dimensional space. Then we describe V^{++} corresponding to Nahm global solutions for the case of SU(2) group.

In three dimensions there is only one type of spinor indices, say α, and vector $x^{\alpha\beta}$ is symmetric in them, $x^{\alpha\beta}=x^{\beta\alpha}$ (the reality condition is $\overline{x_{\alpha\beta}}=x^{\alpha\beta}$). The monopole configuration consists of gauge field A_i ($A_{\alpha\beta}$ in spinor indices) and Higgs field Φ. The self-dual monopole equation reads
$$2\nabla_i\Phi = \varepsilon_{ijk}F_{jk}, \tag{13}$$
where F_{jk} is the field strength of the potential A_i. In spinor indices we have
$$2\nabla_{\alpha\beta}\Phi=[\nabla^{\gamma}{}_{\alpha},\nabla_{\gamma\beta}], \tag{14}$$
where $\nabla_{\alpha\beta}=\partial_{\alpha\beta}+A_{\alpha\beta}$, $\partial_{\alpha\beta}=\partial/\partial x^{\alpha\beta}$. Introducing harmonics and multiplying (14) by $u^{+\alpha}u^{+\beta}$ we get
$$\nabla^{++}\Phi=[\nabla^{++},\nabla^{+-}], \tag{15}$$
where $\nabla^{++}=u^{+\alpha}u^{+\beta}\nabla_{\alpha\beta}$, $\nabla^{+-}=u^{+\alpha}u^{-\beta}\nabla_{\alpha\beta}$; we used the easily verified identity $A^{\alpha}B_{\alpha}=A^{+}B^{-}-A^{-}B^{+}$ for $A^{+}=A_{\alpha}u^{+\alpha}$. Now put $\nabla_o=\nabla^{+-}-\Phi$. Then (15) becomes
$$[\nabla^{++},\nabla_o]=0. \tag{16}$$
As before, we have to supply this equation with the information about the dependence of ∇^{++},∇_o on harmonics. The corresponding equations are
$$[D^{++},\nabla^{++}] = 0, \quad [D^{++},\nabla_o] = \nabla^{++}. \tag{17}$$
The equation (16) implies the existence of such gauge transformation h that $\nabla^{++}=h\partial^{++}h^{-1}$, $\nabla=h\partial h^{-1}$. where $\partial^{++}=u^{+\alpha}u^{+\beta}\partial/\partial x^{\alpha\beta}$,

$\partial=u^{+\alpha}u^{-\beta}\partial/\partial x^{\alpha\beta}$. Then we have

$$D^{++} \to \mathcal{D}^{++} = h^{-1}D^{++}h = D^{++}+V^{++} \tag{18}$$

and equations (17) become

$$\partial^{++}V^{++} = 0, \quad \partial V^{++} = 0 \tag{19}$$

In coordinates $x^{++}=u^{+\alpha}u^{+\beta}x_{\alpha\beta}$, $x=u^{+\alpha}u^{-\beta}x_{\alpha\beta}$, $x^{--}=u^{-\alpha}u^{-\beta}x_{\alpha\beta}$ this means simply that V^{++} depends only on x^{++}. Thus we have found the general local solution of self-dual monopole equation in terms of $V^{++}=V^{++}(x^{++},u)$. Conversely, almost any V^{++} gives rise to local solution.

Remarks. (1) The coordinates x^{++},u^{+} parametrize the tangent bundle to $\mathbf{CP}^1$ which is the basic manifold in Hitchin treatment of monopoles [3].

(2) In principle, we could deduce all this from four dimensional self-duality equations by dimensional reduction (three dimensional spinor group SU(2) corresponds to the diagonal SU(2) in four dimensional spinor group SU(2)×SU(2)). However, it is nice to do everything in purely three dimensional way.

Example. The most simple expression for V^{++} for the SU(2) group is $V^{++}{}_a{}^{\beta} = (x^{++})^2u^{-}{}_{\alpha}u^{-\beta}$. One can check that this V^{++} corresponds to one-monopole solution found by Prasad and Sommerfield [12].

To get used to this technique let us explain how to extract the information about the solution out of V^{++}, for example, how to find the energy of the solution. We have another harmonic derivative $D^{--}=u^{-\alpha}\partial/\partial u^{+\alpha}$. Its commutator with D^{++} equals to $D^{(o)}$, $[D^{++},D^{--}]=D^{(o)}$. Under the action of h the derivative D^{--} transforms to

$$\mathcal{D}^{--} = h^{-1}D^{--}h = D^{--}+V^{--} \tag{20}$$

The commutation relation with D^{++} becomes $[\mathcal{D}^{++},\mathcal{D}^{--}] = h^{-1}D^{(o)}h = D^{(o)}$, or

$$D^{++}V^{--}-D^{--}V^{++}+[V^{++},V^{--}] = 0 \tag{21}$$

and this is the equation to express V^{--} in terms of V^{++}, without finding the gauge transformation h [13].

The energy of solution is proportional to $\int \mathrm{tr}(F^{\alpha\beta}F_{\alpha\beta})d^3x$, (where $F_{\alpha\beta}=[\nabla_{\alpha}{}^{\gamma},\nabla_{\beta\gamma}]$), which, in turn, equals to $2\int \mathrm{tr}(F^{++}F^{--}-F^{+-}F^{+-})d^3x$. Using the commutation relations $[D^{--},\nabla^{++}]=2\nabla^{+-}$, $[D^{--},\nabla_{o}]=\nabla^{--}$, $[D^{--},\nabla^{--}]=0$ one finds

$$F^{++}=-\partial^{++2}V^{--},\ F^{+-}=-\partial\partial^{++}V^{--},\ F^{--}=-\partial^{2}V^{--} \tag{22}$$

So

$$E \sim \int tr(\partial^{++2}V^{--}\partial^{2}V^{--} - (\partial\partial^{++}V^{--})^{2})d^{3}x \tag{23}$$

This is obviously the full derivative (as it should be: the energy is proportional to the topological charge for self-dual solutions).

Finally let us describe V^{++} arising in Nahm construction [8]. Recall that Nahm solutions are expressed in terms of $v_{\beta}{}^{\gamma}(z)$, which is a solution of the equation

$$(i\delta_{\alpha}{}^{\beta}d/dz - x_{\alpha}{}^{\beta} + \overline{T}_{\alpha}{}^{\beta})v_{\beta}{}^{\gamma} = 0 \tag{24}$$

on an interval [-a/2,a/2]. Here $T_{\alpha}{}^{\beta}=(T_{\alpha}{}^{\beta})_{i}{}^{j}$ are matrices in indices i,j, $v_{\beta}{}^{\gamma}=v_{i\beta}{}^{\gamma}$ are vectors, $i,j=1,\dots,k$, where k is the topological charge of the monopole to be constructed. Matrices $T_{\alpha}{}^{\beta}$ must satisfy Nahm equations $dT_{\alpha\beta}/dz=[T_{\alpha}{}^{\gamma},T_{\beta\gamma}]$ with appropriate boundary conditions (they must have simple poles at the endpoints of the interval [-a/2.a/2], $T_{\alpha\beta}\sim t_{\alpha\beta}/(z-z_{o})$, $z_{o}=$ a/2 with the residues $t_{\alpha\beta}$ constituting some irreducible representation of the Lie algebra su(2)). Chose a normalized solution, $\int v_{i}{}^{\alpha\beta}v_{i\alpha\gamma}\,dz = \delta^{\beta}{}_{\gamma}$. Then $A_{\alpha\beta}$ and Φ satisfying self-dual monopole equation are given by the formulae

$$\begin{aligned}(A_{\alpha\beta})_{\tau}{}^{\rho} &= \int_{-a/2}^{a/2} v^{\sigma}{}_{\tau}\partial_{\alpha\beta}v_{\sigma}{}^{\rho}\,dz\\ \Phi_{\tau}{}^{\rho} &= \int_{-a/2}^{a/2} v^{\sigma}{}_{\tau}\ z\ v_{\sigma}{}^{\rho}\,dz\end{aligned} \tag{25}$$

It turns out that gauge transformation h and V^{++} are also easily expressed in terms of v.

Theorem. Let $(x_{H})_{\alpha\beta}=x^{++}u^{-}{}_{\alpha}u^{-}{}_{\beta}$. Then

$$h_{\tau}{}^{\rho}(x,u) = \int_{-a/2}^{a/2} v^{\alpha}{}_{\tau}(x,z)e^{ix^{+-}z}\ v_{\alpha}{}^{\rho}(x_{H},z)dz$$

and

$$V^{++}(x^{++},u)_{\alpha}{}^{\rho} = \int_{-a/2}^{a/2} v^{\alpha}{}_{\tau}(x_{H},z)(ix^{++}z+D^{++})v_{\alpha}{}^{\rho}(x_{H},z)dz$$

In conclusion we note that these methods can be applied to supersymmetric monopoles and self-dual monopoles as well. The results will be presented elsewhere.

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