Three Approaches

to

M-theory

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Three approaches to M-theory

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Summary: Since the early days of its discovery, when the term was used to characterise the quantum completion of eleven-dimensional supergravity and to determine the strong-coupling limit of type IIA superstring theory, the idea of M-theory has developed with time into a unifying framework for all known superstring theories. This evolution in conception has followed the progressive discovery of a large web of dualities relating seemingly dissimilar string theories, such as theories of closed and oriented strings (type II theories), of closed and open unoriented strings (type I theory), and theories with built in gauge groups (heterotic theories). In particular the eleven-dimensional Lorentz invariance of M-theory has been further highlighted by the discovery of how it descends by an orbifold compactification to the weakly-coupled heterotic $E_8 \times E_8$ string theory. Furthermore the derivation from M-theory of one-loop corrections to type II and heterotic low-energy effective supergravity has shed new light on our understanding of anomaly cancellation mechanisms which are so crucial for the consistency of superstring theories and are also of phenomenological importance, since the fermion content of the Standard Model is chiral.

Despite these successes, M-theory remains somewhat elusive and is still in want of a complete and closed formulation. In particular, its fundamental degrees of freedom are still unknown, which in principle prevents us from establishing the full quantum theory. Proposals have been made, advocating D0-branes or M2-branes as its elementary states. However, even the most fruitful of these proposals, based on a partonic description of M-theory and known as the BFSS (Bank-Fischler-Seiberg-Susskind) conjecture or M(atrix) theory, has stumbled on a number of issues, one of the most nagging being its failure to correctly reproduce quantum corrections to two- and three-graviton tree-level scattering amplitudes known from supergravity one-loop calculations.

In the absence of a closed and definite description, we are bound to adopt a fragmentary approach to M-theory. This work naturally follows this route. Three facets of M-theory are thus uncovered or further investigated in the course of the present thesis. The first part deals with the matrix model approach to M-theory and studies the relevancy of an alternative to the BFSS proposal, defined in purely algebraic terms, which is desirable from the point of view of a background independent formulation of M-theory. It implements overt eleven-dimensional Lorentz invariance by resorting to a $\mathfrak{osp}(1|32)$ realisation of its group of symmetry, thereby incorporating M2- and M5-branes as fundamental degrees of freedom on the same footing as the original D0-branes of the BFSS model. This seems more in phase with the concept of M-theory which does not appear to give preference to strings over more extended objects. Finally, this construction partially solves the puzzle about the absence of transverse five-brane degrees of freedom in M(atrix) theory and opens the perspective of a description of M-theory in a curved AdS_{11} background.

The second part of this thesis focuses on the determination of non-perturbative effects in low-energy effective heterotic supergravity, investigating some phenomenology related consequences of the Hořava-Witten scenario. The latter provides the M-theory origin of the strongly-coupled heterotic $E_8 \times E_8$ superstring theory in terms of an orbifold compactification on an interval, with twisted sectors given by two copies of E_8 super Yang-Mills multiplets propagating on the ten-dimensional boundary hyperplanes sitting at the orbifold fixed points. This setups can accomodate, under certain conditions, the insertion of parallel fivebranes, allowing for Euclidean membranes to stretch between them and the boundary hyperplanes. In this framework, we investigate a Calabi-Yau compactification of this setup to four dimensions, establishing the effective supergravity and computing the resulting instanton corrections to the interaction Lagrangian. In the condensed phase, this allows us to derive the corresponding gauge threshold corrections and determine the resulting non-perturbative superpotential with the correct dependence in the fivebrane modulus, from a purely supergravity calculation. Confirming results determined earlier by instanton calculations this works also renders them more accurate and sets them on a sounder basis, whence the vacuum structure of the effective supergravity can be addressed.

The third and last part of this thesis tackles the issue of how to recover information about M-theory by studying hidden symmetries of supergravity actions. In particular, we explore the conjecture stating that M-theory possesses an underlying hyperbolic infinite-dimensional Kac-Moody symmetry encapsulated in the split form $\mathfrak{e}_{10|10}$, which occurs as a natural very-extension of the exceptional U-duality algebra of eleven-dimensional supergravity compactified on a seven-torus. However, this conjecture is supposed to carry on beyond the toroidal case and to give clues about uncompactified M-theory. In the present work, we make use of this conjecture to determine the algebraic structure of Z_m orbifold compactifications of M-theory, showing how their untwisted sectors are encoded in a certain class of Borcherds algebras. Furthermore, concentrating on a certain family of Z_2 orbifolds of M-theory descending to non-supersymmetric but nonetheless stable orientifolds of type 0A string theory, we establish a dictionary between tilted Dp-/Dp'-brane configurations required in these models by tadpole cancellation and a certain class of threshold-one roots of $\mathfrak{e}_{10}$. This latter result is of some phenomenological interest insomuch as it gives a deeper algebraic understanding of the magnetised D9-/D9$'$-brane setup of the T-dual type $0'$ string theories, which have been recently shown to play a rôle in moduli stabilisation.

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Contents

Introduction

The discovery of a large web of dualities relating all superstring theories has led theorists to believe in a unifying framework, called M-theory, which would appear as the non-perturbative limit of the five known superstring theories and would contain the $11D$ supergravity of Cremmer, Julia and Scherk [81] as its low-energy limit. In particular, this yet unknown theory should be formulated in eleven dimensions and, by analogy with $11D$ supergravity, be endowed with a supersymmetry algebra of 32 supercharges, which corresponds to $\mathcal{N} = 1$ supersymmetry in eleven dimensions [247, 235].

The known superstring theories appear then as five limits of M-theory, for each of which the perturbative string expansion is different, and thus correspond to diverse corners of its moduli space of vacua. The two ten-dimensional string theories that directly descend from M-theory through compactification of one spatial dimension are the type IIA and the $E_8 \times E_8$ heterotic theory, which are both closed string theories, one with $\mathcal{N} = 2$ and the other with $\mathcal{N} = 1$ supersymmetry in ten dimensions. Indeed, the original 32 supercharges in eleven dimensions are preserved in the type IIA theory by compactifying M-theory on a circle, while the reduced supersymmetry of the heterotic string is achieved by an orbifold compactification which projects out half of the original supercharges [145]. In the latter case, anomaly cancellation in the eleven dimensional low-energy theory requires the presence of (Yang-Mills) gauge supermultiplets on ten-dimensional hyperplanes at the two orbifold fixed points, where they are allowed to propagate. In ten dimensions, these will reconstitute the gauge sector of the heterotic supergravity with 1-loop corrections [144].

In both cases, the radius of the compactified dimension is reinterpreted as the dilaton of the resulting supergravity theory, so that the decompactification limit indeed represents the strong-coupling regime of the corresponding string theory. In addition, the non-perturbative limit of the $E_8 \times E_8$ heterotic string theory reveals another feature of M-theory. It entails that M-theory has the property of generating an E_8 current algebra when embedded in a space with boundaries. This bears a striking resemblance with the anomaly cancellation mechanism in Chern-Simons gauge theory, which also requires similar "edge states" to be present. Such a connection has led to the alternative proposal that M-theory might be described up to the Planck scale by a non-gravitational effective Chern-Simons gauge theory [143], where the three-form of $11D$ supergravity would emerge as a composite field.

These two basic compactification schemes make M-theory the missing link between two, otherwise separate, groups of string theories related by dualities: the type II theories, on the one side, with no gauge group, and the heterotic / type I theories, on the other side, with gauge groups $E_8 \times E_8$ or $SO(32)$ [106, 247].

The dualities mapping these superstring theories among themselves are of two types. One of them is T-duality which relates one theory compactified on a circle to another theory compactified on a circle with inverse radius. Since this transformation constitutes, in string theory, an exact symetry of the conformal theory, it remains valid at all orders in perturbation theory, and is thus usually referred to as a *perturbative* symmetry. Typically, T-duality maps type IIA to IIB string theory [101, 83], and relates the two heterotic theories when, in addition, a Wilson line is turned on in the compactified theories, thereby breaking both the $E_8 \times E_8$ and $SO(32)$ gauge groups to their common $SO(16) \times SO(16)$ subgroup [196, 197, 128].

On the other hand, string theories may be related by non-perturbative dualities which map the weakly-coupled sector of one theory to the strongly-coupled sector of the same or of another theory [219]. Such a transformation relates, for instance, the heterotic theory with gauge group $SO(32)$ to the type I theory (a theory of closed and open unoriented strings with $\mathcal{N} = 1$ supersymmetry) by identifying the coupling constant of the first theory with the inverse coupling of the second [212]. The S-duality symmetry of type IIB string theory is another example, which acts this time as a modular transformation on a complexification of the axion and dilaton fields [78, 217].

When applied to open strings, one of the major by-products of T-duality has been to reveal the existence of extended objects defining hyperplanes on which the open strings of the dual theory are bound to end. These "defects" which are localised in the compact dimensions in which T-duality is performed are known as Dp-branes, by reference to the Dirichlet boundary conditions satisfied by the open strings [142, 83]. It was then realised that these objects could be identified with certain semi-classical solutions, or solitons, of the type I/II supergravity theories, the p-branes [211]. The latter are massive, weakly-coupled dynamical objets which naturally couple electrically or magnetically to the Ramond-Ramond fields of the low-energy type I/II theories, and thus carry a conserved charge (type II and heterotic supergravities also possess an NS5-brane, which couples magnetically to the antisymmetric B-field). Since their mass scales as the inverse of the string coupling constant, these appear as non-perturbative objects. Furthermore, these p-branes preserve a fraction of the original supersymmetry of the theory, in other words, they saturate a Bogomolnyi-Prasad-Sommerfeld (BPS) bound, and are therefore protected against quantum corrections. For this reason, Dp-branes have been extensively used to test duality conjectures between different regimes of superstring theories.

In particular, a study of the non-perturbative spectrum of supergravity theories has been crucial in determining the existence of the group of *U-duality* [149], which is expected to be a symmetry of both perturbative string theories compactified on tori and their conjectured non-perturbative extension, and relates certain theories modulo field redefinition. More precisely, supergravity theories usually exhibit hidden symmetries, such as, for instance, the continuous S-duality symmetry of type IIB supergravity [217]. In the toroidally reduced theory, these combine with the continuous T-duality symmetry to form a bigger group, whose discrete subgroup, the group of U-duality, is conjectured to survive as a symmetry of the full superstring theory. In particular, $11D$ supergravity reduced on a torus T^n, so as type IIA supergravity on T^{n-1}, possess a non-compact continuous $E_{n|n}(\mathbb{R})$ hidden symmetry [77, 79] which is broken to $E_{n|n}(\mathbb{Z})$ by quantum effects [149]. This U-duality algebra has been proposed as describing an exact quantum symmetry of M-theory compactified on T^n, and is conjectured to extend to the hyperbolic $E_{10|10}(\mathbb{Z})$ Kac-Moody algebra in 0+1 dimension [124]. In this case, there has been some evidence that, in the framework of continuous infinite dimensional U-duality symmetries [154, 156], such an infinite dimensional Lie algebra can be lifted to eleven dimensions and might constitute an underlying symmetry of uncompactified M-theory [84, 87].

In any case, the non-perturbative spectrum of supergravity theories has helped identifying the BPS sector of M-theory. In particular, since M-theory is required to contain $11D$ supergravity, the Kaluza-Klein modes of the eleven-dimensional graviton will account for the tower of type IIA D0-branes as excitations with momenta in the eleventh direction. Likewise, M-theory is expected to possess membrane (M2-brane) and fivebrane (M5-brane) states which, when wrapped around the eleventh dimension, reproduce respectively the fundamental string (F1) and the D4-brane of type IIA theory, and, when unwrapped, the type IIA D2-brane and NS5-brane.

Thus, M-theory does not seem to give preference to strings over membranes or fivebranes. Actually, the fundamental degrees of freedom of M-theory are still unknown, but there have been proposals advocating D0-branes [231] or M2-branes [91] as the elementary states of the theory.

The first of these two proposals, that the fundamental *partonic* degrees of freedom of M-theory are described by D0-branes, is at the core of the M(atrix) model approach to M-theory, which we shall consider in Part I of this thesis. This approach is based on a conjecture by Banks, Fischler, Shenker and Susskind [231] and is supposed to give a description of M-theory in (discrete) light-front coordinates. The proposal is that M-theory in the Infinite Momentum Frame (IMF) is defined as a quantum mechanics for $U(N)$ matrices in the large N limit, which results from the reduction of $10D$ super Yang-Mills theory to $0+1$ dimensions. In this perspective, M-theory can be regarded as a pregeometrical theory [198], since the target space coordinates are replaced by matrices, thus implying a kind of non-commutative structure. Furthermore, this description of M-theory on the light front was shown to be equivalent to a particular limit of M-theory compactified on a space-like circle [228], so that this matrix model eventually encodes the low-energy dynamics of type IIA D0-branes, which can then be regarded as the fundamental degrees of freedom of M-theory in the IMF. The theory is second-quantized with a Fock space containing multiple supergraviton states [234]. The other states of M-theory such as membranes and fivebrane appear as particular configurations of D0-branes. However, there is a number of unsolved problems in the matrix model approach to light-front M-theory which will be detailed in the next chapter.

This will then lead us to consider, in Chapter 2 of Part I, an alternative to the BFSS theory. The analysis presented there is based on the publication [14] and investigates a matrix model defined in purely geometrical terms, with manifest $\mathfrak{osp}(1|32)$ symmetry. This superalgebra is the supersymmetric minimal extension of the super-Poincaré algebra and has been put forward as unifying structure for the M-theory, F-theory and type II supersymmetry algebras [36, 28]. The matrix model thus constructed has certain desirable features, such as being background independent and respecting Lorentz invariance in eleven dimensions. Furthermore, it naturally incorporates membranes and transverse fivebranes degrees of freedom and presumably describes M-theory physics in an AdS_{11} background.

In Part II, we will consider the strong coupling regime of $E_8 \times E_8$ heterotic string theory, which is described, in the low-energy limit, by the Hořava-Witten scenario mentioned above. We will review the anomaly cancellation argument which requires the insertion of gauge and gravitational anomaly polynomials supported at the two fixed points of the $S^1/\mathbb{Z}_2$ orbifold of M-theory. This leads to a modification of the Bianchi identity of the four-form field-strength of $11D$ supergravity. We will also recall how M5-branes can be accomodated in this framework.

Then, we will move to studying a compactification to four dimensions of such a theory, preserving four-dimensional $\mathcal{N}=1$ supersymmetry. In particular, Chapter 4 in Part II is based on the publication [63], where we have considered the usual Calabi-Yau compactification of M-theory on $S^1/\mathbb{Z}_2$, with the inclusion of space-time filling M5-branes transverse to the orbifold interval and wrapping a two-cycle in the Calabi-Yau threefold. The presence of M5-branes aligned along the interval allows for open Euclidean membranes to stretch between a pair of M5-branes. The effects due to such open membranes in the four-dimensional effective supergravity are localised in space-time, and can be regarded as instanton-like corrections to the interaction Lagrangian. In particular, we will see how to determine the gauge threshold corrections induced by such membrane instanton effects directly from the reduction of the topological Chern-Simons term, and then derive the resulting non-perturbative effective superpotential with the correct dependence on the positions of the M5-branes along the interval.

In Part III, we will study the hidden exceptional symmetries of $11D$ supergravity reduced on n-dimensional tori. We will be particularly interested in the compactification to $0+1$ dimension, where the continuous U-duality algebra is conjectured to become the hyperbolic Kac-Moody algebra $E_{10|10}(\mathbb{R})$, whose discrete subgroup has been suggested to lift to a symmetry of uncompactified M-theory. As seen above, the continuous symmetry of the reduced supergravity theory is broken to its arithmetic subgroup by quantum effects. However, some results which one can obtain by this algebraic approach depend only on the root system of the algebra, so that working with the continuous or discrete valued version is in these cases irrelevant. In particular, such an infinite-dimensional Lie algebra possesses infinitely many roots, which split into three categories: real, isotropic and imaginary non-isotropic roots, determined by their norm being positive bounded from above, null, or negative and unbounded from below. If one considers the moduli space of M-theory on a ten-dimensional torus, certain positive roots of E_{10} can be related to M-theory instantons, in the large volume limit. Along this line, Ganor et al. [56] have made a recent proposal identifying a certain set of prime isotropic roots of E_{10} with Minkowskian branes and other objects expected in M-theory.

In Chapter 5 of Part III, we have applied this algebraic approach to the study of $\mathbb{Z}_n$ orbifolds of $11D$ supergravity and M-theory. In this chapter, based on the publication [13], we will in particular derive the real $\mathbb{Z}_n$-invariant subalgebras describing the residual continuous U-duality algebras of the untwisted sector of the theory. In $0+1$, these will be shown to be Borcherds algebras modded out by their centres and derivations. In addition, one can in this case find a description of the twisted sectors of $\mathbb{Z}_2$ orbifolds of M-theory or of their descendant type II orientifolds in terms of certain threshold-one roots of E_{10}.

Finally, an appendix collects conventions, and useful formulas and identities used in this thesis.

Part I

M-theory, Matrix models and the $\mathfrak{osp}(1|32, \mathbb{R})$ superalgebra

Introduction

This first part is separated into two main chapters. The first chapter consists in a presentation of the matrix model approach to M-theory, and is ment as an introduction to Chapter 2, based on the publication [14], in which a matrix model defined on the superalgebra $\mathfrak{osp}(1|32,\mathbb{R})$ has been investigated in the context of the infinite momentum frame and large N limit of the BFSS conjecture.

In Chapter 1, we start by reviewing the connection between early works on quantisation and regularisation of the membrane theory and M-theory compactified on a light-like circle. The first results on membrane theory go back to more then twenty years, when it was realised that eleven was the natural dimension in which a supersymmetric theory of membranes could be constructed. It was subsequently found [140] how to quantise this theory in the light-cone gauge. From a methodological and computational point of view, an important consequence is that the regularised theory is expressed as a simple quantum mechanics for $N \times N$ matrices in the large N limit. The analysis has then been extended to include supersymmetry in [91]. This supersymmetric quantum mechanics is usually called matrix-theory, and will also be presented in the following chapter.

At this stage, a quantised supersymmetric theory of membranes appeared as a natural candidate to construct a microscopic description of M-theory, because of the presence in M-theory of a three-form potential which naturally couples to a membrane (or M2-brane) world-volume. Moreover, the classical supersymmetric membrane theory enjoys a local symmetry acting on the fermions called κ-symmetry, which is preserved only when the background fields satisfy the equations of motions of classical $11D$ supergravity. Thus, this low-energy limit of M-theory is somehow latent in supermembrane theory already at the classical level.

However, deWit, Lüscher and Nicolai then showed in [93] that the regularised supermembrane theory possesses a continuous spectrum of energy, so that there is a priori no simple interpretation of the states of the matrix model in terms of discrete particle spectrum. The theory was then believed to be unstable and was abandonned as a candidate for a low-energy description of quantum gravity. A decade later, interest in membrane theory was revived by Banks, Seiberg, Suskind and Fischler (or BFSS for short) which showed how it could be rephrased as a low-energy theory for many D0-branes and thus provide a partonic description of M-theory in light-front coordinates. In addition, the fact that such a theory is second quantised solves the puzzle of the existence of a continuous spectrum of energy for the membrane theory. This constitutes in some sense an advantage of such a matrix model description over the five known superstring theories, which are first-quantised with respect to the target-space, and where, therefore, there is no simple description of extended objects such as NS- and D-branes in terms of the string Hilbert space; these objects appear in fact as solitons in the non-perturbative regime of superstring theories.

In the following chapter, we will thus be led to consider the relation between the Hamiltonian description of supermembranes and the conjecture proposed by BFSS, stating that M-theory in the infinite momentum frame (or IMF) is described by a quantum mechanics for $U(N)$ matrices encoding the low-energy dynamics of N D0-branes, when $N \to \infty$. To distinguish it from the original regularised matrix model for (super)membranes, and to highlight, at the same time, its relevancy to a light-cone description of M-theory, this proposal is generally referred to as M(atrix)-theory. For this purpose, we shall first recall the duality between the low-energy effective action for Dp-branes in IIA theory and the ten-dimensional $\mathcal{N} = 1$ super Yang-Mills reduced to (p+1) dimensions. The connection to the BFSS matrix model is then realised by considering a limit where both the number N of D0-brane and the M-theory radius (or alternatively the type IIA coupling constant) are large, but N grows faster, so that their ratio is also large. In this limit, all degrees of freedom decouple except for D0-branes, which, in this perspective, appear as the microscopic degrees of freedom of M-theory in the IMF limit. We will also present the argument in [228] that the matrix gauge theory provides information on M-theory already for finite N, giving in this case a Discrete Light-Cone Quantisation (or DLCQ for short) description of M-theory. An infinite boost then relates this compactification of M-theory on a light-like circle to a compactification on a vanishing space-like circle, connecting the original matrix model description to type IIA string theory in the presence of D0-branes [218, 221].

This conjecture has passed a certain number of tests. In particular, matrix model loop calculation have been shown to correctly reproduce linear and, to a certain extent, non-linear effects of $11D$ supergravity at tree-level.

Moreover, besides D0-branes which are naturally related to eleven-dimensional supergravitons with momentum in the eleventh dimension, the other known objects of M-theory (wrapped, unwrapped and higher genus M2-branes and longitudinal and transverse M5-branes) can be described by particular configurations of $U(N)$ matrices associated to the elementary D0-branes. In this respect, a thorough investigation of the BFSS supersymmetry algebra [230] is very illuminating, and we shall see how the spectrum of brane charges of the theory can be deduced from such an analysis. In this perspective, the brane charges appear as traces over composite operators of the fundamental matrix degrees of freedom. Nevertheless, a puzzle remains as to the presence or absence of transverse five-branes in this context. In principle, the light-cone frame approach to M-theory leads to the existence of both longitudinal and transverse five-branes in the M(atrix) theory, where longitudinal refers to the direction in which the infinite boost to the IMF is performed. However, there is no sign of transverse five-brane charge in the brane spectrum one recovers from the BFSS supersymmetry algebra, so that one might expect such states to be absent in the IMF limit. This view is contrasted with a proposal made in [123], which considers M(atrix) theory on T^3. This has a dual description as a $(3+1)D$ $\mathcal{N} = 4$ super Yang-Mills theory on the dual torus, with gauge group $U(\infty)$. These authors then find some evidence that in this setup T-duality on the M(atrix) model side corresponds to S-duality on the super Yang-Mills side, so that a transverse five-brane wrapped around the original torus in the M(atrix) model could in principle be given a dual description as an unwrapped membrane on the super Yang-Mills side. However, this requires an explicit realisation of S-duality in four-dimensional super Yang-Mills theory, which is still an open question. In addition, it is also unclear how S-duality behaves in the large N limit, which, according to the BFSS prescription, should be taken before performing S-duality.

In addition to this seeming (and yet unsolved) contradiction, another delicate point in M(atrix) theory: although such a theory was shown to reproduce correctly in the finite N limit the two-graviton and three-graviton scattering amplitudes known from $11D$ supergravity tree-level calculations, there appears a disagreement between the matrix model calculation of the $\mathcal{R}^4$ correction to this process and the supergravity 1-loop calculation [137, 136], so that it is doubtful whether M(atrix) theory can actually give an accurate description of quantum corrections in $11D$ supergravity without having to deal with the subtleties of the large N limit (and perhaps it does not even in this limit).

Finally, eleven-dimensional Lorentz invariance of the model is still an open question and needs to be demonstrated (see [184] for some hints in this direction).

This has led us to investigate an alternative supersymmetric matrix model to the one proposed by BFSS, which we insist on defining in purely algebraic terms, with no reference to a particular target space-time. This is in principle desirable from the point of view of a background independent formulation of M-theory. Since, in addition, we aim at recovering the eleven-dimensional covariance expected from M-theory, a candidate with the required properties is a matrix model inspired by [226, 225, 229] with manifest $\mathfrak{osp}(1|32,\mathbb{R})$ symmetry, which can be viewed as the maximal finite-dimensional (non-central) supersymmetric extension of the AdS_{11} algebra. In a larger perspective, $\mathfrak{osp}(1|32,\mathbb{R})$ has also been shown as the natural structure to unify the ten-dimensional supersymmetry algebras of type IIA/B string theories and the eleven- and twelve-dimensional superalgebras, relevant to M- and F-theory. In this framework, in particular, the T- and S-dualities of these theories appear as redefinitions of the generators of $\mathfrak{osp}(1|32,\mathbb{R})$ [37, 36, 28].

In Chapter 2, we compute explicitely such a matrix model, with a cubic interaction, in eleven- and twelve-dimensions, and analyse the respective supersymmetry algebras. We concentrate in particular on the action in eleven-dimensions and study its connection to the BFSS matrix model. To do so, one has to consider the IMF limit of the T-dual theory, where some original fields now become auxiliary, and can be eliminated by an iterative computation of the effective action. At this stage, the resulting theory nicely incorporates membrane and transverse five-brane degrees of freedom on the same footing as the fundamental D0-branes. The effective action reproduces the BFSS action with a mass term as leading contribution, along with an infinity series of higher interaction terms. In particular, it now contains both transverse five-branes as fundamental degrees of freedom and longitudinal five-branes as the usual configurations of D0-branes constructed from the BFSS theory. Finally, since the bosonic sector of the superalgebra of the model is now the conformal algebra, and not the Lorentz one, in contrast to the BFSS matrix model (this accounts in particular for the appearance of an additional negative definite mass term in the leading contribution of the $\mathfrak{osp}(1|32,\mathbb{R})$ effective action), this opens the possibility of its relation to M-theory in an AdS_{11} background, and leaves it for further investigation.

Chapter 1

M(atrix)-theory, branes and the BFSS conjecture

1.1 A matrix model for membranes

In this section, we will investigate the close connection between the matrix model description of M-theory (also called M(atrix) theory) and a theory of supermembranes in 11 dimensions. It has long been known [140, 141, 120, 91] that a regularised light-cone description of the supersymmetric membrane action in $11D$ leads to a quantum mechanical model where the fundamental degrees of freedom can be described by finite dimensional matrices. Now, it is believed that when the dimension of these matrices is taken to infinity, this regularised action gives a microscopic description of a lightlike compactification of M-theory. This second approach is the purpose of the BFSS conjecture which we will thoroughly discussed in section 1.3.

1.1.1 The bosonic membrane in the light-cone frame

Membrane theory comes as a natural candidate for a microscopic description of M-theory, since membranes couple electrically to the antisymmetric three-form field of the $11D$ supergravity multiplet (for a detailled description of the low-energy limit of M-theory, see section 1.3), in the same guise as strings are electrically charged under the two-form field $B_{(2)}$. One might then expect a quantised theory of supermembranes to produce an acceptable, even though not always very handy, microscopic description of M-theory. The fact that the classical action for supermembranes is consistent in 11 dimensions and its quantum regularised version anomaly free (which is not the case for any other dimensions) also seems to point in this direction.

A relativistic bosonic membrane theory in D flat Minkowskian dimensions can be described by a Nambu-Goto type action [129, 195] for fields $X^\mu = X^\mu(\vec{\sigma})$, $\mu \in \{0,..,D-1\}$, representing the embedding into the membrane target space:

$$S_{\mathcal{M},\mathrm{NG}} = -T_{\mathcal{M}} \int d^3\sigma \sqrt{-h}\,, \qquad h_{\alpha\beta} = \eta_{\mu\nu}\partial_\alpha X^\mu\,\partial_\beta X^\nu\,, \tag{1.1}$$

where we denote the world-volume coordinates by $\vec{\sigma} \in \mathbb{R}^3$ with $\vec{\sigma} = (\tau, \sigma^1, \sigma^2)$, distinguishing between two sets of indices: $\alpha, \beta \in \{0,1,2\}$ and $a, b \in \{1,2\}$. Then $h_{\alpha\beta}$ is the pullback of flat Minkowskian metric to the three-dimensional membrane world-volume, and $T_{\mathcal{M}} = 1/(2\pi)^2 l_P^3$ the membrane tension in units of the Planck scale.

Similarly to string theory, one may introduce a fiducial world-volume metric $g_{\alpha\beta}$ and the related Polyakov action

$$S_{\mathcal{M},\mathrm{P}} = -\frac{T_{\mathcal{M}}}{2} \int d^3\sigma \sqrt{-g}\left(g^{\alpha\beta}\,\partial_\alpha X_\mu\,\partial_\beta X^\mu - 1\right) \doteq \int d^3\sigma \mathcal{L}_{\mathcal{M},\mathrm{P}}\,. \tag{1.2}$$

This action yields the following equations of motion for the fields X^μ

$$\partial_\alpha \left(\sqrt{-g} g^{\alpha\beta}\, \partial_\beta X^\mu \right) = 0$$

together with the energy momentum tensor

$$T^{\alpha\beta} = -\frac{2(2\pi)^2}{\sqrt{-g}} \frac{\delta \mathcal{L}_{\mathcal{M},P}}{\delta g_{\alpha\beta}} = -\frac{1}{l_P^3} \left(\partial^\alpha X_\mu\, \partial^\beta X^\mu - \frac{1}{2} g^{\alpha\beta} \left(\partial^\gamma X_\mu\, \partial_\gamma X^\mu - 1 \right) \right) . \tag{1.3}$$

where use has been made of $\delta g = g g^{\alpha\beta}\, \delta g_{\alpha\beta}$ and $\delta g^{\alpha\beta} = -g^{\alpha\gamma} g^{\beta\delta} \delta g_{\gamma\delta}$, which can be deduced from the identity $\delta(g^{\alpha\beta} g_{\beta\delta}) = \delta(\delta^\alpha{}_\beta) = 0$.

The equation of motion for the metric $g_{\alpha\beta}$ corresponds to setting the energy momentum tensor (1.3) to zero:

$$T^{\alpha\beta} = 0 . \tag{1.4}$$

Lowering the indices in equation (1.4) yields

$$h_{\alpha\beta} = \frac{1}{2} (\mathrm{Tr} h - 1) g_{\alpha\beta} \tag{1.5}$$

with $\mathrm{Tr} h = g^{\gamma\delta} h_{\gamma\delta}$. Finally tracing both sides of eqn.(1.5) gives: $\mathrm{Tr} h = 3$. Plugging this result back in eq. (1.5) sets $h_{\alpha\beta} = g_{\alpha\beta}$, which proves the equivalence:

$$S_{\mathcal{M},\mathrm{P}} = -\frac{T_M}{2} \int d^3\sigma\, \sqrt{-g}\, (\mathrm{Tr} h - 1) = S_{\mathcal{M},\mathrm{NG}} .$$

Since the action (1.2) enjoys both three diffeomorphism symetries, we can use them to gauge-fix as many components of the fiducial metric

$$g_{\alpha\beta} = \begin{pmatrix} -\frac{4}{n^2} \tilde{h} & 0_{1\times 2} \\ 0_{2\times 1} & h_{ab} \end{pmatrix} , \qquad n \in \mathbb{R} \tag{1.6}$$

where $\tilde{h} \doteq \det h_{ab}$ is the reduced determinant. Using (1.4) to set $h_{ab} = g_{ab}$, and the obvious relation $\partial^a X_\mu\, \partial_a X^\mu = \mathrm{Tr} h_{ab} = 2$, action (1.2) can be gauge-fixed by the choice (1.6) to

$$S_{\mathcal{M}} = -T_{\mathcal{M}} n \int d^3\sigma\, \sqrt{-g} \left(\frac{1}{4} \partial_\tau X_\mu\, \partial_\tau X^\mu - \frac{\tilde{h}}{n^2} \right) . \tag{1.7}$$

The three-dimensional membrane world-volume is now constrained to be of the form $\Sigma_2 \times \mathbb{R}$, where Σ_2 is a two-dimensional Riemann surface of fixed topology and volume $\int d^2\sigma = 4\pi$, this being a necessary condition for choosing the gauge (1.6).

Since our final aim is to find a proper way of quantising the membrane action (1.7), we find it convenient to reformulate it in terms of Poisson bracket, which we denote by PB

$$\{A, B\}_{\mathrm{PB}} = \varepsilon^{ab} \partial_a A\, \partial_b B , \qquad \varepsilon^{12} = 1 .$$

It is straightforward to show that the reduce determinant can be rewritten in terms of PB as

$$\tilde{h} = \frac{1}{2} (\varepsilon^{ab} \partial_a X^\mu\, \partial_b X^\nu)^2 = \frac{1}{2} \{X_\mu, X_\nu\}_{\mathrm{PB}} \{X^\mu, X^\nu\}_{\mathrm{PB}} . \tag{1.8}$$

Then taking advantage of the simple form assumed by the inverse of the block diagonal metric (1.6)

$$g^{00} = (g_{00})^{-1} , \qquad h^{ab} \doteq (h^{-1})_{ab} = \frac{1}{\tilde{h}} \begin{pmatrix} h_{22} & -h_{12} \\ -h_{12} & h_{11} \end{pmatrix}$$

one easily shows that

$$\begin{aligned}\partial_a(\tilde{h}\,\partial^a X^\mu) &\equiv \partial_a(\tilde{h}\,h^{ab}\partial_b X^\mu) = \partial_1(h_{22}\partial_1 X^\mu - h_{12}\partial_2 X^\mu) + \partial_2(h_{11}\partial_2 X^\mu - h_{12}\partial_1 X^\mu) \\ &= \{X_\rho, \{X^\rho, X^\mu\}_{\rm PB}\}_{\rm PB}\end{aligned} \tag{1.9}$$

where additional use has been made of the equality

$$\partial_a X^\mu\,\partial_b X_\nu\,(\partial_a\partial_b X^\nu) = \frac{1}{2}\partial_a X^\mu\,\partial_a h_{bb}\,, \qquad \text{no sum on } a \text{ or } b\,.$$

Along the same line, one can also prove

$$\begin{aligned}\tilde{h}h^{ab}\partial_a X^\mu\,\partial_b X^\nu &= h_{11}\partial_2 X^\mu\,\partial_2 X^\nu + h_{22}\partial_1 X^\mu\,\partial_1 X^\nu - h_{12}\,(\partial_1 X^\mu\,\partial_2 X^\nu + \partial_2 X^\mu\,\partial_1 X^\nu) \\ &= \{X^\mu, X^\rho\}_{\rm PB}\{X^\nu, X_\rho\}_{\rm PB}\,.\end{aligned} \tag{1.10}$$

A little check shows that by contracting expression (1.10) on both sides with the flat metric $\eta_{\mu\nu}$ and using $h^{ab}h_{ab} = 2$ one comes up with $2\tilde{h} = \{X^\mu, X^\rho\}_{\rm PB}\{X_\mu, X_\rho\}_{\rm PB}$, which is, as expected, formula (1.8).

By means of equality (1.8), the action (1.7) can be recast into

$$S_{\mathcal{M}} = \frac{T_{\mathcal{M}}n}{4}\int d^3\sigma\,\left(\partial_\tau X_\mu\,\partial_\tau X^\mu - \frac{2}{n^2}\{X_\mu, X_\nu\}_{\rm PB}\{X^\mu, X^\nu\}_{\rm PB}\right) \tag{1.11}$$

yielding the equation of motion for X^μ

$$\begin{aligned}\partial_\tau\left(\sqrt{-g}g^{00}\,\partial_\tau X^\mu\right) + \partial_a\left(\sqrt{-g}g^{ab}\,\partial_b X^\mu\right) &= -\frac{n}{2}\partial_\tau^2 X^\mu + \frac{2}{n}\partial_a(\tilde{h}\,h^{ab}\partial_b X^\mu\,) = 0 \\ \longrightarrow\quad \partial_\tau^2 X^\mu &= \frac{4}{n^2}\{X_\rho, \{X^\rho, X^\mu\}_{\rm PB}\}_{\rm PB}\end{aligned} \tag{1.12}$$

where, in the last line, use has been made of relation (1.9).

These equations of motion have to be complemented by the equations for the metric (1.4) which act as a Virasoro constraint. Detailling the latter system, we have to impose the additional constraints

$$\begin{aligned}T_{00} &= 0 \longrightarrow \partial_\tau X_\mu\,\partial_\tau X^\mu + \frac{2\tilde{h}}{n^2}(h^\alpha{}_\alpha - 1) = 0 \\ &\longrightarrow\ \partial_\tau X_\mu\,\partial_\tau X^\mu = -\frac{4}{n^2}\tilde{h}\,, & (1.13) \\ T_{0a} &= 0 \longrightarrow \partial_\tau X_\mu\,(\partial_a X^\mu) = 0\,, & (1.14) \\ T_{ab} &= 0 \longrightarrow h_{ab} - \frac{1}{2}h_{ab}(h^\alpha{}_\alpha - 1) \stackrel{!}{=} 0\,.\end{aligned}$$

The last equation is trivially satisfied for a three-dimensional world-volume.

One can again reformulate the two remaining constraints (1.13) and (1.14) in terms of the Poisson bracket:

$$\partial_\tau X_\mu\,\partial_\tau X^\mu = -\frac{2}{n^2}\{X_\mu, X_\nu\}_{\rm PB}\{X^\mu, X^\nu\}_{\rm PB}\,, \qquad \{\partial_\tau X_\rho, X^\rho\}_{\rm PB} = 0 \tag{1.15}$$

Actually, the second equation of system (1.15) is not exactly a reformulation of constraint (1.14), but is rather inferred from the latter, as one can check from:

$$\{\partial_\tau X_\rho, X^\rho\}_{\rm PB} = \partial_1(\partial_\tau X_\rho\,\partial_2 X^\rho) - \partial_2(\partial_\tau X_\rho\,\partial_1 X^\rho) \stackrel{(1.14)}{=} 0\,.$$

At this point, we have reformulated the classical bosonic membrane theory as a constrained but still covariant Hamiltonian system, where the Hamiltonian itself can be trivially extracted from the action (1.11) by switching the sign in front of the potential term $(\{X_\mu, X_\nu\}_{\rm PB})^2$. Due to the presence of constraints and to the non-linearity of the equations of motion, the system is not easily quantised. It turns out to be more convient to tackle the question of quantisation in the light-cone frame, where, as we will see, one can deal with the constraints without too much effort.

We start by shifting to coordinates

$$X^0 = \frac{X^+ + X^-}{\sqrt{2}}\,, \qquad X^{D-1} = \frac{X^+ - X^-}{\sqrt{2}}\,,$$

then, expressions such as the LHS of the first constraint in system (1.15) will be recast into

$$\partial_\alpha X_\mu\, \partial_\beta X^\mu = -2\partial_{(\alpha} X^+\, \partial_{\beta)} X^- + \partial_\alpha X_i\, \partial_\beta X^i\,,$$

and the whole system of constraints (1.15) can be solved by gauge-fixing the sytem to the light-front gauge

$$X^+(\vec{\sigma}) = \tau\,.$$

The gauged-fixed action (1.11) reads

$$\begin{aligned} S_{\mathcal{M}}[X_i; \partial_\tau X^-, \partial_\tau X^i] &= -\frac{T_{\mathcal{M}} n}{2} \int d^3\sigma \left(\partial_\tau X^- - \frac{1}{2}\partial_\tau X_i\, \partial_\tau X^i + \frac{1}{n^2}\{X_i, X_j\}_{\rm PB}\{X^i, X^j\}_{\rm PB} \right) \\ &\doteq \int d^3\sigma \mathcal{L}_{\mathcal{M},\rm lc}\,, \end{aligned}$$

and the constraints (1.15) turn into a sytem linear in X^-

$$\begin{aligned} \partial_\tau X^- &= \frac{1}{2}\partial_\tau X_i\, \partial_\tau X^i + \frac{1}{n^2}\{X_i, X_j\}_{\rm PB}\{X^i, X^j\}_{\rm PB}\,, \\ \partial_a X^- &= \partial_\tau X_i\, (\partial_a X^i) \longrightarrow \{\partial_\tau X_i, X^i\}_{\rm PB} = 0\,. \end{aligned} \tag{1.16}$$

Going to the Hamiltonian formalism is straightforward, but for the way to treat the conjugate momentum density P^-. The latter being non-dynamical, it should not appear in the Legendre transform $\mathcal{H}_{\mathcal{M},\rm lc} = -P^+\partial_\tau X^- + P^i\partial_\tau X_i - \mathcal{L}_{\mathcal{M},\rm lc}$. Then

$$H_{\mathcal{M},\rm lc} = \int d^2\sigma \left(\frac{1}{2}\frac{P_i P^i}{P^+} + \frac{P^+}{n^2}\{X_i, X_j\}_{\rm PB}\{X^i, X^j\}_{\rm PB} \right) \doteq \int d^2\sigma \mathcal{H}_{\mathcal{M},\rm lc} \tag{1.17}$$

with light-cone momentum $P^+ = \frac{\delta \mathcal{L}_{\mathcal{M},\rm lc}}{\delta(\partial_\tau X_+)} = -\frac{\delta \mathcal{L}_{\mathcal{M},\rm lc}}{\delta(\partial_\tau X^+)}$. Since the light-front gauge induces $\partial_\tau X^+ = 1$, P_+ is the generator of time (τ) translations, which implies $P_+ = -P^- = -\mathcal{H}_{\mathcal{M},\rm lc}$, where the minus sign is due to the time translation being a passive one.

From the conditions on the Legendre transformation, one easily finds

$$P^+ = \frac{T_{\mathcal{M}} n}{2}, \qquad P^i = P^+\partial_\tau X^i, \qquad P^- = \mathcal{H}_{\mathcal{M},\rm lc}$$

with total transverse momentum and light-cone Hamiltonian given by

$$\mathbf{P}^+ = \int d^2\sigma\, P^+ = 2\pi T_{\mathcal{M}} n, \qquad H_{\mathcal{M},\rm lc} = \int d^2\sigma\, P^-\,.$$

The Hamilton equations for the dynamical variables read

$$\partial_\tau P^i = -\frac{\delta \mathcal{H}_{\mathcal{M},\rm lc}}{\delta X_i} = 4\frac{P^+}{n^2}\{X_j, \{X^j, X^i\}_{\rm PB}\}_{\rm PB}, \qquad \partial_\tau X^i = \frac{\delta \mathcal{H}_{\mathcal{M},\rm lc}}{\delta P_i} = \frac{P^i}{P^+}\,. \tag{1.18}$$

Combining both of them, we recover, as expected, equation (1.12). At this stage, the gauge-fixed Hamiltonian (1.17) still enjoys an overall invariance under reparametrisations which are both time independent and separately preserve the area. Since these diffeomorphisms preserve the symplectic form, they obviously leave the equations (1.18) invariant.

The only remaining constraint among (1.16) is the second one, whose canonical version is

$$\{P_i, X^i\}_{\text{PB}} = 0\,, \tag{1.19}$$

while the first of the two constraints is nothing else but the Hamilton equation[1] for X^-. Although gauge-fixing to the light-cone gauge helps us solve explicitely the Gauss constraints, the equations of motion (1.18) remain highly nonlinear. A additional simplifying choice has to be made to quantise Hamiltonian (1.17), which results, as we will see in the next section, in picking a adequate regularisation scheme for the quantisation procedure.

1.1.2 Regularising the membrane Matrix theory

The regularisation we choose to present in this section is the one initial introduced by [140] which has been shown to lead to a simple but powerful quantisation procedure for the membrane Hamiltonian in light-cone gauge. The Riemann surface Σ_2 included in the three-dimensional world-sheet is taken to be a two-sphere S^2, but can in principle be generalised to membranes of arbitrary topology. The membrane world-sheet becomes $S^2 \times \mathbb{R}$, and can be described by a unit two-sphere at fixed time τ, endowed with a canonical symplectic form invariant under rotations.

The sphere S^2 can be parametrised by a set of coordinates $\vec{l} = (l^1, l^2, l^3)$ satisfying

$$\sum_{I=1}^{3} (l^I)^2 = 1. \tag{1.20}$$

By choosing the following parametrisation with respect to the original variables σ^a of Σ

$$l^1(\vec{\sigma}) = \sqrt{1-(\sigma^1)^2}\,\cos(\sigma^2)\,, \qquad l^2(\vec{\sigma}) = \sqrt{1-(\sigma^1)^2}\,\sin(\sigma^2)\,, \qquad l^3(\vec{\sigma}) = -\sigma^1\,,$$

the coordinates on the unit two-sphere have the following Poisson bracket

$$\{l^I, l^J\}_{\text{PB}} = \varepsilon^{ab}\partial_a l^I\,\partial_b l^J = \varepsilon^{IJK} l^K \tag{1.21}$$

with ε^{IJK} being the three-dimensional Levi-Civita tensor. The algebraic version of equation (1.21) is satisfied by the commutation relations of the $\mathfrak{su}(2)$ Lie algebra. Then a natural regularisation of the theory consists in interpreting the Cartesian coordinates l^I as elements of an N-dimensional representation of $\mathfrak{su}(2)$, *ie* as the $N \times N$ generating matrices thereof. This corresponds precisely to the case where the arbitrary normalisation parameter n in the gauge choice is equal to $n = N \in \mathbb{N}^*$.

The prescription is then

$$l^I \to \frac{2}{N} L^I\,, \qquad \{\bullet,\bullet\}_{\text{PB}} \to -\frac{iN}{2}[\bullet,\bullet] \tag{1.22}$$

with the usual commutation relations for generators L^I of an N-dimensional representation of $\mathfrak{su}(2)$:

$$[L^I, L^J] = i\varepsilon^{IJK} L^K\,. \tag{1.23}$$

Since any function on the membrane world-volume $S^2 \times \mathbb{R}$ can be expanded on spherical harmonics: $A(\vec{l};\tau) = \sum_{l,m} c_{lm}(\tau) Y_{lm}(\vec{l})$, with $Y_{lm}(\vec{l}) = \sum_{I_1,..,I_l} y_{lm}^{I_1,..,I_l} l^{I_1} \cdot \ldots \cdot l^{I_l}$ (the coefficients

[1] Note that the system still possesses an overall invariance under time-independent area-preserving diffeomorphisms, which preserves the symplectic form, and hence the Hamilton equations.

$y_{lm}^{I_1,..,I_l}$ have to be taken symmetric and traceless because of condition (1.20)). Replacing the l^I by the prescription (1.22) provides us with a matrix approximation $\mathcal{Y}_{lm}$ to the original spherical harmonics, which can be regarded as the spherical harmonics on a fuzzy sphere. In the algebraic case, the range of the l index is constrained to be smaller than N, since in an N-dimensional rep. of a Lie algebra only products of less than N generators are linearly independent:

$$A(\vec{l};\tau) \to \mathcal{A}(\vec{L};\tau) = \sum_{l<N,m} d_{lm}(\tau)\mathcal{Y}_{lm}(\vec{L}), \qquad \mathcal{Y}_{lm}(\vec{L}) = \left(\frac{2}{N}\right)^l \sum_{I_1,..,I_l} y_{lm}^{I_1,..,I_l} L^{I_1} \cdot ... \cdot L^{I_l}\,. \quad (1.24)$$

The dimension of the representation equals in fact the number of independent combinations of spherical harmonics: $N^2 = \sum_{l=0}^{N-1}(2l+1)$.

Since spherical harmonics are closed under the action of the Poisson bracket, this feature carries over to their "fuzzy" equivalent $\mathcal{Y}_{lm}$:

$$\{Y_{lm}(\vec{l}), Y_{pq}(\vec{l})\}_{\mathrm{PB}} = f_{lm,pq}^{rs} Y_{rs}(\vec{l}) \longrightarrow [\mathcal{Y}_{lm}(\vec{L}), \mathcal{Y}_{pq}(\vec{L})] = F_{lm,pq}^{rs}\mathcal{Y}_{rs}(\vec{L})$$

where the relation prescription (1.22) dictates the relation

$$-i \lim_{N\to\infty} \frac{N}{2} F_{lm,pq}^{rs} \to f_{lm,pq}^{rs}$$

Finally, since the classical Hamiltonian (1.17) is defined by an integral over the two-dimensional Riemann surface Σ at fixed $\sigma^0 = \tau$, this will now be replaced by a trace over $N \times N$ matrices (1.24), in the large N limit:

$$\frac{1}{4\pi}\int d^2\sigma\; A(\vec{l};\tau) = \lim_{N\to\infty}\frac{1}{N}\mathrm{Tr}\mathcal{A}(\vec{L};\tau) \quad (1.25)$$

Likewise, one can show that if the classical Poisson bracket of any two functions A and B yields

$$\{A(\vec{l};\tau), B(\vec{l};\tau)\}_{\mathrm{PB}} = C(\vec{l};\tau)\,,$$

then for some fuzzy matrix approximation of some smooth test function Φ, one has the following limit

$$\lim_{N\to\infty}\frac{1}{N}\mathrm{Tr}\left(\left(-i\frac{N}{2}[\mathcal{A}(\vec{L};\tau),\mathcal{B}(\vec{L};\tau)] - \mathcal{C}(\vec{L};\tau)\right)\Phi(\vec{L};\tau)\right) = 0\,.$$

In summary, we can now use the dictionary

$$\{\bullet,\bullet\}_{\mathrm{PB}} \to -\frac{iN}{2}[\bullet,\bullet], \qquad \frac{1}{4\pi}\int d^2\sigma \;=\; \lim_{N\to\infty}\frac{1}{N}\mathrm{Tr} \quad (1.26)$$

to convert the classical Hamiltonian (1.17) over continuous functions into its regularised form. Taking implicitely the large N limit:

$$\begin{aligned} H_{\mathcal{M},\mathrm{reg}} &= \frac{1}{2\pi l_P^3}\mathrm{Tr}\left(\frac{1}{2}\partial_\tau\mathcal{X}_i\,\partial_\tau\mathcal{X}^i - \frac{1}{4}[\mathcal{X}_i,\mathcal{X}_j][\mathcal{X}^i,\mathcal{X}^j]\right) \\ &\equiv \mathrm{Tr}\left(2\pi l_P^3\frac{\mathcal{P}_i\mathcal{P}^i}{2} - (2\pi l_P^3)^{-1}\frac{1}{4}[\mathcal{X}_i,\mathcal{X}_j][\mathcal{X}^i,\mathcal{X}^j]\right)\,, \end{aligned} \quad (1.27)$$

for $\mathcal{X}^i$ and $\mathcal{P}^i$ now $N \times N$ matrices. When converting time derivatives into momenta, one should be careful to use the regularised Hamiltonian, *ie* $\mathcal{P}_i = \frac{\delta H_{\mathcal{M},\mathrm{reg}}}{\delta(\partial_\tau\mathcal{X}^i)} = (2\pi l_P^3)^{-1}\partial_\tau\mathcal{X}^i$, and not start from the continuum version of the theory, Legendre transform and then regularise: in this procedure the $\mathcal{P}_i$ inherit an extra unabsorbable factor of N^{-2}.

The equations of motion and the Gauss constraint lift naturally to their regularised form

$$\partial_\tau^2 \mathcal{X}^i = -[\mathcal{X}_j, [\mathcal{X}^j, \mathcal{X}^i]]\,, \qquad [\partial_\tau \mathcal{X}^i, \mathcal{X}_i] = 0\,. \tag{1.28}$$

Once regularised, the Hamiltonian $H_{\mathcal{M},\mathrm{reg}}$ can be readily quantised, even though solving the quantum theory may turn out to involve subtleties. Nevertheless, this regularisation procedure provides us with well-defined quantum theory for membranes in the light-cone gauge. Furthermore, the invariance of the classical Hamiltonian (1.17) under time-independent area-preserving diffeomorphisms translates, in the large N regularised version of the theory, into the $U(N)$ symmetry of $H_{\mathcal{M},\mathrm{reg}}$ (1.27). In short, the diffeomorphism group becomes, for the regularised $N \to \infty$ limit, the $U(N)$ matrix group.

1.1.3 The supermembrane on a curved background

How to supersymmetrise the bosonic membrane action is more readily understood if one starts from a theory where the membrane is allowed to evolve in a general (curved) backround. The flat Minkowskian metric $\eta_{\mu\nu}$ in the pullback (1.1) is now replaced by a general symmetric two-tensor $G_{\mu\nu}(X)$ depending on the embedding X^μ:

$$h_{\alpha\beta} = G_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu\,. \tag{1.29}$$

Furthermore, just as strings couple electrically to a the antisymmetric field $B_{\mu\nu}(X)$, in a general background, membranes are also allowed an electric coupling, but to a three-form potential $A_{\mu\nu\lambda}(X)$. Now such a field is intrinsic to eleven-dimensional supergravity, so we expect superspace methods in use on the supergravity side to be applicable to a theory of supermembranes in $11D$. It will also be pointed out later on that 11 is the natural dimension for a supersymmetric theory of membranes to be well-behaved. We can understand the connection to eleven-dimensional supergravity as a first insight into the conjectured correspondance between M(atrix) theory and a sector (IMF limit) of M-theory.

Thus, the membrane action for a general background is the Nambu-Goto action (1.1) for the generalised pullback (1.29) complemented with an electrical coupling to the threeform potential $A_{(3)}$:

$$S_{\mathcal{M},\mathrm{curv}} = S_{\mathcal{M},\mathrm{NG}} - T_{\mathcal{M}} \int d^3\sigma\, A_{\mu\nu\lambda}(X)\partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\lambda\, \varepsilon^{\alpha\beta\gamma}. \tag{1.30}$$

As before, one can introduce a fiducial metric $g_{\alpha\beta}$, shifting to the Polyakov formalism by replacing

$$\begin{aligned} S_{\mathcal{M},\mathrm{NG}} \longrightarrow S_{\mathcal{M},\mathrm{P}}: \quad S_{\mathcal{M},\mathrm{curv}} \;&=\; -\frac{T_{\mathcal{M}}}{2}\int d^3\sigma \Big[\sqrt{-g}\left(g^{\alpha\beta}\,\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X) - 1\right) \\ &\qquad + 2\, A_{\mu\nu\lambda}(X)\partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\lambda\, \varepsilon^{\alpha\beta\gamma}\Big]\,. \end{aligned} \tag{1.31}$$

in expression (1.30). Similarly to the flat case, one can pick the light-cone gauge, and apply a regularisation and quantisation procedure close to what has been outlined in Section 1.1.2.

1.1.4 Supersymmmetrisation in eleven dimensions

As for bosonic strings, the way to stabilise the bosonic membrane theory consists in adding supersymmetry. But supersymmetry is also required when approaching M-theory from the regularised membrane theory. As we briefly mentioned before, supersymmetrisation of the classical membrane theory can only be carried out in dimensions 3+1, 4+1, 6+1 and 10+1 (we do not take into account dimensions with multiple times), which are the dimensionalities where a certain Fierz identity needed for the invariance of the supermembrane Lagrangian holds true. The existence of this Fierz identity is also responsible for the invariance of the Lagrangian under a novel kind of symmetry called κ-symmetry. However, one can be even more restrictive, since among these four possible dimensionalities, $11D$ appears as the one preferred by the quantum theory: all other cases indeed seem to be plagued by anomalies in the Lorentz algebra.

Supersymmetrisation of the membrane action in a general background has been originally carried out by [29], who have found the proper way of adapting the Green-Schwarz formalism [131] to three-dimensional objects. Thus, the supersymmetric theory automatically enjoys target-space supersymmetry and κ-symmetry. Contrary to superstrings, supermembranes have no NSR description, and it is not yet clear wether a world-volume supersymmetric formulation of membranes is workable (see [105]).

An eleven-dimensional superspace formulation of the theory consists in the original 11 bosonic fields X^μ (describing membrane fluctuations in the transverse directions) and 32 Grassmanian coordinates $\theta^{\bar{\alpha}}$. These 43 fields are now handily collected into a bunch of supercoordinates

$$Z^M = (X^\mu, \theta^{\bar{\alpha}})$$

with $M = (\mu, \bar{\alpha})$ taking 43 different values (we distinguish between the spinor index $\bar{\alpha} = 1, .., 32$ and the index $\alpha = 0, 1, 2$ which still denotes the world-volume coordinates of section 1.1)

The field content of the classical bosonic supermembrane theory in arbitrary background is the same as for $11D$ supergravity: thus, the metric $G_{\mu\nu}(X)$, or alternatively the elfbein e^a_μ, $a = 0, .., 10$, and the three-form field $A_{\mu\nu\lambda}(X)$ of section 1.1.3 are now complemented with a background gravitino field $\psi_\mu(X)$. All these fields can be recast in $11D$ superspace formalism, and appear as the lowest order components in θ of the super-vielbein E^A_M and super-threeform B_{MNP}, where now $A = (a, \omega)$ is the superspace equivalent of the tangent space index of the elfbein (note that the gaugino spinor index runs from $\omega = 1, .., 32$). Thus

$$E^A_M \quad \rightarrow \quad E^A_\mu = (E^a_\mu, E^\omega_\mu) = (e^a_\mu + \mathcal{O}(\theta), \psi^\omega_\mu + \mathcal{O}(\theta))\,, \tag{1.32}$$

$$B_{MNP} \quad \rightarrow \quad B_{\mu\nu\rho} = A_{\mu\nu\rho} + \mathcal{O}(\theta)\,. \tag{1.33}$$

The identification of higher order contributions to the superfields (1.32-1.33) has been carried out only up to $\mathcal{O}(\theta^2)$ explicitely [94].

Supersymmetrising action (1.31) in the Green-Schwarz formalism is a quite involved and technical issue. We will therefore not attempt to present the whole superspace derivation, but rather highlight some conceptual keypoints of the calculation.

From $B_{(3)}$, one can define a four-form field strength H_{MNPQ} such that $H_{(4)} = dB_{(3)}$. Next, defining the pullback of the super 43-bein and the ABC components of $B_{(3)}$ by

$$\Pi^A_\alpha = Z^M_{,\alpha} E^A_M\,, \qquad B_{ACD} = E^M_A E^N_C E^P_D B_{MNP} \tag{1.34}$$

with E^M_A the inverse 43-bein, supersymetrising the curved space Polyakov action (1.31) simply consists in replacing the pullback of the metric and the electric coupling to the threeform by their supersymmetric extension, written in terms of (1.34):

$$S_{\mathcal{M},susy} = -\frac{T_M}{2} \int d^3\sigma \, \left(\sqrt{-g} \left(g^{\alpha\beta}\, \Pi^a_\alpha \Pi^b_\beta \eta_{ab} - 1 \right) + 2\varepsilon^{\alpha\beta\gamma} \Pi^A_\alpha \Pi^C_\beta \Pi^D_\gamma B_{ACD} \right)\,. \tag{1.35}$$

Compared to the action for superstrings in the Green-Schwarz formalism, expression (1.35) containts an extra cosmological constant (the (-1) in the firts part of the RHS) and an electric coupling to a three-form instead of a two-form.

Apart from supersymmetry, the action (1.35) boasts a certain number of symmetries, namely: super-diffeomorphism invariance, world-volume diffeomorphism invariance, super-gauge symmetry, a discrete $\mathbb{Z}_2$ symmetry sending $B_{MNP} \to -B_{MNP}$ and inverting the sign of one of the space-time coordinates and, last but not least, κ-symmetry . We will not spend time discussing these symmetries in detail, and we refer the reader to the account reviewed in [234].

Let us however abide a bit on the κ-symmetry of the supermembrane theory. The transformation rule is given by

$$\delta_\kappa Z^M E_M^A = \left(0^a, \frac{1}{2}(1+\Gamma_\natural)^\omega_\rho \kappa^\rho\right), \qquad \Gamma_\natural = \frac{1}{3!\sqrt{|\det g|}}\varepsilon^{\alpha\beta\gamma}\Pi^a_\alpha \Pi^b_\beta \Pi^c_\gamma \Gamma_{abc},$$

for a transformation parameter κ which is actually an anti-commuting world-volume scalar behaving as constant spinor with 32 components.

There are two key features to κ-symmetry in supermembrane theory: first, it is realised as a symmetry of the theory provided the background valued super-43-bein and the super-B-field satisfy the equations of motions of eleven-dimensional supergravity. Then, $11D$ supergravity somehow appears already in classical supersymmetric membrane theory [29]. Second, since $\Gamma_\natural^2 = 1$, the combination $(1+\Gamma_\natural)/2$ is a projector that can be used to gauge away half of the target-space fermions $\theta^{\bar\alpha}$, thus reducing dynamical fermionic degrees of freedom to 8. This closely matches the 8 propagating bosonic fields X^i, $i = 3,..,10$, left once we choose the static gauge $X^\alpha = \sigma^\alpha$. However, this type of gauge-fixing clearly breaks Lorentz invariance, so that no fully covariant quantisation of the theory has yet been found. An attempt has been made by [121, 122] to fix κ-symmetry by breaking the 32-components spinors of $SO(1,10)$ into a 16_L and a 16_R, leaving a residual $SO(1,9)$ symmetry intact. But the initial $11D$ Lorentz symmetry is in this case still incomplete, however minimally reduced.

To conclude this section, we want to show that, in the flat case, the regularised version of the supermembrane action (1.35) can be approximated by supersymmetric quantum mechanics for 8 transverse bosons X^i and 16-components Majorana-Weyl spinor of $SO(1,9)$.

We start by specialising to Minkowski space $G_{\mu\nu}(X) = \eta_{\mu\nu}$ and setting the background three-form $A_{\mu\nu\lambda}(X) = 0$. In flat space, the 43-bein becomes

$$E_M^A = (E_M^a, E_M^\lambda) = ((\delta^a_\mu, (\Gamma^a\theta)_{\bar\alpha}), (0, \delta^\lambda_{\bar\alpha}))\,.$$

The background we have chosen selects and sets as only non-vanishing components of $H_{(4)}$: $H_{\mu\nu\bar\alpha\bar\beta} = (1/3)(\Gamma_{ab})_{\bar\alpha\bar\beta}E^a_\mu E^b_\nu$. With this in hand and the definition $H_{(4)} = dB_{(3)}$, one can deduce the rest of the components of B_{MNP} in terms of polynomials of $(\Gamma_{\mu_1..\mu_m}\theta)_{\bar\alpha}$, for $m < 3$. The details of the derivation are irrelevant to our purpose and can be found in the original paper [29].

Then, the supersymmetric membrane action (1.35) becomes

$$\begin{aligned} S_{\mathcal{M},susy} = -\frac{T_M}{2}\int d^3\sigma \Big(\sqrt{-g}\left(g^{\alpha\beta}\,\Pi^\mu_\alpha \Pi^\nu_\beta \eta_{\mu\nu} - 1\right) \\ -\frac{1}{2}\varepsilon^{\alpha\beta\gamma}\bar\theta\Gamma_{\mu\nu}\partial_\gamma\theta\Big[\partial_\alpha X^\mu \Pi^\nu_\beta + \frac{1}{3}(\bar\theta\Gamma^\mu\partial_\alpha\theta)(\bar\theta\Gamma^\nu\partial_\beta\theta)\Big]\Big)\,. \end{aligned} \tag{1.36}$$

Similarly to κ-gauged GS superstring, we notice the appearance of Wess-Zumino type terms in the second part of action (1.36), which follows naturally from the superspace formalism. The pullback of the 43-bein (1.34) now simplifies to

$$\Pi^\mu_\alpha = \partial_\alpha X^\mu + \bar\theta\Gamma^\mu\partial_\alpha\theta\,. \tag{1.37}$$

One can show that expression (1.36) is invariant under the target-space supersymmetry transformations

$$\delta_\epsilon X^\mu = \bar\theta\Gamma^\mu\epsilon\,, \qquad \delta_\epsilon\theta = \epsilon\,,$$

Obviously $\delta_\epsilon \Pi^\mu_\alpha = 0$, since, for Majorana spinors: $\bar{\theta}\Gamma^\mu\epsilon = -\bar{\epsilon}\Gamma^\mu\theta$, according to eqn.(A.18).

Invariance of Π^μ_α combined with the following Fierz identity in $11D$ for a quadruplet of spinors θ_n:

$$\varepsilon^{mnop}(\bar{\theta}_m\Gamma^\mu\theta_n)(\bar{\theta}_o\Gamma_{\mu\nu}\theta_p) = 0$$

(ε^{mnop} is the Levi-Civita tensor in $4D$) leaves the action (1.36) invariant. This last identity holds true, as mentioned at the beginning of this section, in $D = 4, 5, 7, 11$, and also ensures κ-symmetry of the action. However, the theory in 11D is suspected to be the only one to posses an anomaly-free quantum conterpart. Eleven dimensions is thus a privileged dimensionality for supermembranes, just as $10D$ is for superstrings. As will appear later on, another Fierz identity valid in $10D$ and involving this time an antisymmetrised triplet of spinors is responsible for the supersymmetry invariance of SYM in $10D$ and of the BFSS matrix model.

Finally, the equations of motion for the fiducial metric defines the latter to be the pullback of the flat Minkowskian one:

$$g_{\alpha\beta} = \eta_{\mu\nu}\Pi^\mu_\alpha\Pi^\nu_\beta\,.$$

Repeating the procedure exposed in Section 1.1, we gauge-fix the theory by going to the light-cone gauge:

$$X^+(\vec{\sigma}) = \tau\,.$$

Moreover, we still have the liberty of using κ-symmetry to set $\Gamma^+\theta = 0$. This implies that the following spinor bilinear vanish: $\bar{\theta}\Gamma^i\partial_\alpha\theta = 0$, for $i = 2, .., 10$, $\bar{\theta}\Gamma^{ij}\partial_\alpha\theta = 0$ and $\bar{\theta}\Gamma^{+\mu}\partial_\alpha\theta = 0$, so that the momentum (1.37) now becomes:

$$\Pi^i_\alpha = \partial_\alpha X^i\,.$$

Similar to what has been done for the bosonic membrane theory in Section 1.1.1, one computes the constraints analogous to eqns.(1.15) for the action (1.36), which is now simplified by the choice of light-cone gauge and by fixing κ-symmetry. These constraints lead to:

$$\begin{aligned} \partial_\tau X^- &= \frac{1}{2}(\partial_\tau X^i)^2 + \frac{1}{N^2}(\{X^i, X^j\})^2 - i\bar{\theta}\Gamma^-\partial_\tau\theta\,, \\ \partial_{\sigma^k} X^- &= \partial_{\sigma^k} X_i \partial_\tau X^i - i\bar{\theta}\Gamma^-\partial_{\sigma^k}\theta \quad \text{for } k = 1, 2\,, \end{aligned}$$

where the normalisation constant N is brought into play for the same reasons as the parameter n earlier in bosonic membrane theory.

Eventually, using these constraints the action (1.36) simplifies even further, yielding the following expression:

$$\begin{aligned} S_{\mathcal{M},susy} = \frac{T_\mathcal{M} N}{4}\int d^3\sigma\,\Big(&\mathcal{D}_\tau X_i\,\mathcal{D}_\tau X^i - \frac{2}{N^2}\{X_i, X_j\}_{\text{PB}}\{X^i, X^j\}_{\text{PB}} \\ &- i\bar{\theta}\Gamma^-\partial_\tau\theta + \frac{2}{N}i\bar{\theta}\Gamma^i\{X^i, \theta\}_{\text{PB}}\Big)\,. \end{aligned}$$

Since now half of the components of θ are fixed by κ-symmetry, we are left with 16-dimensional $SO(9,1)$ Majorana spinors. One can then replace the 32×32 Dirac matrices $\Gamma^i = \gamma^i \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ by their 16×16-dimensional $SO(9)$ Dirac submatrices γ^i.

After performing a Legendre transform of the Lagrangian density and integrating it over the spatial world-volume coordinates, we arrive at the Hamitonian:

$$H_{\mathcal{M},susy} = \int d^2\sigma\,\left(\frac{1}{T_\mathcal{M} N}P_i\,P^i + \frac{T_\mathcal{M}}{2N}\{X_i, X_j\}_{\text{PB}}\{X^i, X^j\}_{\text{PB}} - \frac{T_\mathcal{M}}{2}i\bar{\theta}\gamma^i\{X_i, \theta\}_{\text{PB}}\right) \quad (1.38)$$

with momenta $P_i = \frac{NT_\mathcal{M}}{2}\partial_\tau X^i$ and $\Pi_\theta = -\frac{NT_\mathcal{M}}{4}i\bar{\theta}\Gamma^-$ for the fermions.

Finally, the regularisation procedure already outlined in Section 1.1.2 now tells us how to convert the fields X^i, $\partial_\tau X^i$ and θ into $N \times N$ matrices in the adjoint representation of $U(N)$, by means of the correspondence (1.26). If, in addition, we wish to start from a gauge-covariant action, one replaces $\partial_\tau \to \mathcal{D}_\tau = \partial_\tau + iA_0$. After Legendre transforming to the Hamiltonian formalism, this generates a term $-2A_0\big(\{P_i, X^i\}_{\rm PB} - \{\{\Pi_\theta, \theta\}\}_{\rm PB}\big)$ where $\{\{\ ,\ \}\}_{\rm PB}$ denotes symetrised Poisson brackets.

Denoting the regularised fields $\mathcal{X}^i$, $\mathcal{P}_i = (2\pi l_P^3)^{-1}[\mathcal{D}_\tau, \mathcal{X}_i]$, ϑ and $\mathcal{A}_0$, one obtains the quantum supermembrane Hamiltonian in Planck units (as $T_{\mathcal{M}}^{-1} = 4\pi^2 l_P^3$):

$$\begin{aligned} H_{\mathcal{M},\rm reg} \;&=\; 2\pi l_P^3 {\rm Tr}\left(\frac{1}{2}\mathcal{P}_i\mathcal{P}^i - \frac{1}{4(2\pi l_P^3)^2}[\mathcal{X}_i,\mathcal{X}_j][\mathcal{X}^i,\mathcal{X}^j] - \frac{1}{2(2\pi l_P^3)^2}\vartheta^\dagger\Gamma^{0i}[\mathcal{X}_i,\vartheta]\right. \\ &\qquad\left. + \frac{i}{2\pi l_P^3}\mathcal{A}_0\Big([\mathcal{P}_i,\mathcal{X}^i] + \frac{1}{2(2\pi l_P^3)}\{i\bar\vartheta\Gamma^-,\vartheta\}\Big)\right). \end{aligned} \tag{1.39}$$

The term on the second line is the supersymmetric version of the Gauss constraint (1.19), and the gauge field $\mathcal{A}_0$ can be regarded as a Lagrange multiplier implementing the constraint directly in the Hamiltonian formalism.

The Hamiltonian (1.39) plays a central rôle in the M(atrix) approach to M-theory. In Section 1.2.1 indeed, the same Hamiltonian will be derived from $10D$ super Yang-Mills theory toroidally compactified to $9D$, which is a first order approximation to a quantum theory N D0-branes and appears at the basis of the BFSS conjecure.

1.1.5 Instabilities and continuous spectrum of states in the supermembrane model

At the classical level, the classical bosonic membrane theory exhibits instabilities. This phenomenon can best be illustrated by considering, for instance, the simplest membrane configuration, with energy proportional to the area of the membrane times its tension. Such a configuration can start growing long but very narrow "spikes" at a very low energy cost. This would not be possible for a classical string, where such an excrescence would have energy proportional to the string length. In membrane theory, instead, a cylindrical spike of legth l has energy $2\pi T_{\mathcal{M}} R l$, with R the radius of the cylinder, so that the energy can stay small if l is big but $R \ll \frac{1}{2\pi T_{\mathcal{M}} l}$. The fluctuations the growing of these spikes generates prevents us to see the membrane as a pointlike object.

In the regularised theory, these instabilities can be cured, since they now appear as flat directions in the bosonic potential part of quantum Hamiltonian (1.39). They can arise from potential terms such as $(x_1 x_2)^2 \mathbb{1}$, if the only non-zero bosonic matrix fields are for instance $\mathcal{X}^1 = \begin{pmatrix} 0 & x_1 \\ -x_1 & 0 \end{pmatrix}$ and $\mathcal{X}^2 = \begin{pmatrix} 0 & 0 \\ 0 & x_2 \end{pmatrix}$. Then, for either $x_1 = 0$ or $x_1 = 0$ there arises flat directions in the space of solutions of the theory since the other variable is unconstrained, producing marginal instabilities in the classical theory for $N > 1$. These can be cured in the regularised quantum theory. There, the off-diagonal component x_1 now becomes a harmonic oscillator with big mass when x_2 is unconstrained, and its zero-point energy gets large when x_2 increases, giving rise to an effective confining potential which lifts the flat direction. The quantum bosonic membrane theory thus has a discrete spectrum of energy for any N.

In the supersymmetric regularised theory however, deWit, Lüscher and Nicolai have shown in [93] that the matrix theory exhibits a continuous spectrum, since now the zero-point energies of the additional fermions exactly cancel those of the bosons, which were used earlier to remove the flat directions. The absence of mass gap separating the massless from the massive states apparently prevents an interpretation of the states of the theory as particle-like objects described by a discrete spectrum of states in the Hilbert space of a first-quantised theory, like string theory.

At the time, this seemed to compromise the validity of membrane theory as a likely candidate for a microscopic description of M-theory, until the new perspective which the BFSS conjecture [231] opened for this model.

1.2 D-branes and Dirac-Born-Infeld theory

In this section, we will comment on the string origin of the dynamical degrees of freedom of a Dp-brane in a fixed Dp-brane background in $10D$. When attached to a D-brane, the massless open string vector field A_μ can be described as a low-energy theory on the D-brane world-volume. The vector field A_μ then splits into $(p+1)$ components parallel to the D-brane world-volume, giving rise to $U(1)$-gauge fields A_α $(\alpha = 0,..,p)$ on the latter, and leaves $(9-p)$ transverse components X^a $(a = p+1,..,9)$, describing the D-brane fluctuations in the transverse directions, which are identified with the $(9-p)$ components of the brane embedding X^μ $(\mu = 0,..,9)$ in the $10D$ target space.

In a purely bosonic theory, it was shown by [180] that the D-brane equations of motion compatible with the open string theory in the D-brane background being conformally invariant is given by the Dirac-Born-Infeld action. Introducing coordinates y^α $(\alpha = 0,..,p)$ on the brane, and defining the *pull-back* to the brane of a n-form field C as $C_{\alpha_1..\alpha_n} = \partial_{y^{\alpha_1}} X^{\mu_1}(y) \cdots \partial_{y^{\alpha_n}} X^{\mu_n}(y)\, C_{\mu_1..\mu_n}$, this action reads

$$S_{\rm DBI} = -T_p \int d^{p+1}y\, e^{-\phi} \sqrt{-\det(g_{\alpha\beta} + B_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})}\,, \tag{1.40}$$

where $g_{\alpha\beta}$, $B_{\alpha\beta}$, ϕ are the induced space-time metric, antisymmetric tensor and dilaton pulled back to the Dp-brane world-volume, and $F_{\alpha\beta}$ is field-strength of the $U(1)$-gauge field $A_\alpha(y)$. A cross check from string theory (actually a disk diagram computation) gives the precise value for the brane tension T_p in terms of string units

$$\tau_p = g_s^{-1} T_p = (2\pi)^{-p} \alpha'^{-\frac{p+1}{2}} g_s^{-1}\,, \tag{1.41}$$

where $g_s = \exp\langle\phi\rangle$ is the string coupling and α' is given in terms of the string tension as $T_s^{-1} = 2\pi\alpha'$.

To make contact with the full superstring theory, one should of course complement the above action (1.40) by additional fermionic and Chern-Simons terms. Fore instance, since Dp-branes are carriers for R-R charges, their world-volume couples to the corresponding R-R potential through the leading Wess-Zumino term $\mu_p \int_{\Sigma_{p+1}} A_{(p+1)}$, where Σ_{p+1} is the Dp-brane world-volume. A more general coupling will be considered below.

In general, for a collection of N D-branes, the gauge group carried by A_α becomes non-abelian, so that both A_α and X^a are now matrices. An extension of expression (1.40) will then involve traces of products of these fields and their derivatives. How to order the traces in order to obtain a gauge invariant quantity is still kind of an open question beyond the fifth order, even though comparaison with various scattering amplitudes from supergravity and the nature of the solutions for some BPS non-abelian solitons seems to point at a trace which is symmetrised over gauge indices [33, 30, 42, 31, 32]. In the next section we will need only the leading order of such an expansion, which is well established.

In any case, the non-abelian version of the DBI action (1.40) will receive extra corrections from the following topological term, integrated over the world-volume of the Dp-brane $\mathcal{W}_{p+1}$

$$T_p \int_{\mathcal{W}_{p+1}} \sum_q A_{(q+1)} \wedge {\rm Tr} e^{2\pi\alpha' F + B} \tag{1.42}$$

which, at lower order, gives the Wess-Zumino term we mentioned above. In string language, these terms will be dictated by anomaly cancellation in the effective field theory description of the theory, as we will see in Section 3.

The abelian action (1.40) can be cast into a simplified version under the following assumptions:

a) The background $10D$ space-time is flat, with metric $g_{\mu\nu} = \eta_{\mu\nu} \doteq (-,+,..,+)$.

b) The Dp-brane is approximately flat, and we assume static gauge: namely that we can identify the world-volume coordinates on the Dp-brane with the $(p+1)$ space-time coordinates: $y^\alpha = x^\alpha$, $\alpha = 0,..,p$.

c) The antisymmetric tensor is neglected: $B_{(2)} = 0$.

d) We have the following order of magnitude: $\mathcal{O}(\partial_\alpha X^a) \approx \mathcal{O}(2\pi\alpha' F_{\alpha\beta})$ both small.

The induced metric to fourth order in ∂X is then

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \partial_\alpha X^a \partial_\beta X_a + \mathcal{O}((\partial X)^4)\,,$$

leading to the following simplified version for the determinant in expression (1.40)

$$\left|g_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}\right| = -\left(1 + \partial_\alpha X^a \partial^\alpha X_a + \frac{(2\pi\alpha')^2}{2} F_{\alpha\beta}F^{\alpha\beta}\right) + \sum_q \Delta_q\,. \tag{1.43}$$

where the higher order terms Δ_q in expression (1.43) start at $\mathcal{O}((\partial X)^4, (\partial X)^2, (2\pi\alpha' F)^2, (2\pi\alpha' F)^4)$.

A straightforward but tedious computation shows that:

$$\begin{aligned}
\Delta_0 &= 0\,,\\
\Delta_1 &= \frac{1}{2!}\left((\partial_\alpha X^a \partial_\beta X_a)(\partial^\alpha X^b \partial^\beta X_b) - (\partial_\alpha X^a \partial^\alpha X_a)^2\right)\,,\\
\Delta_2 &= \Delta_1 + \frac{1}{3!}\Big(3(\partial_\alpha X^a \partial^\alpha X_a)(\partial_\beta X^b \partial_\gamma X_b)(\partial^\beta X^c \partial^\gamma X_c) + 2(\partial_\alpha X^a \partial_\beta X_a)(\partial^\beta X^b \partial^\gamma X_b)(\partial_\gamma X^c \partial^\alpha X_c)\\
&\quad -(\partial_\alpha X^a \partial^\alpha X_a)^3\Big) - \frac{(2\pi\alpha')^2}{2}\partial_\alpha X^a \partial_\beta X_a\,(\eta^{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta} + 2F^\alpha{}_\delta F^{\delta\beta})\,,
\end{aligned}$$

and so on.

Dropping however these fourth order corrections, the low-energy Dp-brane world-volume action becomes

$$S = -\tau_p V_p + \frac{1}{g_p^2}\int d^{p+1}y\left(-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2(2\pi\alpha')^2}\partial_\alpha X^a \partial^\alpha X_a + \mathcal{O}((\partial X)^4, (\partial X)^2F^2, F^4)\right)\,, \tag{1.44}$$

where V_p is the Dp-brane world-volume, and the coupling constant g_p can be given either in terms of the Dp-brane tension or solely in string units:

$$g_p^2 = \frac{1}{(2\pi\alpha')^2\tau_p} = g_s(\alpha')^{-\frac{1}{2}}(2\pi\sqrt{\alpha'})^{p-2}\,. \tag{1.45}$$

Focusing on the second part of expression (1.44), we notice it to be a $U(1)$ gauge theory in $(p+1)$ dimensions along with $9-p$ scalars. Pushing a bit further by adding fermions and rendering the gauge connection non-abelian, we arrive naturally at non-abelian supersymmetric Yang-Mills theory (SYM), obtained by reduction to $(p+1)$ dimensions of a $10D$ SYM theory, which is essentially unique.

1.2.1 Ten-dimensional super Yang-Mills theory and its toroidal reduction

The abelian Yang-Mills theory (1.44) appearing as the low-energy limit of a single D-brane action carrying $U(1)$ gauge field can be promoted to N parallel D-branes, giving rise to a total $U(1)^N$ gauge group. So much for the diagonal part. However, the possibility now arises of having fields stretching between two different D-branes and carrying Chan-Paton indices (I, J) $(I, J = 1, .., N)$ labelling each one of them. Since such strings are oriented, they build up $N(N-1)$ configurations. As the branes stack themselves together and the strings stretching inbetween become massless, it has been shown in [248] that the gauge group is enhanced to $U(N)$.

Non-abelian SYM has the following action in $10D$, resulting as the low-energy version of a supersymmetric D9-brane theory:

$$S = \frac{1}{g_{\rm YM}^2} \int d^{10}y \,{\rm Tr} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \overline{\psi} \Gamma^\mu [D_\mu, \psi] \right) . \tag{1.46}$$

The field strength is defined by

$$F_{\mu\nu} = -i[D_\mu, D_\nu]$$

with $D_\mu \doteq \partial_\mu + iA_\mu$ being the connection for non-abelian massless $U(N)$ gauge vector field A_μ. The fermionic part of the action is composed of massless 16-component Majorana-Weyl spinors ψ of $SO(9, 1)$ (the gaugino). In the following, we take advantage of the Majorana condition being satisfied to choose the spinors and the Dirac matrices all real. See in particular Appendix A.1 for our conventions of $SO(9, 1)$ and $SO(10, 1)$ Dirac matrices.

In addition both the bosonic gauge fields and the gaugino are in the adjoint representation of $U(N)$ and carry therefore matrix indices $a, b = 1, .., N$.

It can be shown that the action (1.46) is supersymmetric under the following transformations

$$\delta_\epsilon A_\mu = \frac{i}{2} \bar{\epsilon} \Gamma^\mu \psi \, , \qquad \delta_\epsilon \psi = -\frac{1}{4} F_{\mu\nu} \Gamma^{\mu\nu} \epsilon \, , \tag{1.47}$$

where ϵ is also a Majorana-Weyl constant spinor: in $(1{+}9)D$ this again endows the model with 16 independent supercharges.

The equations of motion are

$$[D_\mu, F^{\mu\nu}] = \frac{i}{2} (\Gamma^\nu)_\alpha{}^\beta \{\overline{\psi}^\alpha, \psi_\beta\}, \qquad [\not{D}, \psi] = 0 \tag{1.48}$$

where $\not{D} \doteq \Gamma^\mu D_\mu$. Hence, there are 8 on-shell bosonic degrees of freedom, and 8 fermionic ones after imposing the second equation in (1.48). Note that we can absorb the gauge coupling $g_{\rm YM}$ by a field redefinition $g_{\rm YM} A_\mu$ and $g_{\rm YM}\psi$ and thence by $D_\mu \doteq \partial_\mu + i g_{\rm YM} A_\mu$.

Finally, the supersymmetry of (1.46) can be proven by using properties of bilinears of Majorana fermions in $10D$, which can be found in Appendix A.5, together with the Bianchi identity $[D_{[\mu}, F_{\nu\rho]}] = 0$ and and the Fierz identity valid in $10D$ (which actually also holds in $D = 3, 4, 6$):

$$(\overline{\psi}^{[a} \Gamma^\mu \psi^b)(\Gamma_\mu \psi^{c]})_\alpha = 0 \, . \tag{1.49}$$

The proof of this identity can be found in Appendix A.6.

After this brief review on 10-dimensional SYM, we are now able to show that the non-abelian version of the low-energy action for Dp-branes (1.44) is related to the reduction to $(p + 1)$ dimensions of the original $10D$ SYM theory (1.46). First, we assume that all fields in expression (1.46) are independent of the compact coordinates x^a, $a = p + 1, .., 9$, depending only on the brane world-volume coordinates y^α, $\alpha = 0, .., p$. Then, the action of iD_a reduces to $-A_a$, which we write (after rescaling) as $X_a/(2\pi\alpha')$. These are precisely the expected $(p + 1)$ transverse scalars of the theory (1.44). Moreover, the field strength now splits into purely transverse components: $F_{ab} = i[X_a, X_b]/(2\pi\alpha')^2$ at the origin of the scalar potential, into mixed ones: $F_{\alpha a} = -[\mathcal{D}_\alpha, X_a]/(2\pi\alpha')$, giving rise to kinetic terms for the transverse scalars. In addition, the components $F_{\alpha\beta}$ now constitute the gauge curvature on the brane (note that we now use $\mathcal{D}$ to denote the gauge connection on the D-brane world-volume). We will see later on how this correspondence is related to a T-duality of the theory performed on $(p+1)$ of the original dimensions.

Dimensional reduction of the action (1.46) then yields

$$S = \frac{1}{g_p^2}\int d^{p+1}y\,\mathrm{Tr}\left(-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2(2\pi\alpha')^2}[\mathcal{D}_\alpha, X^a][\mathcal{D}^\alpha, X_a] + \frac{1}{4(2\pi\alpha')^4}([X_a, X_b])^2 \right.$$
$$\left. + \frac{i}{2}\overline{\psi}\Gamma^\alpha[\mathcal{D}_\alpha, \psi] + \frac{1}{2(2\pi\alpha')}\overline{\psi}\Gamma^a[X_a, \psi]\right)\,, \tag{1.50}$$

where we recongnise the non-abelian version of expression (1.44), whose coupling can both be expressed in terms of the original $10D$ SYM gauge constant divided by the volume of the compact space $\mathcal{V}_{9-p}^{\mathrm{cpct}} \doteq (2\pi)^{9-p}\prod_{i=p+1}^{9} R_i$ or in string units as in equation (1.45)

$$g_p^2 = g_{\mathrm{YM}}^2(\mathcal{V}_{9-p}^{\mathrm{cpct}})^{-1} = g_s(\alpha')^{-\frac{1}{2}}(2\pi\sqrt{\alpha'})^{p-2}\,.$$

Moreover, it is worth noting that the potential $([X_a, X_b])^2$ is indeed negative definite, since $[X_a, X_b]^\dagger = -[X_a, X_b]$. So both the kinetic and potential terms in expression (1.50) come out with the expected sign.

Finally, finding a classical vacuum of the theory amounts to looking for a static solution of the equations of motion where the the potential energy of the system is minimized. This is the case when $F_{\alpha\beta} = 0$ and $\psi = 0$ and all transverse fields are covariantly constant and commute with one another: $[\mathcal{D}_\alpha, X_b(y)] = 0 = [X_a(y), X_b(y)]\ \forall y \in \Sigma_{p+1}$. This occurs for a solution of the type

$$X^a = \mathrm{diag}(x_1^a, .., x_N^a)\,, \qquad x_i^a \in \mathbb{R} \quad \forall i = 1, .., N$$

and defines the moduli space of classical vacua

$$\mathbb{R}^{(9-p)N}/S_N\,,$$

where we have to quotient by the permutation group of N objects, since the branes are regarded as indistinguishable.

1.2.2 D-branes and T-duality

Type II string theories are endowed with two remarkable duality symmetries, T- and S-duality (the latter peculiar to type IIB), which carry through the low-energy limit, down to the effective SYM theory. T-duality essentially sends type IIA theory compactified on a circle of radius say R_9 to IIB on the dual circle, of radius $\hat{R}_9 = \alpha'/R_9$. Since this procedure exchanges Neumann with Dirichlet boundary conditions on the compactified target-space field X^9, this will turn Dp-branes in the original theory to D$(p \mp 1)$-branes in the dual theory, depending on wether the original brane is wrapped or not along the compact direction (T-duality wrapping in the dual theory what was unwrapped in the original one, and vice-versa).

This feature can be proven in a precise fashion using the properties of the low-energy description of the Dp-brane action, to wit, the corresponding super Yang-Mills theory. We will start, for simplicity, with the action for N D0-branes in flat space, which is encoded in the following matrix quantum dynamic:

$$\begin{aligned} S_{\rm D0} &= \frac{1}{2g_s\sqrt{\alpha'}}\int dt\,{\rm Tr}\Big(\sum_{a=1}^{9}[\mathcal{D}_t,X^a][\mathcal{D}_t,X_a]+\frac{1}{(2\pi\alpha')^2}\sum_{a<b=1}^{9}([X^a,X^b])^2 \\ &\qquad -i(2\pi\alpha')^2\psi^\dagger[\mathcal{D}_t,\psi]+2\pi\alpha'\overline{\psi}\sum_{a=1}^{9}\Gamma^a[X_a,\psi]\Big)\,, \end{aligned} \tag{1.51}$$

and will compactify it along the X^9 direction. In order to implement periodicity of $X^9(t)$ without increasing the number of degrees of freedom of the system, a cunning way of doing has been resorted to by [104] along the line of orbifold compactifications. In general, if one wishes to describe the dynamics of N objects on $\mathbb{R}^9/\Gamma$, where Γ is the discrete group of the orbifold, one may simply take $|\Gamma|\times N$ copy of the original object and impose that all configurations thereof be invariant under Γ. In our case, it amounts to identifing the compact direction $S^1=\mathbb{R}/\mathbb{Z}$ and considering the dynamics of $\mathbb{Z}\times N$ D0-branes on the convering space $\mathbb{R}$. This translates into a $U(\infty)$ quantum mechanics where the original $N\times N$ $(X^a)_i^{\ j}$ matrices now take $m,\,n\in\mathbb{Z}$ additional indices referring to the mth and nth copy of the i and j branes between which a string is stretching in the ath direction. We write the matrix elements of X^a as $X^a_{mi,nj}\equiv(X^a_{mn})_i^{\ j}$ in order to treat X^a_{mn} as $N\times N$ matrices. Now, translational invariance under $\Gamma=\mathbb{Z}$ implies, for X^9 periodic, the following set of conditions:

$$\begin{aligned} (X^a_{mn})_i^{\ j} &= (X^a_{(m-1)(n-1)})_i^{\ j}\,, \quad a=0,..,8 \\ (X^9_{mn})_i^{\ j} &= (X^9_{(m-1)(n-1)})_i^{\ j}\,, \quad m\neq n\,, \qquad (X^9_{nn})_i^{\ j}=2\pi R_9\delta_i^{\ j}+(X^9_{(n-1)(n-1)})_i^{\ j}\,. \end{aligned} \tag{1.52}$$

A solution to this system of constraints is given by

$$X^9_{00}=X_0\,,\qquad X^9_{nn}=X_0+2\pi nR_9\mathbb{1}\,,\qquad X^9_{nm}=X_{n-m}\,, \tag{1.53}$$

for $N\times N$ matrices.

Going to the Fourier space on the dual circle, where a scalar field f is decomposes as $f(\hat{y}_9)=\sum_{n=-\infty}^{\infty}\tilde{f}_n e^{in\hat{y}_9/\hat{R}_9}$, we notice that the action of X^9 on such a state is reproduced by the operator

$$X^9(\hat{y}_9)=2\pi\alpha' i\widehat{\mathcal{D}}_9=2\pi\alpha'(i\hat{\partial}_9-A_9(\hat{y}_9))\,.$$

In fact, $2\pi\alpha' i\partial_{\hat{9}}$ acts as $2\pi nR_9\delta_{nm}$ for $n,m\in\mathbb{Z}$, yielding the second half of the diagonal part of (1.53), on the Fourier components of

$$f(\hat{y}_9)=\sum_{n=-\infty}^{\infty}\tilde{f}_n\,\hat{e}_n=\begin{pmatrix}\vdots\\ \tilde{f}_2\\ \tilde{f}_1\\ \tilde{f}_0\\ \tilde{f}_{-1}\\ \tilde{f}_{-2}\\ \vdots\end{pmatrix}\,, \tag{1.54}$$

(writing $\hat{e}_n\doteq e^{in\hat{y}_9/R\hat{R}_9}$ as a base such that $\hat{e}_n\cdot\hat{e}_m=\hat{e}_{n+m}$). Since $-2\pi\alpha' A_9(\hat{y}_9)=-2\pi\alpha'\sum_{n=-\infty}^{\infty}\tilde{A}_n\,\hat{e}_n$, the expression $-2\pi\alpha' A_9(\hat{y}_9)f(\hat{y}_9)$ produces the action of the remaining part of matrix (1.53) on vector (1.54), after setting $A_n\doteqdot X_n$. As an illustration, the $\hat{e}_1$ component of (1.54) becomes

$$-2\pi\alpha'\sum_{n=-\infty}^{\infty}(X_{n+1}\tilde{f}_{-n}+X_n\tilde{f}_{1-n})\,,$$

as expected.

Pushing the demonstration a bit further, one can now precisely show the duality between the low-energy description of N D0-branes on $\mathbb{R}^8 \times S^1$ and the low-energy description of N D1-branes on $\mathbb{R}^8 \times \hat{S}^1$ in the static gauge, since terms subject to T-duality in the D0-brane action are mapped as follows to corresponding terms in the D1-brane matrix model, whose Fourier modes are the winding modes of the original D0-branes

$$\begin{aligned} \int dt\, \mathrm{Tr}([X_a, X_b]^2) &\longrightarrow \frac{1}{2\pi \hat{R}_9} \int dt\, d\hat{y}^9\, \mathrm{Tr}([X_a, X_b]^2)\,, \\ \int dt\, \mathrm{Tr}([X_9, X_b]^2) &\longrightarrow -\frac{(2\pi\alpha')^2}{\hat{R}_9} \int dt\, d\hat{y}^9\, \mathrm{Tr}([\widehat{\mathcal{D}}_9, X_b]^2)\,, \\ \int dt\, \mathrm{Tr}([\mathcal{D}_t, X_9]^2) &\longrightarrow \frac{(2\pi\alpha')^2}{\hat{R}_9} \int dt\, d\hat{y}^9\, \mathrm{Tr}(F_{09}^2)\,. \end{aligned} \tag{1.55}$$

The process in (1.55) is the reverse of the compactification procedure in (1.50). In this respect, the $2\pi\hat{R}_9$ factors in (1.55) exactly cancels the corresponding factor of $2\pi R_9$ in (1.50) after applying T-duality: $R_9 \longrightarrow \alpha'/R_9$, and thus the coupling constant $g_0^2 \longrightarrow \hat{g}_1^2$ as expected.

This correspondance can now easily be extended to a system of unwrapped N Dp-branes on $\mathbb{R}^{9-(p+d)} \times T^d$ which become dual to N D$(p+d)$-branes wrapped on $\mathbb{R}^{9-(p+d)} \times \hat{T}^d$, after sending $R_i \longrightarrow \alpha'/R_i$, $i = 9-(d-1), .., 9$. In this case, the only additional term complementing the list (1.55) is

$$\int d^{p+1} y\, \mathrm{Tr}([X_a, X_b]^2) \longrightarrow -\frac{(\alpha')^4}{(2\pi)^{d-4} \prod_{i=9-(d-1)}^{9} R_i} \int d^{p+1} y\, d^d\, \hat{y}\, \mathrm{Tr}(F_{ab}^2)\,.$$

Finally, one can generalise the above orbifold construction (1.52) to dual field configurations with non-trivial boundary conditions. These new sectors can actually be viewed as connections on bundles with twisted boundary conditions. Going back to the D0/D1-brane duality example, one can implement such non-trivial boundary conditions by requiring that the formal translation operator $U = \exp[2\pi i R_9 \hat{y}^{\,9}]$ producing the conditions (1.52) when conjugating the X^a states, generates these conditions up to a gauge transformation $\Omega \in U(N)$. Then, condition (1.52) becomes

$$U \cdot X^a(\hat{y}^{\,9}) \cdot U^{-1} = \Omega \cdot (X^a(\hat{y}^{\,9}) + 2\pi R_9 \delta^{a,9} \mathbb{1}) \cdot \Omega^{-1},$$

which is solved for $X^9(\hat{y}^{\,9}) = 2\pi\alpha' i \hat{\mathcal{D}}_9$ and $U = \Omega \cdot \exp[2\pi i R_9 \hat{y}^{\,9}]$. Concretely, the operator Ω could act as an element of the permutation group switching the D-branes indices, or, when compactifying several dimensions, as a parameter rendering the gauge theory on the dual torus non-commutative.

1.3 The BFSS Conjecture

Before stating the BFSS conjecture, we will start by reviewing some basic facts about the reduction of M-theory to IIA string theory, which should put into a clearer perspective some arguments of the conjecture itself.

1.3.1 M-theory and IIA string theory

Before being extended to the evasive non-perturbative limit of all known string theories, M-theory was in the beginning defined by Witten [247] as a theory in $11D$ possessing eleven-dimensional supergravity as its-low energy limit, which can then be defined as the strong-coupling limit of type IIA string theory when compactified on a circle (which will be taken in the eleventh dimension x^{10}). The radius of this circle is given by $R_{\rm IIA} = \sqrt{\alpha'} g_s$ in string units so that when $R_{\rm IIA} \to 0$ we recover a $10D$ weakly-coupled string theory.

Let us now turn for a moment to $11D$ supergravity, interpreted as the strongly-coupled low-energy classical limit of M-theory. The eleven-dimensional supergravity multiplet [81] contains the following massless fields, labelled by indices $A, B = 0, .., 10$: the symetric traceless tensor g_{AB}, or alternatively an elevenbein e_a^A (with tangent-space index $a = 0, .., 10$), a three form field $C_{(3)}$ and a Majorana gravitino ψ_A. Retaining only the physical degrees of freedom of the theory amounts to computing the rank of tensors in $D-2$ dimensions, which yields the counting: for g_{AB}, $9 \times 10/2 - 1 = 44$ dof, for $C_{(3)}$, $9 \times 8 \times 7/3! = 84$ dof, and for ψ_A, $2^{[11/2]}/2 \times (9-1) = 128$ dof (projecting out spin 1/2 components). This builds up a supermultiplet of 256 components.

Besides, the low-energy limit of type IIA string theory consists of the following bosonic massless sectors:

- an NS-NS sector with: a 10-dimensional metric $g_{\mu\nu}$, a dilaton ϕ and a two-form $B_{(2)}$. The latter can be related, by Hodge-duality of their respective field strength, to a 6-form $\tilde{B}_{(6)}$, which couples electrically to the world-volume of the NS5-brane, the magnetic dual of the fundamental string.

- a R-R-sector with: a 1-form $A_{(1)}$ and a threeform $A_{(3)}$. Such R-R potentials couple electrically to the D0- and D2-branes, and magnetically to the D4- and D6-branes (since Dp-branes are BPS objects, they necessarily carry a conserved quantity, which is the R-R charge). On should also include a 9-form potential, even though it is not associated with a propagating state, since a D8-brane appears in the vertex operator analysis of type IIA theory.

The correspondence between M-theory objects and the field content of type IIA supergravity relates the zero modes of the Fourier expansion of g_{AB}, $C_{(3)}$ and ψ_A on the M-theory circle S^1 to the NS-NS and R-R sectors mentioned above: the zero mode of the metric $g^{(0)}_{10\,10}$ is thus associated to the type IIA dilaton ϕ, whereas the components $g^{(0)}_{\mu\nu}$ yield the ten-dimensional metric, and the $C^{(0)}_{\mu\nu\,10}$ are related to the IIA B-field $B_{(2)}$. So much for the NS-NS sector. As for the R-R form-fields: the Kaluza-Klein photon $g^{(0)}_{\mu\,10}$ gives the type IIA one-form A_μ, while the $C^{(0)}_{\mu\nu\rho}$ components correspond to the type IIA three-form $A_{(3)}$.

As the R-R potentials of type II theories couple to the world-volume of Dp-branes, one expects the eleven-dimensional three-form potential $C_{(3)}$ and its dual $\widetilde{C}_{(6)}$ to be sourced by an (electric) M2-brane and a (magnetic) M5-brane respectively. Such BPS objects have indeed been shown to represent vacua of the eleven dimensional theory.

One can then rephrase the correspondence given above as follows: the $11D$ supergraviton with momentum $p_{10} = R_{\rm IIA}^{-1}$ is related to the type IIA D0-brane, an M2-brane wrapped around the compact dimension to the type IIA F1-brane (or fundamental string), the unwrapped M2-brane to the type IIA D2-brane, while the M5-brane, wrapped and unwrapped, is associated respectively to the D4-brane and the NS5-brane. Obtaining the D6- and D8-brane is slightly more complicated: since the D6-brane is the magnetic dual of the D0-brane, it is expected to descend from a Kaluza-Klein monopole in M-theory. As for the D8-brane, it seems to be somehow related to the ten-dimensional boundary hyperplanes ("M9"-branes) appearing, for instance $E_8 \times E_8$ heterotic supergravity, but the M-theory origine of the D8-brane is still an open problem.

Regarding the fermionic sector, the 32-component gravitino ψ_A of M-theory splits, in type IIA theory, into a pair 16-components Majorana-Weyl gravitinos ψ^a_μ, $a = 1, 2$, of opposite chirality and, when $A = 10$, a pair of 16-components Majorana-Weyl spinors ψ^a_{10}, $a = 1, 2$.

Now, besides the supergraviton with momentum $P_{10} = 1/R_{\rm IIA}$ yielding the IIA D0-brane, there are, in M-theory, Kaluza-Klein modes with momenta: $P_{10} = N/R$, $N \in \mathbb{Z}$. Each each of these states forms, as the fundamental supergraviton, a (short) supermultiplet of $2^8 = 256$ states. However, for $N \neq \pm 1$, such KK modes are massive, being the only contribution to the 11-dimensional mass, and can be interpreted as bound states at threshold of N D0-branes or of N $\overline{\rm D0}$-branes. Since the compactification radius is given by $R_{\rm IIA} = g_s\sqrt{\alpha'}$, when g_s is large (non-perturbative regime), these Kaluza-Klein modes become light BPS objects, and are thus low-energy states of strongly coupled type IIA string theory, which are precisely described by $U(N)$ super Yang-Mills theory reduced to $0+1$ dimensions, in other words a supersymmetric quantum mechanics for $N \times N$ hermitian matrix . Moreover, as $g_s \to \infty$, $R_{\rm IIA}$ is decompactified, and we obtain 11-dimensional supergravity as the expected low-energy limit of strongly-coupled type IIA theory.

For simplicity, and when there is no ambiguity, we will denote the radius $R_{\rm IIA}$ as R in the following.

1.3.2 The BFSS matrix model

We have recalled in the preceding section how an eleventh non-compact dimension appears in the strong-coupling limit of type IIA string theory and how the Kaluza-Klein modes of the eleven-dimensional graviton reproduces in this case N D0-branes states bound at threshold, described by supersymmetric quantum mechanics for $N \times N$ hermitian matrix. With this in hand, we can now discuss the BFSS conjecture [231], which states that in the limit where $N \to \infty$, M-theory expressed in the Infinite Momentum frame (or IMF) is exactly described by such a non-relativistic theory for a system of many D0-branes in type IIA string theory.

The particular Lorentz frame in which one has to consider M-theory in this case, the IMF, makes it possible to relate a quantum mechanics for nine-dimensional target space (matrix) fields (the X^i, i=1,..,9 in the action (1.51)) to a eleven-dimensional low-energy theory such as $11D$ supergravity. Indeed, if one consider such a matrix model as living in 9 dimensions transverse to the eleventh dimension of M-theory in the IMF, or in the light-cone frame (we will see later that there is an equivalent formulation of this conjecture on the light front), this matrix model still knows of the 11-dimensional Lorentz invariance of the full system, as long as we can interpret it in terms of a transverse Hamiltonian $H_\perp$ related to the eleven-dimensional hamiltonian by $H_{(11)} = (P^+)^{-1} H_\perp$ (cf. the Hamiltonian (1.17))

The conjecture can then be stated as:

Conjecture 1.3.1 *The $N \to \infty$ limit of the D0-branes quantum mechanics*

$$S_{\rm BFSS} = \frac{1}{2R}\int dt \sum_{i=1}^{9} Tr\left([\mathcal{D}_t, X^i]^2 + \sum_{j=i+1}^{9}([X_i, X_j])^2 - i\theta^\top[\mathcal{D}_t, \theta] + \bar{\theta}\Gamma^i[X_i, \theta]\right), \qquad (1.56)$$

where the bosonic fields X^i, $i = 1,..,9$, and 16-components $SO(1,9)$ Majorana-Weyl spinors θ are $N \times N$ hermitian matrices in the adjoint representation of $U(N)$, exactly describes M-theory in the IMF, when both $R \to \infty$ and $N/R \to \infty$, where $P_{10} = N/R$ is the longitudinal momentum for a bound state of N D0-branes.

In expression (1.56), we have taken advantage of the Majorana condition to work with real Weyl spinors, and replace $\theta^\dagger \to \theta^\top$. Moreover, we have this time chosen a convention where $2\pi\alpha' = 1$. Since the compactification radius $R = g_s\sqrt{\alpha'}$ in string units, the relation to expression (1.51) is then obvious. In addition, since the Planck length is $l_P = \sqrt{\alpha'}g_s^{1/3}$ in string units, we can compare with the regularised supermembrane Hamiltonian (1.39). To relate it to expression (1.56), we consider its corresponding Lagrangian, which goes as $\mathcal{L}_{\mathcal{M}} = \frac{1}{4\pi l_P^3}\big(([\mathcal{D}_\tau, X^i])^2 + \frac{1}{2}([X^i, X^j])^2 \dots\big)$. Since:

$$\frac{1}{4\pi l_P^3} = \frac{1}{(2\pi\alpha')(2g_s\sqrt{\alpha'})} = \frac{1}{(2\pi\alpha')(2R)}\,,$$

the supersymmetric membrane action is then a $(2\pi\alpha')^{-1}$ rescaled version of the BFSS action, and they are the same precisely when $(2\pi\alpha') = 1$.

Varying expression (1.56), we get the following equations of motion:

$$\begin{aligned} [\mathcal{D}_t, [\mathcal{D}_t, X^i]] &= -[X^k, [X_k, X^i]] - \frac{1}{2}(\Gamma^i)_\alpha{}^\beta\{\overline{\theta}^\alpha, \theta_\beta\}\,, \\ [\mathcal{D}_t, \theta] &= -i\Gamma^{0k}[X_k, \theta]\,. \end{aligned} \tag{1.57}$$

Finally, extremising with respect to the gauge field A_0 yields the constraint

$$\mathcal{R} = [X^k, [\mathcal{D}_t, X_k]] + \frac{i}{2}\{\theta^\dagger, \theta\} = 0\,. \tag{1.58}$$

Thus, the supersymmetric extention of the Gauss Law constraint (1.28) now naturally appears upon covariantisation of the time derivative, the gauge field acting in this perspective as a Lagrange multiplier which implements the constraint. This mirrors exactly the end of the discussion on the regularised supermembrane Hamiltonian (1.39)

From Legendre transforming the Lagrangian (1.56) according to $H = \mathrm{Tr}(P_i\dot{X}_i - \Pi\dot{\theta}) - L$, one obtains the BFSS Hamiltonian

$$\begin{aligned} H_{\mathrm{BFSS}} &= R\sum_{i=1}^{9}\mathrm{Tr}\Big(\frac{1}{2}P_iP^i + \frac{i}{R}\Pi\Gamma^{0i}[X_i, \theta] - \frac{1}{2R^2}\sum_{j=i+1}^{9}[X_i, X_j][X^i, X^j] \\ &\qquad + \frac{i}{R}A_0\big([P_i, X^i] - \{\Pi, \theta\}\big)\Big) \end{aligned} \tag{1.59}$$

with canonical bosonic and fermionic momenta given by

$$P_i = \frac{1}{R}[\mathcal{D}_t, X_i]\,, \qquad \Pi = \frac{i}{2R}\theta^\dagger\,.$$

The last term in expression (1.59) is proportional to the Hamiltonian equivalent of the Gauss constraint $\mathcal{R}$ (1.58).

Preliminary discussion of the conjecture

We will be going into the details of this conjecture in the coming sections, where we will discuss the Infinite Momentum Frame, the fact that the BFSS model is a second quantised theory, the finite N interpretation of the model, etc. . But before, we can already point out a few important arguments in favour of this conjecture.

First, the IMF is a frame where we have performed an infinite boost in particular direction. In our case we choose this longitudinal direction to be x^{10}. As a consequence, all states with P_{10} negative or vanishing have infinite energy and decouple from the theory (see Section 1.3.3 for details). In type IIA theory, states that can carry non-vanishing momentum P_{10} are branes charged under the Ramond-Ramond potential $A_{(1)}$, in other words D0- or $\overline{\text{D0}}$-branes. All other states of the theory such as the F1 and the NS5-brane, which carry no R-R charge, and the D2-brane, which couples to the higher rank $A_{(3)}$ potential, thus have zero P_{10} momentum, and can be integrated out. We are thus left with bound states of N D0-branes carrying momentum $P_{10} = N/R > 0$. In contrast, $\overline{\text{D0}}$-brane states carry negative momentum $P_{10} = -N/R$ and are boosted to infinity in the IMF. However the system still nows somehow of their existence, since eleven-dimensional Lorentz invariance is latent in the resulting light-cone theory. To summarise, at large momentum string theory can be described in terms of elementary partonic degrees of freedom, i.e. D0-branes with one unit of positive transverse momentum $P_{10} = 1/R$.

Second, we now have a non-relativistic low-energy theory with super-Galilean invariance in the dimensions transverse to x^{10}.

Third, the BFSS action (1.56) reproduces the regularised supermembrane action (1.39), even though from a different approach. Thus the quantum light-front supermembrane naturally emerges as a coherent state in the BFSS theory.

Finally, the leading long range interaction between two quanta of D0-branes can computed from the BFSS matrix model [103] and exactly reproduces the two-graviton scattering amplitude known from light-front $11D$ supergravity $V(r) = -\frac{15}{16}\frac{v^4}{r^7}$, where v is the relative velocity of the gravitons.

1.3.3 The Infinite Momentum Frame

In this section, we will come back on the details of how a physical system behaves in the Infinite Momentum Frame, extending it to correspondence stated in the Conjecture 1.3.1. The IMF is a frame in which the physics has been heavily boosted in a preferred direction. It was introduced long ago by [243] to simplify perturbation theory, since resorting to the IMF highly supresses all diagrams where vertices are created out of the vacuum.

We start with a reference frame where the total momentum $\vec{P}_{\text{tot}}$ is very large. Then the individual momenta of the particles read:

$$\vec{P}_n = \zeta_n \vec{P}_{\text{tot}} + \vec{P}_n^{\perp} ,$$

with the following poperties: $\vec{P}_n^{\perp} \cdot \vec{P}_{\text{tot}} = 0$, $\sum_n \vec{P}_n^{\perp} = 0$ and $\sum_n \zeta_n = 1$, the first of these three conditions holding only when $n \geqslant 2$. Now, if the system is sufficiently boosted in the $\vec{P}_{\text{tot}}$ direction, all ζ_n become positive, and further boosting will only increase momenta in the $\vec{P}_{\text{tot}}$ direction without affecting the signs of the ζ_n.

Such a system can be compared to the corresponding 11-dimensional system in the light-cone frame, for

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^{10})$$

by identifying: $\vec{P}_n^{\perp} \doteqdot (0, P_n^1, .., P_n^9, 0)$ and $\zeta_n \vec{P}_{\text{tot}} \doteqdot (0, .., 0, P_n^{10})$.

Hence,

$$P_n^{\pm} = c^{-1} E_n \pm \zeta_n |\vec{P}_{\text{tot}}|$$

and since the conservation law for the energy-momentum tensor becomes: $M_n^2 = P_n^+ P_n^- - |\vec{P}_n^{\perp}|^2$, the energy of each particle reads in the light-cone frame:

$$c^{-1} E_n = P_n^{10} + \frac{|\vec{P}_n^{\perp}|^2 + M_n^2}{P_n^+} . \tag{1.60}$$

Returning, for comparaison to the IMF, we can expand the energy as

$$c^{-1} E_n = \Big[|\vec{P}_n|^2 + M_n^2 \Big]^{1/2} = P_n^{10} + \frac{|\vec{P}_n^{\perp}|^2 + M_n^2}{2 P_n^{10}} + \mathcal{O}((P_n^{10})^{-2}) . \tag{1.61}$$

Noticing that for $P_n^{10} \to \infty$, we have $c^{-1} E_n \simeq P_n^{10}$, and hence $P_n^+ \simeq 2 P_n^{10}$, we come to the conclusion that, in this limit, expressions (1.60) and (1.61) match. Furthermore, going back to QFT, we now have a clearer insight into the triviality of the IMF vacuum: in a theory with energy denominators, diagrams which carry a large internal momentum $\vec{P}_{\text{tot}}$ with ζ_n negative, or alternatively, with large negative P_n^{10}, will be suppressed by a factor of $1/P_n^{10}$. These diagrams precisely correspond to particles created from the vacuum.

Considering the eleven-dimensional theory on a circle: $x^{10} \in S^1$, this involves the transverse momentum P_{10} to quantised:

$$x^{10} \sim x^{10} + 2\pi R \longrightarrow P_{10} = \frac{N}{R}\,.$$

Implementing the IMF implies boosting the system to $P_{10} \to \infty$, but since taking the decompactification limit forces $R \to \infty$, N has to tend to infinity quicker than R, in order to satisfy $N/R \to \infty$. This is the origin of the peculiar limit appearing in the Conjecture 1.3.1. Now, boosting a theory with a compact direction is delicate, since a boost in the compactified direction is not a symmetry of the previously Lorentz invariant theory. This should however not affect the calculation at hand, which deal with the IMF as a way of calculating interaction between states with large longitudinal momenta.

Since there are no 11-dimensional masses coming into play, the energy (1.60) and (1.61), which coincide in the decompactification limit, becomes

$$c^{-1}E_{\text{tot}} = P_{\text{tot}}^{10} + \sum_n \frac{|\vec{P}_\perp^n|^2}{2P_n^{10}} \equiv \frac{P_{\text{tot}}^+}{2} + \sum_n \frac{|\vec{P}_\perp^n|^2}{P_n^+}$$

where Galilean invariance is now manifest. Note that after introducing the supercharges required by the supersymmetry of (1.56), this is enhanced to super-Galilean invariance.

1.3.4 M(atrix) theory: a proposal for DLCQ M-theory

The original formulation of the BFSS conjecture requires taking the limit $N \to \infty$ on the matrix model side of the correpondence. However, this limit can be very delicate to compute concretely. In particular, there is often no clear prescription of how to take it when doing perturbative loop calculations in M(atrix) theory. Furthermore, most of the checks of the validity of the BFSS conjecture have shown that the correspondence between M-theory in the IMF and a supersymmetric matrix model for N D0-branes holds true already at finite N. This has prompted Susskind to investigate the finite N matrix model theory, arguing that such a supersymmetric quantum mechanis gives in fact a description of the discrete light-front quantised sector of M-theory, or DLCQ M-theory, containing states of momentum $P^+ = N/R$ [228]. This relation was made more precise by Seiberg and Sen in [218, 221], where they showed in particular how DLCQ M-theory is related to perturbative type IIA string theory and gave evidence that a low-energy action for D0-branes is the appropriate description for M-theory in the IMF, thereby giving credit to the BFSS conjecture.

Concretely, this finite N correspondence can be established by showing that DLCQ M-theory is the natural framework yielding spacelike compactifications of type IIA string theory where both the string coupling and the string length vanish. Thus, type IIA string theory is in this limit perturbative since weakly coupled; and second, the massive string modes decouple, leaving only D0-branes as fundamental *partonic* objects of the theory.

The starting point of the DLCQ approach takes advantage of the fact that one can view a theory compactified on a lightlike circle as a theory compatified on a spacelike circle in the limit where the latter vanishes. In the following, we will show how the light-front compactification of M-theory can be described by such a limiting process, in the form of a matrix theory encoding the low-energy behaviour of many D0-branes, as shown in [218, 221].

Starting from a lightlike compactification in two dimensions with radius R, we can deform it to a family of spacelike (near lightlike) compactifications by introducing a new radius $\widehat{R}$ with identification:

$$\begin{pmatrix} y \\ ct \end{pmatrix} \sim \begin{pmatrix} y-\sqrt{2}\pi R \\ ct+\sqrt{2}\pi R \end{pmatrix} \overset{\widehat{R}\to 0}{\longleftarrow} \begin{pmatrix} y \\ ct \end{pmatrix} \sim \begin{pmatrix} y-2\pi\sqrt{R^2/2+\widehat{R}^2} \\ ct+\sqrt{2}\pi R \end{pmatrix}. \qquad (1.62)$$

Defining the boost parameter

$$\beta^{-1} = \sqrt{1+2(\widehat{R}/R)^2}\,,$$

it can be shown that the family of space-like compactifications on the left of expression (1.62) can be obtained through the boost

$$\begin{pmatrix} \hat{y} \\ c\hat{t} \end{pmatrix} = \frac{1}{\sqrt{1-\beta^2}} \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} y \\ ct \end{pmatrix},$$

from a compactification with identification

$$\begin{pmatrix} \hat{y} \\ c\hat{t} \end{pmatrix} \sim \begin{pmatrix} \hat{y}-2\pi\widehat{R} \\ c\hat{t} \end{pmatrix}, \qquad (1.63)$$

since we have the (exact) relation:

$$\frac{\sqrt{1-\beta^2}}{\beta} = \sqrt{2}\frac{\widehat{R}}{R}\,.$$

Now identification (1.63) being a mere longitudinal compactification, it in turn relates to a regular type IIA compactification of M-theory with dictionary:

$$g_s = \left(\frac{\widehat{R}}{l_P}\right)^{3/2}, \qquad l_s = \left(\frac{l_P^3}{\widehat{R}}\right)^{1/2}, \qquad (1.64)$$

where l_P is the $11D$ Planck length. So, to summarise, every spacelike compactification of M-theory of the kind (1.62) is related through a boost to type IIA string theory with relationships (1.64). Now taking the $\widehat{R}\to 0$ limit to recover the original lightlike compactification, the string coupling g_s vanishes as desired, however the string scale $l_s^{-1} = (\alpha')^{-1/2}$ vanishes as well; in this limit, the massive states of string theory will not decouple and the higher order corrections from the DBI action (1.40) cannot be neglect, ruling out the SYM approximation for D0-brane dynamics.

The way out, however, is to concentrate on the energy of the particles of interest in this limit. A theory with lightlike compactification scheme (1.62) has quantised momentum in the compact direction $P^+ = N/R$ and light-front Hamiltonian $H = P^-$. Then for a spatial compactification (1.63), the mass energy of the D0-brane quanta is $\widehat{P}^+ = N/\widehat{R}$ and the total energy:

$$\widehat{E} = c\widehat{P}^+ + \Delta E\,.$$

Transforming back to the near lightlike compactification variables of (1.62)

$$\begin{pmatrix} P \\ c^{-1}E \end{pmatrix} = \frac{1}{\sqrt{1-\beta^2}} \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} \widehat{P} \\ c^{-1}\widehat{E} \end{pmatrix},$$

we see that the energy of interest is given by

$$\Delta E \simeq \frac{\widehat{R}}{R}\,cP^-\,, \qquad (1.65)$$

since $P^- = (c^{-1}E - P)/\sqrt{2}$ and then

$$P^- = \sqrt{\frac{1+\beta}{2(1-\beta)}}\frac{\Delta E}{c} \simeq \frac{R}{\widehat{R}}\frac{\Delta E}{c}\,,$$

and obviously $\Delta E \to 0$ in the limit $\widehat{R} \to 0$. Comparing to how fast the string scale l_s^{-1} vanishes in this limit, we notice, using (1.65) and (1.64), that

$$\frac{\Delta E}{l_s^{-1}} \simeq \sqrt{l_P^3 \widehat{R}}\frac{cP^-}{R} \to 0\,.$$

Thus, the lightlike compactification limit of a family of spacelike compactifications leads to an energy scale of interest for D0-branes which is much smaller than the string scale, although the latter vanishes still. Then, one may hope to find new units where both the new string coupling and the new string length go to zero, meeting all the required conditions for the correspondence to hold.

To do so, we define a new Planck length $\tilde{l}_P$ for which the new energy of interest $\Delta\widetilde{E}$ becomes independent of $\widehat{R}$. This can be achieved by setting

$$\tilde{l}_P^2\,\Delta\widetilde{E} = l_P^2\,\Delta E \longrightarrow \Delta\widetilde{E} = \left(\frac{l_P}{\tilde{l}_P}\right)^2 \frac{\widehat{R}}{R}\,cP^- \sim \text{finite} \longrightarrow \tilde{l}_P^{-2}\widehat{R} \sim \text{fixed}\,.$$

Since $\widehat{R} \to 0$, this requires $\tilde{l}_P \to \sqrt{0}$, leading to the new M-theory / IIA dictionary:

$$\tilde{g}_s = \left(\frac{\widehat{R}}{\tilde{l}_P}\right)^{3/2} \to 0, \qquad \tilde{l}_s = \left(\frac{\tilde{l}_P^3}{\widehat{R}}\right)^{1/2} \to 0, \qquad \text{for } \widehat{R} \to 0$$

Going back to the DBI for D0-branes, we see that the limiting process at work relies on the fact that even though the compactification lengths R and $\widehat{R}$ now scale like $\widehat{R}/R = (\tilde{l}_P/l_P)^2$, the transverse coordinate fields behave normally as $X^i/l_P = \widetilde{X}^i/\tilde{l}_P$, then, the approximation to the DBI non-abelian action for D0-branes becomes

$$\begin{aligned} &-\frac{V_0}{\tilde{g}_s\tilde{l}_s} + \frac{1}{2\tilde{g}_s\tilde{l}_s}\int dt\,Tr\left([\mathcal{D}_t, \widetilde{X}^i]^2 + \mathcal{O}\big((\partial_t\widetilde{X}^i)^4\big)\right) \\ &\qquad\Longrightarrow -\frac{V_0}{\widehat{R}} + \frac{1}{2R}\int dt\,Tr\left([\mathcal{D}_t, X^i]^2 + \mathcal{O}(\widehat{R}/R)\right), \qquad \text{for } \widehat{R} \to 0\,. \end{aligned}$$

The first contribution is proportional to the D0-brane energy, which we substract to compare our result with the energy P^- of light-cone compactified M-theory. We also expect higher order corrections from the nonabelian DBI action, collected indiscriminately in $\mathcal{O}(\widehat{R}/R)$, to become negligible in the limit $\widehat{R} \to 0$. The non-relativistic suprsymmetric matrix model (2.63) then gives a complete description of DLCQ M-theory, provided, of course, there exists such a theory endowed with necessary the properties of M-theory, and a well-defined type IIA theory which can be interpreted as its small radius compactification limit, so that that the arguments given above depend on these issues.

Furthermore, whether such a matrix theory really describes DLCQ supergravity in $11D$ in the low-energy limit is, a priori, far from being obvious, pertaining, in particular, to the delicate large N limit one has to perform on the matrix side.

Perturbative and non-perturbative evidence in favour of the well-definedness of this large N limit in connection with (almost) light-like compactifications of type IIA string theory can be found in [40, 41].

A note on interactions in M(atrix) theory

To conclude this section, we outline the comparaison between higher order loop corrections in M(atrix) theory and interactions in $11D$ supergravity.

Interactions in matrix theory involve block matrices which represent general time-dependent configurations combining gravitons, membranes and fivebranes. One can then use perturbative SYM theory in background field gauge to compute loop contributions in the matrix model and compare them to $11D$ supergravity interactions. For a finite N formulation of M(atrix) theory, the one loop results have been shown to generate an effective potential with an infinite series of terms which exactly reproduce, at tree level, supergravity interactions involving the exchange of a one graviton, a quantum of three-form field or a gravitino. One-loop calculations involving a pair of D0-branes can be found in [103], and for a pair of general membrane backgrounds see [157]

The two-loop results are more mixed. On the one hand, the leading two-loop contribution to three particle interaction in M(atrix) theory have been shown to correctly reproduce three-graviton scattering amplitudes in $11D$ supergravity, at tree-level [137]. This is based on the approach by Yoneya and Okawa [207, 206] Other results at two-loops can be found in [19, 20]. Beyond this, however, it is unclear whether higher loop results can still be used to recover more involved nonlinear effects in supergravity. One the other hand, already at the two-loop level in M(atrix) theory, one can try to compare next-to-leading order terms in the two-loop effective action, with the first quantum corrections to three graviton scattering amplitudes induced by $\mathcal{R}^4$ corrections to classical supergravity. In this case, [137] has shown that there is a clear disagrement between the tensorial structure of the M(atrix) theory two-loop computations and the M-theory corrections on the supergravity side. Other works [167, 21] seem to point at the fact that quantum effects in supergravity cannot be accurately recovered from M(atrix) theory loop calculations.

This discrepancy can however be accounted for by first noting that finite N M(atrix) theory describes M-theory on a lightlike circle, which is Lorentz equivalent to M-theory on a vanishing spacelike circle [218], and is perturbative in g_s, while $11D$ supergravity is a good description of M-theory in the large radius / large g_s limit. One then only expects an agreement between amplitudes which are protected when one goes from one regime to the other. This is the case for tree-level amplitudes involving two and three particles [208, 209], while the terms appearing in $\mathcal{R}^4$ corrections to three graviton scattering amplitudes are not subject to a non-renormalisation theorem. More on supersymmetric non-renormalisation theorems in M(atrix) theory can be found in [165, 166]. Thus, a matching between M(atrix) theory calculations and quantum corrections to supergravity can only be hoped to be reached (if it can) in the large N limit, and how to handle this limit is still an open question.

1.3.5 Continuous energy spectrum and second-quantisation in M(atrix) theory

As we have seen in the Section 1.3.2, the supermembrane emerges naturally from the BFSS theory, and appears as a coherent state in the supersymmetric matrix model (1.56). The fact that this matrix model gives rise to a second quantised theory solves in addition the problem of the continuous spectrum of supermembrane theory pointed out in Section 1.1.5.

Classical longitudinal gravitons can be constructed from DLCQ M(atrix) theory by finding a solution which minimises the potential in the action (1.56) and solves the classical equation of motion (1.57). A simple class of solution corresponding to an N-graviton state is given by commuting matrices with linear time dependence:

$$(X^i(t))_a{}^b = (x_a^i + v_a^i t)\delta_a{}^b\,, \quad i = 1,..9\,,\ \text{and}\ a,b = 1,..,N\,. \tag{1.66}$$

Each of the N gravitons composing this state carries a total longitudinal momentum $P^+ = 1/R$ and is massless. Making the gauge choice $A_0 = 0$ in the action (1.56), its light-front kinetic energy reads:

$$H = \frac{1}{2R}\sum_i \mathrm{Tr}(\dot{X}^i)^2 = \frac{1}{2R}\sum_{i,a}(v_a^i)^2 = \frac{1}{2P^+}\sum_i (p^i)^2$$

One can likewise construct a single classical graviton state with $P^+ = N/R$ by setting $x_a^i = x^i$ and $v_a^i = v^i$, $\forall a = 1,..,N$, in eqn.(1.66).

Including supersymmetry brings about some subtleties, so that when considering what we will call an N-supergraviton state, on now has to separate the matrix target space fields $X^i(t)$ into centre of mass and relative coordinates. This corresponds to factoring out the $U(1)$ in the matrices (1.66) according to $U(N) = SU(N) \times U(1)$:

$$X^i(t) = X^i_{\mathrm{rel}}(t) + \langle x^i(t)\rangle \mathbb{1}_{N\times N}\,, \qquad \langle x^i(t)\rangle = \frac{1}{N}\mathrm{Tr}X^i(t) \tag{1.67}$$

with $\mathrm{Tr}X^i_{\mathrm{rel}} = \mathrm{Tr}\dot{X}^i_{\mathrm{rel}} = 0$. The Hamiltonian separates this time in two parts

$$H = H_{\mathrm{rel}} + \frac{1}{2P^+}\sum_i \langle p^i\rangle^2 \tag{1.68}$$

with $\langle p^i\rangle = R^{-1}\langle \dot{x}^i\rangle$ and $P^+ = N/R$. H_{rel} is the BFSS Hamiltonian (1.59) restricted to $SU(N)$ matrices and can be shown to have zero-energy threshold bound states, [248, 222, 213], for which the energy is just the centre of mass energy $\frac{1}{2RN}\sum_i \langle \dot{x}^i\rangle^2$.

We now consider the fermionic sector of the theory. The 16 fermionic variables θ^α have anticommutation relations given by the following Dirac brackets:

$$\{\theta_\alpha, \theta^\beta\}_{\mathrm{DB}} = -iR(\mathcal{P}_+)_\alpha{}^\beta \mathbb{1}_{N\times N}\,,$$

(see eqn.(1.83) below). One can then combine them into 8 pairs of lowering an raising operators:

$$\theta^\pm_\alpha = \frac{i}{\sqrt{2}R}\,(\theta_\alpha \pm \theta_\alpha)$$

which generates the costumary fermion Fock space of dimension $2^8 = 256$. This is exactly the number of states required to form a supermultiplet, of $11D$ supergravity, as seen in Section (1.3.1). To sum up, for $N = 1$, we have a single D0-brane with $P^+ = 1/R$, described by a $U(1)$ matrix Hamiltonian with a spectrum of 256 states, which corresponds to a supergravity multiplet. This is precisely what is usually called a *supergraviton*. For $N > 1$, only states in the $SU(N)$ quantum mechanics theory are relevant, and each of them describes a supergravity multiplet of 256 states. These then correspond to a bound state at threshold.

Besides single supergraviton states, one can show that the Hamiltonian of M(atrix) theory has a Hilbert which actually contains multiple particle states. To see this, consider now the $N \times N$ (or infinite) matrices X^i to be block-diagonal:

$$X^i = \ldots \oplus X^i_{(k-2)} \oplus X^i_{(k-1)} \oplus X^i_{(k)} \oplus X^i_{(k+1)} \oplus X^i_{(k+2)} \oplus \ldots\,. \tag{1.69}$$

Each of these matrices will satisfy the equation of motion (1.57) separately (neglecting the fermions):

$$[\mathcal{D}_t, [\mathcal{D}_t, X^i_{(k)}]] = -[X^l_{(k)}, [X_{(k)l}, X^i_{(k)}]]\,.$$

As in eqn.(1.67), the submatrices $X^i_{(k)}$ can then be related to separate objects with center of mass $\langle x^i_{(k)}\rangle = \frac{1}{N_k}\mathrm{Tr}X^i_{(k)}$, so that the classical theory encodes, even for N finite, configurations with multiple independent objects. Repeating the analysis above, the Hamiltonian of such a configuration splits into a sum of uncoupled Hamiltonians $\sum_{k=1}^{N} H_{(k)}$, where each separate $H_{(k)}$ has a supergraviton in its spectrum. The quantum theory can then be constructed by replacing the classical matrix configuration dsecribing multiple independent supergravitons (1.69) by a wave function which approximates the tensor product of as many bound state wavefunctions, in the limit where the garvitons are taken to be far apart. Finally, letting N tend to infinity, we conclude that M(atrix) theory contains the full Fock space of supergravitons, and can then be viewed as a second-quantised theory with respect to its target space.

Since M(atrix) theory is endowed with a Fock-space allowing multi-supergraviton states, one can build in particular a two-body state by combining two localised ($\mathcal{H} \approx 0$) graviton states, which are far apart and have small relative velocities. This configuration can be chosen to have arbitrary small total energy. In the type IIA picture, where gravitons with one unit of longitudinal momentum $P^{10} = 1/R$ are associated to D0-branes, we may have in this case a continuous spectrum of energy, $\forall N > 1$. This gives an answer to the problem raised in Section 1.1.5.

Note that multiple objects states can also be seen directly in the continuous membrane theory by mapping a classical smooth membrane to a configuration in the target-space representing sphere-like objects connected by tubes. Taking the tubes to be long and thin makes their energy negligible, and we are left with a collection of independent spherical membranes.

1.3.6 The BFSS matrix model and its supersymmetry algebra

In this section we will discuss and verify the supersymmetry of the model (1.56). When going from M-theory to the BFSS M(atrix) model, we start in eleven dimensions with the 32 supercharges Q_A, $A = 1, .., 32$, that transform as a Majorana spinors of $SO(10,1)$. Then, in the IMF limit, the Q_A decompose into two groups of 16 supercharges, each transforming as a Majorana-Weyl spinor of $SO(1,9)$: call them $\widetilde{\mathsf{Q}}_\alpha$ and Q_α, $\alpha = 1, .., 16$. For reasons exposed in the next section, the first ones are called kinematical and the second dynamical. The infinitesimal supersymmetry transformations acting on a field F are then generated by these charges in the following fashion:

$$\delta_\epsilon F = i[\epsilon^\alpha \mathsf{Q}_\alpha, F]\,, \qquad \tilde{\delta}_\eta F = i[\eta^\alpha \widetilde{\mathsf{Q}}_\alpha, F]\,.$$

To be more specific, the *dynamical* susy transformation laws derive from $10D$ SYM theory, provided the required decomposition $A_\mu \to (A_0, X_i)$ as in eqn.(1.47)

$$\begin{aligned} \delta_\epsilon X^i &= \frac{i}{2}\bar{\epsilon}\Gamma^i\theta\,, \qquad \delta_\epsilon A_0 = -\frac{i}{2}\epsilon^\dagger\theta\,, \\ \delta_\epsilon\theta &= \left(\frac{1}{2}\Gamma^{0k}[\mathcal{D}_t, X_k] - \frac{i}{4}[X_i, X_j]\Gamma^{ij}\right)\epsilon\,. \end{aligned}$$

The *kinematical* ones, on the other hand, are peculiar to the BFSS matrix model, and affect only the fermions of the theory, shifting them by a constant valued spinor

$$\delta_\eta X^i = \delta_\eta A_0 = 0\,, \qquad \delta_\eta \theta = \eta \mathbb{1}\,.$$

Commutators of two such transformations yield constant spinor bilinears:

$$
\begin{aligned}
[\delta_{\epsilon^1},\delta_{\epsilon^2}]X^i &= -\frac{1}{2}\bar{\epsilon}^2\Gamma^\mu\epsilon^1[\delta_\mu{}^0 i\mathcal{D}_t+\delta_\mu{}^k X_k, X^i]\,,\\
[\delta_{\epsilon^1},\delta_{\epsilon^2}]A_0 &= -\frac{i}{2}\bar{\epsilon}^2\Gamma^k\epsilon^1[\mathcal{D}_t, X_k]\,,\\
[\delta_{\epsilon^1},\delta_{\epsilon^2}]\theta &= -\frac{1}{2}[\delta_\mu{}^0 i\mathcal{D}_t+\delta_\mu{}^j X_j,\bar{\theta}]\,\Gamma_k\epsilon^{[1}\Gamma^{\mu k}\epsilon^{2]} \\
[\delta_\epsilon,\tilde{\delta}_\eta]X^i &= -\frac{i}{2}\bar{\epsilon}\Gamma^i\eta\,,\qquad [\delta_\epsilon,\tilde{\delta}_\eta]A_0=\frac{i}{2}\epsilon^\dagger\eta\,,\qquad [\delta_\epsilon,\tilde{\delta}_\eta]\theta=0\,,\\
[\tilde{\delta}_\eta,\tilde{\delta}_{\eta'}](\text{any field}) &= 0
\end{aligned}
\tag{1.70}
$$

In order to show invariance of expression (1.56) under dynamical susy transformation, we start by listing each separate term of $\delta_\epsilon S_{\text{BFSS}}$

$$
\begin{aligned}
\delta_\epsilon \text{Tr}\big([\mathcal{D}_t,X^i]^2\big) &= \text{Tr}\big(\epsilon^\dagger[\theta,X^i][\mathcal{D}_t,X_i]\big)+i\text{Tr}\big(\bar{\epsilon}\Gamma^i[\mathcal{D}_t,\theta][\mathcal{D}_t,X_i]\big)=A_1+B_1\,,\\
\delta_\epsilon \text{Tr}\big(\theta^\dagger[\mathcal{D}_t,\theta]\big) &= \text{Tr}\big([\mathcal{D}_t,X_i][\mathcal{D}_t,\bar{\epsilon}\Gamma^i\theta]\big)+\frac{i}{2}\text{Tr}\big([\mathcal{D}_t,\epsilon^\dagger\Gamma^{ij}\theta][X_i,X_j]\big)+\frac{1}{2}\text{Tr}\big(\theta^\dagger[\epsilon^\dagger\theta,\theta]\big)\\
&\quad -\partial_t\Big(\frac{1}{2}\text{Tr}\big(\bar{\epsilon}\Gamma^i\theta[\mathcal{D}_t,X_i]\big)+\frac{i}{4}\text{Tr}\big(\bar{\epsilon}\Gamma^{ij}\theta[X_i,X_j]\big)\Big)\\
&= B_2+A_2+C_1+\partial_t G\,,\\
\delta_\epsilon \text{Tr}\big(\overline{\theta}\Gamma^i[X_i,\theta]\big) &= \frac{1}{2}\text{Tr}\big([\mathcal{D}_t,X_i][X_j,\epsilon^\dagger\Gamma^i\Gamma^j\theta]\big)-\frac{1}{2}\text{Tr}\big([\mathcal{D}_t,X_j][X_i,\theta^\dagger\Gamma^i\Gamma^j\epsilon]\big)\\
&\quad +\frac{i}{4}\text{Tr}\big([X_i,X_j][X_k,\bar{\epsilon}\Gamma^{ij}\Gamma^k\theta]\big)+\frac{i}{4}\text{Tr}\big([X_i,X_j][X_k,\bar{\theta}\Gamma^k\Gamma^{ij}\epsilon]\big)+\frac{i}{2}\text{Tr}\big(\bar{\theta}\Gamma_k[\bar{\epsilon}\Gamma^k\theta,\theta]\big)\\
&= A_3+A_4+D_1+D_2+C_2\,,\\
\delta_\epsilon \text{Tr}\big([X^i,X^j]^2\big) &= 2i\text{Tr}\big([\bar{\epsilon}\Gamma^i\theta,X^j][X_i,X_j]\big)=D_3\,.
\end{aligned}
\tag{1.71}
$$

Notice that in the list (1.71), we have explicitely integrated by part certain terms in the lagrangian, assuming the fermionic parameters ϵ are constant, and keeping for future purpose the total derivative produced thereby. We first notice the cancellation of expression

$$
A_1-iA_2+A_3+A_4=\text{Tr}([\mathcal{D}_t,X_i][\epsilon^\dagger\Gamma^{ij}\theta,X_j]-\epsilon^\dagger\Gamma^{ij}\theta[X_j,[\mathcal{D}_t,X_i]])=0
$$

by using $\Gamma^i\Gamma^j=\Gamma^{ij}+\delta^{ij}\mathbb{1}$, and the following properties for Majorana spinors in $10D$: $\epsilon^\dagger\Gamma^{ij}\theta=\theta^\dagger\Gamma^{ij}\epsilon$ and $\epsilon^\dagger\theta=-\theta^\dagger\epsilon$ according to eqn.(A.19). The next terms that cancel after restoring the coefficients of the action are:

$$
B_1-iB_2=0\,.
$$

Moreover, we also have

$$
-iC_1+C_2=-\frac{i}{2}\text{Tr}(-\bar{\theta}\Gamma^0[\theta,\bar{\epsilon}\Gamma^0\theta]+\bar{\theta}\Gamma_i[\theta,\bar{\epsilon}\Gamma^i\theta])=\frac{1}{2}f_{abc}(\bar{\theta}^a\Gamma^\mu\theta^b)(\bar{\epsilon}\Gamma_\mu\theta^c)=0 \tag{1.72}
$$

which vanishes thanks to Fierz identity in $10D$ for any triplet of right-handed Majorana spinors: $\varepsilon_{abc}(\overline{\psi}^a\Gamma^\mu\psi^b)(\Gamma_\mu\psi^c)_\alpha=0$ (A.21).

Finally, their remains

$$
D_1+D_2+\frac{1}{2}D_3=-\frac{i}{2}\text{Tr}(\bar{\epsilon}\Gamma_{ijk}\theta[X^i,[X^j,X^k]])=0
$$

where we have been using the relations $\Gamma^{ij}\Gamma^k=\Gamma^{ijk}+2\delta^{i[k}\Gamma^{j]}$ and $\Gamma^i\Gamma^{jk}=\Gamma^{ijk}+2\delta^{i[j}\Gamma^{k]}$. This expression vanishes because of the Jacobi identity applying on $\epsilon_{ijk}[X^i,[X^j,X^k]]$.

The left-over term is a total derivative

$$-i\partial_t G = \frac{i}{2}\partial_t \mathrm{Tr}(\bar\epsilon\Gamma^i\theta[\mathcal{D}_t, X_i]) - \frac{1}{4}\partial_t \mathrm{Tr}(\bar\epsilon\Gamma^{ij}\theta[X_i, X_j]) \tag{1.73}$$

from which one can extract Noether charges. Restoring the overall coupling constant from the action and keeping ϵ local, it can be shown that the Noether charges associated to dynamical supersymmetry can be obtained from G by removing the ϵ parameter and multiplying by g_0^{-2}.

Invariance of (1.56) under kinematical supersymmetry follows from varying

$$\begin{aligned}\tilde\delta_\eta \int dt\, \mathrm{Tr}\Big(-i\theta^\dagger[\mathcal{D}_t,\theta] + \bar\theta\Gamma^i[X^i,\theta]\Big)& \\ = \int dt\, \mathrm{Tr}\Big(-i\eta^\dagger[\mathcal{D}_t,\theta] - i\theta^\dagger[\mathcal{D}_t,\eta] + \bar\eta\Gamma^i[X_i,\theta] + \bar\theta\Gamma^i[X_i,\eta]\Big)& \\ = -\int dt\, i\Big(\partial_t \mathrm{Tr}(\eta^\dagger\theta) - 2\mathrm{Tr}\big((\partial_t\eta^\dagger)\theta\big)\Big) = 0&\end{aligned} \tag{1.74}$$

where we used $\bar\eta\Gamma^i\theta = -\bar\theta\Gamma^i\eta$ and $\int dt\, \mathrm{Tr}(\eta^\dagger[\mathcal{D}_t,\theta]) = -\int dt\, \mathrm{Tr}(\partial_t\eta^\dagger\theta)$+total derivative $= \int dt\, \mathrm{Tr}(\theta^\dagger\partial_t\eta)$+total derivative.

From first term of second line of (1.74) one can again extract the Noether charge for kinematical supersymmetry, by defining

$$\widetilde{G} = -\mathrm{Tr}(\eta^\dagger\theta)\,, \tag{1.75}$$

removing the η parameter and normalizing the resulting expression by g_0^{-2}.

1.4 Brane charges and the BFSS supersymetry algebra

In this section, we want to address the issue of the appearance in M(atrix) theory of states consisting of D0-branes bound at threshold. These states behave like BPS Dp-brane and can be systematically studied by starting from the supersymmetry algebra in 11 dimensions, as first advocated in [230].

The eleven-dimensional supersymmetry algebra reads

$$\{\mathsf{Q}_\alpha, \mathsf{Q}^\beta\} = i(C\Gamma^M)_\alpha{}^\beta P_M + \frac{i}{2!}(C\Gamma^{MN})_\alpha{}^\beta Z_{MN} + \frac{i}{5!}(C\Gamma^{MNOPQ})_\alpha{}^\beta Z_{MNOPQ} \tag{1.76}$$

and includes on the right-hand side a rank two and rank five tensors which act as central charges for membranes (M2) and fivebranes (M5). Indices run from $M = 0,..,10$ and the supercharges Q_α, $\alpha = 1,..,32$, form an $SO(1,10)$ spinor of 32 components. The occurence of central charges that are both Lorentz tensors seem to contradict the Coleman Mandula theorem. However, it can be shown (the detailled calculation in matrix theory will eventually shed light on this issue) that such charges either vanish or are infinite, since they are associated with branes with infinite volume, whose charge per unit volume remains finite. But we can avoid dealing with infinite charges by compactifying space on a large but finite-dimensional torus.

Now, it is easy to see that central charges with a timelike index will vanish, since they originate from space integral of conserved currents, which are themselves antisymmetric tensors: $Z^{M_1..M_n} = \int d^{10}x J^{0M_1..M_n}$, then $Z^{0M_1..M_{n-1}} = 0$. This argument can be reinterpreted in the language of the above paragraph by noting that the time direction being kept non-compact, the time components of central charges necessarily vanish.

In order to make contact with the DLCQ M(atrix) theory formalism, one need rewrite the algebra (1.76) in the light-cone frame. According to the conventions for SO(1,9) Dirac matrices in Appendix A.1, we have $C = \Gamma^0$, and $\mathcal{P}_+ = (1/2)(\mathbb{I} + \Gamma^{10})$ and, as mentioned in Section

1.3.6, the former supercharges Q_α split into 16 kinematical supercharges $\widetilde{\mathsf{Q}}_\alpha$ and 16 dynamical supercharges Q_α, which form a Majorana-Weyl spinor representation of $SO(1,9)$:

$$\{\widetilde{\mathsf{Q}}_\alpha, \widetilde{\mathsf{Q}}^\beta\} = i(\mathcal{P}_+)_\alpha{}^\beta \mathrm{Tr} P^+ \,, \tag{1.77}$$

$$\{\widetilde{\mathsf{Q}}_\alpha, \mathsf{Q}^\beta\} = i(\mathcal{P}_\pm \Gamma^0 \Gamma^i)_\alpha{}^\beta \mathrm{Tr} P_i + \frac{i}{2!}(\mathcal{P}_+ \Gamma^{ij})_\alpha{}^\beta Z_{ij} + \frac{i}{5!}(\mathcal{P}_+ \Gamma^{ijklm})_\alpha{}^\beta Z_{ijklm} \,, \tag{1.78}$$

$$\{\mathsf{Q}_\alpha, \mathsf{Q}^\beta\} = i(\mathcal{P}_+)_\alpha{}^\beta \mathrm{Tr} P^- + i(\mathcal{P}_+ \Gamma^0 \Gamma^i)_\alpha{}^\beta Z_i + \frac{i}{4!}(\mathcal{P}_+ \Gamma^{ijkl})_\alpha{}^\beta Z_{ijkl} \,, \tag{1.79}$$

with $\mathrm{Tr}P^- = H$, and since we eventually take the IMF limit: $\mathrm{Tr}P^+ = N/R = p^+$ (note that, to avoid confusion, P^+ is now an operator, and p^+ the longitudinal momentum, which called P^+ previously).

Lastly, going to the IMF selects only those charges carrying a covariant (-) index, namely: $Z^i \doteq Z^{-i}$ and $Z^{ijkl} \doteq Z^{-ijkl}$, which turn up in the algebra (1.79). The latter are precisely switched on by the presence of M2 and M5 brane wrapped around the longitudinal direction, that appear as D4-branes and fundamental (F1) strings in type IIA string theory. As will be checked later on, to keep those tensors finite, one should take the longitudinal direction to be compact. As a consequence, the correponding charges will scale as $1/R$ in our normalisation.

From the IMF supersymmetric algebra (1.77)-(1.79), on can read off the BPS brane-like objects which appear as central charges, namely:

a) $\frac{1}{2}$-BPS objects: they consist of transverse M2- and M5-branes with charges given by the RHS of commutator (1.79), and preserve the biggest portion of the initial supersymmetry, since they are linear combination of both kinematical *and* dynamical generators. In type IIA language, they turn out to be D2- and NS5-branes. Their energy is proportional to the square of the norm of the related tensor: $Z_{ij}Z^{ij}$ and $Z_{i_1..i_5}Z^{i_1..i_5}$, since these excited states behave like particles propagating in a space with a compact transverse direction. In the IMF variables, the energy scales as $E = \frac{e_T^2}{2p^+}$, with $e_T = T_p A = \sqrt{-\mathrm{Tr}P_A P^A}$ the surface tension energy for a p-brane, which has first been verified by [91, 93] for the transverse membrane. We will see how to construct these objects form M(atrix) theory in the next two sections.

b) $\frac{1}{4}$-BPS objects: these are the longitudinal M2- and M5-branes with charges appearing in commutator (1.78), corresponding to the fundamantal string (F1) and the D4-brane of type IIA theory. Since in expression (1.78) all kinematical commutators are broken and only half of the dynamical ones are preserved, we are indeed dealing with $\frac{1}{4}$-BPS states. In this case, they energy is proportional to the central charge. This is due to the fact that a longitudinal object with momentum in the compact direction will develop an internal excitation breaking translational invariance in the compact direction. This, in turn, is the source for the breaking of the extra half of the remaining supersymmetry generators. In addition, since the energy of the internal excitations increases as $p^+ \to \infty$, the energy scales like a constant in the IMF limit [230]

c) more $\frac{1}{2}$-BPS objects: longitudinal M5-branes with charge given in commutator (1.79) can break only half of the supersymmetries, if they become carriers for membrane charges. This requires activating tranverse M2-branes in the commutator (1.78). Such a state will then combine non-zero values for both (type IIA) D2- and D4-brane charges. Turning on, for instance, the D4-brane charge in directions [1;2;3;4] by choosing Z^{1234} to be non-vanishing, one may embed in the D4 two stacks of infinite D2-branes by activating their respective charges Z^{12} and Z^{34}, and by requiring that $\mathrm{Tr}P^+ Z^{1234} = Z^{12}Z^{34}$. From the longitudinal brane condition on the D4, implying that its energy is proportional to Z_{ijkl} and hence to a constant, one infers that $Z^{12}Z^{34} \propto \mathrm{Tr}P^+$.

1.4.1 The BFSS supercharge density algebra

We will now turn to constructing the equivalent of the supersymmetry algebra (1.77)-(1.79) by starting from a canonical quantisation procedure applied to the matrix model (1.56). We interprete untraced products of matrices as supercharge densities, since the analog of the integration is, in the matrix language, the trace.

We define anticommutators of supercharges by means of Dirac brackets defined on the canonical variables of BFSS Hamiltonian (1.59):

$$P_i = \frac{1}{R}[\mathcal{D}_t, X_i]\,, \qquad \Pi = \frac{i}{2R}\theta^\dagger\,. \tag{1.80}$$

As we argued in the preceding section, the Noether charges associated to the kinematical and dynamical susy can be inferred from G and 1.73) and $\widetilde{G}$ (1.75), and, in components, read

$$\begin{aligned} \widetilde{\mathsf{Q}}_\alpha &= -\frac{1}{R}\mathrm{Tr}\theta_\alpha\,, \\ \mathsf{Q}_\alpha &= -\frac{1}{2R}\Big(\bar\theta(\Gamma^i[\mathcal{D}_t, X_i] - \frac{1}{2}\Gamma^{ij}[X_i, X_j])\Big)_\alpha\,. \end{aligned} \tag{1.81}$$

We implement the non-commutative structure on the canonical variables (1.80) by defining the Dirac brackets:

$$\{(X^i)_a^b, (P_j)_c^d\}_{\mathrm{DB}} = \delta^i_j \delta^d_a \delta^b_c\,, \tag{1.82}$$

$$\{(\theta_\alpha)_a^b, (\theta^\beta)_c^d\}_{\mathrm{DB}} = -iR(\mathcal{P}_+)_\alpha{}^\beta \delta^d_a \delta^b_c\,, \tag{1.83}$$

which will be specified by DB, so as not to mistake them for matrix commutators acting on the adjoint of $SU(N)$. Since there is no operator ordering prescription, the algebra (1.82)-(1.83) conveys the full quantum structure of the theory.

Now, if we start by computing the algebra directly from traced operators such as (1.81), we will be faced with subtleties related to the $N \to \infty$ limit of the matrix model. So following [230], we will instead used supercharge densities as a starting point, which are simply the untraced version of (1.81). Furthermore, we *improve* the dynamical supercharge density by symmetrising it, thus ensuring the operator to be hermitian; this *minimal* improvement does not affect the traced operator in the finite N limit, and, as argued in [230], vanishes in the smooth membrane approximation to the theory, when $N \to \infty$.

Defining the supercharge densities to be

$$\begin{aligned} (\tilde{\mathsf{q}}_\alpha)_a^b &= -\frac{1}{R}(\theta_\alpha)_a^b\,, \\ (\mathsf{q}_\alpha)_a^b &= -\frac{1}{4}\Big\{(\Gamma^{0i}P_i + \frac{i}{2R}\Gamma^{ij}[X_i, X_j])_\alpha{}^\beta, \theta_\beta\Big\}_{\mathrm{DB}\,a}^{\;\;b}\,, \end{aligned}$$

we have indeed $\mathsf{Q}_\alpha = \mathrm{Tr}\mathsf{q}_\alpha$ and $\widetilde{\mathsf{Q}}_\alpha = \mathrm{Tr}\tilde{\mathsf{q}}_\alpha$.

In the following, we take advantage of the Majorana condition applying to $SO(1,9)$ spinors to restrict to a real representation thereof:

$$\mathsf{q}^\alpha \doteq \mathsf{q}_\alpha^\dagger = \mathsf{q}_\alpha^\top\,, \qquad \tilde{\mathsf{q}}^\alpha \doteq \tilde{\mathsf{q}}_\alpha^\dagger = \tilde{\mathsf{q}}_\alpha^\top\,.$$

In particular, using anticommutation relations for $\Gamma-$matrices and antisymmetry of Γ^{ij}: $(\mathsf{q}^\alpha)_a^b = (\tilde{\mathsf{q}}_\alpha^\dagger)_a^b = -\frac{1}{4}\{(\Gamma^{0i}P_i - \frac{i}{2R}\Gamma^{ij}[X_i, X_j])_\beta{}^\alpha, \theta^\beta\}_a^b$.

We are now in a position to compute Dirac brackets of supercharges which still encode the index structure of the gauge group (and thus remain sensitive to the tracing procedure). However, this will in turn produce terms antisymmetric in both matrix and spinorial indices, which are similar to Schwinger terms with odd derivative of delta functions. Provided all charges are regularised in the same way in the large N limit, these unwanted terms will disappear when tracing on one of the supercharges, which is what we will eventually do.

The first Dirac brackets including kinematical supercharges only are easy to derive, yielding

$$\{(\widetilde{\mathsf{q}}_\alpha)^b_a, \widetilde{\mathsf{Q}}^\beta\}_{\mathrm{DB}} = -\frac{i}{R}(\mathcal{P}_+)_\alpha{}^\beta \delta^b_a\,. \tag{1.84}$$

Next, the commutator mixing kinematical with dynamical supercharges reads

$$\begin{aligned}\{(\widetilde{\mathsf{q}}_\alpha)^b_a, \mathsf{Q}^\beta\}_{\mathrm{DB}} &= \frac{i}{R}\{(\theta_\alpha)^b_a, (\theta^\gamma)^c_d\}_{\mathrm{DB}}\Big((\Gamma^{0i})_\gamma{}^\beta (P_i)^d_c - \frac{i}{2R}(\Gamma^{ij})_\gamma{}^\beta [X_i, X_j]^d_c\Big)\\ &= -\frac{1}{2}(\mathcal{P}_+\Gamma^{0i})_\alpha{}^\beta (P_i)^b_a - \frac{i}{4R}(\mathcal{P}_+\Gamma^{ij})_\alpha{}^\beta [X_i, X_j]^b_a\,.\end{aligned}$$

Finally, computing the Dirac bracket between two dynamical supercharges is a little more involved. For the sake of clarity, we split the Dirac bracket

$$\{(\mathsf{q}_\alpha)^b_a, \mathsf{Q}^\beta\}_{\mathrm{DB}} = \frac{1}{8}\Big\{\big\{(\Gamma^{0i}P_i + \frac{i}{2R}\Gamma^{ij}[X_i, X_j])_\alpha{}^{\alpha'}, \theta_{\alpha'}\big\}_{\mathrm{DB}}{}^b{}_a, \theta^{\beta'}\big(\Gamma^{0k}P_k - \frac{i}{2R}\Gamma^{kl}[X_k, X_l]\big)_{\beta'}{}^\beta\big)^c_c\Big\}_{\mathrm{DB}} \tag{1.85}$$

into two parts: a part where we compute the Dirac bracket of the two fermionic variables: $(\{\theta_{\alpha'}, \theta^{\beta'}\}_{\mathrm{DB}} \times (...))_\alpha{}^{\beta\, b)}_{\ (a}$ (which will be denoted by a 0 subscript), and a part where only the bosonic variables P_i and X_j are involved in the DB (denoted by a 1 subscript). Since all these expressions should be symmetric in the $U(N)$ indices, this will often be abbreviated $(\bullet)_{(ab)} \doteq \frac{1}{2}\left[(\bullet)_{ab} + (\bullet)_{ba}\right]$. Thus subscript (0) of expression (1.85) reads

$$\begin{aligned}\{(\mathsf{q}_\alpha)^b_a, \mathsf{Q}^\beta\}_{\mathrm{DB}}\Big|_{(0)} &= -\frac{iR}{4}\Big(\mathcal{P}_+\big(\Gamma^{0i}P_i + \frac{i}{2R}\Gamma^{ij}[X_i, X_j]\big)\big(\Gamma^{0k}P_k + \frac{i}{2R}\Gamma^{kl}[X_k, X_l]\big)\Big)_\alpha{}^{\beta\, b)}_{\ (a}\\ &= -\frac{iR}{4}\bigg((P_iP^i)^b_a(\mathcal{P}_+)_\alpha{}^\beta + \frac{1}{2R^2}([X_i, X_j]^2)^b_a(\mathcal{P}_+)_\alpha{}^\beta\\ &\quad + \frac{i}{R}\{[X_i, X^k], P_k\}_{\mathrm{DB}}{}^b{}_a(\mathcal{P}_+\Gamma^{0i})_\alpha{}^\beta + \frac{1}{4R^2}([X_i, X_j][X_k, X_l])^b_a(\mathcal{P}_+\Gamma^{ijkl})_\alpha{}^\beta\bigg)\end{aligned}$$

where getting from the first to the second line requires using such identities as $[\Gamma^{ij}, \Gamma^{0k}] = -2\Gamma^0(\delta^{ki}\Gamma^j - \delta^{kj}\Gamma^i)$ and $\{\Gamma^{ij}, \Gamma^{kl}\} = \Gamma^{ijkl} + \delta^{il}\delta^{jk} - \delta^{ik}\delta^{jl}$. Clearly, in the final result, we have not added extra symetrisation to expressions overtly symmetric in matrix indices.

The subscript (1) part of (1.85) reads, after factorising the left-over fermions

$$\begin{aligned}\{(\mathsf{q}_{(\alpha})^b_a, \mathsf{Q}^{\beta)}\}_{\mathrm{DB}}\Big|_{(1)} &= -\frac{i}{4R}\Big((X_m)^d_{(a}(\theta_\gamma)^{b)}_c(\theta^\delta)^c_d - (X_m)^d_c(\theta^\delta)^c_{(a}(\theta_\gamma)^{b)}_d\Big)\Big((\mathcal{P}_+\Gamma^0\Gamma_l)_{(\alpha}{}^\gamma(\mathcal{P}_+\Gamma^{lm})_\delta{}^{\beta)}\\ &\qquad - (\mathcal{P}_+\Gamma^0\Gamma_l)_\delta{}^{(\beta}(\mathcal{P}_+\Gamma^{lm})_{\alpha)}{}^\gamma\Big)\\ &= \frac{i}{4R}\big(\theta_\gamma\theta_\delta X_m - \theta_\delta X_m\theta_\gamma\big)^{b)}_{(a}\Big((\mathcal{P}_+\Gamma^0\Gamma_l)_{(\alpha}^{\ |\gamma|}(\mathcal{P}_+\Gamma^{lm})^{\beta)\delta}\\ &\qquad + (\mathcal{P}_+\Gamma^0\Gamma_l)_{(\alpha}^{\ |\delta|}(\mathcal{P}_+\Gamma^{lm})^{\beta)\gamma}\Big)\\ &= \frac{i}{4R}\Big((\theta^{\mathsf{T}}[\theta, X_k])^{b)}_{(a}(\mathcal{P}_+\Gamma^{0k})_\alpha{}^\beta - (\theta^{\mathsf{T}}\mathcal{P}_+\Gamma^{0k}[\theta, X_k])^{b)}_{(a}(\mathcal{P}_+)_\alpha{}^\beta\Big)\,.\end{aligned} \tag{1.86}$$

Let us briefly comment on the details of the calculation. To compute the first line of expression (1.86), use

$$\{(P_i)^b_a, [X^j, X^k]^d_c\}_{\mathrm{DB}} = 2\left(\delta^b_c\delta^{[j}_i(X^{k]})^d_a - \delta^d_a\delta^{[j}_i(X^{k]})^b_c\right)\,.$$

Next, going from the first to the second line takes advantage of the Majorana representation where all Dirac matrices can be chosen real (see Appendix A.4). In this basis, the matrices $\Gamma^0\Gamma_l$ are symmetric, while Γ^{lm} are antisymmetric under interchange of spinor indices. The resulting expression turns out to be symmetric for $\gamma \leftrightarrow \delta$, which has been emphasized by this same symbol. Finally, last line of expression (1.86) has been cast in this condensed form thanks to the 10 dimensional Fierz identity in its component form (see Appendix A.6)

$$\begin{aligned}(\mathcal{P}_\pm\Gamma^0\Gamma_l)_{(\alpha}{}^{|\gamma|}(\mathcal{P}_\pm\Gamma^{lm})^{\beta)\delta} + (\mathcal{P}_\pm\Gamma^0\Gamma_l)_{(\alpha}{}^{|\delta|}(\mathcal{P}_\pm\Gamma^{lm})^{\beta)\gamma} \\ = -(\mathcal{P}_\pm\Gamma^0\Gamma^m)_\alpha{}^\beta(\mathcal{P}_\pm)^{\gamma\delta} + (\mathcal{P}_\pm\Gamma^0\Gamma^m)^{\gamma\delta}(\mathcal{P}_\pm)_\alpha{}^\beta\,.\end{aligned} \tag{1.87}$$

To summarise, expressions (1.84), (1.85) and (1.86) generate the algebra

$$\begin{aligned}
\{(\widetilde{\mathsf{q}}_\alpha)^b_a, \widetilde{\mathsf{Q}}^\beta\}_{\rm DB} &= -\frac{i}{R}(\mathcal{P}_+)_\alpha{}^\beta\delta^b_a\,,\\
\{(\widetilde{\mathsf{q}}_\alpha)^b_a, \mathsf{Q}^\beta\}_{\rm DB} &= -\frac{1}{2}(\mathcal{P}_+)_\alpha{}^\gamma\left(i(\Gamma^{0i})_\gamma{}^\beta P_i + \frac{1}{2R}(\Gamma^{ij})_\gamma{}^\beta[X_i,X_j]\right)^b_a\,,\\
\{(\mathsf{q}_{(\alpha})^b_a, \mathsf{Q}^{\beta)}\}_{\rm DB} &= -\frac{iR}{4}\Bigg((P_iP^i)^b_a(\mathcal{P}_+)_\alpha{}^\beta - \frac{1}{2R^2}([X_i,X_j]^2)^b_a(\mathcal{P}_+)_\alpha{}^\beta\\
&\quad+\frac{i}{R}\{[X_i,X^k],P_k\}^b_a(\mathcal{P}_+\Gamma^{0i})_\alpha{}^\beta + \frac{1}{4R^2}([X_i,X_j][X_k,X_l])^b_a(\mathcal{P}_+\Gamma^{ijkl})_\alpha{}^\beta\\
&\quad-\frac{1}{R^2}(\theta^\intercal[\theta,X_k])^{b)}_{(a}(\mathcal{P}_+\Gamma^{0k})_\alpha{}^\beta + \frac{1}{R^2}(\theta^\intercal\Gamma^{0k}[\theta,X_k])^{b)}_{(a}(\mathcal{P}_+)_\alpha{}^\beta\Bigg)\,.
\end{aligned} \tag{1.88}$$

By tracing on supercharge densities, we notice that the coefficient on the RHS of the first PB of the algebra (1.88) can be rewritten as $\mathrm{Tr}\delta^b_a/R = N/R = \mathrm{Tr}P^+$. As for the the last PB of the algebra (1.88), the first, second and sixth terms can be collected in the Hamiltonian (1.59) (with Gauss constraint set to zero), thus yielding the familiar expression

$$\begin{aligned}
\{\widetilde{\mathsf{Q}}_\alpha, \widetilde{\mathsf{Q}}^\beta\}_{\rm DB} &= -i(\mathcal{P}_+)_\alpha{}^\beta\mathrm{Tr}P^+,\\
\{\widetilde{\mathsf{Q}}_\alpha, \mathsf{Q}^\beta\}_{\rm DB} &= -\frac{1}{2}\left(i(\mathcal{P}_+\Gamma^{0i})_\alpha{}^\beta\mathrm{Tr}P_i + \frac{1}{2R}(\mathcal{P}_+\Gamma^{ij})_\alpha{}^\beta\mathrm{Tr}[X_i,X_j]\right)\\
\{\mathsf{Q}_{(\alpha}, \mathsf{Q}^{\beta)}\}_{\rm DB} &= -\frac{i}{2}H(\mathcal{P}_+)_\alpha{}^\beta + \frac{1}{2}\mathrm{Tr}\left(P^k[X_k,X_i] + \Pi[\theta,X_i]\right)(\mathcal{P}_+\Gamma^{0i})_\alpha{}^\beta\\
&\quad-\frac{i}{16R}\mathrm{Tr}([X_i,X_j][X_k,X_l](\mathcal{P}_+\Gamma^{ijkl})_\alpha{}^\beta\,,
\end{aligned} \tag{1.89}$$

which reminds us of the IMF superalgebra (1.77)-(1.79). By identify the central charges on the LHS of expressions (1.77)-(1.79) with the corresponding objects in the superalgebra (1.89), we obtain the M(atrix) theory formulation of some of the branes of M-theory mentioned at the beginning.

Thus we recognise in

$$\begin{aligned}
Z_i &= -\frac{1}{4R}\mathrm{Tr}\left(2i[D_t,X^k][X_k,X_i] - \theta^\top[\theta,X_i]\right) \equiv -\frac{i}{2}\mathrm{Tr}\left(P^k[X_k,X_i] + \Pi[\theta,X_i]\right)\,,\\
Z_{ij} &= \frac{i}{2R}\mathrm{Tr}[X_i,X_j]\,, \qquad\qquad Z_{ijkl} = -\frac{6}{R}\mathrm{Tr}(X_{[i}X_jX_kX_{l]})
\end{aligned} \tag{1.90}$$

the (type IIA) fundamental string (longitudinal M2-brane) along with the (type IIA) D2-brane (transverse M2) and the (type IIA) D4-brane (longitudinal M5).

We now restore the originial normalisation of [230], which is appropriate for discriminating wrapped from unwrapped branes. This can be done by rescaling fields and derivative appearing in the brane charges (1.90) adequately: $X_i \to \sqrt{R}X_i$, $i\mathcal{D}_t \to \sqrt{R}i\mathcal{D}_t$ and $\theta \to R^{3/4}\theta$, leading to:

$$\begin{aligned} Z_i &\propto R\mathrm{Tr}\left(2i[\mathcal{D}_t, X^k][X_k, X_i] - \theta^\top[\theta, X_i]\right), && (1.91)\\ Z_{ij} &\propto i\mathrm{Tr}[X_i, X_j], && (1.92)\\ Z_{ijkl} &\propto R\mathrm{Tr}(X_{[i}X_jX_kX_{l]}), && (1.93)\end{aligned}$$

and we are happy to notice that objects wrapped along the eleventh dimension such as Z_i and Z_{ijkl} indeed scale as R, as pointed out earlier, while Z_{ij} does not exhibit any R dependence.

Since the first charge is proportional to the Gauss constraint and the last two are traces of commutators, they obviously vanish for N finite. However, when taking the large N limit, these expressions can be non-zero, as a consequence of matrices becoming infinite dimensional. In the low energy supergravity description of the theory, these central charges would indeed appear as integrals over total derivatives, and therefore would be non-vanishing only on topologically non-trivial configuration of fields. This interpretation of central charges as topological objects is advocated in [231].

The algebric method used throughout this section seems to give a systematic analysis of the brane configurations in M(atrix) theory. Still, if we go through the brane scan given at the beginning of this section, we notice a striking anomaly in the list (1.90): among the transverse $\frac{1}{2}$-BPS states only the transverse M2-brane Z_{ij} appears, whereas the transverse five-brane is absent. We will came back to this puzzle below.

The construction of brane charges from traces over products of matrix commutators can be extended to include couplings of extended objects of matrix theory to supergravity fields. These new object resulting from tracing over such configurations correspond to multipole moments. More details can be found in [157, 194].

One can for example construct the multipole moments of the membrane

$$I^{ij(a_1\dots a_l)} = \int d^2\sigma\, \{X^i, X^j\}\, X^{a_1}\cdot\ldots\cdot X^{a_l}$$

in terms of matrix moments by tracing the charge density defined from (1.92)

$$Z^{ij(a_1\dots a_l)} = -2\pi i \mathrm{STr}\big([X^i, X^j]\, X^{a_1}\cdot\ldots\cdot X^{a_l}\big)\,. \tag{1.94}$$

The choice of symmetrised trace STr is a prescription dictated by explicite calculations of interactions in matrix theory.

1.5 M-theory branes from M(atrix) theory

In this section, we will show how the central charges (1.90) are related to extended objects in M(atrix) theory and how one can construct the various $\frac{1}{2}$- or $\frac{1}{4}$-BPS branes given in the list at the end of Section 1.4. A more thorough analysis can be found in [233, 234], where matrix model prediction for the energy and charges of various configurations of branes are put in perspective with the results of the DBI theory.

1.5.1 Transverse membranes

Consider a membrane in compact space, wrapped on a torus T^2 and described by the appropriate reduction of the $U(N)$ SYM theory (1.50). It can carry in general $k \in \mathbb{Z}$ units of magnetic flux, given by the first Chern number of the $U(N)$ bundle over T^2:

$$C_1 = \int c_1(F) = \frac{1}{2\pi}\int F = k \in \mathbb{Z}\,.$$

This Chern number is non-zero for a non-trivial bundle. Indeed, decomposing $U(N)$ according to its normal subgroup $U(1)$:

$$U(N) = \frac{U(1) \times SU(N)}{\mathbb{Z}_N}$$

the gauge curvature F has its trace determined from the $U(1)$ part of this decomposition, and the total field strength with $C_1 = k$ is given by $F = \frac{k}{N} \mathbb{I}_{N\times N}$.

T-duality teaches us that we can construct a D($p+2n$)-branes from a sytem of lower-dimensional interacting Dp-branes by representing the transverse coordinates by non-commutative matrices. In particular, we are interested here in constructing the D2-brane of type IIA string theory appearing in BFSS theory, from N D0-branes on T^2. We start from a 1-brane (a string) wrapping the torus diagonally along (N, k), with N and k relatively prime. Taking the surface of the torus to by given by $A_{T^2} = (2\pi R_m) \times (2\pi R_n)$. Then, working in the BFSS coordinates where $2\pi\alpha' = 1$, the diagonal 1-brane configuration satisfies [133]:

$$[iD_m, X^n] = \frac{iR_n k}{R_m N} \mathbb{I}_{N\times N} \,.$$

T-dualising along x^n, sending $R_n \to \widehat{R}_n = \frac{\alpha'}{R_n} = \frac{1}{2\pi R_n}$, we obtain in the dual picture a configuration of N D2-branes with k units of flux:

$$[iD_m, iD_n] = -iF_{jk} = \frac{ik}{2\pi R_m \widehat{R}_n N} \mathbb{I}_{N\times N}$$

On the other hand, one can instead T-dualise along x^m:

$$[X^m, X^n] = \frac{2\pi i \widehat{R}_m R_n k}{N} \mathbb{I}_{N\times N}$$

which gives us a sytem of N D0-branes carrying an overall D2 charge k.

Then, a membrane wrapped on a two-torus can be constructed from M(atrix) theory by choosing a system of N D0-branes described by two matrices statisfying:

$$\mathrm{Tr}[X^m, X^n] = \frac{iA_{T^2} k}{2\pi} \tag{1.95}$$

with $A_{T^2} = 4\pi^2 \widehat{R}_m R_n$ the area of the torus. The expression (1.95) gives the surface charge Z_{mn} (1.92) of the longitudinal membrane on T^2.

Fixing the two directions of the T^2 to be $(m, n) = (1, 2)$, one can use the BFSS Hamiltonian (1.59) to calculate the potential energy for such a (stationary) configuration:

$$H = -\frac{1}{2R} \mathrm{Tr}([X^1, X^2])^2 = \frac{A_{T^2}^2 k^2}{8\pi^2 RN} \,. \tag{1.96}$$

We may now compare this result with the energy predicted from the DBI theory (1.40). The energy for a bound state of N_p p-branes and N_{p+2} $(p+2)$-branes is given by $E_{\mathrm{DBI}} = \sqrt{(N_p E_p)^2 + (N_{p+2} E_{p+2})^2}$ (see [233] for instance), so that, in our case, we have:

$$E_{DBI} = \sqrt{(N\tau_0)^2 + (A_{T^2} k \tau_2)^2} = N\tau_0 + \frac{(A_{T^2} k \tau_2)^2}{2N\tau_0} + \mathcal{O}(N^{-3}) \,.$$

The second term in this expansion exactly reproduces the static D2-brane energy (1.96) (remember that we work in the conventions $2\pi\alpha' = 1$ where $R = g_s\sqrt{\alpha'}$):

$$E_{D2} = \frac{(A_{T^2} k)^2}{2(2\pi)^4 NR(\alpha')^2} = \frac{A_{T^2}^2 k^2}{8\pi^2 RN}$$

Setting $k = 1$ and using the relation $R = 2\pi l_P^3$, which can be obtained by matching the coefficients of the Lagrangian version of the regularised supermembrane Hamiltonian (1.39) with those of the BFSS Lagrangian (1.56), one can recast expression (1.96) into the form:

$$E_{D2} = \frac{(T_2 A_{T^2})^2 R}{2N} \equiv \frac{e_T^2}{2\mathrm{Tr}P^+}$$

with $T_2^{-1} = (2\pi)^2 l_P^3$ the membrane tension and e_T the surface tension energy (1.41). This is the scaling law for the energy of a $\frac{1}{2}$-BPS transverse membrane, as explained in paragraph a) at the end of Section 1.4.

1.5.2 Spherical membranes

Another illustration of $\frac{1}{2}$-BPS transverse membrane with light-front energy $E = \frac{e_T^2}{2\mathrm{Tr}P^+}$ is the spherical membrane [158]. The matrix approximation to a membrane in $11D$ embedded in the sphere: $\sum_{i=1}^3 (x^i)^2 = R_{S^2}$, is given, according to eqn.(1.22), by the identification

$$X^i = \frac{2R_{S^2}}{N} L^i \,, \qquad i = 1, 2, 3\,, \tag{1.97}$$

where the L^i are the generators $SU(2)$ for its N-dimensional representation, and satisfy the equation:

$$\sum_{i=1}^{3} (X^i)^2 = \left(\frac{2R_{S^2}}{N}\right)^2 C_2(N)\mathbb{1} = R_{S^2}^2\Big(1 - \frac{1}{N^2}\Big)\mathbb{1}\,. \tag{1.98}$$

Here, $C_2(N) = \frac{N^2-1}{4}$ is the quadratic Casimir of the N-dimensional representation of $SU(2)$. We see in particular from the above equation that the D0-branes forming the membrane bound state can be seen as somehow *localised* on a (fuzzy) sphere of radius $R_{S^2} + \mathcal{O}(N^{-2})$.

The tension energy of the stationary membrane is given this time by:

$$e_T = T_2 A_{S^2} = \frac{4\pi R_{S^2}^2}{(2\pi)^2 l_P^3} = \frac{R_{S^2}^2}{\pi l_P^3} = \frac{2R_{S^2}^2}{R}\,. \tag{1.99}$$

We work again in string units where $R = 2\pi l_P^3$. Then, the light-front energy $E = \frac{e_T^2}{2\mathrm{Tr}P^+}$ perfectly matches the leading term in Hamiltonian for the (stationary) bosonic membrane (1.27):

$$H = -\frac{1}{4R}\sum_{i,j} \mathrm{Tr}\big([X_i, X_j][X^i, X^j]\big) = +\frac{2R_{S^2}^4}{NR}\Big(1 - \frac{1}{N^2}\Big)\,.$$

To compute this last expression, we have used the $SU(2)$ commutation relations (1.23) and the equation for the fuzzy sphere (1.98).

Finally, to illustrate formula (1.94), let us compute the first dipole moment for the spherical membrane:

$$Z^{12(3)} = \frac{4\pi (R_{S^2})^3}{3}\big(1 - \frac{1}{N}\big)\,, \qquad \text{cyclicly on the indices 1,2,3}\,.$$

This matches the smooth spherical membrane dipole moment up to $\mathcal{O}(N^{-2})$.

1.5.3 Longitudinal fivebranes

Proceeding as for the transverse membrane, we can construct a D4-brane of type IIA theory on a T^4 by choosing a configuration of D0-branes in the matrix model which satisfy the charge volume relation:

$$\mathrm{Tr}\big(\varepsilon_{ijkl}X^iX^jX^kX^l\big) = \frac{V_{T^4}}{2\pi^2}$$

where ε_{ijkl} is Levi-Civita tensor on the D4-brane world-volume, with $\varepsilon^{1234} = -1$. This corresponds to a longitudinal fivebrane in M(atrix) theory with charge Z_{ijkl} (1.93)

As mentioned in paragraph b) at the end of Section (1.4), some of these longitudinal fivebranes preserve only one fourth of the supersymmetry. These objects can be directly constructed from the supersymmetry variation for a static solution of the matrix model:

$$\delta_\epsilon\theta = -\frac{i}{4}[X_i, X_j]\Gamma^{ij}\epsilon\,, \qquad \tilde{\delta}_\eta\theta = 0\,.$$

Since we are looking for a $\frac{1}{4}$-BPS state, we require half of the dynamical variation $\delta_\epsilon\theta$ to vanish. This is achieved for a self-dual solution of the form

$$[X_i, X_j] = \frac{1}{2!}\varepsilon_{ijkl}[X^k, X^l]\,. \tag{1.100}$$

Contracting both sides with $[X^i, X^j]$ and using $[X^i, X^j][X_i, X_j] = 4X_{[i}X_jX_kX_{l]}$, we obtain the longitudinal fivebrane charge Z_{ijkl} (1.93). Clearly, when the system is compactified on a T^4, such an object becomes, after T-duality (1.55), an instanton solution of a $(4+1)$-dimensional gauge theory on the dual torus.

We now consider the $\frac{1}{2}$-BPS longitudinal fivebranes mentioned in paragraph c) at the end of Section (1.4). We start by considering solutions carrying no membrane charge. In order to preserve one half of the supersymmetry, a static solution should cancel its kinematical variation against its dynamical one. Since the kinematical supersymmetry variation is now non zero

$$\tilde{\delta}_\eta\theta = \eta\mathbb{1}\,,$$

a static solution with the same dynamical variation as in expression (1.100) will compensate it for

$$[X_i, X_j] = -iF_{ij}\mathbb{1} \tag{1.101}$$

where the gauge curvature F_{ij} satisfies $F_{ij}\Gamma^{ij}\epsilon = 4\eta$.

Both these solutions can indeed be shown to solve the static classical equations of the matrix model, and exist only in the large N limit, as expected from their central charge (1.90).

Finally, $\frac{1}{2}$-BPS longitudinal fivebranes carrying membrane charge can be constructed classically by two orthogonal transverse two-planes embedded in the D4-brane world-volume. In fact, this configuration is not limited to wrapped five-banes (transverse D4 in type IIA theory), but also extends to transverse D6 and D8-branes (which are difficult to interpret in M-theory, since there are no *M7 and *M8-branes and "M9"-branes are the rigid hyperplanes used for anomaly cancellation in $E_8 \times E_8$ heterotic string theory), where three and four orthogonal transverse two-planes can be embedded. Returning now to our longitudinal fivebrane, the construction with two embedded orthogonal transverse two-planes requires four fields X_i which, according to a uniqueness theorem for irreducible reps of the canonical algebra, must be elements of $\mathcal{H}_1 \otimes \mathcal{H}_2$ and satisfy

$$\begin{aligned} X_{2i-1} &= \mathsf{q}_i\,, \qquad X_{2i} = \mathsf{p}_i\,, \quad \text{for } i = 1, 2\,, \\ [\mathsf{p}_i\,, \mathsf{q}_j] &= -iF_i\delta_{ij}, \quad \text{for } i, j = 1, 2\,, \end{aligned} \tag{1.102}$$

with $F_i \propto (\sqrt{N})^{-1}$. As a consequence, the D4-brane now carries two charges Z_{12} and Z_{34} which must fulfill the condition $\mathrm{Tr}P^+Z^{1234} = Z^{12}Z^{34}$. Our transverse D4-brane may thus be interpreted as a wrapped M5-brane, *ie* a longitudinal five-brane with stacks of membranes embedded in it.

Finally, we will briefly discuss the fate of the transverse five-brane in M(atrix) theory, since its absence seems incompatible with Lorentz invariance of the light-cone version of the theory (this discussion will be more detailed in the next section). Two seemingly incompatible interpretations have been put forward:

I) Relying on a construction where the initial transverse five-brane is wrapped around a transverse T^3, reference [123] proposes that these states can be described by the S-dual of unwrapped membrane states in the four-dimensional $\mathcal{N}=4$ super Yang-Mills theory on the dual $\widehat{T}^3$. This construction will be made more precise in the following subsection.

II) On the other hand, reference [230] rather interprets this absence as an artefact of the light-cone description of the matrix model. Since transverse D-branes are not translational invariant in the longitudinal direction, they cannot be static objects when boosted to IMF. They should therefore disappear from the IMF algebra (1.89) too. One may wonder from this argument why then membranes should be allowed as transverse objects. A tentative answer could be that, as has been pointed out in section 1.1.3, membranes are elementary degrees of freedom of the matrix theory, while five-branes appear as D-brane objects [2].

Obviously, the two arguments above clash, and a solution should be found to reconcile the evidence that dynamical transverse five-branes might appear in the large N limit of M(atrix) theory and restore full Lorentz invariance (this covariant generalisation of the matrix model having yet to be found). In the next section, a tentative solution to this conendrum due to [123, 232] will be proposed.

1.5.4 Transverse fivebranes

To conclude this discussion on p-brane states in M(atrix) theory, we review in a condensed form the argument outlined in I) above. As seen in the last section, there are two ways in which an M-theory five-brane can appear in M(atrix) theory. If it is wrapped along the longitudinal direction in which the infinite boost is performed, it gives rise to a D4-brane, while if unwrapped in this direction, it appears as a transverse five-brane in M(atrix) theory. In the preceding section we have seen how to construct longitudinal five-branes concretely in terms of matrix variables. For transverse five-branes unfortunately, we only have indirect evidence of their existence.

Ref.[123] gives a proposal which uses the duality between M-theory compactified on T^3 in directions $\{x^8; x^9; x^{10}\}$ and M-theory on the dual torus $\widehat{T}^3$, summarised by:

$$\frac{\text{M-theory}}{\{g_{AB}; R_7, R_8, R_9\}} = \frac{\text{M-theory}}{\{M_P^2 V_3^{2/3} g_{AB}\,;\, M_P^{-2} V_3^{-2/3} R_8\,;\, M_P^{-2} V_3^{-2/3} R_7\,;\, M_P^{-2} V_3^{-2/3} R_9\}}\,, \qquad (1.103)$$

with $V_3 = R_7 R_8 R_9$ the volume of the original T^3. Since M-theory on T^3 descends to type IIA string theory on T^2, the duality transformation above corresponds to a double T-duality in type IIA theory, and thus to an automorphism of the latter. The longitudinal direction is taken to be x^{10}.

The construction above can be generalised to include non-vanishing vacuum expection value of the three-form in the torus directions, leading to the transformation:

$$\tau = i M_P^3 V_3 + C_{789} \xrightarrow{\mathcal{T}_{789}} -\frac{1}{\tau}\,. \qquad (1.104)$$

[2]In this respect, a parallel can be drawn with string theory, where large transverse strings are allowed in light-cone formalism, contrary to D-branes. Technically, the impossibility of D-branes in the light-cone frame stems from the Virasoro condition $\partial_\sigma X^- = \partial_\tau X_i\, \partial_\sigma X^i$, which always induces Neumann boundary conditions for transverse variable, whatever (Neumann or Dirichlet) boundary conditions we choose for the longitudinal coordinate.

Following the arguments of the previous sections, the LHS of eqn.(1.103) can be described by a regular M(atrix) model on T^3, while the RHS can be captured by $\mathcal{N} = 4$ super-Yang-Mills theory with gauge group $U(\infty)$ on the dual torus $\widehat{T}^3$. Its dynamics is given by the Hamiltonian version of expression (1.50), with the convention $2\pi\alpha' = 1$. The coupling constant is $g_3^2 = 2\pi g_s$ and we have rescaled the covariant time derivatives as $\mathcal{D}_t \to g_3^2 \mathcal{D}_t$:

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\int d^3\hat{x}\,\mathrm{Tr}\Big(g_3^2 \overrightarrow{E}^2 + \frac{1}{g_3^2}\overrightarrow{B}^2 + \frac{\theta}{8\pi^2}\overrightarrow{E}\cdot\overrightarrow{B} + g_3^2\sum_{a=1}^{6}[\mathcal{D}_t, X^a]^2 \\ &\qquad + \frac{1}{g_3^2}\sum_{a=1}^{6}[\mathcal{D}_i, X^a]^2 + \frac{1}{2g_3^2}\sum_{a,b=1}^{6}[X^a, X^b]^2 + \text{ fermions}\Big)\end{aligned} \tag{1.105}$$

with coupling constant:

$$\tau = \frac{4\pi i}{g_3^2} + \frac{\theta}{8\pi^2} = iM_P^3 V_3 + C_{789}\,.$$

Then, the (generalised) T-duality of M-theory (1.104) corresponds to an S-duality: $\tau \xrightarrow{S} -\frac{1}{\tau}$ on the SYM side. The sides of the dual torus $\widehat{T}^3$ are then rescaled by $(\mathrm{Im}\,\tau)^{2/3}$ without change of shape, according to eqn.(1.103), so that one can use conformal invariance of the SYM theory to fix the volume of T^3 to 1, with appropriate rescaling of the scalars $X^a \to M_P^{-1}(V_3)^{-\frac{1}{3}}X^a$.

In particular the authors of [123] have shown that configurations with one unit of magnetic or electric flux in the $\mathcal{N} = 4$ super-Yang-Mills theory

$$\frac{1}{2\pi}\int_{\widehat{T}^3}\mathrm{Tr}B^{ij} = 1\,, \qquad \frac{1}{2\pi}\int_{\widehat{T}^3}\mathrm{Tr}E_i = 1\,, \tag{1.106}$$

can be reproduced by matrices acting on the twisted $U(N)$ bundle. For instance:

$$[X^i, X^j] = iB^{ij} = \frac{2\pi}{N} i\mathbb{1}_{N\times N}$$

so that the magnetic field in the super-Yang-Mills theory is dual to the BFSS membrane when $N \to \infty$, according to eqn.(1.95). Likewise, the electric flux in (1.106) is reproduced by D0-brane momentum in the BFSS theory. To sum up, the correspondence is:

$$\mathrm{Tr}[X^i, X^j] = i\int_{\widehat{T}^3}\mathrm{Tr}B^{ij}\,, \qquad \mathrm{Tr}[\mathcal{D}_t, X_i] = g_3^2\mathrm{Tr}P_i = \int_{\widehat{T}^3}\mathrm{Tr}E_i\,.$$

Then, the electric-magnetic duality in the super Yang-Mills theory corresponds to a T-duality transformation on the M(atrix) theory side, exchanging Kaluza-Klein momentum modes with membrane winding modes.

Finally, one can get the complete U-duality group of M-theory on T^3 by combining the $SL(2,\mathbb{Z})$ S-duality group of SYM theory with the modular group of the dual torus $\widehat{T}^3$, namely $SL(3,\mathbb{Z})$.

Consider now an infinite M2-brane extending along the non-compact directions $\{x^5; x^6\}$ on the LHS of eqn.(1.103), the triple T-duality in directions $\{x^7; x^8; x^9\}$ will turn it into a M5-brane extending along $\{x^5; x^6\}$ and wrapped around $\{\hat{x}^7; \hat{x}^8; \hat{x}^9\}$, which corresponds to a transverse five-brane (T5) in M(atrix) theory. In the small radius limit $M_P R_9 \to 0$, the M2-brane descends to an infinite D2-brane of type IIA theory along $\{x^5; x^6\}$, while the transverse M5-brane gives a D4-brane of type IIA theory wrapped around $\{\hat{x}^7; \hat{x}^8\}$. This is summarised in the following commuting diagram:

$$\begin{array}{ccc} \text{M-theory: M2 } [5,6] & \xrightarrow{\mathcal{T}_{789}} & \text{M-theory: M5 } [5,6,\tilde{7},\tilde{8},\tilde{9}] \\ \Big\downarrow{\scriptstyle M_P R_9 \to 0} & & \Big\downarrow{\scriptstyle M_P R_9 \to 0} \\ \text{type IIA: D2 } [5,6] & \xrightarrow{\mathcal{T}_{78}} & \text{M-theory: D4 } [5,6,\tilde{7},\tilde{8}] \end{array} \tag{1.107}$$

Now, the unwrapped M2-brane in the diagram (1.107) corresponds to a configuration in the $\mathcal{N} = 4$ $U(\infty)$ super-Yang-Mills theory (1.105) given by expression (1.101):

$$[X_5, X_6] = -iF_{56}\mathbb{1}$$

where the scalars X^5 and X^6 have condensed in the form $X^5 = \mathsf{p}+$(oscillators) and $X^6 = \mathsf{q}+$(oscillators), with p and q being $\infty \times \infty$ canonical matrices (1.102). Then, the wave-function $|M2; 5, 6\rangle$ corresponding to this configuration is related by S-duality to the wave-function for the transverse T5-brane wrapped on T^3 (1.107):

$$|T5; 5, 6, \hat{7}, \hat{8}, \hat{9}\rangle = S\,|M2; 5, 6\rangle$$

where S is the S-duality operator.

As a first remark, this construction is subject to finding an explicit representation of S-duality in $4D$ super-Yang-Mills theory and its generalisation to $U(\infty)$ gauge group, which are still open problems. Only then can one hope to have a concrete realisation of the transverse five-brane state, and decompactify it to $11D$ M-theory. Note that in the decompactification limit, we take $N \to \infty$ before sending $V_3 \to \infty$ or alternatively $g_3^2 \to 0$ (1.104), so that the effective coupling Ng_3^2 is never perturbative.

This limiting process calls for another remark, namely that we must take $N \to \infty$ before performing S-duality. However, as mentioned before, it is unclear how S-duality in super-Yang-Mills theory behaves in this large N limit, so that this whole construction must be considered with great caution.

With this in mind, it can be shown [232] that the object thus constructed couples correctly to the supergravity fields even in the absence of a transverse fivebrane charge. This would in principle give a solution to the puzzle mentioned at the end of the last section. But it also tend to suggest that transverse fivebranes appear more like solitons in M(atrix) theory, and thus are local objects which do not carry any independent conserved quantum number.

1.6 Outlook

In this chapter, we have shown how a regularisation of the supermembrane Hamiltonian could be achieved in the light-cone gauge, by replacing the target-space fields by $N \times N$ matrices and letting $N \to \infty$. This theory is automatically second-quantised, and we have pointed out that the resulting quantum membrane supersymmetric action was similar to the low-energy action for infinitely many D0-branes. We have then shown how this supersymmetric quantum mechanics for $U(N)$ matrices has been related to M-theory on the (discrete) light-front for N finite, and thus describes a low-energy limit of type IIA string theory where all degrees of freedom except D0-branes have decoupled.

In the large N limit, this M(atrix) theory is conjectured to describe all of M-theory in the Infinite Momentum Frame. Thus the single matrix degrees of freedom, describing D0-branes or bound states at threshold of N D0-branes, have many of the expected properties of the physical supergraviton with longitudinal momentum $P_{10} = N/R$. Since the theory is second quantized, the Fock space of M(atrix) theory contains, in addition, multiple supergraviton.states in its spectrum.

Moreover, the elementary matrix degrees of freedom can be chosen appropriately to build configuration of D0-branes which describe extended objects, such as transverse and longitudinal supermembrane and longitudinal fivebrane of M-theory. However, longitudinal fivebranes are known only for very special geometries, and no complete description for these objects exists yet, even at the classical level. In this perspective, progress in the field of fuzzy geometry (including fuzzy spheres, fuzzy projective spaces etc.) and its implementation in matrix theory, along the line of [170, 214, 2, 12, 15] could be a lead for future work. Moreover, the puzzle remains as to the existence in M(atrix) theory of the transverse fivebrane, which would reproduce the NS5-brane of type IIA string theory by the Seiberg-Sen limiting argument. If it can be shown that there is no good reason why the transverse fivebrane should disappear when taking the IMF limit, while transverse membranes survive, one might question whether D0-branes are truely fundamental degrees of freedom in this case, and whether the BFSS theory really gives a complete description of this sector of M-theory.

We have also mentioned that, for finite N, perturbative loop calculations in M(atrix) theory are in perfect agreement with tree-level graviton scattering amplitudes in DLCQ $11D$ supergravity, up to a reasonable loop order. However, they do not seem to correctly reproduce M-theory quantum corrections to these amplitudes. So, until one is able to take the large N limit in such amplitudes and deal with all the subtleties attached to this procedure, it remains unclear whether M(atrix) theory is correct. For finite N, the BFSS matrix model thus seems incomplete, and it would be desirable to find an alternative matrix model which exactly reproduces the extra terms one needs to add to the dimensionally reduced SYM theory to find agreement with M-theory. This might even prove to be necessary if the large N limits fails to reproduce quantum corrected DLCQ $11D$ supergravity.

Finally, up to now, M(atrix) theory has only been formulated on simple backgrounds, such as the flat background and the pp-wave [26] and deSitter [181, 134] background, which gives but a glimpse of what the (presumably) very rich and complicated vacuum structure of M-theory can be. An extension of the matrix model description to more involved backgrounds is a path that must be further investigated if this theory is to claim relevancy to M-theory. In addition, how to compactify M(atrix) theory on arbitrary compact spaces is still an open question, mainly due to the fact that a simple toroidal compactification of the theory is much more complex than for ordinary field or string theory, since, already for the basic T^d compactification, the quantum mechanics for $U(N)$ matrices becomes, in the large N limit, a supersymmetric nonabelain gauge theory in $d+1$ dimensions.

In any case, until more evidence is found in favour of M(atrix) theory, we still can say that the M(atrix) model approach gives a well-defined and practical framework in which to access at least some portion of M-theory on the light-front, giving a concrete and closed mathematical formulation of the theory from which one can carry out ab intio computations, even though these perturbative loop calculations still remain short of being able to tackle the fundamental issues of M-theory. In this perspective, M(atrix) theory should be considered one more step toward a background independent and microscopic formulation of M-theory and, more generally, of a quantum theory of gravity, and give us some hints of how to think about the fundamental degrees of freedom of M-theory.

In the next chapter, we will turn to a more constructive approach to M-theory based on a matrix theory defined in purely algebraic terms. When reduced to eleven dimensions, this matrix model will be shown to reproduce, in the IMF limit, the BFSS theory with extra corrective terms, and give possible answers to some of the questions raised above.

Chapter 2

Supermatrix models for M-theory based on $\mathfrak{osp}(1|32,\mathbb{R})$

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2.1 Introduction

In the absence of a microscopic description of M-theory, some of its expected features can be obtained by looking at the eleven-dimensional superalgebra [238], whose central charges correspond to the extended objects, i.e. membranes and five-branes present in M-theory. Relations with the hidden symmetries of eleven-dimensional supergravity [81] and its compactifications and associated BPS configurations (see e.g. [92, 246] and references therein) underlined further the importance of the algebraic aspects. It has been conjectured [37] that the large superalgebra $\mathfrak{osp}(1|32)$ may play an important and maybe unifying rôle in M and F theory [240].

In this chapter, we will explore further this possible unifying rôle and study its implications for matrix models. One of the main motivations is to investigate the dynamics of extended objects such as membranes and five-branes, when they are treated on the same footing as the "elementary" degrees of freedom. In order to see eleven and twelve-dimensional structures emerge, we have to embed the $SO(10,2)$ Lorentz algebra and the $SO(10,1)$ Poincaré algebra into the large $\mathfrak{osp}(1|32)$ superalgebra. This will yield certain deformations and extensions of these algebras which nicely include new symmetry generators related to the charges of the extended objects appearing in the eleven and twelve-dimensional theories. The supersymmetry transformations of the associated fields also appear naturally.

Besides these algebraic aspects, we are interested in the dynamics arising from matrix models derived from such algebras. Following ideas initially advocated by Smolin [226], we start with matrices $M \in \mathfrak{osp}(1|32)$ as basic dynamical objects, write down a very simple action for them and then decompose the result according to the different representations of the eleven and twelve-dimensional algebras. In the eleven-dimensional case, we expect this action to contain the scalars X_i of the BFSS matrix model and the associated fermions together with five-branes. In ten dimensions, cubic supermatrix models have already been studied by Azuma, Iso, Kawai and Ohwashi [229] (more details can be found in Azuma's master thesis [11]) in an attempt to compare it with the IIB matrix model of Ishibashi, Kawai, Kitazawa and Tsuchiya [151].

To test the relevance of our model, we try to exhibit its relations with the BFSS matrix model. For this purpose, we perform a boost to the infinite momentum frame (IMF), thus reducing the explicit symmetry of the action to $SO(9)$. Then, we integrate out conjugate momenta and auxiliary fields and calculate an effective action for the scalars X_i, the associated fermions, and higher form fields. What we obtain in the end is the BFSS matrix model with additional terms. In particular, our effective action explicitly contains couplings to 5-brane degrees of freedom, which are thus naturally incorporated in our model as fully dynamical entities. Moreover, we also get additional interactions and masslike terms. This should not be too surprising since we started with a larger theory. The interaction terms we obtain are somewhat similar to the higher-dimensional operators one expects when integrating out (massive) fields in quantum field theory. This can be viewed as an extension of the BFSS theory describing M-theoretical physics in certain non-Minkowskian backgrounds.

The outline of this paper is the following: in the next section we begin by recalling the form of the $\mathfrak{osp}(1|32)$ algebra and the decomposition of its matrices. In section 3 and 4, we study the embedding of the twelve-, resp. eleven-dimensional superalgebras into $\mathfrak{osp}(1|32)$, and obtain the corresponding algebraic structure including the extended objects described by a six-resp. five-form. We establish the supersymmetry transformations of the fields, and write down a cubic matrix model which yields an action for the various twelve- resp. eleven-dimensional fields. Finally, in section 5, we study further the eleven-dimensional matrix model, compute an effective action and do the comparison with the BFSS model.

2.2 The $\mathfrak{osp}(1|32, \mathbb{R})$ superalgebra

We first recall some definitions and properties of the unifying superalgebra $\mathfrak{osp}(1|32, \mathbb{R})$ which will be useful in the following chapters. The superalgebra is defined by the following three equations:

$$\begin{aligned} [Z_{AB}, Z_{CD}] &= \Omega_{AD} Z_{CB} + \Omega_{AC} Z_{DB} + \Omega_{BD} Z_{CA} + \Omega_{BC} Z_{DA} , \\ [Z_{AB}, Q_C] &= \Omega_{AC} Q_B + \Omega_{BC} Q_A , \\ \{Q_A, Q_B\} &= Z_{AB} , \end{aligned} \tag{2.1}$$

where Ω_{AB} is the antisymmetric matrix defining the $\mathfrak{sp}(32, \mathbb{R})$ symplectic Lie algebra. Let us now give an equivalent description of elements of $\mathfrak{osp}(1|32, \mathbb{R})$. Following Cornwell [70], we call $\mathbb{R}B_L$ the real Grassmann algebra with L generators, and $\mathbb{R}B_{L0}$ and $\mathbb{R}B_{L1}$ its even and odd subspace respectively. Similarly, we define a $(p|q)$ supermatrix to be even (degree 0) if it can be written as:

$$M = \begin{pmatrix} A & B \\ F & D \end{pmatrix} .$$

where A and D are $p \times p$, resp. $q \times q$ matrices with entries in $\mathbb{R}B_{L0}$, while B and F are $p \times q$ (resp. $q \times p$) matrices, with entries in $\mathbb{R}B_{L1}$. On the other hand, odd supermatrices (degree 1) are characterized by 4 blocks with the opposite parities.

We define the supertranspose of a supermatrix M as[1]:

$$M^{ST} = \begin{pmatrix} A^\top & (-1)^{deg(M)} F^\top \\ -(-1)^{deg(M)} B^\top & D^\top \end{pmatrix} .$$

If one chooses the orthosymplectic metric to be the following 33×33 matrix:

$$G = \begin{pmatrix} 0 & -\mathbb{1}_{16} & 0 \\ \mathbb{1}_{16} & 0 & 0 \\ 0 & 0 & i \end{pmatrix} ,$$

[1]We warn the reader that this is not the same convention as in [11].

(where the i is chosen for later convenience to yield a hermitian action), we can define the $\mathfrak{osp}(1|32,\mathbb{R})$ superalgebra as the algebra of (32|1) supermatrices M satisfying the equation:

$$M^{ST}\cdot G+(-1)^{deg(M)}G\cdot M=0\ .$$

From this defining relation, it is easy to see that an even orthosymplectic matrix should be of the form:

$$M=\begin{pmatrix} A & B & \Phi_1 \\ F & -A^\top & \Phi_2 \\ -i\Phi_2^\top & i\Phi_1^\top & 0\end{pmatrix}=\begin{pmatrix} m & \Psi \\ -i\Psi^\top C & 0\end{pmatrix}\ , \tag{2.2}$$

where A,B and F are 16×16 matrices with entries in $\mathbb{R}B_{L0}$ and $\Psi=(\Phi_1,\Phi_2)^\top$ is a 32-components Majorana spinors with entries in $\mathbb{R}B_{L1}$. Furthermore, $B=B^\top$, $F=F^\top$ so that $m\in\mathfrak{sp}(32,\mathbb{R})$ and C is the following 32×32 matrix:

$$C=\begin{pmatrix} 0 & -\mathbb{1}_{16} \\ \mathbb{1}_{16} & 0\end{pmatrix}\ , \tag{2.3}$$

and will turn out to act as the charge conjugation matrix later on.

Such a matrix in the Lie superalgebra $\mathfrak{osp}(1|32,\mathbb{R})$ can also be regarded as a linear combination of the generators thereof, which we decompose in a bosonic and a fermionic part as:

$$H=\begin{pmatrix} h & 0 \\ 0 & 0\end{pmatrix}+\begin{pmatrix} 0 & \chi \\ -i\chi^\top C & 0\end{pmatrix}=h^{AB}Z_{AB}+\chi^A Q_A \tag{2.4}$$

where Z_{AB} and Q_A are the same as in (2.1). An orthosymplectic transformation will then act as:

$$\delta^{(1)}_H=[H,\bullet]=h^{AB}[Z_{AB},\bullet]+\chi^A[Q_A,\bullet]=\delta^{(1)}_h+\delta^{(1)}_\chi\ . \tag{2.5}$$

This notation allows us to compute the commutation relations of two orthosymplectic transformations characterized by $H=(h,\chi)$ and $E=(e,\epsilon)$. Recalling that for Majorana fermions $\chi^\top C\epsilon=\epsilon^\top C\chi$, we can extract from $[\delta^{(1)}_H,\delta^{(1)}_E]$ the commutation relation of two symplectic transformations:

$$[\delta^{(1)}_h,\delta^{(1)}_e]_A{}^B=\begin{pmatrix} [h,e]_A{}^B & 0 \\ 0 & 0\end{pmatrix}\ , \tag{2.6}$$

the commutation relation between a symplectic transformation and a supersymmetry:

$$[\delta^{(1)}_h,\delta^{(1)}_\chi]_A{}^B=\begin{pmatrix} 0 & h_A{}^D\chi_D \\ i(\chi^\top C)^D h_D{}^B & 0\end{pmatrix}\ , \tag{2.7}$$

and the commutator of two supersymmetries:

$$[\delta^{(1)}_\epsilon,\delta^{(1)}_\chi]_A{}^B=\begin{pmatrix} -i(\chi_A(\epsilon^\top C)^B-\epsilon_A(\chi^\top C)^B) & 0 \\ 0 & 0\end{pmatrix}\ . \tag{2.8}$$

2.3 The twelve-dimensional case

In order to be embedded into $\mathfrak{osp}(1|32,\mathbb{R})$, a Lorentz algebra must have a fermionic representation of 32 real components at most. The biggest number of dimensions in which this is the case is 12, where Dirac matrices are 64×64. As this dimension is even, there exists a Weyl representation of 32 complex components. We need furthermore a Majorana condition to make them real. This depends of course on the signature of space-time and is possible only for signatures $(10,2)$, $(6,6)$ and $(2,10)$, when (s,t) are such that $s-t=0 \mod 8$. Let us concentrate in this paper on the most physical case (possibly relevant for F-theory) where the number of timelike dimensions is 2. However, since we choose to concentrate on the next section's M-theoretical case, we will not push this analysis too far and will thus restrict ourselves to the computation of the algebra and the cubic action.

To express the $\mathfrak{osp}(1|32,\mathbb{R})$ superalgebra in terms of 12-dimensional objects, we have to embed the $SO(10,2)$ Dirac matrices into $\mathfrak{sp}(32,\mathbb{R})$ and replace the fundamental representation of $\mathfrak{sp}(32,\mathbb{R})$ by $SO(10,2)$ Majorana-Weyl spinors. A convenient choice of 64×64 Gamma matrices is the following:

$$\Gamma^0 = \begin{pmatrix} 0 & -\mathbb{1}_{32} \\ \mathbb{1}_{32} & 0 \end{pmatrix}, \Gamma^{11} = \begin{pmatrix} 0 & \tilde{\Gamma}^0 \\ \tilde{\Gamma}^0 & 0 \end{pmatrix}, \Gamma^i = \begin{pmatrix} 0 & \tilde{\Gamma}^i \\ \tilde{\Gamma}^i & 0 \end{pmatrix} \ \forall\, i = 1,\dots,10, \tag{2.9}$$

where $\tilde{\Gamma}^0$ is the 32×32 symplectic form:

$$\tilde{\Gamma}^0 = \begin{pmatrix} 0 & -\mathbb{1}_{16} \\ \mathbb{1}_{16} & 0 \end{pmatrix}$$

which, with the $\tilde{\Gamma}^i$'s, builds a Majorana representation of the $10+1$-dimensional Clifford algebra $\{\tilde{\Gamma}^\mu,\tilde{\Gamma}^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_{32}$ for the mostly + metric . Of course, $\tilde{\Gamma}^{10} = \tilde{\Gamma}^0\tilde{\Gamma}^1\dots\tilde{\Gamma}^9$. This choice has $(\Gamma^0)^2 = (\Gamma^{11})^2 = -\mathbb{1}_{64}$, while $(\Gamma^i)^2 = \mathbb{1}_{64}$, $\forall i = 1\dots 10$, and gives a representation of $\{\Gamma^M,\Gamma^N\} = 2\eta^{MN}\mathbb{1}_{64}$ for a metric of the type $(-,+,\dots,+,-)$. As we have chosen all Γ's to be real, this allows to take $B=\mathbb{1}$ in $\Psi^* = B\Psi$, which implies that the charge conjugation matrix $C = \Gamma^0\Gamma^{11}$, i.e.

$$C = \begin{pmatrix} -\tilde{\Gamma}^0 & 0 \\ 0 & \tilde{\Gamma}^0 \end{pmatrix}.$$

This will then automatically satisfy:

$$C\Gamma^M C^{-1} = (\Gamma^M)^\top , \ C\Gamma^{MN}C^{-1} = -(\Gamma^{MN})^\top \tag{2.10}$$

and more generally:

$$C\Gamma^{M_1\dots M_n}C^{-1} = (-1)^{n(n-1)/2}(\Gamma^{M_1\dots M_n})^\top . \tag{2.11}$$

The chirality matrix for this choice will be:

$$\Gamma_* = \Gamma^0\dots\Gamma^{11} = \begin{pmatrix} -\mathbb{1}_{32} & 0 \\ 0 & \mathbb{1}_{32} \end{pmatrix}.$$

We will identify the fundamental representation of $\mathfrak{sp}(32,\mathbb{R})$ with positive chirality Majorana-Weyl spinors of $SO(10,2)$, i.e. those satisfying: $\mathcal{P}_+\Psi = \Psi$, for:

$$\mathcal{P}_+ = \frac{1}{2}(1+\Gamma_*) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{32} \end{pmatrix}.$$

Decomposing the 64 real components of the positive chirality spinor Ψ into $32+32$ or $16+16+16+16$, we can write: $\Psi^\top = (0,\Phi^\top) = (0,0,\Phi_1^\top,\Phi_2^\top)$. Because $\overline{\Psi} = \Psi^\dagger\Gamma^0\Gamma^{11} = \Psi^\top C$, this choice for the charge conjugation matrix C is convenient since it will act as C in equation (2.3) (though with a slight abuse of notation), and thus:

$$(0,0,-i\Phi_2^\top, i\Phi_1^\top) = (0,-i\Phi^\top\tilde{\Gamma}^0) = -i\Psi^\top C = -i\overline{\Psi}.$$

2.3.1 Embedding of $SO(10,2)$ in $OSp(1|32,\mathbb{R})$

We would now like to study how the Lie superalgebra of $OSp(1|32,\mathbb{R})$ can be expressed in terms of generators of the super-Lorentz algebra in 10+2 dimensions with additional symmetry generators. In other words, if we separate the $\mathfrak{sp}(32,\mathbb{R})$ transformations h into a part sitting in the Lorentz algebra and a residual $\mathfrak{sp}(32,\mathbb{R})$ part, we can give an explicit description of the orthosymplectic algebra (2.1) in the form of an enhanced super-Lorentz algebra, where the central charges of the super-Lorentz algebra appear as new generators of the enhanced superalgebra.

To do so, we need to expand a symplectic matrix in irreducible tensors of $SO(10,2)$. This can be done as follows:

$$h_A{}^B = \frac{1}{2!}(\mathcal{P}_+\Gamma^{MN})_A{}^B h_{MN} + \frac{1}{6!}(\mathcal{P}_+\Gamma^{M_1\dots M_6})_A{}^B h^+_{M_1\dots M_6} \tag{2.12}$$

where the + on $h_{M_1\dots M_6}$ recalls its self-duality, and the components of h in the decomposition in irreducible tensors of $SO(10,2)$ are given by $h_{MN} = -\frac{1}{32}Tr_{\mathfrak{sp}(32,\mathbb{R})}(h\Gamma_{MN})$ and $h^+_{M_1\dots M_6} = -\frac{1}{32}Tr_{\mathfrak{sp}(32,\mathbb{R})}(h\Gamma_{M_1\dots M_6})$. Indeed, a real symplectic 32×32 matrix satisfies $m\tilde{\Gamma}^0 = -\tilde{\Gamma}^0 m^\top$, and C acts like $\tilde{\Gamma}^0$ on $\mathcal{P}_+\Gamma^{M_1\dots M_n}$. Furthermore, (2.11) indicates that:

$$C(1+\Gamma_*)\Gamma^{M_1\dots M_n} = (-1)^{n(n-1)/2}((1+(-1)^n\Gamma_*)\Gamma^{M_1\dots M_n})^T C\ . \tag{2.13}$$

Thus, $\mathcal{P}_+\Gamma^{M_1\dots M_n}$ is symplectic iff n is even and $(-1)^{n(n-1)/2} = -1$. For $0\le n\le 6$, this is only the case if $n=2$ or 6. As a matter of fact, the numbers of independent components match since: $12\cdot 11/2 + 1/2\cdot 12!/(6!)^2 = 528 = 16\cdot 33$.

The symplectic transformation δ_h may then be decomposed into irreducible 12-dimensional tensors of symmetry generators, namely the $\mathfrak{so}(10,2)$ Lorentz algebra generator J^{MN} and a new 6-form symmetry generator $J^{M_1\dots M_6}$. To calculate the commutation relations of this enhanced Lorentz algebra, we will choose the following representation of the symmetry generators:

$$J^{MN} = \frac{1}{2!}\,\mathcal{P}_+\,\Gamma^{MN}\,, \qquad J^{M_1\dots M_6} = \frac{1}{6!}\,\mathcal{P}_+\,\Gamma^{M_1\dots M_6}\ .$$

so that a symplectic transformation will be given in this base by:

$$h = h_{MN}\,J^{MN} + h_{M_1\dots M_6}\,J^{M_1\dots M_6}\ .$$

We will now turn to computing the superalgebra induced by the above bosonic generators and the supercharges for $D=10+2$. The bosonic commutators may readily be computed using:

$$[\Gamma_{M_1\dots M_k},\Gamma_{N_1\dots N_l}] = \begin{cases} \displaystyle\sum_{j=0}^{\lfloor (min(k,l)-1)/2\rfloor} (-1)^{k-j-1}\,2\cdot(2j+1)!\binom{k}{2j+1}\binom{l}{2j+1}\times \\ \qquad \times\eta_{[M_1[N_1}\cdots\eta_{M_{2j+1}N_{2j+1}}\Gamma_{M_{2j+2}\dots M_k]N_{2j+2}\dots N_l]} \text{ if } k\cdot l \text{ is even and,} \\ \displaystyle\sum_{j=0}^{(min(k,l)-1)/2}(-1)^j\,2\cdot(2j)!\binom{k}{2j}\binom{l}{2j}\times \\ \qquad \times\eta_{[M_1[N_1}\cdots\eta_{M_{2j}N_{2j}}\Gamma_{M_{2j+1}\dots M_k]N_{2j+1}\dots N_l]} \qquad \text{if } k\cdot l \text{ is odd.} \end{cases} \tag{2.14}$$

On the other hand, for the commutation relations involving fermionic generators, we proceed as follows. We expand equation (2.7) of the preceding chapter in irreducible tensors of $SO(10,2)$:

$$[\delta_\chi,\delta_h] = -\frac{1}{2!}\,\chi^A\,h_{MN}(\mathcal{P}_+\,\Gamma^{MN})^B{}_A Q_B - \frac{1}{6!}\,\chi^A\,h_{M_1\dots M_6}(\mathcal{P}_+\,\Gamma^{M_1\dots M_6})^B{}_A Q_B\ ,$$

which is also given by:

$$[\delta_\chi, \delta_h] \;=\; \chi^A\, h_{MN}[Q_A, J^{MN}] + \chi^A\, h_{M_1\ldots M_6}[Q_A, J^{M_1\ldots M_6}] \,. \tag{2.15}$$

Comparing terms pairwise, we see that the supercharges transform as:

$$[J^{MN}, Q_A] \;=\; \frac{1}{2!}\,(\mathcal{P}_+\,\Gamma^{MN})^B{}_A Q_B \;, \qquad [J^{M_1\ldots M_6}, Q_A] \;=\; \frac{1}{6!}\,(\mathcal{P}_+\, J^{M_1\ldots M_6})^B{}_A Q_B \;.$$

Finally, in order to obtain the anti-commutator of two supercharges, we expand the RHS of (2.8) in the bosonic generators J^{MN} and $J^{M_1\ldots M_6}$:

$$-\chi^A \epsilon_B\{Q_A, Q^B\} \;\equiv\; [\delta_\chi, \delta_\epsilon] \;=\; \frac{i}{16}(\chi^\top C\Gamma_{MN}\epsilon)J^{MN} + \frac{i}{16}(\chi^\top C\Gamma_{M_1\ldots M_6}\epsilon)J^{M_1\ldots M_6} \;, \tag{2.16}$$

and match the first and the last term of the equation.

Summarizing the results of this section, we get the following 12-dimensional realization of the superalgebra $\mathfrak{osp}(1|32,\mathbb{R})$[2]:

$$\begin{aligned}
[J^{MN}, J^{OP}] \;&=\; -4\eta^{[M[O}J^{N]P]} \\
[J^{MN}, J^{M_1\ldots M_6}] \;&=\; -12\,\eta^{[M[M_1}\,J^{N]M_2\ldots M_6]} \\
[J^{N_1\ldots N_6}, J^{M_1\ldots M_6}] \;&=\; -4!\,6!\,\eta^{[N_1[M_1}\,\eta^{N_2\,M_2}\,\eta^{N_3\,M_3}\,\eta^{N_4\,M_4}\,\eta^{N_5\,M_5}\,J^{N_6]M_6]} \\
&\quad+\;2\cdot 6^2\,\eta^{[N_1[M_1}\,\varepsilon^{N_2\ldots N_6]M_2\ldots M_6]}{}_{AB}\,J^{AB} \\
&\quad+\;4\left(\frac{6!}{4!}\right)^3\,\eta^{[N_1[M_1}\,\eta^{N_2\,M_2}\,\eta^{N_3\,M_3}\,J^{N_4\ldots N_6]M_4\ldots M_6]}
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
[J^{MN}, Q_A] \;&=\; \frac{1}{2}\,(\mathcal{P}_+\,\Gamma^{MN})^B{}_A Q_B \\
[J^{M_1\ldots M_6}, Q_A] \;&=\; \frac{1}{6!}\,(\mathcal{P}_+\,\Gamma^{M_1\ldots M_6})^B{}_A Q_B \\
\{Q_A, Q^B\} \;&=\; -\frac{i}{16}(C\Gamma_{MN})_A{}^B J^{MN} - \frac{i}{16}(C\Gamma_{M_1\ldots M_6})_A{}^B J^{M_1\ldots M_6} \;,
\end{aligned}$$

where antisymmetrization brackets on the RHS are meant to match the anti-symmetry of indices on the LHS.

2.3.2 Supersymmetry transformations of 12D matrix fields

In the following, we will construct a dynamical matrix model based on the symmetry group $\mathfrak{osp}(1|32,\mathbb{R})$ using elements in the adjoint representation of this superalgebra, i.e. matrices in this superalgebra. We can write such a matrix as:

$$M \;=\; \begin{pmatrix} m & \Psi \\ -i\Psi^\top C & 0 \end{pmatrix} \;, \tag{2.18}$$

[2]Notice that the second term appearing on the right-hand side of the third commutator is in fact proportional to $\Gamma^{M_1\ldots M_{10}}$, which, in turn, can be reexpressed as $\Gamma^{M_1\ldots M_{10}} = -(1/2)\varepsilon^{AB\,M_1\ldots M_{10}}\,\Gamma_{AB}\Gamma_*$. Indeed, in $10+2$ dimensions, we always have:

$$\Gamma^{M_1\ldots M_k} = \frac{(-1)^{\frac{(k-1)k}{2}}}{(12-k)!}\varepsilon^{M_1\ldots M_k M_{k+1}\ldots M_{12}}\,\Gamma_{M_{k+1}\ldots M_{12}}\Gamma_*$$

where m is in the adjoint representation of $\mathfrak{sp}(32,\mathbb{R})$ and Ψ is in the fundamental one. Since M belongs to the adjoint representation, a SUSY will act on it in the following way:

$$\delta_\chi^{(1)} M_A{}^B = \chi^D [Q_D, M]_A{}^B = \begin{pmatrix} -i(\chi_A (\Psi^\top C)^B - \Psi_A (\chi^\top C)^B) & -m_A{}^D \chi_D \\ -i(\chi^\top C)^D m_D{}^B & 0 \end{pmatrix} \tag{2.19}$$

In our particular 12D setting, m gives rise to a 2-form field C (with $SO(10,2)$ indices, not to be confused with the charge conjugation matrix with $\mathfrak{sp}(32,\mathbb{R})$ indices) and a self-dual 6-form field Z^+, as follows:

$$m_A{}^B = \frac{1}{2!}(\mathcal{P}_+\Gamma^{MN})_A{}^B C_{MN} + \frac{1}{6!}(\mathcal{P}_+\Gamma^{M_1\dots M_6})_A{}^B Z^+_{M_1\dots M_6} \ . \tag{2.20}$$

We can extract the supersymmetry transformations of C, Z^+ and Ψ from (2.19) and we obtain:

$$\begin{aligned} \delta_\chi^{(1)} C_{MN} &= \frac{i}{16}\overline{\chi}\,\Gamma_{MN}\Psi \ , \\ \delta_\chi^{(1)} Z^+_{M_1\dots M_6} &= \frac{i}{16}\overline{\chi}\,\Gamma_{M_1\dots M_6}\Psi \ , \\ \delta_\chi^{(1)}\Psi &= -\frac{1}{2}\Gamma^{MN}\chi\, C_{MN} - \frac{1}{6!}\Gamma^{M_1\dots M_6}\chi\, Z^+_{M_1\dots M_6} \ . \end{aligned} \tag{2.21}$$

These formulæ allow us to compute the effect of two successive supersymmetry transformations using (2.11) and (2.14):

$$\begin{aligned} [\delta_\chi^{(1)},\delta_\epsilon^{(1)}]\Psi &= \frac{i}{16}\Big\{(\overline{\epsilon}\Psi)\,\chi - (\overline{\chi}\,\Psi)\,\epsilon\Big\} \ , \\ [\delta_\chi^{(1)},\delta_\epsilon^{(1)}]C_{MN} &= \frac{i}{4}\overline{\chi}\left\{\Gamma_{[M}{}^P C_{N]P} + \frac{1}{5!}\Gamma_{[M}{}^{M_1\dots M_5} Z^+_{N]M_1\dots M_5}\right\}\mathcal{P}_+\,\epsilon \ , \\ [\delta_\chi^{(1)},\delta_\epsilon^{(1)}]Z^+_{M_1\dots M_6} &= \overline{\chi}\left\{\frac{3i}{4}\Gamma_{[M_1\dots M_5}{}^N C_{M_6]N} + \frac{3i}{2}\Gamma_{[M_1}{}^N Z^+_{M_2\dots M_6]N} - \right. \\ &\quad \left. -\frac{5i}{12}\Gamma_{[M_1M_2M_3}{}^{N_1N_2N_3} Z^+_{M_4M_5M_6]N_1N_2N_3}\right\}\mathcal{P}_+\,\epsilon \ , \end{aligned} \tag{2.22}$$

where we used the self-duality[3] of Z^+. At this stage, we can mention that the above results are in perfect agreement with the adjoint representation of $[\delta_\chi^{(1)},\delta_\epsilon^{(1)}]$ (viz. (2.8)) on the matrix fields.

2.3.3 $\mathfrak{sp}(32,\mathbb{R})$ transformations of the fields and their commutation relation with supersymmetries

To see under which transformations an $\mathfrak{osp}(1|32,\mathbb{R})$-based matrix model should be invariant, one should look at the full transformation properties including the bosonic $\mathfrak{sp}(32,\mathbb{R})$ transformations. In close analogy with equation (2.19), we have the following full transformation law of M:

$$\delta_H^{(1)} M_A{}^B = \left[\begin{pmatrix} h & \chi \\ -i\overline{\chi} & 0\end{pmatrix}, \begin{pmatrix} m & \Psi \\ -i\overline{\Psi} & 0\end{pmatrix}\right]_A{}^B \ , \tag{2.23}$$

implying the following transformation rules:

$$\delta_H^{(1)} m_A{}^B = [h,m]_A{}^B - i(\chi_A\overline{\Psi}^B - \Psi_A\overline{\chi}^B) \ , \tag{2.24}$$

$$\delta_H^{(1)}\Psi_A = h_A{}^C\Psi_C - m_A{}^C\chi_C \ . \tag{2.25}$$

[3] Z^+ satisfies $Z^+_{M_1\dots M_6} = \frac{1}{6!}\varepsilon_{M_1\dots M_6}{}^{N_1\dots N_6} Z^+_{N_1\dots N_6}$

We then want to extract from the first of the above equations the full transformation properties of C_{MN} and $Z^+_{M_1 \ldots M_6}$. From (2.17) and (2.22) or directly using (2.14) and the cyclicity of the trace, the bosonic transformations are:

$$\begin{aligned} \delta^{(1)}_h C_{MN} =& \; 4h^P{}_{[N} C_{M]P} + \frac{4}{5!} h^{N_1 \ldots N_5}{}_{[N} Z^+_{M]N_1 \ldots N_5} \,, \\ \delta^{(1)}_h Z^+_{M_1 \ldots M_6} =& \; 12 h_{[M_1 \ldots M_5}{}^P C_{M_6]P} - 24\, h^N{}_{[M_1} Z^+_{M_2 \ldots M_6]N} - \\ & + \frac{20}{3} h^{N_1 N_2 N_3}{}_{[M_1 M_2 M_3} Z^+_{M_4 M_5 M_6] N_1 N_2 N_3} \,, \end{aligned} \tag{2.26}$$

while the fermionic part is as in (2.21). If one uses (2.26) to compute the commutator of a supersymmetry and an $\mathfrak{sp}(32, \mathbb{R})$ transformation, the results will look very complicated. On the other hand, the commutator of two symmetry transformations may be cast in a compact form using the graded Jacobi identity of the $\mathfrak{osp}(1|32, \mathbb{R})$ superalgebra, which comes into the game since matrix fields are in the adjoint representations of $\mathfrak{osp}(1|32, \mathbb{R})$.

Such a commutator acting on the fermionic field Ψ yields:

$$\begin{aligned} [\delta^{(1)}_\chi, \delta^{(1)}_h]\, \Psi =& \; -hm\chi + [h, m]\chi \; = \; -mh\chi \; = \\ =& \; -\frac{1}{2!} (\mathcal{P}_+ \Gamma^{MN} h\chi) C_{MN} - \frac{1}{6!} (\mathcal{P}_+ \Gamma^{M_1 \ldots M_6} h\chi) Z^+_{M_1 \ldots M_6} \,. \end{aligned} \tag{2.27}$$

The same transformation on m leads to:

$$[\delta^{(1)}_\chi, \delta^{(1)}_h]\, m_A{}^B = i \left(\Psi_A (\chi^\top h^\top C)^B - (h\chi)_A (\Psi^\top C)^B \right) \,, \tag{2.28}$$

which in components reads:

$$[\delta^{(1)}_\chi, \delta^{(1)}_h]\, C_{MN} = \frac{i}{16} \chi^\top C h \Gamma_{MN} \Psi \,, \tag{2.29}$$

$$[\delta^{(1)}_\chi, \delta^{(1)}_h]\, Z^+_{M_1 \ldots M_6} = \frac{i}{16} \chi^\top C h \Gamma_{M_1 \ldots M_6} \Psi \,. \tag{2.30}$$

In eqns. (2.27), (2.29) and (2.30), one could write h in components as in (2.12) and use:

$$\Gamma_{M_1 \ldots M_k} \Gamma_{N_1 \ldots N_l} = \sum_{j=0}^{min(k,l)} (-1)^{j/2(2k-j-1)}\, j! \binom{k}{j} \binom{l}{j} \eta_{[M_1 [N_1} \cdots \eta_{M_j N_j} \Gamma_{M_{j+1} \ldots M_k] N_{j+1} \ldots N_l]} \tag{2.31}$$

to develop the products of Gamma matrices in irreducible tensors of $SO(10, 2)$ and obtain a more explicit result. The final expression for (2.27) and (2.30) will contain Gamma matrices with an even number of indices ranging from 0 to 12, while in (2.29) the number of indices will stop at 8. Since we won't use this result as such in the following, we won't give it here explicitly.

2.3.4 A note on translational invariance and kinematical supersymmetries

At this point, we want to make a comment on the so-called kinematical supersymmetries that have been discussed in the literature on matrix models ([151], [229]). Indeed, in the IIB matrix model, commutation relations of dynamical supersymmetries do not close to give space-time translations, i.e. they do not shift the target space-time fields X^M by a constant vector.

However, as was pointed out in [151] and [229], if one introduces so-called kinematical supersymmetry transformations, their commutator with dynamical supersymmetries yields the expected translations by a constant vector, as we explained in subsection 4.2.2. By kinematical supersymmetries, one simply means translations of fermions by a constant Grassmannian odd parameter. In our case, this assumes the form:

$$\delta^{(2)}_\xi C_{MN} = \delta^{(2)}_\xi Z^+_{M_1\dots M_6} = 0 \,, \qquad \delta^{(2)}_\xi \Psi = \xi \,, \tag{2.32}$$
$$\Longrightarrow [\delta^{(2)}_\xi, \delta^{(2)}_\zeta] M = 0$$

Since there is no vector field to be interpreted as space-time coordinates in this 12-dimensional setting, it is interesting to look at the interplay between dynamical and kinematical supersymmetries (which we denote respectively by $\delta^{(1)}$ and $\delta^{(2)}$) when acting on higher-rank tensors. In our case:

$$[\delta^{(1)}_\chi, \delta^{(2)}_\xi] C_{MN} = -\frac{i}{16}(\chi^\top C\Gamma_{MN}\xi) \,, \qquad [\delta^{(1)}_\chi, \delta^{(2)}_\xi] Z^+_{M_1\dots M_6} = -\frac{i}{16}(\chi^\top C\Gamma_{M_1\dots M_6}\xi) \,. \tag{2.33}$$

Thus, $[\delta^{(1)}_\chi, \delta^{(2)}_\xi]$ applied to p–forms closes to translations by a constant p-form, generalizing the vector case mentioned above.

For fermions, we have as expected:

$$[\delta^{(1)}_\chi, \delta^{(2)}_\xi]\Psi = 0 \,. \tag{2.34}$$

It is however more natural to consider dynamical and kinematical symmetries to be independent. We would thus expect them to commute. With this in mind, we suggest a generalized version of the translational symmetries introduced in (2.32):

$$\delta^{(2)}_K \Psi = \xi, \quad \delta^{(2)}_K C_{MN} = k_{MN}, \quad \delta^{(2)}_K Z^+_{M_1\dots M_6} = k^+_{M_1\dots M_6} \,. \tag{2.35}$$

It is then natural that the matrix

$$K = \begin{pmatrix} k & \xi \\ -i\xi^\top C & 0 \end{pmatrix} \tag{2.36}$$

should transform in the adjoint of $\mathfrak{osp}(1|32,\mathbb{R})$, which means that:

$$\delta^{(1)}_H k_A{}^B = [h,k]_A{}^B - i(\chi_A(\xi^\top C)^B - \xi_A(\chi^\top C)^B) \tag{2.37}$$
$$\delta^{(1)}_H \xi_A = h_A{}^C \xi_C - k_A{}^C \chi_C. \tag{2.38}$$

We can now compute the general commutation relations between translational symmetries $M \to M + K$ and $\mathfrak{osp}(1|32,\mathbb{R})$ transformations and conclude that these operations actually commute:

$$[\delta^{(1)}_H, \delta^{(2)}_K] M = 0 \,. \tag{2.39}$$

2.3.5 Twelve-dimensional action for supersymmetric cubic matrix model

We will now build the simplest gauge- and translational-invariant $\mathfrak{osp}(1|32,\mathbb{R})$ supermatrix model with $U(N)$ gauge group. For this purpose, we promote each entry of the matrix M to a hermitian matrix in the Lie algebra of $\mathfrak{u}(N)$ for some value of N. We choose the generators $\{t^a\}_{a=1,\dots,N^2}$ of $\mathfrak{u}(N)$ so that: $[t^a, t^b] = if^{abc}t^c$ and $Tr_{\mathfrak{u}(N)}(t^a \cdot t^b) = \delta^{ab}$.

In order to preserve both orthosymplectic and gauge invariance of the model, it suffices to write its action as a supertrace over $\mathfrak{osp}(1|32,\mathbb{R})$ and a trace over $\mathfrak{u}(N)$ of a polynomial of $\mathfrak{osp}(1|32,\mathbb{R})\otimes\mathfrak{u}(N)$ matrices. Following [226], we consider the simplest model containing interactions, namely: $STr_{\mathfrak{osp}(1|32,\mathbb{R})} Tr_{\mathfrak{u}(N)}(M[M,M]_{\mathfrak{u}(N)})$. For hermiticity's sake one has to multiply such an action by a factor of i. We also introduce a coupling constant g^2. This cubic action takes the following form:

$$\begin{aligned} I = & \frac{i}{g^2} STr_{\mathfrak{osp}(1|32,\mathbb{R})} Tr_{\mathfrak{u}(N)}(M[M,M]_{\mathfrak{u}(N)}) = -\frac{1}{g^2} f^{abc} STr_{\mathfrak{osp}(1|32,\mathbb{R})}(M^a M^b M^c) = \\ = & -\frac{1}{g^2} f^{abc} \left(Tr_{\mathfrak{sp}(32,\mathbb{R})}(m^a m^b m^c) + 3i \Psi^{a\top} C m^b \Psi^c \right) \end{aligned} \tag{2.40}$$

which we can now express in terms of 12-dimensional representations, where the symplectic matrix m is given by (2.20).

Let us give a short overview of the steps involved in the computation of the trace in (2.40). It amounts to performing traces of triple products of m^a's over $\mathfrak{sp}(32,\mathbb{R})$, i.e. traces of products of Dirac matrices. We proceed by decomposing such products into their irreducible representations using (2.31). The only contributions surviving the trace are those proportional to the unit matrix. Thus, the only terms left in (2.40) will be those containing traces over triple products of 2-forms, over products of a 2-form and two 6-forms, and over triple products of 6-forms, while terms proportional to products of two 2-forms and a 6-form will yield zero contributions.

The two terms involving Z^+'s (to wit CZ^+Z^+ and $Z^+Z^+Z^+$) require some care, since $\Gamma^{A_1 \ldots A_{12}}$ is proportional to $\Gamma_\star$ in $12D$, and hence $Tr(\mathcal{P}^+ \, \Gamma^{A_1\ldots A_{12}}) \propto Tr(\Gamma_\star^2) \neq 0$. Since double products of six-indices Dirac matrices decompose into $\mathbb{1}$ and Dirac matrices with 2, 4 up to 12 indices, their trace with Γ^{MN} will keep terms with 2, 10 or 12 indices (the last two containing Levi-Civita tensors) while their trace with $\Gamma^{M_1 \ldots M_6}$ will only keep those terms with 6, 8, 10 and 12 indices.

Finally, putting all contributions together, exploiting the self-duality of Z^+ and rewriting cubic products of fields contracted by f^{abc} as a trace over $\mathfrak{u}(N)$, we get:

$$\begin{aligned} I = & \frac{32i}{g^2} Tr_{\mathfrak{u}(N)} \Bigg(C_M{}^N \, [C_N{}^O, C_O{}^M]_{\mathfrak{u}(N)} - \frac{1}{20} C_A{}^B \, [Z_B^{+M_1\ldots M_5}, Z^{+}_{M_1\ldots M_5}{}^{A}]_{\mathfrak{u}(N)} + \\ & + \frac{61}{2(3!)^3} Z^{+}_{ABC}{}^{DEF} \, [Z^{+}_{DEF}{}^{GHI}, Z^{+}_{GHI}{}^{ABC}]_{\mathfrak{u}(N)} + \\ & + \frac{3i}{64} \Psi^\top C \mathcal{P}_+ \Gamma^{MN} [C_{MN}, \Psi]_{\mathfrak{u}(N)} + \frac{3i}{32 \cdot 6!} \Psi^\top C \mathcal{P}_+ \Gamma^{M_1 \ldots M_6} [Z^{+}_{M_1\ldots M_6}, \Psi]_{\mathfrak{u}(N)} \Bigg) \end{aligned}$$

where we have chosen: $\varepsilon^{0\ldots 11} = \varepsilon_{0\ldots 11} = +1$, since the metric contains two time-like indices. Similarly, one can decompose invariant terms such as $STr_{\mathfrak{osp}(1|32,\mathbb{R})} Tr_{\mathfrak{u}(N)}(M^2)$ and $STr_{\mathfrak{osp}(1|32,\mathbb{R})} Tr_{\mathfrak{u}(N)}([M,M]_{\mathfrak{u}(N)} [M,M]_{\mathfrak{u}(N)})$, etc. While it might be interesting to investigate further the $12D$ physics obtained from such models and compare it to F-theory dynamics, we will not do so here. We will instead move to a detailed study of the better known $11D$ case, possibly relevant for M-theory.

2.4 Study of the 11D M-theory case

We now want to study the $11D$ matrix model more thoroughly. Similarly to the 12 dimensional case, we embed the $SO(10,1)$ Clifford algebra into $\mathfrak{sp}(32,\mathbb{R})$ and replace the fundamental representation of $\mathfrak{sp}(32,\mathbb{R})$ by $SO(10,1)$ Majorana spinors. A convenient choice of 32×32 Gamma matrices are the $\tilde{\Gamma}$'s we used in the $12D$ case. We choose them as follows:

$$\tilde{\Gamma}^0 = \begin{pmatrix} 0 & -\mathbb{1}_{16} \\ \mathbb{1}_{16} & 0 \end{pmatrix}, \; \tilde{\Gamma}^{10} = \begin{pmatrix} 0 & \mathbb{1}_{16} \\ \mathbb{1}_{16} & 0 \end{pmatrix}, \; \tilde{\Gamma}^i = \begin{pmatrix} \gamma^i & 0 \\ 0 & -\gamma^i \end{pmatrix} \;\; \forall i = 1, \ldots, 9, \tag{2.41}$$

where the γ^i's build a Majorana representation of the Clifford algebra of $SO(9)$, $\{\gamma^i, \gamma^j\} = 2\delta^{ij} \mathbb{1}_{16}$. As before, we have $\tilde{\Gamma}^{10} = \tilde{\Gamma}^0 \tilde{\Gamma}^1 \cdots \tilde{\Gamma}^9$ provided $\gamma^1 \cdots \gamma^9 = \mathbb{1}_{16}$, since we can define γ^9 to be $\gamma^9 = \gamma^1 \cdots \gamma^8$. This choice has $(\tilde{\Gamma}^0)^2 = -\mathbb{1}_{32}$, while $(\tilde{\Gamma}^M)^2 = \mathbb{1}_{32}$, $\forall M = 1 \ldots 10$ and gives a representation of $\{\tilde{\Gamma}^M, \tilde{\Gamma}^N\} = 2\eta^{MN}\mathbb{1}_{32}$ for the choice $(-,+,\ldots,+)$ of the metric. As we have again chosen all $\tilde{\Gamma}$'s to be real, this allows to take $\tilde{B} = \mathbb{1}$ in $\Psi^* = \tilde{B}\Psi$, which implies that the charge conjugation matrix is $\tilde{C} = \tilde{\Gamma}^0$. Moreover, we have the following transposition rules for the $\tilde{\Gamma}$ matrices:

$$\tilde{C}\tilde{\Gamma}^{M_1\ldots M_n}\tilde{C}^{-1} = (-1)^{n(n+1)/2}(\tilde{\Gamma}^{M_1\ldots M_n})^\top \tag{2.42}$$

We will identify the fundamental representation of $\mathfrak{sp}(32,\mathbb{R})$ with a 32-component Majorana spinor of $SO(10,1)$. Splitting the 32 real components of the Ψ into 16+16 as in: $\Psi^\top = (\Phi_1^\top, \Phi_2^\top)$, we can use the following identity:

$$(-i\Phi_2^\top, i\Phi_1^\top) = -i\Psi^\top\tilde{\Gamma}^0 = -i\Psi^\top\tilde{C} = -i\overline{\Psi}$$

to write orthosymplectic matrices again as in (2.2).

2.4.1 Embedding of the 11D super-Poincaré algebra in $\mathfrak{osp}(1|32,\mathbb{R})$

In $11D$, we can also express the $\mathfrak{sp}(32,\mathbb{R})$ transformations in terms of translations, Lorentz transformations and new 5-form symmetries, by defining:

$$h = h_M P^M + h_{MN}J^{MN} + h_{M_1\ldots M_5}J^{M_1\ldots M_5} \, . \tag{2.43}$$

With the help of (2.14), we can compute this enhanced super-Poincaré algebra as in dimension 12, using the following explicit representation of the generators:

$$P^M = \tilde{\Gamma}^M \, , \quad J^{MN} = \frac{1}{2}\tilde{\Gamma}^{MN} \, , \quad J^{M_1\ldots M_5} = \frac{1}{5!}\tilde{\Gamma}^{M_1\ldots M_5} \tag{2.44}$$

In order to express everything in terms of the above generators, we need to dualize forms using the formula: $(-1)^{\frac{k(k-1)}{2}}\varepsilon^{M_1\ldots M_{11}}\tilde{\Gamma}_{M_{k+1}\ldots M_{11}} = -(11-k)!\tilde{\Gamma}^{M_1\ldots M_k}$. This leads to the following superalgebra:

$$\begin{aligned}
[P^M, P^N] &= 4J^{MN} \\
[P^M, J^{OP}] &= 2\eta^{M[O}P^{P]} \\
[J^{MN}, J^{OP}] &= -4\eta^{[M[O}J^{N]P]} \\
[P^M, J^{M_1\ldots M_5}] &= -\frac{2}{5!}\varepsilon^{MM_1\ldots M_5}{}_{N_1\ldots N_5}J^{N_1\ldots N_5} \\
[J^{MN}, J^{M_1\ldots M_5}] &= -10\,\eta^{[M[M_1}J^{N]M_2\ldots M_5]} \\
[J^{M_1\ldots M_5}, J^{N_1\ldots N_5}] &= -\frac{2}{(5!)^2}\varepsilon^{M_1\ldots M_5 N_1\ldots N_5}{}_A P^A + \frac{1}{(3!)^2}\eta^{[M_1[N_1}\eta^{M_2\,N_2}\varepsilon^{M_3\ldots M_5]N_3\ldots N_5]}{}_{O_1\ldots O_5}J^{O_1\ldots O_5} + \\
&\quad + \frac{1}{3!}\eta^{[M_1[N_1}\eta^{M_2\,N_2}\eta^{M_3\,N_3}\eta^{M_4\,N_4}J^{M_5]N_5]} \\
[P^M, Q_A] &= (\tilde{\Gamma}^M)^B{}_A Q_B \\
[J^{MN}, Q_A] &= \frac{1}{2}(\tilde{\Gamma}^{MN})^B{}_A Q_B \\
[J^{M_1\ldots M_5}, Q_A] &= \frac{1}{5!}(\tilde{\Gamma}^{M_1\ldots M_5})^B{}_A Q_B \\
\{Q_A, Q^B\} &= \frac{i}{16}(\tilde{C}\tilde{\Gamma}_M)_A{}^B P^M - \frac{i}{16}(\tilde{C}\tilde{\Gamma}_{MN})_A{}^B J^{MN} + \frac{i}{16}(\tilde{C}\tilde{\Gamma}_{M_1\ldots M_5})_A{}^B J^{M_1\ldots M_5} \, .
\end{aligned} \tag{2.45}$$

Note that this algebra is the dimensional reduction from $12D$ to $11D$ of (2.17). In particular, the first three lines build the $\mathfrak{so}(10,2)$ Lie algebra, but appear in this new 11-dimensional context as the Lie algebra of symmetries of AdS_{11} space (it is of course also the conformal algebra in 9+1 dimensions). We may wonder whether this superalgebra is a minimal supersymmetric extension of the AdS_{11} Lie algebra or not. If we try to construct an algebra without the five-form symmetry generators, the graded Jacobi identity forbids the appearance of a five-form central charge on the RHS of the $\{Q_A, Q^B\}$ anti-commutator. The number of independent components in this last line of the superalgebra will thus be bigger on the LHS than on the RHS. This is not strictly forbidden, but it has implications on the representation theory of the superalgebra. The absence of central charges will for example forbid the existence of shortened representations with a non-minimal eigenvalue of the quadratic Casimir operator $C = -1/4 P_M P^M + J_{MN} J^{MN}$ ("spin") of the AdS_{11} symmetry group (see [199]). More generally, in 11D, either all objects in the RHS of the last line are central charges (this case corresponds simply to the $11D$ Super-Poincaré algebra) or they should all be symmetry generators. Thus, although it is not strictly-speaking the minimal supersymmetric extension of the AdS_{11} Lie algebra, it is certainly the most natural one. That's why some authors [37] call $\mathfrak{osp}(1|32,\mathbb{R})$ the super-AdS algebra in $11D$. Here, we will stick to the more neutral $\mathfrak{osp}(1|32,\mathbb{R})$ terminology. Furthermore, $\mathfrak{osp}(1|32,\mathbb{R})$ is also the maximal finite-dimensional (non-central) $\mathcal{N}=1$ extension of the AdS_{11} algebra. In principle, one could consider even bigger superalgebras, but we will not investigate them in this article.

It is also worth remarking that similar algebras have been studied in [35] where they are called topological extensions of the supersymmetry algebras for supermembranes and super-5-branes.

2.4.2 The supersymmetry properties of the 11D matrix fields

Let us now look at the action of supersymmetries on the fields of an $\mathfrak{osp}(1|32,\mathbb{R})$ eleven-dimensional matrix model. We expand once again the bosonic part of our former matrix M on the irreducible representations of $SO(10,1)$ in terms of 32-dimensional Γ matrices:

$$m = X_M \widetilde{\Gamma}^M + \frac{1}{2!} C_{MN} \widetilde{\Gamma}^{MN} + \frac{1}{5!} Z_{M_1 \dots M_5} \widetilde{\Gamma}^{M_1 \dots M_5} ,$$

where the vector, the 2- and 5-form are given by:

$$X_M = \frac{1}{32} Tr_{\mathfrak{sp}(32,\mathbb{R})}(m \widetilde{\Gamma}_M) , \; C_{MN} = -\frac{1}{32} Tr_{\mathfrak{sp}(32,\mathbb{R})}(m \widetilde{\Gamma}_{MN}) , \; Z_{M_1 \dots M_5} = \frac{1}{32} Tr_{\mathfrak{sp}(32,\mathbb{R})}(m \widetilde{\Gamma}_{M_1 \dots M_5}) .$$

Let us give the whole $\delta_H^{(1)}$ transformation acting on the fields (using the cyclic property of the trace, for instance: $Tr([h,m]\widetilde{\Gamma}^M) = Tr(h[m,\widetilde{\Gamma}^M])$):

$$\begin{aligned}
\delta_H^{(1)} X^M &= 2\left(h^{MQ} X_Q + h^Q C_Q{}^M - \frac{1}{(5!)^2} \varepsilon^{M M_1 \dots M_5}{}_{N_1 \dots N_5} h^{N_1 \dots N_5} Z_{M_1 \dots M_5}\right) - \frac{i}{16} \chi^\top \widetilde{\Gamma}^0 \widetilde{\Gamma}^M \Psi , \\
\delta_H^{(1)} C^{MN} &= -4\left(h^{[M} X^{N]} - h^{[M}{}_Q C^{N]Q} + \frac{1}{4!} h_{M_1 \dots M_4}{}^{[M} Z^{N] M_1 \dots M_4}\right) + \frac{i}{16} \chi^\top \widetilde{\Gamma}^0 \widetilde{\Gamma}^{MN} \Psi , \\
\delta_H^{(1)} Z^{M_1 \dots M_5} &= 2\left(\frac{1}{5!} \varepsilon^{M_1 \dots M_5}{}_{N_1 \dots N_5 Q} h^{N_1 \dots N_5} X^Q + 5 h_Q{}^{[M_1 \dots M_4} C^{M_5]Q} - 5 h_Q{}^{[M_1} Z^{M_2 \dots M_5]Q} + \right. \\
&\quad \left. + \frac{1}{5!} \varepsilon^{M_1 \dots M_5}{}_{O N_1 \dots N_5} h^O Z^{N_1 \dots N_5} - \frac{1}{3 \cdot 4!} h^{O_1 \dots O_5} \varepsilon_{O_1 \dots O_5 N_1 N_2 N_3}{}^{[M_1 M_2 M_3} Z^{M_4 M_5] N_1 N_2 N_3}\right) - \\
&\quad - \frac{i}{16} \chi^\top \widetilde{\Gamma}^0 \widetilde{\Gamma}^{M_1 \dots M_5} \Psi , \\
\delta_H^{(1)} \Psi &= \left(h_M \widetilde{\Gamma}^M + h_{MN} \widetilde{\Gamma}^{MN} + h_{M_1 \dots M_5} \widetilde{\Gamma}^{M_1 \dots M_5}\right) \Psi - \\
&\quad - \widetilde{\Gamma}^M \chi X_M - \frac{1}{2} \widetilde{\Gamma}^{MN} \chi C_{MN} - \frac{1}{5!} \widetilde{\Gamma}^{M_1 \dots M_5} \chi Z_{M_1 \dots M_5} ,
\end{aligned}$$

where the part between parentheses describes the symplectic transformations, while the remainder represents the supersymmetry variations. Note that we used $(-1)^{\frac{k(k-1)}{2}}\varepsilon^{M_1\ldots M_{11}}\tilde{\Gamma}_{M_{k+1}\ldots M_{11}} = -(11-k)!\tilde{\Gamma}^{M_1\ldots M_k}$ in $\delta^{(1)}_H\, Z^{M_1\ldots M_5}$ to dualize the Dirac matrices when needed.

2.4.3 Eleven-dimensional action for a supersymmetric matrix model

As in the $12D$ case, we will now consider a specific model, invariant under $U(N)$ gauge and $\mathfrak{osp}(1|32,\mathbb{R})$ transformations. The simplest such model containing interactions and "propagators" is a cubic action along with a quadratic term. Hence, we choose:

$$I = STr_{\mathfrak{osp}(1|32,\mathbb{R})\otimes\mathfrak{u}(N)}\left(-\mu M^2 + \frac{i}{g^2}\,M[M,M]_{\mathfrak{u}(N)}\right)\,. \tag{2.46}$$

Contrary to a purely cubic model, one loses invariance under $M\to M+K$ for a constant diagonal matrix K, which contains the space-time translations of the BFSS model. In contrast with the BFSS theory, our model doesn't exhibit the symmetries of flat $11D$ Minkowski space-time, so we don't really expect this sort of invariance. However, the symmetries generated by P^M remain unbroken, as well as all other $\mathfrak{osp}(1|32,\mathbb{R})$ transformations. Indeed, the related bosonic part of the algebra (2.45) contains the symmetries of AdS_{11} as a subalgebra, and as was pointed out in [125] and [65], massive matrix models with a tachyonic mass-term for the coordinate X's fields appear in attempts to describe gravity in de Sitter spaces (an alternative approach can be found in [181]). Note that we take the opposite sign for the quadratic term of (2.46), this choice being motivated by the belief that AdS vacua are more stable than dS ones, so that the potential energy for physical bosonic fields should be positive definite in our setting.

The computation of the 11-dimensional action for this supermatrix model is analogous to the one performed in 12 dimensions. We remind the reader that each entry of the matrix M now becomes a hermitian matrix in the Lie algebra of $\mathfrak{u}(N)$ for some large value of N whose generators are defined as in the $12D$ case.

After performing in (2.46) the traces on products of Gamma matrices, it comes out that the terms of the form XXX, XXZ, XCC, CCZ and XCZ have vanishing trace (since products of Gamma matrices related to these terms have decomposition in irreducible tensors that do not contain a term proportional to $\mathbb{1}_{32}$) so that only terms of the form XXC, XZZ, CZZ, CCC, ZZZ will remain from the cubic bosonic terms. As for terms containing fermions and the mass terms, they are trivial to compute. Using (2.31) and the usual duality relation for Dirac matrices in $11D$, one finally obtains the following result:

$$\begin{aligned} I = &-32\mu\, Tr_{\mathfrak{u}(N)}\left\{X_M X^M - \frac{1}{2!}\,C_{MN}C^{MN} + \frac{1}{5!}\,Z_{M_1\ldots M_5}Z^{M_1\ldots M_5} + \frac{i}{16}\overline{\Psi}\Psi\right\} + \\ &+ \frac{32i}{g^2}\,Tr_{\mathfrak{u}(N)}\Bigg(3\,C_{NM}\,[X^M,X^N]_{\mathfrak{u}(N)} - \varepsilon^{M_1\ldots M_{11}}\left\{\frac{3}{(5!)^2}\,Z_{M_1\ldots M_5}\,[X_{M_6},Z_{M_7\ldots M_{11}}]_{\mathfrak{u}(N)} - \right. \\ &\left. - \frac{2^3\,5^2}{(5!)^3}\,Z_{M_1\,M_2\,M_3}{}^{AB}\,[Z_{AB\,M_4\,M_5\,M_6},Z_{M_7\ldots M_{11}}]_{\mathfrak{u}(N)}\right\} + \frac{3}{4!}\,C_{MN}\,[Z_{A_1\ldots A_4}{}^N, Z^{A_1\ldots A_4\,M}]_{\mathfrak{u}(N)} + \\ &+ C_{MN}\,[C^N{}_O, C^{OM}]_{\mathfrak{u}(N)} + \frac{3i}{32}\left\{\overline{\Psi}\,\widetilde{\Gamma}^M[X_M,\Psi]_{\mathfrak{u}(N)} + \frac{1}{2!}\,\overline{\Psi}\,\widetilde{\Gamma}^{MN}\,[C_{MN},\Psi]_{\mathfrak{u}(N)} + \right. \\ &\left. + \frac{1}{5!}\,\overline{\Psi}\,\widetilde{\Gamma}^{M_1\ldots M_5}\,[Z_{M_1\ldots M_5},\Psi]_{\mathfrak{u}(N)}\right\}\Bigg)\,. \end{aligned} \tag{2.47}$$

2.5 Dynamics of the $11D$ supermatrix model and its relation to BFSS theory

Now, we will try to see to what extent our model may describe at least part of the dynamics of M-theory. Since the physics of the BFSS matrix model and its relationships to 11D supergravity and superstring theory are relatively well understood, if our model is to be relevant to M-theory, we expect it to be related to BFSS theory at least in some régime. To see such a relationship, we should reduce our model to one of its ten-dimensional sectors and turn it into a matrix quantum mechanics.

2.5.1 Compactification and T-duality of the $11D$ supermatrix action

If we want to link (2.47) to BFSS, which is basically a quantum mechanical supersymmetric matrix model, we should reduce the eleven-dimensional target-space spanned by the X^M's to 10 dimensions, and, at the same time, let a "time" parameter t appear. At this stage, the world-volume of the theory is reduced to one point. We start by decompactifying it along two directions, following the standard procedure outlined in [233]. Namely, we compactify the target-space coordinates X_0 and X_{10} on circles of respective radii $R_0 = R$ and $R_{10} = \omega R$. We introduce the rescaled field $X'_{10} \equiv X_{10}/\omega$ which has the same $2\pi R$ periodicity as X_0. We can then perform T-dualities on X_0 and X'_{10} to circles of dual radii $\widehat{R} \equiv l_{11}^2/R$ (parametrized by τ and y), where l_{11} is some scale, typically the 11-dimensional Planck length. The fields of our theory, for simplicity denoted here by Y, now depend on the world-sheet coordinates τ and y as follows:

$$Y(\tau, y) \;=\; \sum_{m,n} Y_{mn}\, e^{i(m\tau+ny)/\widehat{R}} \;. \qquad (2.48)$$

As a consequence, we now need to average the action over τ and y with the measure $d\tau dy/(2\pi\widehat{R})^2$. Finally, one should identify under T-duality:

$$X_0 \sim 2\pi l_{11}^2 \Big(i\partial_\tau - A_\tau(\tau, y)\Big) \triangleq i\widehat{\mathcal{D}}_\tau \;, \quad X_{10} \equiv \omega X'_{10} \sim 2\pi\omega l_{11}^2 \Big(i\partial_y - A_y(\tau, y)\Big) \triangleq i\omega\widehat{\mathcal{D}}_y \;, \quad (2.49)$$

where A_τ and A_y are the connections on the $U(N)$ gauge bundle over the world-sheet. For notational convenience, we rewrite $\phi \triangleq C_{0\,10}$, $F_{\tau y} \triangleq -i\,[\widehat{\mathcal{D}}_\tau, \widehat{\mathcal{D}}_y]$ and $\widetilde{\Gamma}_* \triangleq \widetilde{\Gamma}_{10}$ and encode the possible values of the indices in the following notation:

$$A,\ B \;=\; 0,\ldots,10\;, \qquad i,\ j,\ k \;=\; 1,\ldots,9\;,$$
$$\alpha \;=\; 1,\ldots,10\;, \qquad \beta \;=\; 0,\ldots,9\;.$$

Then, the compactified version of (2.47) reads:

$$
\begin{aligned}
I_c = \frac{32i}{g^2} \int \frac{d\tau dy}{(2\pi \widehat{R})^2}\, Tr_{\mathrm{u}(N)} \Bigg(& - 6\, C_{i0}\, i[\widehat{\mathcal{D}}_\tau, X_i] \,+\, 6\omega\, C_{i10}\, i[\widehat{\mathcal{D}}_y, X_i] \,+\, \frac{3}{32}\, \overline{\Psi}\, \widetilde{\Gamma}_0\, [\widehat{\mathcal{D}}_\tau, \Psi] - \\
& - \frac{3\omega}{32}\, \overline{\Psi}\, \widetilde{\Gamma}_*\, [\widehat{\mathcal{D}}_y, \Psi] \,-\, \frac{3}{(5!)^2}\, \varepsilon_{\alpha_1 \cdots \alpha_{10} 0}\, Z_{\alpha_1 \cdots \alpha_5}\, i[\widehat{\mathcal{D}}_\tau, Z_{\alpha_6 \cdots \alpha_{10}}] \,+\, \frac{3\omega}{(5!)^2}\, \varepsilon^{\beta_1 \cdots \beta_{10} 10}\, Z_{\beta_1 \cdots \beta_5}\, i[\widehat{\mathcal{D}}_y, Z_{\beta_6 \cdots \beta_{10}}] + \\
& + 6i\omega\, \phi\, F_{\tau y} \,+\, 3\, C_{ij}\, [X_j, X_i] \,+\, \frac{3}{(5!)^2}\, \varepsilon^{A_1 \cdots A_{10}}{}_j\, Z_{A_1 \cdots A_5}\, [X_j, Z_{A_6 \cdots A_{10}}] - \\
& - \frac{2^3 5^2}{(5!)^3}\, \varepsilon^{A_1 \cdots A_{11}}\, Z_{A_1 A_2 A_3}{}^{B_1 B_2}\, [Z_{B_1 B_2 A_4 A_5 A_6}, Z_{A_7 \cdots A_{11}}] \,+\, \frac{3}{4!} \Big\{ C_{ij}\, [Z_{j\, A_1 \cdots A_4}, Z_i{}^{A_1 \cdots A_4}] - \\
& - 2\, C_{i0}\, [Z_{0\, \alpha_1 \cdots \alpha_4}, Z_{i\, \alpha_1 \cdots \alpha_4}] \,+\, 2\, C_{i10}\, [Z_{10\, \beta_1 \cdots \beta_4}, Z_i{}^{\beta_1 \cdots \beta_4}] \,-\, 2\, \phi\, [Z_{10\, i_1 \cdots i_4}, Z_{0\, i_1 \cdots i_4}] \Big\} + \\
& + C_{ij}\, [C_{jk}, C_{ki}] \,+\, 3\, C_{i0}\, [C_{k0}, C_{ki}] \,-\, 3\, C_{i10}\, [C_{k10}, C_{ki}] \,+\, 6\, \phi\, [C_{k10}, C_{k0}] + \\
& + \frac{3i}{32} \Big\{ \overline{\Psi}\, \widetilde{\Gamma}_i\, [X_i, \Psi] \,+\, \frac{1}{2!}\, \overline{\Psi}\, \widetilde{\Gamma}_{ij} [C_{ij}, \Psi] \,-\, \overline{\Psi}\, \widetilde{\Gamma}_i \widetilde{\Gamma}_0\, [C_{i0}, \Psi] \,+\, \overline{\Psi}\, \widetilde{\Gamma}_i \widetilde{\Gamma}_*\, [C_{i10}, \Psi] \,-\, \overline{\Psi}\, \widetilde{\Gamma}_0 \widetilde{\Gamma}_*\, [\phi, \Psi] + \\
& + \frac{1}{5!}\, \overline{\Psi}\, \widetilde{\Gamma}^{A_1 \cdots A_5}\, [Z_{A_1 \cdots A_5}, \Psi] \Big\} \,+\, i\mu g^2 \Big(\widehat{\mathcal{D}}_\tau \widehat{\mathcal{D}}_\tau \,-\, \omega^2\, \widehat{\mathcal{D}}_y \widehat{\mathcal{D}}_y \,+\, X_i X_i \,+\, \frac{i}{16} \overline{\Psi} \Psi \,+\, \phi^2 - \\
& - \frac{1}{2!}\, C_{ij} C_{ij} \,+\, C_{i0} C_{i0} \,-\, C_{i10} C_{i10} \,+\, \frac{1}{5!} Z_{A_1 \cdots A_5} Z^{A_1 \cdots A_5} \Big) \Bigg) \,.
\end{aligned}
\tag{2.50}
$$

Repeated indices are contracted, and when they appear alternately up and down, minkowskian signature applies, whereas euclidian signature is in force when both are down.

2.5.2 Ten-dimensional limits and IMF

Since the BFSS matrix model is conjectured to describe M-theory in the infinite momentum frame, we shall investigate our model in this particular limit. For this purpose, let's define the light-cone coordinates $t_+ \equiv (\tau + y)/\sqrt{2}$ and $t_- \equiv (\tau - y)/\sqrt{2}$ and perform a boost in the y direction. In the limit where the boost parameter u is large, the boost acts as $(t_+, t_-) \overset{\sim}{\longrightarrow} (ut_+, u^{-1} t_-)$, or as $(\tau, y) \overset{\sim}{\longrightarrow} \sqrt{2}(ut_+, ut_+)$ on the original coordinates. In particular, when $u \to \infty$, the t_- dependence disappears from the action and we can perform the trivial t_- integration. The dynamics is now solely described by the parameter $t \equiv \sqrt{2} u t_+$, which is decompactified through this procedure. In particular, both $\widehat{\mathcal{D}}_\tau$ and $\widehat{\mathcal{D}}_y$ are mapped into $\widehat{\mathcal{D}}_t$.

So far, the ratio of the compactification radii ω is left undetermined and it parametrizes a continuous family of frames. It affects the kinetic terms as:

$$
\begin{aligned}
I_c = \frac{32i}{g^2} \lim_{u \to \infty} \int_{-\pi \widehat{R} u}^{\pi \widehat{R} u} \frac{dt}{2\sqrt{2}\pi \widehat{R} u}\, Tr_{\mathrm{u}(N)} \Bigg(& - 6 \Big(C_{i0} - \omega\, C_{i10} \Big)\, i[\widehat{\mathcal{D}}_t, X_i] \,+\, \frac{3}{32}\, \overline{\Psi} \Big(\widetilde{\Gamma}_0 - \omega\, \widetilde{\Gamma}_* \Big)\, [\widehat{\mathcal{D}}_t, \Psi] - \\
- \frac{3}{(5!)^2}\, \varepsilon_{\alpha_1 \cdots \alpha_{10} 0}\, Z_{\alpha_1 \cdots \alpha_5}\, i[\widehat{\mathcal{D}}_t, Z_{\alpha_6 \cdots \alpha_{10}}] \,+\, & \frac{3\omega}{(5!)^2}\, \varepsilon^{\beta_1 \cdots \beta_{10} 10}\, Z_{\beta_1 \cdots \beta_5}\, i[\widehat{\mathcal{D}}_t, Z_{\beta_6 \cdots \beta_{10}}] \,+\, \ldots \Bigg)
\end{aligned}
\tag{2.51}
$$

In order to have a non-trivial action, as in the BFSS case, we must take the limit $u \to \infty$ together with $N \to \infty$ in such a way that $N/(\widehat{R}u) \to \infty$. In the following, we will write $\overline{R} \equiv \widehat{R}u$, implicitly take the limit $(\overline{R}, N) \to \infty$ and let t run from $-\infty$ to ∞.

In the usual IMF limit, one starts from an uncompactified X_0. In our notation, this corresponds to $R \to \infty$, i.e. to the particular choice $\omega = R_{10}/R \to 0$. So, in the IMF limit, all terms proportional to ω drop out of (2.51). In the following chapters, we will restrict ourselves to this case, since we are especially interested in the physics of our model in the infinite momentum frame.

2.5.3 Dualization of the mass term

Let us comment on the meaning of the $\widehat{\mathcal{D}}_t^2$ term arising from the T-dualization of the mass term $Tr((X_0)^2)$, which naively breaks gauge invariance. To understand how it works, we should recall that the trace is defined by the following sum:

$$Tr_{\mathfrak{u}(N)}(-\widehat{\mathcal{D}}_t^2) \;=\; -\sum_a \langle u_a(t)|\widehat{\mathcal{D}}_t^2|u_a(t)\rangle \;=\; \sum_a \|i\widehat{\mathcal{D}}_t|u_a(t)\rangle\|^2 \quad . \tag{2.52}$$

for a set of basis elements $\{|u_a(t)\rangle\}_a$ of $\mathfrak{u}(N)$, which might have some t-dependence or not. If the $|u_a(t)\rangle$ are covariantly constant, the expression (2.52) is obviously zero. Choosing the $|u_a(t)\rangle$ to be covariantly constant seems to be the only coherent possibility. Such a covariantly constant basis is:

$$|u_a(t)\rangle \;\triangleq\; e^{-i\int_{t_0}^t A_0(t')\,dt'}|u_a\rangle \quad ,$$

(where the $|u_a\rangle$'s form a constant basis, for instance, the generators of $\mathfrak{u}(N)$ in the adjoint representation). Now, t lives on a circle and the function $\exp(\int_{t_0}^t A_0(t')\,dt')$ is well-defined only if the zero-mode $A_0^{(0)} = 2\pi n$, $n \in \mathbb{Z}$. But we can always set $A_0^{(0)}$ to zero, since it doesn't affect the behaviour of the system, as it amounts to a mere constant shift in "energy". With this choice, we can integrate $\widehat{\mathcal{D}}_t$ by part without worrying about the trace.

2.5.4 Decomposition of the five-forms

In (2.50), the only fields to be dynamical are the X_i, the $Z_{\alpha_1\cdots\alpha_5}$ and the Ψ. The remaining ones are either the conjugate *momentum*-like fields when they multiply derivatives of dynamical fields, or *constraint*-like when they only appear algebraically.

Thus, the C_{i0} and $\overline{\Psi}$ have a straightforward interpretation as *momenta* conjugate respectively to the X_i and to Ψ. For the 5-form fields $Z_{A_1\cdots A_5}$ however, the matter is a bit more subtle, due to the presence of the $11D$ ε tensor in the kinetic term for the 5-form fields. Actually, the real degrees of freedom contained in $Z_{A_1\cdots A_5}$ decompose as follows, when going down from $(10+1)$ to 9 dimensions:

$$\Omega^5(\mathcal{M}_{10,1},\mathbb{R}) \;\longrightarrow\; 3\times\Omega^4(\mathcal{M}_9,\mathbb{R})\oplus\Omega^3(\mathcal{M}_9,\mathbb{R}) \quad . \tag{2.53}$$

To be more specific (as in our previous convention, $i_k = 1,\ldots,9$ are purely spacelike indices in $9D$), the 3-form fields on the RHS of (2.53) are $Z_{i_1i_2i_30,10} \triangleq B_{i_1i_2i_3}$, while the 4-form fields are $Z_{i_1i_2i_3i_410} \triangleq Z_{i_1i_2i_3i_4}$, $Z_{i_1i_2i_3i_40} \triangleq H_{i_1i_2i_3i_4}$ and[4] $\Pi^{i_1\cdots i_4} \triangleq 1/5!\,\varepsilon^{j_1\cdots j_5 i_1\cdots i_4 0,10}Z_{j_1\cdots j_5}$; these conventions allow us to cast the kinetic term for the 5-form fields into the expression $6/4!\,\Pi^{i_1\cdots i_4}\,[\widehat{\mathcal{D}}_t, Z_{i_1\cdots i_4}]$, while B and H turn out to be *constraint*-like fields, the whole topic being summarized in Table 1.

---

[4] Using

$$\varepsilon^{j_1\cdots j_N i_{N+1}\cdots i_9 0,10}\,\varepsilon_{k_1\cdots k_N i_{N+1}\cdots i_9 0,10} \;=\; -(9-N)!\sum_\pi \sigma(\pi)\prod_{n=1}^N \delta^{j_n}_{k_{\pi(n)}} \quad ,$$

where π is any permutation of $(1,2,..,N)$ and $\sigma(\pi)$ is the signature thereof, this relation can be inverted: $Z_{i_1\cdots i_5} = \frac{1}{4!}\,\varepsilon_{i_1\cdots i_5 j_6\cdots j_9}\,\Pi^{j_6\cdots j_9} \quad ,$

dynamical var.	*number of real comp.*	*conjugate momenta*	*constraint-like*	*number of real comp.*
X_i	9	C_{i0}	C_{ij} C_{i10} ϕ	36 9 1
$Z_{i_1\cdots i_4}$	126	$\Pi_{i_1\cdots i_4}$	$H_{i_1\cdots i_4}$ $B_{i_1 i_2 i_3}$	126 84
Ψ	32	$\overline{\Psi}$		

Table 1: *Momentum*-like and *constraint*-like auxiliary fields

We see that longitudinal 5-brane degrees of freedom are described by the 4-form $Z_{i_1\cdots i_4}$, while transverse 5-brane fields $Z_{i_1\cdots i_5}$ appear in the definition of the conjugate momenta. As they are dual to one another, we could also have exchanged their respective rôles. Both choices describe the same physics. We can thus interpret these degrees of freedom as transverse 5-branes, completing the BFSS theory, which already contains longitudinal 5-branes as bound states of D0-branes.

Choosing the $\varepsilon_{i_1\cdots i_9}$ tensor in 9 spatial dimensions to be:

$$\varepsilon_{i_1\cdots i_9} \triangleq \varepsilon_{i_1\cdots i_9}{}^{0,10} = -\varepsilon_{i_1\cdots i_9 0,10} \quad ,$$

we can express the action I_c in terms of the degrees of freedom appearing in Table 1 (note that from now on all indices will be down, the signature for squared expressions is Euclidean and we write $\mathcal{D}_t$ instead of $\widehat{\mathcal{D}}_t$):

$$\begin{aligned}
I_c \,=\, & \frac{8\sqrt{2}i}{\pi g^2 \overline{R}} \int dt\, Tr_{\mathrm{u}(N)} \Bigg(-6i\, C_{i0}\,[\mathcal{D}_t, X_i] \,-\, \frac{i}{4}\,\Pi_{i_1\cdots i_4}\,[\mathcal{D}_t, Z_{i_1\cdots i_4}] \,+\, \frac{3}{32}\,\overline{\Psi}\,\widetilde{\Gamma}_0\,[\mathcal{D}_t, \Psi] \,+\, 3\,C_{ij}\,[X_j, X_i] - \\
& + \left(\Pi_{i_1 i_2 i_3 j}\,[X_j, B_{i_1 i_2 i_3}] \,-\, \frac{1}{4\cdot 4!}\,\varepsilon_{i_1\cdots i_8\, j} Z_{i_1\cdots i_4}\,[X_j, H_{i_5\cdots i_8}] \right) \,+\, \frac{1}{3!\cdot 4!}\, W(Z,\Pi,H,B) + \\
& + \frac{1}{2} \Bigg\{ C_{ij}\, K_{ij}(Z,\Pi,H,B) \,-\, 2\,C_{i0} \left(\frac{1}{4\cdot 4!} \varepsilon_{i\, j_1\cdots j_4 k_1\cdots k_4} [H_{j_1\cdots j_4}, \Pi_{k_1\cdots k_4}] \,+\, [Z_{i\, j_1 j_2 j_3}, B_{j_1 j_2 j_3}] \right) + \\
& +\, 2\,C_{i10} \left(\frac{1}{4\cdot 4!} \varepsilon_{i\, j_1\cdots j_4 k_1\cdots k_4} [Z_{j_1\cdots j_4}, \Pi_{k_1\cdots k_4}] \,-\, [H_{i\, j_1 j_2 j_3}, B_{j_1 j_2 j_3}] \right) \,-\, \frac{1}{2}\,\phi\,[Z_{i_1\cdots i_4}, H_{i_1\cdots i_4}] \Bigg\} + \\
& + C_{ij}\,[C_{jk}, C_{ki}] \,+\, 3\,C_{i0}\,[C_{k0}, C_{ki}] \,-\, 3\,C_{i10}\,[C_{k10}, C_{ki}] \,+\, 6\,\phi\,[C_{k10}, C_{k0}] + \\
& + \frac{3i}{32} \Bigg\{ \overline{\Psi}\,\widetilde{\Gamma}_i\,[X_i, \Psi] \,+\, \frac{1}{2!}\,\overline{\Psi}\,\widetilde{\Gamma}_{ij}[C_{ij}, \Psi] \,-\, \overline{\Psi}\,\widetilde{\Gamma}_i \widetilde{\Gamma}_0\,[C_{i0}, \Psi] \,+\, \overline{\Psi}\,\widetilde{\Gamma}_i \widetilde{\Gamma}_*\,[C_{i10}, \Psi] - \\
& - \overline{\Psi}\,\widetilde{\Gamma}_0 \widetilde{\Gamma}_*\,[\phi, \Psi] \,+\, \frac{1}{4!}\,\overline{\Psi}\,\widetilde{\Gamma}_{i_1\cdots i_4} \widetilde{\Gamma}_* [Z_{i_1\cdots i_4}, \Psi] \,+\, \frac{1}{4!}\,\overline{\Psi}\,\widetilde{\Gamma}_{i_1\cdots i_4} \widetilde{\Gamma}_0 \widetilde{\Gamma}_* [\Pi_{i_1\cdots i_4}, \Psi] + \\
& - \frac{1}{4!}\,\overline{\Psi}\,\widetilde{\Gamma}_{i_1\cdots i_4} \widetilde{\Gamma}_0 [H_{i_1\cdots i_4}, \Psi] \,-\, \frac{1}{3!}\,\overline{\Psi}\,\widetilde{\Gamma}_{i_1 i_2 i_3} \widetilde{\Gamma}_0 \widetilde{\Gamma}_* [B_{i_1 i_2 i_3}, \Psi] \Bigg\} \,+\, \mu g^2 i \Bigg\{ (X_i)^2 \,+\, \frac{i}{16}\,\overline{\Psi}\Psi \,+\, \phi^2 - \\
& - \frac{1}{2!}\,(C_{ij})^2 \,+\, (C_{i0})^2 \,-\, (C_{i10})^2 \,+\, \frac{1}{4!} \Big((Z_{i_1\cdots i_4})^2 \,+\, (\Pi_{i_1\cdots i_4})^2 \,-\, (H_{i_1\cdots i_4})^2 \,-\, 4\,(B_{i_1 i_2 i_3})^2 \Big) \Bigg\} \Bigg) \,.
\end{aligned} \tag{2.54}$$

We have abbreviated two lengthy expressions in the result above to make it shorter: on one hand, the term coupling the various 5-form components to the C_{ij}:

$$K_{ij}(Z,\Pi,H,B) \triangleq [Z_{j\, k_1 k_2 k_3}, Z_{i\, k_1 k_2 k_3}] + [\Pi_{j\, k_1 k_2 k_3}, \Pi_{i\, k_1 k_2 k_3}] - 3[B_{j\, k_1 k_2}, B_{i\, k_1 k_2}] - [H_{j\, k_1 k_2 k_3}, H_{i\, k_1 k_2 k_3}] \,,$$

on the other hand, the trilinear couplings amongst the 5-form components:

$$\begin{aligned} W(Z,\Pi,H,B) \;&\triangleq\; \varepsilon_{i_1\cdots i_9}\Big\{B_{i_1 i_2 j}\ \left(2\,[\Pi_{j\,i_3 i_4 i_5},\Pi_{i_6\cdots i_9}] - [Z_{j\,i_3 i_4 i_5},Z_{i_6\cdots i_9}] - [H_{j\,i_3 i_4 i_5},H_{i_6\cdots i_9}]\right) + \\ &+\frac{2}{3}\,B_{i_1 i_2 i_3}\ \left([B_{i_4 i_5 i_6},B_{i_7 i_8 i_9}] + [Z_{i_4 i_5 i_6\,j},Z_{j\,i_7 i_8 i_9}] - [H_{i_4 i_5 i_6\,j},H_{j\,i_7 i_8 i_9}]\right)\Big\} \\ &+(3!)^2\,\Pi_{i_1 i_2 j_1 j_2}\,[Z_{j_1 j_2 k_1 k_2},H_{k_1 k_2 i_1 i_2}] \quad . \end{aligned}$$

2.5.5 Computation of the effective action

We now intend to study the effective dynamics of the X_i and Ψ fields, in order to compare it to the physics of D0-branes as it is described by the BFSS matrix model. For this purpose, we start by integrating out the 2-form momentum-like and constraint-like fields, which will yield an action containing the BFSS matrix model as its leading term with, in addition, an infinite series of couplings between the fields. Similarly, one would like to integrate out the Z-type momenta and constraints Π, H and B, to get an effective action for the 5-brane (described by Z_{ijkl}) coupled to the D0-branes. We will however not do so in the present paper, but leave this for further investigation.

To simplify our expressions, we set:[5]

$$\beta \triangleq \mu g^2 \quad , \qquad\qquad \gamma \triangleq \frac{8\sqrt{2}}{\pi g^2 \overline{R}} \quad ,$$

and write (2.54) as (after taking the trace over $\mathfrak{u}(N)$):

$$I_c = \gamma \int dt\ \left(\beta(\mathbf{C}^a_i)^\intercal(\mathcal{J}^{ab}_{ij} + \Delta^{ab}_{ij})\mathbf{C}^b_j + \mathbf{C}^a_i\cdot\mathbf{F}^a_i + \mathcal{L}_C + \mathcal{L}_\phi + \widehat{\mathcal{L}}\right) \quad . \tag{2.55}$$

For convenience, we have resorted to a very compact notation, where:

$$\mathbf{C}^a_i \triangleq \begin{pmatrix} C^a_{i0} \\ C^a_{i10}\end{pmatrix} \ , \qquad \mathcal{J}^{ab}_{ij} \triangleq \begin{pmatrix} -\delta^{ab}\delta_{ij} & 0 \\ 0 & \delta^{ab}\delta_{ij}\end{pmatrix} \ , \qquad \Delta^{ab}_{ij} \triangleq \frac{3f^{abc}}{\beta}\begin{pmatrix} C^c_{ij} & \phi^c\delta_{ij} \\ -\phi^c\delta_{ij} & -C^c_{ij}\end{pmatrix} \quad ,$$

and where the components of the vector $\mathbf{F}^a_i = \begin{pmatrix} F^a_i \\ G^a_i \end{pmatrix}$, are given by the following expressions:

$$\begin{aligned} F_i &\triangleq 6\,[\mathcal{D}_t,X_i] - \frac{i}{4\cdot 4!}\varepsilon_{i\,j_1\cdots j_4 k_1\cdots k_4}\,[H_{j_1\cdots j_4},\Pi_{k_1\cdots k_4}] - i\,[Z_{i\,j_1 j_2 j_3},B_{j_1 j_2 j_3}] - \frac{3}{32}\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_0\Psi\} \quad , \\ G_i &\triangleq \frac{i}{4\cdot 4!}\varepsilon_{i\,j_1\cdots j_4 k_1\cdots k_4}[Z_{j_1\cdots j_4},\Pi_{k_1\cdots k_4}] - i\,[H_{i\,j_1 j_2 j_3},B_{j_1 j_2 j_3}] + \frac{3}{32}\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_*\Psi\} \quad . \end{aligned}$$

Note that we have written $if^{abc}\overline{\Psi}^b\widetilde{\Gamma}_{\ldots}\Psi^c$ as $\{\overline{\Psi},\widetilde{\Gamma}_{\ldots}\Psi\}^a$ with a slight abuse of notation. The remaining terms in the action (2.55) depending on C_{ij} and ϕ are contained in

$$\begin{aligned} \mathcal{L}_C &\triangleq \frac{\beta}{2}\,(C^a_{ij})^2 + E^a_{ij}C^a_{ij} - f^{abc}C^a_{ij}C^b_{jk}C^c_{ki} \quad , \\ \mathcal{L}_\phi &\triangleq -\beta(\phi^a)^2 + J^a\phi^a \quad , \end{aligned}$$

with the following definitions

$$\begin{aligned} E_{ij} &\triangleq \frac{i}{2}K_{ij} + 3i\,[X_i,X_j] + \frac{3}{64}\{\overline{\Psi},\widetilde{\Gamma}_{ij}\Psi\} \quad , \\ J &\triangleq \frac{-i}{4}[Z_{i_1\cdots i_4},H_{i_1\cdots i_4}] - \frac{3}{32}\{\overline{\Psi},\widetilde{\Gamma}_0\widetilde{\Gamma}_*\Psi\} \quad , \end{aligned}$$

[5] If we consider X and hence C, Z and Ψ to have the engineering dimension of a length, then so has β, while γ has dimension (length)$^{-4}$.

and finally $\widehat{\mathcal{L}}$ is the part of I_c in (2.54) independent of C_{ij}, C_{i10}, C_{i0} and ϕ. in other words the part containing only dynamical fields (fermions Ψ and coordinates X_i) as well as all fields related to the 5-brane (the dynamical ones: Z and Π, as well as the constrained ones: B and H).

Now, (2.55) is obviously bilinear in the $\mathbf{C}_i^a$ (note that Δ_{ij}^{ab} *is* symmetric, since C_{ij} is actually antisymmetric in i and j). So one may safely integrate them out, after performing a Wick rotation such as

$$t \to \tau = it \quad , \qquad C_{i10} \to \overline{C}_{i10} = \pm i C_{i10} \quad .$$

The indeterminacy in the choice of the direction in which to perform the Wick rotation will turn out to be irrelevant after the integration of C_{i10} (indeed, this $\pm$ sign appears in each factor of ϕ and each factor of G, which always come in pairs).

We then get the Euclidean version of (2.55):

$$I_{\mathrm{E}} = \gamma \int d\tau \,\Big(\beta(\overline{\mathbf{C}}_i^a)^{\mathsf{T}}(\mathbb{1}_{ij}^{ab} + \overline{\Delta}_{ij}^{ab})\overline{\mathbf{C}}_j^b + (\overline{\mathbf{C}}_i^a)^{\mathsf{T}}\overline{\mathbf{F}}_i^a - \mathcal{L}_C - \mathcal{L}_\phi - \widehat{\mathcal{L}}\Big) \quad ,$$

where the new rotated fields assume the following form:

$$\overline{\mathbf{C}}_i^a \triangleq \begin{pmatrix} C_{i0}^a \\ \overline{C}_{i10}^a \end{pmatrix} , \qquad \overline{\mathbf{F}}_i^a \triangleq \begin{pmatrix} -F_i^a \\ \pm i G_i^a \end{pmatrix} ,$$

$$\mathbb{1}_{ij}^{ab} \triangleq \begin{pmatrix} \delta^{ab}\delta_{ij} & 0 \\ 0 & \delta^{ab}\delta_{ij} \end{pmatrix} , \qquad \overline{\Delta}_{ij}^{ab} \triangleq \frac{3f^{abc}}{\beta}\begin{pmatrix} -C_{ij}^c & \pm i\phi^c\delta_{ij} \\ \mp i\phi^c\delta_{ij} & C_{ij}^c \end{pmatrix} .$$

The gaussian integration is straightforward, and yields, after exponentiation of the non trivial part of the determinant:

$$\int D\overline{C}_{i10}\, DC_{i0} \,\exp\Big\{-I_{\mathrm{E}}\Big\}$$
$$\propto \exp\Big\{ -\frac{1}{2}Tr\Big(\ln(\mathbb{1}_{ij}^{ab} + \overline{\Delta}_{ij}^{ab})\Big) - \gamma\int d\tau\,\Big(-\frac{1}{4\beta}(\overline{\mathbf{F}}_i^a)^{\mathsf{T}}(\mathbb{1}_{ij}^{ab} + \overline{\Delta}_{ij}^{ab})^{-1}\overline{\mathbf{F}}_j^b - \mathcal{L}_C - \mathcal{L}_\phi - \widehat{\mathcal{L}}\Big)\Big\} .$$

The term quadratic in $\mathbf{F}$ is obviously tree-level, whereas the first one is a 1-loop correction to the effective action. The 1-loop "behaviour" is encoded in the divergence associated with the trace of an operator, since

$$Tr\widehat{O} = \int d\tau\, O^i{}_i(\tau)\langle\tau|\tau\rangle = \Lambda\int d\tau\, O^i{}_i(\tau) \quad , \tag{2.56}$$

where the integration in Fourier space is divergent, and has been replaced by the cutoff Λ. Transforming back to real Minkowskian time t, we obtain the following effective action

$$I_{\mathrm{eff}} = \gamma\int dt\,\Big(\widehat{\mathcal{L}} + \mathcal{L}_C + \mathcal{L}_\phi + \frac{1}{4\beta}(\overline{\mathbf{F}}_i^a)^{\mathsf{T}}(\mathbb{1}_{ij}^{ab} + \overline{\Delta}_{ij}^{ab})^{-1}\overline{\mathbf{F}}_j^b - \frac{\Lambda}{2\gamma}\big(\ln(\mathbb{1} + \overline{\Delta}(t))\big)_{ii}^{aa}\Big) . \tag{2.57}$$

2.5.6 Analysis of the different contributions to the effective action

The natural scale of (2.57) is β, which is proportional to the mass parameter μ. We therefore expand (2.57) in powers of $1/\beta$, which amounts to expanding (2.57) in powers of $\overline{\Delta}$. Now, this procedure must be regarded as a formal expansion, since we don't want to set β to a particular value. However, this formal expansion in $1/\beta$ actually conceals a true expansion in $[X_i, X_j]$, which should be small to minimize the potential energy, as will become clear later on.

First of all, let us consider the expansion of the tree-level term up to $\mathcal{O}(1/\beta^3)$. The first order term is given by:

$$\frac{1}{\beta}\int dt\,(\overline{\mathbf{F}}_i^a)^\intercal\overline{\mathbf{F}}_i^a \;=\; \frac{1}{\beta}\int dt\,Tr\Big((F_i)^2-(G_i)^2\Big)\,.$$

Since F_i contains $[\mathcal{D}_t,X_i]$ and $\{\overline{\Psi},\Psi\}$, while G_i contains only $\{\overline{\Psi},\Psi\}$ (ignoring Z-type contributions), this term will generate a kinetic term for the X^i's as well as trilinear and quartic interactions.

The second-order term is:

$$\frac{1}{\beta}\int dt\,(\overline{\mathbf{F}}_i^a)^\intercal\overline{\Delta}_{ij}^{ab}\overline{\mathbf{F}}_j^b \;=\; \frac{3i}{\beta^2}\int dt\,Tr\bigg(C_{ij}\Big\{[F_i,F_j]-[G_i,G_j]\Big\}-2\,\phi\,[F_i,G_i]\bigg).$$

All vertices generated by this term contain either one C, with 2 to 4 X or Ψ, or one ϕ, with 3 or 4 X or Ψ.

Finally, the third-order contibution is as follows:

$$\begin{aligned}\frac{1}{\beta}\int dt\;\;(\overline{\mathbf{F}}_i^a)^\intercal(\overline{\Delta}^2)_{ij}^{ab}\overline{\mathbf{F}}_j^b \;=\; -\frac{3^2}{\beta^3}\int dt\;\;Tr\bigg(&[F_i,C_{ij}][C_{jk},F_k]\;-\;[G_i,C_{ij}][C_{jk},G_k]+\\ &+[F_i,\phi][\phi,F_i]\;-\;[G_i,\phi][\phi,G_i]\;+\;2\,[G_i,C_{ij}][\phi,F_j]\;-2\,[F_i,C_{ij}][\phi,G_j]\bigg)\,,\end{aligned}$$

producing vertices with 2 ϕ's or 2 C's, together with 2 to 4 X or Ψ, as well as vertices with 1 ϕ or 1 C, with 3 to 4 X or Ψ.

Next we turn to the 1-loop term, where we expand the logarithm up to $\mathcal{O}(1/\beta^3)$. Because of the total antisymmetry of both f^{abc} and C_{ij}, one has $Tr\overline{\Delta}=0$, so that the first term cancels. Now, keeping in mind that

$$f^{abc}f^{bad}\;=\;-C_2(\mathfrak{ad})\delta^{cd}\qquad\text{and}\qquad f^{amn}f^{bno}f^{com}\;=\;\frac{1}{2}C_2(\mathfrak{ad})f^{abc}\quad,$$

$C_2(\mathfrak{ad})$ referring to the quadratic Casimir operator in the adjoint representation of the Lie algebra, one readily finds:

(i). $Tr\overline{\Delta}^2\;=\;\left(\frac{3}{\beta}\right)^2 2iC_2(\mathfrak{ad})\Lambda\int dt\,Tr\Big((C_{ij})^2-9(\phi)^2\Big)\,,$

(ii). $Tr\overline{\Delta}^3\;=\;-\left(\frac{3}{\beta}\right)^3 C_2(\mathfrak{ad})\Lambda\int dt\,Tr\Big(C_{ij}[C_{jk},C_{ki}]\Big)\,.$

In other words, the 1-loop correction (i) renormalizes the mass terms for C_{ij} and ϕ in $\widetilde{I}_c$ as follows:

- Mass renormalization for C_{ij}: $\frac{1}{2}\gamma\beta\longrightarrow\frac{1}{2}\gamma\beta\left(1+\frac{3^2}{\gamma\beta^3}C_2(\mathfrak{ad})\Lambda\right)$
- Mass renormalization for ϕ: $\gamma\beta\longrightarrow\gamma\beta\left(1+\frac{3^4}{2\gamma\beta^3}C_2(\mathfrak{ad})\Lambda\right)$

Whereas the 1-loop correction (ii) renormalizes the trilinear coupling between the C_{ij} in I_c:

- Renormalization of the $C_{ij}[C_{jk},C_{ki}]$ coupling: $\gamma\longrightarrow\gamma\left(1-\frac{3^2}{2\gamma\beta^3}C_2(\mathfrak{ad})\Lambda\right)$

Up to $Tr\overline{\Delta}^3$, the 1-loop corrections actually only renormalize terms already present in I_c from the start. This is not the case for the higher order subsequent 1-loop corrections: there is an infinite number of such corrections, each one diverging like Λ. A full quantization of (2.57) is obviously a formidable task, which we will not attempt in the present paper. A sensible regularization of the divergent contributions should take into account the symmetries of the classical action, which are not explicit anymore after performing T-dualities and the IMF limit. However, since our model is quantum-mechanical, we believe it to be finite even if we haven't come up with a fully quantized formulation.

Summing up the different contributions computed in this section, one gets the following 1-loop effective action up to $\mathcal{O}(1/\beta^3)$:

$$\begin{aligned}\frac{1}{\gamma}\quad I_{\text{eff}} \;&= \int dt\,\left(\mathcal{L}_C + \mathcal{L}_\phi + \widehat{\mathcal{L}}\right) + \frac{\gamma}{4\beta}\int dt\; Tr\left(F_i^2 - G_i^2\right) - \\ &- \frac{3i\gamma}{4\beta^2}\int dt\,Tr\Big(C_{ij}\Big([F_i,F_j]-[G_i,G_j]\Big) - 2\,\phi\,[F_i,G_i]\Big) + \frac{\gamma\lambda}{2\beta^2}\int dt\,Tr\Big(C_{ij}^2 - 9\phi^2\Big) - \\ &- \frac{9\gamma}{4\beta^3}\int dt\,Tr\Big([F_i,C_{ij}][C_{jk},F_k] - [G_i,C_{ij}][C_{jk},G_k] + [F_i,\phi][\phi,F_i] - [G_i,\phi][\phi,G_i] + \\ &+ 2\,[G_i,C_{ij}][\phi,F_j] - 2\,[F_i,C_{ij}][\phi,G_j]\Big) - \frac{i\lambda\gamma}{2\beta^3}\int dt\,Tr\Big(C_{ij}\,[C_{jk},C_{ki}]\Big) + \mathcal{O}(1/\beta^4) \end{aligned} \tag{2.58}$$

where λ is proportional to the cutoff Λ:

$$\lambda \triangleq \frac{9\,C_2(\mathfrak{ad})\Lambda}{\gamma} \quad .$$

Note that the $\mathcal{O}(1/\beta^4)$ terms that we haven't written contain at least three powers of C_{ij} or ϕ.

2.5.7 Iterative solution of the constraint equations

The 1-loop corrected action (2.58) still contains the constraint fields C_{ij} and ϕ, which should in principle be integrated out in order to get the final form of the effective action. Since I_{eff} contains arbitrarily high powers of C_{ij} and ϕ, we cannot perform a full path integration. We can however solve the equations for C_{ij} and ϕ perturbatively in $1/\beta$. This allows to replace these fields in (2.58) with the solution to their equations of motion. Thus, in contrast with the preceeding subsection, here we remain at tree-level.

The equation of motion for C_{ij} may be computed from (2.58), and reads:

$$\begin{aligned} &C_{ij} + \frac{1}{\beta}\Big(E_{ij} + 3i[C_{jk},C_{ki}]\Big) + \frac{1}{\beta^3}\left(\frac{3}{4}i\Big\{[G_i,G_j]-[F_i,F_j]\Big\} + \lambda C_{ij}\right) + \\ &+ \frac{1}{\beta^4}\frac{9}{2}\Big(\Big\{[F_{[i},[C_{j]k},F_k]] - [G_{[i},[C_{j]k},G_k]] + [G_{[i},[\phi,F_{j]}]] - [F_{[i},[\phi,G_{j]}]]\Big\} + \\ &- \frac{i\lambda}{3}[C_{jk},C_{ki}]\Big) + \mathcal{O}(1/\beta^5) = 0\,, \end{aligned} \tag{2.59}$$

while the equation of motion for ϕ is:

$$\begin{aligned} &\phi - \frac{1}{2\beta}J - \frac{3}{\beta^3}\left(\frac{i}{4}[F_i,G_i] - 3\lambda\phi\right) + \frac{3^2}{4\beta^4}\Big([F_i,[F_i,\phi]] - \\ &\qquad - [G_i,[G_i,\phi]] + [F_i,[C_{ij},G_j]] - [G_i,[C_{ij},F_j]]\Big) + \mathcal{O}(1/\beta^5) = 0\,. \end{aligned} \tag{2.60}$$

By solving the coupled equations of motion (2.59) and (2.60) recursively, one gets C_{ij} and ϕ up to $\mathcal{O}(1/\beta^5)$. We can safely stop at $\mathcal{O}(1/\beta^5)$, because the terms contributing to that order in (2.59) and (2.60) are, on the one hand, $\beta^{-1}\Lambda(\delta/\delta C_{ij})Tr\overline{\Delta}^4$ and $\beta^{-1}\Lambda(\delta/\delta\phi)Tr\overline{\Delta}^4$, whose lowest order is $\mathcal{O}(1/\beta^8)$, and on the other hand $\beta^{-2}(\delta/\delta C_{ij})\mathbf{F}^{\intercal}\overline{\Delta}^3\mathbf{F}$ and $\beta^{-2}(\delta/\delta\phi)\mathbf{F}^{\intercal}\overline{\Delta}^3\mathbf{F}$, whose lowest order is $\mathcal{O}(1/\beta^7)$, so that the eom don't get any corrections from contributions of $\mathcal{O}(1/\beta^4)$ coming from I_{eff}.

Subsequently, the $1/\beta$ expansion for C_{ij} reads

$$\begin{aligned} C_{ij} \; = & -\frac{1}{\beta}E_{ij} + \frac{3i}{\beta^3}\Big([E_{ik},E_{kj}] + \frac{1}{4}[F_i,F_j] - \frac{1}{4}[G_i,G_j]\Big) + \frac{\lambda}{\beta^4}E_{ij} + \\ & + \frac{9}{\beta^5}\Big(-2[E_{[ik},[E_{kl},E_{lj]}]] + \frac{1}{2}[E_{[ik},[F_k,F_{j]}]] - \frac{1}{2}[E_{[ik},[G_k,G_{j]}]] + \frac{1}{2}[[E_{[ik},F_k],F_{j]}] - \\ & - \frac{1}{2}[[E_{[ik},G_k],G_{j]}] + \frac{1}{4}[G_{[i},[F_{j]},J]] - \frac{1}{4}[F_{[i},[G_{j]},J]]\Big) + \mathcal{O}(1/\beta^6) \quad , \end{aligned} \tag{2.61}$$

and the expansion for ϕ:

$$\begin{aligned} \phi \; = \; & \frac{1}{2\beta}J + \frac{3i}{4\beta^3}[F_i,G_i] - \left(\frac{3}{2}\right)^2\frac{\lambda}{\beta^4}J - \\ & - \frac{9}{8\beta^5}\Big([F_i,[F_i,J]] - [G_i,[G_i,J]] - 2[[F_i,E_{ij}],G_j] + 2[[G_i,E_{ij}],F_j]\Big) + \mathcal{O}(1/\beta^6) \, . \end{aligned} \tag{2.62}$$

Now, plugging the result for C_{ij} and ϕ into I_{eff}, one arrives at the "perturbative" effective action, which we have written up to and including $\mathcal{O}(1/\beta^5)$, since the highest order ($\mathcal{O}(1/\beta^3)$) we calculated in I_{eff} is quadratic in C and ϕ[6], and since the $\mathcal{O}(1/\beta^4)$-terms in (2.58) only generate $\mathcal{O}(1/\beta^7)$ - terms. This effective action takes the following form:

$$\begin{aligned} \frac{1}{\gamma}I_{\text{eff}} \; = \; & \int dt \Bigg(\widehat{\mathcal{L}} + \frac{1}{4\beta}Tr\left(F_i^2 - G_i^2 + J^2 - 2\,(E_{ij})^2\right) + \\ & + \frac{i}{\beta^3}Tr\Big(- E_{ij}[E_{jk},E_{ki}] + \frac{3}{4}E_{ij}\Big\{[F_i,F_j] - [G_i,G_j]\Big\} + \frac{3}{4}J[F_i,G_i]\Big)\Bigg) + \\ & + \frac{\lambda}{2\beta^4}Tr\Big((E_{ij})^2 - \frac{9}{4}J^2\Big) + \frac{9}{2\beta^5}Tr\Big(([E_{ik},E_{kj}])^2 + \frac{1}{16}([F_i,F_j] - [G_i,G_j])^2 - \\ & + \frac{1}{2}[E_{ik},E_{kj}]\big([F_i,F_j] - [G_i,G_j]\big) - \frac{1}{8}([F_i,G_i])^2 - \frac{1}{2}\Big\{([F_i,E_{ij}])^2 - ([G_i,E_{ij}])^2\Big\} + \\ & + \frac{1}{4}\Big\{([F_i,J])^2 - ([G_i,J])^2\Big\} - \frac{1}{2}[G_i,E_{ij}][J,F_j] + \frac{1}{2}[F_i,E_{ij}][J,G_j]\Big)\Bigg) + \mathcal{O}(1/\beta^6)\,. \end{aligned}$$

At that point, we can replace the aliases E, F, G and J by their expression in terms of the fundamental fields X, Ψ, Z, Π, B and H. The result of this lengthy computation (already to order $1/\beta$) is presented in the Appendix. Here, we will only display the somewhat simpler result obtained by ignoring all 5-form induced fields. Furthermore, we will remove the parameter β from the action, since it was only useful as a reminder of the order of calculation in the perturbative approach. To do so, we absorb a factor of $1/\beta$ in every field, as well as in $\mathcal{D}_t$ (so that the measure of integration scales with β). Thus, β only appears in the prefactor in front of the action, at the 4^{th} power. This is similar to the case of Yang-Mills theory, where one can choose either to have a factor of the coupling constant in the covariant derivatives or have it as a prefactor in front of the action. To be more precise, we set:

$$\Theta \, = \, \frac{1}{4\sqrt{6}\beta}\,\Psi\,, \quad \widetilde{X}_i \, = \, \frac{1}{\beta}\,X_i\,, \quad \widetilde{A}_0 \, = \, \frac{1}{\beta}\,A_0\,, \quad G \, = \, 9\beta^4\,\gamma\,, \quad \tilde{t} \, = \, \beta t\,,$$

and similarly for the Z sector: $(Z,\Pi,H,B) \to (Z/\beta,\Pi/\beta,H/\beta,B/\beta)$.

[6] note that their expansion starts at $\mathcal{O}(1/\beta)$

With this redefinition, it becomes clear that our developpment is really an expansion in higher commutators and not in β. It makes thus sense to limit it to the lowest orders since the commutators should remain small to minimize the potential energy. To get a clearer picture of the final result, we will put all the 5-form-induced fields (Z, Π, H, B) to zero. For convenience we will still write $\widetilde{X}$ as X and $\tilde{t}$ as t in the final result, which reads:

$$
\begin{aligned}
I(X,\Theta) =& \frac{1}{G}\int dt\, Tr_{\mathfrak{u}(N)}\Bigg(([\mathcal{D}_t, X_i])^2 + \frac{1}{2}([X_i,X_j])^2 + i\overline{\Theta}\tilde{\Gamma}_0[\mathcal{D}_t,\Theta] - \overline{\Theta}\tilde{\Gamma}_i[X_i,\Theta] - \\
& -\frac{1}{9}(X_i)^2 - \frac{2i}{3}\overline{\Theta}\Theta - 3[\mathcal{D}_t, X_i]\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_0\Theta\} - \frac{3i}{2}[X_i,X_j]\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\} + \\
& +\frac{9}{4}(\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_0\Theta\})^2 - \frac{9}{4}(\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_*\Theta\})^2 + \frac{9}{4}(\{\overline{\Theta},\tilde{\Gamma}_0\tilde{\Gamma}_*\Theta\})^2 - \frac{9}{8}(\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\})^2 + \\
& +3[X_i,X_j][[X_j,X_k],[X_k,X_i]] - 9[X_i,X_j][[\mathcal{D}_t,X_i],[\mathcal{D}_t,X_j]] - \\
& -\frac{3^3 i}{2}\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}[[X_j,X_k],[X_k,X_i]] + \frac{3^4}{2^2}[X_i,X_j][\{\overline{\Theta},\tilde{\Gamma}_{jk}\Theta\},\{\overline{\Theta},\tilde{\Gamma}_{ki}\Theta\}] - \\
& -\frac{3^4 i}{2^3}\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}[\{\overline{\Theta},\tilde{\Gamma}_{jk}\Theta\},\{\overline{\Theta},\tilde{\Gamma}_{ki}\Theta\}] + \frac{3^3 i}{2}\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}[[\mathcal{D}_t,X_i],[\mathcal{D}_t,X_j]] + \\
& +3^3[X_i,X_j][[\mathcal{D}_t,X_i],\{\overline{\Theta},\tilde{\Gamma}_j\tilde{\Gamma}_0\Theta\}] - \frac{3^4 i}{2^2}\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}[[\mathcal{D}_t,X_i],\{\overline{\Theta},\tilde{\Gamma}_j\tilde{\Gamma}_0\Theta\}] - \\
& -\frac{3^4}{2^2}[X_i,X_j][\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_0\Theta\},\{\overline{\Theta},\tilde{\Gamma}_j\tilde{\Gamma}_0\Theta\}] + \frac{3^5 i}{2^3}\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}[\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_0\Theta\},\{\overline{\Theta},\tilde{\Gamma}_j\tilde{\Gamma}_0\Theta\}] + \\
& +\frac{3^4}{2^2}[X_i,X_j][\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_*\Theta\},\{\overline{\Theta},\tilde{\Gamma}_j\tilde{\Gamma}_*\Theta\}] - \frac{3^5 i}{2^3}\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}[\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_*\Theta\},\{\overline{\Theta},\tilde{\Gamma}_j\tilde{\Gamma}_*\Theta\}] - \\
& -\frac{3^4 i}{2}\{\overline{\Theta},\tilde{\Gamma}_0\tilde{\Gamma}_*\Theta\}[[\mathcal{D}_t,X_i],\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_*\Theta\}] +, \frac{3^5 i}{2}\{\overline{\Theta},\tilde{\Gamma}_0\tilde{\Gamma}_*\Theta\}[\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_0\Theta\},\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_*\Theta\}]\Bigg) + \\
& + \text{eighth-order interactions.}
\end{aligned}
$$

We see that the first four terms in this action correspond to the BFSS matrix model, but with a doubled number of fermions. So, in order to maintain half of the original supersymmetries (i.e. $\mathcal{N} = 1$ in $10D$), one could project out half of the original fermions with $\mathcal{P}_- \xrightarrow{\text{IMF}} (1 + \tilde{\Gamma}_*)/2$. Finally, in addition to the BFSS-like terms, we have mass terms and an infinite tower of interactions possibly containing information about the behaviour of brane dynamics in the non-perturbative sector.

2.6 Discussion

After a general description of $\mathfrak{osp}(1|32)$ and its adjoint representation, we have studied its expression as a symmetry algebra in $12D$. We have described the resulting transformations of matrix fields and their commutation relations. Finally, we have proposed a matrix theory action possessing this symmetry in $12D$. We have then repeated this analysis in the 11-dimensional case, where $\mathfrak{osp}(1|32)$ is a sort of super-AdS algebra. Compactification and T-dualization of two coordinates produced a one-parameter family of singular limiting procedures that shrink the world-sheet along a world-line. We have then identified one of them as the usual IMF limit, which gave rise to a non-compact dynamical evolution parameter that has allowed us to distinguish dynamical from auxiliary fields. Integrating out the latter and solving some constraints recursively, we have obtained a matrix model with a highly non-trivial dynamics, which is similar to the BFSS matrix model when both X^2 and multiple commutators are small. The restriction to a low-energy sector where both X^2 and $[X, X]$ are small seems to correspond to a space-time with weakly interacting (small $[X, X]$) D-particles that are nevertheless not far apart (small X^2). The stable classical solutions correspond to vanishing matrices, i.e. to D-particles stacked at the origin, which displays some common features with matrix models in pp-wave backgrounds (see for instance [26, 89, 49]).

Since the promotion of the membrane charges in the $11D$ super-Poincaré algebra to symmetry generators implied the non-commutativity of the P's, and thus the AdS_{11} symmetry, the membranes are responsible for some background curvature of the space-time. Indeed, since the C_{MN} don't appear as dynamical degrees of freedom, their rôle is to produce the precise tower of higher-order interactions necessary to enforce such a global symmetry on the space-time dynamically generated by the X_i's. The presence of mass terms is thus no surprise since they were also conjectured to appear in matrix models aimed at describing gravity in de Sitter spaces, albeit with a tachyonic sign reflecting the unusual causal structure of de Sitter space ([125, 65]). One might also wonder whether the higher interaction terms we get are somehow related to the high energy corrections to BFSS one would obtain from the non-abelian Dirac-Born-Infeld action. Another question one could address is what kind of corrections a term of the form $STr_{\mathfrak{osp}(1|32)\otimes\mathfrak{u}(n)}([M, M][M, M])$ would induce.

It would also be interesting to investigate the dynamics of the 5-branes degrees of freedom more thoroughly by computing the effective action for Z (from I_{eff} of the Appendix) and give a definite proposal for the physics of 5-branes in M-theory. Note that there is some controversy about the ability of the BFFS model to describe transverse 5-branes (see e.g. [123, 230] and references therein for details). Our model would provide an interesting extension of the BFSS theory by introducing in a very natural way transverse 5-branes (through the fields dual to Z_{ijkl}) in addition to the D0-branes bound states describing longitudinal 5-branes, which are already present in BFSS theory.

2.7 Appendix

We give here the complete effective action at order $1/\beta$.

$$
\begin{aligned}
I_{\text{eff}} \;=\;& \frac{1}{G}\int dt\, Tr_{\mathrm{u}(N)} \Bigg(-\beta\Big\{ (X_i)^2 + \frac{i}{16}\,\overline{\Psi}\Psi + \frac{1}{4!}\Big((Z_{i_1\cdots i_4})^2 + (\Pi_{i_1\cdots i_4})^2 - (H_{i_1\cdots i_4})^2 - 4\,(B_{i_1i_2i_3})^2\Big)\Big\} + \\
+\;& \Big\{ \frac{1}{4}\Pi_{i_1\cdots i_4}[\mathcal{D}_t, Z_{i_1\cdots i_4}] + \frac{3i}{32}\overline{\Psi}\widetilde{\Gamma}_0[\mathcal{D}_t,\Psi] + i\Pi_{i_1i_2i_3\,j}[X_j, B_{i_1i_2i_3}] - \frac{i}{4\cdot 4!}\varepsilon_{i_1\cdots i_8 j} Z_{i_1\cdots i_4}[X_j, H_{i_5\cdots i_8}] + \\
+\;& \frac{i}{3!\cdot 4!}\varepsilon_{i_1\cdots i_9}\Big(B_{i_1i_2\,j}\Big(2\,[\Pi_{j\,i_3i_4i_5},\Pi_{i_6\cdots i_9}] + [Z_{j\,i_3i_4i_5}, Z_{i_6\cdots i_9}] - [H_{j\,i_3i_4i_5}, H_{i_6\cdots i_9}]\Big) + \\
+\;& \frac{2}{3}\,B_{i_1i_2i_3}\Big([B_{i_4i_5i_6}, B_{i_7i_8i_9}] + [Z_{i_4i_5i_6\,j}, Z_{j\,i_7i_8i_9}] - [H_{i_4i_5i_6\,j}, H_{j\,i_7i_8i_9}]\Big)\Big) + \\
+\;& \frac{i}{4}\,\Pi_{i_1i_2j_1j_2}\,[Z_{j_1j_2k_1k_2}, H_{k_1k_2i_1i_2}] - \frac{3}{32}\Big(\overline{\Psi}\,\widetilde{\Gamma}_i\,[X_i,\Psi] + \frac{1}{4!}\,\overline{\Psi}\,\Big(\widetilde{\Gamma}_{i_1\cdots i_4}\widetilde{\Gamma}_*[Z_{i_1\cdots i_4},\Psi] + \\
+\;& \widetilde{\Gamma}_{i_1\cdots i_4}\widetilde{\Gamma}_0\widetilde{\Gamma}_*[\Pi_{i_1\cdots i_4},\Psi] - \widetilde{\Gamma}_{i_1\cdots i_4}\widetilde{\Gamma}_0[H_{i_1\cdots i_4},\Psi] - 4\,\widetilde{\Gamma}_{i_1i_2i_3}\widetilde{\Gamma}_0\widetilde{\Gamma}_*[B_{i_1i_2i_3},\Psi]\Big)\Big)\Big\} + \\
+\;& \frac{1}{4\beta}\Big\{36\,([\mathcal{D}_t, X_i])^2 - \frac{i}{8}\varepsilon_{ij_1\cdots j_8}[\mathcal{D}_t,X_i][H_{j_1\cdots j_4},\Pi_{j_5\cdots j_8}] - 12i\,[\mathcal{D}_t,X_i][Z_{ij_1\cdots j_3}, B_{j_1\cdots j_3}] - \\
-\;& \frac{9}{8}[\mathcal{D}_t,X_i]\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_0\Psi\} - \frac{1}{16}[H_{i_1\cdots i_4},\Pi_{j_1\cdots j_4}]\Big([H_{i_1\cdots i_4},\Pi_{j_1\cdots j_4}] - 16[H_{i_1i_2i_3j_4},\Pi_{j_1j_2j_3i_4}] + \\
+\;& 36[H_{i_1i_2j_3j_4},\Pi_{j_1j_2i_3i_4}] - 16[H_{i_1j_2j_3j_4},\Pi_{j_1i_2i_3i_4}] + [H_{j_1j_2j_3j_4},\Pi_{i_1i_2i_3i_4}]\Big) - \\
-\;& \frac{1}{2\cdot 4!}\varepsilon_{ij_1\cdots j_8}[H_{j_1\cdots j_4},\Pi_{j_5\cdots j_8}][Z_{ik_1\cdots k_3},B_{k_1\cdots k_3}] + \frac{i}{2^9}\varepsilon_{ij_1\cdots j_8}[H_{j_1\cdots j_4},\Pi_{j_5\cdots j_8}]\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_0\Psi\} - \\
-\;& ([Z_{ij_1\cdots j_3},B_{j_1\cdots j_3}])^2 + \frac{3i}{16}[Z_{ij_1\cdots j_3},B_{j_1\cdots j_3}]\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_0\Psi\} + \frac{9}{2^{10}}(\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_0\Psi\})^2 + \\
+\;& \frac{1}{16}\,[Z_{i_1\cdots i_4},\Pi_{j_1\cdots j_4}]\Big([Z_{i_1\cdots i_4},\Pi_{j_1\cdots j_4}] - 16[Z_{i_1i_2i_3j_4},\Pi_{j_1j_2j_3i_4}] + 36[Z_{i_1i_2j_3j_4},\Pi_{j_1j_2i_3i_4}] - \\
-\;& 16[Z_{i_1j_2j_3j_4},\Pi_{j_1i_2i_3i_4}] + [Z_{j_1j_2j_3j_4},\Pi_{i_1i_2i_3i_4}]\Big) - \frac{1}{2\cdot 4!}\varepsilon_{ij_1\cdots j_8}[Z_{j_1\cdots j_4},\Pi_{j_5\cdots j_8}][H_{ik_1\cdots k_3},B_{k_1\cdots k_3}] - \\
-\;& \frac{i}{2^9}\varepsilon_{ij_1\cdots j_8}[Z_{j_1\cdots j_4},\Pi_{j_5\cdots j_8}]\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_*\Psi\} + ([H_{ij_1\cdots j_3},B_{j_1\cdots j_3}])^2 + \frac{3i}{16}[H_{ij_1\cdots j_3},B_{j_1\cdots j_3}]\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_*\Psi\} - \\
-\;& \frac{9}{2^{10}}(\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_*\Psi\})^2 - \frac{1}{16}([Z_{i_1\cdots i_4},H_{i_4\cdots i_1}])^2 + \frac{3i}{2^6}[Z_{i_1\cdots i_4},H_{i_4\cdots i_1}]\{\overline{\Psi},\widetilde{\Gamma}_0\widetilde{\Gamma}_*\Psi\} + \frac{9}{2}([B_{ik_1k_2},B_{jk_1k_2}])^2 + \\
+\;& \frac{1}{2}([Z_{ik_1k_2k_3},Z_{jk_1k_2k_3}])^2 + [Z_{ik_1k_2k_3},Z_{jk_1k_2k_3}][\Pi_{il_1l_2l_3},\Pi_{jl_1l_2l_3}] + \frac{1}{2}([\Pi_{ik_1k_2k_3},\Pi_{jk_1k_2k_3}])^2 - \\
-\;& 3[Z_{ik_1k_2k_3},Z_{jk_1k_2k_3}][B_{il_1l_2},B_{jl_1l_2}] - [Z_{ik_1k_2k_3},Z_{jk_1k_2k_3}][H_{il_1l_2l_3},H_{jl_1l_2l_3}] + \frac{9}{2^{10}}(\{\overline{\Psi},\widetilde{\Gamma}_0\widetilde{\Gamma}_*\Psi\})^2 - \\
-\;& 3\,[\Pi_{ik_1k_2k_3},\Pi_{jk_1k_2k_3}][B_{il_1l_2},B_{jl_1l_2}] - [\Pi_{ik_1k_2k_3},\Pi_{jk_1k_2k_3}][H_{il_1l_2l_3},H_{jl_1l_2l_3}] + \\
+\;& 3\,[B_{ik_1k_2},B_{jk_1k_2}][H_{il_1l_2l_3},H_{jl_1l_2l_3}] + \frac{1}{2}([H_{ik_1k_2k_3},H_{jk_1k_2k_3}])^2 - 6\,[Z_{ik_1k_2k_3},Z_{jk_1k_2k_3}][X_i,X_j] + \\
+\;& \frac{3i}{32}[Z_{ik_1k_2k_3},Z_{jk_1k_2k_3}]\{\overline{\Psi},\widetilde{\Gamma}_{ij}\Psi\} - 6[\Pi_{ik_1k_2k_3},\Pi_{jk_1k_2k_3}][X_i,X_j] + \frac{3i}{32}[\Pi_{ik_1k_2k_3},\Pi_{jk_1k_2k_3}]\{\overline{\Psi},\widetilde{\Gamma}_{ij}\Psi\} + \\
+\;& 18\,[B_{ik_1k_2},B_{jk_1k_2}][X_i,X_j] - \frac{9i}{32}[B_{ik_1k_2},B_{jk_1k_2}]\{\overline{\Psi},\widetilde{\Gamma}_{ij}\Psi\} + 6\,[H_{ik_1k_2k_3},H_{jk_1k_2k_3}][X_i,X_j] - \\
-\;& \frac{3i}{32}[H_{ik_1k_2k_3},H_{jk_1k_2k_3}]\{\overline{\Psi},\widetilde{\Gamma}_{ij}\Psi\} + 18\,([X_i,X_j])^2 - \frac{9i}{16}[X_i,X_j]\{\overline{\Psi},\widetilde{\Gamma}_{ij}\Psi\} - \frac{9}{2^{11}}(\{\overline{\Psi},\widetilde{\Gamma}_{ij}\Psi\})^2\Big\} + \\
+\;& \mathcal{O}(1/\beta^3)\Bigg)\,.
\end{aligned}
$$

Part II

Anomaly cancellation in heterotic M-theory and membrane instantons

Introduction

In this second part, we will be interested in the limit in which M-theory descends to the heterotic $E_8 \times E_8$ string theory, which differs from the purely circular compactification to IIA theory in that the theory we now want to recover only has $\mathcal{N}_{10} = 1$ supersymmetry. A compactification on a circle S^1 would indeed preserve all 32 original supercharges present in the eleven-dimensional theory, and thus lead to $\mathcal{N}_{10} = 2$ supersymmetry required by type II string theories. Consequently, to get rid of half of these supersymmetries in ten-dimensions, on must consider a compactification on a singular space. In the $E_8 \times E_8$ heterotic case, the appropriate space was shown to be the orbifold $S^1/\mathbb{Z}_2$ [148, 147], which can be pictured as an interval with end-points corresponding to the two fixed points of the orbifold action.

In general, compactifying a string theory on orbifolds generates anomalies which have to be cancelled, for the quantum theory to be consistent. This entails the existence of twisted states charged under the appropriate gauge group and localised at the orbifold fixed points. But while twisted sectors of superstring theories can be determined by a standard procedure, when working in the eleven-dimensional M-theory limit however, one must resort to indirect arguments to reconstitute the equivalent of "twisted states". In particular since, in this approach, one is ultimately bound to work in the low-energy limit of the eleven-dimensional theory, the consistency criterion there is the absence of anomalies in local symmetries, in the resulting field theory description. But since the anomalies are determined by the massless modes of the spectrum, their absence in the low-energy field theory guarantees automatically that its string theory limit is also well behaved.

Thus, anomaly cancellation has proved a powerful tool which Hořava and Witten have used to show that recovering the strongly-coupled $E_8 \times E_8$ heterotic string from M-theory requires a factorization of the gauge and gravitational anomaly polynomial of the heterotic theory into two separate terms, each being associated to one of the E_8 factor. Local anomaly cancellation in the M-theory setup then involves the existence of E_8 gauge multiplets propagating on the ten-dimensional boundary hyperplanes located at each of the $\mathbb{Z}_2$ fixed points. In the small radius limit of the orbifold, one recovers the low-energy heterotic supergravity with one-loop corrections. In particular, the fundamental string of the heterotic theory descends from M2-brane states in the presence of boundary hyperplanes, while the dependence on the two separate E_8 gauge group is now merged into a single semisimple $E_8 \times E_8$ dependence, expected for the gauge bosons of heterotic string theory.

For this purpose, we also review in Chapter 3 anomaly cancellation in string theory in general, since the discovery that all known string theories are anomaly-free was a major step in reviving the interest in them, and led to the so-called first string revolution. In particular, the heterotic string requires an additional mechanism to cancel mixed gravitational and gauge anomaly, which necessitates the introduce of a higher order type correction to the supergravity lagrangian, namely the Green-Schwarz term.

In eleven dimensions, a similar term is known to exist, from string dualities [241] and anomaly cancellation of the five-brane in eleven dimensions [107, 251], and has to be taken into account in heterotic M-theory, in addition to the topological Chern-Simons term predicted by eleven-dimensional supergravity. More concretely, the presence of boundaries in the eleven-dimensional setup modifies the Bianchi identity for the four-form field strength of eleven-dimensional supergravity, which is now neither exact nor closed anymore.

Anomaly cancellation is then achieved by inflow from both the topological Chern-Simons interaction and the additional Green-Schwarz term, both depending on a four-form field strength now satisfying a Bianchi identity modified by a non-zero gauge and gravitational source term. This construction requires care. In particular, when working in a formalism where instead of considering the orbifold $S^1/\mathbb{Z}_2$, one works alternatively on a boundary free-circle S^1, and then imposes $\mathbb{Z}_2$ projection, one must insist on requiring periodicity of all fields of the theory in the circle coordinate [43]. In addition, [44] have shown that that the Chern-Simons term should be modified by a redefinition of the three-form potential, similar to what is done to treat the normal bundle anomaly of the M5-brane [119].

Finally, the solution to the modified Bianchi identity for the four-form field strength depends on an arbitrary integration parameter, which, in principles, defines a whole family of solution. However, it has been pointed out [43] that demanding invariance under large gauge and Lorentz transformation of the four-form modified by boundary contribution fixes this parameter uniquely. This, in turn, determines the value of the ratio between the $11D$ gravitational constant and the gauge coupling, which was put forward by [147] as a prediction of M-theory, and guarantees that all eleven-dimensional fields can be safely truncated to the massless modes of the heterotic theory.

At the end of Chapter 3, we will move to considering how the Hořava-Witten scenario is affected by the presence of five-branes. Since they are described by a tensor multiplet containing chiral fermions and a self-dual threeform and break $11D$ lorentz invariance, five-branes are potential sources for both tangent space and normal bundle anomalies, which we will carefully review. In particular, when introduced in the Hořava-Witten setup five-branes wrapping the orbifold circle do not influence the results obtained in their absence, while five-brane perpendicular to the interval can be shown to produce an additional non-vanishing anomaly inflow from the Chern-Simons term, which are localised on the world-volume of the five-brane and lead to new interesting effects in the four-dimensional theory.

These five-brane contribution are indeed central to the discussion in Chapter 4, where we consider further compactification of M-theory on $S^1/\mathbb{Z}_2$ to four dimensions, which preserves $\mathcal{N}_4 = 1$ supersymmetry. For this purpose, we choose the additional six-dimensional compact space to be a Calabi-Yau three-fold, and the five-branes to be space-time filling and transverse to the $S^1/\mathbb{Z}_2$ interval, and having two directions wrapping a two-cycle in the Calabi-Yau. Their fluctuation along the orbifold direction is parametrized by a modulus field, independent of the precise geometry of the Calabi-Yau space, which is thus referred to as *universal.* In the effective supergravity realization, this modulus can be incorporated either in a chiral or (in a dual picture) in a linear multiplet, depending on the formulation we choose for the effective theory.

The effective supergravity including these space-time filling M5-branes can then be obtained by using a superfield formulation which takes into account the modification of the Bianchi identity of the four-form field strength, incorporates the topological contribution from the anomaly polynomials located on the fixed planes and respects the self-duality of the three-form living on the five-brane. In this framework, one can in particular study configurations with open membrane with Euclidean world-volume connecting $10D$ hyperplanes or five-branes together. In particular, taking pairs of five-branes transverse to the interval, one can consider an Euclidean membrane localized in space-time stretching between both of them. Such effects lead to instanton corrections to the interaction Lagrangian and, after gaugino condensation occurs, contribute to the non-perturbative superpotential of the four-dimensional theory.

More precisely, in this effective supergravity approach, such contributions originate from the topological Chern-Simons term, which generates an interaction between the gauge fields on the $10D$ hyperplanes and the masless orbifold modes of the three-form located on the world-volume of the fivebranes. These interactions depend explicitely on the position of the five-branes along the interval, and can be viewed, in this framework, as gauge threshold corrections. One can then use gaugino condensation to determine the non-perturbative superpotential resulting from these interaction terms. In particular, we recover the exact exponential dependence on the universal moduli of the fivebranes as it is obtained by standard membrane instanton calculations [193, 183], but by a completely different route and without the limitations of validity imposed by the latter. This work, done in collaboration with J.P. Derendinger, appears in [63].

Chapter 3

M-theory on an orbifold, anomaly cancellation and membrane instantons

3.1 Eleven-dimensional supergravity à la Cremmer-Julia-Scherk

In this section, we will review some basic facts about the $11D$ supergravity theory found by Cremmer-Julia-Scherk [81]. This introduction is motivated by the fact that the so-called Hořava-Witten scenario, which gives a proposal for the strong coupling limit of the $E_8 \times E_8$ heterotic string, results from a modification of the Bianchi identity of the four-form field strength of $11D$ supergravity. In $11D$ supergravity, it is an exact form, derived from a three-form potential: $G_4 = dC_3$. Hence, its Bianchi identity is trivial: $dG_4 = 0$. In constrast, the low-energy $E_8 \times E_8$ supergravity with 1-loop corrections is obtained from the 11D theory when the RHS of this Bianchi identity is now proportional to a source term, reproducing upon compactification on the interval $I = S^1/\mathbb{Z}_2$, the correct anomaly counterterms of the heterotic effective theory. As a result, G_4 is in this case neither closed nor exact anymore.

We have already introduced qualitatively the M-theory limit of IIA string theory in Section 1.3.1, thereby making contact with $11D$ supergravity à la Cremmer-Julia-Scherk. In the following, we will make these statement mathematically more precise. The bosonic sector of $11D$ supergravity comprises, along with the Einstein-Hilbert action in $11D$ and the kinetic term for the three-form field, a topological, or Chern-Simons, term, transforming into a total derivative under a gauge transformation of C_3. In the absence of the fermionic sector, we leave, following [44], the coefficients of each term unfixed (the original choice of Cremmer-Julia-Scherk [81] is $a = -1$, $b = 2\kappa^2$, $c = -2\sqrt{2}\kappa^3$):

$$S_{bos} = \frac{1}{2\kappa^2} \int_{\mathcal{M}_{11}} d^{11}x \, e \Big(aR(\omega_1) - \frac{b}{48} G_{ABCD} G^{ABCD} + \frac{c}{(144)^2 e} \epsilon^{A_1..A_{11}} C_{A_1A_2A_3} G_{A_4..A_7} G_{A_8..A_{11}} \Big) . \tag{3.1}$$

We have defined $\epsilon^{A_1..A_{11}} = -1$, and we lower and raise the indices of the epsilon tensor with the metric, so that: $\epsilon_{A_1..A_{11}} = -\det g^{-1} = e^{-2}$. The integration is performed over a (yet unspecified) $11D$ space $\mathcal{M}_{11}$ with signature $(-1, +1, .., +1)$. Like in Chapter 2, we choose capital roman indices (A, B, C...) for $11D$ space-time indices, and small case greek letters (μ, ν, ρ...) for the $10D$ space-time ones. As mentioned above, G_4 is the field-strength associated to C_3: $G_{ABCD} = 4\partial_{[A} C_{BCD]}$, and the metric is given in terms of the elfbein as: $g_{AB} = \eta_{ab} e_A{}^a e_B{}^b$, with the lower-case a, b.. referring to $SO(10,1)$ tangent-space indices (the $SO(9,1)$ ones will be denoted m, n, ..)

We see that the theory has a unique parameter κ^2, and no scalars ("dilatons") whose background value would generate additional Lorentz invariant parameters. This supergravity is the maximally supersymmetric theory in $11D$, with $2^5 = 32$ supercharges. We recall the counting of massless physical states of the theory already outlined in Section 1.3.1. The massless states of the three-form C_3 are classified according to the helicity group, or little group, which is $SO(11-D)$. In $11D$, an antisymmetric 3-index tensor has $(9\times 8\times 7)/3! = 84$ dof. This, together with the traceless symmetric tensor g_{AB} with $(9\times 10)/2! - 1 = 44$ physical degrees of freedom, makes 128 bosonic fields. The $11D$ supergravity multiplet will thus contain, in addition, a gravitino, given by the physical components of spin-$\frac{3}{2}$ Lorentz spinor ψ_A. In $11D$, this leads precisely $(11-3)\times 2^{[\frac{9}{2}]} = 128$ physical massless fermionic fields[1].

From their tensorial nature, one can deduce how an infinitesimal local Lorentz transformations will act on the vielbein and the Rarita-Schwinger spinor now defined in the tangent frame: $\psi_a = e^A{}_a\psi_A$. While the the vielbein transforms as $\delta_{\Lambda_{\rm L}} e_A{}^a = -(\Lambda_{\rm L})^a{}_b e_A{}^b$ (where $(\Lambda_{\rm L})^a{}_b$ are the infinitesimal parameters of the Lorentz transformation), the variation of the $\frac{3}{2}$-spin field combines the transformations of both a vector and a spinor according to:

$$\delta_{\Lambda_{\rm L}}\psi_a = -(\Lambda_{\rm L})_a{}^b\psi_b - \frac{1}{4}(\Lambda_{\rm L})_{bc}\Gamma^{bc}\psi_a\,.$$

The supersymmetrisation of the theory can now be considered. It will, in particular, fix the ratios b/a and c/a in the action (3.1), which have up to now been left undetermined.

Our conventions for the remain of this section are: $a = 1$, $b = 1$, $c = 1$, so that the $11D$ gravitational constant only appears as factor κ^{-2} in front of the action. This corresponds to the rescaling $\kappa \to \kappa/\sqrt{2}$ in the original supersymmetric action of [81], followed by a redefinition of the fields: $\psi_A \to \kappa^{-1}\psi_A$ and $C_3 \to (\sqrt{2}\kappa)^{-1}C_3$, and a change of sign of $R(\omega_1)$.

We also choose a Majorana representation for the Clifford algebra where all the Dirac matrices are real. With respect to [81], this corresponds to the redefinition $\Gamma^A \to -i\Gamma^A$.

$$\begin{aligned} S_{\rm CJS} \;=\;& \frac{1}{2\kappa^2}\int d^{11}x\, e\, R(\omega_1) - \frac{1}{2\kappa^2}\int\Big(\frac{1}{2}G_4\wedge *G_4 + \frac{1}{6}C_3\wedge G_4\wedge G_4\Big)\\ &+\frac{1}{2\kappa^2}\int d^{11}x\, e\,\Big[i\overline{\psi}_A\Gamma^{ABC}D_B\Big(\frac{\omega_1+\widehat{\omega}_1}{2}\Big)\psi_C \\ &-\frac{1}{192}\overline{\psi}_{A_1}\left[\Gamma^{A_1..A_6} + 12\Gamma^{A_3A_4}g^{A_1A_2}g^{A_5A_6}\right]\psi_{A_6}\left(\frac{G_{A_2..A_5}+\widehat{G}_{A_2..A_5}}{2}\right)\Big]\,. \end{aligned} \tag{3.2}$$

We see in particular that supersymmetry requires the introduction of a covariant derivative for the spinors, which contains the $11D$ spin connection one-form: $\omega_1 = \frac{1}{2}\omega_{A\,ab}T^{ab}dx^A$, where $(T^{ab})_{cd} = 2\delta^a_{[c}\delta^b_{d]}$ are the Lorentz generators in the vector representation. In components, it reads:

$$\omega_A{}^{ab} = e^{Ba}\partial_{[A}e_{B]}{}^b - e^{Bb}\partial_{[A}e_{B]}{}^a - e^{Ba}e^{Cd}\partial_{[B}e_{C]f}e_A{}^f \equiv \omega_A{}^{ab}(e)\,,$$

which defines the curvature two-form as $R_2 = d\omega_1 + \omega_1\wedge\omega_1$. Writing $R_2 = \frac{1}{4}R_{ab\,AB}T^{ab}dx^A\wedge dx^B$, the Riemann tensor is obtained from the components of the curvature two-form as $R^{AB}{}_{CD} = e^A{}_a e^B{}_b R^{ab}{}_{CD}$.

The covariant derivative is given by:

$$D_A(\omega_1)\psi_B = (\partial_A + \frac{1}{4}\omega_{A\,ab}\Gamma^{ab})\psi_B$$

[1]The spin-$\frac{3}{2}$ representation can be viewed as the tensor product $\mathbf{1}\otimes\frac{\mathbf{1}}{\mathbf{2}}$ of a spinor and a vector. It reduces under the Lorentz group to the sum of gravitino and a spinor. In order to eliminate the (non-physival) spinor part, one must project $\Gamma^a\psi_a = 0$. For massless particles, this keeps $\alpha(D-3)\times 2^{[\frac{D-2}{2}]}$ degrees of freedom, where $\alpha = 1$ for a Majorana spinor, which is our case here (we would have $\alpha = 2$ for a Dirac spinor and $\alpha = 1/2$ for a Majorana-Weyl one). For more details see [179].

where ω_1 is the spin connection with non-vanishing torsion. For a Majorana gravitino, this is equal to:

$$\omega_A{}^{ab} = \omega_A{}^{ab}(e) - \frac{1}{8}\psi_B(\Gamma^{BC}{}_A)^{ab}\psi_C - \frac{1}{4}\big(\psi^{[a}\Gamma_A\psi^{b]} + \psi_A\Gamma^{[a}\psi^{b]} + \psi^{[a}\Gamma^{b]}\psi_A\big)\,.$$

In addition, supersymmetry requires the connection to be modified by a term bilinear in the fermions. The field strength G_4 is shifted accordingly so that the hatted components in the action (3.2) read:

$$\begin{aligned}\widehat{\omega}_{A\,bc} &= \omega_{A\,bc} - \frac{1}{8}\overline{\psi}_D(\Gamma_{Abc})^{DE}\psi_E\,,\\ \widehat{G}_{A_1A_2A_3A_4} &= G_{A_1A_2A_3A_4} + 3\overline{\psi}_{[A_1}\Gamma_{A_2A_3}\psi_{A_4]}\,.\end{aligned} \tag{3.3}$$

Then, the action (3.2) is invariant under the $\mathcal{N}=1$ supersymmetry variations, for an infinitesimal Grassmanian transformation parameter ϵ

$$\delta_\epsilon e_A{}^a = -\frac{1}{\sqrt{2}}\bar{\epsilon}\Gamma^a\psi_A\,, \tag{3.4}$$

$$\delta_\epsilon C_{A_1A_2A_3} = -\frac{3}{\sqrt{2}}\bar{\epsilon}\Gamma_{[A_1A_2}\psi_{A_3]}\,,$$

$$\delta_\epsilon\psi_A = \left[\sqrt{2}D_A(\widehat{\omega}_1) - \frac{1}{144\sqrt{2}}\Big(\Gamma^{A_1..A_4}{}_A - 8\delta_A^{A_1}\Gamma^{A_2A_3A_4}\Big)\widehat{G}_{A_1A_2A_3A_4}\right]\epsilon\,.$$

This can be proven thanks to a particular Fierz identity in $11D$, similar in spirit to the $10D$ one found in Appendix A.6. The closure of the supersymmetry algebra is realised on shell only, so that commutators of two supercharges have to be evaluated on solutions of the equations of motion of eleven-dimensional supergravity.

In addition, under a general infinitesimal variation of the (curved) coordinates system $\{x^A\}$: $\delta x^A = -\zeta^A(x)$, the gravity and three-form fields transform as

$$\delta_\zeta e_A{}^a = \zeta^B\partial_B e_A{}^a + e_B{}^a\partial_A\zeta^B\,, \qquad \delta_\zeta g_{AB} = \zeta^C\partial_C g_{AB} + 2g_{C(A}\partial_{B)}\zeta^C\,,$$

$$\delta_\zeta C_{ABC} = \zeta^D\partial_D C_{ABC} + 3C_{D[AB}\partial_{C]}\zeta^D\,, \quad \delta_\zeta\psi_A = \zeta^B\partial_B\psi_A + \psi_B\partial_A\zeta^B\,,$$

The action (3.2) can be shown to covariant under such general $11D$ coordinate transformation. Invariance of the action under gauge transformation of the three-form potential will be discussed in the next section.

3.2 Dualising the three-form and the introduction of M5-branes

In this section, we will concentrate on the bosonic sector of $11D$ supergravity (3.1) and show how, in the context of M-theory, it can accomodate the inclusion of dynamical five-branes, or M5-branes. In contrast, the fermionic sector (3.2) will be important for us when discussing the possible anomalies in the compactified M-theory, so we will leave it aside for the moment being and come back to it later in this chapter.

We consider the bosonic part of the action (3.2):

$$S_{bos} = \frac{1}{2\kappa^2}\int d^{11}x\,e\,R^{(11)} - \frac{1}{2\kappa^2}\int\left(\frac{1}{2}G_4\wedge *G_4 + \frac{1}{6}C_3\wedge G_4\wedge G_4\right) \tag{3.5}$$

which is invariant under the gauge transformation of C_3 by a closed two-form

$$\delta C_3 = d\Lambda_2\,. \tag{3.6}$$

The equation of motion for the three-form and the Bianchi identity for G_4 are

$$d * G_4 + \frac{1}{2} G_4 \wedge G_4 = 0\,, \qquad dG_4 = 0\,, \tag{3.7}$$

Now we can in particular consider the picture where the identity $G_4 = dC_3$ is expressed by means of a Lagrange multiplier. This requires introducing a six-form C_6, whose field-strength is the dual of G_4, namely: $dC_6 = G_7$ with $G_7 = *G_4$. Then, we can regard the field G_4 as independent and constrain it by the addition to the action (3.5) of the term

$$S_{\text{constraint}} = \frac{1}{2\kappa^2} \int (G_4 - dC_3) \wedge G_7$$

where now G_7 plays the rôle of a Lagrange multiplier. Now, this new action reads:

$$S_{bos} = \frac{1}{2\kappa^2} \int d^{11}x\, e\, R_{(11)} - \frac{1}{2\kappa^2} \int \left(\frac{1}{2} G_4 \wedge *G_4 + \frac{1}{3!} C_3 \wedge dC_3 \wedge dC_3 \right) + S_{\text{constraint}} \tag{3.8}$$

and the equations of motion for C_3 namely $dG_7 + \frac{1}{2} dC_3 \wedge dC_3 = 0$, can be solved by

$$G_7 = -dC_6 - \frac{1}{2} C_3 \wedge dC_3$$

which is gauge invariant if

$$\delta C_6 = -\frac{1}{2} dC_3 \wedge \Lambda_2 + d\Lambda_5\,. \tag{3.9}$$

At this stage one can now rewrite the constraint in the action (3.8) in terms of C_6. This is particularly useful when introducing additional terms sourced by M5-branes, since the world-volume of such an object naturally couples to a six-form potential. After this substitution we have:

$$\begin{aligned} S_{bos} \;=\; & \frac{1}{2\kappa^2} \int d^{11}x\, e\, R^{(11)} - \frac{1}{4\kappa^2} \int G_4 \wedge *G_4 - \frac{1}{4\kappa^2} \int C_3 \wedge dC_3 \wedge \left(G_4 - \frac{2}{3} dC_3 \right) \\ & + \frac{1}{2\kappa^2} \int dG_4 \wedge C_6 \end{aligned} \tag{3.10}$$

This form of the Lagrangian will in a particular be used in Chapter 4. There, the analysis of the effective heterotic M-theory resulting from the orbifold compactification of action (3.10) on $\mathcal{M}_{11} = \mathcal{M}_4 \times K_6 \times S^1/Z_2$, where $\mathcal{M}_4$ is the $4D$ Minkowski space and K_6 a Calabi-Yau threefold, will necessitate a modification of the Bianchi identity for G_4 to $dG = \sum_a J_{a,5}$, where the $J_{a,5}$ are source terms induced by anomaly inflow from the chiral or self-dual fields of either the heterotic supergravity multiplet or from the $D = 6$ $\mathcal{N} = 2$ five-brane tensor multiplet. This modification is then easily implemented in expression (3.10) by a simple shift: $G_4 \to G_4' = G_4 - \sum_a J_{a,4}$, with $J_{a,5} = dJ_{a,4}$.

In addition, when considering dynamical five-branes in M-theory, one has to include the world-volume bosonic action for the $D = 6$ $\mathcal{N} = 2$ five-brane tensor multiplet, containing five massless scalar fields and a chiral two-form, whose field-strength is self-dual on the five-brane world-volume $\mathcal{W}_6$. In general, constructing a covariant action for such a rank-two tensor is notably difficult, because of the usual clash between incorporating the self-duality condition and preserving manifest Lorentz invariance. This problem has been solved for the $11D$ five-brane in [210, 16], by introducing an auxiliary scalar field a which ensures world-volume covariance and first class constraints on the model (this latter point becomes important when quantising the theory). In the absence of any supergravity fields, the kinetic action for the field strength

$\mathcal{F}_3 = d\mathcal{B}_2$, where $\mathcal{B}_2$ is the two-form living on $\mathcal{W}_6$, is then given, for a free five-brane, by a modification of the DBI action (1.40):

$$S_{\rm M5}^{free} = -T_5 \int_{\mathcal{W}_6} d^6\hat{x} \left(\sqrt{-\left| \widehat{g}_{\alpha\beta} + i\mathcal{F}^*_{\alpha\beta\gamma} V^\gamma \right|} + \frac{1}{4} \hat{e}\, V^\alpha \mathcal{F}^*_{\alpha\beta\gamma} \mathcal{F}^{\beta\gamma\delta} V_\delta \right) . \tag{3.11}$$

As already seen in Section 1.2, all the quantities in expression (3.11) depend on the brane coordinates $\{y^\alpha\}_{\alpha=0,..,5}$. The tensors already present in the $11D$ supergravity Lagrangian such as g_{AB} and C_{ABC} have to be pulled back to the five-brane word volume. To avoid confusion, we denote the pullback $\widehat{Y}_{\alpha_1..\alpha_p} = \partial_{\alpha_1} X^{A_1} \cdots \partial_{\alpha_p} X^{A_p} Y_{A_1..A_p}$ of a p-form Y_p by a hat. Here in particular, the fields X^A, $A = 0, 1, .., 10$, represent the five-brane embedding in the $11D$ target space, among which the X^i, $i = 6, .., 10$, constitute the five scalar fields of the five-brane multiplet describing the fluctuation of $\mathcal{W}_6$ in the transverse space. In particular, we will see in Chapter 4 that when considering a compactification of M-theory on $\mathcal{M}_{11} = \mathcal{M}_4 \times K_6 \times S^1/\mathbb{Z}_2$, the only transverse scalar surviving the Calabi-Yau compactification is X^{10}, which represents the five-brane position along the interval $S^1/\mathbb{Z}_2$ and is referred to, in this context, as the five-brane *universal modulus.*

In expression (3.11), the auxiliary field $a(y)$ has been introduced as

$$V_\alpha = \frac{\partial_\alpha a}{\sqrt{\partial_\beta a\, \partial^\beta a}} .$$

In addition, we have $(\mathcal{F}^*)^{\alpha_1\alpha_2\alpha_3} = -\frac{1}{3!\hat{e}} \varepsilon^{\alpha_1..\alpha_6} \mathcal{F}_{\alpha_4\alpha_5\alpha_6}$ and $\hat{e} = \sqrt{-|\widehat{g}_{\alpha\beta}|}$. We see in particular, that when expanding expression (3.11) around the flat metric, we get:

$$\begin{aligned} S_{\rm M5}^{free} &\sim -\frac{T_5}{4} \int_{\mathcal{W}_6} d^6\hat{x}\, \hat{e}\, V_\alpha \mathcal{F}^{*\alpha\beta\gamma} \left[\mathcal{F}_{\beta\gamma\delta} - \mathcal{F}^*_{\beta\gamma\delta} \right] V^\delta \\ &\cong -\frac{T_5}{4} \int_{\mathcal{W}_6} d^6\hat{x}\, \hat{e} \left(\frac{1}{3!} \mathcal{F}^2 - \frac{1}{2} V^\alpha (\mathcal{F}^* - \mathcal{F})_{\alpha\beta\gamma} (\mathcal{F} - \mathcal{F}^*)^{\beta\gamma\delta} V_\delta \right) \end{aligned}$$

which is precisely the action for a free self-dual field $\mathcal{B}_2$ with: $\mathcal{F}^*_{\alpha\beta\gamma} = \mathcal{F}_{\alpha\beta\gamma}$. The action (3.11) is invariant under diffeomorphisms and the following local transformations:

$$\delta\mathcal{B}_2 = d\widehat{\Lambda}_1 - \frac{1}{2} \widehat{\Lambda}_1 \wedge da, \qquad \delta a = 0 , \tag{3.12}$$

plus another set of transformations for which, in particular $\delta a = \lambda_0$ (together with another more involved transformation for $\mathcal{B}_2$).

Thanks to the transformations (3.12), one can algebraically gauge-fix some of the components of $\mathcal{B}_2$. Moreover, one can use the other set of local symmetries to eliminate a. Noting that $V_\alpha V^\alpha = 1$ by definition, one can in particular, after identifying $y^\mu = x^\mu$, make the choice:

$$V_\mu = 0 , \qquad V_4 V^4 + V_5 V^5 = 1$$

which preserves four dimensional Lorentz covariance. This will be the choice made in Chapter 4, where we study M5-branes with world-volume $\mathcal{W}_6 = \mathcal{M}_4 \times \mathcal{C}_2$, where $\mathcal{C}_2$ is a two-cycle in an unspecified Calabi-Yau three-fold.

Finally, in the presence of an $11D$ supergravity background, the five-brane two-form $\mathcal{B}_2$ will couple not only to C_3 but also to its Hodge dual C_6, through a Wess-Zumino type interaction (1.42). In addition, this coupling requires shifting $\mathcal{F}_3 \to \mathcal{H}_3 = \mathcal{F}_3 - \widehat{C}_3$ in the action (3.12), where $\widehat{C}_3$ is the pullback of the three-form to $\mathcal{W}_6$. Then, gauge invariance demands that for the usual local gauge transformation $\delta\widehat{C} = d\widehat{\Lambda}_2$ we shift $\delta\mathcal{B}_2 = \widehat{\Lambda}_2$. Furthermore, the original self-duality condition on $\mathcal{F}_3$ is now extended to $\mathcal{H}_3$.

Then, the five-brane action in a non-trivial 11D supergravity background reads:

$$S_{\mathrm{M5}} = -T_5 \int_{\mathcal{W}_6} d^6\hat{x} \left(\sqrt{-\left|\hat{g}_{\alpha\beta} + i\mathcal{H}^*_{\alpha\beta\gamma}V^\gamma\right|} + \frac{1}{4}\hat{e}\, V^\alpha \mathcal{H}^*_{\alpha\beta\gamma}\mathcal{H}^{\beta\gamma\delta}V_\delta \right) - T_5 \int_{\mathcal{W}_6} \left(\widehat{C}_6 + \frac{1}{2}\widehat{C}_3 \wedge \mathcal{F}_3 \right) . \tag{3.13}$$

Clearly, this Wess-Zumino term is invariant under the transformations (3.6) and (3.9) pulled back on the five-brane world-volume, with now the additional variation for $\mathcal{B}_2$ mentioned above

$$\delta\widehat{C}_3 = d\widehat{\Lambda}_2 \, , \qquad \delta\widehat{C}_6 = -\frac{1}{2} d\widehat{C}_3 \wedge \widehat{\Lambda}_2 + d\widehat{\Lambda}_5 \, , \qquad \delta\mathcal{B}_2 = \widehat{\Lambda}_2$$

provided $\int_{\partial\mathcal{W}_6} \Lambda_5 = 0$.

Expression (3.13) will be supersymmetrised, along with the rest of the heterotic M-theory action in Section 4. We will be particularly interested to study, in this setup, how threshold corrections are modified by instanton effects due to the presence of open Euclidean membranes stretching between two M5-branes, and the structure of the resulting superpotential and gaugino condensates, and their dependence on the five-brane universal modulus.

3.3 Anomaly cancellation in ten-dimensional string theories

In this section, we will, as a warm up, review the mechanism of anomaly cancellation in the $E_8 \times E_8$ heterotic supergravity theory, which is precisely reproduced by the small radius limit of the Hořava-Witten scenario. Thus, when studying the anomaly inflows from the Chern-Simons and Green-Schwartz terms in $11D$, we will pay particular attention to retrieving the anomaly counterterms in the same convention as the one chosen here.

Generally speaking, anomalies will appear in chiral theories. We thus expect them only in even dimensions, which are the ones allowing Weyl spinors. Anomalies will break the classical symmetry of the theory at the quantum level, and only depend on the massless modes of the fields, since massive degrees of freedom will, asympotically in the distance, contribute local terms in the path integral, so that their (local) gauge variation can be removed by a counterterm. Now, while global symmetry breakings may be desirable for phenomenological reasons, local gauge and Lorentz anomalies must be cancelled to avoid inconsistencies in the theory, thus a careful study of anomalies is a crucial test for the validity of low-energy effective string theories.

3.3.1 Anomaly polynomials in D dimensions

In dimension $D = 2d$, the simplest Feynman diagram leading to anomalies is a loop with $d+1$ legs. For superstrings in $10D$ this corresponds to the well known hexagonal diagram. The external incoming legs are gauge bosons and gravitons, while the internal ones fermions or self-dual bosons.

We define the gauge field-strength (curvature) in terms of differential forms as $F = dA + A\wedge A$ with[2] $A = (A^i_\mu T_i)dx^\mu$, T_i the generators of the gauge group Lie algebra $\mathfrak{g}$, and ω the spin connection one-form introduced in equation (3.3), *i.e.* an $SO(D)$-valued one-form for a theory in D dimensions. The relevant gauge and Lorentz transformations read:

$$\delta_{\Lambda_g} A = d\Lambda_g + [A, \Lambda_g]_{\mathfrak{g}} \, , \qquad \delta_{\Lambda_{\mathrm{L}}} {e_A}^a = -(\Lambda_{\mathrm{L}})^a{}_b {e_A}^b \, , \qquad \delta_{\Lambda_{\mathrm{L}}} \omega = d\Lambda_{\mathrm{L}} + [\omega, \Lambda_{\mathrm{L}}]_{\mathfrak{so}(D)} \, ,$$

where $(\Lambda_{\mathrm{L}})^a{}_b$ is the parameter of the local infinitesimal Lorentz transformation and Λ_{L} the $\mathfrak{so}(D)$-valued zero-form constructed from it.

[2] In this chapter, The generators of the gauge group are chosen anti-hermitian: $T_i^\dagger = -T_i$, $i = 1, .., \dim G$.

From now on, we will omit to specify the rank of the gauge and gravitational n-forms we are using, writing simply F, A, ω.. for instance. Then, the anomaly is governed by a formal gauge- and Lorentz-invariant $D+2$-form $\mathcal{I}_{D+2}(R,F)$, or anomaly polynomial, subject to the descent equation

$$\mathcal{I}_{D+2} = d\mathcal{I}_{D+1}\,, \qquad \delta\mathcal{I}_{D+1} = d\mathcal{I}^1_D\,, \tag{3.14}$$

which results in the D-form $\mathcal{I}_D$ used to compensate a local gauge or Lorentz transformation of the classical action. Under such variations, the Minkowskian action[3] transforms as $\delta S_{\rm cl} = \int_{\mathcal{M}_D} \Delta_D$. Then, the anomaly-free effective action will incorporates the anomaly polynomial in its variation as $\delta\Gamma = \int_{\mathcal{M}_{2n}} (\Delta_D + \mathcal{I}_D)$, leading to the condition for anomaly cancellation: $\mathcal{I}_D = -\Delta_D$.

An important point, is that $\mathcal{I}_{D+2}$ is uniquely defined, which is not the case of $\mathcal{I}_D$. Eqn.(3.14) is indeed unchanged if we shift $\mathcal{I}_{D+1} \to \mathcal{I}'_{D+1} = \mathcal{I}_{D+1} + d\Lambda_D$. Then, the second descent equation tells us that $\mathcal{I}_D$ is defined up to the transformation $\mathcal{I}^1_D \to \mathcal{I}'^{,1}_D = \mathcal{I}^1_D + \delta\Lambda_D + d\Lambda_{D-1}$. Thus starting form the same polynomial $\mathcal{I}_{D+2}$, we arrive at two different anomalies $\mathcal{I}^1_D$ and $\mathcal{I}'^{,1}_D$ related by the variation of a local counterterm and a possible additional exact form, which does not survive the second eq.(3.14).

In $10D$ supergravity theories, the relevant anomaly polynomials originating from gauge, gravitational and mixed anomalies are given by the Atiyah-Singer index theorem applied to the relevant differential operators. Thus, the anomalous contributions are in general due to spin-$\frac{1}{2}$ fields, spin-$\frac{3}{2}$ fields and self-dual tensor fields. In the following, we will determine all indices and related anomaly polynomials relevant for the known superstring low-energy theories and $11D$ supergravity. In this discussion, we follow in particular [6, 7, 44].

The operator attached to a positive (negative) chirality spin-$\frac{1}{2}$ fermion, or Weyl operator, is defined from the basic Dirac operator

$$\not{D} = e^B{}_a\Gamma^a\left[\partial_B + A_B + \frac{1}{4}\omega_{A\,bc}\Gamma^{bc}\right].$$

by combining it with a chiral projector $\not{D}_{1/2} = \not{D}\mathcal{P}_+$, with $\mathcal{P}_+ = \frac{1}{2}(\mathbb{I} \pm \Gamma_*)$ as defined in Appendix A.3. Then, the anomaly polynomial corresponding to such a spinor is related to the index of its Weyl operator by the Atiyah-Singer theorem, where the index is defined as the difference between the number of zero modes of this operator and the zero modes of its hermitian conjugate. For a Weyl operator in a D dimensional space $\mathcal{M}_D$, in particular, it picks up the D-form contribution of the following expression:

$$\mathrm{Ind}(i\not{D}_{1/2}) = \int_{M_D}\left[\widehat{A}(\mathcal{M}_D)\mathrm{ch}(F)\right]_D\,, \tag{3.15}$$

with $\mathrm{ch}(F) = \mathrm{tr}e^{\frac{i}{2\pi}F}$ the Chern character, and the A-roof genus given by

$$\widehat{A}(\mathcal{M}_D) = \prod_{k=1}^{D/2}\left(\frac{x_k/2}{\sinh(x_k/2)}\right) = 1 + \frac{1}{(4\pi)^2}\frac{1}{12}\mathrm{tr}R^2 + \frac{1}{(4\pi)^4}\left(\frac{1}{288}(\mathrm{tr}R^2)^2 + \frac{1}{360}\mathrm{tr}R^4\right) + .. \tag{3.16}$$

where the $\pm ix_k$ are the two-form "eigenvalues" of the curvature two-form R. Moreover, the trace tr is on the adjoint of $SO(D)$. More precisely, we have used the antisymmetry of the curvature two-form R to bring it to a skew-diagonal form:

$$\frac{1}{2\pi}R = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & 0 & x_2 & \\ & & -x_2 & 0 & \\ & & & & \ddots \end{pmatrix}. \tag{3.17}$$

[3] We will give the results directly for the Minkowskian case, without deriving them first, as on should do, in the Euclidean picture, and then doing the analytic continuation [6].

In order to determine the expression of the two-forms x_k in terms of tr$R$, we start by expanding the total Pontrjagin class for the $SO(D)$-valued Riemann curvature tensor as $p(R) = \det(\mathbb{1} + \frac{1}{2\pi}R) = \sum_{i=0}^{D/2} p_i(R)$.

Now in the basis (3.17), we have: $p(R) = \prod_{k=1}^{D/2}(1+x_k^2) = \sum_{n=0}^{D/2} t^{i_1..i_n} x_{k_1}^2 \cdots x_{k_n}^2$ with $t^{k_1..k_n} = 1$ if $i_1 < .. < i_n$ and 0 otherwise, which implies the relation:

$$\frac{1}{2}\mathrm{tr}\left(\frac{R}{2\pi}\right)^{2l} = (-1)^l \sum_{k=1}^{D/2} (x_k)^{2l} \,. \tag{3.18}$$

This, in turn, fixes the dictionary between the Pontrjagin classes $p_i(R)$, the eigenvalues two-forms x_k and tr$R$, namely:

$$\begin{aligned} p_1(R) &= \sum_k x_k^2 = -\frac{1}{2}\frac{1}{(2\pi)^2}\mathrm{tr}R^2 \,, \\ p_2(R) &= \sum_{k_1<k_2} x_{k_1}^2 x_{k_2}^2 = \frac{1}{8}\frac{1}{(2\pi)^4}\left((\mathrm{tr}R^2)^2 - 2\mathrm{tr}R^4\right) \,, \\ &\vdots \\ p_{D/2}(R) &= \sum_{k_1<..<k_{D/2}} x_{k_1}^2 \cdots x_{k_{D/2}}^2 = \frac{1}{(2\pi)^D}\det R \,. \end{aligned} \tag{3.19}$$

With these relations in hand, one can now rewrite the LHS of the A-roof genus (3.16) in terms of the Pontrjagin classes as:

$$\widehat{A}(\mathcal{M}_D) = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) - \frac{1}{967680}(31p_1^3 - 44p_1p_2 + 16p_3) + ... \tag{3.20}$$

which, as one can check, gives the result on the RHS of eq.(3.16).

The standard procedure [6] relating the index (3.15) to the anomaly polynomial appearing as the variation $\delta\Gamma = \int_{\mathcal{M}_D} \mathcal{I}_D^1$ yields in this case:

$$\mathcal{I}_{D+2}^{(1/2)}(R,F) = 2\pi\left[\widehat{A}(\mathcal{M}_D)\mathrm{ch}(F)\right]_{D+2} \,. \tag{3.21}$$

For the spin-$\frac{3}{2}$ fermion, viewed as the tensor product between a spin-$\frac{1}{2}$ object and a vector, the corresponding index can be computed by substracting the spin-$\frac{1}{2}$ part to the index of a vector field, with index density $\mathrm{tr}e^{\frac{i}{2\pi}R}$, which yields:

$$\mathrm{Ind}(iD_{3/2}) = \int_{M_D}\left[\widehat{A}(\mathcal{M}_D)\left(\mathrm{tr}e^{\frac{i}{2\pi}R} - 1\right)\mathrm{ch}(F)\right]_D \tag{3.22}$$

Since in the supergravity theories we will be dealing with, the gravitino field is not charged under the gauge group, we will drop the gauge-curvature part in the following. In this case, the expansion of the index density in expression (3.22) with $\mathrm{ch}(F) = 1$ is best computed by diagonalizing the matrix (3.17) to $(1/2\pi)R = \mathrm{diag}\{ix_1, -ix_1, .., ix_{D/2}, -ix_{D/2}\}$, so that

$$(e^{\frac{i}{2\pi}R} - 1) = D - 1 + 2\sum_{j,k=1}^{D/2}\frac{1}{(2j)!}(x_k)^{2j} = D - 1 + \sum_{j=1}^{D/2}\frac{1}{(2\pi)^{2j}(2j)!}\mathrm{tr}R^{2j} \,, \tag{3.23}$$

where we have used eq.(3.18).

Putting together expressions (3.16) and (3.23), one arrives at:

$$\begin{aligned}\widehat{A}(\mathcal{M}_D)\left(\mathrm{tr}e^{\frac{i}{2\pi}R}-1\right) &= D-1+\frac{1}{(4\pi)^2}\left(\frac{D-25}{12}\right)\mathrm{tr}R^2 \\ &+\frac{1}{(4\pi)^4}\left(\left(\frac{D-49}{288}\right)(\mathrm{tr}R^2)^2+\left(\frac{D+239}{360}\right)(\mathrm{tr}R^2)^2\right)+..\end{aligned} \tag{3.24}$$

Then, the anomaly polynomial for the spin-$\frac{3}{2}$ particle in the absence of mixed anomalies can once again be directly read off the index (3.22), yielding:

$$\mathcal{I}^{(3/2)}_{D+2}(R)=\left[\widehat{A}(\mathcal{M}_D)\left(\mathrm{tr}e^{\frac{i}{2\pi}R}-1\right)\right]_{D+2}. \tag{3.25}$$

Finally, there remains to determine the anomaly polynomial for the self-dual tensors possibly appearing in the supergravity theories of interest. As already seen, for D even, these objects are described by $\left(\frac{D}{2}-1\right)$-form potential whose field strength are self-dual in a space with Minkowskian signature. In particular, when $D=2n+2$, this $\frac{D}{2}$-form can be constructed from the bilinear $\overline{\chi}\Gamma^{(D/2)}\psi$, where χ and ψ are two spin-$\frac{1}{2}$ fermions with positive chiralities (cf. eqn.(A.5) in Appendix A). Such self-dual antisymmetric tensors are usually not charged under under the gauge group, so their index will only dependend on the curvature two-form. Moreover, since self-dual form fields can be viewed as fermion bilinears, the index will contain a factor $\mathrm{tr}e^{\frac{i}{2\pi}\frac{1}{4}R_{ab}\Gamma^{ab}}$, where Γ^{ab} is the spin rep for $SO(D)$. More precisely, it is given by the following prescription:

$$\mathrm{Ind}(iD_{SD})=\frac{1}{4}\int_{M_D}\left[L(\mathcal{M}_D)\right]_D. \tag{3.26}$$

in terms of the Hirzebruch polynomial $L(M_D)=\widehat{A}(M_D)\mathrm{tr}e^{\frac{i}{2\pi}\frac{1}{4}R_{ab}\Gamma^{ab}}$. The prefactor in expression (3.26) decomposes as $\frac{1}{4}=\frac{1}{2}\times\frac{1}{2}$, where one factor comes from the chirality projector and the other from the reality condition on the self-dual field-strength.

Hirzebruch polynomial has the following expansion

$$L(\mathcal{M}_D)=\prod_{k=1}^{D/2}\left(\frac{x_k}{\tanh(x_k)}\right)=1-\frac{1}{(2\pi)^2}\frac{1}{6}\mathrm{tr}R^2+\frac{1}{(2\pi)^4}\left(\frac{1}{72}(\mathrm{tr}R^2)^2-\frac{7}{180}\mathrm{tr}R^4\right)+..$$

So one can again check, as we have done for expression (3.16), the RHS of the above equation by reexpressing its LHS in terms of the Pontrjagin classes (3.19):

$$L(\mathcal{M}_D)=1+\frac{1}{3}p_1-\frac{1}{45}\left(p_1^2-7p_2\right)+\frac{1}{945}(2p_1^3-13p_1p_2+6p_3)+... \tag{3.27}$$

In contrast to the two preceding cases, the corresponding $(D+2)$-anomaly polynomial should be rescaled by an extra corrective factor taking into account both the bosonic nature of the field and the dimension of the spinor represention in D dimensions. Thus, we have:

$$\mathcal{I}^{(SD)}_{D+2}(R)=2\pi\left(-\frac{1}{8}\right)\left[L(\mathcal{M}_D)\right]_{D+2}. \tag{3.28}$$

All these rather technical results can now be applied to verify that the gauge, gravitational and mixed anomalies of the known superstring theories indeed cancel, which led, when this fact was discovered, to the so-called first string revolution.

3.3.2 The example of IIB string theory

In this section, we will discuss anomaly cancellation in low-energy string theories more precisely. Since we have ultimately the low-energy effective heterotic M-theory in mind, we will just illustrate how anomalies cancel in the IIB setting, as an example up, before treating the more involved $E_8 \times E_8$ heterotic case.

In contrast to type IIA, type IIB string theory is chiral, but since it contains no gauge fields, it exhibits only gravitational anomalies, which can be directly deduced from its massless spectrum

$$\text{IIB}: \ [0]^2 + [2]^2 + [4]_+ + (2) + (\mathbf{8}_c)^2 + (\mathbf{56}_s)^2 \,.$$

In the expression above, we recognise the (NS$_+$,NS$_+$) sector of the $10D$ theory, which decomposes under the little group $SO(8)$ into the following representations: $[0]+[2]+(2) = \mathbf{1}+\mathbf{28}+\mathbf{35}_v$. In supergravity terms, this refers to the dilaton, the antisymmetric tensor B_2 and the symmetric and traceless metric tensor (which accounts for the v index denoting the vector representation). On the other hand, the (R$_+$,R$_+$) sector consists of the following states $[0]+[2]+[4]_+ = \mathbf{1}+\mathbf{28}+\mathbf{35}_c$, in other words a zero-, a two- and a four-form potential. The latter defines a self-dual field-strength in $10D$, and is thus in a left-handed (the c subscript) chiral representation of $SO(8)$. Finally in the mixed (R$_-$,NS$_+$)+(NS$_+$,R$_-$) sector, we have two copies of left-handed spin-$\frac{1}{2}$ fermions and right-handed gravitinos: $(\mathbf{8}_c)^2 + (\mathbf{56}_s)^2$, for a total of 128 fermions, as expected.

Clearly, the source of anomalies comes the two spin-$\frac{1}{2}$ Majorana-Weyl fermions $(\mathbf{8}_c)^2$, the gravitinos $(\mathbf{56}_s)^2$ with opposite chiralities, and four-form potential $[4]_+$. The two pairs of fermions and gravitinos can be grouped together into two complex combinations, yielding one twelve-form anomaly polynomial per chiral field.

Assembling the relevant contributions from eqs.(3.21), (3.25) and (3.28) with R now the ten-dimensional curvature two-form, the total anomaly polynomial

$$\mathcal{I}_{12}^{\text{IIB}}(R) = \mathcal{I}_{12}^{(1/2)}(R) - \mathcal{I}_{12}^{(3/2)}(R) - \mathcal{I}_{12}^{(SD)}(R) = 0$$

vanishes thanks to expressions (3.20), (3.27) and (3.24), rephrased in terms of Pontrjagin classes (3.19) as follows:

$$\begin{aligned}
\frac{1}{2\pi}\mathcal{I}_{12}^{(1/2)}(R) &= -\frac{1}{967680}(31p_1^3 - 44p_1p_2 + 16p_3)\,,\\
\frac{1}{2\pi}\mathcal{I}_{12}^{(3/2)}(R) &= \frac{1}{967680}(225p_1^3 - 1620p_1p_2 + 7920p_3)\,,\\
\frac{1}{2\pi}\mathcal{I}_{12}^{(SD)}(R) &= -\left(\frac{1}{8}\right)\frac{1}{945}(2p_1^3 - 13p_1p_2 + 6p_3)\,.
\end{aligned}$$

This is indeed a beautiful and somehow striking result. In this respect, the stringency of the numerical constraints involved points at the idea that the internal consistency conditions for string theory are so powerful and straightforward that they directly imply that the resulting low-energy theory is anomaly-free.

3.3.3 Anomaly cancellation for the $E_8 \times E_8$ heterotic string

The heterotic and type I string theories, in contrast, have chiral fields which are charged under their respective gauge groups. However, since the only charged spinor is the gaugino, we need to take into account an additional coupling term, which is absent from the minimal supergravity theory, to cancel gauge and mixed anomalies. This is the Green-Schwarz mechanism [130], which will be presented in detail hereafter. This additional Green-Schwarz correction will typically originate from the higher derivative topological term (1.42), and thus receive an interpretation in terms of a coupling of the world-volume of a p-brane of the original string theory, with a lower rank p'-brane, with $p' < p$.

Here we will concentrate on the $E_8 \times E_8$ heterotic string, since it is the only heterotic theory directly related to M-theory via compactification. The heterotic $SO(32)$ theory is then obtained from the $E_8 \times E_8$ heterotic theory by toroidally compactifying one dimension, turning on a Wilson line to break both groups to $SO(16) \times SO(16)$, and then perform T-duality on the compactified dimension. S-dualising the heterotic $SO(32)$ theory we finally get type I string theory, so that all three of these string theories are related by the appropriate chain of dualities. In any case, the analysis of anomaly cancellation for the heterotic $SO(32)$ and type I strings is similar in spirit, and can be easily worked out by following the subsequent discussion.

All these theories are constructed from combining the right-moving side of a superstring of type II with a left-moving bosonic string in 26 flat dimensions, but with 16 of its bosonic degrees of freedom compactified on a self-dual torus T^{16}. The requirement for these gauge bosons to be written as vertex operators and the modular invariance of the partition function demands an even self-dual compactification lattice (which only exist in dimensions which are multiples of eight), leading, in 16 Euclidean dimensions, to the well known $SO(32)$ and $E_8 \times E_8$ gauge groups. The remaining ten non-compact bosons together with the ones from the right-moving type II superstring form the usual 10 world-volume coordinates in the vector rep of the $SO(9,1)$ Lorentz group.

The low-lying heterotic $E_8 \times E_8$ string states have massless spectrum given in terms of $SO(8)_{\text{Spin}} \times E_8 \times E_8$ representations, where again $SO(8)$ is the light-cone little group:

$$(\mathbf{1},\mathbf{1},\mathbf{1})+(\mathbf{28},\mathbf{1},\mathbf{1})+(\mathbf{35},\mathbf{1},\mathbf{1})+(\mathbf{56}_c,\mathbf{1},\mathbf{1})+(\mathbf{8}_s,\mathbf{1},\mathbf{1})+[\mathbf{8}_v+\mathbf{8}_c]\times[(\mathbf{248},\mathbf{1})+(\mathbf{1},\mathbf{248})]\,. \quad (3.29)$$

Denoting by $\widetilde{\text{NS}}$ and $\widetilde{\text{R}}$ the sectors corresponding to the $\mathbf{8}_v$ and $\mathbf{8}_c$ reps of the right-moving II superstring, and by NS the tachyonic $(\mathbf{1},\mathbf{1})$ and massless $(\mathbf{8}_v,\mathbf{1}) + (\mathbf{1},\mathbf{496})$ states from the $26D$ bosonic string side, the spectrum (3.29) decomposes into an $(\widetilde{\text{NS}},\text{NS})$ sector: $(\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{28},\mathbf{1},\mathbf{1})+(\mathbf{35},\mathbf{1},\mathbf{1})+(\mathbf{8}_v,\mathbf{248},\mathbf{1})+(\mathbf{8}_v,\mathbf{1},\mathbf{248})$, and a mixed $(\widetilde{\text{R}},\text{NS})$ one: $(\mathbf{56}_c,\mathbf{1},\mathbf{1})+(\mathbf{8}_s,\mathbf{1},\mathbf{1})+(\mathbf{8}_c,\mathbf{248},\mathbf{1}) + (\mathbf{8}_c,\mathbf{1},\mathbf{248})$.

In the low-energy effective theory, the uncharged states correspond to the dilaton ϕ, the antisymetric tensor $B_{\mu\nu}$, the vielbein ${e_\mu}^a$, the left-handed Majorana-Weyl gravition ψ_μ and the right-handed Majorana-Weyl fermion χ forming the $\mathcal{N}=1$ supergravity multiplet we already now from the type II superstring. The charged part of the spectrum (3.29), on the other hand, is incorporated in a SYM multiplet, containing the ten-dimensional gluon A^i_μ and Majorana-Weyl gluino λ^i, where the i labels the generators of a certain representation of the gauge group (here, the adjoint rep of $E_8 \times E_8$).

Obviously, the chiral fields contributing to the anomaly inflow can be read off the $(\widetilde{\text{R}},\text{NS})$ sector and consist of a spin-$\frac{1}{2}$ fermion $(\mathbf{8}_s,\mathbf{1},\mathbf{1})$, as in type IIB theory, a gravitino $(\mathbf{56}_c,\mathbf{1},\mathbf{1})$ and the gauginos $(\mathbf{8}_c,\mathbf{248},\mathbf{1}) + (\mathbf{8},\mathbf{1},\mathbf{248})$ in the adjoint representation of $E_8 \times E_8$.

Leaving the dimension of the gauge group unfixed for the time being, these (heterotic and type I) supergravity theories thus have the following anomaly twelve-form (on gets a 1/2 factor in front of eveverybody, since all chiral fields are this time Majorana spinors):

$$\begin{aligned}
\mathcal{I}^{\text{het}}_{12}(R,F) &= \frac{1}{2}\left(-\mathcal{I}^{(1/2)}_{12}(R,F)+\mathcal{I}^{(1/2)}_{12}(R)-\mathcal{I}^{(3/2)}_{12}(R)\right)\\
&= \frac{1}{(2\pi)^5 96}\left(\frac{1}{15}\text{Tr}_A F^6-\frac{1}{24}\text{Tr}_A F^4\text{tr}R^2+\frac{1}{960}\text{Tr}_A F^2\left(5(\text{Tr}_A R^2)^2+4\text{tr}R^4\right)\right.\\
&\quad -\frac{1}{32}(\text{tr}R^2)^3-\frac{1}{8}\text{tr}R^2\text{tr}R^4\\
&\quad \left.-\frac{\dim G-496}{72}\left[\frac{1}{105}\text{tr}R^6+\frac{1}{80}\text{tr}R^4\text{tr}R^2+\frac{1}{192}(\text{tr}R^2)^3\right]\right)\,.
\end{aligned} \quad (3.30)$$

The second equality is obtained by using (3.21) and (3.25), and taking the trace on the adjoint representation of the gauge group in $\mathrm{ch}(F)$, denoted here by Tr_A.

As we will see below, the Green-Schwarz mechanism for compensating anomalies relies on the fact that the anomaly twelve-form factorises as $I_4(R,F) \wedge X_8(R,F)$, where I_4 and X_8 are respectively a four- and an eight-form. Now, terms such as $\mathrm{Tr}_A F^6$ and $\mathrm{tr} R^6$ cannot a priori be compensated this way. So $\mathrm{tr} R^6$ must disappear from expression (3.30), which is achieved when $\dim G = 496$ and provided $\mathrm{Tr}_A F^6$ is not an independent Casimir operator. Both conditions are met for the following groups: $SO(32)$, $E_8 \times E_8$, $E_8 \times U(1)^{248}$ and $U(1)^{496}$. However, only the two first ones are related to known $10D$ superstring theories.

We will now go into the details of the anomaly cancellation mechanism. To fix the conventions, we choose to write the classical heterotic supergravity action as

$$S_{\mathrm{het}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x\, e e^{-2\phi} \left[R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} H_3 \wedge * H_3 - \frac{\kappa_{10}^2}{g_{YM}^2} \frac{1}{30} \mathrm{Tr}_A (F \wedge * F) \right] \tag{3.31}$$

where κ_{10} is the ten-dimensional gravitational coupling, and g_{YM} the gauge coupling constant. The factor $1/30$ comes from selecting a normalisation for the trace on the adjoint rep of the gauge group which is compatible both with $SO(32)$ and $E_8 \times E_8$ (for type I, one usually resorts to the vector rep, which does not exist for $E_8 \times E_8$). In other words, we fix $\mathrm{tr} F^2 = \frac{1}{30} \mathrm{Tr}_A F^2$ which is an identity for $SO(32)$ and a definition for $E_8 \times E_8$.

The three-form tensor H_3 is given by

$$H_3 = dB_2 - \frac{\kappa_{10}^2}{g_{YM}^2} (\omega_{\mathrm{YM}} - \omega_{\mathrm{L}})\,. \tag{3.32}$$

It contains the gauge Chern-Simons three-form ω_{YM} and the ten-dimensional Lorentz Chern-Simons three-form ω_{L} built from the spin connection ω (the ten-dimensional version of eq.(3.3)). Both are given by:

$$\omega_{\mathrm{YM}} = \frac{1}{30} \mathrm{Tr}_A \Big(A \wedge \Big[dA + \frac{2}{3} A^2 \Big] \Big)\,, \qquad \omega_{\mathrm{L}} = \mathrm{tr} \Big(\omega \wedge \Big[d\omega + \frac{2}{3} \omega^2 \Big] \Big)\,. \tag{3.33}$$

One can verify that the action (3.31) is invariant under both a Lorentz transformation with parameter Λ_{L} and a gauge transformation, provided:

$$\begin{aligned} \delta_{\Lambda_g} \omega_{\mathrm{YM}} &= \frac{1}{30} d\mathrm{Tr}_A (\Lambda_g dA)\,, \qquad \delta_{\Lambda_{\mathrm{L}}} \omega_{\mathrm{L}} = d\mathrm{tr}(\Lambda_{\mathrm{L}} d\omega)\,, \\ \delta_{(\Lambda_g, \Lambda_{\mathrm{L}}, \Lambda')} B_2 &= \frac{\kappa_{10}^2}{g_{YM}^2} \left[\frac{1}{30} \mathrm{Tr}_A (\Lambda_g dA) - \mathrm{tr}(\Lambda_{\mathrm{L}}\, d\omega) \right] + d\Lambda'_1 \end{aligned} \tag{3.34}$$

As a first try, one considers, in addition, a Chern-Simons type interaction, not present in the minimal version (3.31) since it results from higher derivative effects:

$$S' = \frac{g_{YM}^2}{\kappa_{10}^2} \int_{M_{10}} B_2 \wedge (a \mathrm{Tr}_A F^4 + b (\mathrm{Tr}_A F^2)^2) \tag{3.35}$$

Such a term is invariant under gauge transformation of A and under the linear transformation $\delta_{\Lambda'} B_2 = d\Lambda'_1$, since F closed, so that by integrating by parts in $\delta_{\Lambda'} S'$, F will vanish. Now, crucial to this discussion, under the more involved gauge transformations of B_2 (3.34), the action (3.35) transforms non-trivially as

$$\delta_{\Lambda_g} S' = \frac{1}{30} \int \mathrm{Tr}_A (\Lambda_g dA) \wedge (a \mathrm{Tr}_A F^4 + b (\mathrm{Tr}_A F^2)^2)\,. \tag{3.36}$$

This anomalous gauge variation is the anomaly ten-form $\mathcal{I}_{10}$ at the end of a descent equation (3.14). In fact, it is determined by the following chain of anomaly polynomials:

$$\begin{aligned}
\mathcal{I}_{12} &= \frac{1}{30}(a\mathrm{Tr}_A F^2 \wedge \mathrm{Tr}_A F^4 + b(\mathrm{Tr}_A F^2)^3)\,, \\
\mathcal{I}_{11} &= \frac{1}{30}\mathrm{Tr}_A(A\wedge F^2)\wedge(a\mathrm{Tr}_A F^4 + b(\mathrm{Tr}F^2)^2)\,, \\
\mathcal{I}^1_{10} &= \frac{1}{30}\mathrm{Tr}_A(\Lambda_g dA)\wedge(a\mathrm{Tr}_A F^4 + b(\mathrm{Tr}F^2)^2)\,,
\end{aligned} \tag{3.37}$$

so that the additional term S' (3.35) cancels an anomaly given by the twelve-form $\mathcal{I}_{12}$. This is not the whole story, since we have not yet reproduced the total anomaly (3.30).

To do so, we generalise expression (3.36) by taking into account both gauge and Lorentz transformations. Such a topological expression, or Green-Schwarz term, is defined up to local counterterms, as discussed at the beginning of Section 3.3:

$$S_{GS} = -\frac{1}{(4\pi)^2}\frac{g^2_{YM}}{\kappa^2_{10}}\int_{M_{10}} B_2\wedge X_8(R,F) + \text{ local counterterms}\,. \tag{3.38}$$

Extending the discussion of the simplified case (3.36), a term such as (3.38) will cancel an anomaly of the form:

$$\mathcal{I}_{12} = \frac{1}{(4\pi)^2}\left(\frac{1}{30}\mathrm{Tr}_A F^2 - \mathrm{tr}R^2\right)\wedge X_8(R,F)\,. \tag{3.39}$$

The aim of the rest of this section is to determine the correct expression for $X_8(R,F)$ which reproduces the anomaly (3.30) for the gauge group $E_8\times E_8$.

For this purpose, we first note that: $d\omega_{\mathrm{YM}} = \frac{1}{30}\mathrm{Tr}_A F^2$ and $d\omega_{\mathrm{L}} = \mathrm{tr}R^2$, defining the following three- and four-form polynomials:

$$I_3 = \frac{1}{(4\pi)^2}\left(\omega_{\mathrm{YM}} - \omega_{\mathrm{L}}\right)\,, \qquad I_4 = dI_3 = \frac{1}{(4\pi)^2}\left(\frac{1}{30}\mathrm{Tr}_A F^2 - \mathrm{tr}R^2\right)\,. \tag{3.40}$$

The descent equation is then completed by computing the combined gauge and Lorentz variations: $\delta_{(\Lambda_g,\Lambda_{\mathrm{L}})}I_3 = dI^1_2$, resorting to the previous results (3.34). This determines the end of the descent chain to be:

$$I^1_2 = \frac{1}{(4\pi)^2}\left(\frac{1}{30}\mathrm{Tr}_A\Lambda_g dA - \mathrm{tr}\Lambda_{\mathrm{L}}d\omega\right) \overset{(3.34)}{\equiv} \frac{1}{(4\pi)^2}\frac{g^2_{YM}}{\kappa^2_{10}}\,\delta B\,. \tag{3.41}$$

Now, coming back to the anomaly twelve-form, a solution to the descent $\mathcal{I}_{12} = d\mathcal{I}_{11}$ is

$$\mathcal{I}_{11} = \frac{1}{3}I_3\wedge X_8 + \frac{2}{3}I_4\wedge X_7 + \beta d\left(I_3\wedge X_7\right)$$

where $X_8 = dX_7$, β is an arbitrary constant, and the last total derivative term reflects the ambiguity in defining $\mathcal{I}_{11}$. Then, by further applying the procedure, the solution to $\delta_{(\Lambda_g,\Lambda_{\mathrm{L}})}\mathcal{I}_{11} = d\mathcal{I}^1_{10}$ is

$$\mathcal{I}^1_{10} = \left(\beta + \frac{2}{3}\right)I_4\wedge X^1_6 + \left(\frac{1}{3} - \beta\right)I^1_2\wedge X_8$$

where $\delta_{(\Lambda_g,\Lambda_{\mathrm{L}})}X_7 = dX^1_6$.

Such an anomaly inflow is cancelled by the following coupling:

$$S_{GS} = -\frac{1}{(4\pi)^2}\frac{g^2_{YM}}{\kappa^2_{10}}\int_{M_{10}} B\wedge X_8(R,F) - \left(\beta + \frac{2}{3}\right)\int_{M_{10}} I_3\wedge X_7\,. \tag{3.42}$$

Clearly, this is indeed of the form (3.38) with an additional term which lifts to a vanishing anomaly twelve-form, thereby falling in the category of the *local counterterms* mentioned above.

Up to now, the whole discussion is in fact applicable to any gauge group. But we saw above that the anomaly (3.30) factorises only when $\dim G = 496$ and $\mathrm{Tr}F^6$ is not an independent Casimir of the group in the adjoint rep, which occurs for the four semi-simple groups we mentioned above. The first of these two points is straightforward. The second one will now be presented in details for the case $G = E_8 \times E_8$. For the adjoint rep of E_8, both $\mathrm{Tr}_A F^6$ and $\mathrm{Tr}_A F^4$ can be reexpressed in terms of lower power Casimir operators by using the decomposition of $\mathbf{248} = \mathbf{120} + \mathbf{128}$ under the $SO(16)$ subgroup of E_8:

$$\mathrm{Tr}_{248}F^6 = \frac{1}{7200}(\mathrm{Tr}_{248}F^2)^3 \,, \qquad \mathrm{Tr}_{248}F^4 = \frac{1}{100}(\mathrm{Tr}_{248}F^2)^2 \,. \tag{3.43}$$

Both these relations can be applied to the adjoint rep of $E_8 \times E_8$, with the definition: $\mathrm{tr}_{G_1\times G_2}X^m = \mathrm{tr}_{G_1}X^m + \mathrm{tr}_{G_2}X^m$, to derive:

$$\mathrm{Tr}_A F^6 = \frac{1}{48}\mathrm{Tr}_A F^2 \left(\mathrm{Tr}_A F^4 - \frac{1}{300}(\mathrm{Tr}_A F^2)^2\right) .$$

With this last identity in hand, we can now show that expression (3.30) factorises according to: $\mathcal{I}_{12}^{\mathrm{het}}(R,F) \overset{(3.39)}{=} \mathcal{I}_{12}$, and then be cancelled by the variation of the Green-Schwarz term (3.38) with:

$$X_8(R,F) = \frac{1}{(2\pi)^3 4!8}\left(\mathrm{tr}R^4 + \frac{1}{4}(\mathrm{tr}R^2)^2 - \frac{1}{30}\mathrm{Tr}_A F^4 \mathrm{tr}R^2 + \frac{1}{3}\mathrm{Tr}_A F^4 - \frac{1}{900}(\mathrm{Tr}_A F^2)^2\right) . \tag{3.44}$$

The $SO(32)$ case can be treated similarly, see [132] for details.

From the physical point of view, the addition of the Green-Schwarz term to the low-energy effective action (3.31) brings in higher derivative effects. Indeed, the spin connection is proportional to the derivative of the vielbein, so a gravitational term such as $\omega \wedge (\mathrm{tr}R^2)^2$ will contain three derivatives instead of one.

3.4 M-theory on an orbifold: the Hořava-Witten scenario

The Hořava-Witten scenario proposes that the strong coupling limit of the $E_8 \times E_8$ heterotic string in ten dimensions is given by M-theory on $\mathcal{M}_{11} = \mathcal{M}_{10} \times S^1/\mathbb{Z}_2$ with the insertion of two copies of E_8 gauge multiplets propagating on the boundary of space-time [148, 147]. The dilaton in the $10D$ heterotic string theory is then reinterpreted as the radius R_{het} of the interval. The introduction of massless fields located at the orbifold singularities and charged under a particular gauge group is reminiscent of the twisted sectors occurring in the orbifold compactifications of superstring theories. In both cases, these are required for the quantum consistency of the resulting theory. In the string theory setup, these twisted sectors have to be added at the fixed points to ensure modular invariance of the orbifolded theory and have an explicit representation as closed strings with twisted boundary conditions. In the M-theory picture, since we are working in the strongly coupled but low-energy limit of the theory, we lack such a microscopic description, so that the equivalent of the *twisted sectors* is determined by anomaly-cancellation arguments.

Another consequence of considering a space-time with boundaries is that the formerly exact four-form G_4 of $11D$ supergravity (3.1) is now modified by terms which are neither exact nor closed: $G_4 = dC_3 + ..$, leading to the non-trivial Bianchi identity: $dG_4 \neq 0$, where the LHS of this equation is supported at the fixed points of the orbifold. Properly solving this equation by considering functions with definite periodicity properties under the $\mathbb{Z}_2$ action is at the core of a rigorous verification of local anomaly cancellation in the $11D$ theory. This requires, in particular, a modification of the topological Chern-Simons term [44], similar to the prescription we will encounter in Section 3.6 when treating the normal bundle anomaly of the M5-brane.

3.4.1 Heterotic M-theory: the upstairs and downstairs approaches

When treating such an orbifold of M-theory with end-of-the-world hyperplanes, one can either work in terms of a manifold with boundaries, which is convenient intuitively, or consider the interval as a boundary-free circle S^1 and project out all non $\mathbb{Z}_2$-invariant modes of the fields. These two points of view have been denominated respectively as the *downstairs* and *upstairs* approach, and are defined by:

$$S_{bos} = \int_{\mathcal{M}_{10}\times S_1/\mathbb{Z}_2} d^{11}x\,\mathcal{L}_{down} = \int_{\mathcal{M}_{10}\times S_1} d^{11}x\,\mathcal{L}_{up}\,,$$

where the upstairs Lagrangian density now includes an addition factor of 2 to account for integrating on the whole circle.

In the following, we will work exclusively in the upstairs approach. Making the structure of $\mathcal{L}_{up}$ more precise, it will become clear later on that, in addition to the (modified) topological Chern-Simons term of $11D$ supergravity, one has to include, as in the heterotic case, a Green-Schwarz term, to cancel both gravitational and mixed anomalies:

$$\begin{aligned} 4\kappa^2\mathcal{L}_{up} &= e\,R^{(11)} - \frac{1}{2}G_4\wedge *G_4 - \frac{\kappa^2}{g_{YM}^2}\sum_{i=1,2}F_i\wedge *F_i\wedge\delta_i - \frac{1}{3!}\widetilde{C}_3\wedge G_4\wedge G_4 \\ &\quad -\sqrt[3]{\frac{2\kappa^4}{\pi}}G_4\wedge X_7^{\text{grav}} + .. \\ &= 4\kappa^2\left(\mathcal{L}_{EH} + \mathcal{L}_{kin}^{\text{CJS}} + \mathcal{L}_{kin}^{\text{SYM}} + \mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{GS}}\right) + ..\ . \end{aligned} \tag{3.45}$$

To distinguish the eleven-dimensional from the ten-dimensional curvature two-form, we represent, in the rest of this chapter, the first by $R^{(11)}$ and the second by R.

In the action (3.45), we denote by $i = 1, 2$ the two $10D$ hyperplanes, and include the kinetic terms for the gauge multiplets, which are confined to the hyperplanes by the delta one-forms (3.48) and contain the Yang-Mills connection $\omega_{\text{YM},i}$ as in expression (3.33) (these are now charged separately under the two E_8). In the supersymmetric formulation of the theory, one will include the fermionic part of the supergravity action (3.2) and of the SYM Lagrangian (1.46) plus additional terms required for supersymmetry to hold locally, which we ignore here. Note that we have fixed the two-derivative part of the (modified) Lagrangian density (3.45) by setting $a = b = c = 1$ in expression (3.1).

The three-form $\widetilde{C}_3$ is the modification $\widetilde{C}_3 = C_3 + ...$ resulting from solving the new Bianchi identity for G_4 which now includes contributions from the boundary hyperplanes. The subsequent modification of $\mathcal{L}_{\text{CS}}$ has been pointed out in [44] as necessary to ensure simultaneously local anomaly cancellation in $11D$ and well-defined periodicity properties under $\mathbb{Z}_2$ for all bosonic fields of the theory.

The seven-form X_7^{grav} is a polynomial in the $11D$ curvature two-form resulting from the descent equation

$$X_8^{\text{grav}} = dX_7^{\text{grav}}\,, \qquad \delta X_7^{\text{grav}} = dX_6^{\text{grav},1} \tag{3.46}$$

for the gauge-invariant eight-form:

$$X_8^{\text{grav}}(R^{(11)}) = \frac{1}{(4\pi)^3 4!}\left(\text{tr}(R^{(11)})^4 - \frac{1}{4}(\text{tr}(R^{(11)})^2)^2\right)\,. \tag{3.47}$$

The above polynomial is formally defined as a polynomial in the $11D$ curvature two-form. We will see later on that anomaly cancellation restricts it to the $10D$ boundary hyperplanes. For this purpose, we also introduce a $10D$ spin connection one-form on each plane: ω_i, with $i = 1, 2$, giving rise to two curvature two-forms restricted to the boundaries: $R_i = d\omega_i + \omega_i^2$. These are nothing but the $11D$ Riemann curvature two-form $R^{(11)}$ with components tangent to S^1 suppressed. In addition, one also defines a $10D$ Lorentz Chern-Simons three-form per hyperplane through $\mathrm{tr}R_i^2 = d\omega_{\mathrm{L},i}$.

Finally, denoting $y \in [-\pi R_{\mathrm{het}}, \pi R_{\mathrm{het}}]$ the coordinate on the circle S^1, the delta one-forms appearing in expression (3.45) are defined as

$$\delta_1 = \delta(y)dy\,, \qquad \delta_2 = \delta(y - R_{\mathrm{het}}\pi)dy\,. \tag{3.48}$$

with respect to the fixed points of the interval: $\{0;\, \pi R_{\mathrm{het}}\}$.

A word now on the orbifold projection. Since we are in an odd orbifold of M-theory, invariance of the topological Chern-Simons term $\propto C_3 \wedge G_4 \wedge G_4$ requires the three-form to undergo a global change of sign under the action of the orbifold group:

$$\mathbb{Z}_2 : \quad C_3 \to -C_3$$

so the components $C_{\mu\nu\rho}$ are odd under the $\mathbb{Z}_2$ action, and so are projected out, whereas the $C_{\mu\nu 10}$ are invariant and therefore kept. In the upstairs formalism, the three-form C_3 will then reduce to: $\frac{1}{2!}C_{\mu\iota 10}\, dx^\mu \wedge dx^\nu \wedge dy = \overline{B}_2 \wedge dy$, where the factor 2 is needed to recover, in the upstairs picture, the correct small R_{het} limit to the two-form of heterotic string theory.

Furthermore, when considering the fermionic sector of the theory, the supersymmetries (3.4) commuting with the orbifold action satisfy, at the boundary of $\mathcal{M}_{11}$:

$$(\mathbb{1} - \Gamma^{10})\epsilon = 0\,,$$

from which we deduce the invariance conditions for the gravitino on the boundary space:

$$(\mathbb{1} - \Gamma^{10})\psi_\mu = 0\,, \qquad (\mathbb{1} + \Gamma^{10})\psi_{10} = 0\,. \tag{3.49}$$

A $\mathbb{Z}_2$-invariant truncation will thus project out half of the spinor components, reducing ψ_A to the $10D$ Majorana-Weyl gravitino ψ_μ together with the $10D$ Majorana-Weyl spinor ψ of the low-energy heterotic supergravity. This is supplemented with the gauginos entering expression (1.46), which already live in $10D$.

3.4.2 The modified Bianchi identity

As was first shown in [147], the insertion of boundary hypeplanes in the $11D$ setting calls for a modification of the Bianchi identity (3.50). The precise way in which this modification occurs can be deduced from observing how the basic lagrangian $\mathcal{L}_{\mathrm{CJS}} + \mathcal{L}_{\mathrm{SYM}}$ given by expressions (3.2) and (1.46) has to be modified by additional interactions to become *locally* supersymmetric. This in turn also has the effect of modifying the supersymmetry transformations (3.4). This procedure however only produces the $\mathrm{Tr}_A F_i^2$ part of expression (3.51). The part depending on $\mathrm{tr}R_i^2$ is introduced for the needs of anomaly cancellation (which is a typical $\mathcal{O}(\alpha')$ string correction, cf. Section 3.3).

The Bianchi identity for G_4 has now to be modified to:

$$dG_4 = -\gamma \sum_{i=1,2} I_{4,i} \wedge \delta_i \tag{3.50}$$

by a contribution supported at the $\mathbb{Z}_2$ fixed points. The four-forms $I_{4,i}$ are the analogs of expression (3.40), being exact and gauge/Lorentz invariant, they satisfy the descent equations:

$$I_{4,i} = d\omega_{3,i} = \frac{1}{(4\pi)^2}\left(\frac{1}{30}\mathrm{Tr}_A F_i^2 - \frac{1}{2}\mathrm{tr}R_i^2\right), \qquad \delta\omega_{3,i} = d\omega_{2,i}^1 . \tag{3.51}$$

with the polynomials:

$$\omega_{3,i} = \frac{1}{(4\pi)^2}\left(\omega_{\mathrm{YM},i} - \frac{1}{2}\omega_{\mathrm{L},i}\right), \qquad \omega_{2,i}^1 = \frac{1}{(4\pi)^2}\left(\frac{1}{30}\mathrm{Tr}_A\Lambda_g dA_i - \frac{1}{2}\mathrm{tr}\Lambda_{\mathrm{L}} d\omega_i\right) \tag{3.52}$$

Note that the above expressions contain a 1/2 factor with respect to expression (3.40) and (3.41) in front of the gravitational contributions. Indeed, when performing the small radius limit to recover the weakly coupled heterotic theory, we have $R_{2,1} = R_{2,2} = R_2$, and the gravitational contributions from each hyperplane add up.

The next important issue is the constant appearing in expression (3.50). It is fixed by anomaly cancellation in eleven dimensions, as we will see in Section 3.5, according to:

$$\gamma = (4\pi)^2 \frac{2\kappa^2}{g_{YM}^2} = 2\sqrt[3]{4\pi\kappa^2} . \tag{3.53}$$

The second equality can be determined, for instance, by the variation of the Green-Schwarz term, see eqn.(3.69) later on. This relation is interesting in various respects. First of all, it fixes the ratio $g_{YM}^6/\kappa^2 = (4\pi)^5$ and thereby the dependence of the gauge coupling in terms of the $11D$ gravitational constant. This is expected from the $11D$ theory which has no scalar field controlling the value of this ratio. Furthermore, in the bottom up approach, the strong coupling regime of the resulting string theory does not exhibit any such adjustable parameter, so that the $11D$ theory has to fix it in some way. Consequently, M-theory on $S^1/\mathbb{Z}_2$ can be regarded as predicting this value.

Solving the Bianchi identity (3.50) requires the primitive of the delta one-forms (3.48), which, to be consistent with the orbifold reduction, need to exhibit a definite $\mathbb{Z}_2$ periodicity under $\mathbb{Z}_2$. For this purpose, we choose [43]

$$\epsilon_i(y) = \mathrm{sgn}(y - y_i) - \frac{1}{\pi R}(y - y_i), \qquad \delta_i(y) = \frac{1}{2}\left(d\epsilon_i + \frac{dy}{\pi R_{\mathrm{het}}}\right), \quad i = 1, 2 . \tag{3.54}$$

defined with respect to fixed points of the interval $\{y_1 = 0;\, y_2 = \pi R_{\mathrm{het}}\}$.

When performing the reduction to the heterotic string, we will need in particular the following integrals:

$$\int_{S^1} \epsilon_i(y)\, dy = 0 , \qquad \int_{S^1} \epsilon_i(y)\epsilon_j(y)\, dy = \pi R_{\mathrm{het}}\left(\delta_{ij} - \frac{1}{3}\right) . \tag{3.55}$$

The first one reflects the fact that the ϵ_i are odd under a parity transformation of y. Moreover, one should also choose a sensible regularisation for an expression such as $\int \epsilon_i\epsilon_j\delta_k$ involving the Dirac distributions one-forms δ_k. Ref. [43] has shown that, since in the vicinity of the identity $\epsilon_i(y) \simeq 2\delta(y)$, one can deduce the first of the two following relations:

$$\int \epsilon_i\epsilon_j\delta_k \simeq \frac{1}{3}\int \delta_{i,k}\delta_{i,k}\delta_k , \qquad \int \delta_i\epsilon_j \to 0 \tag{3.56}$$

where the $\delta_{i,k}$ are the Kronecker symbols. The second relation is obvious from the definition of ϵ_i (3.54).

The Bianchi identity (3.50) is then solved by expression

$$G_4 = d\widetilde{C}_3 + \gamma \sum_{i=1,2} \delta_i \wedge \omega_{3,i} \tag{3.57}$$

which contains a free integration parameter b_{HW}, defining the following family of solution:

$$\widetilde{C}_3 = C_3 - b_{HW}\frac{\gamma}{2}\sum_{i=1,2}\epsilon_i\omega_{3,i} \qquad (3.58)$$

The tilded three-form $\widetilde{C}_3$ is precisely the expression modifying the Chern-Simons term in the Lagrangian (3.45). The motivation for rewriting the four-form (3.57) in terms of $\widetilde{C}_3$, stems from expressing it as closed form $G_4 = d\widetilde{C}_3$ in the bulk, which suggests the modification of the Lagrangian (3.45). This has been confirmed [44] to be required by anomaly cancellation. The presence of localised contributions in expression (3.57) will generate interaction terms in the Lagrangian proportional to δ_i. In [147], this has been argued to be attributable to the fact that one is trying to enforce a classical supergravity treatment of a problem which should be considered in quantum M-theory, where a natural cutoff proportional to $1/\sqrt[9]{\kappa^2}$ would presumably replace the singular δ_i one-forms.

In any case, the information we can retrieve from the analysis of anomalies should in principle be reliable, since, being interpretable as an infrared effect, anomalies are independent from the choice of the cutoff, so that anomalous 1-loop contributions can be computed even in non-renormalisable theories.

3.4.3 Gauge invariance and consistent truncation

In the preceding section, we just saw that solving the modified Bianchi identity (3.50) led to a family of solution (3.57) determined by a parameter b_{HW} (3.58). Let us pause a while and consider how gauge invariance and a consistant $\mathbb{Z}_2$-truncation of the four-form constrains the value of b_{HW}.

Since the $\omega_{3,i}$ have non-trivial gauge/Lorentz transformation (3.51), for G_4 to be gauge invariant, the three-form has to transform as

$$\begin{aligned} \delta C_3 &= dB_2^1 + \gamma\sum_i\left(\frac{b_{HW}}{2}\epsilon_i d\omega_{2,i}^1 + \delta_i\wedge\omega_{2,i}^1\right) \\ &\leftrightarrow \quad \delta\widetilde{C}_3 = dB_2^1 + \gamma\sum_i\delta_i\wedge\omega_{2,i}^1 \end{aligned} \qquad (3.59)$$

with B_2^1 linear in the gauge and Lorentz parameters Λ_g and Λ_{L}.

In particular, if $b_{HW} = 1$, which we will see below to be a necessary condition for the $\mathbb{Z}_2$ truncation to be consistent (and later on, for G_4 to be globally well defined), the transformation for C_3 takes the particularly simple form:

$$\delta C_3 = d\Big(B_2^1 + \sum_i\frac{\gamma}{2}\epsilon_i\omega_{2,i}^1\Big) + \frac{\gamma}{2\pi R_{\mathrm{het}}}dy\wedge\sum_i\omega_{2,i}^1$$

where we have used the second relation (3.54).

As seen in Section (3.4), the $\mathbb{Z}_2$ projection resulting from considering the $11D$ theory on an interval eliminates $C_{\mu\nu\rho} = 0$. However, this is a consistent truncation only if the components we project out are gauge-invariant, namely: $\delta C_{\mu\nu\rho} = 0$, which implies $\partial_{[\mu}B_{\nu\rho]}^1 + \gamma\frac{b_{HW}}{2}\sum_i\epsilon_i\partial_{[\mu}\omega_{i\,\nu\rho]}^1 = 0$, and therefore fixes

$$B_2^1 = -\gamma\frac{b_{HW}}{2}\sum_i\epsilon_i\omega_{2,i}^1$$

so that the transformation (3.59) is now rewritten:

$$\begin{aligned} \delta C_3 &= \gamma \sum_i \left((1-b_{HW})\delta_i + \frac{b_{HW}}{2\pi R_{\rm het}} dy \right) \wedge \omega^1_{2,i} \\ &\leftrightarrow \quad \delta \widetilde{C}_3 = \gamma \sum_i \delta_i \wedge \omega^1_{2,i} - \gamma \frac{b_{HW}}{2} \sum_i d(\epsilon_i \omega^1_{2,i}) \,. \end{aligned} \tag{3.60}$$

At this stage, setting $b_{HW} = 1$ cancels the part of the transformation which is localised on the hyperplanes for *both* C_3 and $\widetilde{C}_3$. In any case, δC_3 and $\delta\widetilde{C}_3$ will have non-trivial transformations in the bulk, due to requiring the step functions ϵ_i to be periodic.

We will now show, following [43], that a consistent truncation of the $11D$ fields in the small $R_{\rm het}$ limit actually requires $b_{HW} = 1$. In this limit we make contact with the weakly coupled heterotic string theory, whose field content is recovered from the zero-modes of the Fourier expansion of the M-theory degrees of freedom on the circle. In this procedure, the massive fields corresponding to higher Fourier modes are truncated. In particular, if we expand the $11D$ fields G_4 and C_3 on S^1 using the orthonormal basis

$$e_n(y) = \frac{1}{\sqrt{2\pi R_{\rm het}}} e^{in\frac{y}{R_{\rm het}}} \,, \qquad n \in \mathbb{N}$$

satisfying $\int dy\, e_n(y) e^*_m(y) = \delta_{nm}$, then, the Fourier modes for any tensor field $X(x,y)$ are obtained from $X^{(n)}(x) = \frac{1}{\sqrt{2\pi R_{\rm het}}} \int_{-\pi R_{\rm het}}^{\pi R_{\rm het}} X(x,y)\, e_n(y)\, dy$, leading, in the case of interest, to the following expressions

$$\begin{aligned} G^{(0)}_{\mu\nu\rho\,10} &= 3\partial_{[\mu} C^{(0)}_{\nu\rho]10} - \frac{\gamma}{2\pi R_{\rm het}} \sum_{i=1,2} (\omega_i)_{\mu\nu\rho} \,, \\ G^{(n)}_{\mu\nu\rho\,10} &= 3\partial_{[\mu} C^{(n)}_{\nu\rho]10} - \frac{\gamma}{2\pi R_{\rm het}} (1-b_{HW}) \left[(\omega_1)_{\mu\nu\rho} + (-1)^n (\omega_2)_{\mu\nu\rho} \right] \,. \end{aligned}$$

In principle, a truncation of the $n > 0$ modes can only be consistently performed if the latter are gauge invariant. Now, since G_4 is gauge and local Lorentz invariant, $C^{(n)}_{\nu\rho 10}$ is never so, unless $b_{HW} = 1$. Consequently, the small radius limit to the perturbative heterotic string can be considered safe only in this case, which, in passing, corresponds to a non-singular solution for G_4 (3.57) (where all delta-function contributions have been cancelled). We will see later on when considering the addition of M5-branes to this setup, that b_{HW} is fixed to the same value independently by a cohomology constraint relating the number of five-branes and the non-triviality of the anomaly polynomial $I_{4,i}$. Actually, it will be shown that even in the absence of M5-branes, a global definition of the four-form G_4 requires $b_{HW} = 1$ if the polynomial $I_{4,i}$ is cohomologically non-trivial.

Having set $b_{HW} = 1$, we perform the reduction to the $10D$ heterotic theory. In particular, the B-field from the $(\widetilde{\rm NS,NS})$ sector and its associated field strength are given by the following zero-modes:

$$\begin{aligned} 2B_{\mu\nu} &\doteq \overline{B}^{(0)}_{\mu\nu}(x) = \frac{1}{2\pi R_{\rm het}} \int_{-\pi R_{\rm het}}^{\pi R_{\rm het}} C_{\mu\nu 10}(x,y)\, dy \,, \\ 2H_{\mu\nu\rho} &\doteq G^{(0)}_{\mu\nu\rho 10}(x) = \frac{1}{2\pi R_{\rm het}} \int_{-\pi R_{\rm het}}^{\pi R_{\rm het}} G_{\mu\nu\rho 10}(x,y)\, dy \end{aligned} \tag{3.61}$$

for $x \in \mathbb{R}^{1+9}$.

Note that that in the upstairs approach, taking the $10D$ gravitational constant to be $\kappa^2_{10} = \frac{\kappa^2}{2\pi R_{\rm het}}$ requires both $10D$ fields B_2 and H_3 to be defined with an additional factor of 2, as in expression (3.61)

We also insist on keeping the $\frac{\kappa_{10}^2}{g_{YM}^2}$ factor overt, to compare our results with expression (3.32) and, later on, (3.42). Then, in the weak coupling limit, taking $R_{\text{het}} \to 0$ identifies $\omega_{\text{L},1} = \omega_{\text{L},2} = \omega_{\text{L}}$, so that using eqs. (3.53), (3.33), and integrating G_4 (3.57) over S^1, and truncating to the zero-modes, one obtains:

$$H = dB - \frac{\gamma}{4\pi R_{\text{het}}} \sum_{i=1,2} \omega_{3,i} = dB - \frac{\kappa_{10}^2}{g_{YM}^2} (\omega_{\text{YM}} - \omega_{\text{L}}) \tag{3.62}$$

since we have defined $\kappa_{10}^2 = \frac{\kappa^2}{2\pi R_{\text{het}}}$.

As expected, we recover expression (3.32). We will see later on how the field-strength (3.62) is modified when five-branes are present in the parent $11D$ theory. In the $10D$ heterotic picture, these additional terms descend to contributions from NS five-branes.

3.5 Anomaly cancellation in M-theory compactified on $S^1/\mathbb{Z}_2$

In this section, we carry out the anomaly cancelling procedure in the Hořava-Witten theory. Since, despite a few proposals, the fundamental degrees of freedom of M-theory are still unknown, we will be working in the strongly coupled low-energy supergravity description of the theory, inspired by a modification of the $11D$ Cremmer-Julia-Scherk (CJS) supergravity Lagrangian and given by expression (3.45). As pointed out in Section 3.4.2, since anomalies are infrared effects, their cancellation in the effective theory should in principle suffice to ensure quantum consistency of the original high-energy theory, thereby uncovering some new aspects of the dualities between the known superstring theories and the still evasive M-theory.

In particular, the topological Chern-Simons term of $11D$ supergravity will now play a rôle similar to the Green-Schwarz term (3.38) appearing as a 1-loop correction to classical heterotic supergravity. However, this only cancels part of the $11D$ anomaly, so that one needs yet another counterterm, the $11D$ Green-Schwarz $\int \mathcal{L}_{\text{GS}}$ (3.45), to remove the residual part.

The chiral fields contributing to the anomaly result from the invariant zero-modes of the $11D$ gravitino and from the $10D$ gaugino fields living on the hyperplanes located at the fixed points of the interval (which can be considered, as mentioned before, as part of the "twisted sector" of the orbifold). The $\mathbb{Z}_2$-projection (3.49) removes half the components of the $11D$ (Majorana) gravitino leaving two $10D$ spinors: a (Majorana-Weyl) graviton together with a spin-$\frac{1}{2}$ (Majorana-Weyl) fermion. Adding up the gaugino contributions, one arrives at two separate anomaly twelve-forms attached to each hyperplane:

$$\mathcal{I}_{12,i}^{\text{HW}}(R_i, F_i) = \frac{1}{2}\left(-\mathcal{I}_{12}^{(1/2)}(R_i, F_i) + \frac{1}{2}\left[\mathcal{I}_{12}^{(1/2)}(R_i) - \mathcal{I}_{12}^{(3/2)}(R_i)\right]\right) . \tag{3.63}$$

A few comments about the shape of the anomaly (3.63) are worth making. First, since the two hyperplanes have equal orientations[4], it is natural to assume that they carry the same anomaly. As pointed out before, this implies a factor 1/2 in front of the purely gravitational part (the $11D$ gravitino contribution), since we expect to recover the anomaly polynomial (3.39) in the heterotic limit where $R_1 = R_2 = R$. The inflow from the gauge fields remains the same, since the two separate E_8 eventually combine into a single $E_8 \times E_8$. Finally, similarly to the heterotic case (3.30) there is an overall 1/2 factor coming from the Majorana condition in $10D$. The difference in signs between the three contributions to the anomaly reflects the fact that gravitino and the gauginos are left-handed, while the spin-$\frac{1}{2}$ fermion has opposite chirality.

[4]In contrast to the $E_8 \times \overline{E}_8$ theory of [112] which is non-supersymmetric, and unstable, due to the Casimir force exerted on the two boundary hyperplanes.

Using the eight-form (3.47) and relation (3.43), expression (3.63) factorises as

$$\mathcal{I}^{\rm HW}_{12,i}(R_i,F_i) = \left(X_8^{\rm grav}(R_i) + \frac{\pi}{3} I^2_{4,i}\right) \wedge I_{4,i} \tag{3.64}$$

so that one can check that in the small radius limit, one recovers the anomaly polynomial (3.39) by using the elementary identity $a^3+b^3 = (a+b)(a^2+b^2-ab)$. In this limit, the above expression restricts to

$$\sum_i \left(X_8^{\rm grav}(R_i) + \frac{\pi}{3} I^2_{4,i}\right) \wedge I_{4,i}\Bigg|_{R_1=R_2=R} = \sum_i I_{4,i} \wedge \Big[X_8^{\rm grav}(R) + \frac{\pi}{3}\Big(\sum_j I^2_{4,j} - I_{4,1}\wedge I_{4,2}\Big)\Big]\,. \tag{3.65}$$

We then use the fact that for a semisimple group such as $E_8\times E_8$, the trace splits into a sum of traces over each simple factor: ${\rm tr}X^m = {\rm tr}X_1^m + {\rm tr}X_2^m$, and resort again to relation (3.43), to obtain:

$$300{\rm Tr}_A F^4 - ({\rm Tr}_A F^2)^2 = 2\left[({\rm Tr}_A F_1^2)^2 + ({\rm Tr}_A F_2^2)^2 - {\rm Tr}_A F_1^2\,{\rm Tr}_A F_2^2\right]\,. \tag{3.66}$$

Combining both eqs (3.65) and (3.66), we are able to show that:

$$\sum_{i=1,2} \mathcal{I}^{\rm HW}_{12,i}(R_i,F_i)\Bigg|_{R_1=R_2=R} = \frac{1}{(4\pi)^2}\left(\frac{1}{30}{\rm Tr}_A F^2 - {\rm tr}R^2\right) \wedge X_8(R,F)\,.$$

recovering the result (3.39) known from heterotic string theory.

After this consistency check, we poceed to show how both the Chern-Simons and Green-Schwarz terms in expression (3.45) are needed to cancel the total anomaly inflow (3.64) and how this cancellation occurs *locally* on each plane.

Taking advantage of the analysis made in Section (3.4.3), we set $b_{HW} = 1$ for consistency throughout the calculation. Using the solutions (3.57) and (3.58) and relation (3.55), we can compute the anomaly inflow due to the Chern-Simons:

$$\begin{aligned}
\delta\left[-\frac{1}{24\kappa^2}\int_{\mathcal{M}_{10}\times S^1} \widetilde{C}_3 \wedge G_4 \wedge G_4\right] &= \frac{\gamma^3}{96\kappa^2}\int_{\mathcal{M}_{10}\times S^1} \sum_{i,j,k}\Big(2\epsilon_i\epsilon_j\delta_k \wedge d\omega^1_{2,i}\wedge I_{4,j}\wedge\omega_{3,k} \\
&\qquad -\epsilon_j\epsilon_k\delta_i \wedge \omega^1_{2,i}\wedge I_{4,j}\wedge I_{4,k}\Big) \\
&= -\frac{\gamma^3}{96\kappa^2}\sum_i \int_{M_{10}} \omega^1_{2,i}\wedge (I_{4,i})^2\,.
\end{aligned} \tag{3.67}$$

To determine the first two lines of the contribution (3.67), we have exploited the fact that $C_3 = \overline{B}_2 \wedge dy$ and $\delta_i \wedge \delta_j = 0$. While the last line is obtained after integration by parts (assuming that $\partial M_{10} = \varnothing$) by using the first of the two relations (3.56)

At this stage, we can check the validity of the relation (3.53). Using this relation indeed turns the prefactor of the last line of eqn.(3.67) into $\frac{\gamma^3}{96\kappa^2} = \frac{\pi}{3}$, so that the anomalous variation 3.67 results, by descent, from the anomaly twelve-form:

$$\mathcal{I}^{CS}_{12} = \sum_i \mathcal{I}^{CS}_{12,i}\,, \qquad \text{with } \mathcal{I}^{CS}_{12,i} = -\frac{\pi}{3}(I_{4,i})^3\,, \tag{3.68}$$

and the inflow from the topological Cher-Simons term exactly compensates the second term on the RHS of expression (3.64).

The first term on the RHS of expression (3.64), meanwhile, is cancelled by inflow from the $11D$ Green-Schwarz term $\int \mathcal{L}_{\rm GS}$ (3.45) whose variation yields:

$$\delta\left[-\frac{1}{2\sqrt[3]{4\pi\kappa^2}}\int_{\mathcal{M}_{10}\times S^1} G_4 \wedge X_7^{\rm grav}\right] \overset{(3.53)}{=} -\sum_i \int_{\mathcal{M}_{10}} I_{4,i}\wedge X_6^{\rm grav,1} \tag{3.69}$$

after integration by parts. Once again, fixing the value of the gauge coupling in terms of the gravitational one as in eqn.(3.53), results from requiring the anomalous variation (3.69) to descend from the twelve-form:

$$\mathcal{I}_{12}^{GS} = \sum_i \mathcal{I}_{12,i}^{GS}\,, \qquad \text{with } \mathcal{I}_{12,i}^{GS} = -X_8^{\text{grav}}(R_i) \wedge I_{4,i}\,. \tag{3.70}$$

Summing up both contributions 3.70 and 3.68 will cancel the total anomaly 3.64 *locally* on each $10D$ boundary hyperplanes:

$$\mathcal{I}_{12,i}^{GS} + \mathcal{I}_{12,i}^{CS} = -\mathcal{I}_{12,i}^{\text{HW}}(R_i, F_i)\,.$$

In conclusion, we will make a few comments on what new light the analysis of anomalies performed here sheds on the structure of M-theory.

We first notice that topological Chern-Simons interaction now acts a Green-Schwarz term with respect to anomaly cancellation. Since this term is already present in the minimal (CJS) supergravity, it implies that the classical theory with supergravity multiplets in the bulk and E_8 vector multiplets on the boundary is not gauge invariant, so that the Hořava-Witten construction with end-of-the-world hyperplanes is really consistent only as a quantum theory. This is in particular illustrated by the prediction (3.53) which fixes the order of magnitude between the gauge and gravitational couplings to $g_{YM} \propto \kappa^{2/3}$. Then, the gauge kinetic terms being of $\mathcal{O}(1/g_{YM}^2)$ are higher order corrections compared to the gravity Lagrangian of $\mathcal{O}(1/\kappa^2)$, which confirms that the classical theory is only consistent in the absence of gauge fields.

This is somehow different from the heterotic string case presented in Section 3.3.3, where the Green-Schwarz term is a one-loop corrections, so that the theory is gauge invariant both at the classical (ignoring altogether loop contributions from chiral fields) and the quantum level.

Furthermore, the structure of the $11D$ Green-Schwarz also opens some perspectives in the framework of string dualities. In particular, restricting the term $\int G_7 \wedge X_7^{\text{grav}}$ to the case where $G_4 = dC_3$, we obtain, after integration by parts, an expression proportional to $\int C_3 \wedge X_8^{\text{grav}}$. Upon compactifying on S^1(without $\mathbb{Z}_2$ projection this time) and after taking the small radius limit, one recovers the one-loop correction to IIA string theory, proportional to

$$\int_{\mathcal{M}_{10}} B_{\text{IIA}} \wedge \left(\text{tr}R^4 - \frac{1}{4}(\text{tr}R^2)^2\right) \tag{3.71}$$

with B_{IIA} the antisymetric NSNS two-form of IIA supergravity. Such an interaction has been shown in [241] to be an exact one-loop calculation and is thus expected to be found in $11D$ as well. Moreover, the polynomial X_8^{grav} (3.47) is also proportional, as we will see in Section 3.6, to the Lorentz anomaly of the $11D$ five-brane, so that, in $10D$, inflow from a term such as expression (3.71) cancels the NS five-brane world-sheet anomaly in IIA string theory, as shown in [107, 251]. The fact that the same $11D$ Green-Schwarz term gives rise to both the heterotic anomaly cancelling term, a one-loop exact correction in IIA string theory and ensures consistency of the $11D$ five-brane, and that all these corrective terms are known independently in ten or eleven dimensions seems to indicate that what relates them is their common link through M-theory.

3.5.1 The heterotic limit

Having verified anomaly cancellation in $11D$ for the Hořava-Witten model, it is instructive to take the small radius limit $R_{\text{het}} \to 0$ and check that we indeed recover the heterotic Green-Schwarz term (3.38) from the reduction of both the $11D$ Chern-Simons and Green-Schwarz terms in expression (3.45).

Inserting the solution for G_4 (3.57) and $\widetilde{C}_3$ (3.58) in $\mathcal{L}_{\rm CS}$ (3.45) and setting $b_{HW} = 1$ for consistency, the $11D$ Chern-Simons term yields the following contributions:

$$\begin{aligned}
&\int_{\mathcal{M}_{10}\times S^1} \mathcal{L}_{\rm CS} \\
&= -\frac{1}{24\kappa^2}\int_{\mathcal{M}_{10}\times S^1} \Big(\overline{B}\wedge dy - \frac{\gamma}{2}\sum_i \epsilon_i \omega_{3,i}\Big)\wedge\Big(\Big[d\overline{B} - \frac{\gamma}{2\pi R_{\rm het}}\sum_j \omega_{3,j}\Big]\wedge dy - \frac{\gamma}{2}\sum_k \epsilon_k I_{4,k}\Big)^2 \\
&= -\frac{1}{24\kappa^2}\int_{\mathcal{M}_{10}\times S^1} \sum_{i,j}\epsilon_i\epsilon_j dy\wedge\left[3\left(\frac{\gamma}{2}\right)^2 \overline{B}\wedge I_{4,i}\wedge I_{4,j} - \frac{2}{\pi R_{\rm het}}\left(\frac{\gamma}{2}\right)^3 \omega_{3,i}\wedge I_{4,j}\wedge\sum_k \omega_{3,k}\right].
\end{aligned} \tag{3.72}$$

where we have used the fact that the differential acting on $d\overline{B}$ vanishes on a function $f = f(y)$, when integrated by parts.

One can now perform the integral over S^1, by using the second eqn. (3.55), and truncate to the zero-modes of the invariant $\overline{B}$ (3.61).

Next, in order to compare with expression (3.38), we need to reinstate the dependence on the gauge coupling constant g_{YM}. For this purpose, we use relation (3.53) to reexpress

$$2\pi^2 R_{\rm het}\frac{\gamma^2}{\kappa^2} = \frac{64\pi^3 R_{\rm het}}{\gamma} = \frac{g_{YM}^2}{\kappa_{10}^2}\,.$$

Then, the $10D$ reduction of the Chern-Simons (3.72) term gives the contribution:

$$\begin{aligned}
&-\frac{1}{(4\pi)^2}\frac{g_{YM}^2}{\kappa_{10}^2}\int_{\mathcal{M}_{10}} B\wedge\frac{\pi}{3}\left(\sum_i I_{4,i}^2 - I_{4,1}\wedge I_{4,2}\right) \\
&\quad+\frac{1}{72\pi}\int_{\mathcal{M}_{10}}\Big[\sum_i \omega_{3,i}\wedge I_{4,i} - \frac{1}{2}\Big(\omega_{3,1}\wedge I_{4,2} + \omega_{3,2}\wedge I_{4,1}\Big)\Big]\wedge(\omega_{\rm YM} - \omega_{\rm L})\,.
\end{aligned} \tag{3.73}$$

In the first line of the above expression, we recognise the descent of the second part of the anomaly (3.65), so that this first line reproduces part of the heterotic Green-Schwarz term (3.38), while the second line contributes an irrelevant counterterm, which, due to the identification $R_1 = R_2 = R$, is given in terms of $\omega_{3,i} = \frac{1}{(4\pi)^2}\left(\omega_{{\rm YM},i} - \frac{1}{2}\omega_{\rm L}\right)$ and $I_{4,i} = \frac{1}{(4\pi)^2}\left(\frac{1}{30}{\rm Tr}_A F_i^2 - \frac{1}{2}{\rm tr}R^2\right)$, where R is the $10D$ curvature tensor.

The rest of the Green-Schwarz term (3.38) is retrieved from the $S^1/\mathbb{Z}_2$ reduction of the $11D$ Green-Schwarz action (3.45). Paying attention to the position of the one-form dy for the overall sign, this contribution reads:

$$\begin{aligned}
\int_{\mathcal{M}_{10}\times S^1}\mathcal{L}_{GS} &= \frac{1}{\gamma}\int_{\mathcal{M}_{10}\times S^1} dy\wedge\left[d\overline{B} - \frac{\gamma}{2\pi R_{\rm het}}\sum_i \omega_{3,i}\right]\wedge X_7^{\rm grav}(R^{(11)}) \\
&= \left(\frac{1}{(4\pi)^2}\frac{64\pi^3 R_{\rm het}}{\gamma}\int_{M_{10}} dB - \int_{M_{10}}\sum_i \omega_{3,i}\right)\wedge X_7^{\rm grav}(R) \\
&= -\frac{1}{(4\pi)^2}\frac{g_{YM}^2}{\kappa_{10}^2}\int_{M_{10}} B\wedge X_8^{\rm grav}(R) - \frac{1}{(4\pi)^2}\int_{M_{10}}(\omega_{\rm YM} - \omega_{\rm L})\wedge X_7^{\rm grav}(R)\,.
\end{aligned} \tag{3.74}$$

The first term on the second line (3.74) is what we need to cancel the first part of the anomaly (3.65). Again, there comes along in the process a counterterm which descends from a vanishing twelve-form. Using the definition (3.40) for I_3, it exactly reproduces the counterterm computed in expression (3.42) for $\beta = \frac{1}{3}$. Alternatively, the whole expression (3.74) can be rephrased as:

$$\frac{1}{(4\pi)^2}\frac{g_{YM}^2}{\kappa_{10}^2}\int_{M_{10}} H_3\wedge X_7^{\rm grav}(R)$$

were now H_3 is gauge invariant, but not X_7^{grav}, so that the anomaly inflow will now result from applying the descent equation to the latter.

If we now add up expressions (3.73) and (3.74), we reproduce the Green-Schwarz term (3.38) with a sum of irrelevant counterterms determined by the residual parts of these expressions, namely:

$$S_{\text{counter}} = \int_{M_{10}} \Big[\frac{2\pi}{9}\sum_i \omega_{3,i}\wedge I_{4,i} - \frac{\pi}{9}\Big(\omega_{3,1}\wedge I_{4,2} - \omega_{3,2}\wedge I_{4,1}\Big) + X_7^{\text{grav}}(R)\Big]\wedge I_3$$

where I_3 is given by expression (3.40).

3.6 Eleven-dimensional five-branes

In this section we will consider the anomalies resulting from the inclusion of five-branes in the Hořava-Witten scenario. We will see in particular what new interaction terms are implied in the topological part of the Lagrangian (3.45) by the resulting modification of the Bianchi identity and what constraints the presence of $11D$ five-branes sets on the free parameters of the theory, in particular the parameter b_{HW} seen previously and a new integration constant b_{M5} related to the five-brane solution.

3.6.1 Five-branes and membrane solutions in $11D$

Before giving a detailed treatment of $11D$ five-brane anomalies, we briefly review some basic facts necessary to our discussion. As we have seen in Section 3.2, M5-branes couple magnetically to the six-form potential whose field-strength is the Hodge dual of the three-form C_3 of $11D$ supergravity, through the Wess-Zumino term: $Q_5 \int_{\mathcal{W}_6} \widehat{C}_6$. Integration is over the five-brane world volume $\mathcal{W}_6$. For later purpose, we now distinguish the M5-brane charge Q_5 from its surface tension T_5.

In particular, since an M5-brane produces long-range gravitational and other fields, its charge Q_5 can be detected as a surface integral at infinity. The M5-brane is a solitonic solution of the equations of motion in $11D$. The spatial part of the metric (see eqn.(3.78) below) for an infinite planar M5-brane is then topologically equivalent to $\mathbb{R}^5\times S^4$ asympotically. Translational invariance in the directions parallel to the brane reduces integrals for total charges to integrals over the transverse S^4. Then Q_5 is determined by the magnetic flux of the four-form G_4 at infinity, which we give in units relevant for $11D$ supergravity:

$$Q_5 = \frac{1}{2\kappa^2}\int_{S^4} G_4 \tag{3.75}$$

For $\frac{1}{2}$BPS objects, Q_5 is equal to T_5 . In particular, it has been set to this value in the corresponding WZ term of expression (3.13). To see how this identification comes about, consider the (simplified) M5-brane action in a $11D$ supergravity background (see eq.3.11):

$$S_{\text{M5}} = -T_5\int_{\mathcal{W}_6}\left(\text{vol}(\widehat{g}) - \frac{Q_5}{T_5}\widehat{C}_6\right) .$$

We now recall that an M5-brane contibutes a central charge term in the $11D$ supersymmetry algebra (1.76), given by the integral of current density:

$$Z^{MNOP} = Q_5\int_{\mathcal{C}_4} dX^M\wedge dX^N\wedge dX^O\wedge dX^P . \tag{3.76}$$

where $\mathcal{C}_4$ is a non-contractible four-cycle occupied by the M5-brane is space-time.

Now choosing, for instance, a Majorana rep of the Dirac matrices where $C = -i\Gamma^0$. Then, for a static M5-brane, a four-cycle $\mathcal{C}_4$ in directions $\{y^1; y^2; y^3; y^4\}$ and in the absence of M2-brane charges, expression (1.76) becomes [238]:

$$\{\mathsf{Q}_\alpha, \mathsf{Q}^\beta\} = P^0 \delta_\alpha{}^\beta + (\Gamma^{01234})_\alpha{}^\beta Z_{1234}\,. \tag{3.77}$$

As already mentioned in Section 1.4, P^0 and Z_{1234} are infinite, but the tension T_5 and the charge Q_5 remain nonetheless finite.

Since the Q are real, the LHS of the expression (3.77) is positive semi-definite. When $P^0 = 0$, we have the vacuum. Otherwise, since $P^0 \geqslant 0$, the energy must satisfy the Witten-Olive bound $P^0 \geqslant |Z_{1234}|$, as taking anti-M5-brane solutions switches the sign $Z_{1234} \to -Z_{1234}$. This bound can be reformulated using eqn.(3.76) as:

$$T_5 \geqslant |Q_5|$$

When this bound is saturated $T_5 = |Q_5|$, expression (3.77) becomes

$$\{\mathsf{Q}_\alpha, \mathsf{Q}^\beta\} = 2P^0 (\mathcal{P}_\pm^{01234})_\alpha{}^\beta$$

where $\mathcal{P}_\pm^{01234} = \frac{1}{2}\left[\mathbb{I} \pm (\Gamma^{01234})\right]$ acts as a projector on spinors satisfying $(\Gamma^{01234})\epsilon = \pm\epsilon$, which are then eigenspinors of this operator with zero eigenvalue. Since Γ^{MNOPQ} is traceless and idempotent, we have $\mathrm{Tr}\mathcal{P}_\pm^{01234} = \frac{1}{2} \times 32$, so that the dimension of the eigenspace with zero-eigenvalue is 16. The (anti)-M5-brane solution preserves therefore 1/2 of the supersymmetry for $T_5 = \pm|Q_5|$.

A p-brane whose mass saturates the BPS bound is called extremal. For ten and eleven-dimensional supergravities, extremality coincides with supersymmetry, which ensures that these objects are protected against quantum corrections. Second, there is in addition a no-force theorem guaranteeing that the static force between two parallel p-branes cancels due to the compensation between the gravitational attraction and the electrostatic repulsion resulting from branes being charged under the fields of the theory (under RR-fields in the case of type II supergravities).

In complement to the extremal *magnetic* M5-brane, $11D$ supergravity also possesses an electric M2-brane solution which couples to C_3, as already seen in Section 1.4. Its electric charge also satisfies a BPS bound $T_2 \geqslant |Q_2|$. The whole string of argument above can be repeated by exchanging $Z_{1234} \leftrightarrow Z_{12}$ and $\Gamma^{01234} \leftrightarrow \Gamma^{012}$. In addition, there exists two more extremal solitonic solution: the *pp*-wave (or Kaluza-Klein particle) and the KK7M-brane (or also Kaluza-Klein monopole), which couple to the vielbein and its dual. We will not need them further here.

We now focus on the M2- and M5-branes. They represent vacua of $11D$ supergravity for a warped metric which splits into a world-volume part $ds^2(\mathbb{E}^{(1,p)})$ (with Minkoswkian signature) and a transverse Euclidean space $ds^2(\mathbb{E}^{(10-p)})$, preserving $(\text{Poincaré})_{p+1} \times SO(10-p)$ symmetry:

$$\text{M5-brane:} \qquad ds^2_{M5} = H(r)^{-\frac{1}{3}} ds^2(\mathbb{E}^{(1,5)}) + H(r)^{\frac{2}{3}} ds^2(\mathbb{E}^{(5)})\,, \tag{3.78}$$

$$\text{M2-brane:} \qquad ds^2_{M2} = H(r)^{-\frac{2}{3}} ds^2(\mathbb{E}^{(1,2)}) + H(r)^{\frac{1}{3}} ds^2(\mathbb{E}^{(8)})\,, \tag{3.79}$$

$H(r)$ being a harmonic function with single pole[5]:

$$H(r) = 1 + \frac{k_p}{r^{8-p}}$$

where, in polar coordinates, r is the radius of the transverse space $ds^2(\mathbb{E}^{(10-p)})$.

[5]It can have more than one pole for multi-centered p-brane solutions, which we will not consider here.

Given the metrics (3.78) and (3.79), Einstein's equations with non-vanishing Energy momentum tensor computed from the 11D supergravity Lagrangian (3.5)

$$T_{AB} = \frac{1}{12}\left(G_{AD_1D_2D_3}G_B{}^{D_1D_2D_3} - \frac{1}{12}g_{AB}G_{D_1D_2D_3D_4}G^{D_1D_2D_3D_4}\right)$$

and the equations of motion of the four-form field strength (3.7) are solved for:

$$G_4 = \begin{cases} *_5 dH(r) & , \quad \text{for the M5-brane}\,, \\ \text{vol}(\mathbb{E}^{(1,2)}) \wedge d(H(r)^{-1}) & , \quad \text{for the M2-brane}\,, \end{cases} . \tag{3.80}$$

These solutions satisfy $G_4 \wedge G_4 = 0$, so that the equations of motion (3.7) reduce to $\partial_i(eG^{ijkl}) = 0$, which is solved by expressions (3.80), since $H(r)$ is radial. Denoting by α, β, $\gamma = 0, 1, .., p$, the coordinates on the p-brane and i, j, k, $l = p{+}1, .., 10$, the coordinates of the transverse Euclidean space, and restricting to flat space for $\mathbb{E}^{(10-p)}$ and flat space-time for $\mathbb{E}^{(1,p)}$, then $r = \sqrt{x^i x^i}$, and the M5-brane solution gives in components

$$G_{ijkl} = \frac{3k_5}{r^5}\varepsilon_{ijkl}x^l \longleftrightarrow G_{\theta^1\theta^2\theta^3\theta^4} = 3k_5\sqrt{|g_{S^4}|}$$

with $\varepsilon_{1234} = +1$. The θ^i, $i = 1, 2, 3, 4$, parametrize the solide angle for the transverse space $ds^2(\mathbb{E}^5) = dr^2 + r^2 d\Omega_4^2$ in spherical coordinates, with determinant $\sqrt{|g_{S^4}|}$.

The surface charge (3.75) is then easily determined in these coordinates

$$Q_5 = \frac{3k_5}{2\kappa^2}\Omega_4 = \frac{4\pi^2 k_5}{\kappa^2}$$

since the volume of a unit n-sphere is given by:

$$\Omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} .$$

In units of the Planck length l_P, the 11D gravitational constant reads $2\kappa^2 = (2\pi)^8 l_P^9$, so that the M5-brane charge can be reexpressed as $Q_5 = \frac{2k_5 M_P^9}{(2\pi)^6}$, with $M_P = l_P^{-1}$ the Planck mass. Then, if one is dealing with a stack of N M5-branes, the BPS bound is modified to $NT_5 = |Q_5|$, where $T_5 = \frac{M_P^6}{(2\pi)^5}$ is the tension of a single M5-brane. This equation then fixes the parameter k_5 to

$$k_5 = \pi N l_P^3 .$$

Following the same reasoning, one can determine the electric charge

$$Q_2 = \frac{1}{2\kappa^2}\int_{S^7} G_7 .$$

From expression (3.80), on obtains with $G_{\theta^1\ldots\theta^7} = -6k_2\sqrt{|g_{S^7}|}$, with and the electric charge:

$$Q_2 = -\frac{3k_2}{\kappa^2}\Omega_7 = -\frac{\pi^4 k_2}{\kappa^2} .$$

Considering N M2-branes with individual tension $T_2 = \frac{M_P^3}{(2\pi)^2}$ and total charge $NT_5 = |Q_2|$, one can again determine the parameter k_2 :

$$k_2 = 32\pi^2 N l_P^6 .$$

Finally, their exists a Dirac-like charge quantization condition relating the tensions of an electric brane and its magnetic dual, which fixes their product to a constant value:

$$T_5 T_2 = \frac{\pi}{\kappa^2} \,.$$

This relation is clearly verified for the tensions of two single M2- and M5-branes given above. It can be used to rephrase the results for the M5-brane in units of the M2-brane tension.

Finally, their is yet another relation between these two quantities, given in [90]: $(T_2)^2 T_5^{-1} = 2\pi$, which allows to write $T_{2,5}$ in units of κ^2 alone:

$$T_2 = \sqrt[3]{\frac{2\pi^2}{\kappa^2}} \,, \qquad T_5 = \sqrt[3]{\frac{\pi}{2\kappa^4}} \,, \tag{3.81}$$

In the next section, a singular and smoothed out solution of the above equation will be derived to show how the anomalies due to the chiral fields in the five-brane mutiplet cancel in the $11D$ theory.

3.6.2 Anomalies of the eleven-dimensional five-brane

As seen in Section 3.2, five-branes have an even dimensional world-volume, thereby constituting a potential source for anomalies, contrary to membranes, whose world-volume is even[6]. In eleven dimensions, the presence of a five-brane world-volume $\mathcal{W}_6 \subset \mathcal{M}_{11}$ reduces the Lorentz symmetry according to $SO(10,1) \to SO(5,1) \times SO(5)$, which reflects the decomposition of the tangent space $T\mathcal{M}_{11}$ restricted to $\mathcal{W}_6$ into a sum of two orthogonal vector spaces: $T\mathcal{W}_6 \oplus N\mathcal{W}_6$, where $N\mathcal{W}_6 = \cup_{p\in\mathcal{W}_6} N_p\mathcal{W}_6$ has the structure of a fiber bundle over $\mathcal{W}_6$, and is therefore referred to as the *normal bundle*. Clearly, its structure group is the $SO(5)$ resulting from the breaking of $SO(10,1)$ into simple factors.

For a theory admiting five-branes as solutions to its equations of motion, the group of diffeomorphisms mapping the five-brane to itself should be a symmetry of the theory. In the present case, these diffeomorphisms split into vector fields acting on the world-volume W_6 and into gauge transformations on the connection of the $SO(5)$-normal bundle mentioned above. Both sets of diffeomorphisms are potentially afflicted by anomalies.

More precisely, as we have seen in Section 3.2, the $D = 6$ $\mathcal{N} = 2$ five-brane tensor multiplet contains as chiral fields a two-form potential $\mathcal{B}_2$, with anti-self-dual field-strength, and two spinors with negative chirality, whose zero-modes will be sources for anomalies. The two chiral fermions are in the spinor representation of $SO(5)$, which, as we will see below, can be regarded as a sort of "gauge" group, and will thus contribute both world-volume and "gauge" anomalies. The dual three-form field strength $\mathcal{F}_3$ (cf. Section 3.2) being a singlet under $SO(5)$ will only contribute gravitational anomalies in the five-brane world-volume $\mathcal{W}_6$.

Because of the direct sum decomposition $T\mathcal{W}_6 \oplus N\mathcal{W}_6$ of $T\mathcal{M}_{11}|_{\mathcal{W}_6}$, the restriction of the spin connection to $\mathcal{W}_6$ can be shown to split into a tangent space and a normal bundle part: $\omega|_{\mathcal{W}_6} = \omega^{\rm t} + \omega^{\rm n} \equiv \frac{1}{2}(\omega^{\rm t}_{\alpha\, ab}T^{ab} + \omega^{\rm n}_{\alpha\, st}T^{st})dy^\alpha$, with $\{dy^\alpha\}_{\alpha=0,1,..,5}$ the basis of one-forms on $T\mathcal{W}_6$ and T^{ab} and T^{st} the generators of respectively $SO(5,1)$ and $SO(5)$. As in the preceding section, the coordinates on $\mathcal{W}_6$ are labelled by $\alpha, \beta = 0, 1, .., 5$ and their tangent space equivalent are denoted by $a, b = 0, 1, .., 5$, while the normal bundle indices are represented by $s, t, u, v = 6, .., 10$. We see now that the space $N\mathcal{W}_6$ has clearly the structure of a principal bundle with gauge connection one-form $A_1 = (\omega_N)_{\alpha\, st}T^{st}dy^\alpha$. This splitting of the spin connection extends to the curvature two-form $R^{(11)}|_{\mathcal{W}_6} = R_N + R_T$ by definition with $R_{N/T} = d\omega_{N/T} + \omega^2_{N/T}$.

[6] To be more precise, the zero-modes of the membrane do not lead to any perturbative anomalies. There may however be global anomalies associated to them. In M-theory, their cancellation has been established in [254].

Computing the anomaly polynomials

We now turn to the computation of the five-brane anomaly as given from the index formulae presented in Section 3.3.1. We start by considering the anomaly inflow from the chiral fermions in the five-brane multiplet. When acting on spin-$\frac{1}{2}$ particles, the spin-connection and curvature on the normal bundle are expanded on the generators Γ^{st} of the spin representation of $SO(5)$ as $\omega_N = \frac{1}{4}(\omega_N)_{st}\Gamma^{st}$ and $R_N = \frac{1}{4}(R_N)_{st}\Gamma^{st}$, so that using the identity $[\Gamma^{st}, \Gamma^{uv}] = 8\delta^{[s[u}\Gamma^{v]t]}$.we have the relation:

$$\frac{1}{4}(R_N)_{st}\Gamma^{st} = \frac{1}{4}(d\omega_N)_{st}\Gamma^{st} + \frac{1}{16}(\omega_N)_{st}\wedge(\omega_N)_{uv}\frac{1}{2}[\Gamma^{st},\Gamma^{uv}] = \frac{1}{4}\left((d\omega_N)_{st} + (\omega_N)_{su}\wedge(\omega_N)^u{}_t\right)\Gamma^{st}.$$

As already mentioned, the relevant Dirac operator for the chiral fermions of the five-brane tensor multiplet will then contain a part behaving as a "gauge" connection and given by:

$$A_\alpha = \frac{1}{4}(\omega_N)_{\alpha\, st}\Gamma^{st}, \quad \alpha = 0, 1, .., 5$$

which interacts with the fermions through the operator:

$$\not{D} = \hat{e}^\alpha{}_a\Gamma^a\left(\partial_\alpha + A_\alpha + \frac{1}{4}(\omega_T)_{\alpha\, bc}\Gamma^{bc}\right).$$

Since the brane world-volume is six-dimensional, the above operator will contribute a mixed anomaly characterized by an eight-form polynomial in both R^{t} and R^{n}, the latter playing the rôle of the gauge curvature F. The index for such an operator is given by replacing $F \to R_N = \frac{1}{4}(R_N)_{st}\Gamma^{st}$ so that the total Chern class is now evaluated over the spin bundle over NW_6, denoted by $S(NW_6)$ [253]:

$$\mathrm{Ind}(i\not{D}_{1/2}) = \int_{\mathcal{W}_6}\left[\widehat{A}(\mathcal{W}_6)\,\mathrm{ch}(S(NW_6))\right]_6.$$

The anomaly due to the chiral fermions on the five-brane is then given by the following eight-form anomaly polynomial, with a factor $-\frac{1}{2}$ accounting for the projection onto negative chirality spinors:

$$\begin{aligned}\mathcal{I}_8^{M5,ferm} &= -\pi\left[\widehat{A}(\mathcal{W}_6)\,\mathrm{ch}(S(NW_6))\right]_8 \\ &= -\frac{1}{(4\pi)^3}\frac{1}{8}\left(\frac{1}{45}\mathrm{tr}R_T^4 + \frac{1}{36}(\mathrm{tr}R_T^2)^2 - \frac{1}{6}\mathrm{tr}R_T^2\,\mathrm{tr}R_N^2 - \frac{1}{3}\mathrm{tr}R_N^4 + \frac{1}{4}(\mathrm{tr}R_N^2)^2\right).\end{aligned} \tag{3.82}$$

where we have used the following relations for gamma-matrices:

$$\begin{aligned}\Gamma^{st}\Gamma^{uv} &= \Gamma^{stuv} + 4\delta^{[s[u}\Gamma^{v]t]} + 2\delta^{[s[u}\delta^{v]t]}\,\mathbb{I}, \\ \Gamma^{rs}\Gamma^{tuvw} &= \Gamma^{rstuvw} + 8\delta^{[r[t}\Gamma^{uvw]s]} - 12\delta^{[t[r}\delta^{s]u}\Gamma^{vw]}.\end{aligned}$$

to determine the two relevant terms in the expansion of $\mathrm{ch}(S(NW_6)) = \mathrm{tr}\ \exp\left[\frac{i}{2\pi}\frac{1}{4}(R_N)_{st}\Gamma^{st}\right]$, namely:

$$\begin{aligned}[\mathrm{ch}(S(NW_6))]_2 &= -\frac{1}{(4\pi)^2 4}\mathrm{tr}R_N^2\cdot\mathrm{tr}\mathbb{I}, \\ [\mathrm{ch}(S(NW_6))]_4 &= -\frac{1}{(4\pi)^4}\left[\frac{1}{24}\mathrm{tr}R_N^4 - \frac{1}{32}(\mathrm{tr}R_N^2)^2\right]\cdot\mathrm{tr}\mathbb{I},\end{aligned} \tag{3.83}$$

with $\mathrm{tr}\mathbb{I} = 4$ since we are working with 2^2-dimensional $SO(5)$ gamma-matrices. The expansion of $\widehat{A}(\mathcal{W}_6)$ has been given in eqn.(3.16) so that combining it with eqn.(3.83), one arrives at expression (3.82).

There is an additional contribution to the anomaly coming from the three-form $\mathcal{F}_3$ of the $D=6$ $\mathcal{N}=2$ five-brane multiplet, which is purely gravitational. Resorting to the anomalous variation for the Euclidean action of the three-form field strength, we can use expression (3.28) to compute its corresponding anomaly polynomial. Since the Minkowskian three-form $\mathcal{F}_3$ is self-dual, its Euclidean version is anti-self dual. This accounts for the minus sign in its contribution to the anomaly:

$$\mathcal{I}_8^{M5,SD} = \frac{\pi}{4}\left[L(T\mathcal{W}_6)\right]_8 = -\frac{1}{(4\pi)^3}\frac{1}{36}\left(\frac{7}{5}\mathrm{tr}R_T^4 - \frac{1}{2}(\mathrm{tr}R_T^2)^2\right) . \tag{3.84}$$

Summing up contributions (3.82) and (3.84), we arrive at the total anomaly eight-form for the five-brane, which can be decomposed into a purely gravitational part, depending on R^{t}, also called tangent bundle anomaly, and a mixed anomaly, which involves both R^{n} and R^{t}:

$$\mathcal{I}_8^{M5} = \mathcal{I}_8^{M5,ferm} + \mathcal{I}_8^{M5,SD} = -X_8^{\mathrm{grav}}(R^{\mathrm{t}}) + Y_8(R^{\mathrm{n}},R^{\mathrm{t}}) \tag{3.85}$$

The polynomial $X_8^{\mathrm{grav}}(R^{\mathrm{t}})$ is the usual eight-form (3.47) produced by anomaly inflow of the Green-Schwarz term, while the normal bundle anomaly can be determined to be:

$$Y_8(R^{\mathrm{n}},R^{\mathrm{t}}) = \frac{1}{(4\pi)^3 8}\left(\frac{1}{3}\mathrm{tr}R_N^4 - \frac{1}{4}(\mathrm{tr}R_N^2)^2 + \frac{1}{6}\mathrm{tr}R_T^2\,\mathrm{tr}R_N^2\right)$$

If the decomposition (3.85) is useful in discriminating purely gravitational contributions from mixed ones (implying SO(5) "gauge" contributions from the structure group of the normal bundle), the appropriate decomposition with respect to anomaly cancellation will be shown to be expression (3.86), where one makes visible the eight-form polynomial (3.47) defined on the curvature $R^{(11)}|_{\mathcal{W}_6} = R^{\mathrm{t}} + R^{\mathrm{n}}$ of the total M5-brane tangent space $T\mathcal{M}_{11}|_{\mathcal{W}_6} = T\mathcal{W}_6 \oplus N\mathcal{W}_6$ [253]:

$$\begin{aligned}\mathcal{I}_8^{M5} &= -X_8^{\mathrm{grav}}(R^{(11)}|_{\mathcal{W}_6}) + \left[X_8^{\mathrm{grav}}(R^{(11)}|_{\mathcal{W}_6}) - X_8^{\mathrm{grav}}(R^{\mathrm{t}})\right] + Y_8(R^{\mathrm{n}},R^{\mathrm{t}}) \\ &= -X_8^{\mathrm{grav}}(R^{(11)}|_{\mathcal{W}_6}) - \frac{\pi}{12}p_2(N\mathcal{W}_6)\end{aligned} \tag{3.86}$$

with $p_2(N\mathcal{W}_6)$ the second Pontrjagin class (3.19) defined on the normal space curvature R^{n}. We will refer to the later as the *normal bundle anomaly.*

3.6.3 Anomaly cancellation mechanisme for the five-brane in $11D$

As just mentioned, the choice of decomposing the anomaly eight-form (3.86) as in expression (3.86) is not at all arbitrary. Indeed, the part $X_8^{\mathrm{grav}}(R^{(11)}|_{\mathcal{W}_6})$ is cancelled by inflow from the Green-Schwarz term, while cancellation of the normal bundle anomaly is achieved by the anomalous various of the M5-brane contributions to the Chern-Simons term (3.45). In this discussion, we consider only a single M5-brane, but the results which follow can readily be extended to a setup with multiple five-branes. Since we are considering the embedding of $\mathcal{W}_6$ in a general space $\mathcal{M}_{11}$, we work with usual normalisation for the toplogical terms:

$$\begin{aligned}S_{CS} &= -\frac{1}{12\kappa^2}\int_{\mathcal{M}_{11}} C_3\wedge G_4\wedge G_4\,, \\ S_{GS} &= -\frac{T_2}{2\pi}\int_{\mathcal{M}_{11}} G_4\wedge X_7 = -\frac{2}{\gamma}\int_{\mathcal{M}_{11}} G_4\wedge X_7\,,\end{aligned}$$

where, on the second line, we have used relations (3.53) and (3.81) to rewrite $T_2 = 4\pi/\gamma$. In this case, since we have not yet considered the insertion of gauge multiplets in the theory, γ is simply defined as $\gamma = 2\sqrt[3]{4\pi\kappa^2}$, with no reference to the gauge coupling g_{YM}.

To see how this comes about, first note that a four-form carrying a non-vanishing flux (3.75) corresponds to a solution of the modified Bianchi identity. As a first approximation, the term sourcing the four-form flux can be modelled by a singular delta-like five-form in the modified Bianchi identity (since we are aiming at introducing five-branes in the Hořava-Witten scenario, we adopt here the upstairs approach):

$$dG_4 = \pm\frac{\gamma}{2}\delta^{(5)}(\mathcal{W}_6)\,. \tag{3.87}$$

The $\pm$-sign corresponds to M5-/anti-M5-brane solutions. The five-brane current is given by the five-form

$$\delta^{(5)}(W_6) = \prod_{k=1}^{5}\delta(x^{i_k} - x_0^{i_k})\,dx^{i_1}\wedge..\wedge dx^{i_5} \tag{3.88}$$

with delta function support on the five-brane world-volume, located, in transverse space, at $\{x_0^{i_k}\}_{k=1,..,5}$. Furthermore, we have used eqns.(3.81) and (3.53) to rephrase the coefficient in expression (3.75) in terms of γ.

The above expression is only valid locally, but is sufficient to treat anomaly inflow from the Green-Schwarz term, where C_3 enters only linearly. In this case, considering a single (anti-)M5-brane in eqn.(3.87), we have the anomalous variation:

$$\delta S_{GS} = -\frac{2}{\gamma}\int G_4\wedge dX_6^{\mathrm{grav},1}(R^{(11)}) = (\pm)\int_{\mathcal{W}_6} X_6^{\mathrm{grav},1}(R^{(11)})\,. \tag{3.89}$$

Thanks to the descent equation (3.46), this variation cancels the $(\mp)X_8^{\mathrm{grav}}(R^{(11)}|_{\mathcal{W}_6})$ part of the (anti-)M5-brane anomaly (3.86)[7].

Cancellation of the remaining part $-\frac{\pi}{12}p_2(N\mathcal{W}_6)$ will by achieved by anomaly inflow from the Chern-Simons coupling in expression (3.45). However, such a term is now cubic in C_3, and a well-defined and nonsingular result in such a case requires smoothing out the delta-function contribution on the RHS of eqn.(3.87) together with covariantizing it under the structure group of the normal bundle, $SO(5)$. The prescription is in this case [119] to switch to spherical coordinates $ds^2(\mathbb{E}^5) = dr^2 + r^2d\Omega_4^2$ for the transverse space, as in Section 3.6.1. Then define a smooth function $\rho(r)$ of the radial coordinate, with transverse compact support centered at the position of the M5-brane world-volume in transverse space, such that it interpolates between the two values $\rho(0) = -1$ and $\lim_{r\to\infty}\rho(r) = 0$, while keeping $\int_{\mathbb{R}_+} d\rho = 1$. After introducing, in addition, a global angular four-form e_4 which is closed, $SO(5)$-invariant, and integrates to $\int_{S^4}\frac{e_4}{2} = 1$, thus satisfying the descent equations:

$$e_4 = de_3\,, \qquad \delta e_3 = de_2^1\,,$$

one arrives at a smoothed and gauge invariant version of the modified Bianchi identity (3.87):

$$dG_4 = \pm\frac{\gamma}{2}d\rho(r)\wedge\frac{e_4(\mathcal{W}_6)}{2}\,. \tag{3.90}$$

This expression reduces to (3.87) for a flat infinite M5-brane. To sum up, we have smoothed out the magnetic charge (3.75) to a four-sphere connecting to the horizon.

In order to evaluate the variation (3.95), one needs to solve the Bianchi identity (3.90), which leads to the general solution:

$$G_4 = d\Big(C_3 + (\pm)a_{\mathrm{M5}}\frac{\gamma}{2}\,\rho(r)\frac{e_3(\mathcal{W}_6)}{2}\Big) - (\pm)\frac{\gamma}{2}d\rho(r)\wedge\frac{e_3(\mathcal{W}_6)}{2} \tag{3.91}$$

[7]The anomaly for an anti-M5-brane.is the same as for the M5-brane but with overall sign flipped.

where again C_3 can be viewed locally as a small fluctuation field about the M5-brane and $a_{\rm M5}$ is an integration parameter. Since $\frac{e_4(\mathcal{W}_6)}{2}$ is singular at the location of the M5-brane, a smooth and nonsingular solution for both C_3 and G_4 fixes $a_{M5}=0$. In the next section, we will see how this parameter is set to a different value if the requirement for smoothness and nonsingularity of the solution is relaxed, but Witten's cohomology condition [252] for M5-brane aligned along an interval is taken into account.

In any case, for G_4 to be gauge invariant, one has to enforce the following anomalous gauge-variation:

$$\delta C_3 = -(\pm)\frac{\gamma}{2} d\rho(r)\wedge(1-a_{M5})\frac{e_2^1(\mathcal{W}_6)}{2}\,. \tag{3.92}$$

In the following, we will require regularity of the solutions at $r=0$, which imposes $a_{\rm M5}=0$.

If one, in addition, takes into account the constraints on $\mathcal{L}_{CS}$ dictated by gauge invariance under diffeomorphism and three-form gauge transformation, one should modify the Chern-Simons term according as $\mathcal{L}_{CS}\sim\widetilde{C}\wedge\widetilde{G}\wedge\widetilde{G}$ [119] by shifting the four-form (3.91) back to an exact form:

$$\widetilde{G}_4 \doteq G_4-(\pm)\frac{\gamma}{2}\rho(r)\frac{e_4(\mathcal{W}_6)}{2}=d\widetilde{C}_3\,,\qquad \widetilde{C}_3\doteq C_3-(\pm)\frac{\gamma}{2}\rho(r)\frac{e_3(\mathcal{W}_6)}{2}$$

with the modified gauge variation resulting from eqn.(3.92):

$$\delta\widetilde{C}_3 = -(\pm)\frac{\gamma}{2}d\Big(\rho(r)\frac{e_2^1(\mathcal{W}_6)}{2}\Big)\,, \tag{3.93}$$

so that $\widetilde{G}_4$ is still gauge invariant.

Now the singularity at $r=0$ can be dealt with by cutting a disc of radius η around the M5-brane position, which corresponds to removing a tube of corresponding radius starting from the origine $r=0$. We then construct the regularized Chern-Simons action as the bulk integral:

$$\widetilde{S}_{CS} = -\frac{1}{12\kappa^2}\lim_{\eta\to0}\int_{\mathcal{M}_{11}\backslash D_\varepsilon(\mathcal{W}_6)}\widetilde{C}_3\wedge\widetilde{G}_4\wedge\widetilde{G}_4\,,$$

where $D_\varepsilon(\mathcal{W}_6)$ denotes the disc bundle, with base $\mathcal{W}_6$ and fiber the discs of radius ε. In particular, the space $\mathcal{M}_{11}\backslash D_\varepsilon(\mathcal{W}_6)$ has boundary $\partial(\mathcal{M}_{11}\backslash D_\varepsilon(\mathcal{W}_6))\doteq S_\varepsilon(\mathcal{W}_6)$, which is an S^4-bundle with base $\mathcal{W}_6$ and radius ε.

We use once again eqn.(3.53).Then, the variation (3.93) followed by an integration by parts which is reexpressed as an integration over the boundary space $S_\varepsilon(\mathcal{W}_6)$ yields

$$\begin{aligned}\delta\widetilde{S}_{CS} &= \frac{4\pi}{3\gamma^3}\lim_{\varepsilon\to0}\int_{\mathcal{M}_{11}\backslash D_\varepsilon(\mathcal{W}_6)}(\pm)\gamma d\Big(\rho(r)\frac{e_2^1}{2}\Big)\wedge\widetilde{G}_4\wedge\widetilde{G}_4\\ &= (\mp)\frac{4\pi}{3\gamma^2}\lim_{\varepsilon\to0}\int_{S_\varepsilon(\mathcal{W}_6)}\rho(r)\frac{e_2^1}{2}\wedge d\widetilde{C}_3\wedge d\widetilde{C}_3 = (\pm)\frac{\pi}{3}\lim_{\varepsilon\to0}\int_{S_\varepsilon(\mathcal{W}_6)}\frac{e_2^1}{2}\wedge\frac{e_4}{2}\wedge\frac{e_4}{2}\end{aligned} \tag{3.94}$$

where we have used the fact that C_3 is a smooth function of r, and $d\rho(r)\to0$ as $\varepsilon\to0$, so that all terms including dC_3 or $d\rho$ drop out of the integral. The sign on the second line of eqn.(3.94) is due to the choice of profile function with $\rho(0)=-1$.

Then, using a result of [53], the integral over the angular forms yields

$$\delta\widetilde{S}_{GS} = (\pm)\frac{\pi}{12}\int_{W_6}p_2(N\mathcal{W}_6)^1 \tag{3.95}$$

which cancels, in expression (3.86), the normal bundle anomaly for an (anti-)M5-brane due to the eight-form $\mathcal{I}_8=(\mp)\frac{\pi}{12}p_2(N\mathcal{W}_6)$.

More concretely, the anomalous gauge variation of the second Pontrjagin class is given by

$$p_2(N\mathcal{W}_6)^1 = \frac{2}{(4\pi)^4}\left(\mathrm{tr}(\Lambda_{\mathrm{L}} d\omega^{\mathrm{n}})d\omega^{\mathrm{n}}_{\mathrm{L}} - 2(\Omega^{\mathrm{n}})^1_6\right)$$

where ω^{n} and $\omega^{\mathrm{n}}_{\mathrm{L}}$ are the spin connection and the Lorentz three-form of the normal bundle space, and $d\Omega^{\mathrm{n}}_7 = \mathrm{tr}(R^{\mathrm{n}})^4$ with variation $\delta\Omega^{\mathrm{n}}_7 = d[(\Omega^{\mathrm{n}})^1_6]$.

3.7 Five-branes in $11D$ in a space with boundaries

In this final section, we study the effect of including multiple M5/$\overline{\mathrm{M5}}$-branes in the Hořava-Witten scenario. We work this time in the upstairs formalism and use the Lagrangian (3.45). We now assume that cancellation of the normal bundle anomaly can be taken care of by the procedure outlined in Section 3.6.3, so we content ourselves with working with the (more naive) singular solution (3.88). In principle, one could consider a better-behaved smoothed out solution of the type (3.91), but this would only obscure the discussion by complicating the integration procedure without changing much to the discussion.

3.7.1 Global definition of G_4 and five-brane contributions

Since in Chapter 4, we are ultimately interested in M5-brane configuration preserving $\mathcal{N} = 1$ supersymmetry in $4D$, we will concentrate on M5-branes positioned along the orbifold circle S^1 and wrapping an even cycle in space. For obvious reasons, one now singles out the S^1 coordinate in expression (3.88), and introduces a sum on the brane currents, where n_5 is the total number of M5/$\overline{\mathrm{M5}}$-branes. To preserve the $\mathbb{Z}_2$ symmetry of the system, five-branes will be introduced by pairs, one at position y_I and the other (the "mirror" brane) at position $-y_I$.

This configuration generates a current:

$$\delta^{(5)}(\mathcal{W}_6) = \sum_{I=1}^{n_5} q_I \left[\delta(y - y_I) + \delta(y + y_I)\right]\, dy \wedge \delta^{(4)}(\mathcal{W}_{6,I}) \tag{3.96}$$

where $\delta^{(4)}(\mathcal{W}_{6,I})$ is similar to expression (3.88) with lesser rank, and $q_I = \pm 1$ for an (anti-)M5-brane. This expression is now substituted in eqn.(3.87) for (+) sign, and the doubled contributions resulting from considering pairs of five-branes will cancel the factor 2 from the upstairs formalism in expression (3.45).

By defining a Heavyside-type threeform such that:

$$\delta^{(4)}(\mathcal{W}_{6,I}) = d\theta^{(3)}(\mathcal{W}_{6,I})$$

one solves the modified Bianchi identity (3.87) with (we leave aside anti-M5-branes):

$$G_4 = d\Big(C_3 + \frac{a_{\mathrm{M5}}}{4}\gamma \sum_{I=1}^{n_5} q_I\, \widehat{\epsilon}_I(y)\, \theta^{(3)}(\mathcal{W}_{6,I})\Big) - \frac{\gamma}{2}\sum_{I=1}^{n_{\mathrm{M5}}} \left[\delta(y - y_I) + \delta(y + y_I)\right]\, dy \wedge \theta^{(3)}(\mathcal{W}_{6,I})\,. \tag{3.97}$$

where a_{M5} is an integration constant and

$$\widehat{\epsilon}_I(y) = \epsilon_1(y - y_I) + \epsilon_1(y + y_I)\,, \qquad \frac{1}{2}d\widehat{\epsilon}_I(y) = \delta(y - y_I) + (y + y_I) - \frac{1}{\pi R_{\mathrm{het}}}dy\,.$$

Since we do not require this solution to be non-singular, the parameter a_{M5} is in principle free, contrary to what we have seen in Section 3.6.3. The condition on the integration constant b_{HW} (3.58) is now also likely to be modified. It can however be fixed [43] by demanding invariance of G_4 under large gauge and Lorentz transformations. Let us review the argument here. For simplicity, we consider only M5-branes (the conclusions of the discussion remain unchanged even if we also introduce $\overline{\text{M5}}$-branes), then $q_I = +1 \; \forall I$ in expression (3.97).

Invariance of G_4 under large gauge and Lorentz transformations can be rephrased, according to the cohomological criterion in [252], as the requirement that dG_4 be exact, in other words $\int_{\mathcal{C}_5} dG_4 = 0$, for any five-cycle $\mathcal{C}_5$. Then, choosing $\mathcal{C}_5$ along the orbifold circle: $\mathcal{C}_5 = \mathcal{C}_4 \times S^1$, where $\mathcal{C}_4$ is an arbitrary four-cycle at fixed y, and combining eqns.(3.50) and (3.87), dG_4 is exact if:

$$\sum_{i=1,2} \int_{\mathcal{C}_4} I_{4,i} = -\sum_I [\mathcal{C}_{4,I}] = -n_5 \tag{3.98}$$

where $[\mathcal{C}_{4,I}]$ is the Poincaré dual cohomology class to the four-cycles $\mathcal{C}_{4,I}$ wrapped by each five-brane I (the contributions of the M5-brane and its mirror brane add up to one). It counts the total number n_5 of five-branes intersecting $\mathcal{C}_4 \times S^1$ at a point (note that, in this case, $[\mathcal{C}_{4,I}]$ always yields a factor $+1$ since the five-branes and Yang-Mills instantons contributions to the irreducible part of the gravitational anomaly are the same [108]).

Restricting for a moment to a subinterval of S^1 containing the fixed point $y = 0$, namely $\Sigma_5 = \mathcal{C}_4 \times I$, with $I = [y'_1, y'_2] \subset S^1/\mathbb{Z}_2$, we are now dealing with a surface with boundary and not a closed five-cycle and we can apply Stokes' theorem:

$$\int_{\Sigma_5} dG_4 = \left(\int_{\mathcal{C}_4(y'_2)} - \int_{\mathcal{C}_4(y'_1)} \right) G_4 \,. \tag{3.99}$$

The RHS of this expression is evaluated by using again the modified Bianchi identity eqns.(3.50) together with (3.87). For the LHS, we add up (3.57) and (3.97) (without the three-form C_3) and keep only the $G_{\mu\nu\rho\sigma}$ components. Using eqn.(3.98), one then reexpresses $I_{4,2}$ in terms of $I_{4,1}$ and n_5, the total number of M5-branes, and denotes $m_5(I)$ the number of M5-branes intersecting the interval I. Eqn.(3.99) yields:

$$\int_{\mathcal{C}_4} I_{4,1} + m_5(I) = b_{HW} \left(\int_{\mathcal{C}_4} I_{4,1} - \frac{1}{2\pi R_{\text{het}}} (y'_1 - y'_2) n_5 \right) + a_{\text{M5}} \left(m_5(I) + \frac{1}{2\pi R_{\text{het}}} (y'_1 - y'_2) n_5 \right) ,$$

where we have taken advantage of the fact that the $I_{4,i}$ are independent of y and that $\left(\int_{\mathcal{C}_4(y'_2)} - \int_{\mathcal{C}_4(y'_1)} \right) \delta^{(5)}(W_{6,I}) \epsilon_1(y - y_I)$ is equal to $\frac{1}{\pi R_{\text{het}}}(y'_1 - y'_2)$ if an M5-brane does not intersect the interval and to $2 + \frac{1}{\pi R_{\text{het}}}(y'_1 - y'_2)$ if it does [43]. Then since the y'_i are arbitrary, we can deduces the following set of equations for b_{HW} and a_{M5}:

$$(b_{HW} - a_{\text{M5}}) n_5 = 0 \,, \qquad (1 - a_{\text{M5}}) m_5(I) + (1 - b_{HW}) \int_{\mathcal{C}_4} I_{4,1} = 0 \tag{3.100}$$

In the trivial case where there are no M5-branes, $n_5 = m_5(I) = 0$, if $I_{4,1}$ is cohomologically non-trivial, we must set $b_{\text{HW}} = 1$ and G_4 (3.57) becomes non-singular. If $n_5 \neq 0$, since the function $m_5(I)$ can vary, eqns.(3.100) are satisfied only when $b_{\text{HW}} = a_{\text{M5}} = 1$, which implies that the hyperplanes contribution (3.57) to the four-form is non-singular. To sum up, b_{HW} is unconstrained only if $I_{4,i}$ are cohomologically trivial and there are no M5-branes present.

In all other cases, a global definition of G_4 imposes $b_{\text{HW}} = a_{\text{M5}} = 1$, and coincides with expression (3.57) being nonsingular. Fixing $a_{\text{M5}} = 1$, however, only removes the singularity along the interval in the five-brane contribution (3.97), but not the one at the five-brane world-volume in $\delta^{(4)}(\mathcal{W}_{6,I})$. In this respect, the analysis carried out in this section is different from Section 3.6.3, where the singular behaviour of the solution G_4 along the radial direction is smoothed out from the start by an appropriate choice of function with compact support, so that the singularity at the five-brane world-volume can be made to disappear by setting $a_{\text{M5}} = 0$.

3.7.2 Additional terms from five-branes in the Hořava-Witten scenario

Summarizing the results (3.57), (3.97) and (3.60), we can now write the solution for the four-form field strength anf the gauge variation for the modified three-form in the presence of $10D$ boundary hyperplanes, together with n_5 M5-branes not wrapping the orbifold circle. In this case, the previous analysis has shown that global definition of G_4 requires fixing the integration parameters to $b_{\rm HW} = a_{\rm M5} = 1$. Then:

$$\begin{aligned} G_4 &= dC_3 + \frac{\gamma}{2}\sum_{i=1}^{2}\left[\frac{1}{\pi R_{\rm het}} dy\wedge\omega_{3,i} - \epsilon_i I_{4,i}\right] \\ &\quad -\frac{\gamma}{2}\sum_{I=1}^{n_{\rm M5}}\left[\frac{1}{\pi R_{\rm het}} dy\wedge\theta^{(3)}(\mathcal{W}_{6,I}) - \frac{1}{2}\widehat{\epsilon}_I(y)\,\delta^{(4)}(\mathcal{W}_{6,I})\right], \qquad (3.101) \\ \delta\widetilde{C}_3 &= \frac{\gamma}{2\pi R_{\rm het}}\sum_{i=1}^{2}\left[dy\wedge\omega^1_{2,i} - \epsilon_i\, d\omega^1_{2,i}\right]. \qquad (3.102) \end{aligned}$$

In the alternative case where the world-volume of the M5-branes are allowed to extend along the circle S^1, we have a family of solutions parametrised by $b_{\rm HW}$. In addition, the second term on the RHS of eqn.(3.101) is now replaced by a formal solution of the equation $\delta^{(5)}(\mathcal{W}_{6,I}) = d\theta^{(4)}(\mathcal{W}_{6,I})$, where $\delta^{(5)}(\mathcal{W}_{6,I})$ are sources for the G_4-flux induced by the presence the M5-branes. Such a contribution generates a term in G_4 which is trivially gauge and Lorentz invariant We then have the solutions:

$$\begin{aligned} G_4 &= dC_3 + \gamma\sum_{i=1}^{2}\left[\left((1-b_{\rm HW})\,\delta_i + \frac{b_{\rm HW}}{2\pi R_{\rm het}}\,dy\right)\wedge\omega_{3,i} - \frac{b_{\rm HW}}{2}\,\epsilon_i(y) I_{4,i}\right] \\ &\quad +\gamma\sum_{I=1}^{n_5}\theta^{(4)}(\mathcal{W}_{6,I}). \qquad (3.103) \\ \delta\widetilde{C}_3 &= \gamma\sum_{i=1}^{2}\left[\left((1-b_{\rm HW})\,\delta_i + \frac{b_{\rm HW}}{2\pi R_{\rm het}}\,dy\right)\wedge\omega^1_{2,i} - \frac{b_{\rm HW}}{2}\,\epsilon_i(y)\,d\omega^1_{2,i}\right], \qquad (3.104) \end{aligned}$$

As shown in Section 3.6.3, both solutions (3.101) and (3.103) cancel the gravitational anomaly $-X_8^{\rm grav}(R^{(11)}|_{\mathcal{W}_6})$ (3.86) by inflow from the GS term (3.89), while their smoothed versions compensate the normal bundle anomaly by contributing to the anomalous variation of the CS term (3.67). This was in the case where the $11D$ manifold $\mathcal{M}_{11}$ does not possess singularities which call for the presence of twisted sectors at the (orbifold) fixed points. When $10D$ hyperplanes carrying gauge multiplets are present, as in the Horava-Witten scenario, we now expect *mixed* contributions from the CS term, where by *mixed* we mean terms that contain both $E_8\times E_8$ gauge fields contributions, gravitational ones and ones coming from the M5-brane world-volumes.

For instance, considering the case of M5-branes wrapping the circle S^1, anomalous variation of the modified CS term for the solution (3.103)-(3.104) generates a mixed contribution of the form:

$$\begin{aligned} \delta\int\mathcal{L}_{\rm CS} &= -\frac{1}{12\kappa^2}\sum_{I=1}^{n_5}\int\delta\widetilde{C}_3\wedge G_4|_{\text{no M5}}\wedge\gamma\theta^{(4)}(\mathcal{W}_{6,I}) \\ &= \frac{4\pi b_{\rm HW}}{3}\sum_{I=1}^{n_5}\sum_{i,j=1}^{2}\int_{S^1}\left[(1-b_{\rm HW})\,\delta_i + \frac{b_{\rm HW}}{2\pi R_{\rm het}}dy\right]\epsilon_i(y) \\ &\qquad \cdot\int_{\mathcal{M}_{10}}\left(\omega^1_{2,i}\wedge I_{4,j} - d\omega^1_{2,i}\wedge\omega_{3,j}\right)\wedge\theta^{(4)}(\mathcal{W}_{6,I}). \end{aligned}$$

where we have used the fact that both $\delta^{(5)}(\mathcal{W}_{6,I})$ and $\theta^{(4)}(\mathcal{W}_{6,I})$ dependent on the space transverse to the M5-branes' world-volumes and are thus independent of the circle coordinate. The integral over the S^1 then vanishes thanks to eqns.(3.55) and (3.56), and a term such as the one above does not affect the anomaly cancelling procedures for the gauge or M5-brane multiplets outlined in the preceding sections.

3.8 Outlook

In this chapter, we have presented the mechanism of anomaly cancellation for M-theory on a space with boundaries, namely $\mathcal{M}_{11} = \mathcal{M}_{10} \times (S^1/\mathbb{Z}_2)$, which gives the strong coupling limit of the heterotic $E_8 \times E_8$ theory. We have seen in particular that if we insist on working with functions with a definite periodicity under the $\mathbb{Z}_2$ action, one finds a one parameter family of solutions to the modified Bianchi identity for the four-form field-strength.

Then, all gauge and gravitational anomalies in the $11D$ theory cancel for any value of this parameter, provided we modify the original Chern-Simons coupling of CJS $11D$ supergravity by shifting the three-form potential by gauge and gravitational contributions living on the boundary hyperplanes. Anomaly cancellation thus does not set this extra integration parameter to a definite value, and the latter is fixed by requiring the truncation to the ten-dimensional theory to be consistent.

Furthermore, this scenario for heterotic M-theory allows for the insertion of M5-branes, which modify the Bianchi identity for the four-form field strength by extra source terms. This modification introduces an additional free parameter in the solution to the Bianchi identity, which can this time be fixed by demanding invariance of the four-form under large Lorentz and gauge transformations. When the gauge and gravitational four-form polynomials on the boundary hyperplanes are cohmological non-trivial, this fixes both parameters to be equal to one, which is the value required for the small radius limit to the heterotic string theory to be consistent.

We have seen in particular that the anomalies for these M5-branes cancel in this setup, and that the introduction of M5-branes extending in the orbifold direction does not spoil the anomaly cancellation argument of the Hořava-Witten scenario.

In contrast, the presence of M5-branes that do not extend in the direction of the S^1 modifies the solution of the four-form (3.101), and will now induce contributions of the form $\sim \int \delta\widetilde{C}_3 \wedge G_4|_{\text{no M5}} \wedge \widehat{\epsilon}_I(y)\, \delta^{(4)}(\mathcal{W}_{6,I})$ in the anomalous variation of the CS term. The study of these new non-vanishing effects will be at the centre of Chapter 4.

In particular, these contributions are associated to an invariant eight-form polynomial:

$$\mathcal{I}_8 \sim \sum_{I=1}^{n_5} \sum_{i,j=1}^{2} c_{ij} f_i(y_I)\, I_{4,j} \wedge I_{4,i} \tag{3.105}$$

where the c_{ij} are constants which are easily determined, and the $f_i(y_I)$ are functions of the position of the M5-branes along the orbifold circle:

$$f_1(y_I) = \left(\frac{y_I}{2\pi R_{\text{het}}}\right)^2 - \frac{y_I}{2\pi R_{\text{het}}} + \frac{1}{6}\,, \qquad f_2(y_I) = \left(\frac{y_I}{2\pi R_{\text{het}}}\right)^2 - \frac{1}{12}\,. \tag{3.106}$$

In Chapter 4, the rôle of these functions will be studied in a compactification to four dimensions of the Hořava-Witten scenario with space-time filling M5-branes, transverse to the orbifold circle. We will consider in particular a compactification of the type $\mathcal{M}_{11} = \mathcal{M}_4 \times K_6 \times S^1/\mathbb{Z}_2$, where K_6 is a Calabi-Yau threefold. In this case, when integrated over the Calabi-Yau space, contributions from the CS action related to the eight-form (3.105) such as $\sim \int \widetilde{C}_3 \wedge G_4|_{\text{no M5}} \wedge \widehat{\epsilon}_I(y)\, \delta^{(4)}(\mathcal{W}_{6,I})$

generate a coupling of the axionic partner $\mathrm{Im}T$ of the Calabi-Yau volume modulus to the expression $F \wedge F$ involving the gauge fields. The dependence of this gauge coupling on the position of the M5-branes along the orbifold circle is then precisely dictated by the functions (3.106). Now, in the effective supergravity description of the theory presented in the coming chapter, these interactions can be understood as threshold corrections due to instanton contributions to the non-perturbative superpotential, from Euclidean membranes stretching between two M5-branes. In this perspective, the precise dependence of these threshold corrections on the modulus of the M5-branes parametrizing their position on the interval can be determined without resorting to an involved instanton calculations as in [193, 182, 183].

Chapter 4

Five-Brane Thresholds and Membrane Instantons in Four-Dimensional Heterotic M-Theory

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4.1 Introduction

Heterotic $E_8 \times E_8$ strings compactified to four dimensions on a six-dimensional space K_6 are also described by M-theory compactified on $K_7 \equiv S^1/Z_2 \times K_6$ [252, 148]. In particular, it is straightforward to verify [200, 201, 9, 188, 187, 146, 110, 178, 111] that the effective $\mathcal{N} = 1$ supergravity found in Calabi-Yau or orbifold compactifications of perturbative heterotic strings [250, 97] is reproduced by brane-free M-theory configurations with compact space K_7. A novelty of the M-theory approach lies in the possibility to concretely analyse physical effects of non-perturbative brane configurations. In the low-energy effective supergravity approximation, configurations with five-branes and/or membranes (two-branes) [107, 23, 39, 252] of compactified M-theory can be studied from simple modifications of the field equations predicted by eleven-dimensional supergravity [81].

An obvious distinction in the nature of five-brane and membrane effects follows from the alignement conditions applying to their respective world-volumes if one requires that the configuration admits (exact or spontaneously broken) $\mathcal{N}_4 = 1$ supersymmetry (four supercharges). Each five-brane world-volume is the product of four-dimensional space-time and a holomorphic two-cycle in the Calabi-Yau threefold and conditions apply on the respective cycles of pairs of world-volumes [23, 39, 252]. Five-brane massless excitations [127, 162], which belong to six-dimensional chiral supersymmetry multiplets expanded in modes of the two-cycle, lead then to new four-dimensional fields to be included in the effective supergravity description. Some of these modes do not depend on the detail of the Calabi-Yau geometry: the five-brane modulus describing fluctuations along the S^1/Z_2 direction, the two-index antisymmetric tensor $\hat{B}_{\mu\nu}$ with self-dual field strength and their fermionic $\mathcal{N}_4 = 1$ partner. These states can be assembled either in a chiral supermultiplet which we will call $\hat{S}$ or, in a dual version, in a linear multiplet. The effective supergravity for this "universal five-brane modulus" supermultiplet has been studied in ref. [100] (see also ref. [193])[1]. Firstly, the Kähler potential of the theory with this new superfield has been obtained and the absence of direct contributions to the (perturbative) superpotential has been demonstrated. Secondly, on the basis of the four-dimensional superspace structure only, the possible appearance of new threshold corrections has been emphasized.

In contrast, open membrane euclidean world-volumes include the S^1/Z_2 direction and a cycle in K_6 [23, 39, 227, 237, 54, 177]. They stretch between the S^1/Z_2 fixed planes, or between a fixed plane and a five-brane, or between pairs of five-branes. Their effects in the four-dimensional effective supergravity are then localized in space-time, they can be viewed as instanton-like corrections to the interaction Lagrangian. While open membrane stretching between the fixed hyperplanes correspond in the string approach to world-sheet instantons, membranes ending on a five-brane describe forces acting on this brane. Their contributions to the effective supergravity are then expected to lead to new (non-derivative since the world-volume includes S^1/Z_2) interactions involving the five-brane modulus.

The corrections to the effective four-dimensional supergravity induced by the various types of membranes have been studied in refs. [193, 183, 182, 82]. They were found to contribute to the chiral F–density part of the Lagrangian density, in the form of a non-perturbative superpotential. Specifically, an interaction bilinear in the five-brane fermion in superfield $\hat{S}$ has been computed in the four-dimensional background with the five-brane and open membranes ending on it. The resulting non-perturbative superpotential shows an exponential dependence on the five-brane universal modulus typical of instanton calculus. To isolate the membrane contributions from other possible non-perturbative sources, a specific regime is chosen.[2] As a consequence, even if the instanton calculation clearly establishes the existence of an exponential dependence on $\hat{S}$, it does not allow to infer how this exponential term would combine with other non-perturbative contributions which, like gauge instantons, are expected as well.[3]

The relevance to physics of the M-theory system with five-branes and membranes strongly relies upon the structure of superpotentials generated by fluxes, gaugino condensates and open membrane instantons. Phenomenological questions addressed in the literature include supersymmetry breaking and gaugino condensation [189, 82], five-brane stabilization (stabilization of the modulus $\hat{S}$) [193, 82], stabilization of all moduli [60, 57], the existence of stable de Sitter vacua [60, 57, 58, 22], inflationary phases and potentials [58, 22] and cosmic strings [59]. These analyses use in general simplifying assumptions, in the Kähler metric which shows a severe mixing of all moduli when five-brane fields are present, or in the superpotential which is assumed to be a simple sum of non-perturbative contributions.

In the present paper, we use the anomaly-cancelling terms of the eleven-dimensional theory on the orbifold S^1/Z_2, as modified when five-branes are present, to derive new interactions involving the five-brane universal modulus supermultiplet which describes fluctuations along the S^1/Z_2 orbifold direction. These new interactions are then shown to induce, in the condensed phase, the effective non-perturbative instanton superpotential expected from membranes stretching between a fixed hyperplane and a five-brane. This superpotential correctly reduces to the results of refs. [193, 183, 182, 82] in the regime considered in these articles, but its derivation does not require choosing a particular limiting regime. This approach provides then direct information on the non-perturbative superpotential with combined gauge and membrane instantons effects.

The fact that these four-dimensional interactions can be obtained by considering seemingly unrelated arguments (membrane instanton calculus or gauge anomaly-cancelling terms) is a con-

[1] And, as a function of a non-trivial background value of the five-brane modulus, ref. [190, 189].

[2] For instance, Moore, Peradze and Saulina [193] select a regime where "open membrane instanton effects are the leading source of non-perturbative effects".

[3] Writing the complete non-perturbative superpotential as a sum of contributions, as for instance in ref. [193], is an assumption which needs to be justified.

sequence of the superfield structure of the four-dimensional theory. We use an effective superfield formulation [98, 100] which fully respects the symmetry and supersymmetry ingredients defining the microscopic system: the modifications of the Bianchi identities and of the topological term induced by fixed planes and five-branes. It also respects the symmetries of the five-brane multiplet, with its self-dual three-form field. Thus, a given superspace contribution in the effective Lagrangian describes various aspects of the microscopic theory, related by the superfield structure of the effective theory. This method has been applied to the derivation of the Kähler potential [100], including non-linear couplings of the five-brane superfield $\hat{S}$, and we will see in paragraph 4.4.1 that these kinetic terms can be derived from (at least) two quite distinct sectors of the microscopic theory. They can certainly be derived from the Calabi-Yau reduction of the five-brane Born-Infeld Lagrangian [100]. But they can also be derived from a universal correction to gauge kinetic terms, quadratic in $\hat{S}$, induced by S^1/Z_2 anomaly-cancellation.

Schematically, our argument goes as follows. Since we confine ourselves to the effective four-dimensional supergravity with up to two derivatives, for which a (superconformal) superspace formulation exists, counterterms cancelling Lorentz anomalies will be irrelevant to our discussion.[4] Gauge anomaly-cancelling terms are then entirely due to the "topological term" of eleven-dimensional supergravity [81]

$$-\frac{1}{24\kappa^2}\int C_3 \wedge G_4 \wedge G_4.$$

Sources for the Bianchi identity verified by G_4 are provided by the two fixed hyperplanes of the S^1/Z_2 orbifold and by the aligned five-branes, so that

$$G_4 = dC_3 + \Delta G_{4,planes} + \Delta G_{4,branes}.$$

The contribution $\Delta G_{4,planes}$ depends on the gauge curvatures living on the planes, and both corrections explicitly depend on the S^1/Z_2 coordinate and respect the Z_2 symmetry used to define the orbifold projection. The topological term leads then to a gauge interaction of the form

$$-\frac{1}{12\kappa^2}\int C_3 \wedge \Delta G_{4,planes} \wedge \Delta G_{4,branes}.$$

This term gives rise in particular to a gauge interaction of the massless orbifold modes of C_3 located on the five-brane world-volumes and depending explicitly on their position along S^1. And, after integration over the Calabi-Yau space, it produces a coupling to $F \wedge F$ of the axionic partner $\mathrm{Im}\,T$ of the Calabi-Yau volume modulus[5] $\mathrm{Re}\,T$ which depends on the five-brane locations along S^1. The superfield structure developed for the effective supergravity of the universal five-brane modulus [100] can then be used to understand this interaction as a threshold correction with a calculable dependence on the five-brane modulus. In particular, the universal part of these contributions can be derived from the Dirac-Born-Infeld kinetic Lagrangian. The non-perturbative effective superpotential follows then from standard gaugino condensation. Its dependence on the five-brane modulus is precisely the one expected from membrane instanton calculations, as performed for instance in ref. [193].

Along similar lines, a description of some new charged matter contributions arising from M-theory anomaly-cancellation can be given. An interesting feature is that the structure organizing five-brane threshold corrections is carried over to these matter interactions.

[4] They would however lead to similar phenomena.

[5] For the bulk moduli T and S, we use the terminology familiar from string compactifications in which T is the volume modulus and S the dilaton or string coupling modulus. The terminology commonly used in the context of M-theory, as for instance in refs. [193, 190], is unfortunately different. Our conventions are precisely stated in the appendix.

The paper is organised as follows. The anomaly counterterm at the origin of the five-brane-related gauge interactions is derived in Section 4.2. The reduction to four dimensions and the identification of the obtained terms as superfield densities are then discussed in Sections 4.3 and 4.4. Then, in Section 4.5, condensation is performed to derive the non-perturbative superpotential, compare it with membrane instanton results and discuss some physical consequences. Conclusions and comments are added in Section 4.6 and an appendix collects conventions, notations and technical details.

4.2 The anomaly counterterm

Ten-dimensional hyperplanes located at the Z_2 fixed points along S^1 and five-branes act as sources of the Bianchi identity verified by the four-form field G_4 of eleven-dimensional supergravity. As a consequence of these contributions, the topological term acquires anomalous variations under local symmetries. Together with the variations of the Green-Schwarz term [130, 252, 148, 107], of order four in the Riemann curvature, these anomalous variations are precisely those required to cancel perturbative gauge and Lorentz anomalies generated by the Z_2 orbifold projection of the eleven-dimensional theory and by the chiral gauge multiplets living on the hyperplanes.

The modification of the Bianchi identity and of its solution G_4 of course leads to modifications of the effective action. All modifications generated by the topological term would have more than two derivatives in the heterotic, ten-dimensional, small S^1 radius limit. But compactifying further to four dimensions on the Calabi-Yau space K_6 also generates modifications of the effective action at the level of two-derivative gauge terms, because of the non-trivial background values of $\langle \mathrm{tr}\, R^2\rangle$ and $\langle \mathrm{tr}\, F^2\rangle$. The purpose of this section is to precisely derive some of these terms which arise whenever five-branes are present. We then begin by recalling some aspects of the description of M-theory on the orbifold S^1/Z_2. It should be noted that some ambiguities remain in our understanding of this description. The gauge sector relevant to our problem escapes however these ambiguities.

The explicit formulation of the modified Bianchi identity uses two types of sources, associated with hyperplanes supporting Yang-Mills ten-dimensional supermultiplets and five branes world-volumes. On both hyperplanes (labelled by $i = 1, 2$) live gauge supermultiplets with curvature two-forms F_i. The quantity appearing in the Bianchi identity is the gauge invariant four-form

$$I_{4,i} = \frac{1}{(4\pi)^2}\left[\frac{1}{30}\mathrm{Tr}\, F_i^2 - \frac{1}{2}\mathrm{tr}\, R^2\right], \qquad dI_{4,i} = 0, \qquad (i = 1, 2), \tag{4.1}$$

where R is the Lorentz curvature two-form. Five-branes compatible with $\mathcal{N}_4 = 1$ (four supercharges) supersymmetry wrap space-time M_4 and a two-cycle in K_6. The transverse Dirac distribution $\delta^{(5)}_{\mathcal{W}_{6,I}}$ for five-brane number I with world-volume $\mathcal{W}_{6,I}$ is the five-brane source in the Bianchi identity, which then reads[6]

$$dG_4 = 2(4\pi\kappa^2)^{1/3}\left[\frac{1}{2}\sum_I \delta^{(5)}(\mathcal{W}_{6,I}) - \sum_i \delta_i I_{4,i}\right]. \tag{4.2}$$

The one-forms

$$\delta_1 = \delta(y)\, dy, \qquad \delta_2 = \delta(y-\pi)\, dy \tag{4.3}$$

localize the gauge sources on the Z_2–fixed hyperplanes.[7] In order to respect the Z_2 symmetry used in the orbifold projection, we actually label with index I *pairs* of five-branes located at $\pm y_I$.[8]

[6]Supersymmetry forbids that both five-branes and anti-five-branes couple to the S^1/Z_2 orbifold.
[7]The appendix collects our conventions.
[8]And one may then choose $0 \le y_I \le \pi$.

The procedure to resolve ambiguities in the solution of the Bianchi identity has been discussed in detail in the literature.[9] The general solution of eq. (4.2) includes several arbitrary integration constants which are constrained by consistency conditions: the four-form field G_4 should be gauge-invariant and globally well-defined, its action should be well-defined, the fields eliminated by the Z_2 truncation should be gauge-invariant, as well as the massive modes of the S^1 expansion of the surviving states. These conditions should be verified for any number of five-branes at arbitrary locations. Writing

$$G_4 = dC_3 + \Delta G_{4,branes} + \Delta G_{4,planes}, \tag{4.4}$$

the consistency constraints point to a unique solution for the hyperplane contribution[10]:

$$\begin{aligned} \Delta G_{4,planes} &= -(4\pi\kappa^2)^{1/3} \textstyle\sum_i \big(\epsilon_i\, I_{4,i} - \frac{dy}{\pi} \wedge \omega_{3,i}\big), \\ &= -(4\pi\kappa^2)^{1/3} \textstyle\sum_i \big(d[\epsilon_i\, \omega_{3,i}] - 2\delta_i \wedge \omega_{3,i}\big), \end{aligned} \tag{4.5}$$

where, as discussed in detail in the appendix,

$$\epsilon_i(y) = \mathrm{sgn}\,(y - y_i) - \frac{y - y_i}{\pi}, \qquad y_1 = 0, \qquad y_2 = \pi, \qquad y \in [-\pi, \pi] \tag{4.6}$$

and the Chern-Simons three-forms are defined by

$$d\omega_{3,i} = I_{4,i}, \qquad\qquad i = 1, 2. \tag{4.7}$$

A similar discussion can be made for the five-brane contribution to G_4. As already mentioned, the five-branes are space-time filling and wrap a holomorphic two-cycle in K_6. This implies that one can certainly write

$$\delta^{(5)}(\mathcal{W}_{6,I}) = [\delta(y - y_I) + \delta(y + y_I)]dy \wedge \delta^{(4)}(\mathcal{W}_{6,I}), \tag{4.8}$$

where $\delta^{(4)}(\mathcal{W}_{6,I})$ is now a four-form Dirac distribution in K_6 such that

$$\int_{M_4\times K_6} I_6 \wedge \delta^{(4)}(\mathcal{W}_{6,I}) = \int_{\mathcal{W}_{6,I}} I_6$$

for any six-form I_6. The natural solution of the Bianchi identity is then to include in G_4 the brane contribution

$$\begin{aligned} \Delta G_{4,branes} &= (4\pi\kappa^2)^{1/3} \textstyle\sum_I \big[\frac{1}{2}\hat\epsilon_{y_I}(y)\delta^{(4)}(\mathcal{W}_{6,I}) - \frac{1}{\pi}dy \wedge \theta^{(3)}(\mathcal{W}_{6,I})\big] + d\mathcal{C}_3 \\ &= -(4\pi\kappa^2)^{1/3} \textstyle\sum_I \big\{\delta(y - y_I) + \delta(y + y_I)\big\} dy \wedge \theta^{(3)}(\mathcal{W}_{6,I}) \\ &\quad + d\big\{\mathcal{C}_3 + \frac{1}{2}(4\pi\kappa^2)^{1/3} \textstyle\sum_I \hat\epsilon_{y_I}(y)\theta^{(3)}(\mathcal{W}_{6,I})\big\}, \end{aligned} \tag{4.9}$$

where $d\theta^{(3)}(\mathcal{W}_{6,I}) = \delta^{(4)}(\mathcal{W}_{6,I})$ and

$$\begin{aligned} &\hat\epsilon_{y_I}(y) = \mathrm{sgn}\,(y - y_I) + \mathrm{sgn}\,(y + y_I) - \tfrac{2y}{\pi}, \qquad\qquad 0 \le y_I \le \pi, \\ &d\hat\epsilon_{y_I}(y) = 2\left[\delta(y - y_I) + \delta(y + y_I) - \tfrac{1}{\pi}\right] dy. \end{aligned} \tag{4.10}$$

Notice that $\epsilon_1(y) = \frac{1}{2}\hat\epsilon_0(y)$ and $\epsilon_2(y) = \frac{1}{2}\hat\epsilon_\pi(y)$. The addition in eq. (4.9) of the term $d\mathcal{C}_3$ allows for the introduction of brane modes contributions into the topological term, if necessary [44]: it is

[9] For instance in refs. [43] and [44], and references therein.
[10] This is the solution with "$b = 1$" in refs. [43, 44].

known [236, 34, 1, 251, 253] that a gauging by C_3 of the three-form field of five-brane fluctuations is induced by consistent coupling of five-branes to eleven-dimensional supergravity. We will not elaborate on this point here since in four dimensions the needed terms would automatically appear in the supersymmetrization of the effective theory.[11]

To summarize, one can write

$$\begin{aligned} G_4 &= d\tilde{C}_3 + 2(4\pi\kappa^2)^{1/3} \sum_i \delta_i \wedge \omega_{3,i} \\ &\quad -(4\pi\kappa^2)^{1/3} \sum_I \{\delta(y-y_I) + \delta(y+y_I)\} dy \wedge \theta^{(3)}(\mathcal{W}_{6,I}), \\ \tilde{C}_3 &= C_3 - (4\pi\kappa^2)^{1/3} \left[\sum_i \epsilon_i\, \omega_{3,i} - \tfrac{1}{2}\sum_I \hat{\epsilon}_{y_I}(y)\theta^{(3)}(\mathcal{W}_{6,I})\right] + \mathcal{C}_3, \end{aligned} \tag{4.11}$$

and G_4 and $d\tilde{C}_3$ only differ at locations where hyperplanes or five-branes sit.

The gauge transformation of C_3 is completely fixed by consistency of the Z_2 orbifold projection. The topological term indicates that C_3 is intrinsically odd. The components C_{ABC}, $A, B, C = 0 \ldots, 9$, are then projected out and should then for consistency be gauge invariant. This condition implies

$$\begin{aligned} \delta C_3 &= (4\pi\kappa^2)^{1/3} \tfrac{1}{\pi}\, dy \wedge \sum_i \omega^1_{2,i}\,, \\ \delta\tilde{C}_3 &= (4\pi\kappa^2)^{1/3} \sum_i \left[2\delta_i \wedge \omega^1_{2,i} - d\{\epsilon_i \omega^1_{2,i}\}\right], \end{aligned} \tag{4.12}$$

where $\delta\omega_{3,i} = d\omega^1_{2,i}$. The correct modified topological term, which cancels anomalies locally, is then [44]

$$-\frac{1}{24\kappa^2} \int \tilde{C}_3 \wedge G_4 \wedge G_4. \tag{4.13}$$

Our goal is to infer from this modified topological term four-dimensional interactions of the massless $S^1/Z_2 \times K_6$ modes. The substitution of eqs. (4.11) leads to two classes of terms. Firstly, contributions involving the massless mode C_{ABy} ($A, B = 0, \ldots, 9$). This mode corresponds to the antisymmetric tensor B_{AB} of ten-dimensional sixteen-supercharge supergravity. From gauge-transformation (4.12), one deduces that the appropriate definition with $\delta B = (4\pi)^2 \sum_i \omega^1_{2,i}$ is[12]

$$B_{AB} = (4\pi\kappa^2)^{-1/3}(4\pi)^2\, \pi R_{S^1}\, C^{(0)}_{ABy}, \qquad C^{(0)}_{ABy} = \frac{1}{2\pi}\int_{-\pi}^{\pi} dy\, C_{ABy}, \tag{4.14}$$

where R_{S^1} is the S^1 radius. In terms of this massless field, the topological term produces the following four-dimensional interactions

$$-\frac{1}{32\pi^2}\int_{S^1\times K_6} B_2 \wedge \left[\sum_{i,j} \epsilon_i\,\epsilon_j\, I_{4,i}\wedge I_{4,j} - \sum_{i,I} \epsilon_i\,\hat{\epsilon}_{y_I}\, I_{4,i}\wedge \delta^{(4)}(\mathcal{W}_{6,I})\right], \tag{4.15}$$

with $\int_{S^1} = \int_{-\pi}^{\pi} dy$. The first contribution, when integrated over S^1 only, generates the Green-Schwarz gauge anomaly-cancelling terms expected for the $E_8 \times E_8$ heterotic string [130]. Its consequences for Calabi-Yau compactifications have been studied long ago [150, 97]. When integrated over $S^1 \times K_6$, it leads to gauge threshold corrections depending on the Calabi-Yau volume modulus as well as dilaton-dependent charged-matter terms. The second contribution, which also depends on the S^1 position of the five-branes, is of interest to us.

[11] See Section 4.4.

[12] Taking into account the factor $(4\pi)^{-2}$ in the definitions of $I_{4,i}$ and $\omega_{3,i}$. This is the definition of the two-form field commonly used in ten dimensions, with dimension $(\text{mass})^2$.

The modified topological term also produces the following C_3–independent gauge interactions:

$$
\begin{aligned}
-\frac{1}{3}\int_{S^1\times K_6}\Big[&-\sum_{i,j,k}\epsilon_i\epsilon_j\,\omega_{3,i}\wedge I_{4,j}\wedge\omega_{3,k}+\frac{1}{2}\sum_{i,J,k}\epsilon_i\hat\epsilon_{y_J}\,\omega_{3,i}\wedge\delta^{(4)}(\mathcal{W}_{6,J})\wedge\omega_{3,k}\\
&+\frac{1}{2}\sum_{i,J,k}\epsilon_i\hat\epsilon_{y_J}I_{4,i}\wedge\theta^{(3)}(\mathcal{W}_{6,J})\wedge\omega_{3,k}+\sum_{i,j,K}\epsilon_i\epsilon_j\omega_{3,i}\wedge I_{4,j}\wedge\theta^{(3)}(\mathcal{W}_{6,K})\Big].
\end{aligned}
\tag{4.16}
$$

After integration over y, the first contribution is a local counterterm allowed by anomaly cancellation [44][13]. The next three terms are non-trivial brane contributions depending on their positions along S^1.

The overlap integrals over S^1 give an interesting result. First introduce the numbers

$$
d_{ij} \;=\; \int_{-\pi}^{\pi}dy\,\epsilon_i\,\epsilon_j \;=\; \frac{\pi}{3}\,(3\delta_{ij}-1) \tag{4.17}
$$

for the first integrals in contributions (4.15) and (4.16). For those depending on the brane position, define the variables

$$
\Delta_{I,1}=\frac{y_I}{\pi},\qquad \Delta_{I,2}=1-\frac{y_I}{\pi},\qquad\qquad \Big(0<y_I<\pi;\quad 0<\Delta_{I,j}<1\Big), \tag{4.18}
$$

the distances from brane I to the fixed planes, with normalization $\Delta_{I,1}+\Delta_{I,2}=1$. Then,

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}dy\,\epsilon_1(y)\hat\epsilon_{y_I}(y) &= (\Delta_{I,2})^2-\frac{1}{3} = \frac{y_I^2}{\pi^2}-2\frac{y_I}{\pi}+\frac{2}{3},\\
\frac{1}{2\pi}\int_{-\pi}^{\pi}dy\,\epsilon_2(y)\hat\epsilon_{y_I}(y) &= (\Delta_{I,1})^2-\frac{1}{3} = \frac{y_I^2}{\pi^2}-\frac{1}{3}.
\end{aligned}
\tag{4.19}
$$

It will be important for the supersymmetrization of the four-dimensional interactions that the terms quadratic in y_I are identical in both integrals. With these results, contributions (4.15) lead to

$$
\begin{aligned}
-\frac{1}{16\pi}\int_{K_6}B_2\wedge\Big[&\frac{1}{3}(I_{4,1}^2+I_{4,2}^2-I_{4,1}I_{4,2})\\
&-\sum_I\delta^{(4)}(\mathcal{W}_{6,I})\wedge\Big\{\Big(\Delta_{I,2}^2-\frac{1}{3}\Big)I_{4,1}+\Big(\Delta_{I,1}^2-\frac{1}{3}\Big)I_{4,2}\Big\}\Big],
\end{aligned}
\tag{4.20}
$$

while expressions (4.16) give

$$
\begin{aligned}
\frac{\pi}{3}\int_{K_6}\Big\{&\omega_{3,1}\wedge\omega_{3,2}\wedge(I_{4,1}-I_{4,2})+\frac{3}{2}\sum_I(\Delta_{I,1}-\Delta_{I,2})\delta^{(4)}(\mathcal{W}_{6,I})\wedge\omega_{3,1}\wedge\omega_{3,2}\\
&-\sum_I\theta^{(3)}(\mathcal{W}_{6,I})\wedge\Big[(\Delta_{I,2}\,\omega_{3,1}-\Delta_{I,1}\,\omega_{3,2})\wedge(\Delta_{I,2}\,I_{4,1}-\Delta_{I,1}\,I_{4,2})\\
&-\omega_{3,1}\wedge I_{4,1}-\omega_{3,2}\wedge I_{4,2}+\frac{1}{2}\omega_{3,1}\wedge I_{4,2}+\frac{1}{2}\omega_{3,2}\wedge I_{4,1}\Big]\Big\},
\end{aligned}
\tag{4.21}
$$

after some partial integrations.

[13] The anomaly twelve-form obtained from descent equations vanishes.

The anomalous gauge variation of contributions (4.20) and (4.21) can be written as

$$
\begin{aligned}
&-\frac{\pi}{3}(\omega^1_{2,1}\, I^2_{4,1} + \omega^1_{2,2}\, I^2_{4,2}) \\
&+\frac{2\pi}{3}\sum_I \delta^{(4)}(\mathcal{W}_{6,I}) \wedge [\Delta_{I,2}\,\omega^1_{2,1} - \Delta_{I,1}\,\omega^1_{2,2}] \wedge [\Delta_{I,2}\, I_{4,1} - \Delta_{I,1}\, I_{4,2}].
\end{aligned}
\tag{4.22}
$$

Applying descent equations to these variations leads to the formal anomaly twelve-form

$$
\begin{aligned}
\mathcal{I}_{12} &= -\tfrac{\pi}{3}\Big[I_{4,1} + I_{4,2} - \textstyle\sum_I \delta^{(4)}(\mathcal{W}_{6,I})\Big] \\
&\quad \wedge\Big[I^2_{4,1} + I^2_{4,2} - I_{4,1}I_{4,2} + \textstyle\sum_I \delta^{(4)}(\mathcal{W}_{6,I}) \wedge \{(1-3\Delta^2_{I,2})I_{4,1} + (1-3\Delta^2_{I,1})I_{4,2}\}\Big] \\
&= \mathcal{I}_{12,het.} + \textstyle\sum_I \delta^{(4)}(\mathcal{W}_{6,I})\,\mathcal{I}_{8,I}\,,
\end{aligned}
\tag{4.23}
$$

since for four-dimensional space-time-filling five-branes, $\delta^{(4)}(\mathcal{W}_{6,I}) \wedge \delta^{(4)}(\mathcal{W}_{6,J}) = 0$. The contribution of each five-brane is encoded in the eight-form

$$
\mathcal{I}_{8,I} = \pi[\Delta_{I,2}I_{4,1} - \Delta_{I,1}I_{4,2}]^2, \tag{4.24}
$$

while the heterotic contribution is as usual $\mathcal{I}_{12,het.} = -\frac{\pi}{3}[I^3_{4,1} + I^3_{4,2}]$.

The form of $\mathcal{I}_{8,I}$, eq. (4.24), clearly shows that the distance from the brane to the first plane acts as the (inverse squared) coupling of the gauge fields living on the second plane. Similarly, the distance from the brane to the second plane acts as the (inverse squared) coupling of the gauge fields living on the first plane.

At this point, the conclusion is that the effective, four-dimensional supergravity includes gauge contributions due to five-branes which arise from the Calabi-Yau reduction of expressions (4.20) and (4.21), as derived from the modified topological term (4.13). In the next two sections, we perform this reduction keeping only the "universal" massless modes unrelated to geometrical details of K_6, but including the five-brane modulus along the S^1 direction, and we write the effective four-dimensional supergravity using superconformal tensor calculus.

4.3 Reduction to four dimensions

In the reduction to four dimensions, we use the freedom to rescale moduli fields to set the S^1 circumference $2\pi R_{S^1}$ and the Calabi-Yau volume V_6 to unity.[14]

As usual, the massless modes of the metric tensor expanded on $M_4 \times K_7$ include $g_{\mu\nu}$, the scalar field g_{yy} and massless modes of the internal metric $g_{i\overline{k}}$. Among these, we only keep the universal, Kähler-metric volume modulus. Similarly, the antisymmetric tensor C_{ABy} leads to a massless $B_{\mu\nu}$ and we only keep the universal massless mode of the internal tensor $B_{i\overline{k}}$. With these bosonic modes and their fermionic partners, the reduction of eleven-dimensional supergravity can be described by two chiral multiplets S and T with the familiar Kähler potential [250, 97][15]

$$
\mathcal{K} = -\ln(S+\overline{S}) - 3\ln(T+\overline{T}). \tag{4.25}
$$

Following eq. (4.14), we define

$$
B_{\mu\nu} = \frac{(4\pi)^2}{2}(4\pi\kappa^2)^{-1/3}\, C_{\mu\nu y}\,, \qquad B_{i\bar{j}} = \frac{i}{\kappa_4^2}\,\mathrm{Im}\, T\,\delta_{i\bar{j}} \tag{4.26}
$$

[14]The four-dimensional gravitational constant is then $\kappa_4^2 = \kappa^2$, but we nevertheless use different symbols since their mass dimensions differ.

[15]In general, we use the same notation for a chiral supermultiplet and for its lowest complex scalar component.

and $B_{\mu\nu}$ is dual to $\mathrm{Im}\, S$.

With five-branes, vector fields on the two fixed hyperplanes gauge an algebra $G_1 \times G_2$ which is further broken by the Calabi-Yau compactification. Embedding the $SU(3)$ holonomy into $G_1 \times G_2$ defines the four-dimensional gauge group $G_{(4)}$ as the stabilizer of this $SU(3)$ in $G_1 \times G_2$. Calabi-Yau reduction of the ten-dimensional gauge fields[16] $\mathcal{A}_B^{\alpha(i)}$ leads then to the corresponding gauge fields $\mathcal{A}_\mu^{a(i)}$. It also produces a set of $SU(3)$–singlet complex scalar fields $\mathcal{A}^{m(i)}$ in some representation of $G_{(4)}$.

With up to two derivatives, Riemann curvature contributions in counterterms (4.20–4.21) can be omitted. The Calabi-Yau reduction of $I_{4,i}$ delivers then:

$$
\begin{aligned}
I_{i,\mu\nu\rho\sigma} &= \frac{3!}{(4\pi)^2} \sum_\alpha F^{\alpha(i)}_{[\mu\nu} F^{\alpha(i)}_{\rho\sigma]} \,, \\
I_{i,\mu\nu k\bar{l}} &= -\frac{4}{(4\pi)^2} \sum_m (\mathcal{D}_{[\mu} \mathcal{A}^{m(i)})(\mathcal{D}_{\nu]} \overline{\mathcal{A}}^{m(i)})\, \delta_{k\bar{l}} \\
&= \frac{2i}{(4\pi)^2} \sum_m \partial_{[\mu} \left(\mathcal{A}^{m(i)} \mathcal{D}_{\nu]} \overline{\mathcal{A}}^{m(i)} - \overline{\mathcal{A}}^{m(i)} \mathcal{D}_{\nu]} \mathcal{A}^{m(i)} \right) i\delta_{k\bar{l}} \,, \\
I_{i,\mu jkl} &= \frac{2}{(4\pi)^2}\, \partial_\mu \left(\lambda_{i,mnp} \mathcal{A}^{m(i)} \mathcal{A}^{n(i)} \mathcal{A}^{p(i)} \right) \epsilon_{jkl} \,.
\end{aligned}
\tag{4.27}
$$

In the last expression, $\lambda_{i,mnp}$ is the symmetric tensor invariant under $G_{(4)}$ that may arise from the internal Chern-Simons term $\omega_{i,\,jkl}$. We will use the notations

$$
\lambda \mathcal{A}^3 = \sum_i \lambda \mathcal{A}_i^3, \qquad \lambda \mathcal{A}_i^3 \equiv \lambda_{i,mnp} \mathcal{A}^{m(i)} \mathcal{A}^{n(i)} \mathcal{A}^{p(i)} \tag{4.28}
$$

to denote this cubic holomorphic couplings which also appear in the superpotential

$$
W = c + \lambda \mathcal{A}^3. \tag{4.29}
$$

Finally, $I_{4,i}$ has a non-trivial background value $\langle I_{4,i} \rangle_{ij\overline{kl}}$.

With these results, the reduction to four dimensions of the first line in expression (4.20), which depends on B_2 and exists without five-brane can be written

$$
\begin{aligned}
\Delta \mathcal{L}_{plane} = \frac{1}{2(4\pi)^4} \sum_{i,j} d_{ij} \Big\{ & \frac{1}{\kappa_4^2}\, e_4 \langle \mathcal{I}_j \rangle \, \mathrm{Im}\, T\, F^{\alpha(i)}_{\mu\nu} \widetilde{F}^{\alpha(i)\,\mu\nu} \\
& - i \langle \mathcal{I}_j \rangle\, \epsilon^{\mu\nu\rho\sigma} (\partial_\mu B_{\nu\rho}) \sum_m [\mathcal{A}^{m(i)} (\mathcal{D}_\sigma \overline{\mathcal{A}}^{m(i)}) \,-\, \overline{\mathcal{A}}^{m(i)} (\mathcal{D}_\sigma \mathcal{A}^{m(i)})] \\
& - \frac{i}{(4\pi)^2} \epsilon^{\mu\nu\rho\sigma} (\partial_\mu B_{\nu\rho})\, (\lambda \mathcal{A}_i^3\, \partial_\sigma (\overline{\lambda \mathcal{A}_j^3}) - \partial_\sigma (\lambda \mathcal{A}_i^3)\, \overline{\lambda \mathcal{A}_j^3}) \Big\}.
\end{aligned}
\tag{4.30}
$$

The background value of $I_{4,i}$ is encoded in the integral over the Calabi-Yau manifold

$$
\langle \mathcal{I}_i \rangle = V_6^{-1} \int_{K_6} \langle I_i \rangle_{kl\overline{kl}}\, \delta_{j\bar{j}}\, \epsilon^{jkl} \epsilon^{\overline{jkl}}. \tag{4.31}
$$

[16]We find useful to keep track of the plane index $i = 1, 2$ and $\alpha(i)$ is then an index in the adjoint representation of G_i. Similarly, $a(i)$ will be used for the adjoint of $G_{(4)}$ and $m(i)$ for the representation spanned by complex scalar fields.

In expression (4.30), the first term is a threshold correction depending on the volume modulus already well-known from the heterotic strings [97, 150]. The second and third contributions describe interactions of matter scalars with the string coupling multiplet. They have been considered in detail in refs. [200, 201, 188, 187, 98].

The reduction of the five-brane contribution in expression (4.20) leads to the following Lagrangian terms:

$$\begin{aligned}\Delta\mathcal{L}_{brane} \;=\; & \frac{1}{8(4\pi)^3}\sum_{I,i} a_I\Big(\Delta^2_{I,i}-\frac{1}{3}\Big)\Big[\frac{1}{\kappa_4^2}\,e_4\,\mathrm{Im}\,T\,F^{\alpha(i)}_{\mu\nu}\widetilde{F}^{\alpha(i)\,\mu\nu} \\ & -i\,\epsilon^{\mu\nu\rho\sigma}\partial_\mu B_{\nu\rho}\sum_m[\mathcal{A}^{m(i)}(\mathcal{D}_\sigma\overline{\mathcal{A}}^{m(i)}) \,-\, \overline{\mathcal{A}}^{m(i)}(\mathcal{D}_\sigma\mathcal{A}^{m(i)})]\Big].\end{aligned} \tag{4.32}$$

The area of the Calabi-Yau two-cycle (with coordinate z) wrapped by the five-brane world-volume can be written

$$a_I = \int_{\mathcal{C}_{2,I}} dz\,d\overline{z}\,\frac{\partial z^m}{\partial z}\frac{\partial\overline{z}^{\overline{n}}}{\partial\overline{z}}\,\delta_{m\overline{n}}\,. \tag{4.33}$$

The first term in (4.32) is the five-brane contribution to gauge threshold corrections. Its existence has been demonstrated in an explicit background calculation by Lukas, Ovrut and Waldram [190, 189]. The second term is again a matter interaction with the string coupling multiplet. Both terms depend on the positions y_I of the five-branes. Hence, they depend on the S^1/Z_2 five-brane modulus.

The terms collected in expression (4.21) are somewhat ambiguous since they are defined up to contributions which, like the first one or any counterterm of the form $\theta^{(3)}(\mathcal{W}_{6,I})\wedge I_7$, do not contribute to the gauge-invariant anomaly twelve-form. To reduce the first term, introduce the four-dimensional Chern-Simons forms

$$\begin{aligned}&\partial_{[\mu}\omega_{i,\nu\rho\sigma]} = \frac{1}{4}\,F_{i,[\mu\nu}\,F_{i,\rho\sigma]}\,, \qquad\qquad \partial_{[\mu}\omega_{i,\nu]j\bar{k}} = \frac{1}{2}\,I_{i,\mu\nu j\bar{k}}\,, \\ &\partial_\mu\omega_{i,jkl} = I_{i,\mu jkl}\,.\end{aligned} \tag{4.34}$$

The first term then generates couplings of charged matter scalars to gauge fields:

$$\begin{aligned}\Delta\mathcal{L}_{plane} \;=\; & \frac{i}{3(4\pi)^5}\epsilon^{\mu\nu\rho\sigma}\omega_{1,\mu\nu\rho}\Big[\lambda\mathcal{A}^3_2\,\partial_\sigma(\overline{\lambda\mathcal{A}}^3_2) - \overline{\lambda\mathcal{A}}^3_2\,\partial_\sigma(\lambda\mathcal{A}^3_2) +\, \lambda\mathcal{A}^3_1\,\partial_\sigma(\overline{\lambda\mathcal{A}}^3_2) \\ & -\overline{\lambda\mathcal{A}}^3_1\,\partial_\sigma(\lambda\mathcal{A}^3_2) \,+\, 2[\overline{\lambda\mathcal{A}}^3_2\,\partial_\sigma(\lambda\mathcal{A}^3_1) - \lambda\mathcal{A}^3_2\,\partial_\sigma(\overline{\lambda\mathcal{A}}^3_1)]\Big] + (1\leftrightarrow 2) \\ & -\frac{i}{12(4\pi)^3}\epsilon^{\mu\nu\rho\sigma}\,\omega_{1,\mu\nu\rho}\textstyle\sum_m(\mathcal{A}^{m(2)}\mathcal{D}_\sigma\overline{\mathcal{A}}^{m(2)} - \overline{\mathcal{A}}^{m(2)}\mathcal{D}_\sigma\mathcal{A}^{m(2)})\,(\langle\mathcal{I}_1\rangle - \langle\mathcal{I}_2\rangle) \\ & +(1\leftrightarrow 2).\end{aligned} \tag{4.35}$$

As we will see in the next section, these terms do not have a natural supersymmetric extension in general, a fact which may have some relation to their ambiguous character. Notice however that in the minimal embedding of the Calabi-Yau background into one plane only, most of these mixing terms are absent and a natural supersymmetrization exists.

Likewise, the second term in eq. (4.21) yields gauge–matter interactions depending on the five-brane positions along S^1:

$$\Delta\mathcal{L}_{brane} = \frac{i}{8(4\pi)^3}\sum_I a_I\ (\Delta_{I,1}-\Delta_{I,2})\ \epsilon^{\mu\nu\rho\sigma}\omega_{1,\mu\nu\rho} \times\sum_m[\mathcal{A}^{m(2)}(\mathcal{D}_\sigma\overline{\mathcal{A}}^{m(2)})\ -\ \overline{\mathcal{A}}^{m(2)}(\mathcal{D}_\sigma\mathcal{A}^{m(2)})]+(1\leftrightarrow 2). \tag{4.36}$$

In the next section, we will derive the effective four-dimensional supergravity in the particular case of a single five-brane. To simplify, we will omit the index I and the corresponding sums. We will however use a formulation in which restoring contributions of several branes is straightforward.

4.4 The effective supergravity

The universal S^1/Z_2 five-brane modulus describing fluctuations along the interval direction has a supersymmetric bosonic partner arising from the mode expansion of the five-brane antisymmetric tensor $\hat{B}_{mn}$. Six-dimensional (world-volume) supersymmetry of the five-brane multiplet requires that the three-form curvature $\hat{H}_{mnp}$ of this tensor is self-dual. For the massless four-dimensional universal mode, self-duality is the condition[17]

$$\hat{H}_{\mu\nu\rho}\equiv 3\partial_{[\mu}\hat{B}_{\nu\rho]}=e_4\epsilon_{\mu\nu\rho}{}^{\sigma}\partial_\sigma\hat{B}_{i\bar{j}}\equiv e_4\epsilon_{\mu\nu\rho}{}^{\sigma}\hat{H}_{\sigma i\bar{j}}. \tag{4.37}$$

Then, clearly, the four-dimensional supersymmetric description of the five-brane universal modulus uses either a linear multiplet $\hat{L}$ with the tensor $\hat{H}_{\mu\nu\rho}$ and a real scalar $\hat{C}$ for the modulus, or a chiral multiplet with complex scalar $\hat{S}$ and $\mathrm{Im}\,\hat{S}$ related to $\hat{B}_{i\bar{j}}$. The supersymmetric extension of condition (4.37) is chiral-linear duality, the duality transformation exchanging superfields $\hat{S}$ and $\hat{L}$ [113, 223].

The fact that the chiral multiplet $\hat{S}$ is dual to a linear multiplet has three important consequences for its supergravity couplings. Firstly, the Kähler potential is a function of $\hat{S}+\overline{\hat{S}}$ only. Secondly, the holomorphic gauge kinetic function can only depend linearly on $\hat{S}$. These two consequences follow from the intrinsic gauge invariance of $\hat{B}_{\mu\nu}$, which translates into axionic shift symmetry of $\hat{S}$ in the chiral formulation. Thirdly, the superpotential does not depend on $\hat{S}$. In supergravity, in contrast to global supersymmetry, this statement is ambiguous and directly linked to the first consequence above. The superpotential W and the Kähler potential $\mathcal{K}$ are not independent: the entire theory depends only on $\mathcal{G}=\mathcal{K}+\ln|W|^2$. Terms can then be moved from or into $\mathcal{K}$ provided they are harmonic functions of the complex chiral fields. The correct statement is then that $\mathcal{G}$ may only depend on $\hat{S}+\overline{\hat{S}}$. Moving terms from $\mathcal{K}$ to the superpotential can artificially generate a dependence on $\hat{S}$ of the form

$$W_{new}=e^{b\hat{S}}W(z^i), \tag{4.38}$$

where b is a real constant and z^i denotes all other chiral multiplets, and a new Kähler potential $\mathcal{K}_{new}(\hat{S}+\overline{\hat{S}},z^i,\overline{z}_i)$ such that the function $\mathcal{G}$ remains unchanged, $\mathcal{K}+\ln|W|^2=\mathcal{K}_{new}+\ln|W_{new}|^2$. Notice that adding a $\hat{S}$–independent term to the superpotential (4.38) is not allowed. Non-perturbative exponential superpotentials generated, for instance, by gaugino condensation, and *added* to a perturbative superpotential are then incompatible with chiral-linear duality.[18]

The effective four-dimensional supergravity depends on three moduli multiplets coupled to supergravity, gauge and charged matter superfields. Each of the three moduli scalars has a

[17] Omitting fermion and covariantization contributions.
[18] See however ref. [61, 62] for an analysis.

component of an antisymmetric tensor as supersymmetry partner. More precisely, the string coupling modulus is in the multiplet describing $G_{\mu\nu\rho y}$, the K_6 volume modulus is paired with $G_{\mu i\overline{k}y}$ and the five-brane S^1/Z_2 modulus is the partner of the components $\hat{H}_{\mu\nu\rho}$ or $\hat{H}_{\mu i\overline{k}}$ of the self-dual antisymmetric tensor living on the brane world-volume. We find, as explained in ref. [98], more efficient to formulate the theory using superconformal tensor calculus and to introduce three moduli vector superfields to describe these moduli multiplets:[19]

V	$(w = 2, n = 0)$:	$G_{\mu\nu\rho y}$, string coupling modulus, ...
V_T	$(w = n = 0)$:	$G_{\mu i\overline{k}y}$, Calabi-Yau volume modulus, ...
$\hat{V}$	$(w = n = 0)$:	$\hat{H}_{\mu\nu\rho}$, five-brane S^1/Z_2 modulus, ...

The components of the antisymmetric tensors are identified with the vector fields in these multiplets, the moduli scalars with their real lowest components. These vector multiplets are then submitted to Bianchi identities obtained from the K_7 reduction of the eleven-dimensional Bianchi identity for G_4, eq. (4.2), and the self-duality condition of the five-brane tensor. In each case, the supersymmetrized Bianchi identity also reduces the number of off-shell states from $8_B + 8_F$ in a vector multiplet to $4_B + 4_F$. These Bianchi identities are imposed as the field equations of three Lagrange-multiplier superfields:

S	$(w = n = 0)$:	a chiral multiplet for the Bianchi identity verified by V,
L_T	$(w = 2, n = 0)$:	a linear multiplet for the Bianchi identity verified by V_T,
$\hat{S}$	$(w = n = 0)$:	a chiral multiplet for the self-duality condition of the brane tensor, applied to $\hat{V}$.

Eliminating these Lagrange multiplier superfields defines the three vector multiplets in terms of the physical fields solving Bianchi identities. The important advantage of this procedure is that supersymmetrizing the theory before eliminating Lagrange multipliers automatically delivers the correct non-linear couplings of source terms (brane and plane contributions) to moduli and then the correct Kähler potential. Alternatively, equivalent (dual) versions of the theory can be obtained by eliminating some vector multiplets instead of the Lagrange multipliers.

These six multiplets describing bulk and brane states are supplemented by states living on the fixed hyperplanes. In the notation defined in the previous section, these states include gauge chiral superfields $\mathcal{W}^{\alpha(i)}$ ($i = 1, 2$ as usual, $w = n = 3/2$) and charged matter chiral multiplets $(w = n = 0)$ in some representation of the gauge group. They will be collectively denoted by M and they contain the complex scalar components $\mathcal{A}^{m(i)}$. Finally, we need the compensating chiral multiplet S_0 $(w = n = 1)$ to gauge fix the superconformal theory to super-Poincaré symmetry only.

With this set of superfields, the Lagrangian nicely splits in a sum of five terms with well-defined higher-dimensional interpretations:

$$\mathcal{L} = \mathcal{L}_{bulk} + \mathcal{L}_{Bianchi} + \mathcal{L}_{kinetic} + \mathcal{L}_{thresholds} + \mathcal{L}_{superpotential}. \tag{4.39}$$

The bulk Lagrangian [64]

$$\mathcal{L}_{bulk} = -\left[(S_0\overline{S}_0V_T)^{3/2}(2V)^{-1/2}\right]_D \tag{4.40}$$

can be directly obtained from the K_7 reduction of eleven-dimensional supergravity, expressed in terms of G_4. It depends on V (string coupling multiplet) and V_T (K_6 volume modulus multiplet), and of the compensator S_0. In eq. (4.40), $[\ldots]_D$ denotes the invariant real density formula of superconformal calculus, as reviewed and developed in for instance refs. [176]. Similarly, $[\ldots]_F$ will below denote the chiral density formula.[20]

The coupling of plane and brane fields ($\mathcal{W}^{\alpha(i)}$, M and $\hat{V}$) to bulk multiplets is entirely

[19] The Weyl weight is w and n is the chiral weight.

[20] In global supersymmetry, $[\ldots]_D$ and $[\ldots]_F$ would be $\int d^4\theta[\ldots]$ and $\int d^2\theta[\ldots] + \text{h.c.}$

encoded in $\mathcal{L}_{Bianchi}$, which reads [100]

$$\mathcal{L}_{Bianchi} = \Big[-(S+\overline{S})(V+2\Omega_1+2\Omega_2) + L_T(V_T + 2\overline{M}e^{2A}M) + \frac{1}{2}\tau(\hat{S}+\overline{\hat{S}})V\hat{V}\Big]_D, \qquad (4.41)$$

where Ω_1 and Ω_2 are the Chern-Simons multiplets ($w = 2, n = 0$) for the gauge algebra arising on each hyperplane, defined by[21] $\sum_\alpha \mathcal{W}^{\alpha(i)}\mathcal{W}^{\alpha(i)} = 16\,\Sigma(\Omega_i)$, and $\overline{M}e^{2A}M \equiv \sum_{m,i}\overline{M}^{m(i)}[e^{2A}M]^{m(i)}$ is the Wess-Zumino Lagrangian superfield. This contribution is gauge-invariant since $[(S+\overline{S})(\delta\Omega_1+\delta\Omega_2)]_D$ is a derivative. The dimensionless number τ is proportional to the five-brane tension T_5. In our units, it reads

$$\tau = \frac{2}{(4\pi)^3}\,a, \qquad (4.42)$$

where a is the area of the two-cycle wrapped by the brane in K_6, as defined in eq. (4.33). Notice shift symmetries $\delta S = ic$, $\delta\hat{S} = id$ (c, d real).

The kinetic terms of the five-brane fields arise from reduction of the Dirac-Born-Infeld (DBI) Lagrangian:

$$\mathcal{L}_{kinetic} = -\tau\big[VV_T\hat{V}^2\big]_D. \qquad (4.43)$$

They are quadratic in $\hat{V}$, a consequence of the form of the DBI action, and the prefactor VV_T is the coupling to the supergravity background. Notice that since this term is linear in V, it will naturally assemble with the contribution in $S+\overline{S}$ in eq. (4.41).

At this point, the contributions from bulk, plane and five-brane kinetic Lagrangians have been considered, with tensor fields in V, V_T and $\hat{V}$ verifying Bianchi identities modified by plane and brane sources. But we still have to consider further contributions from the topological term with modified G_4, as obtained in the previous section. These terms will be collected in $\mathcal{L}_{threshold}$. The symmetries of expressions (4.40)–(4.43) allow the introduction of the following corrections to gauge kinetic terms [100] (*threshold corrections*):

$$\begin{aligned}\mathcal{L}_{thresholds} \;=\; & \Big[-2\textstyle\sum_i\beta_i\Omega_i(V_T+2\overline{M}e^{2A}M)\Big]_D \\ & +\tfrac{\tau}{4}\Big[\hat{S}\textstyle\sum_i\hat{\beta}_i\mathcal{W}_i\mathcal{W}_i\Big]_F + \Big[V\Big\{\epsilon|\alpha M^3|^2 - 2\tau\textstyle\sum_i g_i(\hat{V})\overline{M}^i e^{2A}M^i\Big\}\Big]_D.\end{aligned} \qquad (4.44)$$

The first contribution corresponds to threshold corrections depending on the volume modulus [97, 150, 95]. Gauge invariance of the full Lagrangian with this term is obtained by postulating the appropriate variation of the linear multiplet L_T in $\mathcal{L}_{Bianchi}$:

$$\delta L_T = 2\sum_i\beta_i\delta\Omega_i.$$

The second and third contributions are threshold corrections depending on the S^1/Z_2 location of the five-brane and/or, for the last one, on matter multiplets. The coefficients $\beta_i, \hat{\beta}_i, \epsilon$ and the functions $g_i(\hat{V})$ can be obtained, as explained below, from Calabi-Yau reduction of the topological term with anomaly-cancelling modifications. Symmetries of the theory leave these functions unconstrained but the terms considered here only require linear functions, $g_i(\hat{V}) = \hat{\gamma}_i\hat{V} + \hat{\delta}_i$. Finally, the quantity αM^3 in expression (4.44) denotes the holomorphic cubic invariant also present in the matter superpotential [250, 97].

The superpotential arises from the components G_{ijky} of G_4. They also verify a non-trivial Bianchi identity which is not modified by five-branes since three holomorphic directions cannot be transverse to their world-volumes. The superpotential contribution to the Lagrangian is then

$$\mathcal{L}_{superpotential} = \Big[S_0^3\,W\Big]_F, \qquad \text{with} \qquad W = c + \alpha M^3, \qquad (4.45)$$

[21] The operation $\Sigma(\ldots)$ is the superconformal analog of $\frac{1}{8}\overline{\mathcal{D}}\overline{\mathcal{D}}$ in global superspace.

The constant c being the 'flux' of the heterotic three-form in direction ϵ_{ijk} [250, 96, 102].

This formulation of the effective supergravity, with six superfields to describe three moduli supermultiplets leads to several equivalent forms, depending on the choice made when eliminating the three superfluous multiplets. The Lagrange multipliers S, L_T and $\hat{S}$ imply the following constraints on the vector multiplets:

$$\begin{array}{lrcll} S: & V &=& L - 2\sum_i \Omega_i, & L \text{ linear } (w=2, n=0), \\ L_T: & V_T &=& T + \overline{T} - 2\overline{M}e^{2A}M, & T \text{ chiral } (w=n=0), \\ \hat{S}: & \hat{V} &=& V^{-1}\left(\hat{L} + 4\sum_i \hat{\beta}_i\Omega_i\right), & \hat{L} \text{ linear } (w=2, n=0). \end{array} \tag{4.46}$$

Eliminating S, L_T and $\hat{S}$ leads then to a formulation where moduli are described by L, T and $\hat{L}$, two linear and one chiral multiplets:

$$\begin{aligned} \mathcal{L} \;=\; & \Big[-\tfrac{1}{\sqrt{2}}\,[S_0\overline{S}_0(T+\overline{T}-2\overline{M}e^{2A}M)]^{3/2}\,(L-2\textstyle\sum_i\Omega_i)^{-1/2} \\ & -\tau(T+\overline{T}-2\overline{M}e^{2A}M)\,(L-2\textstyle\sum_i\Omega_i)^{-1}\,(\hat{L}+4\sum_j\hat{\beta}_j\Omega_j)^2 \\ & +(L-2\textstyle\sum_i\Omega_i)\left\{\epsilon|\alpha M^3|^2 - 2\tau\sum_j\hat{\delta}_j\overline{M}^j e^{2A}M^j\right\} \\ & -2\tau(\hat{L}+4\textstyle\sum_i\hat{\beta}_i\Omega_i)\sum_j\hat{\gamma}_j\overline{M}^j e^{2A}M^j\Big]_D \\ & +\Big[S_0^3W + \tfrac{1}{4}\textstyle\sum_i\beta_i T\mathcal{W}_i\mathcal{W}_i\Big]_F. \end{aligned} \tag{4.47}$$

Component expressions for this apparently complicated theory can be obtained using superconformal tensor calculus [176]. Notice that plane contributions (superfields Ω_i, $\mathcal{W}_i$ and M^i) now appear in the bulk Lagrangian (first line) and also in the five-brane DBI contribution (second line). Five-brane contributions (multiplet $\hat{L}$) appear in gauge kinetic terms (a "plane term") while threshold corrections (third and fourth lines) involve plane and five-brane fields. These mixings are induced by the modified Bianchi identities, eqs. (4.46), and by threshold corrections required by anomaly cancellation.

The kinetic term quadratic in the five-brane modulus superfield $\hat{L}$ appears in the second line. It can clearly be derived from the DBI Lagrangian, as done in ref. [100]. But the superfield structure implied by the modified Bianchi identity leading to the first eq. (4.46) also implies that the same kinetic term can be obtained from gauge threshold corrections which follow from Calabi-Yau reduction of the (modified) topological term. This point will be explicitly verified in the next paragraph.

This unfamiliar supergravity theory is particularly useful to study its scalar potential and vacuum structure since linear multiplets do not have auxiliary fields. We will come back to this point later on and especially when studying condensation.

It is however more common to formulate the supergravity theory with chiral moduli S and T, and then to eliminate V and L_T using their field equations. One obtains:

$$\mathcal{L} = -\frac{3}{2}\Big[S_0\overline{S}_0\,e^{-\mathcal{K}/3}\Big]_D + \Big[S_0^3\,W + \frac{1}{4}\sum_i(S+\beta_i T+\tau\hat{\beta}_i\hat{S})\,\mathcal{W}^i\mathcal{W}^i\Big]_F, \tag{4.48}$$

with the Kähler potential

$$\mathcal{K} = -\ln(S+\overline{S}-\Delta) - 3\ln(T+\overline{T}-2\overline{M}e^{2A}M) \tag{4.49}$$

and

$$\Delta = -\tau(T+\overline{T}-2\overline{M}e^{2A}M)\hat{V}^2 + \frac{1}{2}\tau(\hat{S}+\overline{\hat{S}})\hat{V} + \epsilon|\alpha M^3|^2 - 2\tau\sum_i g_i(\hat{V})\overline{M}^i e^{2A}M^i\,. \qquad (4.50)$$

The field equation of the vector multiplet $\hat{V}$ implies then

$$\hat{V} = (4V_T)^{-1}\Big(\hat{S}+\overline{\hat{S}} - 4\sum_i \hat{\gamma}_i\overline{M}^i e^{2A}M^i\Big), \qquad (4.51)$$

with V_T as in the second eq. (4.46). The fully chiral formulation of the effective supergravity is then defined by Kähler potential (4.49) with now

$$\begin{aligned}\Delta \;=\;& \tfrac{\tau}{16}(T+\overline{T}-2\overline{M}e^{2A}M)^{-1}\Big(\hat{S}+\overline{\hat{S}}-4\textstyle\sum_i\hat{\gamma}_i\overline{M}^i e^{2A}M^i\Big)^2 \\ & +\epsilon|\alpha M^3|^2 - 2\tau\textstyle\sum_i\hat{\delta}_i\overline{M}^i e^{2A}M^i\,,\end{aligned} \qquad (4.52)$$

gauge kinetic functions

$$f^i = S + \beta^i T + \tau\hat{\beta}^i\hat{S}, \qquad (4.53)$$

and superpotential (4.45). The presence of the five-brane then introduces mixing of the three chiral multiplets S, T and $\hat{S}$ and the off-diagonal elements of the Kähler metric severely complicate the analysis of the theory.

Before returning to the analysis of the effective supergravity, we need a precise identification of the supergravity fields in terms of massless modes of the K_7 compactification.

The notation we use for component fields is as follows. Vector multiplets V, $\hat{V}$ and V_T have respectively vector fields V_μ, $\hat{V}_\mu$ and T_μ and (lowest) real scalar C, $\hat{C}$ and C_T. And we use the same notation for chiral multiplets S, T and $\hat{S}$ and for their complex scalar lowest components. Eqs. (4.46) indicate then that

$$C_T = 2(\mathrm{Re}\,T - \overline{M}M), \qquad\qquad T_\mu = -2\partial_\mu\,\mathrm{Im}\,T - 2i\overline{M}(\mathcal{D}_\mu M) + 2i(\mathcal{D}_\mu\overline{M})M,$$

that the lowest scalar component of the string coupling linear multiplet L is also C and that the five-brane linear multiplet $\hat{L}$ has a real scalar $\hat{\ell} = C\hat{C}$. Relation (4.51) also implies that

$$\begin{aligned}\mathrm{Re}\,\hat{S} \;&=\; 4\hat{C}(\mathrm{Re}\,T-\overline{M}M) + 2\textstyle\sum_i\hat{\gamma}_i\overline{M}^iM^i\,,\\ \partial_\mu\,\mathrm{Im}\,\hat{S} \;&=\; -4\hat{V}_\mu(\mathrm{Re}\,T-\overline{M}M) + 4\hat{C}(\partial_\mu\,\mathrm{Im}\,T + i\overline{M}\mathcal{D}_\mu M - iM\mathcal{D}_\mu\overline{M}) \\ &\quad\; -2i\textstyle\sum_i\hat{\gamma}_i(\overline{M}^i\mathcal{D}_\mu M^i - M^i\mathcal{D}_\mu\overline{M}^i).\end{aligned} \qquad (4.54)$$

The scalar field $\hat{C}$ has background value proportional to y, the five-brane location along S^1. In our units with $2\pi R_{S^1} = 1 = V_6$, the four-dimensional gravitational constant is $\kappa_4 = \kappa$ and

$$\langle\hat{C}\rangle = \frac{1}{\kappa_4}yR_{S^1} = \frac{y}{2\pi\kappa}. \qquad (4.55)$$

In order to derive the identification of the matter scalars $\mathcal{A}^{m(i)}$ of eq. (4.27) and the superconformal multiplets $M^{m(i)}$, we note that the component $G_{\mu i\bar{j}y}$ of the four-form is related to the vector component of the real multiplet V_T, which is

$$(V_T)_\mu = -2[\partial_\mu\,\mathrm{Im}\,T + i\overline{M}(\mathcal{D}_\mu M) - i(\mathcal{D}_\mu\overline{M})M].$$

On the other hand, using eqs. (4.26) and (4.27), we also find

$$\begin{aligned} G_{\mu i \bar{j} y} &= \partial_\mu C_{i\bar{j}y} - \frac{(4\pi\kappa^2)^{1/3}}{\pi} \sum_k \omega_{k,\mu i \bar{j}} \\ &= \frac{1}{2\pi}\,(4\pi\kappa^2)^{-2/3} \left(\partial_\mu \,\mathrm{Im}\, T - i\frac{\kappa^2}{2\pi}\left[\mathcal{A}^{(i)}\mathcal{D}_\mu\overline{\mathcal{A}}^{(i)} - \overline{\mathcal{A}}^{(i)}\mathcal{D}_\mu\mathcal{A}^{(i)}\right]\right)\delta_{i\bar{j}}\,. \end{aligned}$$

By comparison, one obtains

$$\mathcal{A}^{(i)} = \frac{\sqrt{2\pi}}{\kappa}\, M^{(i)}, \tag{4.56}$$

with an irrelevant sign choice.

Finally, the gauge fields in Ω_i or $\mathcal{W}^{a(i)}$ are the massless modes $\mathcal{A}_\mu^{a(i)}$.

4.4.1 Gauge coupling constants

The effective supergravity Lagrangian predicts a very specific moduli dependence in the super-Yang-Mills sector relevant to the determination of the effective superpotential with non-perturbative configurations. The field-dependent gauge couplings can of course be obtained in any formulation of the theory. But the closest relation to the higher-dimensional origin of the effective supergravity is realized with supermultiplets L (for the string dilaton multiplet), T (Calabi-Yau volume modulus) and $\hat{L}$ (for the five-brane S^1/Z_2 modulus). This is the theory defined by eq. (4.47) in which tensor calculus leads to the following (inverse squared) gauge couplings:

$$\begin{aligned} \frac{1}{g_i^2} &= \frac{1}{2}\left(\frac{z_0\overline{z}_0(T+\overline{T}-2\overline{M}M)}{2C}\right)^{3/2} + \frac{\tau}{2}(T+\overline{T}-2\overline{M}M)[\hat{C}^2 + 4\hat{\beta}_i\hat{C}] \\ &\quad +\beta_i \,\mathrm{Re}\, T + \tfrac{1}{2}\left[\epsilon|\alpha M^3|^2 - 2\tau \textstyle\sum_j (\hat{\delta}_j - \hat{\beta}_i\hat{\gamma}_j)\overline{M}^j M^j\right]. \end{aligned} \tag{4.57}$$

The complex scalar z_0 is the lowest component of the compensating multiplet S_0. In the Poincaré theory, it is a function of the physical scalars chosen to obtain a specific "gravity frame". The Einstein frame where the gravity Lagrangian is $-\frac{1}{2\kappa_4^2}e_4 R_4$ corresponds to[22]

$$2\kappa_4^2 C = \left(\frac{z_0\overline{z}_0(T+\overline{T}-2\overline{M}M)}{2C}\right)^{-3/2}. \tag{4.58}$$

Without branes or threshold corrections the dimensionless field $4\kappa_4^2 C$ is then the (universal) gauge coupling g_i^2.

On the other hand, the chiral version of the theory has gauge kinetic functions (4.53) and then:

$$\frac{1}{g_i^2} = \mathrm{Re}(S + \beta_i T + \tau\hat{\beta}_i\hat{S}). \tag{4.59}$$

The equality of these two expressions is encoded in the duality transformations exchanging linear multiplets L and $\hat{L}$ with S and $\hat{S}$:

$$\begin{aligned} \tfrac{1}{2}(\hat{S}+\overline{\hat{S}}) &= 2(T+\overline{T}-2\overline{M}e^{2A}M)\hat{V} + 2\textstyle\sum_i \hat{\gamma}_i \overline{M}^i e^{2A} M^i, \\ \tfrac{1}{2}(S+\overline{S}) &= \tfrac{1}{2}\left(\tfrac{S_0\overline{S}_0(T+\overline{T}-2\overline{M}e^{2A}M)}{2V}\right)^{3/2} + \tfrac{1}{2}\left[\epsilon|\alpha M^3|^2 - 2\tau\textstyle\sum_i \hat{\delta}_i \overline{M}^i e^{2A} M^i\right] \\ &\quad + \tfrac{\tau}{2}(T+\overline{T}-2\overline{M}e^{2A}M)\hat{V}^2, \end{aligned} \tag{4.60}$$

[22] See for instance [99, 100].

with V and $\hat{V}$ as in eqs. (4.46). The lowest scalar components of these two superfield equations show the equality of (4.57) and (4.59).

The "natural" definition of the dilaton φ with kinetic Lagrangian $(\partial_\mu \ln\varphi)^2/4$ is to identify

$$-\ln(S+\overline{S}-\Delta) \quad \longleftrightarrow \quad -\ln(2\varphi)$$

in the Kähler potential (4.49), *i.e.*

$$\begin{aligned}\varphi &= \frac{1}{2}\left(\frac{z_0\overline{z}_0(T+\overline{T}-2\overline{M}M)}{2C}\right)^{3/2} \\ &= \operatorname{Re} S - \frac{\tau}{32}\frac{[\hat{S}+\overline{\hat{S}}-4\sum_i\hat{\gamma}_i\overline{M}^i M^i]^2}{T+\overline{T}-2\overline{M}M} + \tau\sum_i \hat{\delta}_i\overline{M}^i M^i - \frac{1}{2}\epsilon|\alpha M^3|^2 .\end{aligned} \tag{4.61}$$

From now on, we omit charged matter terms, as we expect $\langle M\rangle = 0$. In terms of the dilaton, the gauge couplings read then

$$\begin{aligned}\frac{1}{g_i^2} &= \varphi + \operatorname{Re} T\left[\tau\hat{C}^2 + 4\tau\hat{\beta}_i\hat{C} + \beta_i\right] \\ &= \varphi + \operatorname{Re} T\Big[\frac{\tau}{16}\left(\frac{\operatorname{Re}\hat{S}}{\operatorname{Re} T}\right)^2 + \tau\hat{\beta}_i\frac{\operatorname{Re}\hat{S}}{\operatorname{Re} T} + \beta_i\Big].\end{aligned} \tag{4.62}$$

They display a universal[23] correction quadratic in the five-brane location, as well as gauge group-dependent corrections linear in $\hat{C}$ or constant. The chiral version has only terms linear in S, T and $\hat{S}$: the universal quadratic correction has been absorbed in the non-harmonic redefinition turning the dilaton φ into $\operatorname{Re} S$. And of course the quadratic term reappears in the Kähler potential for S [see eqs. (4.49) and (4.52)].

We now restore the summation over several five-branes and split the coefficients β_i according to $\beta_i = \beta_i^{(pl.)} + \beta_i^{(br.)}\sum_I \tau_I$ since they receive in general contributions from both planes and five-branes.[24] Using the identification (4.55), one obtains

$$\frac{1}{g_i^2} = \varphi + \beta_i^{(pl.)}\operatorname{Re} T + \operatorname{Re} T\sum_I \tau_I\Big[\beta_i^{(br.)} + \frac{4}{\kappa}\hat{\beta}_i\left(\frac{y_I}{2\pi}\right) + \frac{1}{\kappa^2}\left(\frac{y_I}{2\pi}\right)^2\Big]. \tag{4.63}$$

Notice that since the exchange $1 \leftrightarrow 2$ of the plane indices is equivalent to moving the five-brane from y_I to $\pi - y_I$, we expect

$$\hat{\beta}_1 + \hat{\beta}_2 = -\frac{1}{4\kappa}, \qquad\qquad \beta_2^{(br.)} = \beta_1^{(br.)} + \frac{2}{\kappa}\hat{\beta}_1 + \frac{1}{4\kappa^2}.$$

The next step is to compare these results with the terms obtained from the reduction of the topological term and especially with the brane contributions described by the first term in eq. (4.32):

$$\begin{aligned}e_4^{-1}\,\Delta\mathcal{L}_{brane} &= \frac{1}{4}\sum_I\frac{\tau_I}{\kappa^2}\operatorname{Im}T\Bigg\{\left[\left(\frac{y_I}{2\pi}\right)^2 - \frac{y_I}{2\pi} + \frac{1}{6}\right]\operatorname{Tr}F^{(1)}\widetilde{F}^{(1)} \\ &\qquad + \left[\left(\frac{y_I}{2\pi}\right)^2 - \frac{1}{12}\right]\operatorname{Tr}F^{(2)}\widetilde{F}^{(2)}\Bigg\}.\end{aligned} \tag{4.64}$$

[23] *i.e.* identical for all group factors, all values of index i.

[24] The constants $\beta_i^{(pl.)}$, $\beta_i^{(br.)}$ and $\hat{\beta}_i$ should not depend on I.

The terms of order $(y_I)^2$ have identical coefficient for both planes. If our effective supergravity is correct, this contribution should appear in the DBI term appearing in the second line of Lagrangian (4.47). The vector component of the Chern-Simons multiplet Ω is

$$[\Omega]_{vector} \equiv \Omega^\mu = \frac{1}{8e_4} \varepsilon^{\mu\nu\rho\sigma} \omega_{\nu\rho\sigma} + ...$$

and the component expansion of (4.47) includes then

$$-4e_4 \, \tau_I \, \hat{C}_I^2 \, (\partial_\mu \, \mathrm{Im} \, T) \sum_i \Omega_i^\mu + \dots$$

Integrating by part for constant values $\hat{C}_I = \langle \hat{C}_I \rangle = y_I/(2\pi\kappa)$, we obtain

$$\frac{1}{4} e_4 \frac{\tau_I}{\kappa^2} \left(\frac{y_I}{2\pi} \right)^2 \, \mathrm{Im} T \sum_i \mathrm{Tr} \; F_{\mu\nu}^{(i)} \widetilde{F}^{(i)\mu\nu} + \dots \tag{4.65}$$

which fits correctly the quadratic term in (4.64). It is then not needed to perform the Calabi-Yau reduction of the five-brane DBI Lagrangian to find the kinetic terms of the five-brane modulus superfield: knowledge of the superfield structure, eq. (4.47), and of the gauge terms implied by the topological term is sufficient. Similarly, the terms of order y_I and $(y_I)^0$ in the effective supergravity and in the reduction of the topological term can be used to find the values of the coefficients $\beta_i^{(2)}$ and $\hat{\beta}_i$.

The second line of Lagrangian (4.47) indicates that the DBI contribution also includes gauge terms of order $\hat{C}$, which after partial integration read

$$\frac{1}{\kappa} \, \mathrm{Im} \, T \sum_I \tau_I \, \frac{y_I}{2\pi} \sum_i \hat{\beta}_i \; \mathrm{Tr} \; F_{\mu\nu}^{(i)} \widetilde{F}^{(i)\mu\nu}. \tag{4.66}$$

By comparison with (4.64), we find

$$\hat{\beta}_1 = -\frac{1}{4\kappa}, \qquad \hat{\beta}_2 = 0. \tag{4.67}$$

Finally, comparison of the y_I–independent terms in eq. (4.64) with the first term of (4.44), which includes

$$-2 \sum_i \left[\beta_i \Omega_i V_T \right]_D = \frac{1}{4} \, \mathrm{Im} \, T \sum_i \beta_i \, \mathrm{Tr} \, F_{\mu\nu}^{(i)} \widetilde{F}^{(i)\mu\nu} + \dots, \tag{4.68}$$

indicates that

$$\beta_1^{(br.)} = \frac{1}{6\kappa^2}\,, \qquad \beta_2^{(br.)} = -\frac{1}{12\kappa^2}\,. \tag{4.69}$$

As expected, exchanging planes $1 \leftrightarrow 2$ is equivalent to $y_I \leftrightarrow \pi - y_I$.

Finally, as usual, the coefficients $\beta_i^{(pl.)}$ can be read directly from the first line of eq. (4.30), which includes contributions to the topological terms arising from the hyperplanes only:

$$\beta_i^{(pl.)} = \frac{2}{(4\pi)^4 \kappa^2} \sum_j d_{ij} \, \langle \mathcal{I}_j \rangle, \tag{4.70}$$

or

$$\beta_1^{(pl.)} = \frac{1}{6(4\pi)^3 \kappa^2} \Big(2\langle \mathcal{I}_1 \rangle - \langle \mathcal{I}_2 \rangle \Big), \qquad \beta_2^{(pl.)} = \frac{1}{6(4\pi)^3 \kappa^2} \Big(2\langle \mathcal{I}_2 \rangle - \langle \mathcal{I}_1 \rangle \Big). \tag{4.71}$$

Notice that

$$\beta_1^{(pl.)} + \beta_2^{(pl.)} = \frac{1}{6(4\pi)^3 \kappa^2} \Big(\langle \mathcal{I}_1 \rangle + \langle \mathcal{I}_2 \rangle \Big)$$

vanishes in the minimal embedding without five-brane [97, 150].

To summarize, in terms of the dilaton, the gauge couplings on both planes, as calculated from the modified topological term, read:

$$\begin{aligned} \frac{1}{g_1^2} &= \varphi + \beta_1^{(pl.)}\,\mathrm{Re}\,T + \frac{1}{\kappa^2}\,\mathrm{Re}\,T\sum_I \tau_I\left[\left(\frac{y_I}{2\pi}\right)^2 - \left(\frac{y_I}{2\pi}\right) + \frac{1}{6}\right], \\ \frac{1}{g_2^2} &= \varphi + \beta_2^{(pl.)}\,\mathrm{Re}\,T + \frac{1}{\kappa^2}\,\mathrm{Re}\,T\sum_I \tau_I\left[\left(\frac{y_I}{2\pi}\right)^2 - \frac{1}{12}\right]. \end{aligned} \tag{4.72}$$

A nicer expression reminiscent of eqs. (4.20) or (4.23) uses the distance from the brane to the planes:

$$\begin{aligned} \frac{1}{g_1^2} &= \varphi + \beta_1^{(pl.)}\,\mathrm{Re}\,T + \frac{1}{4\kappa^2}\,\mathrm{Re}\,T\sum_I \tau_I\left[(\Delta_{I,2})^2 - \frac{1}{3}\right], \\ \frac{1}{g_2^2} &= \varphi + \beta_2^{(pl.)}\,\mathrm{Re}\,T + \frac{1}{4\kappa^2}\,\mathrm{Re}\,T\sum_I \tau_I\left[(\Delta_{I,1})^2 - \frac{1}{3}\right]. \end{aligned} \tag{4.73}$$

where $\Delta_{I,1} = y_I/\pi$, and $\Delta_{I,2} = 1 - y_I/\pi$, as in eq. (4.18). The contribution of a five-brane to the gauge couplings on one hyperplane decreases quadratically from a maximum value

$$\frac{1}{6\kappa^2}\tau_I\,\mathrm{Re}\,T$$

when the brane lies on the plane, to a minimal value

$$-\frac{1}{12\kappa^2}\tau_I\,\mathrm{Re}\,T$$

when the five-brane lies on the opposite plane. For a five-brane in the middle of the interval, both gauge couplings receive the correction

$$-\frac{1}{48\kappa^2}\tau_I\,\mathrm{Re}\,T.$$

With however $(\Delta_{I,i})^2 = -\Delta_{I,1}\Delta_{I,2} + \Delta_{I,i}$, $(i = 1, 2)$, the term quadratic in y is necessarily universal and the two (inverse squared) gauge couplings differ only by a contribution linear in y:

$$\frac{1}{g_1^2} - \frac{1}{g_2^2} = \frac{\mathrm{Re}\,T}{2(4\pi)^3\kappa^2}\left[\langle\mathcal{I}_1\rangle - \langle\mathcal{I}_2\rangle + \sum_I a_I\left(1 - \frac{2y_I}{\pi}\right)\right], \tag{4.74}$$

with $\Delta_{I,2} - \Delta_{I,1} = 1 - 2y_I/\pi$ and in terms of the area a_I of the two-cycle wrapped by the five-brane in K_6 [see eq. (4.42)].

The normalization of the four-forms $I_{4,1}$ and $I_{4,2}$ is such that their integrals over a four-cycle in K_6 are half-integers. Similary a_I is an intersection number of the two-cycle wrapped by the brane with the four-cycle, in units of the Calabi-Yau volume.[25] These statements follow from the integrated Bianchi identity verified by G_4 and from rewriting eqs. (4.31) and (4.33) in the form

$$a_I = -i\int_{K_6}\delta^{(4)}(\mathcal{W}_{6,I})\wedge dz\wedge d\overline{z}, \qquad \langle\mathcal{I}_i\rangle = -i\int_{K_6}\langle I_i\rangle\wedge dz\; d\overline{z}.$$

Eq. (4.74) matches then nicely the idea that a five-brane moved to the hyperplane at $y = 0$ decreases the instanton number on this plane, or on the second plane when moved to $y = \pi$.

[25] We have chosen $V_6 = 1$.

In the chiral version of the theory, the gauge kinetic functions are

$$\begin{aligned} f_2 &= S + \beta_2^{(pl.)} T - \frac{1}{12\kappa^2} \sum_I \tau_I T, \\ f_1 &= f_2 + \left[\beta_1^{(pl.)} - \beta_2^{(pl.)}\right] T + \frac{1}{4\kappa^2} \sum_I \tau_I \left[T - \kappa \hat{S}\right]. \end{aligned} \tag{4.75}$$

Since

$$\langle \mathrm{Re}\, T\rangle - \kappa \langle \mathrm{Re}\, \hat{S}\rangle = \langle \mathrm{Re}\, T\rangle \Big(1 - \frac{2y}{\pi}\Big),$$

the difference is again eq. (4.74).

4.4.2 Discussion of some matter terms

We have seen that the gauge part of the five-brane-induced contributions to the topological terms are due, in the effective supergravity, either to the effect of the modified Bianchi identities on the Dirac-Born-Infeld Lagrangian, or to threshold corrections. Since charged matter multiplets M arise from the fields living on the fixed hyperplanes, as do gauge multiplets, we may expect that some or all matter contributions obtained from the reduction of the topological term can also be derived from the DBI effective Lagrangian.

As an illustration, we will establish that the charged matter term in expression (4.32) arises for its part quadratic in y_I from the DBI Lagrangian, while the terms linear and constant in y_I originate from threshold corrections. Since the vector component of V is related to the string antisymmetric tensor B_{AB} by

$$v^\mu = -\frac{2\pi}{8e_4} \epsilon^{\mu\nu\rho\sigma} \partial_\nu B_{\rho\sigma}\,, \tag{4.76}$$

the string–matter coupling term in eq. (4.32) takes the form

$$\begin{aligned} &-\frac{i}{8(4\pi)^3} \sum_{I,i} a_I \left(\Delta^2_{I,i} - \frac{1}{3}\right) \epsilon^{\mu\nu\rho\sigma} \partial_\mu B_{\nu\rho} \sum_m [\mathcal{A}^{m(i)}(\mathcal{D}_\sigma \overline{\mathcal{A}}^{m(i)}) - \overline{\mathcal{A}}^{m(i)}(\mathcal{D}_\sigma \mathcal{A}^{m(i)})] \\ &\qquad = -\frac{2i}{\kappa^2} e_4 \sum_I \tau_I v^\mu \Big\{ \left(\frac{y_I}{2\pi}\right)^2 [M(\mathcal{D}_\mu \overline{M}) - \overline{M}(\mathcal{D}_\mu M)] \\ &\qquad\qquad + \sum_i \tilde{g}_i(y_I)[M^i(\mathcal{D}_\mu \overline{M}^i) - \overline{M}^i(\mathcal{D}_\mu M^i)] \Big\}. \end{aligned} \tag{4.77}$$

The functions $\tilde{g}_i(y_I)$ are at most linear in y_I:

$$\tilde{g}_1(y_I) = -\frac{y_I}{2\pi} + \frac{1}{6}\,, \qquad \tilde{g}_2(y_I) = -\frac{1}{12}\,. \tag{4.78}$$

The first term in the *r.h.s.* of eq. (4.77) is universal and can clearly be retrieved from the DBI Lagrangian in the second line of theory (4.47) by selecting the matter contribution inside V_T,

$$-2ie_4\tau_I\, \hat{C}_I^2\, v^\mu [M(\mathcal{D}_\mu \overline{M}) - \overline{M}(\mathcal{D}_\mu M)] + \dots\,, \tag{4.79}$$

and identifying as usual $\hat{C}^2$ with $y_I^2(2\pi\kappa)^{-2}$.

The second term in the *r.h.s.* of eq. (4.77) originates from matter threshold corrections in the supergravity Lagrangian (4.47). The relevant term is:

$$-2ie_4\tau_I v^\mu \sum_i g_i(\hat{C})[M^i(\mathcal{D}_\mu \overline{M}^i) - \overline{M}^i(\mathcal{D}_\mu M^i)] + \dots \tag{4.80}$$

Comparison with eq. (4.77) indicates that

$$\hat{\gamma}_1 = -\frac{1}{\kappa} \qquad \hat{\gamma}_2 = 0\,, \qquad \hat{\delta}_1 = \frac{1}{6\kappa^2}\,, \qquad \hat{\delta}_2 = -\frac{1}{12\kappa^2}\,. \tag{4.81}$$

Interestingly enough, $\hat{\delta}_i = \beta_i^{(br.)}$ and $\hat{\gamma}_i = 4\hat{\beta}_i$.

Finally, we briefly return to the issue of tracing back the supersymmetric origin of interactions such as expressions (4.35) and (4.36). As already mentioned in Section 3, eq. (4.35) seems in general hard to cast in a supersymmetric form because of the complicated mixing of matter contributions from both hyperplanes. This feature is also present in eq. (4.36), forbidding by the same token its supersymmetrization for a generic background. We however have little information on the nature of four-dimensional matter counter-terms which could be added to anomaly-cancelling corrections and could radically change the picture. At this level of understanding, this discussion cannot be conclusive.

Nevertheless, eq. (4.35) allows a supersymmetric formulation for the standard embedding in the gauge group in which chiral matter multiplets only appear on one plane, say $\mathcal{A} \equiv \mathcal{A}_2$. Then, eq. (4.35) reduces to

$$\begin{aligned} \Delta\mathcal{L}_{plane} &= i\,\epsilon^{\mu\nu\rho\sigma}\,\omega_{1,\mu\nu\rho}\Big\{\frac{1}{3(4\pi)^5}\,[\lambda\mathcal{A}^3\,\partial_\sigma(\overline{\lambda\mathcal{A}}^3) - \overline{\lambda\mathcal{A}}^3\,\partial_\sigma(\lambda\mathcal{A}^3)] \\ &\quad -\frac{1}{12(4\pi)^3}\sum_m(\mathcal{A}^m\mathcal{D}_\sigma\overline{\mathcal{A}}^m - \overline{\mathcal{A}}^m\mathcal{D}_\sigma\mathcal{A}^m)\,(\langle\mathcal{I}_1\rangle - \langle\mathcal{I}_2\rangle)\Big\}\,, \end{aligned} \tag{4.82}$$

which extends to the supersymmetric density

$$\frac{1}{3(4\pi)^2\kappa^6}\,\left[\Omega_1\,|\alpha M^3|^2\right]_D - \frac{1}{3(4\pi)^2\kappa^2}\,(\langle\mathcal{I}_1\rangle - \langle\mathcal{I}_2\rangle)\left[\Omega_1\,\overline{M}e^{2A}M\right]_D. \tag{4.83}$$

The second term could in principle correspond to the first contribution appearing in the threshold correction (4.44).

4.5 Condensation, the non-perturbative superpotential and membrane instantons

The non-perturbative superpotential arises from the F–density in the supergravity Lagrangian (4.39) when some or all gauge fields condense. It can be evaluated using a standard two-step procedure: first obtain the effective action for condensates and then eliminate the condensate by solving (approximately in general) its field equation. Computing the effective action amounts in principle to couple the superfield $\mathcal{WW}$ to an external source, integrate the gauge fields and perform the Legendre transformation exchanging the source field with the (classical) condensate field. It is well known that the symmetry content of super-Yang-Mills theory and anomaly-matching are sufficiently restrictive to accurately describe condensation [242].

As usual, we assume that the gauge multiplet which first condenses does not couple to matter multiplets M. We then split the gauge group into $\mathcal{G}_0 \times \prod_a \mathcal{G}_a$, where the simple group $\mathcal{G}_0$ condenses and matter multiplets only tranform under $\prod_a \mathcal{G}_a$. The terms involving $\mathcal{G}_0$ gauge fields in the Lagrangian reduce then to

$$\frac{1}{4}\left[(S + \beta_0 T + \tau\hat{\beta}_0\hat{S})\mathcal{W}_0\mathcal{W}_0\right]_F. \tag{4.84}$$

Following ref. [242], these contributions are simply replaced in the effective action for condensates by

$$\frac{1}{4}\left[(S + \beta_0 T + \tau\hat{\beta}_0\hat{S})U + \frac{b_0}{24\pi^2}\left\{U\ln\left(\frac{U}{\mu^3}\right) - U\right\}\right]_F + \left[S_0\overline{S}_0\,\widetilde{K}\left(US_0^{-3}, \overline{US_0}^{-3}\right)\right]_D, \tag{4.85}$$

where U is the (classical) chiral superfield ($w = n = 3$) describing the gaugino condensate, $\langle U\rangle = \langle\lambda\lambda\rangle$. The coefficient of the Veneziano-Yankielowicz superpotential is dictated by the anomaly of the superconformal chiral $U(1)$. It is proportional to the one-loop beta function, $b_0 = 3C(\mathcal{G}_0)$, and the scale parameter μ is the energy at which gauge couplings in expressions (4.73) are defined. Finally, the leading contribution to the Kähler potential $\widetilde{K}$ is controlled by the scaling dimension (Weyl weight) of U.

The effective Lagrangian with condensate U can be obtained by collecting all terms in the "microscopic" Lagrangian (4.39), with contributions (4.84) replaced by the effective terms (4.85):

$$\begin{aligned}\mathcal{L}_{eff.} \;=\;& \Big[-\{S_0\overline{S}_0(T+\overline{T}-2\overline{M}e^{2A}M)\}^{3/2}(2V)^{-1/2} - (S+\overline{S})V \\ & -\tau V\hat{V}^2(T+\overline{T}-2\overline{M}e^{2A}M) + \tfrac{\tau}{2}(\hat{S}+\overline{\hat{S}})V\hat{V} \\ & +V\{\epsilon|\alpha M^3|^2 - 2\tau\textstyle\sum_i g_i(\hat{V})\overline{M}^i e^{2A}M^i\} + S_0\overline{S}_0\,\widetilde{K}\left(US_0^{-3},\overline{US_0^{-3}}\right)\Big]_D \\ & +\tfrac{1}{4}\Big[\textstyle\sum_a(S+\beta_aT+\tau\hat{\beta}_a\hat{S})\mathcal{W}_a\mathcal{W}_a\Big]_F \\ & +\Big[S_0^3\,(c+\alpha M^3) + \tfrac{1}{4}(S+\beta_0T+\tau\hat{\beta}_0\hat{S})U + \tfrac{C(\mathcal{G}_0)}{32\pi^2}\{U\ln(U/\mu^3)-U\}\Big]_F.\end{aligned} \tag{4.86}$$

As before, vector multiplets V and $\hat{V}$ are constrained by the field equations of Lagrange multipliers S and $\hat{S}$, which impose modified Bianchi identities. Rewriting their contributions in the form

$$\Big[2S\Big\{\Sigma(V)+\frac{1}{8}U+\frac{1}{8}\sum_a\mathcal{W}_a\mathcal{W}_a\Big\} - \tau\hat{S}\Big\{\Sigma(V\hat{V})-\frac{1}{4}\hat{\beta}_0U-\frac{1}{4}\sum_a\hat{\beta}_a\mathcal{W}_a\mathcal{W}_a\Big\}\Big]_F, \tag{4.87}$$

multiplier S implies

$$U = -8\Sigma(V_0), \qquad\qquad V = V_0 - 2\sum_a\Omega_a, \tag{4.88}$$

with a real vector multiplet V_0 (8 bosons + 8 fermions). Then, multiplier $\hat{S}$ requires

$$V\hat{V} = \hat{L}_0 - 2\hat{\beta}_0V_0 + 4\sum_a\hat{\beta}_a\Omega_a, \tag{4.89}$$

with a linear multiplet $\hat{L}_0$ and V as given in eqs. (4.88). These solutions can be compared with eqs. (4.46), which apply before condensation of $\mathcal{W}_0\mathcal{W}_0$. Clearly, the real vector multiplet V_0 describes $L-2\Omega_0|_{cond.}$, *i.e.* it includes the string coupling linear multiplet L and the condensate field [61, 62]. Similarly, the linear multiplet $\hat{L}_0$ replaces $\hat{L}+2\hat{\beta}_0L$.

To obtain the non-perturbative superpotential however, one first chooses the formulation of the theory with chiral multiplets only. The elimination of vector multiplets V and $\hat{V}$ is as in the previous section. Omitting from here on gauge fields $\mathcal{W}_a$ and charged matter fields M, the chiral formulation of the effective Lagrangian is

$$\begin{aligned}\mathcal{L}_{eff.} \;=\;& \Big[-\tfrac{3}{2}S_0\overline{S}_0\,e^{-\mathcal{K}/3} + S_0\overline{S}_0\widetilde{K}\Big]_D \\ & +\Big[S_0^3\,(c+\alpha M^3) + \tfrac{1}{4}(S+\beta_0T+\tau\hat{\beta}_0\hat{S})U + \tfrac{C(\mathcal{G}_0)}{32\pi^2}\{U\ln(U/\mu^3)-U\}\Big]_F.\end{aligned} \tag{4.90}$$

The Kähler potential $\mathcal{K}$ is as in eqs. (4.49) and (4.52), with full mixing of multiplets S, T and $\hat{S}$. To derive the non-perturbative effective potential, neglect $\widetilde{K}$.[26] The field equation of the

[26]Disregarding the Kähler potential $\widetilde{K}$ is the same as considering U as a constant background field with value chosen to extremize the action.

condensate field U implies then

$$U = \mu^3 \exp\left(-\frac{8\pi^2}{C(\mathcal{G}_0)}[S + \beta_0 T + \tau\hat{\beta}_0\hat{S}]\right) \equiv \mathcal{U}, \qquad (4.91)$$

and the effective superpotential becomes

$$W_{np} = c + \alpha M^3 - \frac{C(\mathcal{G}_0)}{32\pi^2}\mathcal{U}\, S_0^{-3}. \qquad (4.92)$$

This superpotential is the *sum* of the 'microscopic' superpotential and the non-perturbative contribution of the gauge condensate. The non-perturbative contribution is the *exponential of the sum* of the string coupling, Calabi-Yau volume and five-brane moduli contributions. With several condensates, the non-perturbative piece would be replaced by a *sum of similar terms* over all gauge simple factors which condense.

To get a qualitative picture of the effect of the five-brane, use eqs. (4.75) and (4.74) to rewrite the condensate as a function of the five-brane position along S^1, assuming first that the condensate arises on the hyperplane at $y = 0$:

$$\begin{aligned} |U|_{1st\ plane} &= |\mu|^3 \exp\left(-\tfrac{8\pi^2}{C(\mathcal{G}_0)}\left[\mathrm{Re}\, S + \tfrac{1}{(4\pi)^3\kappa^2}\,\mathrm{Re}\, T\left\{\tfrac{1}{3}\langle\mathcal{I}_1\rangle - \tfrac{1}{6}\langle\mathcal{I}_2\rangle + \tfrac{1}{3}a - \tfrac{y}{\pi}a\right\}\right]\right) \\ &= |\mu|^3 \exp\left(-\tfrac{8\pi^2}{C(\mathcal{G}_0)g_1^2}\right). \end{aligned} \qquad (4.93)$$

If the condensate arises on the hyperplane at $y = \pi$:

$$\begin{aligned} |U|_{2nd\ plane} &= |\mu|^3 \exp\left(-\tfrac{8\pi^2}{C(\mathcal{G}_0)g_2^2}\right) \\ &= |U|_{1st\ plane} \times \exp\left(\tfrac{4\pi^2}{C(\mathcal{G}_0)}\tfrac{\mathrm{Re}\, T}{(4\pi)^3\kappa^2}\left[\langle\mathcal{I}_1\rangle - \langle\mathcal{I}_2\rangle + a(1 - \tfrac{2y}{\pi})\right]\right). \end{aligned} \qquad (4.94)$$

The non-perturbative superpotential (4.92) and the condensates (4.93) and (4.94) display the dependence on the five-brane location on S^1 expected from explicit calculations of membrane instanton corrections in the four-dimensional effective theory [193, 183, 182]. We have obtained this dependence from the analysis of the fundamental Bianchi identity of M-theory and from the (modified) topological term, showing in this way that open membrane instanton contributions find their higher-dimensional origin in anomaly-cancellation in the presence of five-branes.

This observation has a second consequence. The non-perturbative superpotential is not a sum of exponential terms generated by gaugino condensates and membrane instantons, a structure which is not in any case expected to appear in the effective supergravity. Instead, we find a non-perturbative term which is the exponential of a sum of terms linear in the chiral fields, a structure characteristic of threshold corrections induced by anomaly-cancellation in higher dimensions.

In our reduction scheme, the "microscopic" superpotential $c + \alpha M^3$ is moduli-independent. It is however known that T-duality induces a holomorphic dependence on T compatible with our supermultiplet structure as described in Lagrangian (4.86). The existence of dual descriptions of moduli S and $\hat{S}$ in terms of constrained vector multiplets V and $\hat{V}$ or in terms of linear multiplets L and $\hat{L}$ implies that the "microscopic" superpotential cannot depend on S or $\hat{S}$.

4.5.1 The scalar potential

Because of the mixing of the three moduli multiplets S, T and $\hat{S}$ in the Kähler metric, the scalar potential present in the component expansion of the effective Lagrangian (4.86) is not positive and analysing its vacuum structure is a severe problem. This mixing is due to the kinetic terms of the five-brane massless modes, it is unavoidable whenever five-branes, and then whenever $\hat{S}$, are present.

We may however gain insight by deriving the scalar potential directly in terms of the components of the constrained vector multiplets. This version of the theory is indeed more tractable than the chiral one since the mixing of moduli fields is simpler. The relevant multiplets are then T, V, $\hat{V}$ for the moduli and U for the condensate. Charged matter terms are as before omitted since we are interested in vacua where they vanish. In the Einstein frame (4.58), the relation between the dilaton (4.61) and the lowest scalar component C of V is

$$\varphi = \frac{1}{4\kappa_4^2 C} \tag{4.95}$$

and the scalar potential is eventually expressed in terms of the physical dilaton. As explained in eqs. (4.88), eliminating S with Ω_a omitted generates the modified Bianchi identity $U = -8\Sigma(V)$, where V is the vector field describing $L - 2\Omega_0|_{cond.}$. To derive the potential, we only need the scalar components of V and U,

$$\begin{aligned} V &= (C, 0, H, K, 0, 0, d), \qquad & U &= (u, 0, f_U), \\ u &= -4(H - iK), & f_U &= 4d. \end{aligned} \tag{4.96}$$

Since a non-zero condensate u also switches on the field $H - iK$ of the dilaton multiplet V, the gaugino condensate clearly breaks supersymmetry in this sector, as expected in a theory where the dilaton couples to gauge fields. Eliminating $\hat{S}$ defines the five-brane (effective) linear multiplet $V\hat{V} = \hat{L}_0 - 2\hat{\beta}_0 V$ and, since linear multiplets do not have auxiliary fields, we may simply write

$$\hat{L}_0 = (\, C\hat{C} + 2\hat{\beta}_0 C \,, 0, 0, 0, 0, 0, 0\,),$$

in terms of the lowest scalar component $\hat{C}$ of $\hat{V}$, when deriving the scalar potential. The resulting scalar potential is then a function of the physical scalars C (the dilaton, see. eq. 4.95), the S^1/Z_2 modulus $\hat{C} = y/(2\pi\kappa)$, the Calabi-Yau volume modulus T and the gaugino condensate u. It is also a function of the auxiliary fields d, f_T and f_0 (in the compensating multiplet S_0) which can be easily eliminated.

The Kähler potential $\widetilde{K}$ generates a term quadratic in d. We will write the scalar potential by restricting $\widetilde{K}$ to its leading term [242]

$$\widetilde{K}(US_0^{-3}, \overline{US_0}^{-3}) S_0 \overline{S}_0 = A(U\overline{U})^{1/3}\,, \tag{4.97}$$

with an arbitrary normalisation constant $A \geq 0$. The scalar potential as a function of d, C, $\hat{C}$, T and u reads then:

$$\begin{aligned} V_{eff.} \;=\; & -\frac{32}{9} A(u\overline{u})^{-2/3} d^2 - Bd \\ & + \frac{u\overline{u}}{16C}\left[\frac{3}{2}\frac{1}{4\kappa_4^2 C} + \tau(\hat{C} + 2\hat{\beta}_0)^2 (T + \overline{T})\right] \\ & + \frac{2\kappa_4^2 C}{\kappa_4^4 (T+\overline{T})^3}\left\{ -2\left| W + \frac{1}{8}\kappa_4^3\, u \left(\frac{T+\overline{T}}{2\kappa_4^2 C}\right)^{3/2}\right|^2 \right. \\ & \left. + \frac{(T+\overline{T})^2}{3}\left| W_T - \frac{3}{T+\overline{T}} W + \frac{1}{4}\kappa_4^3\, u \frac{(T+\overline{T})^{3/2}}{(2\kappa_4^2 C)^{1/2}} \left(\beta_0 + \tau[\hat{C}^2 + 4\hat{\beta}_0 \hat{C}]\right)\right|^2 \right\}. \end{aligned} \tag{4.98}$$

The first two terms arise respectively from the condensate Kähler potential term $[S_0\overline{S}_0\widetilde{K}]_D = A[(U\overline{U})^{1/3}]_D$ and from the condensate F–density

$$[W_{cond.}]_F \equiv \Big[\frac{1}{4}(S+\beta_0 T+\tau\hat{\beta}_0\hat{S})U+\frac{b_0}{96\pi^2}\{U\ln(U/\mu^3)-U\}\Big]_F.$$

The coefficient B relates the gaugino condensate field u and the gauge coupling g_0^2, as defined in eq. (4.62):

$$B=\frac{2}{g_0^2}+\frac{b_0}{24\pi^2}\ln\left(\frac{u\overline{u}}{\mu^6}\right)=8\,\mathrm{Re}\,\frac{\partial}{\partial u}W_{cond.}\,. \tag{4.99}$$

The standard field-theory value of the condensate,

$$|u|=\mu^3\exp\left(-\frac{24\pi^2}{b_0 g_0^2}\right), \tag{4.100}$$

is obtained if $B=0$ is part of the scalar potential vacuum equations.

In eq. (4.98), the second line is proportional to $|H-iK|^2$ and the fourth line to $|f_T|^2$, and we have included the possibility of a T–dependent perturbative superpotential, as often implied by T–duality, even if our reduction scheme predicts $W_T=\frac{\partial W}{\partial T}=0$. The dependence of the potential on the five-brane position $\hat{C}=y/(2\pi\kappa)$ is best understood by defining the *distance* Δ_c *from the brane to the condensate*:

$$\begin{array}{ll}\text{For a condensate on plane 1:} & \Delta_c=y/\pi=2\kappa\,\hat{C}.\\[2ex] \text{For a condensate on plane 2:} & \Delta_c=1-y/\pi=1-2\kappa\,\hat{C}.\end{array} \tag{4.101}$$

Using then the values of the threshold coefficients found in eqs. (4.67) and (4.69), we find in both cases:

$$\begin{aligned}\beta_0+\tau[\hat{C}^2+4\hat{\beta}_0\hat{C}] &= \beta_0^{(pl.)}+\frac{\tau}{4\kappa^2}\Big[(1-\Delta_c)^2-\frac{1}{3}\Big],\\ (\hat{C}+2\hat{\beta}_0)^2 &= \frac{1}{4\kappa^2}(1-\Delta_c)^2.\end{aligned} \tag{4.102}$$

These results agree with the dependence on the five-brane location found in gauge couplings (4.73).

At this stage, we have two options. We may neglect the Kähler potential $\widetilde{K}$ and assume $A=0$. Then, the auxiliary field d imposes the field equation $B=0$ and the correct value of the gaugino condensate. This procedure is equivalent to the derivation of the effective non-perturbative superpotential (4.92). The field equation $B=0$ allows to eliminate u and to express the effective potential, which does not include the first line in expression (4.98), as a function of the moduli scalars C, $\hat{C}$ and T only.

Instead, with a non-zero Kähler potential $\widetilde{K}$ (*i.e.* with $A>0$), solving for the auxiliary d turns the first two terms of the scalar potential (4.98) into

$$\frac{9}{128}\frac{B^2}{A}(u\overline{u})^{2/3}, \tag{4.103}$$

and a generic (non supersymmetric) stationary point of the potential does not necessarily lead to $B=0$ and to the standard gaugino condensate (4.100). But since B appears quadratically in the potential, the same stationary points with $B=0$ would exist in both cases $A=0$ and $A>0$. Notice that the condensate term (4.103) can also be written

$$\frac{1}{2}(\widetilde{K}_{u\overline{u}})^{-1}\Big(\mathrm{Re}\,\frac{\partial}{\partial u}W_{cond.}\Big)^2, \qquad \widetilde{K}_{u\overline{u}}=\frac{\partial^2}{\partial u\,\partial\overline{u}}A(u\overline{u})^{1/3}.$$

This is the potential term due to the auxiliary field f_u of the condensate chiral superfield U, with non-standard Weyl weight $w = 3$. The imaginary part of $\frac{\partial}{\partial u} W_{cond.}$ does not contribute to the potential because of the constraint $U = -8\Sigma(V)$ [61, 62].

A complete analysis of the stationary values of the scalar potential (4.98) cannot be performed analytically. In the absence of five-branes, the potential can be written in the form

$$V = \frac{1}{2}(\widetilde{K}_{u\overline{u}})^{-1}\Big(\mathrm{Re}\,\frac{\partial}{\partial u}W_{cond.}\Big)^2 + \frac{1}{\kappa_4^2}\Big[(2\kappa_4^2 C)^2|f_S|^2 + 3(T+\overline{T})^{-2}|f_T|^2\Big] - \frac{3}{\kappa_4^4}e^{\mathcal{K}}|W|^2,$$

in terms of the Kähler potential $\mathcal{K} = -\ln(S+\overline{S}) - 3\ln(T+\overline{T})$ with diagonal metric. A relatively simple study of the stationary points of the potential with for instance $B = 0$ can be performed as a function of the auxiliary fields f_S and f_T of the chiral dilaton S and the volume modulus T respectively.

But the introduction of the five-brane mode leads to mixings of the chiral superfields. For instance, according to the second superfield eq. (4.60), the auxiliary field in the chiral dilaton multiplet S reads

$$\begin{aligned} f_S &= \kappa_4^{-1}(2\kappa_4^2 C)^{-1/2}(T+\overline{T})^{-3/2}\left[\overline{W} - \tfrac{1}{4}\left(\tfrac{T+\overline{T}}{2\kappa_4^2 C}\right)^{3/2}\kappa_4^3\overline{u}\right] \\ &\quad +\tau\left[\hat{C}^2 f_T - \tfrac{1}{2}(T+\overline{T})(\hat{C}^2 + 2\hat{\beta}_0\hat{C})(2\kappa_4^2 C)^{-1}\kappa_4^2\overline{u}\right]. \end{aligned} \tag{4.104}$$

The second term is due to the five-brane and it involves the auxiliary field f_T, which is proportional to the last line in the potential (4.98). Similarly, the auxiliary component of the five-brane multiplet $\hat{S}$ is

$$f_{\hat{S}} = 4\hat{C}f_T - (T+\overline{T})(\hat{C} + 2\hat{\beta}_0)(2\kappa_4^2 C)^{-1}\kappa_4^2\overline{u}. \tag{4.105}$$

Mixings of the auxiliary fields then arise whenever $\hat{C} \neq 0$, *i.e.* whenever the five-brane does not lie on the fixed hyperplane at $y = 0$.

4.6 Conclusion

In this paper, we have studied the Calabi-Yau reduction of the low-energy limit of M-theory on the interval S^1/Z_2, with five-branes aligned to preserve four supercharges. We have used a fully consistent, four-dimensional supergravity and superfield setup and included the modulus field describing five-brane fluctuations in the interval direction. The gauge anomaly-cancelling topological term is modified as a consequence of the five-brane contributions to M-theory Bianchi identities. We have derived the new four-dimensional interactions induced by these five-brane modifications and shown that they lead to new gauge threshold corrections with a calculable dependence on the five-brane position along S^1. In particular, these threshold corrections fit nicely the change in the instanton number expected when a five-brane is moved onto one of the fixed hyperplanes. Of course, when gauge condensation occurs, these threshold corrections explicitly appear in the effective non-perturbative superpotential, with an exponential dependence of the five-brane location.

The same five-brane-dependent contributions to the non-perturbative superpotential can be obtained from a different perspective. It is expected that open membranes wrapping a two-cycle in the Calabi-Yau threefold and extending from a five-brane to a ten-dimensional hyperplane generate, in the four-dimensional effective field theory, instanton-like contributions. Instanton calculus allows to explicitly compute these instanton corrections and the resulting contributions to the non-perturbative effective superpotential. Strictly speaking however, instanton calculus only applies in specific limits, which in the case under scrutiny restricts the understanding of the global structure of the superpotential and of the interplay of the various moduli fields. It is precisely here that our effective supergravity Lagrangian, as derived from M-theory, adds important new information. In particular, since membrane instanton corrections actually originate from threshold corrections related to ten-dimensional anomaly cancellation, the non-perturbative superpotential is the exponential of a sum of terms linear in moduli chiral fields.

4.7 Appendix: conventions and notations

Our conventions are as in refs. [44] and [43]. We use the *upstairs* picture $M_4 \times K_6 \times S^1/Z_2$, where the S^1 coordinate is $x^{10} = yR$, with a (2π)–periodic angular variable y. We use $-\pi < y \leq \pi$ when explicit values are needed because of the natural action of Z_2 in this interval. Our indices convention for the $M_4 \times K_6 \times S^1/Z_2$ reduction is

$$x^M = (x^A, yR) = (x^\mu, z^i, \overline{z}^{\overline{k}}, yR), \quad M = 0, \ldots, 10, \quad A = 0, \ldots, 9, \quad i, \overline{k} = 1, 2, 3.$$

For bulk moduli, we use the terminology familiar from string compactifications: the moduli $s = \mathrm{Re}\, S$ and $t = \mathrm{Re}\, T$ with Kähler potential (4.25) are respectively related to the dilaton (or string coupling) and to the Calabi-Yau volume. This convention follows from the metric

$$ds_{11}^2 = e^{-2\phi/3} g_{AB}^{(10)}\, dx^A dx^B + e^{4\phi/3} dy^2 \,, \tag{4.106}$$

which defines the string frame and the string coupling $e^{-2\phi}$ with $R = e^{2\phi/3}$, together with

$$g_{AB}^{(10)}\, dx^A dx^B = g_{\mu\nu}\, dx^\mu dx^\nu + V^{1/3} \delta_{i\bar{j}}\, dz^i d\overline{z}^{\bar{j}} \tag{4.107}$$

which defines the Calabi-Yau volume. Rescaling $g_{\mu\nu}$ to the four-dimensional Einstein frame leads to

$$ds_{11}^2 = e^{4\phi/3} \left[V^{-1} g_{\mu\nu}\, dx^\mu dx^\nu + dy^2\right] + e^{-2\phi/3} V^{1/3} \delta_{i\bar{j}}\, dz^i dz^{\bar{j}} \,. \tag{4.108}$$

Comparison with the standard eleven-dimensional metric used to diagonalize kinetic terms in the four-dimensional $\mathcal{N} = 1$ supergravity Lagrangian,

$$ds_{11}^2 = s^{-2/3} \left[t^{-1} g_{\mu\nu}\, dx^\mu dx^\nu + t^2 dy^2\right] + s^{1/3} \delta_{i\bar{j}}\, dz^i dz^{\bar{j}} \,, \tag{4.109}$$

leads to the identifications

$$(\mathrm{Re}\, T)^3 = V \,, \qquad \mathrm{Re}\, S = V e^{-2\phi} \,. \tag{4.110}$$

Hence, t is the volume modulus while s is the dilaton or string coupling modulus.

The terminology often used in the context of M-theory defines instead another Calabi-Yau volume $\hat{V}$ in units specified by the metric (4.108), with then

$$\hat{V} \equiv V e^{-2\phi} = \mathrm{Re}\, S, \qquad\qquad (\mathrm{Re}\, T)^3 = \hat{V} e^{2\phi} = \hat{V} R^3.$$

It seemingly exchanges the respective roles of the bulk moduli.

In order to avoid duplication of contributions due to Z_2 periodicity, our eleven-dimensional supergravity action and Green-Schwarz terms are multiplied by 1/2 with respect to standard conventions in use for M_{11}:

$$\mathcal{L}_{C.J.S.} + \mathcal{L}_{G.S} = \frac{1}{4\kappa^2} \left[eR - \frac{1}{2} G_4 \wedge *G_4 - \frac{1}{6} C_3 \wedge G_4 \wedge G_4 \right] - \frac{T_2}{4\pi} G_4 \wedge X_7 \,, \tag{4.111}$$

with membrane tension $T_2 = 2\pi(4\pi\kappa^2)^{-1/3}$, and

$$dX_7 = X_8 = \frac{1}{(2\pi)^3 4!} \left[\frac{1}{8} \mathrm{tr}\, R^4 - \frac{1}{32} (\mathrm{tr}\, R^2)^2 \right] . \tag{4.112}$$

To respect Z_2 symmetry, we assume that a five-brane with world-volume located at some y_0 has a Z_2–mirror at $-y_0$. The Dirac distribution transverse to its world-volume $M_4 \subset \mathcal{W}_6 \subset M_4 \times K_6$ is then defined by the condition

$$\int_{M_4 \times K_6 \times S^1/Z_2} I_6 \wedge \delta^{(5)}(\mathcal{W}_6) = 2 \int_{\mathcal{W}_6} I_6 \qquad (4.113)$$

for any 6–form I_6, since it takes both copies into account.

The membrane and five-brane tensions T_2 and T_5 are related by the Dirac-Zwanziger quantization condition

$$2\kappa^2 T_2 T_5 = 2\pi$$

and also by [90]

$$(T_2)^2 = 2\pi T_5.$$

We then express all constants in terms of κ^2, with

$$(4\pi\kappa^2)^{1/3} = \frac{2\pi}{T_2} = 2\kappa^2 T_5. \qquad (4.114)$$

With these conventions, the Bianchi identity is

$$dG_4 = \frac{4\pi}{T_2} \left(\frac{1}{2} \sum_I q_I \, \delta^{(5)}(\mathcal{W}_{6,I}) - \sum_i \delta_i I_{4,i} \right), \qquad (4.115)$$

where the index I labels the Z_2–symmetric pairs of five-branes and the charge q_I is $+1$ for a five-brane, -1 for an anti-five-brane.

One subtlety when integrating the Bianchi identity is that one cannot find a Z_2–odd function $\epsilon(y)$ such that $d\epsilon = \delta(y - y_0)dy$. As in ref. [43], we then use $y, y_0 \in]-\pi, \pi]$ and

$$\begin{aligned}
\epsilon_{y_0}(y) &= \text{sgn}\,(y - y_0) - \frac{y - y_0}{\pi}, & d\epsilon_{y_0}(y) &= \Big(2\delta(y - y_0) - \frac{1}{\pi}\Big)dy, \\
\epsilon_1(y) &= \epsilon_0(y) = \text{sgn}\,(y) - \frac{y}{\pi}, & d\epsilon_1(y) &= 2\delta_1 - \frac{1}{\pi}dy, \\
\epsilon_1(y) &= \epsilon_\pi(y) = \text{sgn}\,(y - \pi) - \frac{y - \pi}{\pi}, & d\epsilon_2(y) &= 2\delta_2 - \frac{1}{\pi}dy.
\end{aligned} \qquad (4.116)$$

The sign function is

$$\text{sgn}\,(r) = \frac{r}{|r|} \quad \text{if} \quad r \neq 0, \qquad \text{sgn}\,(0) = 0, \qquad \text{sgn}\,(r) = -\text{sgn}\,(-r). \qquad (4.117)$$

With this definition, ϵ_1 and ϵ_2 are odd functions while $\epsilon_{y_0}(-y) = -\epsilon_{-y_0}(y)$. The function

$$\epsilon_{y_0}(y) + \epsilon_{-y_0}(y) = \text{sgn}\,(y - y_0) + \text{sgn}\,(y + y_0) - \frac{2y}{\pi} \qquad (4.118)$$

$(0 < y_0 < \pi)$ is then odd with

$$d[\epsilon_{y_0}(y) + \epsilon_{-y_0}(y)] = 2\Big(\delta(y - y_0) + \delta(y + y_0)\Big)dy - \frac{2dy}{\pi}. \qquad (4.119)$$

This function is useful to insert five-brane sources in the Bianchi identity. Since the five-brane world-volumes $\mathcal{W}_{6,I}$ are of the form $M_4 \times \mathcal{C}_2$ ($\mathcal{C}_2$ a holomorphic cycle in K_6), $\mathcal{W}_{6,I}$ is located at $y = y_I$ with a "Z_2–mirror five-brane" at $-y_I$. We then use

$$\begin{aligned}
&\delta^{(5)}(\mathcal{W}_{6,I}) = \Big(\delta(y - y_I) + \delta(y + y_I)\Big)dy \wedge \delta^{(4)}(\mathcal{W}_{6,I}), \\
&d\Big([\epsilon_{y_I}(y) + \epsilon_{-y_I}(y)]\,\delta^{(4)}(\mathcal{W}_{6,I})\Big) = 2\,\delta^{(5)}(\mathcal{W}_{6,I}) - \frac{2}{\pi}\,dy \wedge \delta^{(4)}(\mathcal{W}_{6,I}),
\end{aligned} \qquad (4.120)$$

to integrate five-brane contributions to the Bianchi identity.

Part III

Orbifolds of M-theory and $\mathfrak{e}_{10}$

Introduction

In this last part of the thesis, we want to adress the question of what the underlining symmetries of M-theory could be. String theories are known to enjoy duality transformations which map one theory to another theory (such as T-duality), or relate the weakly-coupled sector to the strongly-coupled sector of the same or of another theory. Taking the example of T-duality, which, performed once, maps type IIA theory compactified on a circle to type IIB on the circle with inverse radius, we see that an even number of T-duality transformation becomes a symmetry of type II theories, and of their M-theory extension. So while string dualities constitute a rephrasing of the same physics in different languages, the study of symmetries may prove useful in a better understanding of M-theory.

Hidden symmetries in toroidally compactified maximal supergravity theories have been known for a long time, since the foundating works of Cremmer and Julia [77, 75, 76, 155, 152]. In particular, the bosonic part of $11D$ supergravity compactified on T^{11-D} for $10 \geqslant D \geqslant 3$ was shown to possess a continuous non-compact $E_{11-D|11-D}(\mathbb{R})$ global symmetry, provided all RR- and NS-NS fields are dualized to scalars whenever possible [79, 80]. This series of exceptional groups also appear as symmetries of the action of classical type IIA supergravity compactified on T^{10-D}, and the BPS states of the compactified theory turn out to arrange into multiplets of the Weyl group of $E_{11-D|11-D}(\mathbb{R})$ [186]. From the point of view of perturbative type IIA string theory, this continuous symmetry does not preserve the weak coupling regime in g_s, and is thus expected to be broken by quantum effects. Nevertheless, the authors of [149] have advocated that a discrete version thereof, namely $E_{11-D|11-D}(\mathbb{Z})$ remains as an exact quantum symmetry of both $11D$ supergravity and type II theories, and thus might provide a guideline for a better understanding of M-theory.

This exceptional series of arithmetic groups can alternatively be obtained as the closure of the T-duality symmetry of IIA theory compactified on T^{10-D}, namely $SO(10-D, 10-D, \mathbb{Z})$, with the discrete modular group of the $(11-D)$-torus of M-theory, namely $SL(11-D, \mathbb{Z})$. The T-duality symmetry $SO(10-D, 10-D, \mathbb{Z})$ is an exact symmetry of the conformal theory, and is thus valid at every order of perturbation theory. As for the $SL(11-D, \mathbb{Z})$ symmetry of the torus, it turns in $D = 9$, into the expected S-duality symmetry of type IIB string theory, which acts as a modular transformation on the complex parameter $\tau = C_0 + ie^{-\phi}$, with C_0 the expectation value of the scalar of the theory, and ϕ the dilaton. Under this transformation, in addition, the NS three-form, H_3, and the Ramond one, F_3, transform as a doublet (H_3, F_3). Again, the supergravity theory is invariant under the continuous group $SL(2, \mathbb{R})$, which is broken, in the quantum regime, to its discrete subgroup $SL(2, \mathbb{Z})$, due to the existence of the fundamental string, charged under this symmetry. One is then led to conjecture the existence of a solitonic string multiplet with integer charges (p, q) with respect to the fields (H_3, F_3), and to suppose a non-perturbative extension of type IIB superstring theory. In the massless limit, these charged states are absent, and we recover the continuous symmetry of the low-energy supergravity.

Moreover, this continuous hidden symmetry also appears in the global $SO(6, 22, \mathbb{R}) \times SL(2, \mathbb{R})$ symmetry of the low energy effective supergravity of the heterotic string compactified on T^6. For reasons similar as before, this semi-simple group is conjectured to extend, in its discrete version, to a full symmetry of the quantum theory. Thus, both the heterotic and IIB string theories share, although at different levels of compactification, an S-duality transformation which acts on the dilaton and axion fields, and in particular interchanges strong and weak coupling regimes, mapping perturbative to non-perturbative effects, and vice versa.

Such hidden symmetries appear in several cases and combine with T-duality symmetry to form a larger group, called the *U-duality* group, acting in the conjectured non-perturbative extension of the known string theories and relating different theories via field redefinitions.

In this perspective, the global $E_{11-D|11-D}(\mathbb{Z})$ can be regarded as a unifying group encoding both a target-space symmetry, which relates apparently different string backgrounds endowed with isometries, and a rigid symmetry of the maximally symmetric space of compactification which naturally contains a non-perturbative symmetry of type IIB string theory (which is also shared by the heterotic string compactified to four dimensions).

Furthermore, this U-duality symmetry has been conjectured to extend to the moduli space of M-theory compactified on T^{11-D} for $10 \geqslant D \geqslant 3$. In particular, it was shown in [204] how to retrieve exact R^4 and $R^4 H^{4g-4}$ corrections as well as topological couplings [205], from M-theory $E_{11-D|11-D}(\mathbb{Z})$-invariant mass formulae.

In $D = 2, 1$, the dualization procedure mentioned above is not enough to derive the full U-duality symmetry, which has been conjectured, already some time ago [153, 154, 156] to be described by the Kac-Moody affine $\mathfrak{e}_{9|10}$ and hyperbolic $\mathfrak{e}_{10|10}$ split forms, that are characterized by an infinite number of positive (*real*) and negative/null (*imaginary*) norm roots. In a more recent perspective, $\mathfrak{e}_{10|10}$ and the split form of $\mathfrak{e}_{11}$ have been more generally put forward as symmetries of the uncompactified $11D$, type II and type I supergravity theories, and possibly as a fundamental symmetry of M-theory itself, containing the whole chain of Cremmer-Julia split algebras, and hinting at the possibility that M-theory might prove intrinsically algebraic in nature.

Along this line, [84] have proposed a tantalizing correspondence between $11D$ supergravity (composite) operators at a given point close to a spacelike singularity and the coordinates of a one-parameter sigma-model based on the coset $\mathfrak{e}_{10|10}/\mathfrak{k}(\mathfrak{e}_{10|10})$ and describing the dynamics of a hyperbolic cosmological billiard. In particular, a class of real roots of $\mathfrak{e}_{10}$ have been identified, using a BKL expansion [24, 25], as multiple gradients of $11D$ supergravity fields reproducing the truncated equations of motion of the theory. More recently [87], imaginary roots were shown to correspond to 8th order $R^m(DF)^n$ type M-theory corrections to the classical $11D$ supergravity action.

Chapter 5

Hidden Borcherds symmetries in $\mathbb{Z}_n$ orbifolds of M-theory and magnetized D-branes in type 0' orientifolds

This chapter is based on the publication [13] which is a first step in the study of the behaviour of the infinite dimensional U-duality symmetry $\mathfrak{e}_{10|10}$ on singular backgrounds. More precisely, the present chapter concentrates on the study of the residual U-duality symmetry that survives the orbifold projection and the way the shift vectors defining the orbifold action are related to the extended objects necessary for the theory to reduce to a well-defined string theory.

In [55], an algebraic analysis of a certain class of orbifolds of M-theory has already been carried out in a compact version of the setup of [84]. Their work is based on a previous investigation of the relation between the moduli space of M-theory in the neighbourhood of a spacelike singularity and its possible hyperbolic billiard description [17]. For their analysis, these authors took advantage of a previous work [56] which helped establishing a dictionary between null roots of $\mathfrak{e}_{10}$ and certain Minkowskian branes and other objects of M-theory on T^{10}. Let us briefly recall this correspondence.

In [205, 202, 203], a systematic description of the relation between a subset of the positive roots of $E_{11-D|11-D}$ and BPS objects in type II string theories and M-theory has been given. In particular, they were shown to contribute to instanton corrections to the low-energy effective theory. In $D = 1$, this suggests a correspondence between certain positive real roots of $\mathfrak{e}_{10}$ and extended objects of M-theory totally wrapped in the compact space (such as Euclidean Kaluza-Klein particles, Euclidean M2 and M5-branes, and Kaluza-Klein monopoles). In the hyperbolic billiard approach to the moduli space of M-theory near a cosmological singularity, these real roots appear in exponential terms in the low-energy effective Hamiltonian of the theory [86, 87]. Such contributions behave as sharp wall potentials in the BKL limit, interrupting and reflecting the otherwise free-moving Kasner metric evolution. The latter can be represented mathematically by the inertial dynamics of a vector in the Cartan subalgebra of $\mathfrak{e}_{10}$ undergoing Weyl-reflections when it reaches the boundary of a Weyl chamber. In the low energy $11D$ supergravity limit, these sharp walls terms can be regarded as fluxes, which are changed by integer amounts by instanton effects. This description, however, is valid only in a regime where all compactification radii can all become simultaneously larger than the Planck length. In this case, the corresponding subset of positive real roots of $\mathfrak{e}_{10}$ can safely be related to instanton effect. As shown in [17], the regions of the moduli space of M-theory where this holds true are bounded by the (approximate) Kasner solution mentioned above. A proper description of these regions calls for a modification of the

Kasner evolution by introducing matter, which leads to a (possibly) non-chaotic behaviour of the system at late time (or large volume). The main contribution of [56] was to give evidence that these matter contributions have a natural description in terms of imaginary roots of $\mathfrak{e}_{10}$. More precisely, these authors have shown that extended objects such as Minkowskian Kaluza-Klein particles, M2-branes, M5-branes, and Kaluza-Klein monopoles (KK7M-branes) can be related to prime isotropic imaginary roots of $\mathfrak{e}_{10}$ that, interestingly enough, are all Weyl-equivalent. These results, although derived in a compact setting, are amenable to the non-compact case [84, 87].

Ref.[55] only considers a certain class of orbifolds of M-theory, namely: $T^{10-q} \times T^q/\mathbb{Z}_2$ for $q = 1, 4, 5, 8, 9$. After orbifold projection, the residual U-duality algebra $\mathfrak{g}_{\text{inv}}$ describing the untwisted sectors of all these orbifolds was shown to possess a root lattice isomorphic to the root lattice of the over-extended hyperbolic $\mathfrak{de}_{10|10}$ algebra. However, a careful root-space analysis led the authors to the conclusion that $\mathfrak{g}_{\text{inv}}$ was actually bigger than its hyperbolic counterpart, and contained $\mathfrak{de}_{10|10}$ as a proper subalgebra. Furthermore, in the absence of flux, anomaly cancellation in such orbifolds of M-theory is known to require the insertion of 16 M$(10-q)$-branes, for $q = 5, 8$, Kaluza-Klein particles/monopoles or other BPS objects (the $S^1/\mathbb{Z}_2$ has to be treated from a type IA point of view, where 16 D8-branes are required to compensate the charges of the two O8-planes) extending in the directions transverse to the orbifold [249, 88]. In [55], such brane configurations were shown to be nicely incorporated in the algebraic realization of the corresponding orbifolds. It was proven that the root lattice automorphism reproducing the $\mathbb{Z}_2$ action on the metric and the three-form field of the low effective M-theory action could always be rephrased in terms of a prime isotropic root, playing the rôle of the orbifold shift vector and describing precisely the transverse Minkowskian brane required for anomaly cancellation.

This construction in terms of automorphisms of the root lattice is however limited to the $\mathbb{Z}_2$ case, where, in particular, the diagonal components of the metric play no rôle. In order to treat the general $\mathbb{Z}_{n>2}$ orbifold case, we are in need of a more elaborate algebraic approach, which operates directly at the level of the generators of the algebra. In this regard, the works of Kac and Peterson on the classification of finite order automorphisms of Lie algebras, have inspired a now standard procedure [163, 164, 66] to determine the residual invariant subalgebra of a given finite dimensional Lie algebras, under a certain orbifold projection. This has in particular been used to study systematically the breaking patterns of the $E_8 \times E_8$ gauge symmetry of the heterotic string [164, 118]. The method is based on choosing an eigenbasis in which the orbifold charge operator can be rephrased as a Cartan preserving automorphism $\text{Ad}e^{\frac{2i\pi}{n}H_\Lambda}$, where Λ is an element of the weight lattice having scalar product $(\Lambda|\theta_G) \leqslant n$ with the highest root of the algebra θ_G, and H_Λ is its corresponding Cartan element. The shift vector Λ then determines by a standard procedure the invariant subalgebra $\mathfrak{g}_{\text{inv}}$ for all $\mathbb{Z}_n$ projections. However, the dimensionality and the precise set of charges of the orbifold have to be established by other means. This is in particular necessary to isolate possible degenerate cases. Finally, this method relies on the use of extended (not affine) Dynkin diagrams, and it is not yet known how it can be generalized to affine and hyperbolic Kac-Moody algebras.

Here we adopt a novel point of view based on the observation that the action of an orbifold on the symmetry group of any theory that possesses at least global Lorentz symmetry can be represented by the rigid action of a formal rotation operator in any orbifolded plane. In algebraic terms, the orbifold charge operator will be represented by a non-Cartan preserving, finite-order automorphism acting on the appropriate complex combinations of generators. These combinations are the components of tensors in the complex basis of the orbifolded torus which diagonalize the automorphism. Thus, they reproduce the precise mapping between orbifolded generators in $\mathfrak{e}_{10|10}/\mathfrak{k}(\mathfrak{e}_{10|10})$ and charged states in the moduli space of M-theory on T^{10}. It also enables one to keep track of the reality properties of the invariant subalgebra, provided we work with a Cartan decomposition of the original U-duality algebra. This is one reason which prompted us to choose the *symmetric gauge* (in contrast to the *triangular* Iwasawa gauge) to parametrize the physical fields of the theory. For this gauge choice, the orbifold charge operator is expressible as $\prod_{\alpha\in\Delta_+} \mathrm{A}de^{\frac{2\pi Q_\alpha}{n}(E_\alpha - F_\alpha)}$, for $E_\alpha - F_\alpha \in \mathfrak{k}(\mathfrak{e}_{10|10})$, and Δ_+ a set of positive roots reproducing the correct orbifold charges $\{Q_\alpha\}$. The fixed point subalgebra $\mathfrak{g}_{\text{inv}}$ is then obtained by truncating to the $Q_\alpha = kn$ sector, $k \in \mathbb{Z}$.

This method is general and can in principle be applied to the U-duality symmetry of any orbifolded supergravities and their M-theory generalization. In this paper, we will restrict ourselves to the $\mathfrak{g}^U = \mathfrak{e}_{11-D|11-D}$ U-duality chain for $8 \leqslant D \leqslant 1$. We will also limit our detailed study to a few illustrative examples of orbifolds, namely: $T^{11-D-q} \times T^q/\mathbb{Z}_{n>2}$ for $q = 2, 4, 6$ and $T^{11-D-q} \times T^q/\mathbb{Z}_2$ for $q = 1, \ldots, 9$. In the $\mathbb{Z}_2$ case, we recover, for $q = 1, 4, 5, 8, 9$, the results of [55]. In the other cases, the results are original and lead, for $D = 1$, to several examples where we conjecture that $\mathfrak{g}_{\text{inv}}$ is obtained by modding out either a Borcherds algebra or an indefinite (not affine) Kac-Moody algebra, by its centres and derivations. As a first check of this conjecture, we study in detail the $T^8 \times T^2/\mathbb{Z}_n$ case, and verify its validity up to level $l = 6$, investigating with care the splitting of the multiplicities of the original $\mathfrak{e}_{10}$ roots under the orbifold projection. We also show that the remaining cases can be treated in a similar fashion. From a different perspective [138, 139], truncated real super-Borcherds algebras have been shown to arise already as more general symmetries of various supergravities expressed in the doubled formalism and compactified on square tori to $D = 3$. Our work, on the other hand, gives other explicit examples of how Borcherds algebras may appear as the fixed-point subalgebras of a hyperbolic Kac-Moody algebra under a finite-order automorphism.

Subsequently, we engineer the relation between our orbifolding procedure which relies on finite order non-Cartan preserving automorphism, and the formalism of Kac-Peterson [161]. We first show that there is a new *primed* basis of the algebra in which one can derive a class of shift vectors for each orbifold we have considered. Then, we prove that these vectors are, for a given n, conjugate to the shift vector expected from the Kac-Peterson formalism. We show furthermore that, in the primed basis, every such class contains a positive root of $\mathfrak{e}_{10}$, which can serve as class representative. This root has the form $\Lambda'_n + n\tilde{\delta}'$, where $\tilde{\delta}'$ is in the same orbit as the null root δ' of $\mathfrak{e}_9$ under the Weyl group of $SL(10)$, and is minimal, in the sense that Λ'_n is the minimal weight leading to the required set of orbifold charges. We then list all such class representatives for all orbifolds of the type $T^{q_1}/\mathbb{Z}_{n_1} \times \cdots \times T^{q_m}/\mathbb{Z}_{n_m}$ with $\sum_{i=1}^m q_i \leqslant 10$, where the $\mathbb{Z}_{n_i}$ actions act independently on each T^2 subtori.

In particular, for the $T^{10-q} \times T^q/\mathbb{Z}_2$ orbifolds of M-theory with $q = 2, 3, 6, 7$ that were not considered in [55], we find that a consistent physical interpretation requires to consider them in the bosonic M-theory that descend to type 0A strings. In such cases, we find class representatives that are either positive real roots of $\mathfrak{e}_{10}$, or positive non-isotropic imaginary roots of norm -2. We then show that these roots are related to the twisted sectors of some particular non-supersymmetric type 0' orientifolds carrying magnetic fluxes. Performing the reduction to type 0A theory and T-dualizing appropriately, we actually find that these roots of $\mathfrak{e}_{10}$ descend to magnetized D9-branes in type 0' orientifolds carrying $(2\pi)^{-[q/2]} \int \mathrm{Tr}\, \underbrace{F \wedge .. \wedge F}_{[q/2]}$ units of flux. This gives a partial characterization of open strings twisted sectors in non-supersymmetric orientifolds in terms of roots of $\mathfrak{e}_{10}$. Moreover, the fact that these roots can be identified with Minkowskian D-branes even though none of them is prime isotropic, calls for a more general algebraic characterization of Minkowskian objects than the one propounded in [55]. A new proposal supported by evidence from the $\mathbb{Z}_{n>2}$ case will be presented in Section 5.9.2.

More precisely, the orbifolds of M-theory mentioned above descend to orientifolds of type 0A string theory by reducing on one direction of the orbifolded torus for q even, and on one direction outside the torus for q odd. T-dualizing to type 0B, we find cases similar to those studied in [46, 45], where specific configurations of D-branes and D'-branes were used to cancel the two 10-form RR tadpoles. Here, in contrast, we consider a configuration in which the branes are tilted with respect to the orientifold planes in the 0A theory. This setup is T-dual to a type 0' orientifold with magnetic fluxes coupling to the electric charge of a D$(10-q)$-brane embedded in the space-filling D9-branes in the spirit of [47, 48]. In this perspective, the aforementioned roots of $\mathfrak{e}_{10}$ which determine the $\mathbb{Z}_2$ action also possess a dual description in terms of tilted D-branes of type 0A string theory. In the original 11-dimensional setting, these roots are related to exotic objects of M-theory and thereby provide a proposal for the M-theory origin of such configurations.

Finally, we will also comment on the structure of the $\mathfrak{e}_{10}$ roots that appear as class representatives for shift vectors of $\mathbb{Z}_n$ orbifolds of M-theory, and hint at the kind of flux configurations these roots could be associated to.

5.1 Generalized Kac-Moody algebras

In this section, we introduce recent mathematical constructions from the theory of infinite-dimensional Lie algebras. Indeed, it is well-known in Lie theory that fixed-point subalgebras of infinite-dimensional Lie algebras under certain algebra automorphisms are often interesting mathematical objects in their own right and might have quite different properties. Of particular interest here is the fact that fixed-point subalgebras of Kac-Moody algebras are not necessarily Kac-Moody algebras, but can belong to various more general classes of algebras like extended affine Lie algebras [5, 4, 3], generalized Kac-Moody algebras [50, 51, 52], Slodowy intersection matrices [224] or Berman's generalized intersection matrices [38]. Indeed, invariant U-duality symmetry subalgebras for orbifolds of M-theory are precisely fixed-point subalgebras under a finite-order automorphism and can be expected (at least in the hyperbolic and Lorentzian cases) to yield algebras that are beyond the realm of Kac-Moody algebras.

5.1.1 Central extensions of Borcherds algebras

Since they are particularly relevant to our results, we will focus here on the so-called generalized Kac-Moody algebras, or GKMAs for short, introduced by Borcherds in [50] to extend the Kac-Moody algebras construction to infinite-dimensional algebras with imaginary simple roots. We define here a number of facts and notations about infinite-dimensional Lie algebras which we will need in the rest of the paper, starting from very general considerations and then moving to more particular properties. This will eventually prompt us to refine the approach to GKMAs with a degenerate Cartan matrix, by providing, in particular, a rigorous definition of how scaling operators should by introduced in this case in accordance with the general definition of GKMAs (see for instance Definitions 5.1.3 and 5.1.4 below). This has usually been overlooked in the literature, but turns out to be crucial for our analysis of fixed point subalgebras of infinite KMAs under a finite order automorphism, which occur, as we will see, as hidden symmetries of the untwisted sector of M-theory under a given orbifold.

In this perspective, we start by defining the necessary algebraic tools. Let $\mathfrak{g}$ be a (possibly infinite-dimensional) Lie algebra possessing a Cartan subalgebra $\mathfrak{h}$ (a complex nilpotent subalgebra equal to its normalizer) which is *splittable*, in other words, the action of adH on $\mathfrak{g}$ is trigonalizable $\forall H \in \mathfrak{h}$. The derived subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ possesses an r-dimensional Cartan subalgebra $\mathfrak{h}' = \mathfrak{g}' \cap \mathfrak{h}$ spanned by the basis $\Pi^\vee = \{H_i\}_{i\in I}$, with indices valued in the set $I = \{i_1, .., i_r\}$.

We denote by $\mathfrak{h}'^*$ the space dual to $\mathfrak{h}'$. It has a basis formed by r linear functionals (or 1-forms) on $\mathfrak{h}'$, the simple roots of $\mathfrak{g}$: $\Pi = \{\alpha_i\}_{i\in I}$. Suppose we can define an indefinite scalar product: $(\alpha_i|\alpha_j) = a_{ij}$ for some real $r \times r$ matrix a, then:

Definition 5.1.1 The matrix a is called a generalized symmetrized Cartan matrix, if it satisfies the conditions:

i) $a_{ij} = a_{ji}\,,\ \forall i,j \in I$.

ii) a has no zero column.

iii) $a_{ij} \leq 0\,$, for $i \neq j$ and $\forall i,j \in I$.

iv) $\left.\begin{array}{ll} \text{if } a_{ii} \neq 0: & 2\frac{a_{ij}}{a_{ii}} \\ \text{if } a_{ii} = 0: & a_{ij} \end{array}\right\} \in \mathbb{Z}\,$, for $i \neq j$ and $\forall i,j \in I$.

From integer linear combinations of simple roots, one constructs the root lattice $Q = \sum_{i\in I} \mathbb{Z}\alpha_i$. The scalar product $(\,|\,)$ is then extended by linearity to the whole $Q \subset \mathfrak{h}'^*$. Furthermore, by defining fundamental weights $\{\Lambda^i\}_{i\in I}$ satisfying $(\Lambda^i|\alpha_j) = \delta^i{}_j\,,\ \forall i,j \in I$, we introduce a duality relation with respect to the root scalar product. Then, from the set $\{\Lambda^i\}_{i\in I}$ we define the lattice of integral weights $P = \sum_{i\in I} \mathbb{Z}\Lambda^i$ dual to Q, such that $Q \subseteq P$.

Let us introduce the duality isomorphism ν: $\mathfrak{h}' \to \mathfrak{h}'^*$ defined by

$$\nu(H_i) = \left\{ \begin{array}{ll} 2\frac{\alpha_i}{|a_{ii}|} & , \text{ if } a_{ii} \neq 0\,. \\ \alpha_i & , \text{ if } a_{ii} = 0\,, \end{array} \right. . \tag{5.1}$$

We may now promote the scalar product $(\alpha_i|\alpha_j) = a_{ij}$ to a symmetric bilinear form B on $\mathfrak{h}'$ through:

$$b_{ij} = B(H_i, H_j) = (\nu(H_i)|\nu(H_j))\,,\ \forall i,j \in I\,.$$

Suppose next that the operation $\mathrm{ad}(H)$ is diagonalizable $\forall H \in \mathfrak{h}$, from which we define the following:

Definition 5.1.2 We call root space an eigenspace of $\mathrm{ad}(H)$ defined as

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}\,|\,\mathrm{ad}(H)X = \alpha(H)X,\ \forall H \in \mathfrak{h}\} \tag{5.2}$$

which defines the root system of $\mathfrak{g}$ as $\Delta(\mathfrak{g},\mathfrak{h}) = \{\alpha \neq 0\,|\,\mathfrak{g}_\alpha \neq \{0\}\}$, depending on the choice of basis for $\mathfrak{h}$.

The multiplicities attached to a root $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$ are then given by $m_\alpha = \dim \mathfrak{g}_\alpha$. As usual, the root system splits into a positive root system and a negative root system. The positive root system is defined as

$$\Delta_+(\mathfrak{g},\mathfrak{h}) = \Big\{\alpha \in \Delta(\mathfrak{g},\mathfrak{h})\,|\,\alpha = \sum_{i\in I} n_i\alpha_i, \text{with } n_i \in \mathbb{N}\,, \forall i \in I\Big\}$$

and the negative root one as $\Delta_-(\mathfrak{g},\mathfrak{h}) = -\Delta_+(\mathfrak{g},\mathfrak{h})$, so that $\Delta(\mathfrak{g},\mathfrak{h}) = \Delta_+(\mathfrak{g},\mathfrak{h}) \cup \Delta_-(\mathfrak{g},\mathfrak{h})$. We call $\mathrm{ht}(\alpha) = \sum_{i\in I} n^i$ the height of α. From now on, we shall write $\Delta \equiv \Delta(\mathfrak{g},\mathfrak{h})$ for economy, and restore the full notation $\Delta(\mathfrak{g},\mathfrak{h})$ or partial notation $\Delta(\mathfrak{g})$ when needed.

Finally, since $(\alpha|\alpha)$ is bounded above on Δ, α is called real if $(\alpha|\alpha) > 0$, isotropic imaginary if $(\alpha|\alpha) = 0$ and (non-isotropic) imaginary if $(\alpha|\alpha) < 0$. Real roots always have multiplicity one, as is the case for finite-dimensional semi-simple Lie algebras, while (non-simple) isotropic roots have a multiplicity equal to $\mathrm{rk}a(\hat{\mathfrak{g}})$ for some affine subalgebra $\hat{\mathfrak{g}} \subset \mathfrak{g}$, while (non-simple) non-isotropic imaginary roots can have very big multiplicities.

Generalized Kac-Moody algebras are usually defined with all *imaginary simple* roots of multiplicity one, as well. One could in principle define a GKMA with simple roots of multiplicities bigger than one, but then the algebra would not be completely determined by its generalized

Cartan matrix. In this case, one would need yet another matrix with coefficients specifying the commutation properties of all generators in the same simple root space. Here, we shall not consider this possibility further since it will turn out that all fixed point subalgebras we will be encountering in the framework of orbifold compactification of $11D$ supergravity and M-theory possess only isotropic simple roots of multiplicity one.

We now come to specifying the rôle of central elements and scaling operators in the case of GKMAs with degenerate generalized Cartan matrix.

Definition 5.1.3 If the matrix a does not have maximal rank, define the centre of $\mathfrak{g}$ as $\mathfrak{z}(\mathfrak{g}) = \{c \in \mathfrak{h} \,|\, B(H_i, c) = 0\,,\, \forall i \in I\}$. In particular, if $l = \dim \mathfrak{z}(\mathfrak{g})$, one can find l linearly independent null root lattice vectors $\{\delta_i\}_{i=1,..,l}$ (possibly roots, but not necessarily) satisfying $(\delta_i|\nu(H_j)) = 0\,$, $\forall i = 1, \dots, l$, $\forall j \in I$. One then defines l linearly independent Cartan generators $\{d_i\}_{i=1,..,l}$ with $d_i \in \mathfrak{h}/\mathfrak{h}'$ thus extending the bilinear form B to the whole Cartan algebra $\mathfrak{h}$ as follows:

- $B(c_i, d_j) = \delta_{ij}\,,\, \forall i, j = 1, .., l\,.$
- $B(d_i, d_j) = 0\,,\, \forall i, j = 1, .., l\,.$
- $B(H, d_i) = 0\,,\, \forall i = 1, .., l$ and for $H \in \mathfrak{h}'/\text{Span}\{c_i\}_{i=1,..,l}\,.$

Then, we have $\text{rk}(a) = r - l$ and $\dim \mathfrak{h} = r + l$.

This definition univocally fixes the i-th level k_i of all roots $\alpha \in \Delta$ to be $k_i = B(\nu^{-1}(\alpha), d_i)$, using the decomposition of $\nu^{-1}(\alpha)$ on orthogonal subspaces in $\mathfrak{h}' = (\mathfrak{h}'/\text{Span}\{c_1, \dots, c_l\}) \oplus \text{Span}\{c_1\} \oplus \dots \oplus \text{Span}\{c_n\}$.

We are now ready to define a GKMA by its commutation relations. Definitions of various levels of generality exist in the literature, but we choose one that is both convenient (though seemingly complicated) and sufficient for our purpose, neglecting the possibility that $[E_i, F_j] \neq 0$ for $i \neq j$ (see, for example, [126, 18] for such constructions), but taking into account the possibility of degenerate Cartan matrices, a generic feature of the type of GKMA we will be studying later on in this paper.

Definition 5.1.4 The universal generalized Kac-Moody algebra associated to the Cartan matrix a is the Lie algebra defined by the following commutation relations (Serre-Chevalley basis) for the simple root generators $\{E_i, F_i, H_i\}_{i\in I}$:

(i). $[E_i, F_j] = \delta_{ij} H_i, \qquad [H_i, H_j] = [H_i, d_k] = 0\,, \qquad \forall i, j \in I, \quad k = 1, .., l\,.$

(ii). $$[H_i, E_j] = \begin{cases} \frac{2a_{ij}}{|a_{ii}|} E_j\,, & \text{if } a_{ii} \neq 0 \\ a_{ij} E_j\,, & \text{if } a_{ii} = 0 \end{cases}, \qquad [H_i, F_j] = \begin{cases} -\frac{2a_{ij}}{|a_{ii}|} F_j\,, & \text{if } a_{ii} \neq 0 \\ -a_{ij} F_j\,, & \text{if } a_{ii} = 0 \end{cases}, \qquad \forall i, j \in I\,.$$

(iii). If $a_{ii} > 0$: $\quad (\text{ad}E_i)^{1-2\frac{a_{ij}}{a_{ii}}} E_j = 0, \qquad (\text{ad}F_i)^{1-2\frac{a_{ij}}{a_{ii}}} F_j = 0\,, \qquad \forall i, j \in I\,.$

(iv). $\forall i, j \in I$ such that $a_{ii} \leq 0$, $a_{jj} \leq 0$ and $a_{ij} = 0$: [1] $\qquad [E_i, E_j] = 0, \qquad [F_i, F_j] = 0\,,$

(v). $[d_i, [E_{j_1}, [E_{j_2}, .., E_{j_n}]..]] = k_i [E_{j_1}, [E_{j_2}, .., E_{j_n}]..]\,,$
$[d_i, [F_{j_1}, [F_{j_2}, .., F_{j_n}]..]] = -k_i [F_{j_1}, [F_{j_2}, .., F_{j_n}]..]\,,$
where k_i is the $i-th$ level of $\alpha = \alpha_{j_1} + \ldots + \alpha_{j_n}$, as defined above.

[1] Note that there is no a priori limit to the number of times one can commute the generator E_i for α_i imaginary with any other generator E_j in case $a_{ij} \neq 0$.

Since a generalized Kac-Moody algebra can be graded by its root system as: $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$, the indefinite scalar product B can be extended to an ad($\mathfrak{g}$)-invariant bilinear form satisfying: $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ except if $\alpha + \beta = 0$, which we call the generalized Cartan-Killing form. It can be fixed uniquely by the normalization

$$B(E_i, F_j) = \begin{cases} \frac{2}{|a_{ii}|}\delta_{ij} & , \text{ if } a_{ii} \neq 0 \\ \delta_{ij} & , \text{ if } a_{ii} = 0 \end{cases} ,$$

on generators corresponding to simple roots. Then ad($\mathfrak{g}$)- invariance naturally implies: $B(H_i, H_j) = (\nu(H_i)|\nu(H_j))$.

The GKMA $\mathfrak{g}$ can be endowed with an antilinear Chevalley involution ϑ_C acting as $\vartheta_C(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ and $\vartheta_C(H) = -H$, $\forall H \in \mathfrak{h}$, whose action on each simple root space $\mathfrak{g}_{\alpha_i}$ is defined as as usual as $\vartheta_C(E_i) = -F_i$, $\forall i \in I$. The Chevalley involution extends naturally to the whole algebra $\mathfrak{g}$ by linearity, for example:

$$\vartheta_C([E_i, E_j]) = [\vartheta_C(E_j), \vartheta_C(E_j)] = [F_i, F_j] \,. \tag{5.3}$$

This leads to the existence of an almost positive-definite contravariant form $B_{\vartheta_C}(X, Y) = -B(\vartheta_C(X), Y)$. More precisely, it is positive-definite everywhere outside $\mathfrak{h}$.

Note that there is another standard normalization, the Cartan-Weyl basis, given by:

$$e_{\alpha_i} = \begin{cases} \sqrt{\frac{|a_{ii}|}{2}} E_i & , \text{ if } a_{ii} \neq 0 \\ E_i & , \text{ if } a_{ii} = 0 \end{cases} , \qquad f_{\alpha_i} = \begin{cases} \sqrt{\frac{|a_{ii}|}{2}} F_i & , \text{ if } a_{ii} \neq 0 \\ F_i & , \text{ if } a_{ii} = 0 \end{cases} ,$$

$$h_{\alpha_i} = \begin{cases} \frac{|a_{ii}|}{2} H_i & , \text{ if } a_{ii} \neq 0 \\ H_i & , \text{ if } a_{ii} = 0 \end{cases} ,$$

and characterized by: $B(e_\alpha, f_\alpha) = 1$, $\forall \alpha \in \Delta_+(\mathfrak{g})$.

We will not use this normalization here, but we will instead write the Cartan-Weyl relations in a Chevalley-Serre basis, as follows:

Definition 5.1.5 For all $\alpha \in \Delta_+(\mathfrak{g})$ introduce $2m_\alpha$ generators: E^a_α and F^a_α, $a = 1, .., m_\alpha$. Generators corresponding to roots of height ± 2 are defined as:

$$E_{\alpha_i+\alpha_j} = \mathcal{N}_{\alpha_i,\alpha_j}[E_i, E_j], \quad F_{\alpha_i+\alpha_j} = \mathcal{N}_{-\alpha_i,-\alpha_j}[F_i, F_j], \, \forall i, j \in I \,,$$

for a certain choice of structure constants $\mathcal{N}_{\alpha_i,\alpha_j}$. Then, higher heights generators are defined recursively in the same way through:

$$[E^a_\alpha, E^b_\beta] = \sum_c (\mathcal{N}_{\alpha,\beta})^{ab}{}_c E^c_{\alpha+\beta} \,. \tag{5.4}$$

The liberty of choosing the structure constants is of course limited by the anti-commutativity of the Lie bracket: $(\mathcal{N}_{\alpha,\beta})^{ab}{}_c = -(\mathcal{N}_{\beta,\alpha})^{ba}{}_c$ and the Jacobi identity, from which we can derive several relations. Among these, the following identity, valid for finite-dimensional Lie algebras, will be useful for our purposes:

$$\mathcal{N}_{\alpha,\beta}\mathcal{N}_{-\alpha,-\beta} = -(p+1)^2, \, p \in \mathbb{N} \,, \text{ s.t. } \{\beta - p\alpha, \ldots, \beta + \alpha\} \subset \Delta(\mathfrak{g}, \mathfrak{h}) \,.$$

Note that this relation can be generalized to the infinite-dimensional case if one chooses the bases of root spaces $\mathfrak{g}_\alpha$ with $m_\alpha > 1$ in a particular way such that there is no need for a sum in (5.4). Imposing in addition $(\mathcal{N}_{\alpha,\beta})^{ab}{}_c = -(\mathcal{N}_{-\alpha,-\beta})^{ab}{}_c$ gives sign conventions compatible with

$\vartheta_C(E^a_\alpha) = -F^a_\alpha\,,\ \forall\alpha\in\Delta_+(\mathfrak{g}), a=1\ldots,m_\alpha$, not only for simple roots. In the Serre-Chevalley normalization, this furthermore ensures that: $\mathcal{N}_{\alpha,\beta}\in\mathbb{Z}, \forall\alpha,\beta\in\Delta$. Here lies our essential reason for sticking to this normalization, and we will follow this convention throughout the paper. In the particular case of *simply-laced* semi-simple Lie algebras, we always have $p=0$, and we can choose $\mathcal{N}_{\alpha,\beta} = \pm 1\,,\ \forall\alpha,\beta\in\Delta$ (note, however, that this is not true for infinite-dimensional simply-laced algebras).

Another important consequence of the Jacobi identity in the finite case, which will turn out to be useful is the following relation

$$\mathcal{N}_{\alpha,-\beta} = \mathcal{N}_{\beta-\alpha,\alpha} \quad \forall\alpha,\beta\in\Delta\,.$$

5.1.2 Kac-Moody algebras as a special case of GKMA

Standard symmetrized Kac-Moody algebras (KMA) can be recovered from the preceding section by imposing $a_{ii}>0\,,\ \forall i\in I$ in all the above definitions. In addition, one usually rephrases the dual basis $\Pi^\vee$ in terms of coroots, by setting $\alpha_i^\vee \equiv H_i$. Their image under the duality isomorphism reads

$$\nu(\alpha_i^\vee) = \frac{2}{(\alpha_i|\alpha_i)}\alpha_i\,,\ \forall i\in I\,,$$

so that instead of the symmetrized Cartan matrix a, one generally resorts to the following non-symmetric version, defined as a realization of the triple $\{\mathfrak{h},\Pi,\Pi^\vee\}$ with $\Pi^\vee=\{\alpha_i^\vee\}_{i\in I}\subset\mathfrak{h}^*$:

$$A_{ij} = \frac{2a_{ij}}{a_{ii}} \equiv \langle\alpha_i^\vee,\alpha_j\rangle\,. \tag{5.5}$$

where the duality bracket on the RHS represents the standard action of the one-form α_j on the vector $\alpha_i^\vee$.

The matrix a is then called the symmetrization of the (integer) Cartan matrix A. As a consequence of having introduced l derivations in Definition 5.1.3, the contravariant form $B_{\vartheta_C}(.,.)$ now becomes non-degenerate on the whole of $\mathfrak{g}$, even in the case of central extensions of multi-loop algebras, which are the simplest examples of extended affine Lie algebra (EALA, for short).

For the following, we need to introduce the Weyl group of $\mathfrak{g}$ as

Definition 5.1.6 The Weyl group of $\mathfrak{g}$, denoted $W(\mathfrak{g})$, is the group generated by all reflections mapping the root system into itself:

$$\begin{array}{rccl} r_\alpha: & \Delta(\mathfrak{g}) & \rightarrow & \Delta(\mathfrak{g}) \\ & \beta & \mapsto & \beta - \langle\alpha^\vee,\beta\rangle\alpha\,. \end{array}$$

The set $\{r_{i_1},..,r_{i_r}\}$, where $r_i\doteq r_{\alpha_i}$ are called the fundamental reflections, is a basis of $W(\mathfrak{g})$. Since $r_i^{-1}=r_{-\alpha_i}$, $W(\mathfrak{g})$ is indeed a group.

The real roots of any finite Lie algebra or KMA can then be defined as being Weyl conjugate to a simple root. In other words, $\alpha\in\Delta(\mathfrak{g},\mathfrak{h})$ is real if $\exists w\in W(\mathfrak{g})$ such that $\alpha=w(\alpha_i)$ for $i\in I$ and $\mathfrak{g}$ *is* a KMA.

A similar formulation exists for imaginary roots of a KMA, which usually turns out to be useful for determining their multiplicities, namely (see [159]):

Theorem 5.1.1 *Let $\alpha=\sum_{i\in I}k_i\alpha_i\in Q\backslash\{0\}$ have compact support on the Dynkin diagram of $\mathfrak{g}$, and set:*

$$K=\{\alpha\,|\,\langle\alpha_i^\vee,\alpha\rangle\leqslant 0\,,\ \forall i\in I\}\,,$$

Then denoting by Δ_{im} *the set of imaginary roots of* $\mathfrak{g}$*, we have:*

$$\Delta_{im}(\mathfrak{g}) = \bigcup_{w \in W(\mathfrak{g})} w(K)\,.$$

It follows from Theorem 5.1.1 that, in the affine case, every isotropic root α is Weyl-equivalent to $n\delta$ (with $\delta = \alpha_0 + \theta$ the null root) for some $n \in \mathbb{Z}^*$, which is another way of showing that all such roots have multiplicity $m_\alpha = r$. All isotropic roots which are Weyl-equivalent to δ are usually called *prime* isotropic. Note, finally, that statements similar to Theorem 5.1.1 holds for non-isotropic imaginary roots of hyperbolic KMAs. For instance, all positive roots with $(\alpha|\alpha) = -2$ can in this case be shown to be Weyl-equivalent to Λ^0, the weight dual to the extended root α_0.

Intersection matrix algebras are even more general objects that allow for positive non-diagonal elements in the Cartan matrix. Slodowy intersection matrices allow such positive diagonal metric elements, while Berman generalized intersection matrices give the most general framework by allowing imaginary simple roots, as well, as in the case of Borcherds algebras. Such more complicated algebras will not appear in the situations considered in this paper, but it is not impossible that they could show up in applications of the same methods to different situations.

5.1.3 A comment on $\mathfrak{sl}(10)$ representations in $\mathfrak{e}_{10}$ and their outer multiplicity

Of particular significance for Kac-Moody algebras beyond the affine case are of course the root system and the root multiplicities, which are often only partially known. Fortunately, in the case of $\mathfrak{e}_{10}$, we can rely on the work in [173, 171, 114] to obtain information about a large number of low-level roots, enough to study $\mathbb{Z}_n$ orbifolds up to $n = 6$. These works rely on decomposing Lorentzian algebras in representations of a certain finite subalgebra. However, the set of representations is not exactly isomorphic to the root system (modulo Weyl reflections). Indeed, the multiplicity of a representation in the decomposition is in general smaller than the multiplicity of the root corresponding to its highest weight vector. Typically, the m-dimensional vector space corresponding to a root of (inner) multiplicity m will be split into subspaces of several representations of the finite subalgebra. Typically, a root α of multiplicity $m_\alpha > 1$ will appear $n_{\text{o}}(\alpha)$ times as the highest weight vector of a representation, plus several times as a weight of other representations. The number $n_{\text{o}}(\alpha)$ is called the outer multiplicity, and can be 0. For a representation $\mathcal{R}$ of $\mathfrak{g}$ it shall be denoted by a subscript as: $\mathcal{R}_{[n_{\text{o}}]}$ when needed. Even though the concept of outer multiplicity is of minor significance for our purpose, it is important to understand the mapping between the results of [115, 171], based on representation of finite subalgebras, and ours, which focuses on tensorial representations with definite symmetry properties.

5.2 Hidden symmetries in M-theory: the setup

As a start, we first review some basic facts about hidden symmetries of $11D$ supergravity and ultimately M-theory, ranging from the early non-linear realizations of toroidally compactified $11D$ supergravity [79, 80] to the conjectured hyperbolic $\mathfrak{e}_{10}$ hidden symmetry of M-theory.

Then in Sections 5.2.1-5.2.5, we do a synthesis of the algebraic approach to U-duality symmetries of $11D$ supergravity on T^q and the moduli space of M-theory on T^{10}, presenting in detail the physical material and mathematical tools that we will need in the subsequent sections, and justify our choice of parametrization for the coset element (algebraic field strength) describing the physical fields of the theory. The reader familiar with these topics may of course skip the parts of this presentation he will judge too detailed.

The global $E_{11-D|11-D}$ symmetry of classical $11D$ supergravity reduced on T^{11-D} for $10 \leqslant D \leqslant 3$ can be best understood as arising from a simultaneous realization of the linear non-perturbative symmetry of the supergravity Lagrangian where no fields are dualized and the perturbative T-duality symmetry of type IIA string theory appearing in $D = 10$ and below. Actually, the full $E_{11-D|11-D}$ symmetry has a natural interpretation as the closure of both these groups, up to shift symmetries in the axionic fields.

Type IIA string theory compactified on T^{10-D} enjoys a $SO(10-D, 10-D, \mathbb{Z})$ symmetry[2] which is valid order by order in perturbation theory. So, restricting to massless scalars arising from T-duality in $D \leqslant 8$, all inequivalent quantum configurations of the scalar sector of the bosonic theory are described by the moduli space

$$\mathcal{M}_D = SO(10-D, 10-D, \mathbb{Z}) \backslash SO(10-D, 10-D)/(SO(10-D) \times SO(10-D)),$$

where the left quotient by the arithmetic subgroup corrects the over-counting of perturbative string compactifications. In contrast to the NS-NS fields B_2 and $g_{\mu\nu}$ which, at the perturbative level, couple to the string worldsheet, the R-R fields do so only via their field strength. So a step towards U-duality can be achieved by dualizing the R-R fields while keeping the NS-NS ones untouched. It should however be borne in mind that such a procedure enhances the T-duality symmetry only when dualizing a field strength to an equal or lower rank one. Thus, Hodge-duals of R-R fields start playing a rôle when $D \leqslant 8$, those of NS-NS fields when $D \leqslant 6$. However, when perturbative symmetries are concerned, we will not dualize NS-NS fields.

This enlarged T-duality symmetry can be determined by identifying its discrete Weyl group $W(D_{10-D})$ [186], which implements the permutation of field strengths required by electric-magnetic duality. In $D \leqslant 8$ it becomes now necessary to dualize R-R field strengths in order to form Weyl-group multiplets. This results in 2^{9-D} R-R axions, all exhibiting a shift symmetry, that enhances the T-duality group to:

$$\widetilde{G} = SO(10-D, 10-D) \ltimes \mathbb{R}^{2^{9-D}}, \tag{5.6}$$

the semi-direct product resulting from the fact that the R-R axions are now linearly rotated into one another under T-duality. The (continuous) scalar manifold is now described by the coset $\widetilde{G}/SO(10-D) \times SO(10-D)$, whose dimension matches the total number of scalars if we include the duals of R-R fields only. The symmetry (5.6) can now be enlarged to accomodate non-perturbative generators, leading to the full global symmetry $E_{11-D|11-D}$. However, this can only be achieved without dualizing the NS-NS fields in the range $9 \geqslant D \geqslant 7$. When descending to lower dimensions, indeed, the addition of non-perturbative generators rotating R-R and NS-NS fields into one another forces the latter to be dualized.

To evade this problem arising in low dimensions, we might wish to concentrate instead on the global symmetry of the $11D$ supergravity Lagrangian for $D \leqslant 9$, whose scalar manifold is described by the coset

$$GL(11-D) \ltimes \mathbb{R}^{(11-D)!/((8-D)!3!)}/O(11-D).$$

The corresponding group $G_{SG} = GL(11-D) \ltimes \mathbb{R}^{(11-D)(10-D)(9-D)/6}$ encodes the symmetry of the totally undualized theory including the $(11-D)(10-D)(9-D)/6$ shift symmetries coming from the axions produced by toroidal compactification of the three-form C_3. Again, the semi-direct product reflects the fact that these axions combine in a totally antisymmetric rank three representation of $GL(11-D)$. Since NS-NS and R-R fields can be interchanged by $GL(11-D)$,

[2] We consider $SO(10-D, 10-D)$ instead of $O(10-D, 10-D)$, as is sometimes done, because the elements of $O(10-D, 10-D)$ connected to $-\mathbb{1}$ flip the chirality of spinors in the type IIA/B theories. As such, this subset of elements is not a symmetry of the R-R sector of the type IIA/B supergravity actions, but dualities which exchange both theories.

the arithmetic subgroup of $GL(11-D) \ltimes \mathbb{R}^{(11-D)(10-D)(9-D)/6}$ constitutes an acceptable non-perturbative symmetry of type II superstring theory in $D \leqslant 9$. The price to pay in this case is to sacrifice T-duality, since the subgroup of the linear group preserving the NS-NS and R-R sectors separately is never big enough to accomodate $SO(10-D, 10-D)$.

Eventually, the full non-perturbative symmetry $E_{11-D|11-D}$ can only be achieved when both NS-NS and R-R fields are dualized, and may be viewed as the closure of its $GL(11-D)$ and $SO(10-D, 10-D)$ subgroups. However, the number of shift symmetries in this fully dualized version of the theory is given by $\{3, 6, 10, 16, 27, 44\}$ for $8 \geqslant D \geqslant 3$. Since, in $D \leqslant 5$, these numbers are always smaller or equal to $(11-D)(10-D)(9-D)/6$ and 2^{9-D}, neither $\widetilde{G}$ nor G_{SG} are subgroups of $E_{11-D|11-D}$ in low dimensions.

This exceptional symmetry is argued to carry over, in its discrete version, to the full quantum theory. Typically, the conjectured U-duality group of M-theory on T^{11-D} can be rephrased as the closure

$$E_{11-D|11-D}(\mathbb{Z}) = SO(10-D, 10-D, \mathbb{Z}) \bar{\times} SL(11-D, \mathbb{Z}) , \tag{5.7}$$

where the first factor can be viewed as the perturbative T-duality symmetry of IIA string theory, while the second one is the modular group of the torus T^{11-D}. In $D = 9$, the latter can be rephrased in type IIB language as the expected S-duality symmetry.

The moduli space of M-theory on T^{11-D} is then postulated to be

$$\mathcal{M}_D = E_{11-D|11-D}(\mathbb{Z}) \backslash E_{11-D|11-D} / K(E_{11-D|11-D}) . \tag{5.8}$$

It encodes both the perturbative NS-NS electric p-brane spectrum and the spectrum of non-perturbative states, composed of the magnetic dual NS-NS $(9-D-p)$-branes and the R-R D-branes of IIA theory for $10 \leqslant D \leqslant 3$.

In dimensions $D < 3$, scalars are dual to themselves, so no more enhancement of the U-duality group is expected from dualization. However, an enlargement of the hidden symmetry of the theory is nevertheless believed to occur through the affine extension $E_{9|10}(\mathbb{Z})$ in $D = 2$, the over-extended $E_{10|10}(\mathbb{Z})$, generated by the corresponding hyperbolic KMA, in $D = 1$, and eventually the very-extended $E_{11}(\mathbb{Z})$ in the split form, whose KMA is Lorentzian, for the totally compactified theory.

Furthermore, there is evidence that the latter two infinite-dimensional KMAs are also symmetries of unreduced $11D$ supergravity [172, 246], viewed as a non-linear realization (in the same spirit as the Monstruous Moonshine [192] has been conjectured to be a symmetry of uncompactified string theory) and are believed to be more generally symmetries of uncompactified M-theory itself [198].

5.2.1 The exceptional E_r series: conventions and useful formulæ

Before going into more physical details, we need to introduce a few mathematical properties of the U-duality groups and their related algebras. To make short, we will denote $G^U \doteq \text{Split}(E_{11-D})$, for $D = 1, \dots, 10$, and their defining Lie or Kac-Moody algebras $\mathfrak{g}^U \doteq \text{Split}(\mathfrak{e}_{11-D})$.

Except in $D = 10, 9$, the exceptional series E_r, with $r = 11-D$, possesses a physical basis for roots and dual Cartan generators:

Definition 5.2.1 Let the index set of Definition 5.1.1 be chosen as $I = 9-r, \dots, 8$, for $3 \leq r \leq 10$ [3], then in the *physical basis* of $\mathfrak{h}'^*$: $\{\varepsilon_{11-r} \doteq (1, 0, \dots, 0), \dots, \varepsilon_{10} \doteq (0, \dots, 0, 1)\}$, the set

[3] The physical basis makes sense only in cases where there are scalars coming from compactification of the 3-form, which excludes the first two algebras of the series.

$\Pi = \{\alpha_{9-r}, \dots, \alpha_8\}$ of simple roots of the semi-simple KMAs $\mathfrak{e}_r$ reads:

$$\begin{aligned} \alpha_{9-r} &= \varepsilon_{11-r} - \varepsilon_{12-r} = (1, -1, 0, \dots, 0)\,, \\ &\vdots \\ \alpha_7 &= \varepsilon_9 - \varepsilon_{10} = (0, \dots, 0, 1, -1)\,, \\ \alpha_8 &= \varepsilon_8 + \varepsilon_9 + \varepsilon_{10} = (0, \dots, 0, 1, 1, 1)\,. \end{aligned} \tag{5.9}$$

The advantage of such a basis is to give a rank-, and hence dimension-, independent formulation of Π, which is not the case for an orthogonal basis e_i. Preserving the scalar product on the root system requires the physical basis to be endowed with the following scalar product:

$$(\alpha|\beta) = \sum_{i=11-r}^{10} n^i m^i - \frac{1}{9} \sum_{i,j=11-r}^{10} n^i m^j \tag{5.10}$$

for $\alpha = \sum_{i=11-r}^{10} n^i \varepsilon_i$ and $\beta = \sum_{i=11-r}^{10} m^i \varepsilon_i$ (note that the basis elements satisfy $(\varepsilon_i|\varepsilon_j) = \delta_{ij} - (1/9)$ and have length $2\sqrt{2}/3$).

In fact, writing this change of basis as $\alpha_i = (R^{-1})_i^{\ j} \varepsilon_j$, we can invert this relation (which leads to the matrix R given in Appendix A.i)and obtain the metric corresponding to the scalar product (5.10), given in terms of the Cartan matrix as:

$$g_\varepsilon = RAR^\top\,, \tag{5.11}$$

in the simply-laced case we are interested in.

As seen in Section 5.1.2, the Cartan matrix of the $\mathfrak{e}_r$ series is a realization of $(\mathfrak{h}, \Pi, \Pi^\vee)$, where now $\Pi^\vee = \{\alpha_{9-r}^\vee, \dots, \alpha_8^\vee\} \cong \Pi$. Then, from $A_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle = (\alpha_i|\alpha_j)$ we have:

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & \cdots & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} \tag{5.12}$$

and A corresponds to a Dynkin diagram of the type:

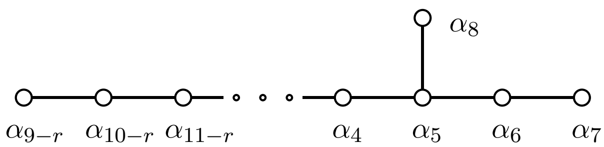


Figure 5.1: Dynkin diagram of the E_r series

The simple coroots, in turn, form a basis of the derived Cartan subalgebra $\mathfrak{h}'$, and we may choose (or alternatively define) $H_i \doteq \alpha_i^\vee$, $\forall i = 9 - r, \dots, 8$. Since we consider simply laced-cases only, the relation $\alpha_i^\vee = (AR^\top)_{ij}\varepsilon^{\vee j}$ determines the dual physical basis for $r \neq 9$, *i.e.* $D \neq 2$:

$$\begin{aligned}
H_{9-r} &= \varepsilon^{\vee\, 11-r} - \varepsilon^{\vee\, 12-r} = (1, -1, 0, \dots, 0) \\
&\vdots \\
H_7 &= \varepsilon^{\vee\, 9} - \varepsilon^{\vee\, 10} = (0, \dots, 0, 1, -1)\,, \\
H_8 &= -\frac{1}{3}\sum_{i=11-r}^{7} \varepsilon^{\vee\, i} + \frac{2}{3}\sum_{i=8}^{10}\varepsilon^{\vee\, i} = \frac{1}{3}(-1, \dots, -1, 2, 2, 2)\,.
\end{aligned} \tag{5.13}$$

In the same spirit as before, this dual basis is equipped with a scalar product given in terms of the metric $g_\varepsilon^\vee = (g_\varepsilon)^{-1}$ as:

$$B(H, H') = \sum_{i=11-r}^{10} h_i h_i' + \frac{1}{9-r}\sum_{i,j=11-r}^{10} h_i h_j'\,, \qquad \text{for } r \neq 9 \tag{5.14}$$

for two elements $H = \sum_{i=11-r}^{10} h_i \varepsilon^{\vee\, i}$ and $H' = \sum_{i=11-r}^{10} h_i' \varepsilon^{\vee\, i}$. In the affine case $r = 9$, the Cartan matrix is degenerate. In order to determine $B(H, H')$, one has to work in the whole Cartan subalgebra $\mathfrak{h}$, and not only in the derived one, and include a basis element related to the scaling operator d. Consequently, there is no meaningful physical basis in this case.

Not surprisingly, we recognize in (5.14) the Killing form of $\mathfrak{e}_r$ restricted to $\mathfrak{h}(\mathfrak{e}_r)$. Since the dual metric is the inverse of $g_\varepsilon^\vee$, the duality bracket is defined as usual as $\langle \varepsilon_i, \varepsilon^{\vee\, j}\rangle = \varepsilon_i(\varepsilon^{\vee\, j}) = \delta_i^{\ j}$, so that consistently:

$$\langle \alpha^\vee, \beta\rangle = (\nu(\alpha^\vee)|\beta) = \beta(H_\alpha)\,. \tag{5.15}$$

Since $\mathfrak{e}_r$ is simply-laced, $\nu(\alpha^\vee) = \alpha$, and we then have various ways of expressing the Cartan matrix:

$$A_{ij} \doteq \alpha_i(H_j) \equiv (\alpha_i|\alpha_j) \equiv B(H_i, H_j)\,. \tag{5.16}$$

A choice for structure constants

We now fix the conventions for the E_r series that will hold throughout the paper. For obvious reasons of economy, we introduce the following compact notation to characterize $\mathfrak{e}_r$ generators:

Notation 5.2.2 Let X_α be a generator of the root space $(\mathfrak{e}_r)_\alpha$, or of the dual subspace $\mathfrak{h}_\alpha \subset \mathfrak{h}$ for some root $\alpha = \sum_{i=9-r}^{8} k^i\alpha_i \in \Delta(\mathfrak{e}_r)$. We write the corresponding generator as

$$X_{(9-r)^{k^{9-r}}\dots 8^{k^8}} \quad \text{instead of} \quad X_{k^{9-r}\alpha_{9-r}+\dots+k^8\alpha_8}\,.$$

For example we will write:

$$E_{45^2678} \quad \text{instead of} \quad E_{\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_8}\,,$$

and similarly for F and H. Sometimes, we will also write $\alpha_{(9-r)^{k^{9-r}}\dots 8^{k^8}}$ instead of $\sum_{i=9-r}^{8} k^i\alpha_i$.

Furthermore, δ always refers to the isotropic root of $\mathfrak{e}_9$, namely $\delta = \delta_{E_9} = \alpha_{01^22^33^44^55^66^47^28^3}$, $c = H_\delta$ to its center, and d to its usual derivation operator $d = d_{E_9}$. Possible subscripts added to δ, c and d will be used to discriminate $\mathfrak{e}_9$ objects from objects belonging to its subalgebras.

Moreover, we use for $\mathfrak{e}_9$ the usual construction based on the loop algebra $\mathcal{L}(\mathfrak{e}_8) \doteq \mathbb{C}[z, z^{-1}] \otimes \mathfrak{e}_8$, $z \in \mathbb{C}$. The affine KMA $\mathfrak{e}_9$ is then obtained as a central extension thereof:

$$\mathfrak{e}_9 = \mathcal{L}(\mathfrak{e}_8) \oplus \mathbb{C}c \oplus \mathbb{C}d\,.$$

and is spanned by the basis of vertex operators satisfying

$$\begin{aligned}
[z^m \otimes H_i, z^n \otimes H_j] &= m\delta_{ij}\delta_{m+n,0}\, c\,, \\
[z^m \otimes H_i, z^n \otimes E_\alpha] &= \langle \alpha_i^\vee, \alpha\rangle z^{m+n} \otimes E_\alpha\,, \\
[z^m \otimes E_\alpha, z^n \otimes E_\beta] &= \left\{ \begin{array}{ll} \mathcal{N}_{\alpha,\beta} z^{m+n} \otimes E_{\alpha+\beta}\,, & \text{if } \alpha+\beta \in \Delta(\mathfrak{e}_9) \\ z^{m+n} \otimes H_\alpha + m\delta_{m+n,0}\, c\,, & \text{if } \alpha = -\beta \\ 0\,, & \text{otherwise} \end{array} \right. .
\end{aligned} \tag{5.17}$$

In addition, the Hermitian scaling operators $d = z\frac{d}{dz}$ defined from $z \in S^1$ normalizes $\mathcal{L}(\mathfrak{e}_8)$: $[d, z^n \otimes X] = nz^n \otimes X$, $\forall X \in \mathfrak{e}_8$.

In $\mathfrak{e}_{10}$, for which there is no known vertex operator construction yet, we rewrite the $\mathfrak{e}_9$ subalgebra according to the usual prescriptions for KMAs by setting: $d = -H_{-1}$, $E^a_{n\delta} = z^n \otimes H_a$, with $a = 1,..,8$ its multiplicity, $E_{\alpha+n\delta} = z^n \otimes E_\alpha$ and $E_{-\alpha+n\delta} = z^n \otimes F_\alpha$, and similarly for negative-root generators.

Finally, there is a large number of mathematically acceptable sign conventions for the structure constants $\mathcal{N}_{\alpha,\beta}$, as long as one satisfies the anti-commutativity and Jacobi identity of the Lie bracket, as explained in Definition 5.1.5. If one decides to map physical fields to generators of a KMA, which will eventually be done in this paper, one has to make sure that the adjoint action of a rotation with positive angle leads to a positive rotation of all physical tensors carrying a covariant index affected by it. This physical requirement imposes more stringent constraints on the structure constants. Though perhaps not the most natural choice from a mathematical point of view, we fix signs according to a lexicographical ordering for level 0 ($\mathfrak{sl}(r,\mathbb{R})-$) roots, but according to an ordering based on their height for roots of higher level in α_8. More concretely, if $\alpha = \alpha_{j\cdots k}$ has level 0, we set:

$$\mathcal{N}_{\alpha_i,\alpha} = \left\{ \begin{array}{l} 1 \text{ if } i < j \\ -1 \text{ if } k < i \end{array} \right. .$$

On the other hand, we fix $\mathcal{N}_{5,8} = +1$, and always take the positive sign when we lengthen a chain of simple roots of level $l > 0$ by acting with a positive simple root generator from the left, i.e.:

$$\mathcal{N}_{\alpha_i,\alpha} = 1\,,\ \forall \alpha \text{ s.t. } l(\alpha) > 0\,.$$

Structure constants for two non-simple and/or negative roots are then automatically fixed by these choices.

5.2.2 Toroidally reduced $11D$ supergravity: scalar fields and roots of E_{11-D}

In this section, we rephrase the mapping between scalar fields of 11D supergravity on T^q, $q \geqslant 3$, and the roots of its finite U-duality algebras, in a way that will make clear the extension to the infinite-dimensional case. We start with $\mathcal{N} = 1$ classical $11D$ supergravity, whose bosonic sector is described by the Lagrangian:

$$S_{11} = \frac{1}{l_P^9} \int d^{11}x\, e\Big(R - \frac{l_P^6}{2\cdot 4!}(G_4)^2\Big) + \frac{1}{2\cdot 3!} \int C_3 \wedge G_4 \wedge G_4 \tag{5.18}$$

where the four-form field strength is exact: $G_4 = dC_3$. There are various conventions for the coefficients of the three terms in the Lagrangian (5.18), which depend on how one defines the fermionic sector of the theory. In any case, the factors of Planck length can be fixed by dimensional analysis. Here, we adopt the conventions of [203], where we have, in units of length

$$[g_{AB}] = 2\,, \quad [C_{ABC}] = 0\,, \quad [d] = [dx] = 0\,.$$

As a consequence of the above: $[R] = -2$.

The action (5.18) rescales homogeneously under:

$$g_{AB} \to M_P^2 g_{AB}\,, \quad C_{ABC} \to M_P^3 C_{ABC} \tag{5.19}$$

which eliminates all l_P terms from the Einstein-Hilbert and gauge Lagrangian while rescaling the Chern-Simons (CS) part by l_P^{-9}, and renders, in turn, g_{AB} dimensionless. This is the convention we will adopt in the following which will fix the mapping between the E_{11-D} root system and the fields parametrizing the scalar manifold of the reduced theory for $D \geqslant 3$. How to extend this analysis to the conjectured affine and hyperbolic U-duality groups E_9 and E_{10} will be treated in the next section.

The Kaluza-Klein reduction of the theory to $D \geqslant 3$ dimensions is performed according to the prescription

$$ds_{11}^2 = e^{\frac{\sqrt{2}}{D-2}(\rho_D|\varphi)} ds_D^2 + \sum_{i=D}^{10} e^{-\sqrt{2}(\varepsilon_i|\varphi)} \left(\tilde{\gamma}^i{}_j dx^j + \mathcal{A}_1^i\right)^2 \,, \tag{5.20}$$

with $\tilde{\gamma}^i{}_j = (\delta^i{}_j + \mathcal{A}^i_{0\,j})$ with $i<j$ for $\mathcal{A}^i_{0\,j}$. The compactification vectors ε_i are the ones defining the physical basis of Definition 5.2.1, and can be expressed in the orthonormal basis $\{e_i\}_{i=1}^{11-D}$, $e_i \cdot e_j = \delta_{ij}$, as

$$\varepsilon_k = -\sum_{i=1}^{10-k} \frac{1}{\sqrt{(10-i)(9-i)}} e_i + \sqrt{\frac{k-2}{k-1}} e_{11-k}\,. \quad \text{for } k \leqslant 8\,.$$

In the $D = 2, 1$ cases, the additional vectors completing the physical basis are defined formally, without reference to the compactification procedure.

Accordingly, the vector of dilatonic scalars can be expanded as $\varphi = \sum_{i=1}^{11-D} \varphi_i e_i$. We will however choose to stick to the physical basis. The expression of ε_k in terms of the orthonormal basis will help to connect back to the prescription of [79] and [168, 169]. In this respect, the scalar product $(\,|\,)$ used in expression (5.20) is precisely the product on the root system (5.10). Finally, we also introduce the "threshold" vector

$$\rho_D = \sum_{i=D}^{10} \varepsilon_i \tag{5.21}$$

which will be crucial later on when studying the structure of Minkowskian objects in E_{10}.

From expression (5.20), we see that the elfbein produces $(11-D)$ one-forms $\mathcal{A}_1^i$ and $(11-D)(12-D)/2$ scalars $\mathcal{A}^i_{0\,j}$, whereas the three form generates the following two-, one- and zero-form potentials: $(11-D)$ $C_{2\,i}$, $(11-D)(10-D)/2$ $C_{1\,ij}$ and $(11-D)(10-D)(9-D)/6$ $C_{0\,ijk}$. The reduction of the $11D$ action (5.18) to any dimension greater than two reads:

$$\begin{aligned} e^{-1}\mathcal{L} \;=\; & R - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2\cdot 4!} e^{\sqrt{2}(\kappa|\varphi)} (\underline{G}_4)^2 - \frac{1}{2\cdot 3!}\sum_i e^{\sqrt{2}(\kappa_i|\varphi)} (\underline{G}_{3\,i})^2 \\ & -\frac{1}{2\cdot 2!}\sum_{i<j} e^{\sqrt{2}(\kappa_{ij}|\varphi)} (\underline{G}_{2\,ij})^2 - \frac{1}{2\cdot 2!}\sum_i e^{\sqrt{2}(\lambda_i|\varphi)} (\underline{\mathcal{F}}_2^i)^2 \\ & -\frac{1}{2}\sum_{i<j<k} e^{\sqrt{2}(\kappa_{ijk}|\varphi)} (\underline{G}_{1\,ijk})^2 - \frac{1}{2}\sum_{i<j} e^{\sqrt{2}(\lambda_{ij}|\varphi)} (\underline{\mathcal{F}}^i_{1\,j})^2 + e^{-1}\mathcal{L}_{CS} \end{aligned} \tag{5.22}$$

where $\mathcal{L}_{CS}$ is the reduction of the CS-term $C_3 \wedge G_4 \wedge G_4$, and again indices run according to $i, j, k = D, .., 10$. The field strengths appearing in the above kinetic term exhibit the exterior derivative of the corresponding potentials as leading term, but contain additional non-linear Kaluza-Klein modifications. For instance:

$$\begin{aligned}
&\underline{G}_4 = G_4 - \gamma^i{}_j G_{3\,i} \wedge \mathcal{A}_1^j + \gamma^i{}_k \gamma^j{}_l G_{2\,ij} \wedge \mathcal{A}_1^k \wedge \mathcal{A}_1^l + \dots\,, && \underline{\mathcal{F}}_2^i = \mathcal{F}_2^i - \gamma^j{}_k \mathcal{F}_{1\,j}^i \wedge \mathcal{A}_1^k\,,\\
&\vdots && \underline{\mathcal{F}}_{1\,j}^i = \gamma^k{}_j \mathcal{F}_{1\,k}^i\,.\\
&\underline{G}_{2\,ij} = \gamma^m_i \gamma^n_j G_{2\,mn} - \gamma^l{}_i \gamma^m_j \gamma^n{}_k G_{1\,lmn} \wedge \mathcal{A}_1^k\,, &&\\
&\underline{G}_{1\,ijk} = \gamma^l{}_i \gamma^m_j \gamma^n{}_k G_{1\,lmn}\,, &&
\end{aligned} \tag{5.23}$$

with $\gamma^i{}_j = (\tilde{\gamma}^{-1})^i{}_j$, the not-underlined field strengths being total derivatives: $G_{(n)\,i_1\cdots i_l} = dC_{(n-1)\,i_1\cdots i_l}$ and $\mathcal{F}^i_{(n)\,i_1\cdots i_l} = d\mathcal{A}^i_{(n-1)\,i_1\cdots i_l}$, where n is the rank of the form. The whole set of field strengths and the details of the reduction of the CS term are well known and can be found in [185].

The global symmetry of the scalar manifold which, upon quantization, is conjectured to become the discrete U-duality symmetry of the theory is encoded in the compactification vectors $\overline{\Delta} = \{\kappa; \kappa_i; \kappa_{ij}; \kappa_{ijk}; \lambda_i; \lambda_{ij}\}$ appearing in the Lagrangian (5.22). As pointed out previously, this global symmetry will only be manifest if potentials of rank $D-2$ are dualized to scalars, thereby allowing gauge symmetries to be replaced by internal ones. In each dimension D, this will select a subset of $\overline{\Delta}$ to form the positive root system of E_{11-D}. One has to keep in mind, however, that in even space-time dimensions, this rigid symmetry is usually only realized on the field strengths themselves, and not on the potentials. This is attributable to the customary difficulty of writing a covariant lagrangian for self-dual fields [4]. In even dimensional cases then, the E_{11-D} symmetry appears as a local field transformation on the solution of the equations of motion.

We now give the whole set $\overline{\Delta}$ in the physical basis. One has to bear in mind that some of these vectors become roots only in particular dimensions, and thus do not, in general, have squared length equal to 2. In constrast, λ_{ij} and κ_{ijk} are always symmetries of the scalar manifold, and can therefore be directly translated into positive roots of E_{11-D} for the first two levels $l = 0, 1$ in α_8. We have for $i < j < k$:

$$\begin{aligned}
l = 0: &\quad W_{\mathrm{KKp}} \ni \lambda_{ij} = \varepsilon_i - \varepsilon_j\\
l = 1: &\quad W_{\mathrm{M2}} \ni \kappa_{ijk} = \varepsilon_i + \varepsilon_j + \varepsilon_k
\end{aligned}$$

In fact, they build orbits of E_{11-D} under the Weyl group of $SL(11-D, \mathbb{R})$, which we denote by W_{KKp} and W_{M2}, anticipating results from M-theory on T^{10} which associates λ_{ij} with Euclidean KK particles and κ_{ijk} with Euclidean M2-branes. So, in $10 \geqslant D \geqslant 6$, since no dualization occurs, the root system of the U-duality algebra is completely covered by W_{KKp} and W_{M2}, with the well known results $G^U = \{SO(1,1); GL(2,\mathbb{R}); SL(3,\mathbb{R}) \times SL(2,\mathbb{R}); SL(5,\mathbb{R}); SO(5,5)\}$, for $D = \{10; 9; 8; 7; 6\}$.

For $D = 5$, we dualize $\underline{G}_4 = e^{-\kappa\cdot\varphi} * \underline{\widetilde{G}}_1$, with $-\kappa = \theta_{E_6} = ((1)^6)$ the highest root of E_6, which constitutes a Weyl orbit all by itself. For highest roots of the Lie algebra relevant to our purpose, we refer the reader to Appendix 5.12 ii). In $D = 4$, we dualize $(\underline{G}_{3\,i}) = e^{-\kappa_i\cdot\varphi} * \underline{\widetilde{G}}_{1\,i}$, with $-\kappa_i = ((1)^{i-1}, 0_i, (1)^{7-i})$ forming the Weyl orbit of θ_{E_7} (which contains θ_{E_6}). Finally, in $D = 3$, dualizing $\underline{G}_{2\,ij} = e^{-\kappa_{ij}\cdot\varphi} * \underline{\widetilde{G}}_{1\,ij}$ and $\underline{\mathcal{F}}^i_2 = e^{-\lambda_i\cdot\varphi} * \underline{\widetilde{\mathcal{F}}}^i_1$ increases the size of the former θ_{E_7} Weyl orbit and creates the remaining θ_{E_8} orbit:

$$\begin{aligned}
D = 5 \quad l = 2: &\quad W_{\mathrm{M5}} \ni -\kappa = \tfrac{3}{D-2}\rho_D\\
D = 4 \quad l = 2: &\quad W_{\mathrm{M5}} \ni -\kappa_i = \tfrac{2}{D-2}\rho_D - \varepsilon_i\\
D = 3 \quad l = 2: &\quad W_{\mathrm{M5}} \ni -\kappa_{ij} = \tfrac{1}{D-2}\rho_D - \varepsilon_i - \varepsilon_j\\
l = 3: &\quad W_{\mathrm{KK7M}} \ni -\lambda_i = \tfrac{1}{D-2}\rho_D + \varepsilon_i
\end{aligned} \tag{5.24}$$

[4] In some cases -for the $11D$ five-brane for instance-, this can be achieved by resorting to the Pasti-Sorokin-Tonin formalism.

For the same reason as before, we denote these two additional orbits $W_{\rm M5}$ and $W_{\rm KK7M}$ since they will be shown to describe totally wrapped Euclidean M5-branes and KK monopoles. For $D=3$, for instance, we can check that $\dim W_{\rm KKp} = \dim W_{\rm M5} = 28$, $\dim W_{\rm M2} = 56$ and $\dim W_{\rm KK7M} = 8$, which reproduces the respective number of scalars coming from the KK gauge fields, 3-form, and their magnetic duals, and verifies $\dim \Delta_+(E_8) = \sum_i \dim W_i$.

For what follows, it will turn out useful to take advantage of the dimensionless character of the vielbein, resulting from the rescaling (5.19), to rewrite the internal metric in terms of the duality bracket (5.15):

$$ds^2_{11-D} = \sum_{i=D}^{10} e^{2\langle H_R,\varepsilon_i\rangle}\delta_{ij}\tilde{\gamma}^i{}_k\tilde{\gamma}^j{}_l dx^k\otimes dx^l \tag{5.25}$$

with $H_R = \sum_{i=D}^{10}\ln(M_P R_i)\varepsilon^{\vee i}$. Thus in particular: $e^{-\sqrt{2}(\varepsilon_i|\varphi)} = (M_P R_i)^2$. In this convention, the scalar Lagrangian for $D=3$ reads

$$\begin{aligned}
&-eg^{AB}(g^\vee_\varepsilon)^{ij}\left(\frac{\partial_A R_i}{R_i}\right)\left(\frac{\partial_B R_j}{R_j}\right) - \frac{1}{2}e\sum_{i<j<k}\frac{1}{(M_P^3 R_i R_j R_k)^2}(\underline{G}_{1\,ijk})^2\\
&-\frac{1}{2}e\sum_{i<j}\left(\frac{R_j}{R_i}\right)^2(\underline{\mathcal{F}}^i_{1\,j})^2 - \frac{1}{4}e\sum_{i<j}\left(\frac{R_iR_j}{M_P^6 V_8}\right)^2(\underline{\widetilde{G}}_{1\,ij})^2 - \frac{1}{4}e\sum_i\left(\frac{1}{M_P^9 R_i V_8}\right)^2(\underline{\widetilde{\mathcal{F}}}^i_1)^2
\end{aligned} \tag{5.26}$$

with the dual metric $(g^\vee_\varepsilon)^{ij} = \delta^{ij} + (D-2)^{-1}\sum_{k,l}\delta^{ik}\delta^{lj}$ (5.14) and the internal volume $V_8 = \prod_{i=3}^{10} R_i$. Clearly, the coefficients[5] in front of the one-form kinetic terms reproduce the inverse squared tensions for totally wrapped Euclidean KK particles, M2-branes, M5-branes and KK monopoles (KK7M-branes). This will be the touchstone of our analysis, and in the next section we will present, following [56], how the corresponding Minkowskian branes arise in $D=1$ in the framework of $E_{10|10}/K(E_{10|10})$ hyperbolic billiards.

5.2.3 Non-linear realization of supergravity: the triangular and symmetric gauges

The final step towards unfolding the hidden symmetry of the scalar manifold of the reduced theory consists in showing that one can construct its Langrangian density as a coset σ-model from a non-linear realization. Here, we rederive, in the formalism we use later on, only the most symmetric $D=3$ case, since the $D\geqslant 4$ constructions are obtainable as restriction thereof by referring to table (5.24). For the detailed study of the $D>3$ cases, see [79].

Furthemore, the use of a parametrization of the coset sigma-model based on the Borel subalgebra of the U-duality algebra, called triangular gauge, is crucial to this type of non-linear realization. In contrast, we will show in the second part of this section, that the most natural setup to treat orbifolds of the corresponding supergravities is given by a parametrization of the coset based on the Cartan decomposition of the U-duality algebra. We refer to this choice as the symmetric gauge.

In $D=3$, the non-linear realization of the scalar manifold is based on the group element:

$$\begin{aligned}
g \;=\; &\exp\Big[-\frac{1}{\sqrt{2}}\sum_i \ln(M_P R_i)\varepsilon^{\vee i}\Big]\cdot\Big(\prod_{i<j} e^{\mathcal{A}^i{}_j K^{+j}_i}\Big)\cdot\exp\Big[\sum_{i<j<k} C_{ijk}Z^{+\,ijk}\Big]\cdot\\
&\cdot\exp\Big[\sum_{i_1<..<i_6}\widetilde{C}_{i_1..i_6}\widetilde{Z}^{+\,i_1..i_6}\Big]\cdot\exp\Big[\sum_{i_1<..<i_8,j}\widetilde{\mathcal{A}}^j{}_{i_1..i_8}\widetilde{K}^{+\,i_1..i_8}_j\Big],
\end{aligned} \tag{5.27}$$

[5] Note that the interpretation of these coefficients is somewhat different than in [203], since here we are working in the conformal Einstein frame.

where the dual potentials are reformulated to exhibit a tensorial rank that would generalize to $D < 3$ (we drop all 0 subscripts since we are only dealing with scalars). In particular, they are related to the $D = 3$ dual potentials as $\widetilde{C}_{i_1..i_6} = \frac{1}{2!}\epsilon_{i_1..i_6kl}\widetilde{C}_0{}^{kl}$ and $\widetilde{\mathcal{A}}^j{}_{i_1..i_8} = \epsilon_{i_1.i_8}\widetilde{\mathcal{A}}^j_0$, by the totally antisymmetric rank 8 tensor of $SL(8,\mathbb{R})$, $\epsilon_{i_1..i_8}$. These entered expression (5.26) as $\widetilde{G}_{1\,ij} = d\widetilde{C}_{0\,ij}$ and $\widetilde{\mathcal{F}}^i_1 = d\widetilde{\mathcal{A}}^i_0$.

The group element (5.27) is built out of the Borel subalgebra of E_8, which is spanned by the following raising operators

$$\begin{aligned}
[K_i^{+\,j}, K_k^{+\,l}] &= \delta^j_k K_i^{+\,l} - \delta^l_i K_k^{+\,j}\,, \\
[K_i^{+\,j}, Z^{+\,k_1k_2k_3}] &= -3\delta^{[k_1}_i Z^{+\,|j|k_2k_3]}\,, & [K_i^{+\,j}, \widetilde{Z}^{+\,k_1..k_6}] &= -6\delta^{[k_1}_i \widetilde{Z}^{+\,|j|k_2..k_6]}\,, \\
[K_i^{+\,j}, \widetilde{K}_k^{+\,k_1..k_8}] &= \delta^j_k \widetilde{K}_i^{+\,k_1..k_8}\,, \\
[Z^{+\,i_1i_2i_3}, Z^{+\,i_4i_5i_6}] &= -\widetilde{Z}^{+\,i_1..i_6}\,, & [Z^{+\,i_1i_2i_3}, \widetilde{Z}^{+\,i_4..i_9}] &= -3\widetilde{K}^{+\,[i_1|i_2i_3]i_4..i_9}\,,
\end{aligned} \tag{5.28}$$

and the Cartan subalgebra, acting on the former as (without implicit summations on repeated indices)

$$\begin{aligned}
[\varepsilon^{\vee i}, K_j^{+\,k}] &= \lambda_{jk}(\varepsilon^{\vee i})\, K_j^{+\,k}\,, & [\varepsilon^{\vee i}, Z^+_{jkl}] &= \kappa_{jkl}(\varepsilon^{\vee i})\, Z^+_{jkl}\,, \\
[\varepsilon^{\vee i}, \widetilde{Z}^+_{j_1..j_6}] &= -\sum_{l<m} \epsilon_{j_1..j_6}{}^{lm}\,\kappa_{lm}(\varepsilon^{\vee i})\widetilde{Z}^+_{j_1..j_6}\,, & [\varepsilon^{\vee i}, \widetilde{K}_j^{+\,k_1..k_8}] &= -\kappa_j(\varepsilon^{\vee i})\, \widetilde{K}_j^{+\,k_1..k_8}\,.
\end{aligned}$$

Anticipating the extension to $D = 2, 1$, we redefine positive roots $\kappa_{i_1..i_6} \doteq \sum_{l<m} \epsilon_{i_1..i_6}{}^{lm}\kappa_{lm}$ and $\lambda_{j|i_1..i_8} \doteq \epsilon_{i_1..i_8}\kappa_j$ corresponding to the generators $\widetilde{Z}$ and $\widetilde{K}$. The scalar Lagrangian in D dimensions is then expressible as a coset sigma-model, obtained from the algebraic field strength $\mathcal{G} \doteq g^{-1}dg$

$$\begin{aligned}
\mathcal{G} \;=\; & -\frac{1}{\sqrt{2}}\sum_i \ln(M_P R_i)\varepsilon^{\vee i} + \sum_{i<j<k} e^{\langle H_R, \kappa_{ijk}\rangle}\underline{G}_{1\,ijk} Z^{+\,ijk} + \sum_{i<j} e^{\langle H_R, \lambda_{ij}\rangle}\underline{\mathcal{F}}^i_{1\,j} K_i^{+\,j} \\
& + \sum_{i_1<..<i_6} e^{-\langle H_R, \kappa_{i_1..i_6}\rangle}\underline{\widetilde{G}}_{1\,i_1..i_6}\widetilde{Z}^{+\,i_1..i_6} + \sum_{i_1<..<i_8, j} e^{-\langle H_R, \lambda_{j|i_1..i_8}\rangle}\underline{\widetilde{\mathcal{F}}}_1{}^j{}_{i_1..i_8}\widetilde{K}_j^{+\,i_1..i_8}\,.
\end{aligned} \tag{5.29}$$

This particular parametrization of the coset $\mathfrak{e}_{8|8}/\mathfrak{k}(\mathfrak{e}_{8|8})$ is known as the Iwasawa decomposition. In other words, the split form $\mathfrak{g}^U = \mathfrak{e}_{11-D|11-D}$ decomposes as a sum of closed factors $\mathfrak{g}^U = \mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$, where $\mathfrak{k}$ is its maximal compact subalgebra. Then, the coset $\mathfrak{g}^U/\mathfrak{k} = \mathfrak{a}\oplus\mathfrak{n}$ is parametrized by the direct sum of an abelian and a nilpotent subalgebra. This can be interpreted as a "gauge" choice, where the coset elements are either diagonal (Cartan generators) or upper triangular (Borel, or positive root, generators). In the following, this choice will be referred to as the *triangular gauge*.

The negative root generators can be retrieved from the Borel subalgebra by defining the appropriate transposition operation. Since we want it to be applicable to $\mathfrak{g}^U = \mathfrak{e}_{11-D|11-D}\ \forall D$, and not only to U-duality algebras with orthogonal maximal compact subalgebra, we construct it as in [79] out of the Cartan involution ϑ as $T(X) = -\vartheta(X)$, $\forall X \in \mathfrak{g}^U$. This induces a corresponding generalized transposition [79, 80] on the group level denoted by: $T(\mathcal{X}) = \Theta(\mathcal{X}^{-1})$, $\forall \mathcal{X} \in G^U$. In the present case, since $\mathfrak{g}^U$ is the split form, we have $\vartheta = \vartheta_C$, the latter being the Chevalley involution.

By requiring the following normalizations:

$$\begin{aligned}
&\mathrm{Tr}(\varepsilon^{\vee i}\varepsilon^{\vee j}) = 2(g^\vee_\varepsilon)^{ij}\,, \qquad \mathrm{Tr}\left(K_i{}^j T(K_k{}^l)\right) = \delta_{ik}\delta^{jl}\,, \\
&\mathrm{Tr}\left(X^{i_1..i_p|j_1..j_q} T(X^{k_1..k_p|l_1..l_q})\right) = p!q!\delta^{[i_1}_{k_1}\cdots\delta^{i_p]}_{k_p}\delta^{[j_1}_{l_1}\cdots\delta^{j_q]}_{l_q}\,,
\end{aligned}$$

where $X^{i_1..i_p|j_1..j_q}$ stand for the remaining generators of (5.28), the bosonic scalar Lagrangian (5.26) is readily obtained from the coset sigma-model

$$\mathcal{L} = -\frac{1}{2} e \mathrm{Tr}\left[g^{-1}\partial g(\mathbb{I}+T)g^{-1}\partial g\right] \equiv \frac{1}{4} e \mathrm{Tr}(\partial\mathcal{M}^{-1}\partial\mathcal{M}) \tag{5.30}$$

where $\mathcal{M} = gT(g)$ is the internal σ-model metric. The equations of motion for the moduli of the theory are then summarized in the Maurer-Cartan equation: $d\mathcal{G} = \mathcal{G}\wedge\mathcal{G}$. By adding the negative root generators, we restore the $K(E_{11-D})$ local gauge invariance, and enhance the coset to the full continuous U-duality group. In $D = 3$, for instance, we thus recover the dimension of E_8 as:

$$\begin{aligned} 248 &= 8(H_i) + 28(K_i^{+j}) + 56(Z^{+ijk}) + 28(\widetilde{Z}^{+i_1..i_6}) + 8(\widetilde{K}_j^{+i_1..i_8}) \\ &+\overline{28}\left(T(K_i^{+j})\right) + \overline{56}\left(T(Z^{+ijk})\right) + \overline{28}\left(T(\widetilde{Z}^{+i_1..i_6})\right) + \overline{8}\left(T(\widetilde{K}_j^{+i_1..i_8})\right) \end{aligned}$$

Note that the triangular gauge is not preserved by a rigid left transformation U from the symmetry group G^U: $g(x) \to Ug(x)$ for $g \in G^U/K(G^U)$. This leaves $\mathcal{G}$ invariant but will generally send g out of the positive root gauge. We will then usually need a local compensator $h(x) \in K(G^U)$ to bring it back to the original gauge. So the Lagrangian (5.30) is kept invariant by the compensated transformation $g(x) \to Ug(x)h(x)^{-1}$ which sends: $\mathcal{M} \to U\mathcal{M}T(U)$, provided $hT(h) = \mathbb{I}$.

If the triangular gauge is the natural choice to obtain a closed non-linear realization of a coset sigma-model, it will show to be quite unhandy when trying to treat orbifolds of reduced $11D$ supergravity and M-theory. In this case, a parametrization of the coset based on the Cartan decomposition into eigenspace of the Chevalley involution $\mathfrak{g}^U = \mathfrak{k}\oplus\mathfrak{p}$ is more appropriate. In other words, one starts from an algebraic field strength valued in $\mathfrak{g}^U$:

$$\widetilde{g}^{-1}d\widetilde{g} = \mathcal{P} + \mathcal{Q} \quad , \tag{5.31}$$

so that the coset is parametrized by:

$$\mathcal{P} = \frac{1}{2}(\mathbb{I}+T)\,g^{-1}dg\,. \tag{5.32}$$

and $\mathcal{Q}$ ensures local (now unbroken) $K(G^U)$ invariance of the model. Note that the Lagrangian (5.30) is, as expected, insensitive to this different parametrization since $\frac{1}{4}e\,\mathrm{Tr}(\partial\mathcal{M}^{-1}\partial\mathcal{M}) \equiv -e\,\mathrm{Tr}\,[\mathcal{P}T(\mathcal{P})]$.

We then associate symmetry generators to the moduli of compactified $11D$ supergravity / M-theory in the following fashion: for economy, we will denote all the Borel generators of $\mathfrak{sl}(11-D,\mathbb{R}) \subset \mathfrak{g}^U$ by K_i^{+j} for $i \leqslant j$, by setting in particular $K_i^{\pm i} = K_i{}^i \doteq \varepsilon^{\vee i}$. Using relation (5.13), the Cartan generators can now be reexpressed as

$$H_i = K_{i+2}{}^{i+2} - K_{i+3}{}^{i+3}\,, \quad i = 1,..,7\,, \qquad H_8 = -\frac{1}{3}\sum_{i=1}^{5} K_{i+2}{}^{i+2} + \frac{2}{3}\left(K_8{}^8 + K_9{}^9 + K_{10}{}^{10}\right)\,.$$

The dictionary relating physical moduli and coset generators can then be established for all moduli fields corresponding to real roots of level $l = 0,1,2,3$, generalizing to $D = 2,1$ the previous result (5.24). We will denote the generalized transpose of a Borel generator X^+ as

$X^- \doteq T(X^+)$.

modulus	generator	physical basis
$\ln(M_P R_i)$	$K_i^{\ i}$	$\varepsilon^{\vee i}$
$\mathcal{A}^i_{\ j}$	$K_i^{\ j} = \frac{1}{2}\left(K_i^{+\,j} + K_i^{-\,j}\right)$	$\varepsilon_i - \varepsilon_j$
C_{ijk}	$Z^{ijk} = \frac{1}{2}\left(Z^{+\,ijk} + Z^{-\,ijk}\right)$	$\varepsilon^i + \varepsilon^j + \varepsilon^k$
$\widetilde{C}_{i_1..i_6}$	$Z^{i_1..i_6} = \frac{1}{2}\left(Z^{+\,i_1..i_6} + Z^{-\,i_1..i_6}\right)$	$\sum_{l=1}^{6} \varepsilon_{i_l}$
$\widetilde{\mathcal{A}}^j_{\ i_1..i_8},\ j \in \{i_1,..,i_8\}$	$\widetilde{K}_j^{\ i_1..i_8} = \frac{1}{2}\left(\widetilde{K}_j^{+\,i_1..i_8} + \widetilde{K}_j^{-\,i_1..i_8}\right)$	$\sum_{l=1}^{8}(1+\delta_j^{i_l})\varepsilon_{i_l}$

(5.33)

This list exhausts all highest weight $\mathfrak{sl}(11-D,\mathbb{R})$ representations present for $D = 3$. In the infinite-dimensional case, there is an infinite number of other $\mathfrak{sl}(11-D,\mathbb{R})$ representations. The question of their identification is still a largely open question. Progresses have been made lately in identifying some roots of E_{10} as one-loop corrections to $11D$ supergravity [87] or as Minkowskian M-branes and additional solitonic objects of M-theory [56]. These questions will be introduced in the next section, and will become one of the main topics of the last part of this paper.

However, it is worth noting that for $D \leqslant 2$, the 8-form generator is now subject to the Jacobi identity

$$\widetilde{K}^{[i_1|i_2..i_9]} = 0\,. \tag{5.34}$$

In $D = 2$, this reflects the fact that the would-be totally antisymmetric generator $\widetilde{K}^{[i_1 i_2..i_9]}$ attached to the null root δ is not the dual of a supergravity scalar, but corresponds to the root space $\{z \otimes H_i\}_{i=1,..,8}$, and reflects the localization of the U-duality symmetry.

In addition, we denote the compact generators by $\mathcal{K}_i^{\ j} = K_i^{+\,j} - K_i^{-\,j}$, $\mathcal{Z}^{ijk} = Z^{+\,ijk} - Z^{-\,ijk}$ and similarly for $\mathcal{Z}^{i_1..i_6}$ and $\widetilde{\mathcal{K}}_j^{i_1..i_8}$. Then:

$$K(G^U) = \text{Span}\left\{\mathcal{K}_i^{\ j};\mathcal{Z}^{ijk};\mathcal{Z}^{i_1..i_6};\widetilde{\mathcal{K}}_j^{\ i_1..i_8}\right\}$$

Fixing the normalization of the compact generators to 1 has been motivated by the algebraic orbifolding procedure we will use in the next sections, and ensures that automorphism generators and the orbifold charges they induce have the same normalization.

In particular, the compact Lorentz generators $\mathcal{K}_i^{\ i+1} \equiv E_{\alpha_{i-2}} - F_{\alpha_{i-2}}$, $\forall i = D, \ldots, 9$ clearly generate rotations in the $(i, i{+}1)$-planes, so that a general rotation in the (i,j)-plane is induced by $\mathcal{K}_i^{\ j}$. One can check that, as expected, $[\mathcal{K}_k^{\ j}, X_{i_1..j..i_p}] = X_{i_1..k..i_p}$ $\forall X \in \mathfrak{g}^U/\mathfrak{k}(\mathfrak{g}^U)$ in Table (5.33). For instance, the commutator $[\mathcal{K}_i^{\ i+1}, Z_{i+1\,jk}]$ for $i+1<j<k$ belongs to the root space of:

$$\alpha = \alpha_{i-2} + .. + \alpha_{j-3} + 2(\alpha_{j-2} + .. + \alpha_{k-3}) + 3(\alpha_{k-2} + .. + \alpha_5) + 2\alpha_6 + \alpha_7 + \alpha_8$$

that defines $Z_{ijk} = (1/2)(E_\alpha + F_\alpha)$.

As a final remark, note that the group element $\widetilde{g}$ with value in G^U can be used to reinstate local $K(G^U)$-invariance of the algebraic field strength $\widetilde{g}^{-1}d\widetilde{g} = \mathcal{P} + \mathcal{Q}$ under the transformation as $\widetilde{g}(x) \to U\widetilde{g}(x)h(x)^{-1}$ for $h(x) \in K(G^U)$ and a rigid U-duality element $U \in G^U$. In this case, $\mathcal{Q}$ transforms as a generalized connection:

$$\mathcal{Q} \to h(x)\mathcal{Q}h(x)^{-1} - h(x)\,dh(x)^{-1}\,,$$

and $\mathcal{P}+\mathcal{Q}$ as a generalized field-strength: $\mathcal{P}+\mathcal{Q} \to U^{-1}(\mathcal{P}+\mathcal{Q})U$. Performing a level expansion of $\mathcal{Q}$:

$$\mathcal{Q} = \frac{1}{2}dx^A\left(\omega_{Aj}^{\ \ \ i}\mathcal{K}_i^{\ j} + \omega_A^{\ \ ijk}\mathcal{Z}_{ijk}\right) + \ldots$$

we recognize for $l = 0$ the Lorentz connection, for $l = 1$ the 3-form gauge connection, etc.

Actually our motivation for working in the symmetric gauge comes from the fact that, at the level of the algebra, the orbifold charge operator acting as $\text{Ad}h$ preserves this choice. Indeed $h \in K(G^U)$ is in this case a rigid transformation, so that one can drop the connection part $\mathcal{Q}$ in expression (5.31), and $\text{Ad}h$ normalizes $\mathcal{P}$.

Along this line, a non-linear realization where only local Lorentz invariance is implemented has been used extensively in [246, 245, 244] to uncover very-extended Kac-Moody hidden symmetries of various supergravity theories. This has led to the conjecture that $\mathfrak{e}_{11}$ is a symmetry of $11D$ supergravity, and possibly M-theory, as this very-extended algebra can be obtained as the closure of the finite Borel algebra of a non-linear realization similiar to the one we have seen above, with the $11D$ conformal algebra.

5.2.4 M-theory near a space-like singularity as a $E_{10|10}/K(E_{10|10})$ σ-model

In the preceding section, we have reviewed some basic material about $11D$ supergravity compactified on square tori, which we will need in this paper to derive the residual U-duality symmetry of the untwisted sector of the theory when certain compact directions are taken on a orbifold. The extension of this analysis to the orbifolded theory in $D = 2, 1$ dimensions, where KM hidden symmetries are expected to arise, will require a generalization of the low-energy effective supergravity approach. The proper framework to treat hidden symmetries in $D = 1$ involves a σ-model based on the infinite coset $E_{10|10}/K(E_{10|10})$. In the vicinity of a space-like singularity, this type of model turns out to be a generalization of a Kasner cosmology, leading to a null geodesic motion in the moduli space of the theory, interrupted by successive reflections against potential walls. This dynamics is usual referred to as a cosmological billiard, where by billiard, we mean a convex polyhedron with finitely many vertices, some of them at infinity.

In [84, 86] the classical dynamics of M-theory near a spacelike singularity has been conjectured to possess a dual description in terms of this chaotic hyperbolic cosmological billiard. In particular, these authors have shown that, in a small tension limit $l_p \to 0$ corresponding to a formal BKL expansion, there is a mapping[6] between (possibly composite) operators[7] of the truncated equations of motion of $11D$ supergravity at a given spatial point, and one-parameter quantities (coordinates) in a formal σ-model over the coset space $E_{10|10}/K(E_{10|10})$. More recently, [87] has pushed the analysis even further, and shown how higher order M-theory corrections to the low-energy $11D$ supergravity action (similar to α' corrections in string theory) are realized in the σ-model, giving an interpretation for certain negative imaginary roots of E_{10}.

In particular, the regime in which this correspondence holds is reached when at least one of the diagonal metric moduli is small, in the sense that $\exists i$ s.t. $R_i \ll l_P$. In this case, the contributions to the Lagrangian of $11D$ supergravity (with possible higher order corrections) coming from derivatives of the metric and p-form fields can be approximated by an effective potential, with polynomial dependence on the diagonal metric moduli. In the BKL limit, these potential terms become increasingly steep, and can be replaced by sharp walls or cushions, which, on the $E_{10|10}/K(E_{10|10})$ side of the correspondence, define a Weyl chamber of E_{10}. The dynamics of the model then reduces to the time evolution of the diagonal metric moduli which, in the coset, map to a null geodesic in the Cartan subalgebra of E_{10} deflected by successive bounces against the billiard walls. In the leading order approximation, one can restrict his attention to the dominant walls, *i.e.* those given by the simple roots of E_{10}, so that the billiard motion is confined to the fundamental Weyl chamber of E_{10}. As mentioned before, [84, 86, 87] have shown how to extend this analysis to other Weyl chambers by considering higher level non-

[6]Which has been worked out up to order $l = 6$ and $\text{ht}(\alpha) = 29$.

[7]Constructed from the vielbein, electric and magnetic components of the four-form field-strength, and their multiple spatial gradients.

simple roots of E_{10}, and how the latter can be related, on the supergravity side, to composite operators containing multiple gradients of the supergravity fields and to M-theory corrections. These higher order terms appear as one considers smaller and smaller corrections in l_P as we approach the singularity $x^0 \to \infty$. These corrections are of two different kind: they correspond either to taking into account higher and higher spatial gradients of the supergravity fields in the truncated equations of motion of $11D$ supergravity at a given point of space, or to considering M-theory corrections to the classical two-derivative Lagrangian.

In the following, since we ultimately want to make contact with [56, 55], we will consider the more restrictive case in which the space is chosen compact, and is in particular taken to be the ten-dimensional torus T^{10}, with periodic coordinates $0 \leqslant x^i < 2\pi$, $\forall i = 1, \ldots, 10$. This, in principle, does not change anything to the non-compact setup of [85], since there the mapping relates algebraic quantities to supergravity fields at a *given point* in space, regardless in principle of the global properties of the manifold.

Before tackling the full-fledged hyperbolic E_{10} billiard and the effective Hamiltonian description of 11D supergravity dual to it, it is instructive to consider the toy model obtained by setting all fields to zero except the dilatons. This leads to a simple cosmological model characterized by a space-like singularity at constant time slices t. This suggests to introduce a lapse function $N(t)$. The proper time σ is then defined as $d\sigma = -N(t)dt$, and degenerates ($\sigma \to 0^+$) at the singularity $N(t) = 0$. This particular limit is referred to as the BKL limit, from the work of Belinskii, Khalatnikov and Lifshitz [24, 25]. As one approaches the singularity, the spatial points become causally disconnected since the horizon scale is smaller than their spacelike distance.

In this simplified picture, the metric (5.20) reduces to a Kasner one, all non-zero fields can be taken to depend only on time (since the space points are fixed):

$$ds^2 = -(N(t)dt)^2 + \sum_{i,j=1}^{10} e^{2\langle H_R(t), \varepsilon_i \rangle} \delta_{ij} dx^i \otimes dx^j \,. \tag{5.35}$$

In addition to proper time σ, we introduce an "intermediate time" coordinate u defined as

$$du = -\frac{1}{\sqrt{g}} d\sigma = \frac{N(t)}{M_P^{10} V(t)} dt \,, \tag{5.36}$$

where $\sqrt{g} = \sqrt{\det g_{ij}} = e^{\langle H_R, \rho_1 \rangle}$ and $V(t) = \prod_{i=1}^{10} R_i(t)$, where ρ_1 is the "threshold" vector (5.21). In this frame, one approaches the singularity as $u \to +\infty$.

Extremizing $\int eR$ with respect to the R_i and $N/\sqrt{g}$, we get the equations of motion for the compactification radii and the zero mass condition:

$$\frac{d}{du}\left(\frac{1}{R_i}\frac{dR_i}{du}\right) = 0 \,, \qquad\qquad \sum_i \left(\frac{\dot{R}_i}{R_i}\right)^2 - \left(\sum_i \frac{\dot{R}_i}{R_i}\right)^2 = 0 \,, \tag{5.37}$$

where the dot denotes $\frac{d}{dt}$. Setting $R_i(u_0) = R_i(s_0) = M_P^{-1}$, one obtains R_i in terms either of u or of σ

$$M_P R_i = e^{-v_i(u-u_0)} = \left(\frac{\sigma}{\sigma_0}\right)^{\frac{v_i}{\sum_j v_j}} \tag{5.38}$$

since $u = -\frac{1}{\sum_j v_j} \ln(\sigma + const) + const'$. Then, the evolution of the system reduces to a null geodesic in $\mathfrak{h}(E_{10})$. In the u-frame in particular, the vector $H_R(u) = \sum_i \ln(M_P R_i(u)) \varepsilon^{\vee i}$ can be regarded as a particle moving along a straight line at constant velocity $-\vec{v}$. In the u-frame, it is convenient to define $\vec{p} = (\sum_j v_j)^{-1} \vec{v}$, whose components are called Kasner exponents. These satisfy in particular:

$$\sum_i p_i^2 - \Big(\sum_i p_i\Big)^2 = 0 \,, \qquad\qquad \sum_i p_i = 1 \,. \tag{5.39}$$

The first constraint originates from the zero mass condition (5.37) and implies $\vec{p} \in \mathfrak{h}^*(E_{10})$, while the second one comes from the very definition of the p_i's. These two conditions result in at least one of the p_i's being positive and at least another one being negative, which leads, as expected, to a Schwartzschild type singularity in the far past and far future.

In the general case, we reinstate off-diagonal metric elements in the line-element (5.35) by introducing the vielbein (5.25) in triangular gauge:

$$\delta_{ij} dx^i \otimes dx^j \to \delta_{ij} \tilde{\gamma}^i{}_p \tilde{\gamma}^j{}_q dx^p \otimes dx^q \,, \tag{5.40}$$

with $\tilde{\gamma}^i{}_p = (\delta^i{}_p + \mathcal{A}^i{}_p)$, and $\mathcal{A}^i{}_p$ defined for $i < p$. For reasons of clarity, we discriminate this time the flat indices (i, j, k, l) from the curved ones (p, q, r, s).

In this more general case, it can be shown [85], that asymptotically (when approaching the singularity), the log of the scale factors $\ln M_P R_i$ are still linear functions of u, while the off-diagonal terms $\mathcal{A}^i{}_j$ tend to constants: in billiard language, they *freeze* asymptotically.

To get the full supergravity picture, one will in addition turn on electric 3-form and magnetic 6-form fields and the duals to the Kaluza-Klein vectors, and possibly other higher order corrective terms. Provided we work in the Iwasawa decomposition (5.20), one can show that, similarly to the off-diagonal metric components, these additional fields and their multiple derivatives also freeze as one approaches the singularity. In particular, all $(p+1)$-form field strengths will tend to constants in this regime, and therefore behave like potential terms for the dynamical scale factors.

An effective Hamiltonian description of such a system has been proposed [85, 87]:

$$\mathcal{H}(H_R, \partial_u H_R, F) = B(\partial_u H_R, \partial_u H_R) + \frac{1}{2} \sum_A e^{2w_A(H_R)} c_A(F) \,, \tag{5.41}$$

For later convenience, we want to keep the dependence on conformal time apparent, so that we use $\partial_u H_R$ to represent the canonical momenta given by $\pi^i = 2(g_\varepsilon^\vee)^{ij} \, \partial_u \ln M_P R_i$. In units of proper time (5.57), the Hamiltonian is then given by the integral:

$$H = \int d^{10}x \, \frac{N}{\sqrt{\bar{g}}} \mathcal{H} \,. \tag{5.42}$$

Let us now discuss the structure of $\mathcal{H}$ (5.41) in more details. First, the Killing form B is defined as in eqn. (5.14) and is alternatively given by the metric $g_\varepsilon^\vee$. It determines the kinetic energy of the scale factors. The second term in expression (5.41) is the effective potential generated by the frozen off-diagonal metric components, the p-form fields, and multiple derivatives of all of them, which are collectively denoted by F. The (possibly) infinite sum over A includes the basic contributions from classical $11D$ supergravity (5.26), plus higher order terms related to quantum corrections coming from M-theory. In the vicinity of a spacelike singularity, the dependence on the diagonal metric elements factorizes, so that these contributions split into an exponential of the scale factors, $e^{2w_A(H_R)}$, and a part that freezes in this BKL limit, generically denoted by $c_A(F)$.

These exponential factors $e^{2w_A(H_R)}$ behave as sharp wall potentials, now interrupting the former straight line null geodesics $H_R(u)$ and reflecting its trajectory, while conserving the energy of the corresponding virtual particle and the components of its momentum parallel to the wall. In contrast, the perpendicular components change sign. Despite these reflections, the dynamics remains integrable and leads to a chaotic billiard motion. The reflections off the walls happen to be Weyl-reflections in $\mathfrak{h}(E_{10})$, and therefore conserve the kinetic term in $\mathcal{H}$ (5.41). However, since the Weyl group of E_{10} is a subgroup of the U-duality group, it acts non-trivially on the individual potential terms of $\mathcal{H}$. As the walls represent themselves Weyl reflections, they will be exchanged under conjugation by the Weyl group. More details on the action of the

U-duality group in the general case, and in relation with hyperbolic billiard dynamics can be found in Appendix 5.13.

In the BKL limit then, the potential terms $e^{2w_A(H_R)}$ can be mimicked by theta-functions: $\Theta\left(w_A(H_R)\right)$ so that the dynamics is confined to a billiard table defined by the inequalities $w_A(H_R) \leqslant 0$. If one can isolate, among them, a finite set of inequalities $I = \{A_1, .., A_n\}$, $n < \infty$, which imply all the others, the walls they are related to are called *dominant*.

The contributions to the effective potential in $\mathcal{H}$ (5.41) arising from classical supergravity can be described concretely, and we can give to the corresponding walls an interpretation in terms of roots of E_{10}. As a first example, we give the reduction on T^{10} of the kinetic energy for the 3-form potential, and write it in terms of the momenta conjugate to the C_{ijk}:

$$\frac{1}{2}\sum_{i<j<k} e^{2w_{ijk}(H_R)}(\underline{\pi}^{ijk})^2 = \frac{1}{2}\sum_{i<j<k} (M_P^3 R_i R_j R_k)^2 \left[\tilde{\gamma}^i{}_p \tilde{\gamma}^j{}_q \tilde{\gamma}^k{}_r \, \pi^{pqr}\right]^2 . \tag{5.43}$$

As pointed out above, the momenta $\underline{\pi}^{ijk}$ freeze in the BKL limit. Their version in curved space can be computed to be

$$\pi^{p_1p_2p_3} = \sum_{i<j<k} e^{-2w_{ijk}(H_R)} \gamma^{p_1}{}_i \gamma^{p_2}{}_j \gamma^{p_3}{}_k \gamma^{q_1}{}_i \gamma^{q_2}{}_j \gamma^{q_3}{}_k \partial_u C_{q_1q_2q_3} ,$$

with $\partial_u C_{q_1q_2q_3} = \sqrt{g}\, G_{0q_1q_2q_3}$, since the flat time-index is defined by: $dx^0 = N(t)dt$. From expression (5.43), one identifies the walls related to the three-form effective potential, and referred to as "electric" in [85], with $l = 1$ positive roots of E_{10}, namely: $w_{ijk} = \varepsilon_i + \varepsilon_j + \varepsilon_k \in W_{\mathrm{M2}}(E_{10})$.

Note, in passing, that the exponential in eqn.(5.43) has the opposite sign compared to the reduced Lagrangian (5.26) for $D \geqslant 3$. This is a consequence of opting for the Hamiltonian formalism, where the Legendre transform inverts the sign of the phase factor $e^{2w_A(H_R)}$ for the momenta π^{pqr}. In this respect, the latter are defined with upper curved indices (flattened by $\tilde{\gamma}^i{}_p$), as in expression (5.43), while their conjugate fields carry lower curved ones (flattened by the inverse $\gamma^i{}_p \doteq (\tilde{\gamma}^{-1})^i{}_p$, see (5.44) below). For more details, see [85]. In any case, one can simultaneously flip all signs in the wall factors for both the Lagrange and Hamiltonian formalisms, by choosing a lower triangular parametrization for the vielbein (5.40), which corresponds to an Iwasawa decomposition with respect to the set of negative roots of the U-duality group.

Similarly, there will be a potential term resulting from the dual six-form $\widetilde{C}_6$ kinetic term (the second term in the second line of expression (5.26)). In contrast to eqn. (5.26), we rewrite the electric field energy for $\widetilde{C}_6$ as the magnetic field energy for C_3:

$$\begin{aligned} &\frac{1}{2}\sum_{i<j<k<l} e^{2w_{ijkl}(H_R)}(\underline{G}_{ijkl})^2 = \\ &\quad \frac{1}{2}\sum_{i_1<\ldots<i_6}\sum_{i_7<\ldots<i_{10}} (M_P^6 R_{i_1}\cdots R_{i_6})^2 \left[\gamma^p{}_{i_7}\gamma^q{}_{i_8}\gamma^r{}_{i_9}\gamma^s{}_{i_{10}} G_{pqrs}\, \epsilon^{i_1\ldots i_{10}}\right]^2 \end{aligned} \tag{5.44}$$

Again, the components $\underline{G}_{ijkl}$ freeze in the BKL limit, leaving a dependence on the "magnetic" walls given by $l = 2$ roots of E_{10}: $w_{ijkl} = \sum_{m\notin\{i,j,k,l\}} \varepsilon_m \in W_{\mathrm{M5}}(E_{10})$. Dualizing this expression with respect to the ten compact directions, we can generate Chern-Simons terms resulting from the topological couplings appearing in the definition of $\underline{G_2}_{\,ij}$ in eqn. (5.23), namely: $2\gamma^{p_1}_{i_1}\gamma^{p_2}_{i_2}\gamma^{p_3}{}_{i_3}\gamma^{p_4}_{i_4}\gamma^{p_5}_{i_5} G_{p_3p_4p_5[p_1}\mathcal{A}^{i_5}{}_{p_2]}$. However, such contributions are characterized by the same walls as expression (5.44), and thus have no influence on the asymptotic billiard dynamics, but only modify the constraints.

The off-diagonal components of the metric $\mathcal{A}^i{}_j$ will also contribute a potential term in $\mathcal{H}$ (5.41). Inspecting the second line of expression (5.26), we recognize it as the frozen kinetic part of the first term on this second line:

$$\frac{1}{2}\sum_{i<j} e^{2w_{ij}(H_R)}(\underline{\pi}^i{}_j)^2 = \frac{1}{2}\sum_{i<j}\left(\frac{R_i}{R_j}\right)^2 \left[\tilde{\gamma}^i{}_p \pi^p{}_j\right]^2 \tag{5.45}$$

where the momentum with curved indices is defined as

$$\pi^p{}_j = \sum_k e^{-2w_{kj}(H_R)}\gamma^p{}_k\gamma^r{}_j\partial_u\mathcal{A}^k{}_r\,, \qquad \text{with } k<j\,.$$

The sharp walls appearing in this case are usually called *symmetry* (or *centrifugal*) walls and correspond to $l=0$ roots of E_{10}, namely: $w_{ij}=\varepsilon_i-\varepsilon_j \in W_{\text{KKp}}(E_{10})$.

Finally, the curvature contribution to the potential in $\mathcal{H}$ (5.41) produces two terms:

$$\begin{aligned} &\frac{1}{2}\sum_{j<k}\sum_{i\neq\{j,k\}} e^{2\widetilde{w}_{ijk}(H_R)}(\underline{\mathcal{F}}^i{}_{jk})^2 - \sum_i e^{2w_i(H_R)}(\underline{\mathcal{F}}_i)^2 \\ &= 2\sum_{i_1<\ldots<i_7,i_8}\sum_{i_9<i_{10}} (M_P^9 R_{i_1}\cdots R_{i_7}R_{i_8}^2)^2(\gamma^p{}_{i_9}\gamma^q{}_{i_{10}}\partial_{[p}\mathcal{A}^{i_8}{}_{q]}\epsilon^{i_1\ldots i_{10}})^2 \\ &- \sum_{i_1<\ldots<i_9,i_{10}} (M_P^9 R_{i_1}\cdots R_{i_9})^2(\underline{\mathcal{F}}_{i_{10}}\epsilon^{i_1\ldots i_{10}})^2\,. \end{aligned} \tag{5.46}$$

The first one is already present as the third term on the second line of expression (5.26), the $\underline{\mathcal{F}}^i{}_{jk}$ being related to the spatial gradients of the metric, or, alternatively, to the structure functions of the Maurer-Cartan equation for the vielbein (5.40):

$$\underline{\mathcal{F}}^i{}_{jk} = 2\gamma^p{}_j\gamma^q{}_k\partial_{[p}\mathcal{A}^i{}_{q]} \tag{5.47}$$

As for expression (5.44), one can generate Chern-Simons couplings $\gamma^p{}_j\gamma^q{}_k\gamma^r{}_l\mathcal{F}^i{}_{r[p}\mathcal{A}^l{}_{q]}$ by dualizing the above expression in the ten compact directions. This again will not generate a new wall, and, as for expression (5.47), corresponds to $l=3$ roots of E_{10} given by $\widetilde{w}_{ijk}=\sum_{l\notin\{i,j,k\}}\varepsilon_l+2\varepsilon_i \in W_{\text{KK7M}}(E_{10})$.

The $\underline{\mathcal{F}}_i$ on the other hand are some involved expressions depending on the fields R_i, ∂R_i, $\underline{\mathcal{F}}^i{}_{jk}$ and $\partial\underline{\mathcal{F}}^i{}_{jk}$. In eqn.(5.46), they are related to lightlike walls $w_i=\sum_{k\neq i}\varepsilon_k$ given by all permutations of the null root $\delta=(0,(1)^9)$. These prime isotropic roots are precisely the ones at the origin of the identity (5.34). Since they can be rewritten as $w_i=(1/2)(\widetilde{w}_{jki}+\widetilde{w}_{kij})$, they are subdominant with respect to the $\widetilde{w}_{ijk}$, and will not affect the dynamics of H_R even for $\vec{p}$ close to the lightlike direction they define. So they are usually neglected in the standard BKL approach. In the next section, we will see that these walls have a natural interpretation as Minkowskian KK-particles [56], and contribute matter terms to the theory.

All the roots describing the billiard walls we have just listed are, except for w_i, real $l\leqslant 3$ roots of E_{10}, and the billiard dynamics constrains the motion of H_R to a polywedge bounded by the hyperplanes: $\langle H_R(t),w_A\rangle=0$, with A spanning the indices of the walls mentioned above. The dominant walls are then the simple roots of E_{10}. In this respect, the orbits $W_{\text{M5}}(E_{10})$ and $W_{\text{KK7M}}(E_{10})$ contains only subdominant walls, which are *hidden* behind the dominant ones, and can, in a first and coarse approximation, be neglected. The condition $\alpha_i(H_R(t))\leqslant 0$, with $\alpha_i\in\Pi(E_{10})\subset W_{\text{KKp}}(E_{10})\cup W_{\text{M2}}(E_{10})$, leads to the constraints:

$$R_1\leqslant R_2\,,\quad R_2\leqslant R_3\,,\ldots,\ R_9\leqslant R_{10}\,,\quad \text{and } R_8R_9R_{10}\leqslant l_P^3 \tag{5.48}$$

and the motion on the billiard is indeed confined to the fundamental Weyl chamber of E_{10}. The order in expression (5.48) depends on the choice of triangular gauge for the metric (5.40), and does not hold for an arbitrary vielbein. In the latter case, the formal E_{10} coset σ-model is more complicated than expression (5.49) below.

At this stage, we can rederive the mapping between geometrical objects of M-theory on T^{10} and the formal coset σ-model on $E_{10|10}/K(E_{10|10})$ proposed by [84], for the first $l = 0, 1, 2, 3$ real positive roots of E_{10}. This geodesic σ-model is governed by the evolution parameter t, which will be identified with the time parameter (5.36). To guarantee reparametrization invariance of the latter, we introduce the lapse function n, different from N. Then, in terms of the rescaled evolution parameter $d\tau = ndt$, the formal σ-model Hamiltonian reads [85]:

$$\mathcal{H}(H_R, \partial_\tau H_R, \nu, \partial_\tau \nu) = n\Big(B(\partial_\tau H_R|\partial_\tau H_R) + \frac{1}{2}\sum_{\alpha\in\Delta_+(E_{10})}\sum_{a=1}^{m_\alpha} e^{2\langle H_R,\alpha\rangle}[P_{\alpha,a}(\nu, \partial_\tau\nu)]^2\Big) \quad (5.49)$$

where $(\nu, \partial_\tau \nu)$ denotes the infinitely-many canonical variables of the system. We again use $\partial_\tau H_R$ to represent the momenta $\pi^i = 2(g_\varepsilon^\vee)^{ij}\,(R_i)^{-1}\partial_\tau R_i$ conjugate to $\ln M_P R_i$. The metric entering the kinetic term is chosen to be $g_\varepsilon^\vee$, which is dictated by comparison with the bosonic sector of toroidally reduced classical 11D supergravity.

Expression (5.49) is obtained by computing the formal Lagrangian density from the algebraic field strength valued in $\mathfrak{a}(E_{10|10}) \oplus \mathfrak{n}(E_{10|10})$ as:

$$g^{-1}\frac{d}{dt}g = -\frac{1}{\sqrt{2}}\sum_i \frac{\dot{R}_i}{R_i}\,\varepsilon^{\vee i} + \sum_{\alpha\in\Delta_+(E_{10})}\sum_{a=1}^{m_\alpha} Y_{\alpha,a}(\nu,\dot{\nu})\,e^{-\langle H_R,\alpha\rangle}E^a_\alpha\,, \quad (5.50)$$

As in eq.(5.30), one starts by calculating $\mathcal{L} = n^{-1}\mathrm{Tr}(\mathcal{P}T(\mathcal{P}))$ with $\mathcal{P}$ given in expression (5.32). One then switches to the Hamiltonian formalism, with momentum-like variables given by the Legendre transform $P_{\alpha,a}(\nu,\dot{\nu}) = \frac{1}{n}e^{-2\langle H_R,\alpha\rangle}Y_{\alpha,a}(\nu,\dot{\nu})$, eventually leading to expression (5.49).

In the BKL limit, the (non-canonical) momenta tend to constant values $P_{\alpha,a}(\nu,\dot{\nu}) \to C_{\alpha,a}$, and the potential terms in expression (5.49) exhibit the expected sharp wall behaviour. One can now try and identifiy the roots $\alpha \in \Delta_+(E_{10})$ of the formal Hamiltonian (5.49) with the wall factors w_A in the effective supergravity Hamiltonian (5.41). With a consistent truncation to $l = 3$, for instance, one recovers the supergravity sector (5.29) on T^{10}. This corresponds to the mapping we have established between real simple roots of E_{10} and the symmetry, electric, magnetic and curvature walls w_{ij} w_{ijk}, w_{ijkl} and $\widetilde{w}_{ijk}$. which are all in $\Delta_+(E_{10})$. The identification of the algebraic coordinates $C_{\alpha,a}$ with geometrical objects in the low energy limit of M-theory given by $c_A(F)$ (as defined in (5.41) and below) can then be carried out.

Proceeding further to $l = 6$, one would get terms related to multiple spatial gradients of supergravity fields appearing in the truncated equations of motion of $11D$ supergravity [84, 87] at a given point. Finally, considering a more general version of the Hamiltonian (5.49) by extending the second sum in the coset element (5.50) to negative roots, *i.e.*: $\alpha \in \Delta_+(E_{10}) \cup \Delta_-(E_{10})$, and pushing the level truncation to the range $l = 10$ to 28, one eventually identifies terms corresponding to 8th order derivative corrections to classical supergravity [87] of the form $R_2^m(DG_4)^n$, where R_2 is the curvature two-form and D is the Lorentz covariant derivative. At eighth order in the derivative, *i.e.* for $(m,n) \in \{(4,0),(2,2),(1,3),(0,4)\}$, they are typically related to $\mathcal{O}(\alpha'^3)$ corrections in $10D$ type IIA string theory, at tree level. In this case however, it may happen that the corresponding subleading sharp walls w_A are negative, which means that they can only be obtained for a non-Borel parametrization of the coset. In addition, they may not even be roots of E_{10}. However, these walls usually decompose into $w_A = -(n+m-1)\rho_1 + \zeta$, for $\zeta \in \Delta_+(E_{10})$, where the first term on the RHS represents the leading R^{n+m} correction. If $n + m = 3\mathbb{N} + 1$, the $R^m(DF)^n$ correction under consideration is compatible with E_{10}, and ζ is regarded as the relative positive root associated to it.

This means that the w_A are not necessarily always roots of $\mathfrak{e}_{10}$, and when this is not the case, a (possibly infinite) subset of them can still be mapped to roots of E_{10}, by following a certain regular rescaling scheme.

5.2.5 Instantons, fluxes and branes in M-theory on T^{10}: an algebraic approach

If we now consider the hyperbolic U-duality symmetry E_{10} to be a symmetry not only of $11D$ supergravity, but also of the moduli space space of M-theory on T^{10}, which is conjectured to be the extension of expression (5.8) to $D = 1$:

$$\mathcal{M}_{10} = E_{10|10}(\mathbb{Z})\backslash E_{10|10}/K(E_{10|10}) \tag{5.51}$$

the real roots appearing in the definition of the cosmological billiard are now mapped to totally wrapped Euclidean objects of M-theory, and can be identified by computing the action:

$$S_\alpha[M_P R_i] = 2\pi e^{\langle H_R,\alpha\rangle}\,, \qquad \alpha \in \Delta_+(E_{10})\,. \tag{5.52}$$

Thus, the roots of E_{10} found in the preceding section, namely: $w_{ij} = \varepsilon_i - \varepsilon_j \in W_{\mathrm{KKp}}(E_{10})$, $w_{ijk} = \varepsilon_i+\varepsilon_j+\varepsilon_k \in W_{\mathrm{M2}}(E_{10})$, $w_{i_1..i_6} = (\epsilon_{i_1}+..+\epsilon_{i_6}) \in W_{\mathrm{M5}}(E_{10})$ and $\widetilde{w}_{ijk} = \sum_{l\notin\{i,j,k\}}\varepsilon_l+2\varepsilon_i \in W_{\mathrm{KK7M}}(E_{10})$ describe totally wrapped Euclidean Kaluza-Klein particles, M2-branes, M5-branes and Kaluza-Klein monopoles. The dictionary relating these roots of E_{10} to the action of extended objects of M-theory can be found in Table 5.1, for the highest weight of the corresponding representation of $\mathfrak{sl}(10,\mathbb{R})$ in $\mathfrak{e}_{10}$.

Now, as pointed out in [17], the (approximated) Kasner solution defines a past and future spacelike singularity. On the other hand, the low-energy limit in which $11D$ supergravity becomes valid requires all eleven compactification radii to be larger than l_P, and consequently the Kasner exponents to satisfy (for a certain choice of basis for $\mathfrak{h}(E_{10})$, which can always be made):

$$0 < p_{10} \leqslant p_9 \leqslant \ldots \leqslant p_1 \tag{5.53}$$

so that the vector $\overrightarrow{p}$ is timelike with respect to the metric $|\overrightarrow{p}|^2 = \sum_i p_i^2 - (\sum_i p_i)^2$ (5.14). Clearly, this does not satisfy the constraints (5.39) which require $\overrightarrow{p}$ to be lightlike. Such a modification of the Kasner solution (5.38) has been argued in [17] to be achieved if one includes matter, which dominates the evolution of the system in the infinite volume limit and thereby changes the solution for the geometry. This does not invalidate the Kasner regime prevailing close to the initial spacelike singularity, since, as pointed out in [17], matter and radiation become negligible when the volume of space tends to zero (even though their density becomes infinite). In the following, we will see how matter, in the form of Minkowskian particles and branes, have a natural interpretation in terms of some class of imaginary roots of $\mathfrak{e}_{10}$, and can thus be incorporated in the hyperbolic billiard approach.

In particular, the inequality (5.53) is satisfied if at late time we have

$$R_1 \gg R_2\,,\quad R_2 \gg R_3\,,\ldots,\ R_9 \gg R_{10}\,,\ \text{ and } R_8R_9R_{10} \gg l_P^3 \tag{5.54}$$

which can be rephrased as: $\langle H_R, \alpha_i\rangle \gg 0$, $\forall \alpha_i \in \Pi(\mathfrak{e}_{10})$. The action (5.52) related to such roots is then large at late time, and the corresponding Euclidean objects of Table 5.1 can then be used to induce fluxes in the background, and thus be related to an instanton effect. This is in phase with the analysis in [17], which states that at large volume, the moduli of the theory become slow variables (in the sense of a Born-Oppenheimer approximation) and can be treated semi-classically.

Let us now make a few remarks on the two different regimes encountered so far, the billiard and semi-classical dynamics. In the semi-classical regime of 5.54, we are well outside the fundamental Weyl chamber (5.48) and higher level roots of $\mathfrak{e}_{10}$ have to be taken into account and given a physical interpretation. In this limit of large radii, the dominance of matter and radiation will eventually render the dynamics non-chaotic at late times, but the vacuum of the theory can be extremely complicated, because of the presence of instanton effects and solitonic backgrounds. In contrast, in the vicinity of the spacelike singularity, matter and radiation play a negligible rôle, leading to the chaotic dynamics of billiard cosmology. On the other hand, the structure of the vacuum is simple in the BKL regime, in which the potential walls appear to be extremely sharp. It is characterized by ten flat directions bounded by infinite potential walls, the dominant walls of the fundamental Weyl chamber of $\mathfrak{e}_{10}$.

Finally, when $\overrightarrow{p}$ is timelike, it has been shown in [17] that the domain (5.53) where $11D$ supergravity is valid can be mapped, after dimensional reduction, to weakly coupled type IIA or IIB supergravity. For instance, the safe domain for type IIA string theory (where all the nine radii are large compared to l_s and $g_{IIA} < 1$, two parameters given in terms of $11D$ quantities in eqns.(5.59)) is given by:

$$p_{10} < 0 < p_{10} + 2p_9\,, \quad \text{and} \quad p_9 \leqslant p_8 \leqslant \ldots \leqslant p_1\,. \tag{5.55}$$

The two domains (5.53) and (5.55) are then related by U-duality transformations (cf. Appendix 5.13).

Let us now discuss the issue of fluxes in this setup. From now on and without any further specification, we assume that the conditions (5.54) are met. Then, in addition to the instanton effects we have just mentioned, one can consider more complicated configurations by turning on some components of the p-form potentials of the theory. In this case, the action (5.52) receives an additional contribution due to the Wess-Zumino coupling of the p-form potential to the world-volume of the corresponding brane-like object. The action (5.52) will now receive a flux contribution which can be rephrased in algebraic terms as [124, 56]:

$$S_{\alpha_{(p)}}[M_P R_i; \mathcal{C}_{\alpha_{(p)}}] = 2\pi e^{\langle H_R, \alpha_{(p)}\rangle} + i\mathcal{C}_{\alpha_{(p)}} = \frac{M_P^{p+1}}{(2\pi)^p}\int_{\mathcal{W}_{p+1}} e\, d^{p+1}x + i\int_{\mathcal{W}_{p+1}} C_{p+1}\,, \tag{5.56}$$

where the $\alpha_{(p)}$ are positive real roots of $\mathfrak{e}_{10}$ given by the second column of Table 5.1, for all possible permutations of components in the physical basis. In particular, we will have three-form and six-form fluxes for non-zero potentials C_3 and $\widetilde{C}_6$ coupling to the Euclidean world-volumes $\mathcal{W}_3/\mathcal{W}_6$ of M2-/M5-branes respectively. For fluxes associated to Kaluza-Klein particle, we have the couplings $\mathcal{C}_{\alpha_{i-2}} = \int_\gamma g_{i\,i+1} g^{-1}_{i+1\,i+1}\, dx^i$, $i = 1, .., 9$, where γ is the KK-particle world-line, and the internal metric g can be written in terms of our variables R_i and $\mathcal{A}^i{}_8$ using eqs (5.20) and (5.25). There is also a similar coupling of the dual potential $\widetilde{\mathcal{A}}^i{}_8$ to the eight-dimensional KK7M world-volume.

The moduli $M_P R_i$, $i = 1, .., 10$, together with the fluxes from p-form potentials (5.56) parametrize the moduli space (5.51). Furthermore, on can define the following function which is harmonic under the action of a certain Laplace operator defined on the variables $\{M_P R_i; \mathcal{C}_\alpha\}^{i=1,..,10}_{\alpha\in\Delta_+(\mathfrak{e}_{10})}$ in the Borel gauge of $\mathfrak{e}_{10|10}$, and which is left-invariant under $E_{10|10}$:

$$\sqrt{N_p}\exp\left[-2\pi N_p\left(e^{\langle H_R, \alpha_{(p)}\rangle} \pm \frac{i}{2\pi}\mathcal{C}_{\alpha_p}\right)\right]\,.$$

In the limit of large radii, N_p is the instanton number and this expression is an extension to $\mathfrak{e}_{10}$ of the usual instanton corrections to string thresholds appearing in the low-energy effective theory. As such, it is conjectured to capture some of the non-perturbative aspects of M-theory [124].

Another kind of fluxes arise from non-zero expectation values of $(p+1)$-form field strengths. If we reconsider the effective Hamiltonian (5.41) in the region (5.54) where instanton effects are present, we notice that the action (5.52) appears in the effective potential as $\frac{1}{2\pi}S_\alpha = e^{2\alpha(H_R)}$. On the other hand, since their coefficients $c_A(F)$ freeze in the BKL limit, we may regard them as fluxes or topology changes provided the $(p+1)$-form field strengths appearing in eqns.(5.43), (5.44), (5.45) and (5.47) have, in this limit, integral background value:

$$\begin{aligned}
\underline{\pi}_{ijk} &\rightarrow (2\pi)^6 \langle (*_{10}\underline{\pi})_{i_1..i_7}\rangle = \frac{1}{2\pi}\int_{\mathcal{C}_{i_1..i_7}} *_{10}\underline{\pi}_3 \in \mathbb{Z}\,,\\
\underline{G}_{ijkl} &\rightarrow (2\pi)^3 \langle \underline{G}_{ijkl}\rangle = \frac{1}{2\pi}\int_{\mathcal{C}_{ijkl}} \underline{G}_4 \in \mathbb{Z}\,,\\
\underline{\pi}^i{}_j &\rightarrow (2\pi)^8 \langle (*_{10}\underline{\pi}^i)_{j_1..j_9}\rangle = \frac{1}{2\pi}\int_{\mathcal{C}_{j_1..j_9}} *_{10}\underline{\pi}^i{}_1 \in \mathbb{Z}\,,\\
\underline{\mathcal{F}}^i{}_{jk} &\rightarrow 2\pi\langle \underline{\mathcal{F}}^i{}_{jk}\rangle = \int_{\mathcal{C}_{jk}} \mathrm{ch}_1\left(\underline{\mathcal{F}}^i{}_2\right) \equiv \mathrm{Ch}_1(\underline{\mathcal{F}}^i{}_2;\mathcal{C}_{jk}) \in \mathbb{Z}\,,
\end{aligned}$$

Where $\mathcal{C}_{i_1..i_{p+1}}$ is a $(p+1)$-cycle chosen along the appropriate spatial directions.

In particular, the coefficients $c_A(F)$ appearing in the potential terms (5.43) and (5.44) are now restricted to be integers: $c_A(F) \rightarrow \left[(2\pi)^6\langle (*_{10}\underline{\pi})_{i_1..i_7}\rangle\right]^2$ and $\left[(2\pi)^3\langle \underline{G}_{ijkl}\rangle\right]^2$ and generate respectively seven-form and four-form fluxes. In this perspective, the instantons encoded in the exponential term $e^{2w_A(H_R)} \equiv e^{2\langle H_R, \alpha_{(p)}\rangle}$ for $\alpha_{\mathrm{M5}} = \sum_{m\notin\{i,j,k,l\}} \varepsilon_m$ and $\alpha_{\mathrm{M2}} = \sum_{n\notin\{i_1,..,i_7\}} \varepsilon_n$ are understood as the process that changes the fluxes by an integral amount.

The wall coefficient $c_A(F) = \left[2\pi\langle \underline{\mathcal{F}}^i{}_{jk}\rangle\right]^2$ (5.47), on the other hand, corresponds to a deformation of the basic torus T^{10} to an S^1 fibration of the ith direction over the two-torus $T^2 = \{x^j, x^k\}$, in other words to the metric:

$$ds^2 = -(Ndt)^2 + \sum_{m\neq i}(M_P R_m)^2 (dx^m)^2 + (M_P R_i)^2\Big[dx^i - \frac{1}{2\pi}\mathrm{Ch}_1(\underline{\mathcal{F}}^i{}_2;\mathcal{C}_{jk})x^k dx^j\Big]^2 .$$

where the periodicity of x^k implies $x^i \rightarrow x^i + \mathrm{Ch}_1(\underline{\mathcal{F}}^i{}_2;\mathcal{C}_{jk})x^j$ for the fiber coordinate, all other coordinates retaining their usual 2π-periodicity. The value of $c_A(F)$ determines the first Chern character (or Chern class, since $\mathrm{ch}_1 = \mathrm{c}_1$) of the fibration, and the instanton effect associated to the root $\alpha_{\mathrm{KK7M}} = \sum_{l\notin\{i,j,k\}} \varepsilon_l + 2\varepsilon_i$ creates an integral jump in this first Chern number.

Minkowskian branes from prime isotropic roots of $\mathfrak{e}_{10}$

As mentioned in the preceding section, when considering the large volume limit (5.54) in the domain (5.53) where $11D$ supergravity is valid, one should in principle start considering higher level roots of $\mathfrak{e}_{10}$ in other Weyl chambers than the fundamental one. These roots, which, in the strict BKL limit, appear as subdominant walls and can be neglected in a first approximation, should now be taken into account as corrective or mass terms. In [56, 55], a program has been proposed to determine the physical interpretation of a class of null roots of $\mathfrak{e}_{10}$. These authors have, in particular, shown the correspondence between prime isotropic roots and Minkowskian extended objects of M-theory, for the first levels $l = 3, 5, 6, 7, 8$. On the other hand, as seen in Section 5.2.4, the autors of [87] have developed a different program where they identify imaginary (both isotropic and non-isotropic) bur also real roots, with $R^m(DF)^n$ type M-theory corrections to classical supergravity. However, these results have been obtained in an intermediate domain of the dynamical evolution, where not only negative roots become leading (accounting for the fact that these higher order corrections are described by negative roots), but where we expect quantum corrections to be visible. In the approach of [56], in contrast, the regime (5.54) allows for effects related to extended objects to become important, pointing, in the line of Section 5.2.5, at an interpretation for certain higher level roots in terms of branes and particles.

In the following, we give a condensed version of the correspondence between prime isotropic roots of $\mathfrak{e}_{10}$ and Minkowskian extended objects of M-theory, which can be found in a much more detailed and ample version in [56], which we follow closely until the end of this section.

First of all, since we now restrict to the region (5.53), we are sufficiently far from the singularity for the lapse function $N(t)$ to have any non-zero value. In particular, we can gauge-fix to $N(t) = M_P$ in expression (5.36), which defines the conformal time:

$$d\tilde{u} = \frac{dt}{M_P^9 V(t)} \,. \tag{5.57}$$

As we will see below, these are the "natural" units to work out the relation between prime isotropic roots of $\mathfrak{e}_{10}$ and Minkowskian particles and branes in M-theory.

Consider, for instance two M5 instantons at times $t_{\beta_{\rm M5}} \ll t_{\alpha_{\rm M5}}$ encoded algebraicly in

$$\alpha_{\rm M5} = ((1)^4, (0)^4, (1)^2), \qquad \beta_{\rm M5} = ((0)^4, (1)^6)$$

Since each of them creates a jump in their associated flux, inverting the time order to $t_{\beta_{\rm M5}} \gg t_{\alpha_{\rm M5}}$ will pass one instanton through the other, thereby creating a Minkowskian M2-brane stretched between them in the interval $[t_{\alpha_{\rm M5}}, t_{\beta_{\rm M5}}]$, where their respective fluxes overlap. This M2-brane will be associated to the root $\gamma_{\rm M2} = \alpha_{\rm M5} + \beta_{\rm M5} = ((1)^8, (2)^2)$. Recalling that we gauge-fixed to conformal time (5.57), the action for such an object has to be expressed in unit of conformal time, then:

$$\frac{d}{d\tilde{u}}\widetilde{S}_\alpha = M_P^9 V \frac{d}{dt} S_\alpha = 2\pi e^{\langle H_R, \alpha\rangle} \;\longrightarrow\; M_\alpha = \frac{1}{2\pi}\frac{d}{dt}S_\alpha = M_P^{-9} V^{-1} e^{\langle H_R,\alpha\rangle} \tag{5.58}$$

This expression for the mass of the object could also be deduced from the rescaling (5.42). Thus, in particular: $M_{\gamma_{\rm M2}} = M_P^3 R_9 R_{10}$ as expected from a membrane wrapped around directions x^9 and x^{10}.

From the supergravity perspective, the instanton described by $\alpha_{\rm M5}$ will create a jump in the flux: $(2\pi)^3\langle \underline{G}_{5678}\rangle \to (2\pi)^3\langle \underline{G}_{5678}\rangle + 1$ when going from $t < t_{\alpha_{\rm M5}}$ to $t_{\alpha_{\rm M5}} < t$, while instanton $\beta_{\rm M5}$ induces $(2\pi)^3\langle \underline{G}_{1234}\rangle \to (2\pi)^3\langle \underline{G}_{1234}\rangle - 1$ when t passes $t_{\beta_{\rm M5}}$. Now the M2-brane flux sourced by $(2\pi)^6\langle \underline{G}_{1234}\rangle\langle \underline{G}_{5678}\rangle$, via the topological term $\int C_3 \wedge G_4 \wedge G_4$ of $11D$ supergravity, has to be counterbalanced by an equal number of anti M2-branes. Thus, going from configuration $t_{\beta_{\rm M5}} \ll t_{\alpha_{\rm M5}}$ to configuration $t_{\beta_{\rm M5}} \gg t_{\alpha_{\rm M5}}$ after setting both initial fluxes to zero produces one unit of Minkowskian anti-M2-brane flux which must be compensated by the creation of a M2-brane in the same directions, as expected.

Another process involves passing an M2-instanton $\alpha_{\rm M2} = ((1)^2, (0)^7, 1)$ through a KK-monopole $\beta_{\rm KK7M} = ((0)^2, (1)^6, 2, 1)$. Since the KK-monopole shifts by one unit the first Chern class of the circular x^9-fibration over the (x^1, x^2)-torus: ${\rm Ch}_1(\underline{\mathcal{F}}^9{}_{(2)}; \mathcal{C}_{12}) \to {\rm Ch}_1(\underline{\mathcal{F}}^9{}_{(2)}; \mathcal{C}_{12}) + 1$, it creates an obstruction that blocks the M2-instanton somewhere in the (x^1, x^2) plane, and, by means of the fibration, produces an object at least wrapped along the x^9 direction. It is not hard to see that the Minkowskian object resulting from this process is a M2-brane wrapped around x^9 and the original x^{10}. One can check that the root $\alpha_{\rm M2} + \beta_{\rm KK7M} = \gamma_{\rm M2} = ((1)^8, (2)^2)$, recovering the same object as before.

Furthermore, by combining an M5-instanton $\alpha_{\rm M5} = (0, (1)^6, (0)^3)$ shifting the magnetic flux $(2\pi)^3\langle \underline{G}_{189\,10}\rangle$ by one unit with an M2-instanton $\beta_{\rm M2} = ((0)^7, (1)^3)$ shifting the electric flux $(2\pi)^6\langle (*_{10}\underline{\pi})_{1\cdots 7}\rangle$ accordingly, one creates a Minkowskian KK-particle $\alpha_{\rm KKp} = (0, (1)^9)$ corresponding to the following contribution the momentum in the x^1 direction: $\mathcal{P}^1 = \int d^{10}x\, G^{189\,10}\pi_{89\,10}$. The mass (5.58) of this object $M_{\gamma_{\rm KKp}} = R_1^{-1}$ is in accordance with this interpretation.

object	real root	S_α	prime isotropic	M_α
KKp	$(0,0,0,0,0,0,0,0,1,-1)$	$2\pi R_9 R_{10}^{-1}$	$(0,1,1,1,1,1,1,1,1,1)$	R_1^{-1}
M2	$(0,0,0,0,0,0,0,1,1,1)$	$2\pi M_P^3 R_8 R_9 R_{10}$	$(1,1,1,1,1,1,1,1,2,2)$	$M_P^3 R_9 R_{10}$
M5	$(0,0,0,0,1,1,1,1,1,1)$	$2\pi M_P^6 R_5 \cdots R_{10}$	$(1,1,1,1,1,2,2,2,2,2)$	$M_P^6 R_5 \cdots R_{10}$
KK7M	$(0,0,1,1,1,1,1,1,1,2)$	$2\pi M_P^9 R_3 \cdots R_9 R_{10}^2$	$(1,1,1,2,2,2,2,2,2,3)$	$M_P^9 R_3 \cdots R_9 R_{10}^2$
KK9M	$(1,1,1,1,1,1,1,1,1,3)$	$2\pi M_P^{12} R_1 \cdots R_9 R_{10}^3$	$(1,2,2,2,2,2,2,2,2,4)$	$M_P^{12} R_2 \cdots R_9 R_{10}^3$

Table 5.1: Euclidean and Minkowskian branes of M-theory on T^{10} and positive roots of E_{10}

Similar combinations of Euclidean objects can be shown, by various brane creation processes, to produce Minkowskian M5-branes and KK7M-branes. To conclude, all four types of time-extended matter fields are summarized in Table 5.1 by their highest weight representative with its mass formula. At present, it is still unclear how matter terms produced by the prime isotropic roots of Table 5.1 should be introduced in the effective Hamiltonian (5.41). Since the corresponding Minkowskian branes originate from creation processes involving two instantons, as explained above, we expect such a contribution to be [56]: $2\pi n_{\gamma_m} e^{\langle H_R, \gamma_m\rangle}$, where the isotropic root γ_m describing the Minkowskian brane decomposes into two real roots related to instantons $\gamma_m = \alpha_e + \beta_e$, and turns on $n_{\gamma_m} = n_{\alpha_e} n_{\beta_e}$ units of flux which compensates for the original n_{α_e} and n_{β_e} units of flux produced by the two instantons. Since γ_m is a root, such a term will never arise as a term in the series (5.49). We then expect the Hamiltonian (5.49) to be modified in the presence of matter. In this respect, a proposal for a corrective term has been made in [56], which reproduces the energy of the Minkowskian brane only up to a 2π-factor. Moreover, it generates additional unwanted contributions for which one should find a cancelling mechanism.

From Table 5.1, one readily obtains the spectrum of BPS objects of type IIA string theory, by compactifying along x^{10}, and taking the limit $M_P R_{10} \to 0$, thereby identifying:

$$R_{10} = \frac{g_A}{M_s}\,, \qquad\qquad M_P = \frac{M_s}{\sqrt[3]{g_A}}\,. \tag{5.59}$$

In this respect, we have included in Table 5.1 the conjectured KK9M-brane as the putative M-theory ascendant of the D8, KK8A and KK9A-branes of IIA string theory, the latter two being highly non-perturbative objects which map, under T-duality, to the IIB S7 and S9-branes.

Since we mention type IIB string theory, we obtain its spectrum after compactification (5.59) by T-dualizing along the toroidal x^9 direction, which maps:

$$R_{9,B} = \frac{1}{M_s^2 R_{9,A}}\,, \qquad\qquad g_B = \frac{g_A}{M_s R_{9,A}}\,.$$

From the E_{10} viewpoint, T-dualizing from type IIA to IIB string theory corresponds to different embeddings of $\mathfrak{sl}(9,\mathbb{R})$ in $\mathfrak{e}_{10|10}$. Going back to the Dynkin diagram 5.1 for $r = 10$, type IIA theory corresponds to the standard embedding of $\mathfrak{sl}(9,\mathbb{R})$ along the preferred subalgebra $\mathfrak{sl}(10,\mathbb{R}) \subset \mathfrak{e}_{10|10}$ (the gravity line), while type IIB is obtained by choosing the the Dynkin diagram of $\mathfrak{sl}(9,\mathbb{R})$ to extend in the α_8 direction. This two embeddings consist in the two following choices of basis of simple roots for the Dynkin diagram of $\mathfrak{sl}(9,\mathbb{R})$:

$$\Pi_A = \{\alpha_{-1}, \alpha_0, \dots, \alpha_5, \alpha_6\}\,, \qquad\qquad \Pi_B = \{\alpha_{-1}, \alpha_0, \dots, \alpha_5, \alpha_8\}\,,$$

which results in two different identifications of the NS-NS sector of both theories. Then, a general T-duality on x^i, $i \neq 10$, can be expressed in purely algebraic language, as the following mapping:

$$\mathcal{T}_i \quad : \quad \mathfrak{h}(E_r) \to \mathfrak{h}(E_r)$$
$$\varepsilon_i \mapsto -\varepsilon_i$$

From the σ-model point of view, the type IIA and IIB theories correspond to two different level truncations of the algebraic field strength (5.50), namely a level decomposition with respect to (l_7, l_8) for type IIA, and (l_6, l_7) for type IIB, l_i being the level in the simple root α_i of E_{10}. The NS-NS and RR sectors of both supergravity theories are then obtained by pushing the decomposition up to level $(l_7, l_8) = (2, 3)$ for IIA, and up to $(l_6, l_7) = (4, 2)$ for IIB. See for instance [174], where the results are directly transposable to E_{10} (all roots considered there are in fact E_{10} roots).

Minkowskian objects from threshold-one roots of $\mathfrak{e}_{10}$

By inspecting the second column of Table 5.1, we observe that all Minkowskian objects extended in p spatial directions, are characterized, on the algebraic side, by adding $\sum_{i=1}^{p} \varepsilon_{k_i}$ to the threshold vector $\rho_1 = ((1)^{10})$. For the Minkowskian KK particle, the corresponding root of $\mathfrak{e}_{10}$ is related to its quantized momentum, and one needs therefore to substract a factor of ε_j to the threshold vector.

The Minkowskian world-volume of these objects naturally couples to the respective $(p+1)$-form potentials (5.52). So, in contrast to the Hamiltonian formalism (5.41), which treats, for a different purpose, the temporal components of the field-strengths as conjugate momenta, one now needs to keep the time index of the tensor potentials apparent, thereby working in the Lagrange formalism. This is similar to what is done in [87] where the authors perform a component analysis of one-loop corrections to classical $11D$ supergravity.

As pointed out in Section 5.2.4, the $(p + 1)$-form components separate into an oscillating part and a part that freezes as $u \to +\infty$, so that we have:

$$C_{0i_1..i_q} = \frac{1}{N(t)} e^{p_1}{}_{i_1} \cdots e^{p_q}{}_{i_q} C_{tp_1\cdots p_q} = e^{-\langle \rho_1 + \sum_{n=1}^{q} \varepsilon_{i_n}, H_R\rangle} \gamma^{p_1}{}_{i_2} \cdots \gamma^{p_q}{}_{i_q} C_{up_1\cdots p_q} \,. \tag{5.60}$$

where we have used $\sqrt{g} = e^{\langle \rho_1, H_R\rangle}$, and the index 0 stands for the flat time coordinate $dx^0 = N(t)dt$. Following the analysis of Section 5.2.4, the component $C_{up_1\cdots p_q}$ can be shown to freeze. Now, by selecting the appropriate basis vectors ε_{j_n}, we observe that all imaginary roots in the second column of Table 5.1 are related to a tensor component of the form (5.60) with the expected value of q. As a side remark, the minus sign appearing in the exponential wall factor in eqn.(5.60) comes from working in the Lagrange formalism, as discussed in Section 5.2.5.

For Minkowskian KK particles and M2-branes, this approach is related to performing the $\mathfrak{e}_{10}$ extension of the last two sets of roots in eqns.(5.24) by setting $D = 1$. When restricting to the roots obtained by this procedure which are highest weights under the Weyl group of $\mathfrak{sl}(10, \mathbb{R})$, one again recovers the two first terms in the second column of Table 5.1. Since we have not performed any Hodge duality in this case, we obtain, as expected, roots characterizing KK particles and M2-branes.

It results from this simple analysis that it is the presence of the threshold vector $\rho_1 = ((1)^{10})$ which determines if an object is time-extended, and not necessarily the fact that the corresponding root is isotropic. We shall see in fact when working out 0B' orientifolds that certain types of magnetized Minkowskian D-branes can be associated to real roots and non-isotropic imaginary roots, provided they decompose as $\alpha = \rho_1 + \vec{q} \in \Delta_+(E_{10})$, where $\vec{q}$ is a positive vector (never a root) of threshold 0, *i.e.* that can never be written as $\vec{q} = n\rho_1 + \vec{q}'$ for $n \neq 0$.

5.3 Orbifolding in a KMA with non-Cartan preserving automorphisms

In this section, we expose the method we use to treat physical orbifolds algebraicly. It is based on the simple idea that orbifolding a torus by $\mathbb{Z}_n$ is geometrically equivalent to a formal $2\pi/n$ rotation. Using the mapping between physical and algebraic objects, one can then translate the geometrical rotation of tensors into purely algebraic language as a formal rotation in the KM algebra. This is given by an adjoint action of the group on its KM algebra given by a finite-order inner automorphism.

More concretely, let us consider an even orbifold T^q/Z_n, acting as a simultaneous rotation of angle $2\pi Q_a/n$, $a = 1, \ldots, q/2$ in each pair of affected dimensions determined by the charges $Q_a \in \{1, \ldots, n-1\}$. A rotation in the (x^i, x^j) plane is naturally generated by the adjoint action of the compact group element $\exp(\frac{2\pi}{n} Q_a \mathcal{K}_i^{\ j}) \equiv \exp(\frac{2\pi}{n} Q_a (E_\alpha - F_\alpha))$ for $\alpha = \alpha_{i-2} + \ldots + \alpha_{j-3}$. In particular, rotations on successive dimensions $(i+2, i+3)$ are generated by $E_{\alpha_i} - F_{\alpha_i}$. We will restrict ourselves to orbifolds acting only on successive pair of dimensions in the following, although everything can be easily extended to the general case. In particular, physically meaningful results should not depend on that choice, since it only amounts to a renumbering of space-time dimensions. For the same reason, we can restrict our attention to orbifolds that are taken on the last q spatial dimensions of space-time $\{x^{11-q}, \ldots, x^{10}\}$. In that case, we have the $q/2$ rotation operators

$$\begin{aligned} V_1 &\doteq e^{\frac{2\pi}{n} Q_1 \mathcal{K}_{11-q\, 12-q}} &&= e^{\frac{2\pi}{n} Q_1 (E_{\alpha_{9-q}} - F_{\alpha_{9-q}})} \\ &\ \ \vdots \\ V_{\frac{q}{2}} &\doteq e^{\frac{2\pi}{n} Q_{\frac{q}{2}} \mathcal{K}_{9\,10}} &&= e^{\frac{2\pi}{n} Q_{q/2} (E_{\alpha_7} - F_{\alpha_7})} \end{aligned} \tag{5.61}$$

that all mutually commute, so that the complete orbifold action is given by:

$$\mathcal{U}_q^{\mathbb{Z}_n} \doteq \prod_{a=1}^{q/2} V_a \,. \tag{5.62}$$

Note that $\mathcal{U}_q^{\mathbb{Z}_n}$ generically acts non-trivially only on generators $\in \mathfrak{g}_{\pm\alpha}$ (and the corresponding $H_\alpha \in \mathfrak{h}$) for which the decomposition of α in simple roots contains at least one of the root $\alpha_{8-q}, \ldots, \alpha_8$ for $q > 2$ (α_6 and α_7 for $q = 2$).

In the particular case of $\mathbb{Z}_2$ orbifolds, the orbifold action leaves the Cartan subalgebra invariant, so that it can be expressed as a chief inner automorphism by some adjoint action $\exp(i\pi H)$, $H \in \mathfrak{h}_{\mathbb{Q}}$ on $\mathfrak{g}$. It indeed turns out that the action of such an automorphism on $\mathfrak{g}/\mathfrak{h} = \bigoplus_{\alpha \in \Delta(\mathfrak{g})} \mathfrak{g}_\alpha$ depends linearly on the root α grading $\mathfrak{g}_\alpha$ and can thus be simply expressed as:

$$\begin{aligned} \mathrm{Ad}(e^{i\pi H}) \mathfrak{g}_\alpha = (-1)^{\alpha(H)} \mathfrak{g}_\alpha = (-1)^{\sum_{i=9-r}^{8} k^i \alpha_i(H)} \mathfrak{g}_\alpha \,, \\ \text{for}\, H \in \mathfrak{h}_{\mathbb{Q}} \ \text{ and } \ \alpha = \textstyle\sum_{i=9-r}^{8} k^i \alpha_i \,. \end{aligned} \tag{5.63}$$

If q is even, both methods are completely equivalent, while, if q is odd, the determinant is negative and it cannot be described by a pure $SO(r)$ rotation. If one combines the action of $E_\alpha - F_\alpha$ with some mirror symmetry, however, one can of course reproduce the action (5.63). Indeed, the last form of the orbifold action in expression (5.63) is the one given in [55]. The results of [55] are thus a subset of those obtainable by our more general method.

5.3.1 Cartan involution and conjugation of real forms

In the preceding sections, we have already been acquainted with the Chevalley involution ϑ_C. Here, we shall introduce just a few more tools we shall need later in this work to deal with real forms in the general sense. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. If it is related to a real Lie algebra $\mathfrak{g}_0$ as $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}} \doteq \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$, $\mathfrak{g}$ will be called a complexification thereof. Reciprocally, $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$ with $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. Next, a semisimple real Lie algebra is called compact if it can be endowed with a Killing form satisfying

$$B(X, X) < 0 \ , \ \ \forall X \in \mathfrak{g}_0 \quad (X \neq 0) \ , \tag{5.64}$$

and non-compact otherwise.

Thus, a non-compact real form can in general be obtained from its complexification $\mathfrak{g}$ by specifying an involutive automorphism ϑ defined on $\mathfrak{g}_0$, such that $B(\vartheta X, \vartheta Y) = B(X, Y)$ and $B_\vartheta(X, Y) = -B(X, \vartheta Y)$, $\forall X, Y \in \mathfrak{g}_0$, is a symmetric and positive definite form. ϑ is called a Cartan involution (the argument here is a generalization of the construction of the almost-positive definite covariant form based on the Chevalley involution ϑ_C in Section 5.1.1). It can be shown that every real semisimple Lie algebra possesses such an involution, and that the latter is unique up to inner automorphisms. This is a corollary of the following theorem:

Theorem 5.3.1 *Every automorphism ψ of $\mathfrak{g}$ is conjugate to a chief automorphism ϑ of $\mathfrak{g}$ through an inner automorphism ϕ, ie:*

$$\psi = \phi^{-1} \circ \vartheta \circ \phi, \qquad \phi \in \mathrm{Int}(\mathfrak{g}) \tag{5.65}$$

Then, it is clear that ψ is involutive iff ϑ is involutive. In this case, the two real forms of $\mathfrak{g}$ generated by ψ and ϑ are isomorphic, so that for every conjugacy class of involutive automorphisms, one needs only consider the chief involutive automorphism (as class representative), which can in turn be identified with the Cartan involution.

The Cartan involution induces an orthogonal $(\pm 1)-$eigenspace decomposition into the direct sum $\mathfrak{g}_0 = \mathfrak{k} \oplus^{\perp} \mathfrak{p}$, called Cartan decomposition of $\mathfrak{g}_0$, with property

$$\vartheta|_{\mathfrak{k}} = 1 \text{ and } \ \vartheta|_{\mathfrak{p}} = -1 \ . \tag{5.66}$$

More specifically, $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}_0$ while $\mathfrak{p}$ is a representation of $\mathfrak{k}$, since: $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. Finally, as our notation for the Cartan decomposition suggested, $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to the Killing form and B_ϑ.

Alternatively, it is sometimes more convenient to define a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ as the fixed point subalgebra of $\mathfrak{g}$ under an involutive automorphism called conjugation τ such that

$$\tau(X) = X \ , \qquad \tau(iX) = -iX \ , \quad \forall X \in \mathfrak{g}^U \tag{5.67}$$

then: $\mathfrak{g}_0 = \{X \in \mathfrak{g} \,|\, \tau(X) = X\}$.

Finally, by Wick-rotating $\mathfrak{p}$ in the Cartan decomposition of $\mathfrak{g}_0$ one obtains the compact Lie algebra $\mathfrak{g}_c = \mathfrak{k} \oplus^{\perp} i\mathfrak{p}$ which is a compact real form of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$.

Because of Theorem 5.3.1, one needs an invariant quantity sorting out involutive automorphisms leading to isomorphic real forms. This invariant is the signature (or character of the real Lie algebra) σ, defined as the difference between the number $d_- = \dim \mathfrak{k}$ of compact generators and the number $d_+ = \dim \mathfrak{p}$ of non-compact generators (the $\pm$-sign recalling the sign of the Killing form):

$$\sigma = d_+ - d_- \ .$$

For *simple* real Lie algebras, σ *uniquely* specifies $\mathfrak{g}_0$. The signature varies between its maximal value for the split form $\sigma = r$ and its minimal one for the compact form $\sigma = -\dim \mathfrak{g}$.

Defining the following linear operator constructed from ϑ (see [71], p.543)

$$\sqrt{\vartheta} = \frac{1}{2}(1 + i)\vartheta + \frac{1}{2}(1 - i)\mathbb{1} \ , \tag{5.68}$$

satisfying $\sqrt{\vartheta} \circ \sqrt{\vartheta}\, X = \vartheta X$, $\forall X \in \mathfrak{g}$, all non-compact real forms of $\mathfrak{g}$ will be obtained through

$$\mathfrak{g}_0 = \sqrt{\vartheta}\, \mathfrak{g}_c \tag{5.69}$$

by selecting the appropriate *chief* involutive automorphism ϑ.

5.3.2 Determining the real invariant subalgebra from its complexification

For a given orbifold $T^{11-q-D} \times T^q/\mathbb{Z}_n$ of eleven-dimensional supergravity/M-theory, the orbifold action on the corresponding U-duality algebra in D dimensions is given by the inner automorphism $\mathcal{U}_q^{\mathbb{Z}_n}$, $\forall D$. This automorphism has a natural extension to the complexification $(\mathfrak{g}^U)^{\mathbb{C}}$ of the split form $\mathfrak{g}^U$, where the appropriate set of generators describing physical fields and duality transformations on a complex space can be properly defined.

The requirement that these new generators diagonalize $\mathcal{U}_q^{\mathbb{Z}_n}$ and are charged according to the index structure of their corresponding physical objects will select a particular complex basis of $(\mathfrak{g}^U)^{\mathbb{C}}$. We will henceforth refer to this algebraic procedure as "orbifolding the theory".

Projecting out all charged states under $\mathcal{U}_q^{\mathbb{Z}_n}$ is then equivalent to an orbifold projection in the U-duality algebra, resulting in the invariant subalgebra $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}} = \text{Fix}_{\mathcal{U}_q^{\mathbb{Z}_n}}(\mathfrak{g}^U)^{\mathbb{C}}$ (the notation $\text{Fix}_V \mathfrak{g}$ stands for the fixed-point subalgebra of $\mathfrak{g}$ under the automorphism V).

Since we expect the untwisted sector of the theory to be expressible from the non-linear realization of $G_{\text{inv}}/K(G_{\text{inv}})$ as a coset sigma-model, we are particularly interested in determining the reality properties of $\mathfrak{g}_{\text{inv}}$, the algebra that describes the residual U-duality symmetry of the theory.

Retrieving the real form $\mathfrak{g}_{\text{inv}}$ from its complexification $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$ can be achieved by restricting the conjugation (5.67) to $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$. Denoting such a restriction $\tau_0 \doteq \tau|_{\mathfrak{g}_{\text{inv}}}$, the real form we are looking for is given by

$$\mathfrak{g}_{\text{inv}} = \text{Fix}_{\tau_0}(\mathfrak{g}_{\text{inv}}{}^{\mathbb{C}})\,.$$

Since $\mathfrak{g}^U$ is naturally endowed with the Chevalley involution ϑ_C, the Cartan involution associated to the real form $\mathfrak{g}_{\text{inv}}$ is then the restriction of ϑ_C to the untwisted sector of the U-duality algebra, which we denote $\phi = \vartheta_C|_{\mathfrak{g}_{\text{inv}}}$. Consequently, the real form $\mathfrak{g}_{\text{inv}}$ possesses a Cartan decomposition $\mathfrak{g}_{\text{inv}} = \mathfrak{k}_{\text{inv}} \oplus \mathfrak{p}_{\text{inv}}$, with eigenspaces $\phi(\mathfrak{k}_{\text{inv}}) = \mathfrak{k}_{\text{inv}}$ and $\phi(\mathfrak{p}_{\text{inv}}) = -\mathfrak{p}_{\text{inv}}$. The whole procedure outlined in this section can be summarized by the following sequence:

$$(\mathfrak{g}^U, \vartheta_C) \xrightarrow{\mathcal{U}_q^{\mathbb{Z}_n}} (\mathfrak{g}^U)^{\mathbb{C}} \xrightarrow{\text{Fix}_{\mathcal{U}_q^{\mathbb{Z}_n}}} (\mathfrak{g}_{\text{inv}})^{\mathbb{C}} \xrightarrow{\text{Fix}_{\tau_0}} (\mathfrak{g}_{\text{inv}}, \phi)\,. \tag{5.70}$$

5.3.3 Non-compact real forms from Satake diagrams

As we have seen before, real forms are described by classes of involutive automorphisms, rather than by the automorphisms themselves. As such, the Cartan involution, which we will refer to as ϑ, can be regarded as some kind of preferred involutive automorphism, and is encoded in the so-called *Satake* diagram of the real form it determines. The Cartan involution splits the set of simple roots Π into a subset of black (invariant) roots ($\vartheta(\alpha_i) = \alpha_i$) we call Π_c, and the subset $\Pi_d = \Pi \,/\, \Pi_c$ of white roots, such as

$$\vartheta(\alpha_i) = -\alpha_{p(i)} + \sum_k \eta_{ik}\alpha_k, \text{ with } \alpha_{p(i)} \in \Pi_d \text{ and } \alpha_k \in \Pi_c$$

where p is an involutive permutation rotating white simple roots into themselves and η_{ik} is a matrix of non-negative integers. A Satake diagram consists in the Dynkin diagram of the complex form of the algebra with nodes painted in white or in black according to the above prescription. Moreover, if two white roots are exchanged under p, they will be joined on the Satake diagram by an arrow.

From the action of ϑ on the root system, one can furthermore determine the Dynkin diagram and multiplicities of the so-called restricted roots, which are defined as follows: for a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p}$, let $\mathfrak{a} \subset \mathfrak{p}$ be maximal abelian. Then, one can define the partition under $\mathfrak{a}$ into simultaneous orthogonal eigenspaces (see [175] for a detailed discussion):

$$(\mathfrak{g}_0)_{\bar{\alpha}} = \{X \in \mathfrak{g}_0 \,|\, \mathrm{ad}(H_{\mathfrak{a}})X = \bar{\alpha}(H_{\mathfrak{a}})X,\ \forall H_{\mathfrak{a}} \in \mathfrak{a}\} \ . \tag{5.71}$$

This defines the restricted roots $\bar{\alpha} \in \mathfrak{a}^*$ as the simultaneous eigenvalues under the commuting family of self-adjoint transformations $\{\mathrm{ad}(H_{\mathfrak{a}})|\forall H_{\mathfrak{a}} \in \mathfrak{a}\}$. Then, we can choose a basis such that $\mathfrak{h}_0 = \mathfrak{t} \oplus \mathfrak{a}$, where $\mathfrak{t}$ is the maximal abelian subalgebra centralizing $\mathfrak{a}$ in $\mathfrak{k}$. The Cartan subalgebra can be viewed as a torus with topology $(S^1)^{n_c} \times (\mathbb{R})^{n_s}$ where $n_s = \dim \mathfrak{a}$ is called the $\mathbb{R}$-rank. Restricted-root spaces are the basic ingredient of the Iwasawa decomposition, so we shall return to them when discussing non-linear realizations (see Section 5.2.2) of orbifolded $11D$ supergravity/M-theory models.

We denote by Σ the set of roots not restricting to zero on $\mathfrak{a}^*$. As an example, one can choose a basis where such a set Σ reads:

$$\Sigma = \left\{ \bar{\alpha} = \frac{1}{2}(\alpha - \vartheta(\alpha)) \in \mathfrak{a}^* \,|\, \bar{\alpha} \neq 0 \right\} \ . \tag{5.72}$$

Then, a real form can be encoded in the triple $(\mathfrak{a}, \Sigma, m_{\bar{\alpha}})$ and $m_{\bar{\alpha}}$ is the function giving the multiplicity of each restricted root, in other words $m_{\bar{\alpha}} = \dim(\mathfrak{g}_0)_{\bar{\alpha}}$. If we denote by $\overline{\Pi}$ a basis of Σ, all non-compact real forms of $\mathfrak{g}$ can be encoded graphically in

I) the Satake diagram of (Π, ϑ);

II) the Dynkin diagram of $\overline{\Pi}$;

III) the multiplicities $m_{\bar{\alpha}_i}$ and $m_{2\bar{\alpha}_i}$ for $\bar{\alpha}_i \in \bar{B}$.

On the other hand, given a Satake diagram, we can determine the real form associated to it as a fixed point algebra under τ. Indeed from the Satake diagram one readily determines ϑ, and since it can be shown that there always exists a basis of $\mathfrak{h}$ such that the "compact" conjugation $\tau^c = \vartheta \circ \tau$ acts as $\tau^c(\alpha) = -\alpha$, $\forall \alpha \in \Delta$, then the conjugation is determined by $\tau = -\vartheta$ on the root lattice.

Finally, in the finite case, the $\mathbb{R}$-rank n_s is given in the Satake diagram by the number of white roots minus the number of arrows, and n_c by the number of black roots plus the number of arrows.

5.4 The orbifolds $T^2/\mathbb{Z}_{n>2}$

From the algebraic method presented in Section 5.3.2, it is evident that a $T^2/\mathbb{Z}_n$ orbifold on the pair of spatial dimensions $\{x^9, x^{10}\}$ is only expected to act non-trivially on the root spaces $(\mathfrak{g}^U)_\alpha \subset \mathfrak{g}^U$, characterized by all roots α containing α_6 and/or α_7, as well as on the corresponding Cartan element H_α.

The basis of $(\mathfrak{g}^U)^{\mathbb{C}}$ diagonalizing the orbifold automorphism $\mathcal{U}_2^{\mathbb{Z}_n}$ with the appropriate set of charges will be derived step by step for the chain of compactification ranging from $D = 8$ to $D = 1$. This requires applying the machinery of Section 5.3.2 to the generators of the root spaces $(\mathfrak{g}^U)_\alpha$ mentioned above and selecting combinations thereof to form a basis of $(\mathfrak{g}^U)^{\mathbb{C}}$ with orbifold charges compatible with their tensorial properties. We will at the same time determine the real invariant subalgebra $\mathfrak{g}_{\mathrm{inv}}$ by insisting on always selecting lowest-height invariant simple roots, which ensures that the resulting invariant subalgebra is maximal. In $D = 2, 1$, subtleties connected with roots of multiplicities greater than one and the splitting of their corresponding root spaces will be adressed.

For a start, we will work out the $D = 8$ case in detail, and then show how this construction can be regularly extended down to the $D = 3$ case. The affine and hyperbolic $D = 2, 1$ cases require more care and will be treated separately. In $D = 8$, then, we consider eleven-dimensional supergravity on T^3, which possesses U-duality algebra $\mathfrak{g}^U = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$, whose complexification is described by the Dynkin diagram of $\mathfrak{a}_2 \oplus \mathfrak{a}_1$. It has positive-root space $\Delta_+ = \{\alpha_6, \alpha_7, \alpha_6 + \alpha_7, \alpha_8\}$, and its Cartan subalgebra is spanned by $\big\{H_6 = \varepsilon_8^\vee - \varepsilon_9^\vee\,;\, H_7 = \varepsilon_9^\vee - \varepsilon_{10}^\vee\,;\, H_8 = (2/3)(\varepsilon_8^\vee + \varepsilon_9^\vee + \varepsilon_{10}^\vee)\big\}$. The $\mathfrak{a}_1$ factor corresponds to transformations acting on the unique scalar $C_{89\,10}$ produced by dimensional reduction of the 3-form field on T^3.

The orbifold action on the two-torus

$$(z, \bar{z}) \to (e^{2\pi i/n} z, e^{-2\pi i/n} \bar{z}) \tag{5.73}$$

induces the following inner automorphism on the U-duality algebra

$$\mathcal{U}_2^{\mathbb{Z}_n} = \mathrm{Ad}(e^{\frac{2\pi}{n}(E_7 - F_7)}) \doteq \mathrm{Ad}(e^{-\frac{2\pi}{n} i\mathcal{K}_{z\bar{z}}})\,, \qquad \text{for } n > 2\,.$$

This automorphism acts diagonally on the choice of basis for $(\mathfrak{g}^U)^{\mathbb{C}}$ appearing in Table 5.2, where both compact and non-compact generators have the charge assignment expected from their physical counterparts.

Q_A	generators
0	$K_{z\bar{z}} = -\frac{1}{6}(2H_6 + H_7) + \frac{1}{2}H_8$ $K_{88} = \frac{1}{3}(2H_6 + H_7) + \frac{1}{2}H_8$ $Z_{8z\bar{z}} = \frac{i}{2}(E_8 + F_8)\,, \quad \mathcal{Z}_{8z\bar{z}} = i(E_8 - F_8)$ $\mathcal{K}_{z\bar{z}} = i(E_7 - F_7)$
± 1	$\left\{ \begin{array}{c} K_{8\bar{z}} \\ K_{8z} \end{array} \right\} = \frac{1}{2\sqrt{2}}\,(E_6 + F_6 \pm i(E_{67} + F_{67}))$ $\left\{ \begin{array}{c} \mathcal{K}_{8\bar{z}} \\ \mathcal{K}_{8z} \end{array} \right\} = \frac{1}{\sqrt{2}}\,(E_6 - F_6 \pm i(E_{67} - F_{67}))$
± 2	$\left\{ \begin{array}{c} K_{\bar{z}\bar{z}} \\ K_{zz} \end{array} \right\} = \frac{1}{2}(H_7 \pm i(E_7 + F_7))$

Table 5.2: Algebraic charges for $S^1 \times T^2/\mathbb{Z}_{n>2}$ orbifolds

The invariant subalgebra $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$ can be directly read off Table 5.2, the uncharged objects building an $\mathfrak{a}_1 \oplus \mathbb{C}^{\oplus^2}$ subalgebra, since the original $\mathfrak{a}_2$ factor of $(\mathfrak{g}^U)^{\mathbb{C}}$ now breaks into two abelian generators $H^{[2]} \doteq 2H_6 + H_7$ and $\widetilde{H}^{[2]} \doteq -\mathcal{K}_{z\bar{z}}$. The total rank (here 3) is conserved, which will appear to be a generic feature of $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$.

The real form $\mathfrak{g}_{\text{inv}}$ is then easily identified by applying the procedure outlined in eq. (5.70). Since $K_{z\bar{z}}$ and K_{88} are already in $\mathrm{Fix}_{\tau_0}(\mathfrak{g}_{\text{inv}}{}^{\mathbb{C}})$ while $\tau_0(Z_{8z\bar{z}}) = -Z_{8z\bar{z}}$, $\tau_0(\mathcal{Z}_{8z\bar{z}}) = -\mathcal{Z}_{8z\bar{z}}$ and $\tau_0(\mathcal{K}_{z\bar{z}}) = -\mathcal{K}_{z\bar{z}}$, a proper basis of the invariant real form is, in terms of $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$ generators: $\mathfrak{g}_{\text{inv}} = \mathrm{Span}\big\{2/3(K_{88} + 2K_{z\bar{z}})\,;\, iZ_{8z\bar{z}}\,;\, i\mathcal{Z}_{8z\bar{z}}\big\} \oplus \mathrm{Span}\{2(K_{88} - K_{z\bar{z}})\} \oplus \mathrm{Span}\{-i\mathcal{K}_{z\bar{z}}\}$. From now on, the last two abelian factors will be replaced by $H^{[2]}$ and $i\widetilde{H}^{[2]}$. Now, how such a basis behaves under the associated Cartan involution ϕ is clear from Section 5.3.2. This determines the invariant real form to be $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(1, 1) \oplus \mathfrak{u}(1)$, with total signature $\sigma = 1$. In general, the signature of the subalgebra kept invariant by a $T^{2n}/\mathbb{Z}_{n>2}$ orbifold will be given by $\sigma(\mathfrak{g}^U) - 2n$ (keeping in mind that some orbifolds are equivalent under a trivial 2π rotation).

The coset defining the non-linear realization of the orbifolded supergravity is obtained in the usual way by modding out by the maximal compact subgroup:

$$\frac{SL(2,\mathbb{R})}{SU(2)} \times \frac{SO(1,1)}{\mathbb{Z}_2} \, .$$

In $D = 7$, there appears an additional simple root α_5, which, in the purely toroidal compactification, enhances and reconnects the U-duality algebra into $\mathfrak{g}^U = \mathfrak{sl}(5,\mathbb{R})$, following the well known $\mathfrak{e}_{n|n}$ serie. The complexification $(\mathfrak{g}^U)^{\mathbb{C}}$ resulting from orbifolding the theory calls for six additional generators: $\{K_{7\bar z}, Z_{7z\bar z}, \mathcal{K}_{7\bar z}, \mathcal{Z}_{7z\bar z}\}$ and the 2 corresponding Hermitian conjugates, produced by acting with $\mathrm{ad}(E_5 \pm F_5)$ on the objects in Table 5.2, all of which, together with the Cartan element K_{77}, have the expected orbifold charges.

Beside these natural combinations, we now have four new types of objects with charge ± 1, namely:

$$\left\{ \begin{array}{c} Z_{78\bar z} \,/\, \frac{1}{2}\mathcal{Z}_{78\bar z} \\ Z_{78z} \,/\, \frac{1}{2}\mathcal{Z}_{78z} \end{array} \right\} = \frac{1}{2\sqrt{2}} \left(E_{5678}(+/-)F_{5678} \pm i(E_{568}(+/-)F_{568}) \right) \, , \tag{5.74}$$

so that the invariant subalgebra $(\mathfrak{g}_{\mathrm{inv}})^{\mathbb{C}}$ is a straightforward extension by α_5 of the $D = 8$ case, as can be seen in Table 5.3. Its real form is obtained from the sequence (5.70) just as in $D = 8$, yielding the expected $\mathfrak{g}_{\mathrm{inv}} = \mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{so}(1,1) \oplus \mathfrak{u}(1)$, where the non-compact abelian factor is now generated by the combination

$$H^{[2]} = 4H_5 + 6H_6 + 3H_7 + 2H_8 = \frac{10}{3}\left(K_7{}^7 + K_8{}^8 - K_z{}^z \right) \, , \tag{5.75}$$

while, as before, $\mathfrak{u}(1) = \mathbb{R}(E_7 - F_7)$. Thus $\sigma(\mathfrak{g}_{\mathrm{inv}}) = 2$, while the total rank is again conserved by the orbifold projection.

The above procedure can be carried out in $D = 6$. In this case however, the invariant combination $H^{[2]}$ which generated earlier the non-compact $\mathfrak{so}(1,1)$ factor is now dual to a root of $\mathfrak{g}^U = \mathfrak{so}(5,5)$, namely:

$$H^{[2]} = H_{\theta_{D_5}} = \frac{2}{3}\left(K_6{}^6 + K_7{}^7 + K_8{}^8 - K_z{}^z \right) \, . \tag{5.76}$$

The abelian factor is thus enhanced to a full $\mathfrak{sl}(2,\mathbb{R})$ subalgebra with root system $\{\pm\theta_{D_5}\}$, while the real invariant subalgebra clearly becomes $\mathfrak{sl}(4,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$. In $D = 5$, θ_{D_5} connects to α_3 giving rise to $\mathfrak{g}_{\mathrm{inv}} = \mathfrak{sl}(6,\mathbb{R}) \oplus \mathfrak{u}(1)$. The extension to $D = 4, 3$ is completely straightforward, yielding respectively $\mathfrak{g}_{\mathrm{inv}} = \mathfrak{so}(6,6) \oplus \mathfrak{u}(1)$ and $\mathfrak{e}_{7|7} \oplus \mathfrak{u}(1)$. The whole serie of real invariant subalgebras appears in Table 5.3, beside their Satake diagram, which encodes the set of simple invariant roots Π_0 and the Cartan involution ϕ.

5.4.1 Affine central product and the invariant subalgebra in $D = 2$

New algebraic features appear in $D = 2$, since, in the purely toroidal case, the U-duality algebra is now conjectured to be the affine $\mathfrak{e}_{9|10} \doteq \mathrm{Split}(\hat{\mathfrak{e}}_8)$.[8]

The invariant subalgebra $(\mathfrak{g}_{\mathrm{inv}})^{\mathbb{C}}$ consists in the affine $\hat{\mathfrak{e}}_7$ together with the Heisenberg algebra $\hat{\mathfrak{u}}(1)^{\mathbb{C}}$ spanned by $\{z^n \otimes (E_7 - F_7), \forall n \in \mathbb{Z}\,; c\,; d\}$. Though both terms commute at the level of loop-algebras, their affine extensions share the same central charge $c = H_\delta$ and scaling operator d. Now, a product of two finite-dimensional algebras possessing (at least partially) a common centre is called a *central product* in the mathematical literature. The present situation is a

[8]Since $\vartheta_C(\delta) = -\delta$ implies $\vartheta_C(d) = -d$ and $\vartheta_C(c) = -c$, the split form of any KMA has signature $\sigma = \dim \mathfrak{h}$.

natural generalization of this construction to the infinite-dimensional setting, where not only the central charge but also the scaling element are in common. Since the latter is a normalizer, we are not strictly dealing with a central product. We will therefore refer to such an operation as an *affine central product* and denote it by the symbol $\bowtie$. Anticipating the very-extended $D=1$ case, we can write the invariant subalgebra as the complexification

$$\hat{\mathfrak{e}}_7 \bowtie \hat{\mathfrak{u}}(1) \equiv \big(\hat{\mathfrak{e}}_7 \oplus \hat{\mathfrak{u}}(1)\big) \, / \, \{\mathfrak{z}, \bar{d}\} \ , \tag{5.77}$$

where $\mathfrak{z} = H_{\delta_{E_7}} - c_{\hat{\mathfrak{u}}(1)}$ is the centre of the algebra and $\bar{d} = d_{\delta_{E_7}} - d_{\hat{\mathfrak{u}}(1)}$ is the difference of scaling operators.

The real form $\mathfrak{g}_{\text{inv}}$ can be determined first by observing that the non-compact and compact generators H_α and $E_\alpha \pm F_\alpha$ (with $(\alpha|\alpha_7)=0$) of the $\mathfrak{e}_{7|7}$ factor in $D=3$ naturally extend to the $(t^n \pm\ t^{-n}) \otimes\ H_\alpha$ and $(t^n \otimes E_\alpha \pm t^{-n} \otimes F_\alpha)$ vertex operators of an affine $\hat{\mathfrak{e}}_{7|9}$ and second, by noting that the remaining factor in the central product $\hat{\mathfrak{e}}_7 \bowtie \hat{\mathfrak{u}}(1)$ is in fact the loop algebra $\mathcal{L}(\mathfrak{u}(1))$ whose tower of generators can be grouped in pairs of one compact and one non-compact generator, according to [9]

$$\phi\big((z^n \pm z^{-n}) \otimes (E_7 - F_7)\big) = \pm (z^n \pm z^{-n}) \otimes (E_7 - F_7) \,, \tag{5.78}$$

in addition to the former compact Cartan generator $i\widetilde{H}^{[2]} = E_7 - F_7$. In short, the $\mathcal{L}(\mathfrak{u}(1))$ factor contributes -1 to the signature of $\mathfrak{g}_{\text{inv}}$, so that in total: $\sigma = 8$. Restoring the central charge and the scaling operator in $\mathcal{L}(\mathfrak{u}(1))$ so as to write $\mathfrak{g}_{\text{inv}}$ in the form (5.77), we will denote the resulting real Heisenberg algebra $\hat{\mathfrak{u}}_{|1}(1)$, so as to render its signature apparent.

For the sake of clarity, we will represent $\mathfrak{g}_{\text{inv}}$ in Table 5.3 by the Dynkin diagram of $\hat{\mathfrak{e}}_{7|9} \oplus \hat{\mathfrak{u}}_{|1}(1)$ supplemented by the signature $\sigma(\mathfrak{g}_{\text{inv}})$, but it should be kept in mind that $\mathfrak{g}_{\text{inv}}$ is really given by the quotient (5.77). In Table 5.3, we have separated the $D=2,1$ cases from the rest, to stress that the Satake diagram of (Π_0, ϕ) describes $\mathfrak{g}_{\text{inv}}$ completely only in the finite case.

Finally, to get some insight into the structure of the algebra $\hat{\mathfrak{e}}_7 \bowtie \hat{\mathfrak{u}}(1)$, it is worthwhile noting that the null root of the original $\mathfrak{e}_9$ is also the null root of $\hat{\mathfrak{e}}_7$, as

$$\begin{aligned} \delta &= \alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 \\ &= \alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_8 + 2\theta_{D_5} = \delta_{E_7} \ . \end{aligned}$$

Although the root space $\mathfrak{g}_\delta \subset \mathfrak{e}_9$ is eight-dimensional, we have $\text{mult}(\delta_{E_7}) = 7$, since the eighth generator $z \otimes H_7$ of $\mathfrak{g}_\delta$ is projected out. The latter is now replaced by the invariant combination $z \otimes (E_7 - F_7)$ whose commutator with itself creates the central charge of the $\hat{\mathfrak{u}}(1)$, whereas the seven remaining invariant generators $\{z \otimes H_{\theta_{D_5}}\, ; z \otimes H_i, \forall i = 1, \cdots, 5, 8\}$ build up the root space $\mathfrak{g}_{\delta_{E_7}}$. In a sense that will become clearer in $D=1$, the multiplicity of δ_{E_9} is thus preserved in $\hat{\mathfrak{e}}_7 \bowtie \hat{\mathfrak{u}}(1)$.

5.4.2 A Borcherds symmetry of orbifolded M-theory in $D = 1$

In $D=1$, finally, plenty of new $\mathfrak{sl}(10,\mathbb{R})$-tensors appear as roots of $\mathfrak{e}_{10}$, so it is now far from obvious whether the invariant subalgebra constructed from $\hat{\mathfrak{e}}_7 \bowtie \hat{\mathfrak{u}}(1)$ by adding the node α_{-1} exhausts all invariant objects. Moreover, the structure of such an algebra, as well as its Dynkin diagram is not a priori clear, since we know of no standard way to reconnect the two factors of the central product through the extended node α_{-1}. As a matter of fact, mathematicians are aware that invariant subalgebras of KMA under finite-order automorphisms might not be KMA, but can be Borcherds algebras or EALA [10, 5, 4]. Despite these preliminary reservations,

[9] Note that the combination used in eqn. (5.78) is well-defined in $\mathfrak{e}_{9|10}$, since it can be rewritten in the following form: $(z^n \otimes E_7 \mp z^{-n} \otimes F_7) \pm (z^{-n} \otimes E_7 \mp z^n \otimes F_7)$.

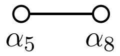
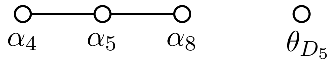
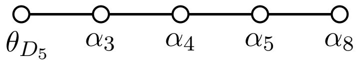
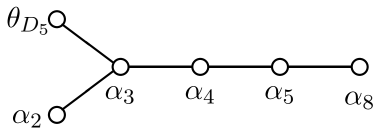
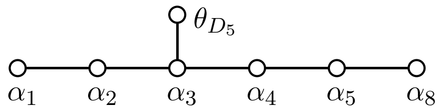
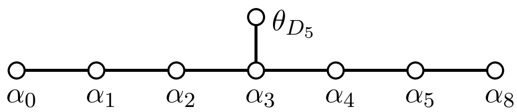
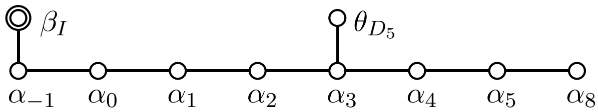
D	(Π_0, ϕ)	$\mathfrak{g}_{\text{inv}}$	$\sigma(\mathfrak{g}_{\text{inv}})$
8	 $\times \quad H^{[2]} \quad \times \quad i\widetilde{H}^{[2]}$	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(1,1) \oplus \mathfrak{u}(1)$	1
7	 $\times \quad H^{[2]} \quad \times \quad i\widetilde{H}^{[2]}$	$\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{so}(1,1) \oplus \mathfrak{u}(1)$	2
6	 $\times \quad i\widetilde{H}^{[2]}$	$\mathfrak{sl}(4,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$	3
5	 $\times \quad i\widetilde{H}^{[2]}$	$\mathfrak{sl}(6,\mathbb{R}) \oplus \mathfrak{u}(1)$	4
4	 $\times \quad i\widetilde{H}^{[2]}$	$\mathfrak{so}(6,6) \oplus \mathfrak{u}(1)$	5
3	 $\times \quad i\widetilde{H}^{[2]}$	$\mathfrak{e}_{7\vert 7} \oplus \mathfrak{u}(1)$	6
2	 $\times \quad \left\{ i\widetilde{H}_n^{[2]} \right\}_{n\in\mathbb{Z}}$	$\hat{\mathfrak{e}}_{7\vert 9} \oplus \mathcal{L}(\mathfrak{u}(1))_{\vert -1}$	8
1	 $\times \quad i\widetilde{H}^{[2]}$	${}^2\mathcal{B}_{10\vert 11} \oplus \mathfrak{u}(1)$	8

Table 5.3: The split subalgebras $\mathfrak{g}_{\text{inv}}$ for $T^{9-D} \times T^2/\mathbb{Z}_{n>2}$ compactifications

we will show that the real invariant subalgebra $\mathfrak{g}_{\text{inv}}$ in $D = 1$ can nevertheless be described in a closed form by a Satake diagram and the Conjecture 5.4.1 below, while its root system and root multiplicities can in principle be determined up to arbitrary height by a proper level decomposition.

Conjecture 5.4.1 *The invariant subalgebra of $\mathfrak{e}_{10}$ under the automorphism $\mathcal{U}_2^{\mathbb{Z}_n}$ is the direct sum of a $\mathfrak{u}(1)$ factor and a Borcherds algebra with degenerate Cartan matrix characterized by one isotropic imaginary simple root β_I of multiplicity one and nine real simple roots, modded out by its centre and its derivation.*

As already mentioned in Section 5.4.1, we choose to represent, in Table 5.3, the real form $\mathfrak{g}_{\text{inv}}$ before quotientation, by the Dynkin diagram of its defining Borcherds algebra ${}^2\mathcal{B}_{10}$. Both are related through

$$\mathfrak{g}_{\text{inv}} = \mathfrak{u}(1) \oplus {}^2\mathcal{B}_{10|11} \,/\, \{\mathfrak{z}, d_I\}\,, \tag{5.79}$$

which is an extension of the affine central product (5.77) encountered in $D = 2$, provided we now set $\mathfrak{z} = H_\delta - H_I$. We also define $H_I \doteq H_{\beta_I}$ and $d_I \doteq d_{\beta_I}$ as the Cartan generator dual to β_I and the derivation counting levels in β_I.

More precisely, ${}^2\mathcal{B}_{10}$ has the following 10×10 degenerate Cartan matrix, with rank $r = 9$

$$A = \left(\begin{array}{ccc|c} 0 & -1 & 0 & \mathbb{O} \\ -1 & 2 & -1 & \\ \hline 0 & -1 & & A(\hat{\mathfrak{e}}_7) \\ & \mathbb{O} & & \end{array} \right),$$

and it can be checked that its null vector is indeed the centre $\mathfrak{z}$ of the Borcherds algebra mentioned above. As for affine KMA, the Cartan subalgebra of Borcherds algebras with a non-maximal $n \times n$ Cartan matrix has to be supplemented by $n - r$ new elements that allow to discriminate between roots having equal weight under $\text{Ad}(H_i)$, $\forall i = 1, .., n$. Here, the Cartan subalgebra of ${}^2\mathcal{B}_{10}$ thus contains a derivation d_I counting the level in β_I, allowing, for example, to distinguish between $2\beta_I$,[10] $\beta_I + \delta$ and 2δ, which all have weights -2 under H_{-1} and 0 under all other Cartan generators dual to simple roots. However, the operator d_I is not in $\mathfrak{e}_{10}$ and consequently not in $\mathfrak{g}_{\text{inv}}$, either. Hence, the quotient by $\{\mathfrak{z}, \bar{d}\}$ in Conjecture 5.4.1 amounts to identifying H_I with H_δ. Furthermore, since the roots α_{-1} and β_I are connected on the Dynkin diagram, $-H_{-1}$ plays, already in ${}^2\mathcal{B}_{10}$, the same rôle as d_I with respect to β_I. So the elimination of d_I by the quotient (5.79) is equivalent to identifying it with $-H_{-1}$, which parallels the treatment of H_I with H_δ.

These two processes reconstruct in $\mathfrak{g}_{\text{inv}}$ the 8-dimensional root space $(\mathfrak{g}_{\text{inv}})_\delta = (\mathfrak{g}_{{}^2\mathcal{B}_{10}})_\delta \oplus (\mathfrak{g}_{{}^2\mathcal{B}_{10}})_{\beta_I}$ inherited from $\mathfrak{e}_{10}$.

Formally, one decomposes:

$$E^a_\delta|_{\mathfrak{g}_{\text{inv}}} = E^a_\delta|_{{}^2\mathcal{B}_{10}}\,, \forall a = 1, \ldots, 7\,, \quad \text{and} \quad E^8_\delta|_{\mathfrak{g}_{\text{inv}}} = E_{\beta_I} \doteq \frac{1}{\sqrt{2}}\,(E_{\delta+\alpha_7} - E_{\delta-\alpha_7})\,.$$

and $F_{\beta_I} = (E_{\beta_I})^\dagger$ in $\mathfrak{e}_{10}$. One should thus pay attention to the fact that although $\beta_I \sim \delta$ in $\mathfrak{g}_{\text{inv}}$, their corresponding ladder operators remain distinct.

We have chosen to depict the Borcherds algebra under scrutiny by the Dynkin diagram displayed in Table 5.3. However such a GKMA, let alone its root multiplicities, is not known in the literature. So at this stage, one must bear in mind that the Dynkin diagram we associate to ${}^2\mathcal{B}_{10}$ is only meant to determine the correct root lattice for $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$. The root multiplicities, on the other hand, have to be computed separately by decomposing root-spaces of $\mathfrak{e}_{10}$ into root-spaces of $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$. So we need *both* the Dynkin diagram of ${}^2\mathcal{B}_{10}$ *and* the root multiplicities listed in Table 5.6 in order to determine $\mathfrak{g}_{\text{inv}}$ completely.

In order to support the conjecture 5.4.1, we start by performing a careful level by level analysis. We proceed by decomposing $\mathfrak{e}_{10}$ with respect to the coefficient of α_8 into tensorial irreducible representations of $\mathfrak{sl}(10, \mathbb{R})$. Such representations, together with the multiplicity of the weights labelling them, are summarized up to level $l = 6$ in α_8 in Tables 5.4 and 5.5. These tables have been deduced from the low-level decomposition of roots of $\mathfrak{e}_{10}$ that can be found up to level 18 in [115]. Since we are more interested in the roots themselves and their

[10] In contrast to real simple roots, we expect for isotropic simple roots of a Borcherds algebra that $n\beta_I \in \Delta$, $\forall n \in \mathbb{Z}$.

l	$\mathcal{R}(\Lambda)$	Λ	$\dim \mathcal{R}(\Lambda)$
0	$K_{(ij)}$	[200000000]	$55 = 45 + 10$ Cartan
1	$Z_{[ijk]}$	[001000000]	120
2	$\widetilde{Z}_{[i_1 \cdots i_6]}$	[000001000]	210
3	$\widetilde{K}_{(i)[j_1 \cdots j_8]}$	[100000010]	$440 = 360 + 8 \times 10_{[0]}$
4	$(\widetilde{K}_{(i)} \otimes Z)_{[j_1 \cdots j_8][k_1 k_2 k_3]}$	[001000001]	$1155 = 840 + 7 \times 45_{[0]}$
	$A_{(ij)}$	[200000000]	$55 = 10 + 45_{[0]}$
5	$(\widetilde{K}_{(i)} \otimes \widetilde{Z})_{[j_1 \cdots j_8][k_1 \cdots k_6]}$	[000001001]	$1848 = 840 + 4 \times 252_{[0]}$
	$B_{(i)[j_1 \cdots j_4]}$	[100100000]	$1848 = 840 + 4 \times 252_{[0]}$
6	$(\widetilde{K}_{(i)} \otimes \widetilde{K}_{(j)})_{[k_1 \cdots k_8][l_1 \cdots l_8]}$	[100000011]	$3200 = 720 + 2 \times 840 + 16 \times 45 + 8 \times 10_{[0]}$
	$(A \otimes \widetilde{Z})_{(ij)[k_1 \cdots k_6]}$	[010001000]	$8250 = 3150 + 5 \times 840 + 20 \times 45$
	$D_{(i)[j_1 \cdots j_7]}$	[100000100]	$1155 = 840 + 7 \times 45$
	$S_{[i_1 \cdots i_8]}$	[000000010]	45

Table 5.4: Representations of $\mathfrak{sl}(10, \mathbb{R})$ in $\mathfrak{e}_{10}$ up to $l = 6$

multiplicities than in the dimension of the corresponding $\mathfrak{sl}(10, \mathbb{R})$ representations, we added, in column $\dim \mathcal{R}(\Lambda)$ of Table 5.4, the way the dimension of each representation decomposes in generators corresponding to different sets of roots, obtained by all reflections by the Weyl group of $\mathfrak{sl}(10, \mathbb{R})$ (i.e. permutations of indices in the physical basis) on the highest weight and possibly other roots. In the first column of Table 5.5, the tensor associated to the highest weight is defined, the highest weight being obtained by setting all indices to their maximal values.

Note that roots that are permutations of the highest weight of no representation, or in other words, have null outer multiplicity, do not appear in Table 5.4, contrary to what is done in [115]. However, these can be found in Table 5.5. The order of the orbits under the Weyl group of $\mathfrak{sl}(10, \mathbb{R})$ is given in column O_w^{10}, in which representations of null outer multiplicity are designated by a [0] subscript. Besides, column m contains the root multiplicities, while column $|\Lambda|^2$ contains the squared length, which, in the particular case of $\mathfrak{e}_{10}$, provide equivalent characterizations.

For example, the representation with Dynkin labels [000001000] at level 3 is composed of the Weyl orbit of its highest weight generator $\widetilde{K}_{(10)[3 \cdots 10]}$ together with 8 Weyl orbits of the (outer multiplicity 0) root $\widetilde{K}_{(2)[3 \cdots 10]}$ for a total size $360 + 8 \times 10$. Similarly, the representation [001000001] at level 4 is composed of the 840 components of the associated tensor, together with 7 copies of the anti-symmetric part of A_{ij} that corresponds to a root of multiplicity 8 and outer multiplicity 0, for a total dimension $840 + 7 \times 45$. The remaining eighth copy combines with the (inner and outer) multiplicity 1 diagonal part $A_{(ii)}$ to form a symmetric tensor [200000000]. Note that the $A_{(ij)}$ representation differs from the $K_{(ij)}$ one, first because the diagonal elements of the latter are given by Cartan elements and not ladder operators as in $A_{(ij)}$, and second because these two representations obviously correspond to roots of totally different level, height and threshold. Clearly, isomorphic irreducible representations of $\mathfrak{sl}(10, \mathbb{R})$ can appear several times in the decomposition of $\mathfrak{e}_{10}$.

Generator	α	Physical basis	O_w^{10}	$m(\alpha)$	$\lvert\alpha\rvert^2$
$K_{9\,10}$	$\begin{smallmatrix} &&&&&&0&& \\ 0&0&0&0&0&0&0&0&1 \end{smallmatrix}$	$(0,0,0,0,0,0,0,0,1,-1)$	45	1	2
$Z_{[89\,10]}$	$\begin{smallmatrix} &&&&&&1&& \\ 0&0&0&0&0&0&0&0&0 \end{smallmatrix}$	$(0,0,0,0,0,0,0,1,1,1)$	120	1	2
$\widetilde{Z}_{[56789\,10]}$	$\begin{smallmatrix} &&&&&&2&& \\ 0&0&0&0&1&2&3&2&1 \end{smallmatrix}$	$(0,0,0,0,1,1,1,1,1,1)$	210	1	2
$\widetilde{K}_{(10)[3\cdots10]}$	$\begin{smallmatrix} &&&&&&3&& \\ 0&0&1&2&3&4&5&3&1 \end{smallmatrix}$	$(0,0,1,1,1,1,1,1,1,2)$	360	1	2
$\widetilde{K}_{(2)[3\cdots10]}$	$\begin{smallmatrix} &&&&&&3&& \\ 0&1&2&3&4&5&6&4&2 \end{smallmatrix}$	$(0,1,1,1,1,1,1,1,1,1)$	$10_{[0]}$	8	0
$(\widetilde{K}_{(2)} \otimes Z)_{[3\cdots10][89\,10]}$	$\begin{smallmatrix} &&&&&&4&& \\ 0&1&2&3&4&5&6&4&2 \end{smallmatrix}$	$(0,1,1,1,1,1,1,2,2,2)$	840	1	2
$A_{(10\,10)}$	$\begin{smallmatrix} &&&&&&4&& \\ 1&2&3&4&5&6&7&4&1 \end{smallmatrix}$	$(1,1,1,1,1,1,1,1,1,3)$	10	1	2
$(\widetilde{K}_{(2)} \otimes Z)_{[3\cdots10][19\,10]}$ $A_{(9\,10)}$	$\begin{smallmatrix} &&&&&&4&& \\ 1&2&3&4&5&6&7&4&2 \end{smallmatrix}$	$(1,1,1,1,1,1,1,1,2,2)$	$45_{[0]}$	8	0
$(\widetilde{K}_{(2)} \otimes \widetilde{Z})_{[3\cdots10][5\cdots\,10]}$	$\begin{smallmatrix} &&&&&&5&& \\ 0&1&2&3&5&7&9&6&3 \end{smallmatrix}$	$(0,1,1,1,2,2,2,2,2,2)$	840	1	2
$B_{(10)[7\cdots\,10]}$	$\begin{smallmatrix} &&&&&&5&& \\ 1&2&3&4&5&6&8&5&2 \end{smallmatrix}$	$(1,1,1,1,1,1,2,2,2,3)$	840	1	2
$(\widetilde{K}_{(2)} \otimes \widetilde{Z})_{[3\cdots10][16\cdots\,10]}$ $B_{(6)[7\cdots\,10]}$	$\begin{smallmatrix} &&&&&&5&& \\ 1&2&3&4&5&7&9&6&3 \end{smallmatrix}$	$(1,1,1,1,1,2,2,2,2,2)$	$252_{[0]}$	8	0
$(\widetilde{K}_{(2)} \otimes \widetilde{K}_{(10)})_{[3\cdots10][3\cdots\,10]}$	$\begin{smallmatrix} &&&&&&6&& \\ 0&1&3&5&7&9&11&7&3 \end{smallmatrix}$	$(0,1,2,2,2,2,2,2,2,3)$	720	1	2
$(A \otimes \widetilde{Z})_{(9\,10)[5\cdots\,10]}$	$\begin{smallmatrix} &&&&&&6&& \\ 1&2&3&4&6&8&10&6&3 \end{smallmatrix}$	$(1,1,1,1,2,2,2,2,3,3)$	3150	1	2
$(\widetilde{K}_{(1)} \otimes \widetilde{K}_{(10)})_{[24\cdots10][3\cdots\,10]}$ $(A \otimes \widetilde{Z})_{[4\,10][5\cdots\,10]}$ $D_{(10)[4\cdots10]}$	$\begin{smallmatrix} &&&&&&6&& \\ 1&2&3&5&7&9&11&7&3 \end{smallmatrix}$	$(1,1,1,2,2,2,2,2,2,3)$	840	8	0
$(\widetilde{K}_{(2)} \otimes \widetilde{K}_{(2)})_{[3\cdots10][3\cdots\,10]}$	$\begin{smallmatrix} &&&&&&6&& \\ 0&2&4&6&8&10&12&8&4 \end{smallmatrix}$	$(0,2,2,2,2,2,2,2,2,2)$	$10_{[0]}$	8	0
$(\widetilde{K}_{(1)} \otimes \widetilde{K}_{(10)})_{[2\cdots9][3\cdots\,10]}$ $(A \otimes \widetilde{Z})_{(34)[5\cdots\,10]}$ $D_{(3)[4\cdots10]}$ $(\widetilde{K}_{(1)} \otimes \widetilde{K}_{(2)})_{[3\cdots10][3\cdots\,10]}$ $S_{[3\cdots10]}$	$\begin{smallmatrix} &&&&&&6&& \\ 1&2&4&6&8&10&12&8&4 \end{smallmatrix}$	$(1,1,2,2,2,2,2,2,2,2)$	45	44	-2

Table 5.5: Decomposition of root spaces of $\mathfrak{e}_{10}$ into $\mathfrak{sl}(10,\mathbb{R})$ representations

Moreover, and more interestingly, weights with different physical interpretations may live in the same representation of $\mathfrak{e}_{10}$. In particular, the third weight $\widetilde{K}_{(10)[3\cdots10]}$ in Table 5.5 is clearly related to the corresponding Euclidean Kaluza-Klein monopole (KK7M), while the fourth weight $\widetilde{K}_{(2)[3\cdots10]}$, though belonging to the same [100000010] representation, corresponds, according to the proposal of [56] (cf. Table 5.1) to the Minkowskian Kaluza-Klein particle (KKp) G_{01}. Similarly, the seventh weight $A_{(10\,10)}$ is associated to the conjectured Euclidean KK9M-brane

$W_{(10\,10)[1\cdots 10]}$, while $A_{(9\,10)}$ is interpreted as the Minkowskian M2-brane $C_{09\,10}$. To complete the list of Minkowskian objects, we have in addition the weights $B_{(6)[7\cdots 10]}$ and $D_{(10)[4\cdots 10]}$ related respectively to the exceptional M5-brane $\widetilde{C}_{06\cdots 10}$ and the Kaluza-Klein monopole (KK7M) $\widetilde{G}_{(10)04\cdots 10}$.

After this short excursion into weights and representations of $\mathfrak{e}_{10}$, let us come back to the characterization of ${}^2\mathcal{B}_{10}$. Observing that objects commuting with $i\widetilde{H}^{[2]} = -i\mathcal{K}_{z\bar{z}}$ have the form $X_{\dots(99)} - X_{\dots(10\,10)}$ or $X_{\dots[9\,10]}$, we are naturally looking for invariant combinations of generators of $\mathfrak{e}_{10}$ with such tensorial properties. The latter can then be decomposed into $\mathfrak{sl}(8,\mathbb{R})$ tensors with Weyl orbits of order O^8_w and identified with a root of ${}^2\mathcal{B}_{10}$. We have carried out such a decomposition up to $l = 6$ in α_8 and listed the corresponding roots of ${}^2\mathcal{B}_{10}$, together with their multiplicities m, in Table 5.6.

In order to make clear how to retrieve the root system of $\mathfrak{g}_{\rm inv}$ from Table 5.6, we give in the third column the expression of a given root of ${}^2\mathcal{B}_{10}$ in a generalized notation for the physical basis, denoted *physical eigenbasis of* $\mathfrak{e}_{10}$. This eigenspace basis is defined by[11]:

$$E'_i = E_i\,,\ \forall i = -1,\dots,5,8\,,\quad E'_6 = \frac{1}{\sqrt{2}}(E_6 + iE_{67})\,,\quad E'_7 = \frac{1}{2}(H_7 - i(E_7 + F_7))\,, \tag{5.80}$$

so that

$$[E_{\alpha'}, E'_7/F'_7] = \mp i(E_{\alpha+\alpha_7} - E_{\alpha-\alpha_7})\,, \tag{5.81}$$

for all α''s satisfying $|\alpha'|^2 \leq 0$ and $\alpha' = \alpha$, where α is a root of $\mathfrak{e}_{10}$ in the original basis. In fact, all invariant generators in $\mathfrak{e}_{10}$ either satisfy $E_{\alpha'} = E_\alpha$, or are of the form (5.81). In Table 5.6, we characterize the former by their root $\alpha' = \alpha$ in the physical eigenbasis, and the latter as the sum of a root $\alpha' = \alpha$ and $-\alpha'_7$, to emphasize the fact that they build separate root spaces of ${}^2\mathcal{B}_{10}$ that will merge in $\mathfrak{g}_{\rm inv}$. Indeed, modding out ${}^2\mathcal{B}_{10}$ by $\{\mathfrak{z};\bar{d}\}$ eliminates the Cartan elements measuring the level in $\beta_I = \delta - \alpha'_7$ in $\mathfrak{g}_{\rm inv}$, thus identifying β_I with δ.

As an example, consider the fourth and sixth root at $l = 4$ in Table 5.6. Both are identified in $(\mathfrak{g}_{\rm inv})^{\mathbb{C}}$:

$$(1,1,1,1,1,1,1,1,2,2)' + (0^8,-1,1)' \sim (1,1,1,1,1,1,1,1,2,2)'$$

so that their respective generators: $A_{99} - A_{10\,10}$ on the one hand, and $A_{9\,10}$ plus 6 combinations of operators of the form $\widetilde{K}_{[1\cdots\hat{i}\cdots 89\,10]} \otimes Z_{[i9\,10]}$, $i = 1,\dots,6$ on the other hand, are now collected in a common 8-dimensional root space $(\mathfrak{g}_{\rm inv})_\alpha$ for $\alpha = \delta + \alpha_{-1} + \dots + \alpha_5 + \alpha_8$. As a result, the root multiplicity of $\delta + \alpha_{-1} + \dots + \alpha_5 + \alpha_8$ is conserved when reducing $\mathfrak{e}_{10}$ to $(\mathfrak{g}_{\rm inv})^{\mathbb{C}}$, even though its corresponding root space is spanned by (partly) different generators in each case. We expect this mechanism to occur for all imaginary roots of $\mathfrak{g}_{\rm inv}$.

On the other hand, the multiplicity of isotropic roots in ${}^2\mathcal{B}_{10}$ splits according to $8 \to 1 + 7$, in which the root space of multiplicity one is of the form (5.81). Likewise, imaginary roots of $\mathfrak{e}_{10}$ of length -2 split in ${}^2\mathcal{B}_{10}$ as $44 \to 8 + 36$, and we expect, though we did not push the analysis that far, that imaginary roots of length -4 will split as $192 \to 44 + 148$. Generally, we predict root multiplicities of ${}^2\mathcal{B}_{10}$ to be 1, 7, 36, 148, 535, 1745,... Although not our initial purpose, the method can thus be exploited to predict root multiplicities of certain Borcherds algebras constructed as fixed-point algebras of KMAs under a finite-order automorphism of order bigger than 2.

Finally, a remark on the real $\mathfrak{g}_{\rm inv}$. As anticipated in eqn.(5.79), the Borcherds algebra involved is actually the split form ${}^2\mathcal{B}_{10|11}$. Its reality properties can be inferred from the affine case, which has been worked out in detail in Section 5.4.1, and the behaviour of the generators $E_{n\beta_I} \doteq (1/\sqrt{2})(E_{n\delta+\alpha_7} - E_{n\delta-\alpha_7})$ and $F_{n\beta_I} = (E_{n\beta_I})^\dagger$ under the restriction ϕ.

Since $\phi(E_{n\beta_I}) = -F_{n\beta_I}$ and $\phi(F_{n\beta_I}) = -E_{n\beta_I}$, both sets of operators combine symmetrically in the usual compact and non-compact operators $E_{n\beta_I} \mp F_{n\beta_I}$. Moreover, the Cartan generator

[11] This basis will be used again for computing shift vectors in Section 5.8.

l	Invariant Tensor	Physical eigenbasis of $\mathfrak{e}_{10}$	O_w^8	$\alpha \in \Delta_+(^2\mathcal{B}_{10})$	$m(\alpha)$
0	$K_{[78]}$	$(0,0,0,0,0,0,1,-1,0,0)'$	28	0 0 0 0 0 0 0 0 1 0	1
1	$Z_{[89\,10]}$	$(0,0,0,0,0,0,0,1,1,1)'$	8	0 0 0 0 0 0 0 0 0 1	1
	$Z_{[678]}$	$(0,0,0,0,0,1,1,1,0,0)'$	56	0 1 0 0 0 0 0 0 0 0	1
2	$\widetilde{Z}_{[56789\,10]}$	$(0,0,0,0,1,1,1,1,1,1)'$	70	0 1 0 0 0 0 1 1 1 1	1
	$\widetilde{Z}_{[345678]}$	$(0,0,1,1,1,1,1,1,0,0)'$	28	0 2 0 0 1 2 3 2 1 0	1
3	$\widetilde{K}_{(9)[2\cdots89]} - \widetilde{K}_{(10)[2\cdots810]}$	$(0,1,1,1,1,1,1,1,1,1)' + (0^8,-1,1)'$	8	1 0 0 0 0 0 0 0 0 0	1
	$\widetilde{K}_{(8)[3\cdots10]}$	$(0,0,1,1,1,1,1,2,1,1)'$	168	0 2 0 0 1 2 3 2 1 1	1
	$\widetilde{K}_{(2)[3\cdots10]}$	$(0,1,1,1,1,1,1,1,1,1)'$	8	0 2 0 1 2 3 4 3 2 1	7
	$\widetilde{K}_{(8)[1\cdots8]}$	$(1,1,1,1,1,1,1,2,0,0)'$	8	0 3 1 2 3 4 5 3 1 0	1
4	$\big((\widetilde{K}\otimes Z)_{[1\cdots9][789]}$ $-(\widetilde{K}\otimes Z)_{[1\cdots8\,10][78\,10]}\big)$	$(1,1,1,1,1,1,2,2,1,1)' + (0^8,-1,1)'$	28	1 1 1 1 1 1 1 0 0 0	1
	$(\widetilde{K}_{(2)}\otimes Z)_{[3\cdots10][89\,10]}$	$(0,1,1,1,1,1,1,2,2,2)'$	56	0 2 0 1 2 3 4 3 2 2	1
	$(\widetilde{K}_{(2)}\otimes Z)_{[3\cdots10][678]}$	$(0,1,1,1,1,2,2,2,1,1)'$	280	0 3 0 1 2 3 4 3 2 1	1
	$(\widetilde{K}_{(2)}\otimes Z)_{[3\cdots10][19\,10]}$ $A_{(9\,10)}$	$(1,1,1,1,1,1,1,1,2,2)'$	1	0 2 1 2 3 4 5 4 3 2	7
	$(\widetilde{K}_{(2)}\otimes Z)_{[3\cdots10][178]}$ $A_{(78)}$	$(1,1,1,1,1,1,2,2,1,1)'$	28	0 3 1 2 3 4 5 3 2 1	7
	$A_{(9\,9)} - A_{(10\,10)}$	$(1,1,1,1,1,1,1,1,2,2)' + (0^8,-1,1)'$	1	1 0 1 1 1 1 1 1 1 1	1
	$A_{(88)}$	$(1,1,1,1,1,1,1,3,1,1)'$	8	0 3 1 2 3 4 5 3 1 1	1
5	$\big((\widetilde{K}\otimes\widetilde{Z})_{[1\cdots9][4\cdots9]}$ $-(\widetilde{K}\otimes\widetilde{Z})_{[1\cdots810][4\cdots8\,10]}\big)$	$(1,1,1,2,2,2,2,2,1,1)' + (0^8,-1,1)'$	56	1 2 1 1 1 2 3 2 1 0	1
	$(\widetilde{K}_{(2)}\otimes\widetilde{Z})_{[3\cdots10][5\cdots\,10]}$	$(0,1,1,1,2,2,2,2,2,2)'$	280	0 3 0 1 2 3 5 4 3 2	1
	$(\widetilde{K}_{(2)}\otimes\widetilde{Z})_{[3\cdots10][3\cdots\,8]}$	$(0,1,2,2,2,2,2,2,1,1)'$	56	0 4 0 1 3 5 7 5 3 1	1
	$B_{(8)[7\cdots\,10]}$	$(1,1,1,1,1,1,2,3,2,2)'$	56	0 3 1 2 3 4 5 3 2 2	1
	$B_{(8)[5\cdots\,8]}$	$(1,1,1,1,2,2,2,3,1,1)'$	280	0 4 1 2 3 4 6 4 2 1	1

5	$B_{(9)[678\,9]} - B_{(10)[678\,10]}$	$(1,1,1,1,1,2,2,2,2,2)' + (0^8,-1,1)'$	56	1 1 1 1 1 1 1 1 1 1	1
	$(\widetilde{K}_{(2)} \otimes \widetilde{Z})_{[3\cdots10][16\cdots10]}$ $B_{(6)[7\cdots10]}$	$(1,1,1,1,1,2,2,2,2,2)'$	56	0 3 1 2 3 4 5 4 3 2	7
	$(\widetilde{K}_{(2)} \otimes \widetilde{Z})_{[3\cdots10][14\cdots8]}$ $B_{(4)[5\cdots8]}$	$(1,1,1,2,2,2,2,2,1,1)'$	56	0 4 1 2 3 5 7 5 3 1	7
6	$\big((\widetilde{K} \otimes \widetilde{K}_{(9)})_{[2\cdots10][2\cdots9]}$ $-(\widetilde{K} \otimes \widetilde{K}_{(10)})_{[2\cdots8\,10][2\cdots8\,10]}\big)$	$(0,2,2,2,2,2,2,2,2,2)' + (0^8,-1,1)'$	8	2 0 0 0 0 0 0 0 0 0	1
	$(\widetilde{K}_{(2)} \otimes \widetilde{K}_{(8)})_{[3\cdots10][3\cdots10]}$	$(0,1,2,2,2,2,2,3,2,2)'$	336	0 4 0 1 3 5 7 5 3 2	1
	$\big((\widetilde{K} \otimes \widetilde{K}_{(9)})_{[1\cdots9][3\cdots10]}$ $-(\widetilde{K} \otimes \widetilde{K}_{(10)})_{[1\cdots8\,10][3\cdots10]}\big)$ $\big((A \otimes \widetilde{Z})_{[89][3\cdots79]}$ $-(A \otimes \widetilde{Z})_{[8\,10][3\cdots7\,10]}\big)$ $D_{(9)[3\cdots9]} - D_{(10)[3\cdots8\,10]}$	$(1,1,2,2,2,2,2,2,2,2)' + (0^8,-1,1)'$	28	2 0 1 0 0 0 0 0 0 0	8
	$(\widetilde{K}_{(2)} \otimes \widetilde{K}_{(8)})_{[3\cdots10][1\cdots8]}$ $(A \otimes \widetilde{Z})_{[78][2\cdots68]}$ $D_{(8)[2\cdots8]}$	$(1,2,2,2,2,2,2,3,1,1)'$	56	0 5 1 3 5 7 9 6 3 1	1
	$(A \otimes \widetilde{Z})_{(9\,10)[5\cdots10]}$	$(1,1,1,1,2,2,2,2,3,3)'$	70	0 3 1 2 3 4 6 5 4 3	1
	$(A \otimes \widetilde{Z})_{(78)[5\cdots10]}$	$(1,1,1,1,2,2,3,3,2,2)'$	420	0 4 1 2 3 4 6 4 3 2	1
	$(A \otimes \widetilde{Z})_{(78)[3\cdots8]}$	$(1,1,2,2,2,2,3,3,1,1)'$	420	0 5 1 2 4 6 8 5 3 1	1
	$\big((A \otimes \widetilde{Z})_{(89)[4\cdots89]}$ $-(A \otimes \widetilde{Z})_{(8\,10)[4\cdots8\,10]}\big)$	$(1,1,1,2,2,2,2,3,2,2)' + (0^8,-1,1)'$	280	1 2 1 1 1 2 3 2 1 1	1
	$\big((\widetilde{K} \otimes \widetilde{K}_{(1)})_{[1\cdots9][2\cdots9]}$ $-(\widetilde{K} \otimes \widetilde{K}_{(1)})_{[1\cdots8\,10][2\cdots8\,10]}\big)$	$(2,2,2,2,2,2,2,2,1,1)' + (0^8,-1,1)'$	8	2 1 2 2 2 2 2 1 0 0	8
	$(\widetilde{K}_{(2)} \otimes \widetilde{K}_{(2)})_{[3\cdots10][3\cdots10]}$	$(0,2,2,2,2,2,2,2,2,2)'$	8	0 4 0 2 4 6 8 6 4 2	7
	$(\widetilde{K} \otimes \widetilde{K}_{(10)})_{[1\cdots9][3\cdots10]}$ $(A \otimes \widetilde{Z})_{(34)[5\cdots10]}$ $D_{(3)[4\cdots10]}$ $(\widetilde{K} \otimes \widetilde{K}_{(2)})_{[13\cdots10][3\cdots10]}$ $S_{[3\cdots10]}$	$(1,1,2,2,2,2,2,2,2,2)'$	28	0 4 1 2 4 6 8 6 4 2	36
	$(\widetilde{K} \otimes \widetilde{K}_{(8)})_{[1\cdots79\,10][1\cdots8]}$ $(A \otimes \widetilde{Z})_{(12)[3\cdots8]}$ $D_{(1)[2\cdots8]}$ $(\widetilde{K} \otimes \widetilde{K}_{(1)})_{[1\cdots9][2\cdots810]}$ $S_{[1\cdots8]}$	$(2,2,2,2,2,2,2,2,1,1)'$	1	0 5 2 4 6 8 10 7 4 1	36

Table 5.6: Decomposition of root spaces of ${}^2\mathcal{B}_{10}$ in representations of $\mathfrak{sl}(8,\mathbb{R})$

H_I has to match the reality property of H_δ to which it gets identified under relation (5.77), and must therefore be non-compact in ${}^2\mathcal{B}_{10}$, which agrees with the definition of the split form ${}^2\mathcal{B}_{10|11}$.

As in the affine case, the signature of $\mathfrak{g}_{\text{inv}}$ remains finite and is completely determined by the reality properties of the Cartan subalgebra. Taking into account the quotient (5.77), the signature is $\sigma = 8$.

5.5 The orbifolds $T^4/\mathbb{Z}_{n>2}$

In this section, we will treat the slightly more involved orbifold $T^{7-D} \times T^4/Z_n$ for $n \geqslant 3$. A new feature appears in this case: the invariant subalgebras will now contain generators that are complex combinations of the original $\mathfrak{e}_{10}$ generators. If the orbifold is chosen to act on the coordinates $\{x^7, x^8, x^9, x^{10}\}$, it will only affect roots containing α_4, α_5, α_6, α_7 or α_8, and the corresponding generators. This orbifold should thus be studied first in $D = 6$ where $\mathfrak{g}^U = \mathfrak{so}(5,5)$, with the following action on the complex coordinates:

$$(z_1, \bar{z}_1) \to (e^{2\pi i/n} z_1, e^{-2\pi i/n} \bar{z}_1)\,, \qquad (z_2, \bar{z}_2) \to (e^{-2\pi i/n} z_2, e^{2\pi i/n} \bar{z}_2)\,. \tag{5.82}$$

In other words, we choose the prescription $Q_1 = +1$ and $Q_2 = -1$ to ensure $\sum_i Q_i = 0$.

The rotation operator $\mathcal{U}_4^{\mathbb{Z}_n} = \prod_{k=1}^{2} e^{-\frac{2\pi i}{n} Q_k \mathcal{K}_{z_k \bar{z}_k}}$ with the above charge prescription leaves invariant the following objects:

$$\begin{array}{l|lcl}
Q_A = 0 & K_{66} & = & \frac{1}{4}(5H_4 + 6H_5 + 4H_6 + 2H_7 + 3H_8)\,, \\
 & K_{z_1\bar{z}_1} & = & \frac{1}{2}(K_5 + K_6) = \frac{1}{4}(H_4 + 4H_5 + 4H_6 + 2H_7 + 3H_8)\,, \\
 & K_{z_2\bar{z}_2} & = & \frac{1}{2}(K_7 + K_8) = \frac{1}{4}(H_4 + 2H_5 + 3H_8)\,, \\
 & \left\{\begin{array}{c} K_{\bar{z}_1\bar{z}_2} \,/\, \frac{1}{2}\mathcal{K}_{\bar{z}_1 bar z_2} \\ K_{z_1 z_2} \,/\, \frac{1}{2}\mathcal{K}_{z_1 z_2} \end{array}\right\} & = & \frac{1}{4}\big(E_{56} - E_{67}(+/-)(F_{56} - F_{67}) \\
 & & & \qquad \pm i\big(E_6 + E_{567}(+/-)(F_6 + F_{567})\big)\big)\,, \\
 & Z_{6z_1\bar{z}_1} \,/\, \frac{1}{2}\mathcal{Z}_{6z_1\bar{z}_1} & = & \frac{i}{2}(E_{45^26^278}(+/-)F_{45^26^278}) \\
 & Z_{6z_2\bar{z}_2} \,/\, \frac{1}{2}\mathcal{Z}_{6z_2\bar{z}_2} & = & \frac{i}{2}(E_{458}(+/-)F_{458})\,, \\
 & \left\{\begin{array}{c} Z_{6\bar{z}_1\bar{z}_2} \,/\, \frac{1}{2}\mathcal{Z}_{6\bar{z}_1\bar{z}_2} \\ Z_{6z_1z_2} \,/\, \frac{1}{2}\mathcal{Z}_{6z_1z_2} \end{array}\right\} & = & \frac{1}{4}\big(E_{45^2678} - E_{4568}(+/-)(F_{45^2678} - F_{4568}) \\
 & & & \qquad \pm i\big(E_{45^268} + F_{45^268}(+/-)(E_{45678} + F_{45678})\big)\big)\,. \\
 & \mathcal{K}_{z_1\bar{z}_1} = i(E_5 - F_5)\,, & & \mathcal{K}_{z_2\bar{z}_2} = i(E_7 - F_7)\,.
\end{array} \tag{5.83}$$

Thus $\mathfrak{g}_{\text{inv}}$ has as before (conserved) rank 5. Note that the invariant diagonal metric elements are in fact linear combinations of the three basic Cartan generators satisfying $\alpha_5(H) = \alpha_7(H) = 0$, namely $\{2H_4 + H_5, H_5 + 2H_8, H_5 + 2H_6 + H_7\}$. Furthermore, we have various charged combinations:

$$\begin{array}{l|lcl}
Q_A = +1 & K_{6\bar{z}_1} \,/\, \frac{1}{2}\mathcal{K}_{6\bar{z}_1} & = & \frac{1}{2\sqrt{2}}\big(E_4(+/-)F_4 + i(E_{45}(+/-)F_{45})\big)\,, \\
 & K_{6z_2} \,/\, \frac{1}{2}\mathcal{K}_{6z_2} & = & \frac{1}{2\sqrt{2}}\big(E_{456}(+/-)F_{456} - i(E_{4567}(+/-)F_{4567})\big)\,, \\
 & Z_{z_1\bar{z}_1z_2} \,/\, \frac{1}{2}\mathcal{Z}_{z_1\bar{z}_1z_2} & = & \frac{1}{2\sqrt{2}}\big(E_{568}(+/-)F_{568} + i(E_{5678}(+/-)F_{5678})\big)\,, \\
 & Z_{\bar{z}_1z_2\bar{z}_2} \,/\, \frac{1}{2}\mathcal{Z}_{\bar{z}_1z_2\bar{z}_2} & = & \frac{1}{2\sqrt{2}}\big(-(E_8(+/-)F_8) + i(E_{58}(+/-)F_{58})\big)\,,
\end{array} \tag{5.84}$$

and their complex conjugates with $Q_A = -1$, along with:

$$
\begin{array}{c|lcl}
Q_A = +2 & K_{\bar{z}_1\bar{z}_1} & = & \frac{1}{2}\big(H_5 + i(E_5 + F_5)\big)\,, \\
\hline
 & K_{z_2 z_2} & = & \frac{1}{2}\big(H_7 - i(E_7 + F_7)\big)\,, \\
 & K_{\bar{z}_1 z_2} \,/\, \frac{1}{2}\mathcal{K}_{\bar{z}_1 z_2} & = & \frac{1}{4}\big(E_{56} + E_{67}(+/-)(F_{56} + F_{67}) \\
 & & & \qquad -i\big(E_{567} - E_6(+/-)(F_{567} - F_6)\big)\big)\,, \\
 & Z_{6\bar{z}_1 z_2} \,/\, \frac{1}{2}\mathcal{Z}_{6\bar{z}_1 z_2} & = & \frac{1}{4}\big(E_{45^2678} + E_{4568}(+/-)(F_{45^2678} + F_{4568}) \\
 & & & \qquad -i\big(E_{45^268} - E_{45678}(+/-)(F_{45^268} - F_{45678})\big)\big)\,,
\end{array}
\tag{5.85}
$$

and complex conjugates ($Q_A = -2$). Note that these five sectors are all different in $T^4/\mathbb{Z}_n$ for $n \geqslant 5$, while the two sectors with $Q_A = \pm 2$ will clearly have the same charge assignment in $T^4/\mathbb{Z}_4$. Finally, the orbifold $T^4/\mathbb{Z}_3$ merges, on the one hand, the two sectors with $Q_A = 2, -1$ and, on the other hand, the two remaining ones with $Q_A = 1, -2$, giving rise to three main sectors instead of five. In string theory, these three cases will lead to different twisted sectors, however, the untwisted sector and the residual U-duality algebra do not depend on n for any $n \geqslant 3$. The $n = 2$ case will again be treated separately.

For clarity, we will start by deriving the general structure of the (complex) invariant subalgebra, leaving aside, for the moment being, the analysis of its reality property. To do so, we perform a change of basis in the $Q_A = 0$ sector, separating raising from lowering operators. Let $X_\alpha = (1/2)(E_\alpha + F_\alpha)$ be the generator of any field element of $\mathfrak{g}^U$, we will resort to the combinations $X_\alpha^+ \doteq X_\alpha + \frac{1}{2}\mathcal{X}_\alpha = E_\alpha$ and $X_\alpha^- \doteq X_\alpha - \frac{1}{2}\mathcal{X}_\alpha = F_\alpha$ to derive $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$. First, the following generators can be shown to form a basis of the non-abelian part of $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$:

$$
\begin{aligned}
E_{\tilde{\alpha}} &= -iE_{458} = Z^+_{6\bar{z}_2 z_2}\,, \qquad\qquad F_{\tilde{\alpha}} \;=\; iF_{458} = Z^-_{6 z_2 \bar{z}_2}\,, \\
E_{\alpha_\pm} &= \frac{1}{2}\left(E_{56} - E_{67} \pm i(E_{567} + E_6)\right) = (K_{\bar{z}_1\bar{z}_2}/K_{z_1 z_2})^+\,, \\
F_{\alpha_\pm} &= \frac{1}{2}\left(F_{56} - F_{67}\right) \pm i(-F_{567} - F_6)) = (K_{z_1 z_2}/K_{\bar{z}_1\bar{z}_2})^-\,.
\end{aligned}
\tag{5.86}
$$

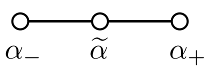
Computing their commutation relations determines the remaining generators of the algebra (for economy, we have omitted the lowering operators, which can be obtained quite straightforwardly by $F_\alpha = (E_\alpha)^\dagger$):

$$
\begin{aligned}
E_{\tilde{\alpha}+\alpha_\pm} &\doteq \pm[E_{\tilde{\alpha}}, E_{\alpha_\pm}] \;=\; \frac{1}{2}\left(E_{45^2678} - E_{4568} \pm i(E_{45678} + E_{45^268})\right) \\
&= (Z_{6\bar{z}_1\bar{z}_2}/Z_{6 z_1 z_2})^+\,, \\
E_{\alpha_- + \tilde{\alpha} + \alpha_+} &\doteq [E_{\alpha_-}, E_{\tilde{\alpha}+\alpha_+}] \;=\; -iE_{45^26^278} = (Z_{6\bar{z}_1 z_1})^+\,, \\
H_{\tilde{\alpha}} &\doteq [E_{\tilde{\alpha}}, F_{\tilde{\alpha}}] \;=\; (H_4 + H_5 + H_8)\,, \\
H_{\alpha_\pm} &\doteq [E_{\alpha_\pm}, F_{\alpha_\pm}] \;=\; \frac{1}{2}\big(H_5 + 2H_6 + H_7 \pm i(F_5 - E_5 + F_7 - E_7)\big)\,, \\
H_{\tilde{\alpha}+\alpha_\pm} &\doteq [E_{\tilde{\alpha}+\alpha_\pm}, F_{\tilde{\alpha}+\alpha_\pm}] \\
&= \frac{1}{2}\big(2H_4 + 3H_5 + 2H_6 + H_7 + 2H_8 \mp i(E_5 - F_5 + E_7 - F_7)\big)\,, \\
H_{\alpha_- + \tilde{\alpha} + \alpha_+} &\doteq [E_{\alpha_- + \tilde{\alpha} + \alpha_+}, F_{\alpha_- + \tilde{\alpha} + \alpha_+}] = H_4 + 2H_5 + 2H_6 + H_7 + H_8\,.
\end{aligned}
\tag{5.87}
$$

which shows that the non-abelian part of the complexified invariant subalgebra is of type $\mathfrak{a}_3 \simeq \mathfrak{d}_3$.

The rest of the $Q_A = 0$ sectors combines into two abelian contributions, so that the whole $D = 6$ $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$ reads

$\mathfrak{d}_3 \oplus \mathbb{C}^{\oplus 2}$:



$\times\ \{H_8 - H_4\} \times\ \{E_5 - F_5 - E_7 + F_7\}$

Concentrating on the non-abelian $\mathfrak{d}_3$ part of the real form $\mathfrak{g}_{\text{inv}}$, we remark that it can be chosen to have Cartan subalgebra spanned by the basis $\{i(H_{\alpha_+} - H_{\alpha_-});\, H_{\widetilde{\alpha}};\, H_{\alpha_+} + H_{\alpha_-}\}$ compatible with the restriction $\text{Fix}_{\tau_0}(\mathfrak{g}_{\text{inv}}{}^{\mathbb{C}})$. Since, in this basis, all ladder operators combine under ϕ into pairs of one compact and one non-compact operator, the signature of the real $\mathfrak{d}_3$ is again completely determined by the difference between non-compact and compact Cartan generators: since $i(H_{\alpha_+} - H_{\alpha_-})$ is compact while the two remaining generators are non-compact, $\sigma(\mathfrak{d}_3) = 1$, which determines the real form to be $\mathfrak{su}(2,2) \simeq \mathfrak{so}(4,2)$. The reality property of the invariant subalgebra is encoded in the Satake diagram of Table 5.7.

In addition, the two abelian factors appearing in the diagram restrict, under Fix_{τ_0} to $H^{[4]} = H_8 - H_4$ and $i\widetilde{H}^{[4]} = (E_5 - F_5 - E_7 + F_7)$ and generate $\mathfrak{so}(1,1) \oplus \mathfrak{u}(1)$, similarly to the $T^2/\mathbb{Z}_{n>2}$ case. Their contributions to the signature cancel out, so $\sigma(\mathfrak{g}_{\text{inv}}) = 1$

If we refer to the Satake diagram of Table 5.7, we note that in contrast to the split case, the arrows now joining the roots α_+ and α_- indeed change the compactness of the Cartan subalgebra without touching the "split" structure of the ladder operators. Moreover, the combinations $i(H_{\alpha_+} - H_{\alpha_-})$ and $H_{\alpha_+} + H_{\alpha_-}$ are now directly deducible from the action of ϕ on the set of simple roots.

Finally, as will be confirmed with the $T^6/\mathbb{Z}_{n>2}$ orbifold, if the chief inner automorphism $\mathcal{U}_q^{\mathbb{Z}_n}$ produces k pairs of Cartan generators in $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$ taking value in $\mathfrak{h}(\mathfrak{e}_{r|r}) \pm i\mathfrak{k}(\mathfrak{e}_{r|r})$, there will be k arrows joining the dual simple roots in the Satake diagram.

Compactifying further to $D = 5$, the additional node α_3 connects to $\widetilde{\alpha}$ forming a $\mathfrak{d}_4$ subalgebra. As in the $T^2/\mathbb{Z}_{n>2}$ case, this extra split $\mathfrak{a}_1$ will increase the total signature by one, yielding the real form $\mathfrak{so}(5,3)$. Since $\alpha_3(H_8 - H_4) \neq 0$, the non-compact Cartan generator $H^{[4]}$ commuting with $\mathfrak{so}(5,3) \oplus \mathfrak{u}(1)$ is now any multiple of $H^{[4]} = 2H_3 + 4H_4 + 3H_5 + 2H_6 + H_7$.

In $D = 4$, a new invariant root $\gamma = \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8 \in \Delta^+(\mathfrak{e}_7)$ appears which enhances the $\mathfrak{so}(1,1)$ factor to $sl(2,\mathbb{R})$. The reality property of the latter abelian factor can be checked by rewriting $\gamma = \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_- + \widetilde{\alpha} + \alpha_+$, which tells us that $\phi(\gamma) = -\gamma$. In $D = 3$, the additional node α_1 extending $\mathfrak{g}_{\text{inv}}$ reconnects γ to the Dynkin diagram, resulting in $\mathfrak{so}(8,6) \oplus \mathfrak{u}(1)$.

5.5.1 Equivalence classes of involutive automorphisms of Lie algebras

Before treating the affine case, we shall introduce a procedure extensively used by [71], par. 14.4, to determine real forms by translating the adjoint action of the involutive automorphism on the generators by an exponential action on the root system directly.

Our concern in this paper will be only with real forms generated from chief *inner* involutive automorphisms, in other words involutions which can be written as $\vartheta = \text{A}d(e^{\overline{H}})$ for $\overline{H} \in \mathfrak{h}^{\mathbb{C}}$. In this case, a matrix realization of the defining chief inner automorphism, denoted $\boldsymbol{\vartheta} = \text{A}d(e^{\overline{\mathbf{H}}})$ will act on the compact real form $\mathfrak{g}_c$ as:

$$\boldsymbol{\vartheta} = \mathbb{1}_r \oplus \sum_{\alpha \in \Delta_+} \begin{pmatrix} \cosh(\alpha(\overline{H})) & i\sinh(\alpha(\overline{H})) \\ -i\sinh(\alpha(\overline{H}) & \cosh(\alpha(\overline{H})) \end{pmatrix} , \qquad \forall \alpha \in \Delta_+ \ . \tag{5.88}$$

Being involutive $\boldsymbol{\vartheta}^2 = \mathbb{1}$ implies $\cosh(2\alpha(\overline{H})) = 1$, leading to

$$e^{\alpha(\overline{H})} = \pm 1, \quad \forall \alpha \in \Delta_+ \ . \tag{5.89}$$

In particular, this should hold for the simple roots: $e^{\alpha_i(\overline{H})} = \pm 1$, $\forall \alpha_i \in \Pi$. Then, how one assigns the $\pm$-signs to the simple roots completely determines the action of ϑ on the whole root lattice (5.89). For a r-rank algebra, there are then 2^r inner involutive automorphisms, but in general far less non-isomorphic real forms of $\mathfrak{g}$.

We are now ready to implement the procedure (5.69), first by splitting the positive root system Δ_+ into two subsets

$$\Delta_{(\pm 1)} \doteq \left\{ \alpha \in \Delta_+ \mid e^{\alpha(\overline{H})} = \pm 1 \right\} \tag{5.90}$$

and then by acting with the linear operator (5.68) in its matrix realisation (5.88) on the base of $\mathfrak{g}_c$. Then, the eigenspaces with eigenvalue (± 1) can be shown to be spanned by

$$\mathfrak{k} = \text{Span}\left\{ iH_{\alpha_j},\, \forall \alpha_j \in \Pi;\, (E_\alpha - F_\alpha) \text{ and } i(E_\alpha + F_\alpha),\ \forall \alpha \in \Delta_{(+1)} \right\} \tag{5.91}$$

and

$$\mathfrak{p} = \text{Span}\left\{ i(E_\alpha - F_\alpha),\, (E_\alpha + F_\alpha),\ \forall \alpha \in \Delta_{(-1)} \right\}\ . \tag{5.92}$$

In this approach, the signature determining all equivalence classes of involutive automorphisms (5.89) takes the handy form

$$-\sigma = \text{Tr}\boldsymbol{\vartheta} = \big(r + 2 \sum_{\alpha \in \Delta^+} \cosh(\alpha(\overline{H}))\big) = r + 2(\dim \Delta_{(+1)} - \dim \Delta_{(-1)})\ . \tag{5.93}$$

5.5.2 A matrix formulation of involutive automorphisms of affine KMAs

This analysis can be extended to real forms of affine extension of Lie algebras. The general method based on a matrix reformulation of the involutive automorphism has been developed in [73] and successfully applied to the $\widehat{A}_r$, $\widehat{B}_r$, $\widehat{C}_r$ and $\widehat{D}_r$ cases in [74, 72, 67, 69, 68]. Here, we will only present the very basics of the method, and refer the reader to these articles for more details.

There are two ways of handling involutive automorphisms of untwisted affine Lie algebras. The first (classical) one is based on the study of Cartan-preserving automorphisms. Since every conjugacy class of the automorphism group contains at least one such automorphism, one can by this means arrive at a first classification of the involutive automorphisms of a given affine KMA. This procedure would be enough for determining all real forms of a finite Lie algebra, but would usually overcount them for affine KMA, because in this case some Cartan-preserving automorphisms can be conjugate via non-Cartan-preserving ones within $\text{Aut}(\widehat{\mathfrak{g}})$. This will obviously reduce the number of conjugacy classes and by the same token the number of real forms of an untwisted affine KMA. A matrix formulation of automorphisms has been proposed in [73] precisely to treat these cases.

The first method takes advantage of the fact that Cartan-preserving automorphisms can be translated into automorphisms of the root system that leave the root structure of $\widehat{\mathfrak{g}}$ invariant. Let us call ϕ such an automorphism acting on $\Delta(\widehat{\mathfrak{g}})$. It can be constructed from an automorphism ϕ_0 acting on the basis of simple roots $\Pi(\mathfrak{g})$, for $\text{rk}\mathfrak{g} = r$, as $\phi_0(\alpha_i) = \sum_{j=1}^{r} (\phi_0)_i^{\ j} \alpha_j$ for $i = 1, .., r$. [12] Define the linear functional $\Omega \in P(\widehat{\mathfrak{g}})$ such that

$$\phi(\delta) = \mu\delta\,, \qquad \phi(\alpha_i) = \phi_0(\alpha_i) - (\phi_0(\alpha_i)|\Omega)\mu\delta\,,\ \forall i = 1, .., r\,. \tag{5.94}$$

with $(\alpha_i|\Omega) = n_i \in \mathbb{Z}$. This automorphism will be root-preserving if

$$\mu = \pm 1\,, \quad \text{and} \quad (\phi_0)_i^{\ j} \in \mathbb{Z}\,.$$

All root-preserving automorphisms can thus be characterized by the triple $\mathcal{D}_\phi = \{\phi_0, \Omega, \mu\}$, with the composition law:

$$\mathcal{D}_{\phi_1}\mathcal{D}_{\phi_2} = \{(\phi_1)_0 \cdot (\phi_2)_0, \mu_2\Omega_1 + (\phi_1)_0(\Omega_2), \mu_1\mu_2\} \tag{5.95}$$

[12] Not to confuse with the Cartan involution acting on the root system, as given from the Satake diagram. In the finite case, if ϕ_0 is non-trivial, it typically corresponds to outer automorphisms of the algebra.

The action of ϕ lifts to an algebra automorphism ϑ_ϕ. The first relation in expression (5.94) implies $\vartheta_\phi(c) = \mu c$, while we have:

$$\begin{aligned}
&\vartheta_\phi(z^n \otimes E_\alpha) = C_{\alpha+n\delta}\, z^{\mu(n-(\phi_0(\alpha_i)|\Omega))} \otimes E_{\phi_0(\alpha)}\,, \\
&\vartheta_\phi(z^n \otimes H_\alpha) = C_{n\delta}\, z^{\mu n} \otimes H_{\phi_0(\alpha)}\,, \qquad \vartheta_\phi(d) = \mu d + H_\Omega - \tfrac{1}{2}|\Omega|^2 \mu c\,.
\end{aligned} \tag{5.96}$$

on the rest of the algebra. By demanding that ϑ_ϕ preserves the affine algebra (5.17), we can derive the relations, for α and $\beta \in \Delta(\mathfrak{g})$: $C_{n\delta}C_{m\delta} = C_{(m+n)\delta}$, $C_{\alpha+n\delta} = C_{n\delta}C_\alpha$, and $\mathcal{N}_{\alpha,\beta}C_{\alpha+\beta} = \mathcal{N}_{\phi_0(\alpha),\phi_0(\beta)}C_\alpha C_\beta$ with $C_0 = 1$ and $C_{-\alpha} = C_\alpha^{-1}$. The condition for ϑ_ϕ to be involutive is analogous to the requirement (5.89), namely

$$e^{\alpha_i(\overline{H})} = \pm 1\,, \qquad \forall i = 0, 1, .., r\,,$$

where $i = 0$ is this time included, and $\overline{H} = \sum c_i H_i + c_d d$, with $c_i, c_d \in \mathbb{C}$.

In particular, for a Cartan-preserving chief inner automorphism of type $\vartheta = e^{\mathrm{ad}(\overline{H})}$, we have:

$$\phi_0 = \mathbb{1}\,, \qquad \Omega = 0\,, \qquad \mu = 1\,, \qquad C_\alpha = e^{\alpha(\overline{H})}\,,\ \forall \alpha \in \Delta(\mathfrak{g})\,. \tag{5.97}$$

Possible real forms of an untwisted affine KMA are then determined by studying conjugacy classes of triples $\mathcal{D}_\phi$, for various involutive automorphisms ϕ. However, from the general structure (5.97), we see that a chief inner automorphism cannot be conjugate through a Cartan-preserving automorphism to an automorphism associated with a Weyl reflection, for instance. They could, however, be conjugate under some more general automorphism (note that this could not happen in the finite context). The above method might thus lead to overcounting the number of equivalence classes of automorphisms, and consequently, of real forms of an affine Lie algebra.

This problem has been solved by a newer approach due to Cornwell, which is based on a matrix reformulation of the set of automorphisms for a given affine KMA. Choosing a faithfull d_Γ-dimensional representation of $\mathfrak{g}$ denoted by Γ, we can represent any element of $\mathcal{L}(\mathfrak{g})$ by $A(z) = \sum_{b=1}^{r}\sum_{n=-\infty}^{\infty} a_n{}^b\, z^n \otimes \Gamma(X_b)$, for $X_b \in \mathfrak{g}$. Then any element of $\widehat{\mathfrak{g}}$ may be written as:

$$\widehat{A}(z) = A(z) + \mu_c c + \mu_d d$$

where the + are clearly not to be taken as matrix additions.

It has been pointed out in [73] that all automorphisms of complex untwisted KMAs are classified in this matrix formulation according to four types, christened: type 1a, type 1b, type 2a and type 2b.

A type 1a automorphism will act on $A(z)$ through an invertible $d_\Gamma \times d_\Gamma$ matrix $U(z)$ with components given by Laurent polynomials in z:

$$\varphi(A(z)) = U(z)A(uz)U(z)^{-1} + \frac{1}{\gamma_\Gamma}\oint \frac{dz}{2\pi i z}\,\mathrm{Tr}\left[\left(\frac{d}{dz}\ln U(z)\right) A(uz)\right] c\,, \tag{5.98}$$

where γ_Γ is the Dynkin index of the representation, and $u \in \mathbb{C}^*$ (this parameter corresponds, in the preceding formulation, to a Cartan preserving automorphism of type $\vartheta = e^{\mathrm{ad}(d)}$). The remaining three automorphisms are defined as above, by replacing $A(uz) \to \{-\widetilde{A}(uz); A(uz^{-1}); -\widetilde{A}(uz^{-1})\}$ on the RHS of expression (5.98) for, respectively, type {1b;2a;2b} automorphisms.

Here the tilde denotes the contragredient representation $-\widetilde{\Gamma}$. The action on c and d is the same for all four automorphisms, namely:

$$\begin{aligned}
\varphi(c) &= \mu c\,, \\
\varphi(d) &= \mu\Phi(U(z)) + \lambda c + \mu d\,,
\end{aligned}$$

with $\mu = 1$ for type 1a and 1b, and $\mu = -1$ for type 2a and 2b, and the matrix:

$$\Phi(U(z)) = -z\frac{d}{dz}\ln U(z) + \frac{1}{d_\Gamma}\operatorname{Tr}\left(z\frac{d}{dz}\ln U(z)\right)\mathbb{1}\,. \tag{5.99}$$

An automorphism φ can then be encoded in the triple: $\mathcal{D}_\varphi = \{U(z), u, \lambda\}$, and, as before, conjugation classes of automorphisms can be determined by studying equivalence classes of triples $\mathcal{D}_\varphi$. In this case, the more general structure of the matrix $U(z)$ as compared to ϕ_0, which acts directly on the generators of $\widehat{\mathfrak{g}}$ in a given representation, allows conjugation of two Cartan-preserving automorphisms via both Cartan-preserving and non-Cartan-preserving ones.

Finally, the conditions for φ to be involutive are, for type 1a:

$$u^2 = 1\,, \tag{5.100}$$

and

$$\begin{aligned} U(z)U(uz) &= \zeta z^k \mathbb{1}\,, \qquad \text{with } k\in\mathbb{N} \text{ and } \zeta\in\mathbb{C}\,,\\ \lambda &= -\frac{1}{2\gamma_\Gamma}\oint\frac{dz}{2\pi i z}\operatorname{Tr}\left[\left(\frac{d}{dz}\ln U(z)\right)\Phi(U(uz))\right]. \end{aligned} \tag{5.101}$$

For a type 1b automorphism, the first condition (5.100) remains the same, while we have to replace $U(uz)\to\widetilde{U}(uz)^{-1}$ and $\Phi(U(uz))\to-\widetilde{\Phi}(U(uz))$ in the two last conditions (5.101).

Involutive automorphisms of type 2a and 2b are qualitatively different since they are already involutive for any value of u (so that condition (5.100) can be dropped), provided the last two conditions (5.101) are met, with the substitutions $U(uz)\to U(uz^{-1})$ in the first and $\Phi(U(uz))\to\Phi(U(uz^{-1}))$ in the second one for type 2a, and $U(uz)\to\widetilde{U}(uz^{-1})^{-1}$ in the first and $\Phi(U(uz))\to-\Phi(\widetilde{U}(uz^{-1}))$ in the second one for type 2b. In both cases, we are free to set $u=1$.

When studying one particular class of involutive automorphisms, one will usually combine both the method based on root-preserving automorphisms and the one using the more elaborate matrix formulation to get a clearer picture of the resulting real form.

5.5.3 The non-split real invariant subalgebra in $D = 2$

The affine extension in $D = 2$ yields a real form of $\hat{\mathfrak{d}}_7 \bowtie \hat{\mathfrak{u}}(1)$. We will show that this real form, obtained from projecting from $\mathfrak{e}_{9|10}$ all charged states, builds a $\widehat{\mathfrak{so}}(8,6)\bowtie\hat{u}_{|1}(1)$, where, by $\widehat{\mathfrak{so}}(8,6)$, we mean the affine real form described by the $D = 2$ Satake diagram of Table 5.7 as determined in [239]. The proof requires working in a basis of $\mathfrak{g}_{\text{inv}}$ in which the Cartan subalgebra is chosen compact. It will be shown that such a basis can indeed be constructed from the restriction $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}\cap\mathfrak{e}_{9|10}$. Then, by determining the action of ϕ on the latter, we will establish that, following [73], the vertex operator (*or* Sugawara) construction of $\mathfrak{g}_{\text{inv}}$ reproduces exactly the Cartan decomposition of $\widehat{\mathfrak{so}}(8,6)$ expected from [69]. Finally, we will show how the reality properties of $\hat{\mathfrak{d}}_7$, entail, through the affine central product, those of the $\hat{\mathfrak{u}}(1)$ factor.

Concentrating first on $\hat{\mathfrak{d}}_7$, we follow for a start the matrix method outlined in the preceding Section 5.5.2. In this case, the automorphism (5.98) restricted to the transformation $A(z)\to U(z)A(z)U(z)^{-1}$ has to preserve the defining condition:

$$A(z)^\intercal G + GA(z) = 0\,,$$

where G is the metric kept invariant by $SO(14)$ matrices in the rep Γ.

We start by choosing, for $\mathfrak{d}_7\subset\hat{\mathfrak{d}}_7$, the 14-dimensional representation given in Appendix 5.14 with Dynkin index $\gamma_\Gamma = 1/\sqrt{42}$, whose generators will be denoted $\Gamma(E_\alpha)$ and $\Gamma(H_i)$. The affine extension of these operators is obtained as usual by the Sugawara construction, and the involutive automorphism $U(z)$ will be represented by a 14×14 matrix. This representation

Γ is in fact equivalent to its contragredient one $-\widetilde{\Gamma}$ in the sense that one can find a 14×14 non-singular matrix C such that:

$$\Gamma(X) = -C\,\widetilde{\Gamma}(X)\,C^{-1}\,, \;\; \forall X \in \mathfrak{d}_7\,.$$

One readily sees from eqn. (5.98) and subsequent arguments that, in this case, type 1b and 2b automorphisms coincide respectively with type 1a and 2a, which leaves us, for $\hat{\mathfrak{d}}_7$, with just two classes of involutive automorphisms, characterizing, roughly, real forms where the central charge c and the scaling operator d are both compact or both non-compact.

Since we do not expect the restriction ϕ of the Chevalley involution to $\mathfrak{g}_{\rm inv}$ to mix levels in δ in this case, this in principle rules out all involutive automorphisms of type 2a, which explicitly depend on z. In turn, it tells us that the central charge and the scaling operator are now both compact in $\mathfrak{g}_{\rm inv}$, contrary to, for instance, the $T^7 \times T^2/\mathbb{Z}_{n>2}$ case analyzed in Section 5.4, and will be written $ic' \doteq iH_{\delta'_{D_7}} = ic$ and id'. Neither is the involution ϕ likely to involve different compactness properties for even and odd levels in δ. These considerations lead us to select $u = +1$. The z-independent automorphism of type 1a with $u = +1$ which seems to be a good candidate, in the sense that it reduces to $\mathfrak{so}(8,6)$ when we restrict to the finite Lie algebra $\mathfrak{d}_7 \subset \hat{\mathfrak{d}}_7$, is

$$U(z) = \mathbb{1}_4 \oplus (-\mathbb{1}_6) \oplus \mathbb{1}_4\,, \tag{5.102}$$

so that eq.(5.98) reduces to $\varphi(A(z)) = U(z)A(z)U(z)^{-1}$.

Obviously, we have $\Phi(U(z)) = 0$ from expression (5.99) and the condition (5.101) for the automorphism to be involutive determines $\lambda = 0$. Now, since both central charge and scaling operator are compact in the new primed basis, we have $\mu = 1$. All these considerations put together lead to:

$$\varphi(ic') = ic'\,, \qquad \varphi(id') = id'\,, \tag{5.103}$$

from which we can determine the two triples:

$$\mathcal{D}_\varphi = \{\mathbb{1}_4 \oplus (-\mathbb{1}_6) \oplus \mathbb{1}_4; +1; 0\} \;\leftrightarrow\; \mathcal{D}_\phi = \{\mathbb{1}_7; 0; +1\}\;,$$

the structure of $\mathcal{D}_\phi$ clearly showing that we are dealing with a chief inner involutive automorphism. A natural choice for the primed basis of the Cartan subalgebra of $\mathfrak{d}_7$ is to pick it compact, so that its affine extension $\tilde{\mathfrak{h}} = \{iH'_1, \ldots, iH'_7, ic', id'\}$ is compact, as well.

We will check that the real form of $\hat{\mathfrak{d}}_7$ generated by the automorphism (5.102) and the one determined by the Cartan involution ϕ are conjugate, and thus lead to isomorphic real forms. Let us, for a start, redefine the basis of simple roots of $\hat{\mathfrak{d}}_7 \subset \mathfrak{g}_{\rm inv}$ appearing in Table 5.7:

$$\beta_1 \equiv \alpha_-\,, \quad \beta_2 \equiv \tilde{\alpha}\,, \quad \beta_3 \equiv \alpha_+\,, \quad \beta_4 \equiv \alpha_3\,, \quad \beta_5 \equiv \alpha_2\,, \quad \beta_6 \equiv \alpha_1\,, \quad \beta_7 \equiv \gamma \quad \beta_0 \equiv \alpha_0\,. \tag{5.104}$$

The lexicographical order we have chosen ensures that the convention for the structure constants is a natural extension of the $D = 6$ case. As for $\mathfrak{e}_9$, we introduce an abbreviated notation $E_{\beta_6+2\beta_5+2\beta_4+\beta_1+2\beta_2+\beta_3} \doteq E_{\underline{65^24^212^23}}$. Conventions and a method for computing relevant structure constants are given in Appendix 5.14.

We can now construct the compact Cartan subalgebra $\tilde{\mathfrak{h}}$ by selecting combinations of elements of $\mathfrak{d}_7 \subset \hat{\mathfrak{d}}_7 \subset \mathfrak{g}_{\rm inv}$ which commute and are themselves combinations of compact generators of $\mathfrak{e}_{9|10}$. The Cartan generators in this new basis are listed below, both in terms of $\mathfrak{d}_7 \subset (\mathfrak{g}_{\rm inv})^{\mathbb{C}}$

and $\mathfrak{e}_8 \subset \mathfrak{g}^U$ generators:

$$\begin{aligned}
iH'_{\underline{1}} &= i(E_{\underline{2}} + F_{\underline{2}}) \equiv E_{458} - F_{458}\,, \\
iH'_{\underline{2}} &= \frac{i}{2}\big(\eta_1(H_{\underline{3}} - H_{\underline{1}}) - E_{\underline{2}} - F_{\underline{2}} - E_{\underline{123}} - F_{\underline{123}}\big) \\
&\equiv \frac{1}{2}\big(\eta_1(E_5 - F_5 + E_7 - F_7) - E_{458} + F_{458} - E_{45^26^278} + F_{45^26^278}\big)\,, \\
iH'_{\underline{3}} &= i(E_{\underline{123}} + F_{\underline{123}}) \equiv E_{45^26^278} - F_{45^26^278}\,, \\
iH'_{\underline{4}} &= \frac{1}{2}\big(\eta_2(E_{\underline{54^212^23}} - F_{\underline{54^212^23}}) - i\eta_1(H_{\underline{3}} - H_{\underline{1}}) - E_{\underline{5}} + F_{\underline{5}}\big) \\
&\equiv -\frac{1}{2}\big(\eta_2(E_{234^25^36^278^2} - F_{234^25^36^278^2}) + \eta_1(E_5 - F_5 + E_7 - F_7) + E_2 - F_2\big)\,, \qquad (5.105) \\
iH'_{\underline{5}} &= E_{\underline{5}} - F_{\underline{5}} \equiv E_2 - F_2\,, \\
iH'_{\underline{6}} &= \frac{1}{2}\big(\eta_3(E_{\underline{76^25^24^212^23}} - F_{\underline{76^25^24^212^23}}) - E_{\underline{7}} + F_{\underline{7}} - E_{\underline{5}} + F_{\underline{5}} - \eta_2(E_{\underline{54^212^23}} - F_{\underline{54^212^23}})\big) \\
&\equiv -\frac{1}{2}\big(\eta_3(E_{\theta_{E_8}} - F_{\theta_{E_8}}) + E_\gamma - F_\gamma + E_2 - F_2 - \eta_2(E_{234^25^36^278^2} - F_{234^25^36^278^2})\big)\,, \\
iH'_{\underline{7}} &= E_{\underline{7}} - F_{\underline{7}} \equiv E_\gamma - F_\gamma\,,
\end{aligned}$$

where the factors $\eta_i = \pm 1$, $\forall i = 1, 2, 3$, determine equivalent solutions.

The Cartan generator attached to the affine root β'_0 is constructed from the above (5.105) in the usual way:

$$\begin{aligned}
iH'_{\underline{0}} &= iH_{\delta'_{D_7}} - \eta_3(E_{\underline{76^25^24^212^23}} - F_{\underline{76^25^24^212^23}}) \\
&= ic' + \eta_3(E_{\theta_{E_8}} - F_{\theta_{E_8}})\,,
\end{aligned}$$

which commutes with $\tilde{\mathfrak{h}}$ (5.105) and is indeed compact, as expected from expression (5.103).

We find the associated ladder operators by solving the set of equations $[H'_{\underline{j}}, E'_{\underline{i}}] = A_{\underline{ij}} E'_{\underline{i}}$, $[E'_{\underline{i}}, F'_{\underline{j}}] = \delta_{\underline{ij}} H'_{\underline{i}}$ and $[E'_{\underline{i}}, E'_{\underline{j}}] = \mathcal{N}_{\underline{i},\underline{j}} E'_{\underline{ij}}$ (the corresponding commutation relations for the lowering operators are then automatically satisfied). Here we write $E'_{\underline{i}} \equiv E_{\beta'_i}$ and $F'_{\underline{i}} \equiv E_{-\beta'_i}$ for short, for the set $\Pi' = \{\beta'_0, \ldots, \beta'_7\}$ of simple roots dual to the Cartan basis (5.105). Thus:

$$\begin{aligned}
E'_{\underline{1}}/F'_{\underline{1}} &= H_{\underline{2}} \mp (E_{\underline{2}} - F_{\underline{2}}) \equiv H_{458} \pm i(E_{458} + F_{458})\,, \\
E'_{\underline{2}} &= E_{\underline{1}} - F_{\underline{3}} - \eta_1(F_{\underline{12}} - F_{\underline{23}}) \equiv E_{\alpha_-} - F_{\alpha_+} - \eta_1(E_{\alpha_- + \tilde{\alpha}} - F_{\tilde{\alpha} + \alpha_+})\,, \\
E'_{\underline{3}}/F'_{\underline{3}} &= H_{\underline{123}} \mp (E_{\underline{123}} - F_{\underline{123}}) \equiv H_{45^26^278} \pm i(E_{45^26^278} + F_{45^26^278})\,, \qquad (5.106) \\
E'_{\underline{4}} &= E_{\underline{4(23\leftrightarrow 12)}} + iE_{\underline{54(23\leftrightarrow 12)}} - \eta_2(F_{\underline{4(12\leftrightarrow 23)}} + iF_{\underline{54(12\leftrightarrow 23)}})\,, \quad \text{for } \eta_1 = \pm 1\,, \\
E'_{\underline{5}}/F'_{\underline{5}} &= H_{\underline{5}} \pm i(E_{\underline{5}} + F_{\underline{5}}) \equiv H_2 \pm i(E_2 + F_2)\,,
\end{aligned}$$

together with $F'_i = (E'_i)^\dagger$, for $i = 2, 4$. In the expression in (5.106) for $E'_{\underline{4}}$, the $\leftrightarrow$ gives the two possible values of the last two indices depending on the choice of $\eta_1 = \pm 1$. It can be checked that $[E'_{\underline{2}}, E'_{\underline{4}}] = 0$.

The raising operator E'_6 is independent of η_1 and takes the form:

$$\begin{aligned}
E'_{\underline{6}} = &\ (E_{\underline{6}} + iE_{\underline{65}} - iE_{\underline{76}} + E_{\underline{765}}) + \eta_2\eta_3(F_{\underline{6}} + iF_{\underline{65}} - iF_{\underline{76}} + F_{\underline{765}}) \\
&- \eta_2(iE_{\underline{654^212^23}} + E_{\underline{65^24^212^23}} + E_{\underline{7654^212^23}} - iE_{\underline{765^24^212^23}}) \qquad (5.107) \\
&- \eta_3(iF_{\underline{654^212^23}} + F_{\underline{65^24^212^23}} + F_{\underline{7654^212^23}} - iF_{\underline{765^24^212^23}})\,,
\end{aligned}$$

the corresponding lowering operator is obtained from the above by hermitian conjugation. Moreover, it can be verified after some tedious algebra that indeed $[E'_{\underline{4}}, E'_{\underline{6}}] = 0$. Note that we have translated the primed generators into $\mathfrak{e}_8$ ones only when the expression is not too lengthy. Hereafter, such substitutions will be made only when necessary.

The two remaining pairs of ladder operator enhancing $\mathfrak{d}_6$ to $\hat{\mathfrak{d}}_7$ are:

$$\begin{aligned} E'_{\underline{7}}/F'_{\underline{7}} &= H_{\underline{7}} \pm i(E_{\underline{7}} + F_{\underline{7}}) \equiv H_\gamma \pm i(E_\gamma + F_\gamma)\,, \\ E'_{\underline{0}}/F'_{\underline{0}} &= -\eta_3\, t^{\pm 1} \otimes \left(H_{\underline{\theta}_{D_7}} \mp i(E_{\underline{\theta}_{D_7}} + F_{\underline{\theta}_{D_7}})\right) \\ &\equiv \eta_3\, t^{\pm 1} \otimes \left(H_{\theta_{E_8}} \mp i(E_{\theta_{E_8}} + F_{\theta_{E_8}})\right), \end{aligned} \tag{5.108}$$

where $\underline{\theta}_{D_7} \doteq \beta_7 + 2(\beta_6 + \beta_5 + \beta_4 + \beta_2) + \beta_1 + \beta_3$.

At this stage it is worth pointing out that the affine real form $\mathfrak{g}_{\rm inv}$ is realized as usual as a central extension of the loop algebra of the finite $(\mathfrak{d}_7)_0$ which may or may not descend to a real form of $\mathfrak{d}_7$ (in our case, it does since it will be shown that $(\mathfrak{d}_7)_0 = \mathfrak{d}_{7|5}$)

$$\mathfrak{g}_{\rm inv}/\mathcal{L}(\mathfrak{u}(1)) = \mathbb{R}[t,t^{-1}] \otimes (\mathfrak{d}_7)_0 \oplus \mathbb{R}ic' \oplus \mathbb{R}id'\,.$$

The difference is that we are now tensoring with an algebra of Laurent polynomials $\mathcal{L} = \mathbb{R}[t,t^{-1}]$ in the (indeterminate) variable t defined as follows

$$t = \frac{1}{2}\big((1-i) + (1+i)\vartheta_C\big)z \equiv \frac{1}{1+i}\big(1 + i\vartheta_C\big)z\,. \tag{5.109}$$

The second term of the equality (5.109) is clearly reminescent from the operator $\sqrt{\vartheta}$ (5.68). The inverse transformation yields:

$$z = \frac{(1+i)t + \sqrt{2i(t^2-2)}}{2}\,, \qquad z^{-1} \equiv \bar{z} = \frac{(1+i)t - \sqrt{2i(t^2-2)}}{2i}\,.$$

On can check that under the Chevalley involution: $\vartheta_C(t) = t$ and $\vartheta_C(t^{-1}) = t^{-1}$. Moreover, using

$$t^n = \frac{1}{(1-i)^n z^n} \sum_{k=0}^{n} \binom{n}{k} (-iz^2)^k$$

one can check that $\vartheta_C(t^n) = t^n\ \forall n \in \mathbb{Z}^*$, as required by the affine extension of the basis (5.105-5.108), which will become clearer when we give the complete realization of the real $\mathfrak{g}_{\rm inv}/\mathcal{L}(u(1))$ (5.114-5.115).

Finally, we may now give the expression of the compact scaling operator in the primed basis:

$$id' = \frac{(1+i\vartheta_C)z}{(1-i\vartheta_C)z}\, id\,,$$

which can be shown to be Hermitian.

Now that we know the structure of the generators $E'_{\underline{i}}$ and $F'_{\underline{i}}$, $i = 0,\ldots,7$, we are in the position of determining $\text{Fix}_{\tau_0}(\hat{\mathfrak{d}}_7)$ and, by acting with ϕ on the latter, are able to reconstruct the eigenvalues of the representation $\phi = \text{Ad}(e^{\overline{H}})$ of the Cartan involution on the basis (5.91)-(5.92), namely $\phi \cdot (\hat{\mathfrak{d}}_7)_{\beta'} = e^{\beta'(\overline{H})}\, (\hat{\mathfrak{d}}_7)_{\beta'}$, with $e^{\beta'_i(\overline{H})} = \pm 1$, $\forall \beta'_i \in \Pi'$. We will then show that the four automorphisms determined by this method corresponding to all possible values of η_i, $i = 1,2,3$, are conjugate to the action of the $U(z)$ given in (5.102) on the representation Γ of $\mathfrak{d}_7$.

Reexpressing, for instance, the second line of the list (5.106) in terms of the original basis (5.86) and (5.87), and taking $\text{Fix}_{\tau_0}(\hat{\mathfrak{d}}_7)$ yields the two following generators of $\mathfrak{g}_{\rm inv}$:

$$\begin{aligned} \frac{1}{2}(E'_{\underline{2}} - F'_{\underline{2}}) &= \frac{1}{2}\Big(E_{56} - F_{56} - E_{67} + F_{67} - \eta_1\big(E_{45^26^278} - F_{45^26^278} - E_{4568} + F_{4568}\big)\Big)\,, \\ \frac{i}{2}(E'_{\underline{2}} + F'_{\underline{2}}) &= \frac{1}{2}\Big(E_{567} - F_{567} + E_6 - F_6 - \eta_1\big(E_{45678} - F_{45678} + E_{45^2678} - F_{45^268}\big)\Big)\,. \end{aligned} \tag{5.110}$$

Both are obviously invariant under ϕ, since they are linear combinations of compact generators. According to Section 5.89, we have $e^{\beta'_2(\overline{H})} = +1$. The same reasoning applies to the pairs $E'_{\underline{4}}/F'_{\underline{4}}$ and $E'_{\underline{6}}/F'_{\underline{6}}$. In contrast to the $E'_{\underline{2}}/F'_{\underline{2}}$ case, these two couples of generators will be alternatively compact or non-compact depending on the sign of η_2 and η_3. In particular, since $E'_{\underline{4}}$ has basic structure $[E_\alpha, E_{\alpha\pm}] - \eta_2[F_{\alpha\mp}, F_\alpha]$, the choice $\eta_2 = +1$ will produce the two compact combinations $2^{-1}(E'_{\underline{4}} - F'_{\underline{4}})$ and $2^{-1}i(E'_{\underline{4}} + F'_{\underline{4}})$, while the opposite choice selects the two non-compact ones, by flipping the reciprocal sign between E and F. From expression (5.107), we see that the $E'_{\underline{6}}/F'_{\underline{6}}$ case is even more straightforward, compactness and non-compactness being selected by $\eta_2\eta_3 = \pm 1$ respectively. At this stage, our analysis thus leads to the four possibilities: $e^{\beta'_4(\overline{H})} = \pm 1$ and $e^{\beta'_6(\overline{H})} = \pm 1$.

Finally, the remaining ladder operators $E'_{\underline{i}}$ and $F'_{\underline{i}}$ for $i = 0, 1, 3, 5, 7$ combine in purely non-compact expressions, for instance

$$\frac{1}{2}(E'_{\underline{1}} + F'_{\underline{1}}) = H_{458}\,, \qquad \frac{i}{2}(E'_{\underline{1}} - F'_{\underline{1}}) = -(E_{458} + F_{458})\,, \tag{5.111}$$

The $E'_{\underline{0}}/F'_{\underline{0}}$ case is a bit more subtle because of the presence of the (t, t^{-1}) loop factors, and requires adding $E_{\theta_{D_7}+\delta}/F_{\theta_{D_7}+\delta} = \eta_3\, t^{\pm 1} \otimes \big(H_{\theta_{E_8}} \pm i(E_{\theta_{E_8}} + F_{\theta_{E_8}})\big)$ into the game. Computing Fix_{τ_0} for all of these four operators results in four non-compact combinations. This is in accordance with θ'_{D_7} which we now know to statisfy $e^{\theta'_{D_7}(\overline{H})} = -1$ for all four involutive automorphisms, and tells us in addition that: $e^{\beta'_0(\overline{H})} = -1$.

Collecting all previous results, the eigenvalues of the four involutive automorphisms $\phi_{(\eta_2,\eta_3)} = \mathrm{Ad}(e^{\overline{H}_{(\eta_2,\eta_3)}})$ are summarized in the table (5.112) below.

η_2	η_3	$e^{\beta'_2(\overline{H})}$	$e^{\beta'_4(\overline{H})}$	$e^{\beta'_6(\overline{H})}$	$e^{\beta'_{i\neq 2,4,6}(\overline{H})}$
$+1$	-1	$+1$	$+1$	$+1$	-1
$+1$	$+1$	$+1$	$+1$	-1	-1
-1	$+1$	$+1$	-1	$+1$	-1
-1	-1	$+1$	-1	-1	-1

(5.112)

The Cartan element $\overline{H}$ defining the involution ϕ can be read off table (5.112). The most general solution is given by $\overline{H} = \pi i \sum_{i=0}^{7} c_i H'_{\underline{i}} + \pi i c_{d'} d'$ with

$$\begin{gathered} c_1 = c_3 = \frac{\kappa+1}{2} + \mathbb{Z}\,, \quad c_4 = \kappa - 1\,, \quad c_5 = \kappa - \frac{\eta_2 - 1}{2} + \mathbb{Z}\,, \quad c_6 = \kappa + \eta_2 - 1\,, \\ c_7 = \frac{\kappa + \eta_2}{2} + \mathbb{Z}\,, \quad c_0 = \frac{\kappa + \eta_2 - \eta_3 + 1}{2} + \mathbb{Z}\,, \quad c_d = \eta_3 - 1 \end{gathered}$$

where $c_2 \doteq \kappa \in \mathbb{C}$ is a free parameter.

Restricted to $\mathfrak{d}_7$, the four inner automorphisms defined in the table (5.112) are all in the same class of equivalence, and thus determine the same real form, namely $\mathfrak{so}(8,6)$ as expected from $\mathfrak{g}_{\mathrm{inv}}$ in $D = 3$. In Appendix 5.14, we have computed the two sets of roots $\Delta_{(+1)}$ and $\Delta_{(-1)}$ (5.90) generating the Cartan decomposition (5.91)-(5.92) of the real form. It can be checked that, in these four cases, the signature $\sigma|_{\mathfrak{d}_7} = -(7 + 2(\dim\Delta_{(+1)} - \dim\Delta_{(+1)})) = 5$, in accordance with $\mathfrak{so}(8,6)$.

The involutive automorphism (5.102) in turn can be shown to split the root system of $\mathfrak{d}_7$ according to

$$\beta' = \varepsilon_i \pm \varepsilon_j \rightarrow \begin{cases} \beta' \in \Delta_{(+1)} & 1 \leqslant i < j \leqslant 4 \text{ and } 4 < i < j \leqslant 7 \\ \beta' \in \Delta_{(-1)} & 1 \leqslant i \leqslant 4 < j \leqslant 7 \end{cases}\,, \tag{5.113}$$

which can be verified by computing $U(z)\gamma(E_\alpha)U(z)^{-1}$ for the representation γ (5.161). We can check that we have again: $\sigma|_{\mathfrak{d}_7} = 5$, for the splitting (5.113), since the automorphism (5.102) corresponds, in our previous formalism to the involution $e^{\beta'_4(\overline{H})} = -1$ and $e^{\beta'_i(\overline{H})} = +1$, $\forall i \neq 4$.

Since they are conjugate at the $\mathfrak{d}_7$ level and all of them preserve the central charge and scaling element, the four automorphisms (5.112) lift to conjugate automorphisms of $\hat{\mathfrak{d}}_7$. All four of them are again clearly conjugate to $U(z)$ defined by properties (5.102), (5.103) and (5.113). These five Cartan preserving inner involutive automorphisms lead to equivalent Cartan decomposition $\mathfrak{k} \oplus^\perp \mathfrak{p}$ (5.66) given by generalizing the basis (5.110) and (5.111) found previously to the affine case:

$$\begin{aligned}
\mathfrak{k}\ :\ &\bullet\ iH'_{\underline{k}}\ (\forall k = 1,..,7);\ ic';\ id'; &(5.114)\\
&\bullet\ \frac{1}{2}(t^n - t^{-n}) \otimes H'_{\underline{k}} \quad \text{and} \quad \frac{i}{2}(t^n + t^{-n}) \otimes H'_{\underline{k}} \qquad (\forall k = 1,..,7;\ n \in \mathbb{N}^*)\,;\\
&\bullet\ \frac{1}{2}(t^n \otimes E_{\beta'} - t^{-n} \otimes F_{\beta'}) \quad \text{and} \quad \frac{i}{2}(t^n \otimes E_{\beta'} + t^{-n} \otimes F_{\beta'})\,,\ n \in \mathbb{Z}\\
&\qquad (\forall \beta' \in \Delta_{(+1)} \text{ defined by (5.113), (5.162), (5.163), (5.164) and (5.165)})\\
\mathfrak{p}\ :\ &\bullet\ \frac{i}{2}(t^n \otimes E_{\beta'} - t^{-n} \otimes F_{\beta'}) \quad \text{and} \quad \frac{1}{2}(t^n \otimes E_{\beta'} + t^{-n} \otimes F_{\beta'}),\ n \in \mathbb{Z} &(5.115)\\
&\qquad (\beta' \in \Delta_{(-1)} = \Delta_+(D_7)\backslash\Delta_{(+1)})\,.
\end{aligned}$$

These decompositions define isomorphic real forms, which we denote by $\widehat{\mathfrak{so}}(8,6)$, encoded in the affine Satake diagram of Table 5.7 (see for instance [239] for a classification of untwisted and twisted affine real forms).

We have checked before the behaviour of the ladder operators of the finite $\mathfrak{d}_7$ subalgebra of $\mathfrak{g}_{\text{inv}}$. The verification can be performed in a similar manner for the level $n \geqslant 1$ roots $\beta' + n\delta'_{D_7}$. Applying for example Fix_{τ_0} to the four generators $2^{-1}(t^{\pm n} \otimes E_{\underline{4}} - t^{\mp n} \otimes F_{\underline{4}})$ and $2^{-1}i(t^{\pm n} \otimes\ E_{\underline{4}} +\ t^{\mp n} \otimes F_{\underline{4}})$ for example, one obtains the following combinations

$$\begin{aligned}
&\tfrac{1}{2}(t^n + t^{-n}) \otimes \big(E_{345^2678} - \eta_2 F_{345^2678} - E_{34568} + \eta_2 F_{34568}\\
&\qquad\qquad -E_{2345678} + \eta_2 F_{2345678} - E_{2345^268} + \eta_2 F_{2345^268}\big)\,,\\
&\tfrac{1}{2}(t^n - t^{-n}) \otimes \big(E_{345678} - \eta_2 F_{345678} + E_{345^268} - \eta_2 F_{345^268}\\
&\qquad\qquad +E_{2345^2678} - \eta_2 F_{2345^2678} - E_{234568} + \eta_2 F_{234568}\big)\,,
\end{aligned}$$

which, since now $\vartheta_C(t^n \pm t^{-n}) = t^n \pm t^{-n}$, are all either non-compact if $\eta_2 = +1$, or compact otherwise, by virtue of table (5.112). This is in accordance with the Cartan decomposition (5.114-5.115). The compactness of the remaining $n \geqslant 1$ ladder operators can be checked in similar and straightforward fashion by referring once again to the table (5.112).

In contrast to the split case, a naive extension of the signature, which we denote by $\hat{\sigma}$, is not well defined since it yields in this case an infinite result:

$$\hat{\sigma} = 3 + 2 \times 5 \times \infty\,. \tag{5.116}$$

In the first finite contribution, we recognize the signature of $\mathfrak{so}(8,6)$ together with the central charge and scaling element, while the infinite towers of vertex operator contribute the second part. As mentioned before in the $D = 4$ case, the signature for the finite $\mathfrak{d}_7$ amounts to the difference between compact and non-compact Cartan generators, for the following alternative choice of basis for the Cartan algebra $\{H_\gamma; H_1; H_2; H_3; H_{\tilde{\alpha}}; H_{\alpha_+} + H_{\alpha_-}; i(H_{\alpha_+} - H_{\alpha_-})\}$. This carries over to the infinite contribution in expression (5.116), where it counts the number of overall compact towers, with an additional factor of 2 coming from the presence of both raising and lowering operators.

Care must be taken when defining the real affine central product $\mathfrak{g}_{\mathrm{inv}}$. The real Heisenberg algebra

$$\hat{\mathfrak{u}}(1)_{|1} = \sum_{n=0}^{\infty} \mathbb{R}(z^n + z^{-n}) \otimes i\widetilde{H}^{[4]} + \sum_{n=1}^{\infty} \mathbb{R}(z^n - z^{-n}) \otimes \widetilde{H}^{[4]} + \mathbb{R}c + \mathbb{R}d$$

is in this case isomorphic to the one appearing in the $T^2/\mathbb{Z}_{n>2}$ orbifold (5.78). Clearly both scaling operators and central charge are, in contrast to $\widehat{\mathfrak{so}}(8,6)$ non-compact. The identification required by the affine central product formally takes place before changing basis in $\widehat{\mathfrak{so}}(8,6)$ to the *primed* operators. The central charge and scaling operators acting on both subspaces of $\mathfrak{g}_{\mathrm{inv}} = \widehat{\mathfrak{so}}(8,6) \bowtie \hat{\mathfrak{u}}_{|1}(1)$ are then redefined as $d \oplus d \to id' \oplus d$ and $c \oplus c \to ic' \oplus c \equiv ic \oplus c$. Then we can write $\mathfrak{g}_{\mathrm{inv}} = \widehat{\mathfrak{so}}(8,6) \oplus \hat{\mathfrak{u}}_{|1}(1)/\{\mathfrak{z}, \bar{d}\}$, with $\mathfrak{z} = c - c'$ and $\bar{d} = d - \frac{\sqrt{2i(t^2-2)}}{(1+i)t} d'$. The signature $\hat{\sigma}$ of $\mathfrak{g}_{\mathrm{inv}}$ is undefined.

5.5.4 The non-split real Borcherds symmetry in $D = 1$

The analysis of the $D = 1$ invariant subalgebra closely resembles the $T^8 \times T^2/\mathbb{Z}_{n>2}$ case. The central product of Section 5.5.3 is extended to a direct sum of a $\mathfrak{u}(1)$ factor with the quotient of a Borcherds algebra by an equivalence relation similar to the one stated in Conjecture 5.4.1. The Borcherds algebra ${}^4\mathcal{B}_{10}$ found here is defined by a 10×10 degenerate Cartan matrix of rank $r = 9$. Its unique isotropic imaginary simple root (of multiplicity one) ξ_I is now attached to the raising operator $E_{\xi_I} = (1/2)(E_{\delta+\alpha_5} - E_{\delta-\alpha_5} - E_{\delta+\alpha_7} + E_{\delta-\alpha_7})$, so that the equivalence relation defining $(\mathfrak{g}_{\mathrm{inv}})^{\mathbb{C}}$ from ${}^4\mathcal{B}_{10} \oplus \mathfrak{u}(1)$ identifies the Cartan generator $H_I \doteq H_{\xi_I}$ with H_δ and removes the derivation operator $d_I \doteq d_{\xi_I}$.

Moreover, the splitting of multiplicities should occur as in the $T^8 \times T^2/\mathbb{Z}_{n>2}$ example, since $\dim(\hat{\mathfrak{d}}_7)_{\delta_{D_7}} = \dim(\hat{\mathfrak{e}}_7)_\delta$. It might a priori seem otherwise from the observation that both $\widetilde{K}_{(7)[2\cdots679\,10]} - \widetilde{K}_{(8)[2\cdots689\,10]}$ and $\widetilde{K}_{(9)[2\cdots89]} - \widetilde{K}_{(10)[2\cdots8\,10]}$ are separately invariant. However, the combination:

$$\widetilde{K}_{(7)[2\cdots679\,10]} - \widetilde{K}_{(8)[2\cdots689\,10]} + \widetilde{K}_{(9)[2\cdots89]} - \widetilde{K}_{(10)[2\cdots8\,10]} = \widetilde{K}_{[2\cdots10]} \otimes i(H_{\alpha_+} - H_{\alpha_-}) \in \mathfrak{d}_7^{\wedge\wedge}$$

contributes to the multiplicity of δ, while we may rewrite

$$\frac{1}{2}(\widetilde{K}_{(7)[2\cdots679\,10]} - \widetilde{K}_{(8)[2\cdots689\,10]} - \widetilde{K}_{(9)[2\cdots89]} + \widetilde{K}_{(10)[2\cdots8\,10]}) = E_{\xi_I} \in {}^4\mathcal{B}_{10}\,,$$

which is the unique raising operator spanning $({}^4\mathcal{B}_{10})_{\xi_I}$. Thus, though root multiplicities remain unchanged, we have to group invariant objects in representations of $\mathfrak{sl}(6,\mathbb{R})$, which are naturally shorter than in the $T^2/\mathbb{Z}_n$ case. We will not detail all such representations here, since they can in principle be reconstructed by further decomposition and/or regrouping of the results of Table 5.6.

The real invariant subalgebra can again be formally realized as

$$\mathfrak{g}_{\mathrm{inv}} = \mathfrak{u}(1) \oplus {}^4\mathcal{B}_{10(Ib)}/\{\mathfrak{z}, d_I\}$$

where $\mathfrak{z} = H_\delta - H_I$. We denote by $\mathcal{B}_{10(Ib)}$ the real Borcherds algebra obtained from $\mathrm{Fix}_{\tau_0}({}^4\mathcal{B}_{10})$ and represented in Table 5.7, choosing Ia to refer to the split form. The disappearance of the diagram automorphism which, in the $D = 2$ case, exchanged the affine root α_0 with γ leads to non-compact H_{-1} and H_δ, in contrast to what happened with the $\widehat{\mathfrak{so}}(8,6) \subset \mathfrak{g}_{\mathrm{inv}}$ factor in $D = 2$. This is reflected by ξ_I being a white node with no arrow attached to it. Note that a black isotropic imaginary simple root connected to a white real simple root would, in any case, be forbidden, since such a diagram is not given by an involution on the root system. Moreover, an imaginary simple root can only be identified by an arrow to another imaginary simple root (and similarly for real simple roots).

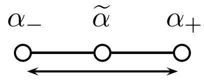
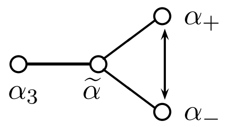
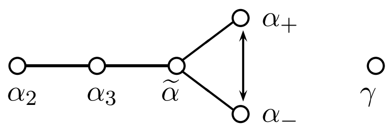
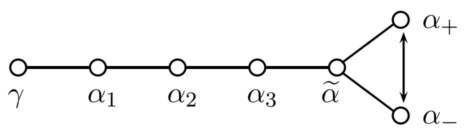
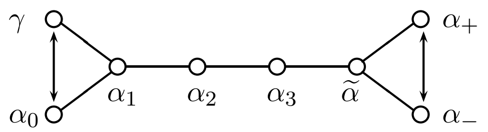
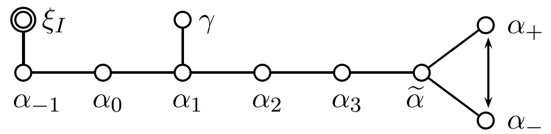
D	(Π_0, ϕ)	$\mathfrak{g}_{\text{inv}}$	$\sigma(\mathfrak{g}_{\text{inv}})$
6	 $\times \quad H^{[4]} \times \quad i\widetilde{H}^{[4]}$	$\mathfrak{so}(4,2) \oplus \mathfrak{so}(1,1) \oplus \mathfrak{u}(1)$	1
5	 $\times \quad H^{[4]} \times \quad i\widetilde{H}^{[4]}$	$\mathfrak{so}(5,3) \oplus \mathfrak{so}(1,1) \oplus \mathfrak{u}(1)$	2
4	 $\times \quad i\widetilde{H}^{[4]}$	$\mathfrak{so}(6,4) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$	3
3	 $\times \quad i\widetilde{H}^{[4]}$	$\mathfrak{so}(8,6) \oplus \mathfrak{u}(1)$	4
2	 $\times \quad \left\{ i\widetilde{H}^{[4]}_n \right\}_{n\in\mathbb{Z}}$	$\widehat{\mathfrak{so}}(8,6) \oplus \mathcal{L}(\mathfrak{u}(1))_{\|-1}$	-
1	 $\times \quad i\widetilde{H}^{[4]}$	${}^4\mathcal{B}_{10(Ib)} \oplus \mathfrak{u}(1)$	-

Table 5.7: The real subalgebras $\mathfrak{g}_{\text{inv}}$ for $T^{7-D} \times T^4/\mathbb{Z}_{n>2}$ compactifications

5.6 The orbifolds $T^6/\mathbb{Z}_{n>2}$

The orbifold compactification $T^{5-D} \times T^6/\mathbb{Z}_n$ for $n \geqslant 3$ can be carried out similarly. We fix the orbifold action in the directions $\{x^5, x^6, x^7, x^8, x^9, x^{10}\}$, so that it will only be felt by the set of simple roots $\{\alpha_2, .., \alpha_8\}$ defining the $\mathfrak{e}_{7|7}$ subalgebra of $\mathfrak{g}^U = \text{Split}(\mathfrak{e}_{11-D})$ from $D = 4$ downward. Thus, we may start again by constructing the appropriate charged combinations of generators for $\mathfrak{g}^U = \mathfrak{e}_{7|7}$, and then extend the result for $D \leq 3$ in a straightforward fashion. Since $\mathfrak{e}_7$ has 63 positive roots, we will restrict ourselves to the invariant subalgebra, and list only a few noteworthy charged combinations of generators. In this case, a new feature appears: the invariant algebra is not independent of n, $\forall n \geqslant 3$, as before. Instead, the particular cases $T^6/\mathbb{Z}_3$ and $T^6/\mathbb{Z}_4$ are non-generic and yield invariant subalgebras larger than the $n \geqslant 5$ one.

More precisely, we start by fixing the orbifold action to be

$$(z_i,\bar{z}_i) \to (e^{2\pi i/n} z_i, e^{-2\pi i/n}\bar{z}_i) \quad \text{for } i=1,2\,, \qquad (z_3,\bar{z}_3) \to (e^{-4\pi i/n} z_3, e^{4\pi i/n}\bar{z}_3)\,, \tag{5.117}$$

in other words, we choose $Q_1 = +1$, $Q_2 = +1$ and $Q_3 = -2$ to ensure $\sum_i Q_i = 0$. Note that for values of n that are larger than four, there are other possible choices, like $Q_1 = 1$, $Q_2 = 2$ and $Q_3 = -3$ for $T^6/\mathbb{Z}_6$ or $Q_1 = 1$, $Q_2 = 3$ and $Q_3 = -4$ for $T^6/\mathbb{Z}_8$ and so on. Indeed, the richness of T^6/Z_n orbifolds compared to T^4/Z_n ones stems from these many possibilities. Though interesting in their own right, we only treat the first of the above cases in detail, though any choice of charges can in principle be worked out with our general method. One has to keep in mind, however, that any other choice than the one we made in expression (5.117) may lead to different non-generic values of n.

5.6.1 The generic $n \geqslant 5$ case

Concentrating on the invariant subalgebra $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$ for $n \geqslant 5$, it turns out that the adjoint action of the rotation operator defining the orbifold charges $\mathcal{U}_6^{\mathbb{Z}_n} = \prod_{k=1}^{3} e^{-\frac{2\pi i}{n} Q_k \mathcal{K}_{z_k\bar{z}_k}}$ leaves invariant the following diagonal components of the metric:

$$\begin{aligned}
K_{44} &= \frac{1}{2}(3H_2 + 4H_3 + 5H_4 + 6H_5 + 4H_6 + 2H_7 + 3H_8)\,,\\
K_{z_1\bar{z}_1} &= \frac{1}{2}(H_2 + 3H_3 + 5H_4 + 6H_5 + 4H_6 + 2H_7 + 3H_8)\,,\\
K_{z_2\bar{z}_2} &= \frac{1}{2}(H_2 + 2H_3 + 3H_4 + 5H_5 + 4H_6 + 2H_7 + 3H_8)\,,\\
K_{z_3\bar{z}_3} &= \frac{1}{2}(H_2 + 2H_3 + 3H_4 + 4H_5 + 2H_6 + H_7 + 3H_8)\,,
\end{aligned} \tag{5.118}$$

as well as various fields corresponding to non-zero roots:

$$\begin{aligned}
K_{z_1\bar{z}_2}/K_{\bar{z}_1 z_2} &= \frac{1}{4}\left(E_{34} + F_{34} + E_{45} + F_{45} \pm i(E_{345} + F_{345} - (E_4 + F_4))\right)\,,\\
Z_{4z_1\bar{z}_1} &= \frac{i}{2}(E_{23^24^35^36^278} + F_{23^24^35^36^278})\,,\\
Z_{4z_2\bar{z}_2} &= \frac{i}{2}(E_{2345^26^278} + F_{2345^26^278})\,,\\
Z_{4z_3\bar{z}_3} &= \frac{i}{2}(E_{23458} + F_{23458})\,,\\
\widetilde{Z}_{z_1\bar{z}_1z_2\bar{z}_2z_3\bar{z}_3} &= -\frac{i}{2}(E_{34^25^36^278^2} + F_{34^25^36^278^2})\,,\\
Z_{4z_1\bar{z}_2}/Z_{4\bar{z}_1z_2} &= \frac{1}{4}\Big(E_{23^24^25^36^278} + F_{23^24^25^36^278} + E_{234^25^26^278} + F_{234^25^26^278}\\
&\quad \pm i\big(E_{23^24^25^26^278} + F_{23^24^25^26^278} - (E_{234^25^36^278} + F_{234^25^36^278})\big)\Big)\,,\\
Z_{z_1z_2z_3}/Z_{\bar{z}_1\bar{z}_2\bar{z}_3} &= \frac{1}{4\sqrt{2}}\Big(E_{345^2678} + F_{345^2678} - E_{34568} - F_{34568} - E_{45^268}\\
&\quad -F_{45^268} - E_{45678} - F_{45678} \pm i\big(-E_{345^268} - F_{345^268} - E_{345678}\\
&\quad -F_{345678} - E_{45^2678} - F_{45^2678} + E_{4568} + F_{4568}\big)\Big)\,,
\end{aligned} \tag{5.119}$$

together with their compact counterparts, supplemented by the generators of the orbifold action $\mathcal{K}_{z_1\bar{z}_1} = -i(E_3 - F_3)$, $\mathcal{K}_{z_2\bar{z}_2} = -i(E_5 - F_5)$ and $\mathcal{K}_{z_3\bar{z}_3} = -i(E_7 - F_7)$, which bring the Cartan subalgebra to rank 7, ensuring rank conservation again.

Note that the 4 invariant combinations in the list (5.119) are in fact spanned by the elementary set of linearly independent Cartan elements satisfying $[H, E_\alpha] = 0$ for $\alpha \in \{\alpha_3, \alpha_5, \alpha_7\}$, namely: $\{2H_2 + H_3; H_3 + 2H_4 + H_5; H_5 + 2H_8; H_5 + 2H_6 + H_7\}$.

Furthermore, let us recall that the objects listed in expression (5.118) form the minimal set of invariant ladder operators for $n \geqslant 5$. In the non-generic cases $n = 3, 4$, this set is enhanced, and so is the size of $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$. We will treat these cases later on, and, for the moment being, focus on the generic invariant subalgebra for $T^6/\mathbb{Z}_{n\geqslant 5}$ only.

As before, we extract the generators corresponding to simple roots of the invariant subalgebra (the negative-root generators are omitted, since they can be obtained in a straighforward manner as $F_\alpha = (E_\alpha)^\dagger$):

$$\begin{aligned}
E_{\tilde{\beta}} &= -iE_{23452^{6}278} = (Z_{4\bar{z}_2 z_2})^+ \,, \\
E_{\beta_\pm} &= \frac{1}{2}\left(E_{34} + E_{45} \pm i(E_{345} - E_4)\right) = (K_{z_1\bar{z}_2}/K_{\bar{z}_1 z_2})^+ \,, \\
E_{\gamma_\pm} &= \frac{1}{2\sqrt{2}}\Big(E_{345^2678} - E_{34568} - E_{45^268} - E_{45678} \\
&\qquad \mp i\big(E_{345^268} + E_{345678} + E_{45^2678} - E_{4568}\big)\Big) = (Z_{z_1 z_2 z_3}/Z_{\bar{z}_1\bar{z}_2\bar{z}_3})^+ \,, \\
E_\epsilon &= -iE_{23458} = (Z_{4\bar{z}_3 z_3})^+ \,,
\end{aligned} \tag{5.120}$$

These generators define a complex invariant subalgebra $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$ of type $\mathfrak{d}_3 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_1 \oplus \mathbb{C}$ with the following root labeling

$\times \quad \{E_3 - F_3 + E_5 - F_5 - 2(E_7 - F_7)\}$

The detailed structure of the $\mathfrak{d}_3 \subset (\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$ is encoded in the following commutation relations:

$$\begin{aligned}
E_{\tilde{\beta}+\beta_\pm} &\doteq \pm[E_{\tilde{\beta}}, E_{\beta_\pm}] = \frac{1}{2}\left(E_{23^24^25^36^278} + E_{234^25^26^278} \pm i(E_{23^24^25^26^278} - E_{234^25^36^278})\right) \\
&= (Z_{4z_1\bar{z}_2}/Z_{4\bar{z}_1 z_2})^+ \,, \\
E_{\beta_-+\tilde{\beta}+\beta_+} &\doteq [E_{\beta_-}, E_{\tilde{\beta}+\beta_+}] = iE_{23^24^35^36^278} = (Z_{4z_1\bar{z}_1})^+ \,, \\
H_{\tilde{\beta}} &\doteq [E_{\tilde{\beta}}, F_{\tilde{\beta}}] = (H_2 + H_3 + H_4 + 2H_5 + 2H_6 + H_7 + H_8) \,, \\
H_{\beta_\pm} &\doteq [E_{\beta_\pm}, F_{\beta_\pm}] = \frac{1}{2}\left(H_3 + 2H_4 + H_5 \pm i(-E_3 + F_3 + E_5 - F_5)\right) \,.
\end{aligned} \tag{5.121}$$

The $\mathfrak{a}_2 \subset (\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$ factor is characterized as follows:

$$\begin{aligned}
E_{\gamma_-+\gamma_+} &\doteq [E_{\gamma_-}, E_{\gamma_+}] = iE_{34^25^36^278^2} = (\widetilde{Z}_{z_1\bar{z}_1 z_2\bar{z}_2 z_3\bar{z}_3})^+ \,, \\
F_{\gamma_-+\gamma_+} &\doteq [F_{\gamma_+}, F_{\gamma_-}] = -iF_{34^25^36^278^2} = (\widetilde{Z}_{z_1\bar{z}_1 z_2\bar{z}_2 z_3\bar{z}_3})^- \,, \\
H_{\gamma_\pm} &\doteq [E_{\gamma_\pm}, F_{\gamma_\pm}] = \frac{1}{2}\left(H_3 + 2H_4 + 3H_5 + 2H_6 + H_7 + 2H_8\right) \\
&\qquad \pm\frac{i}{2}(E_3 - F_3 + E_5 - F_5 - E_7 + F_7) \,.
\end{aligned} \tag{5.122}$$

Finally, the Cartan generator of the remaining $\mathfrak{a}_1 \subset (\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$ is given by $[E_\epsilon, F_\epsilon] = H_2 + H_3 + H_4 + H_5 + H_8$. One can verify that all three simple subalgebras of $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$ indeed commute and that the compact abelian factor $i\widetilde{H}^{[6]} = (E_3 - F_3 + E_5 - F_5 - 2(E_7 - F_7))$ is the centre of $\mathfrak{g}_{\text{inv}}$. The structure of $i\widetilde{H}^{[6]}$ can be retrieved from rewriting the orbifold automorphism as $\mathcal{U}_6^{\mathbb{Z}_n} = \exp\big((2\pi/n)\,(E_3 - F_3 + E_5 - F_5 - 2(E_7 - F_7))\big)$, and noting that it preserves $\mathfrak{g}_{\text{inv}}$.

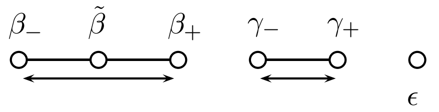
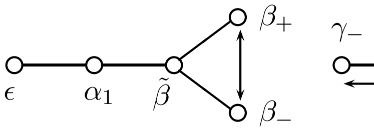
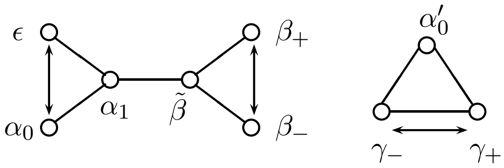
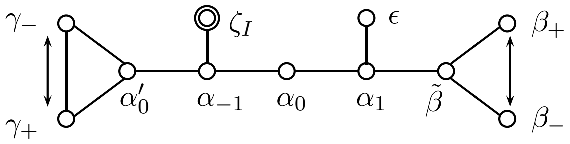
D	(Π_0, ϕ)	$\mathfrak{g}_{\text{inv}}$	$\sigma(\mathfrak{g}_{\text{inv}})$
4	 $\times \quad i\widetilde{H}^{[6]}$	$\mathfrak{su}(2,2) \oplus \mathfrak{su}(2,1)$ $\oplus\, \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$	1
3	 $\times \quad i\widetilde{H}^{[6]}$	$\mathfrak{so}(6,4) \oplus \mathfrak{su}(2,1)$ $\oplus\, \mathfrak{u}(1)$	2
2	 $\times \quad \left\{ i\widetilde{H}_n^{[6]} \right\}_{n\in\mathbb{Z}}$	$\widehat{\mathfrak{so}}(6,4) \oplus \widehat{\mathfrak{su}}(2,1)$ $\oplus\, \mathcal{L}(\mathfrak{u}(1))_{\vert -1}$	-
1	 $\times \quad i\widetilde{H}^{[6]}$	${}^6\mathcal{B}_{11(II)} \oplus \mathfrak{u}(1)$	-

Table 5.8: The real subalgebras $\mathfrak{g}_{\text{inv}}$ for $T^{5-D} \times T^6/\mathbb{Z}_{n\geqslant 5}$ compactifications

We determine the real form $\mathfrak{g}_{\text{inv}}$ by a manner similar to the $T^4/\mathbb{Z}_{\mathfrak{n}\geqslant 3}$ case. Applying procedure (5.70), we find that, in a given basis, the Cartan combinations $i(H_{\beta_+} - H_{\beta_-})$ and $i(H_{\gamma_+} - H_{\gamma_-})$ are compact, resulting, for both $\mathfrak{a}_2$ and the $\mathfrak{d}_3$ subalgebras, in maximal tori $(S^1) \oplus \mathbb{R}^{\oplus^{r-1}}$, for $r = 2, 3$, respectively. Taking into account the remaining $\mathfrak{u}(1)$ factor, it is easy to see that $\mathfrak{g}_{\text{inv}} = \mathfrak{su}(2,2) \oplus \mathfrak{su}(2,1) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$, with overall signature $\sigma(\mathfrak{g}_{\text{inv}}) = 1$.

This and further compactifications of the theory are listed in Table 5.8. In $D = 3$ the roots ϵ and $\tilde{\beta}$ listed in expression (5.120) connect through α_1, producing the invariant real form $\mathfrak{g}_{\text{inv}} = \mathfrak{so}(6,4) \oplus \mathfrak{su}(2,1) \oplus \mathfrak{u}(1)$.

In $D = 2$, the invariant subalgebra is now a triple affine central product $\hat{\mathfrak{d}}_5 \bowtie \hat{\mathfrak{a}}_2 \bowtie \hat{\mathfrak{u}}(1) \equiv \hat{\mathfrak{d}}_5 \bowtie (\hat{\mathfrak{a}}_2 \bowtie \hat{\mathfrak{u}}(1))$, associatively. For convenience, we have once again depicted in Table 5.8 the direct sum before identification of centres and scaling operators. When carrying out the identification, the affine root α'_0 has thus to be understood as a non-simple root in $\Delta_+(E_9)$, enforcing: $\alpha'_0 = \delta - (\gamma_+ + \gamma_-)$. Moreover, it can be checked that $\delta_{D_5} \doteq \alpha_0 + \epsilon + 2(\alpha_1 + \tilde{\beta}) + \beta_+ + \beta_- = \delta$, resulting in $H_{\delta_{D_5}} = H_{\delta_{A_2}} = c_{\hat{\mathfrak{u}}(1)}$, for $\delta_{A_2} = \alpha'_0 + \gamma_+ + \gamma_-$. This identification carries over to the three corresponding scaling operators.

The reality properties of $\mathfrak{g}_{\text{inv}}$ can be inferred from the finite case, by extending the analysis of the $T^5 \times T^4/\mathbb{Z}_{n\geqslant n}$ orbifold in Section 5.5.3 separately to the $\hat{\mathfrak{d}}_5$ and $\hat{\mathfrak{a}}_2$ factors. Since both subalgebras are "next to split", the $\hat{\mathfrak{d}}_5$ case is directly retrievable from the construction exposed in Section 5.5.3 by reducing the rank by two. The real $\widehat{\mathfrak{su}}(2,1)$ factor is also characterized by an automorphism $U(z)$ of type 1a, with $u = 1$, which can be found, along with further specifications, in [72]. The rest of the analysis is similar to the discussion for the $T^5 \times T^4/\mathbb{Z}_{n\geqslant n}$ case. The signatures $\hat{\sigma}$ of both non-abelian factors are infinite again, and can be decomposed as in expression (5.116).

In $D = 1$, the Borcherds algebra ${}^6\mathcal{B}_{11}$ resulting from reconnecting the three affine KMAs appearing in $D = 2$ through the extended root α_{-1} is defined this time by a Cartan matrix of corank 2, with simple imaginary root ζ_I attached to the raising operator $E_{\zeta_I} = (1/2)(E_{\delta+\alpha_3} - E_{\delta-\alpha_3} + E_{\delta+\alpha_5} - E_{\delta-\alpha_5} - 2E_{\delta+\alpha_7} + 2E_{\delta-\alpha_5})$. Since the triple extension is not successive, the ensuing algebra is more involved than an EALA. Writing for short $\delta_2 \doteq \delta_{A_2} = \alpha'_0 + \gamma_+ + \gamma_-$, it possesses two centres, namely $\{\mathfrak{z}_1 = H_\delta - H_I, \mathfrak{z}_2 = H_{\delta_2} - H_\delta\}$ and two scaling elements $\{d_I, d_2\}$ counting the levels in ζ_I and δ_2. The signature $\hat{\sigma}$ of $\mathfrak{g}_{\rm inv}$ is again undefined.

Denoting by ${}^6\mathcal{B}_{11(II)}$ the real Borcherds algebra represented in Table 5.8, the II referring to the two arrows connecting respectively $\gamma_\pm$ and $\beta_\pm$ in the Satake diagram, the real form $\mathfrak{g}_{\rm inv}$ is given by

$$\mathfrak{g}_{\rm inv} = \mathfrak{u}(1) \oplus {}^6\mathcal{B}_{11(II)}/\{\mathfrak{z}_1; \mathfrak{z}_2; d_I; d_2\}\,.$$

By construction, $-H_{-1}$ will replace d_I and d_2 after the quotient is performed.

5.6.2 The non-generic $n = 4$ case

As we mentioned at the beginning of this section, there is a large number of consistent choices for the charges of the $T^6/\mathbb{Z}_n$ orbifold. Moreover, non-generic invariant subalgebras appear for particular periodicities n. For our choice of orbifold charges, the non-generic cases appear in $n = 3, 4$, and are singled out from the generic one by the absence of a $\mathfrak{u}(1)$ factor in the invariant subalgebra. In $D = 1$, this entails the appearance of simple invariant Kac-Moody subalgebras in place of the simple Borcherds type ones encountered up to now. These KMA will be denoted by $\mathcal{KM}$.

The novelty peculiar to the $T^6/\mathbb{Z}_4$ orbifold lies in the invariance of the root α_7, which is untouched by the mirror symmetry $(z_3, \bar{z}_3) \to (-z_3, -\bar{z}_3)$, so that the generators E_7, F_7 and H_7 are now conserved separately. Furthermore, several new invariant generators appear related to $Z_{z_1 z_2 \bar{z}_3}$ and $Z_{\bar{z}_1 \bar{z}_2 z_3}$:

$$\begin{aligned} Z_{z_1 z_2 \bar{z}_3}/Z_{\bar{z}_1 \bar{z}_2 z_3} = &\\ &\frac{1}{4\sqrt{2}}\big(E_{345^2678} + F_{345^2678} + E_{34568} + F_{34568} + E_{45^268} + F_{45^268} - E_{45678} - F_{45678}\\ &\pm i(E_{345^268} + F_{345^268} - E_{345678} - F_{345678} - E_{45^2678} - F_{45^2678} - E_{4568} - F_{4568})\big)\,, \end{aligned}$$

together with the corresponding compact generators.

The invariant subalgebra is now more readily derived by splitting the Z generators into combinations containing or not an overall AdE_7 factor (in other words, we "decomplexify" z_3 into x^9 and x^{10}):

$$\begin{aligned} E_{\lambda_\pm} &= -\frac{i}{2}\big(E_{34568} + E_{45^268} \pm i(E_{345^268} - E_{4568})\big) = (Z_{z_1 z_2 10}/-Z_{\bar{z}_1 \bar{z}_2 10})^+\,,\\ E_{\alpha_7+\lambda_\pm} &= \frac{1}{2}\big(E_{345^2678} - E_{45678} \pm i(E_{345678} + E_{45^2678})\big) = (Z_{\bar{z}_1 \bar{z}_2 9}/Z_{z_1 z_2 9})^+ \end{aligned} \tag{5.123}$$

which verify the following algebra:

$$\begin{aligned} E_{\alpha_7+\lambda_\pm} &\doteq \pm[E_{\alpha_7}, E_{\lambda_\pm}]\,,\\ E_{\lambda_-+\alpha_7+\lambda_+} &\doteq [E_{\lambda_-}, E_{\alpha_7+\lambda_+}] = -E_{34^25^36^278^2} = -i(\widetilde{K}_{z_1\bar{z}_1 z_2\bar{z}_2 z_3\bar{z}_3})^+,\\ H_{\lambda_\pm} &\doteq [E_{\lambda_\pm}, F_{\lambda_\pm}] = \frac{1}{2}\left(H_3 + 2H_4 + 3H_5 + 2H_6 + 2H_8 \pm i(E_3 - F_3 + E_5 - F_5)\right)\,, \end{aligned} \tag{5.124}$$

so that the former $\mathfrak{a}_2$ factor for n generic is now enhanced to $\mathfrak{a}_3$. One compact combination $i(H_{\lambda_+} - H_{\lambda_-})$ results from the action of Fix$_{\tau_0}$ on the algebra formed by the generators in (5.124),

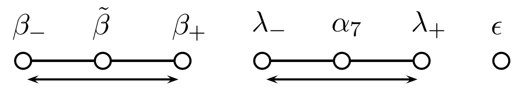
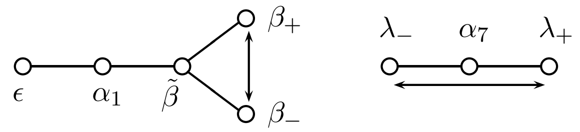
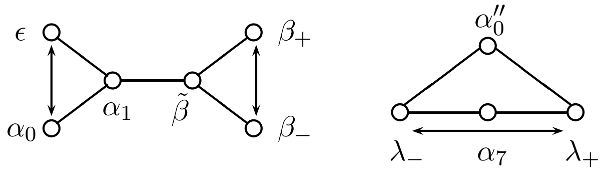
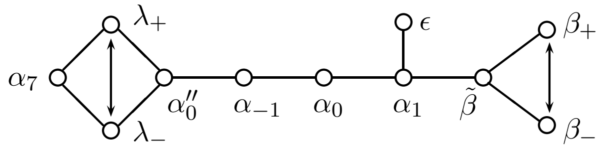
D	(Π_0, ϕ)	$\mathfrak{g}_{\text{inv}}$	$\sigma(\mathfrak{g}_{\text{inv}})$
4		$\mathfrak{su}(2,2)^{\oplus^2} \oplus \mathfrak{sl}(2,\mathbb{R})$	3
3		$\mathfrak{so}(6,4) \oplus \mathfrak{su}(2,2)$	4
2		$\widehat{\mathfrak{so}}(6,4) \oplus \widehat{\mathfrak{su}}(2,2)$	-
1		${}^6\mathcal{KM}_{11(II)}$	-

Table 5.9: The real subalgebras $\mathfrak{g}_{\text{inv}}$ for $T^{5-D} \times T^6/\mathbb{Z}_4$ compactifications

which determines the corresponding real form to be $\mathfrak{su}(2,2)$. The chain of invariant subalgebras resulting from further compactifications follows as summarized in Table 5.9.

In $D = 2$, we have the identification $\alpha_0'' = \delta - (\lambda_+ + \alpha_7 + \lambda_-)$ leading to the now customary affine central product $\widehat{\mathfrak{so}}(6,4) \bowtie \widehat{\mathfrak{su}}(2,2)$, represented for commodity as a direct sum in Table 5.9. The corresponding Satake diagram defines the real form $\mathfrak{g}_{\text{inv}}$. Its signature $\hat{\sigma}$ is infinite with the correspondence $\hat{\sigma}(\mathfrak{g}_{\text{inv}}|_{D=2}) = 2 + 2 \times 4 \times \infty = \sigma(\mathfrak{g}_{\text{inv}}|_{D=3}) - 2 + 2 \times \sigma(\mathfrak{g}_{\text{inv}}|_{D=3}) \times \infty$.

In $D = 1$, $\mathfrak{g}_{\text{inv}}$ is defined by the quotient of a simple KMA: ${}^6\mathcal{KM}_{11}$, by its centre $\mathfrak{z} = H_\delta - H_{\delta_3}$, where $\delta_3 = \alpha_0'' + \lambda_+ + \alpha_7 + \lambda_-$, and by the derivation d_3. As for affine KMAs, ${}^6\mathcal{KM}_{11}$ is characterized by a degenerate Cartan matrix with rank $r = 2 \times 11 - \dim \mathfrak{h} = 10$. However, the principal minors of its Cartan matrix are not all strictly positive, so that ${}^6\mathcal{KM}_{11}$ does not result from the standard affine extension of any finite Lie algebra. The real form $\mathfrak{g}_{\text{inv}}$ is determined from the Satake diagram in Table 5.9 and the relation:

$$\mathfrak{g}_{\text{inv}} = {}^6\mathcal{KM}_{11(II)}/\{\mathfrak{z}; d_3\} \,. \tag{5.125}$$

Our convention denotes by *II* the class of real forms of ${}^6\mathcal{KM}_{11}$ for which the Cartan involution exhibits both possible diagram symmetries, exchanging $\phi(\lambda_\pm) = \mp\lambda_\pm$ and $\phi(\beta_\pm) = \mp\beta_\pm$.

5.6.3 The standard $n = 3$ case

Starting in $D = 4$, the $\mathbb{Z}_3$-invariant subalgebra builds up the semi-simple $(\mathfrak{g}_{\text{inv}})^{\mathbb{C}} = \mathfrak{a}_5 \oplus \mathfrak{a}_2$. The $\mathfrak{a}_5$ part follows from enhancing the $\mathfrak{a}_3 \oplus \mathfrak{a}_1$ semi-simple factor of the generic invariant

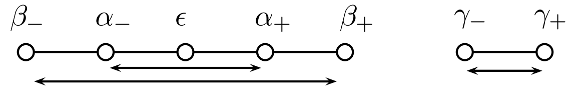
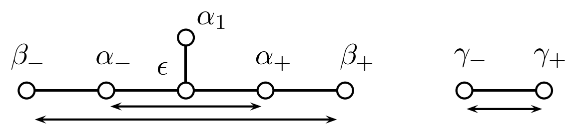
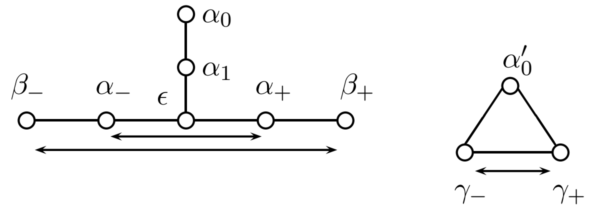
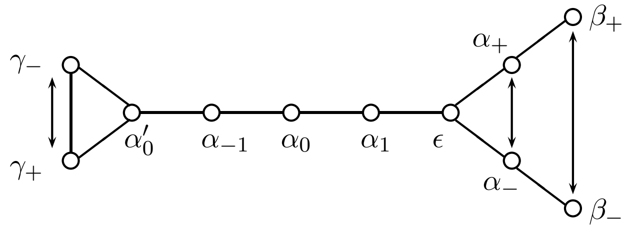
D	(Π_0, ϕ)	$\mathfrak{g}_{\text{inv}}$	$\sigma(\mathfrak{g}_{\text{inv}})$
4		$\mathfrak{su}(3,3) \oplus \mathfrak{su}(2,1)$	1
3		$\mathfrak{e}_{6\vert 2} \oplus \mathfrak{su}(2,1)$	2
2		$\hat{\mathfrak{e}}_{6\vert 2} \oplus \widehat{\mathfrak{su}}(2,1)$	-
1		${}^{6'}\mathcal{KM}_{11(III)}$	-

Table 5.10: The real subalgebras $\mathfrak{g}_{\text{inv}}$ for $T^{5-D} \times T^6/\mathbb{Z}_3$ compactifications

subalgebra (5.120) by the following additional invariant generators:

$$\begin{aligned}
K_{z_1\bar{z}_3}/K_{\bar{z}_1 z_3} &= \frac{1}{4}\big(E_{3456} + F_{3456} + E_{4567} + F_{4567} \\
&\quad \pm i(E_{34567} + F_{34567} - E_{456} - F_{456})\big)\,, \\
K_{z_2\bar{z}_3}/K_{\bar{z}_2 z_3} &= \frac{1}{4}\big(E_{56} + F_{56} + E_{67} + F_{67} \pm i(E_{567} + F_{567} - E_6 - F_6)\big)\,, \\
Z_{4z_1\bar{z}_3}/Z_{4\bar{z}_1 z_3} &= \frac{1}{4}\big(E_{23^24^25^2678} + F_{23^24^25^2678} + E_{234^25^268} + F_{234^25^268} \\
&\quad \pm i(E_{23^24^25^268} + F_{23^24^25^268} - E_{234^25^2678} - F_{234^25^2678})\big)\,, \\
Z_{4z_2\bar{z}_3}/Z_{4\bar{z}_2 z_3} &= \frac{1}{4}\big(E_{2345^2678} + F_{2345^2678} + E_{234568} + F_{234568} \\
&\quad \pm i(E_{2345^268} + F_{2345^268} - E_{2345678} - F_{2345678})\big)\,,
\end{aligned} \tag{5.126}$$

(together with their corresponding compact generators). It becomes clear that one is dealing with an $\mathfrak{a}_5$-type algebra, when recasting the whole system in the basis:

$$
\begin{aligned}
E_{\epsilon} &\doteq [E_2,[E_3,E_{\tilde{\alpha}}]] = -iE_{23458} = (Z_{4z_3\bar{z}_3})^+\,,\\
E_{\alpha_\pm} &\doteq \frac{1}{2}\left(E_{56}+E_{67}\pm i(E_{567}-E_6)\right) = (K_{z_2\bar{z}_3}/K_{\bar{z}_2 z_3})^+\,,\\
E_{\beta_\pm} &\doteq \frac{1}{2}\left(E_{34}+E_{45}\pm i(E_{345}-E_4)\right) = (K_{z_1\bar{z}_2}/K_{\bar{z}_1 z_2})^+\,,\\
E_{\alpha_\pm+\beta_\pm} &\doteq \frac{1}{2}\left(E_{3456}+E_{4567}\pm i(E_{34567}-E_{456})\right) = (K_{z_2\bar{z}_3}/K_{\bar{z}_2 z_3})^+\,,\\
E_{\epsilon+\alpha_\pm} &\doteq \frac{1}{2}\left(E_{2345^2678}+E_{234568}\pm i(E_{2345^268}-E_{2345678})\right)\\
&\quad = (Z_{4z_2\bar{z}_3}/Z_{4\bar{z}_2 z_3})^+\,, \qquad\qquad (5.127)\\
E_{\epsilon+\alpha_\pm+\beta_\pm} &\doteq \frac{1}{2}\left(E_{23^24^25^2678}+E_{234^25^268}\pm i(E_{23^24^25^268}-E_{234^25^2678})\right)\\
&\quad = (Z_{4z_1\bar{z}_3}/Z_{4\bar{z}_1 z_3})^+\,,\\
E_{\alpha_-+\alpha_{23458}+\alpha_+} &\doteq iE_{2345^26^278} = (Z_{4z_2\bar{z}_2})^+\,,\\
E_{\alpha_-+\epsilon+\alpha_++\beta_\pm} &\doteq \frac{1}{2}\left(E_{23^24^25^36^278}+E_{234^25^26^278}\pm i(E_{23^24^25^26^278}-E_{234^25^36^278})\right)\\
&\quad = (Z_{4z_1\bar{z}_2}/Z_{4\bar{z}_1 z_2})^+\\
E_{\beta_-+\alpha_-+\epsilon+\alpha_++\beta_+} &\doteq iE_{23^24^35^36^278} = (Z_{4z_1\bar{z}_1})^+\,,
\end{aligned}
$$

Combining expressions (5.121) and (5.127) reproduces the commutation relations of an $\mathfrak{a}_5$-type algebra. The remaining factor $\mathfrak{a}_3 \subset (\mathfrak{g}_{\text{inv}})^{\mathbb{C}}$ is kept untouched from the generic case. Choosing an appropriate basis for the Cartan subalgebra of $\mathfrak{g}_{\text{inv}}$ produces three compact combinations $i(H_{\gamma_+}-H_{\gamma_-})$, $i(H_{\alpha_+}-H_{\alpha_-})$ and $i(H_{\beta_+}-H_{\beta_-})$, leaving four non-compact ones, which determines $\mathfrak{g}_{\text{inv}} = \mathfrak{su}(3,3)\oplus\mathfrak{su}(2,1)$.

The $T^{5-D}\times T^6/\mathbb{Z}_3$ chain of real invariant subalgebras follows as depicted in Table 5.10, for $D=4,\ldots,1$. In $D=2$, as for the $n=4$ case, we can associate a formal signature $\hat{\sigma}$ to the real form $\mathfrak{g}_{\text{inv}} = \hat{\mathfrak{e}}_{6|2} \bowtie \widehat{\mathfrak{su}}(2,1)$, which is infinite but keeps a trace of the $D=3$ finite case for which $\sigma(\mathfrak{g}_{\text{inv}}) = 2$, as $\hat{\sigma}(\mathfrak{g}_{\text{inv}}|_{D=2}) = \sigma(\mathfrak{g}_{\text{inv}}|_{D=3}) - 2 + 2\times\sigma(\mathfrak{g}_{\text{inv}}|_{D=3})\times\infty = 0 + 2\times 2\times\infty$. In $D=1$, the *III* subscript labeling the real form appearing in Table 5.10 refers, as before, to the number of arrows in its defining Satake diagram, and the real invariant subalgebra is retrieved from modding out ${}^{6'}\mathcal{KM}_{11(III)}$ in a way similar, modulo the required changes, to expression (5.125).

5.7 Non-linear realization of the $\mathbb{Z}_n$-invariant sector of M-theory

In this section, we want to address one last issue concerning the invariant (untwisted) sector of these orbifolds, namely how the residual symmetry $\mathfrak{g}_{\text{inv}}$ can be made manifest in the equations of motion of the orbifolded supergravity in the finite-dimensional case, and in the effective $D=1$ σ-model description of M-theory near a space-time singularity in the infinite-dimensional case. The procedure follows the theory of non-linear σ-models realization of physical theories from coset spaces, more particularly from the conjectured effective Hamiltonian on $E_{10|10}/K(E_{10|10})$ presented in Sections 5.2.2 and 5.2.4.

It is customary in this context to choose the Borel gauge to fix the class representatives in the coset space. To do this in our orbifolded case, we need the Iwasawa decomposition of the real residual U-duality algebra $\mathfrak{g}_{\text{inv}}$, which can be deduced from its restricted-root space decomposition (see Section 5.3.3). For this purpose, we build the set of restricted roots Σ_0 (5.72) for $\mathfrak{g}_{\text{inv}}$ and partition it into a set of positive and a set of negative restricted roots $\Sigma_0 = \Sigma_0^+ \cup \Sigma_0^-$, based on some lexicographical ordering. Then, following definition (5.71), we build:

$$\mathfrak{n}_{\rm inv}^{(\pm)} = \bigoplus_{\bar\alpha\in\Sigma_0^{\pm}} (\mathfrak{g}_{\rm inv})_{\bar\alpha}\,, \qquad \text{with } \phi(\mathfrak{n}_{\rm inv}^{(\pm)}) = \mathfrak{n}_{\rm inv}^{(\mp)}\,. \tag{5.128}$$

Identifying the nilpotent algebra as $\mathfrak{n}_{\rm inv} \equiv \mathfrak{n}_{\rm inv}^{(+)}$, the Iwasawa decomposition of $\mathfrak{g}_{\rm inv}$ is given by $\mathfrak{g}_{\rm inv} = \mathfrak{k}_{\rm inv}\oplus\mathfrak{a}_{\rm inv}\oplus\mathfrak{n}_{\rm inv}$, and the coset parametrization in the Borel gauge by $\mathfrak{g}_{\rm inv}/\mathfrak{k}_{\rm inv} = \mathfrak{a}_{\rm inv}\oplus\mathfrak{n}_{\rm inv}$.

The cases where $\mathfrak{g}_{\rm inv}$ is a finite real Lie algebra are easily handled. For $8 \leqslant D \leqslant 3$, the Satake diagrams listed in Tables 5.3, 5.7, 5.8, 5.9 and 5.10 describing the residual U-duality algebras under $T^q/\mathbb{Z}_n$ orbifolds for $q = 2, 4, 6$ and $n \geqslant 3$ are well known, and the corresponding Dynkin diagrams for the basis $\overline{\Pi}_0$ of restricted roots Σ_0 can be found, together with the associated multiplicities $m_r(\bar\alpha_i)$, in [135]. We will not dwell on the $D = 2$ case, which only serves as a stepping stone to the understanding of the $D = 1$ case. Moreover, all the arguments we present here regarding $D = 1$ also apply, with suitable restrictions, to the $D = 2$ case.

In the $D = 1$ case, since the restricted root space is best determined from the Satake diagram of the real form, we will replace $\mathfrak{g}_{\rm inv}$ in eqn. (5.128) and all ensuing formulae by the real Borcherds and indefinite KM algebras described by the corresponding Satake diagrams in Tables 5.3 and 5.7-5.10. This procedure leads to the Dynkin diagrams and root multiplicities represented in Table 5.11 for the bases of restricted roots $\overline{\Pi}_0$, given for all but the split case of Table 5.3 (for which normal and restricted roots coincide). The multiplicities appear in bold beside the corresponding restricted root, while we denote the multiplicity of $2\bar\alpha$ with a 2 subscript whenever it is also a root of Σ_0.

We may now give a general prescription to compute the algebraic field strength $\mathcal{G} = g^{-1}dg$ of the orbifolded theory, which also applies to the infinite-dimensional case, where $\mathcal{G}$ is the formal coset element (5.50). There are two possible equivalent approaches, depending on whether we set to zero the dual field associated to the possible centres and derivations of the Borcherds/KMA algebras leading to $\mathfrak{g}_{\rm inv}$ before or after the computation of $\mathcal{G}$.

Let us consider an algebra with A pairs of non-compact centres and derivations: $\{\mathfrak{z}_a, d_a\}_{a=1,\ldots,A}$ and a maximal non-compact abelian subalgebra $\mathfrak{a}$ of dimension n_s. We introduce a vector of $n_s - 2A$ scale factors $\bar\varphi$ and a vector of auxiliary fields ψ and develop them on the basis of $\mathfrak{a}(\mathfrak{g})$ as:

$$\bar\varphi = \sum_{\bar\imath=1}^{n_s-2A} \bar\varphi^i H_{\bar\imath}\,, \qquad \psi = \sum_{a=1}^{A} \left(\psi^a \mathfrak{z}_a + \psi^{A+a} d_a\right)\,. \tag{5.129}$$

For example, in the case of the $T^6\times T^4/\mathbb{Z}_n$ orbifold, $\mathfrak{a}_{\rm inv}$ is given by:

$$\begin{aligned}\mathfrak{a}({}^4\mathcal{B}_{10(Ib)}) \;=\;& \mathrm{Span}_{\mathbb{R}}\{H_{\bar\alpha_{-1}};\ldots;H_{\bar\alpha_3};H_{\bar{\tilde\alpha}} = H_4+H_5+H_8;H_{\bar\alpha_+} = H_5+2H_6+H_7;\\ & H_{\bar\gamma} = H_2+2H_3+3H_4+3H_5+2H_6+H_7+H_8;\mathfrak{z};d_I\}\,.\end{aligned}$$

in which the centre is $\mathfrak{z} = H_{\bar\beta_I} - (H_{\bar\alpha_{-1}}+\ldots+H_{\bar\alpha_3}+H_{\bar{\tilde\alpha}}+H_{\bar\alpha_+}+H_{\bar\gamma}) = H_{\beta_I} - H_\delta$. The generator $H_{\bar\imath}$ will be understood to represent the i-th element of the above list, for $\bar\imath = 1,\ldots,8$.

In general, a central element obviously does not contribute much to $\mathcal{G}$, except for a term $\propto d\psi^a/dt$, so that it does not matter whether we impose the physical constraint $\psi^a = 0$, $\forall a = 1,\ldots,A$ before or after the computation of $\mathcal{G}$. The derivations d_a also create terms $\propto d\psi^{A+a}/dt$, but ψ^{A+a} appears in exponentials in front of generators for roots containing an imaginary/affine ($\neq \alpha_0$) simple root, as well. However, there is no difference between setting the auxiliary field ψ to zero directly in g or, later, in the exponentials in $\mathcal{G}$. Indeed, the counting of levels in these imaginary/affine simple roots is taken care of by $H_{\bar\alpha_{-1}}$ anyway. Finally, a term proportional to $\mathfrak{z}_a$ other than $\mathfrak{z}_a\, d\psi^a/dt$ can not be produced either, since we work in the Borel gauge, so that terms containing commutators like $[E_{\beta_I}, F_{\beta_I}]$ are absent. It is thus not necessary to impose $\mathfrak{z}_a = 0\ \forall a$ again in the end.

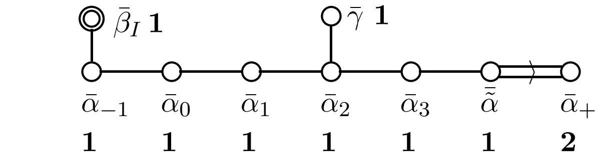
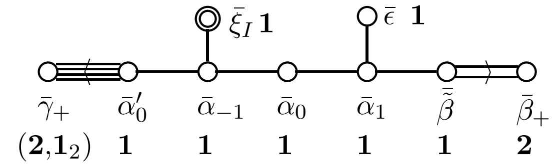
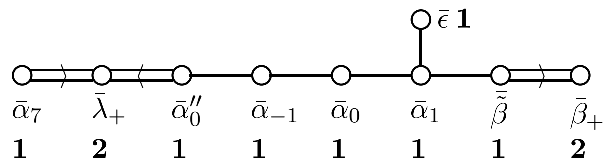
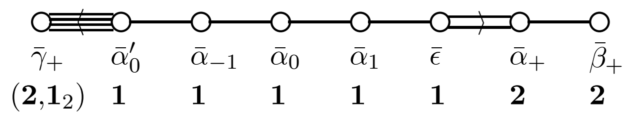
	$\overline{\Pi}_0$
$T^6 \times T^4/\mathbb{Z}_{n>2}$: $^4\mathcal{B}_{10(Ib)}$	
$T^4 \times T^6/\mathbb{Z}_{n\geq 5}$: $^6\mathcal{B}_{11(II)}$	
$T^4 \times T^6/\mathbb{Z}_4$: $^6\mathcal{KM}_{11(II)}$	
$T^4 \times T^6/\mathbb{Z}_3$: $^{6'}\mathcal{KM}_{11(III)}$	

Table 5.11: Restricted Dynkin diagrams for very-extended $\mathfrak{g}_{\text{inv}}$ subalgebras

More generally, let us now set $n_{\text{inv}} = \bigoplus_{\bar{\alpha}\in\Sigma_0^+}(\mathfrak{g}_{\text{inv}})_{\bar{\alpha}}$ with $\dim(\mathfrak{g}_{\text{inv}})_{\bar{\alpha}} = m(\bar{\alpha})\cdot m_r(\bar{\alpha})$, the (formal) group element[13]

$$g = e^{\bar{\varphi}+\psi} \cdot \prod_{\bar{\alpha}\in\Sigma_0^+} \prod_{s=1}^{m_r(\bar{\alpha})} \prod_{a=1}^{m(\bar{\alpha})} e^{C_{\bar{\alpha},(s,a)} E_{\bar{\alpha}}^{(k,a)}}\,, \tag{5.130}$$

[13]Note that we adopt here a perspective that is different from [138] when associating restricted roots to the metric and p-form potential of orbifolded $11D$ supergravity / M-theory. In particular the authors of [139] were concerned with super-Borcherds symmetries of supergravity with non-split U-duality groups, which form the so-called real magic triangle, i.e. which are oxidations of pure supergravity in 4 dimensions with $\mathcal{N} = 0,..,7$ supersymmetries. When doubling the fields of these theories by systematically introducing Hodge duals for all original p-form fields (but not for the metric), the duality symmetry of this enlarged model can be embedded in a larger Borcherds superalgebra. The self-duality equations for all p-forms of these supergravities can be recovered by a certain choice of truncation in the Grassmanian coefficients of the superalgebra. In contrast to our approach however, one positive restricted root was related to one p-form potential term in [139], whereas, we associate a restricted root generator to one component of the potential. This is the reason why these authors drop the sum over $m_r(\bar{\alpha})$ in expression (5.130) (not mentioning the sum over $m(\bar{\alpha})$ in the $D = 1$ case, which we keep since we do not want to discard any higher order contributions to classical 11D supergravity).

can be used to compute the Maurer-Cartan one-form:

$$\mathcal{G} = \left[g^{-1}dg\right]_{\psi^a=0,a=1,\dots,2A}$$

in which the coefficients $\{\bar{\varphi}^i; C_{\bar{\alpha}}\}$ correspond, at levels $l = 0, 1$, to the invariant dilatons and potentials of orbifolded classical 11D supergravity, and, at higher levels, participate to (invariant) contributions from M-theory. Their exact expressions can be reconstructed from the material of Sections 5.4-5.6 and the Satake diagrams of Tables 5.3 and 5.7-5.10.

The Maurer-Cartan equation $d\mathcal{G} = \mathcal{G} \wedge \mathcal{G}$ will then reproduce the equations of motion of the untwisted sector of the reduced supergravity theory in the finite case, of M-theory on $M = \mathbb{R}_+ \times T^{11-D-2p} \times T^{2p}/\mathbb{Z}_n$ in $D = 1$, which will make manifest the residual symmetry $\mathfrak{g}_{\text{inv}}$. Finally, one can write down an effective invariant Hamiltonian as in expression (5.49), by performing a Legendre transform of $\mathcal{L} = \frac{1}{4n}\left[\text{Tr}(\partial\mathcal{M}^{-1}\partial\mathcal{M})\right]_{\psi^a=0,a=1,\dots,2A}$, the orbifolded version of expression (5.30).

5.8 Shift vectors and chief inner automorphisms

We have dedicated the first few chapters of this article to explaining the characterization of fixed-point subalgebras under finite-order automorphisms of U-duality algebras. In physical words, we have computed the residual U-duality symmetry of maximally supersymmetric supergravities compactified on certain toric orbifolds. In $D \geq 3$, the quotient of this residual algebra by its maximal compact subalgebra is in one-to-one correspondence with the physical spectrum of $11D$ supergravity surviving the orbifold projection. In string theory language, this corresponds to the untwisted sector of the orbifolded theory. Extrapolating this picture to $D = 1$, the orbifold spectrum gets enhanced by a whole tower of massive string states and/or non-perturbative states.

Although the interpretation of most of these higher level $\mathfrak{e}_{10}$ roots is still in its infancy, an interesting proposal was made in [55] for a restricted number of them, namely for those appearing as shift vectors describing $\mathbb{Z}_2$ orbifold actions. They were interpreted as the extended objects needed for local anomaly and charge cancellations in brane models of certain M-theory orbifolds and type IIA orientifolds.

In this section, a general method to compute the shift vectors of any $T^p \times T^q/\mathbb{Z}_n$ orbifolds will be given, as well as explicit results for $q = 2, 4, 6$. Then, an empirical technique to obtain $\mathfrak{e}_{10}$ roots that are physically interpretable will be presented, exploiting the freedom in choosing a shift vector from its equivalence class. Our results in particular reproduce the one given in [55] for $T^6 \times T^4/\mathbb{Z}_2$. Note that the method will allow to differentiate, for example, between $T^4/\mathbb{Z}_3$ and $T^4/\mathbb{Z}_4$ despite the fact that they lead to the same invariant subalgebra, which gives a clue on the rôle of the n-dependent part of the shift vectors. Finally, we will see how to extract the roots describing a $\mathbb{Z}_n$ orbifolds from all level $3n$ roots of $\mathfrak{e}_{10}$.

We first remark that the complex combinations of generators corresponding to the complexified physical fields are the eigenvectors of the automorphisms $\mathcal{U}_q^{\mathbb{Z}_n}$ with eigenvalues $\exp(i\frac{2\pi}{n}Q_A)$, for $Q_A \in \mathbb{Z}_n$. Having a basis of eigenvectors suggests that there is a conjugate Cartan subalgebra $\mathfrak{h}'$ inside of the $Q_A = 0$ eigenspace $\mathfrak{g}^{(0)}$ for which the automorphism is diagonal. We can then reexpress the orbifold action as an automorphism that leaves this new Cartan subalgebra invariant, in other words as a chief inner automorphism of the form $\text{A}d(\exp(i\frac{2\pi}{n}H'))$ for some $H' \in \mathfrak{h}'_{\mathbb{Q}}$. As already noticed in the case of $\mathbb{Z}_2$ orbifolds, such a chief inner automorphism simply acts as $\exp(i\frac{2\pi}{n}\alpha'(H'))$ on every root subspace $\mathfrak{g}_{\alpha'}$ where α' is a root defined with respect to $\mathfrak{h}'$. In particular, we can find a (non-unique) weight vector Λ' corresponding to H' so that

$$\text{A}d(e^{i\frac{2\pi}{n}H'})\mathfrak{g}_{\alpha'} = e^{i\frac{2\pi}{n}(\Lambda'|\alpha')}\mathfrak{g}_{\alpha'}\,.$$

Such a weight vector is commonly called shift vector. It turns out that all automorphisms of a given simple Lie algebra can be classified by all weights $\Lambda' = \sum_{i=1}^{r} l_i \Lambda'^i$, $l_i \in \mathbb{Z}_n$ without common prime factor, so that, according to [161],

$$(\Lambda|\theta_G) \leq n \tag{5.131}$$

(here, the fundamental weights Λ'^i are defined to be dual to the new simple roots α'_i, i.e. $(\Lambda'^i|\alpha'_j) = \delta^i_j \ \forall i,j = 9-r,\dots,8)$. Furthermore, there is a simple way, see [66], to deduce the invariant subalgebra from Λ by guessing its action on the extended Dynkin diagram, if Λ satisfies the above condition. Here, we will first show how to obtain the shift vectors in the cases we are interested in and then describe the above-mentioned diagrammatic method with the help of these examples.

5.8.1 A class of shift vectors for $T^2/\mathbb{Z}_n$ orbifolds

Let us start by the particularly simple case of a $T^2/\mathbb{Z}_n$ orbifold in T^3. We can directly use the decomposition in eigensubspaces obtained in equation (5.2). The first task is to choose a new Cartan subalgebra, or equivalently a convenient Cartan-Weyl basis. In other words, we are looking for a new set of simple roots for $\mathfrak{a}_2 \oplus \mathfrak{a}_1$, so that all Cartan generators are in the $Q_A = 0$ eigensubspace $\mathfrak{g}^{(0)}$. Since the $\mathfrak{a}_1$ does not feel the orbifold action, we can simply take $H'_8 = H_8$. On the other hand, we should take for H'_6 and H'_7 some combinations of $2H_6 + H_7$ and $E_7 - F_7$. A particularly convenient choice is given by the following Cartan-Weyl basis:

$$\begin{aligned}
&E'_6 = \tfrac{1}{\sqrt{2}}(E_6 + iE_{67})\,, && F'_6 = \tfrac{1}{\sqrt{2}}(F_6 - iF_{67})\,, && H'_6 = \tfrac{1}{2}(2H_6 + H_7 - i(E_7 - F_7))\,,\\
&E'_7 = \tfrac{1}{2}(H_7 - i(E_7 + F_7))\,, && F'_7 = \tfrac{1}{2}(H_7 + i(E_7 + F_7))\,, && H'_7 = i(E_7 - F_7)\,,\\
&E'_{67} = \tfrac{1}{\sqrt{2}}(E_6 - iE_{67})\,, && F'_{67} = \tfrac{1}{\sqrt{2}}(F_6 + iF_{67})\,, && H'_{67} = \tfrac{1}{2}(2H_6 + H_7 + i(E_7 - F_7))\,,
\end{aligned}$$

This gives the following simple decomposition in eigensubspaces:

$$\begin{aligned}
\mathfrak{g}^{(0)} &= \text{Span}\{H'_6; H'_7; E'_8; F'_8; H'_8\}\,, & \mathfrak{g}^{(n-1)} &= \text{Span}\{E'_{67}; F'_6\}\,,\\
\mathfrak{g}^{(1)} &= \text{Span}\{E'_6; F'_{67}\}\,, & \mathfrak{g}^{(n-2)} &= \text{Span}\{E'_7\}\,,\\
\mathfrak{g}^{(2)} &= \text{Span}\{F'_7\}\,.
\end{aligned}$$

Notice that $\mathfrak{g}^{(n-i)}$ is obtained from $\mathfrak{g}^{(i)}$ by the substitution $E \leftrightarrow F$, so that we will only give the latter explicitly in the following examples. Furthermore, since $\mathcal{U}_2^{\mathbb{Z}_n}$ actually defines a gradation $\mathfrak{g} = \bigoplus_{i=0}^{n-1} \mathfrak{g}^{(i)}$, we have the property $[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subseteq \mathfrak{g}^{(i+j)}$. This implies that if we can find a weight $\Lambda'^{\{2\}}$ that acts as $\exp(i\frac{2\pi}{n}(\Lambda'^{\{2\}}|\alpha'_i))$ on $\mathfrak{g}_{\alpha'_i}$ for all new simple roots α'_i, $i \in I$, it will induce the correct charges for all new generators. Here, we should choose $\Lambda'^{\{2\}}$ so that it has scalar product 0 with α'_8, 1 with α'_6 and $n-2$ with α'_7, which suggests to take:

$$\Lambda'^{\{2\}} = \Lambda'^6 + (n-2)\Lambda'^7\,. \tag{5.132}$$

Note first that the same set of charges can be obtained with all choices of the form: $\Lambda'^{\{2\}} = (c_1 n + 1)\Lambda'^6 + (c_2 n - 2)\Lambda'^7 + c_3 n \Lambda'^8$ for any set of $\{c_i\}_{i=1}^3 \in \mathbb{Z}^3$. In particular, there exists one weight vector that is valid for automorphisms of all finite orders, here $\Lambda'^{\{2\}} = \Lambda'^6 - 2\Lambda'^7$. However, in equation 5.132, we took all coefficients in $\mathbb{Z}_n$ as is required for the Kac-Peterson method to work. Since the $\mathfrak{a}_1$ is obviously invariant, we can restrict our attention to the $\mathfrak{a}_2$ part. One can verify that $\Lambda'^{\{2\}}$ satisfies the condition (5.131) since $\theta_{A_2} = \alpha'_6 + \alpha'_7$ implies $(\Lambda'^{\{2\}}|\theta_{A_2}) = n-1$.

In general, for a U-duality group G, we can define an $(r+1)$-th component of Λ' as $l_9^G = n - (\Lambda', \theta_G)$ ($l_9^{A_1 \times A_2} = 1$ in the above case). On the basis of this extended vector, one can apply the following diagrammatic method to obtain the invariant subalgebra in the finite-dimensional case (a simple justification of this method can be found in [66]):

- Replace the Dynkin diagram of $\mathfrak{g}$ by its extended Dynkin diagram, adding an extra node corresponding to the (non-linearly independent) root $\alpha'_9 = -\theta_G$. We denote the extended diagram by $\mathfrak{g}^+$ to distinguish it from the affine $\hat{\mathfrak{g}}$ in which the extra node $\alpha_0 = \delta - \theta_G$ is linearly independent.

- Discard all nodes corresponding to roots α'_i such that $l_i \neq 0$ and keep all those such that $l_i = 0$, $i \in \{9-r;\dots;9\}$.

- Let p be the number of discarded nodes, the (usually reductive) subalgebra left invariant by the automorphism $\mathcal{U}_2^{\mathbb{Z}_n}$ is given by the (possibly disconnected) remaining diagram times $p-1$ abelian subalgebras.

In particular, for $T^2/\mathbb{Z}_n$ in T^3 for $n \geq 3$, we see that l_6,l_7 and l_9 are non-zero, leaving invariant only $\alpha'_8 = \alpha_8$ which builds an $\mathfrak{a}_1$ diagram. Since $p = 3$, we should add two abelian factors for a total (complexified) invariant subalgebra $\mathfrak{a}_1 \oplus \mathbb{C}^{\oplus^2}$, the same conclusion we arrived at from Table 5.2. On the other hand, for $T^2/\mathbb{Z}_2$ in T^3, $l_7 = n-2 = 0$ and we have one more conserved node, leaving a total (complexified) invariant subalgebra $\mathfrak{a}_1^{\oplus^2} \oplus \mathbb{C}$.

Since the orbifold is not acting on other space-time directions, it seems logical to extend this construction by taking $\alpha'_i = \alpha_i$ and $l_i = 0\ \forall i < 6$ for all $T^p \times T^2/\mathbb{Z}_n$ orbifolds. Indeed, for $p \leq 4$, we obtain $(\Lambda'^{\{2\}}|\theta_{A_4}) = n-1 \to l_9^{A_4} = 1$, $(\Lambda'^{\{2\}}|\theta_{D_5}) = n \to l_9^{D_5} = 0$ and $(\Lambda'^{\{2\}}|\theta_{E_6}) = n \to l_9^{E_6} = 0$, giving the results in Figure 5.2. Comparing Figure 5.2 with Table 5.3 in $D = 6$, we can identify $\alpha'_9 = -\theta_{D_5}$.

However, looking at the respective invariant subdiagrams in $D = 5$, it is clear that one should not choose $\alpha'_i = \alpha_i\ \forall i < 6$. Looking at Table 5.3, one guesses that $\alpha'_3 = -\theta_{D_5}$, $\alpha'_4 = -\alpha_3$ and $\alpha'_5 = -\alpha_4$. Since there is only one element in the eigensubspace $\mathfrak{g}^{(n-2)}$, we also have to take $E'_7 = \frac{1}{2}(H_7 - i(E_7 + F_7))$, as before. On the other hand, there are now plenty of objects in $\mathfrak{g}^{(1)}$, all of them not commuting with E'_7. One should find one that commutes with $F_{\theta_{D_5}}$ and F_{α_3} but not with F_{α_4}. This suggests to set $E'_6 = \frac{1}{\sqrt{2}}(E_{34^25^268} - iE_{34^25^2678})$. Finally, we also take $E'_8 = F_5$. Computing the expression of the generator corresponding to the highest root in this new basis gives $E'_9 = F_{\theta_{E_6}} = iF_8$, as one would expect. Since the shift vector simply takes the same form $\Lambda'^{\{2\}} = \Lambda'^6 + (n-2)\Lambda'^7$ on a new basis of fundamental weights, the naive guess above was correct.

From $D = 4$ downwards, this ceases to be true, since naively $(\Lambda'^{\{2\}}|\theta_{E_7}) = 2n-1 > n$. In fact, we should again change basis in $\mathfrak{e}_7$. Comparing again Figure 5.2 with Table 5.3, we see that there are 2 different equivalent ways to choose the 2 roots to be discarded, on the left (α'_2 and α'_3) or on the right (α'_7 and α'_9). We choose the latter, since it will be easier to generalize to $\mathfrak{e}_8$. Indeed, in the extended diagram of $\mathfrak{e}_8$, the Coxeter label of α'_9 will be the only one to be 1, making $l_9^{E_8} = n-2$ the only possible choice. Further inspection of Figure 5.2 and Table 5.3 suggests to take the new basis as follows: $\alpha'_2 = -\alpha_8$, $\alpha'_3 = -\alpha_5$, $\alpha'_4 = -\alpha_4$, $\alpha'_5 = -\alpha_3$, $\alpha'_6 = -\theta_{D_5}$, $\alpha'_8 = -\alpha_2$ and finally

$$E'_7 = \frac{1}{\sqrt{2}}(E_{23^24^35^46^378^2} - iE_{23^24^35^46^37^28^2}) \in \mathfrak{g}^{(1)}\,.$$

A lengthy computation allows to show that this choice leads to
$E'_9 = F_{\theta'_{E_7}} = (i/2)(H_7 - i(E_7 + F_7)) \in \mathfrak{g}^{(n-2)}$ as it should, giving the shift vector: $\Lambda = \Lambda'^7$ with $l_9^{E_7} = n-2$.

In $\mathfrak{e}_8$, a similar game leads to $\alpha'_2 = -\alpha_8$, $\alpha'_3 = -\alpha_5$, $\alpha'_4 = -\alpha_4$, $\alpha'_5 = -\alpha_3$, $\alpha'_6 = -\alpha_2$, $\alpha'_7 = -\alpha_1$, $\alpha'_8 = -\theta_{D_5}$ and finally

$$E'_1 = \frac{1}{\sqrt{2}}(E_{12^23^34^45^56^378^3} - iE_{12^23^34^45^56^37^28^3}) \in \mathfrak{g}^{(1)}\,,$$

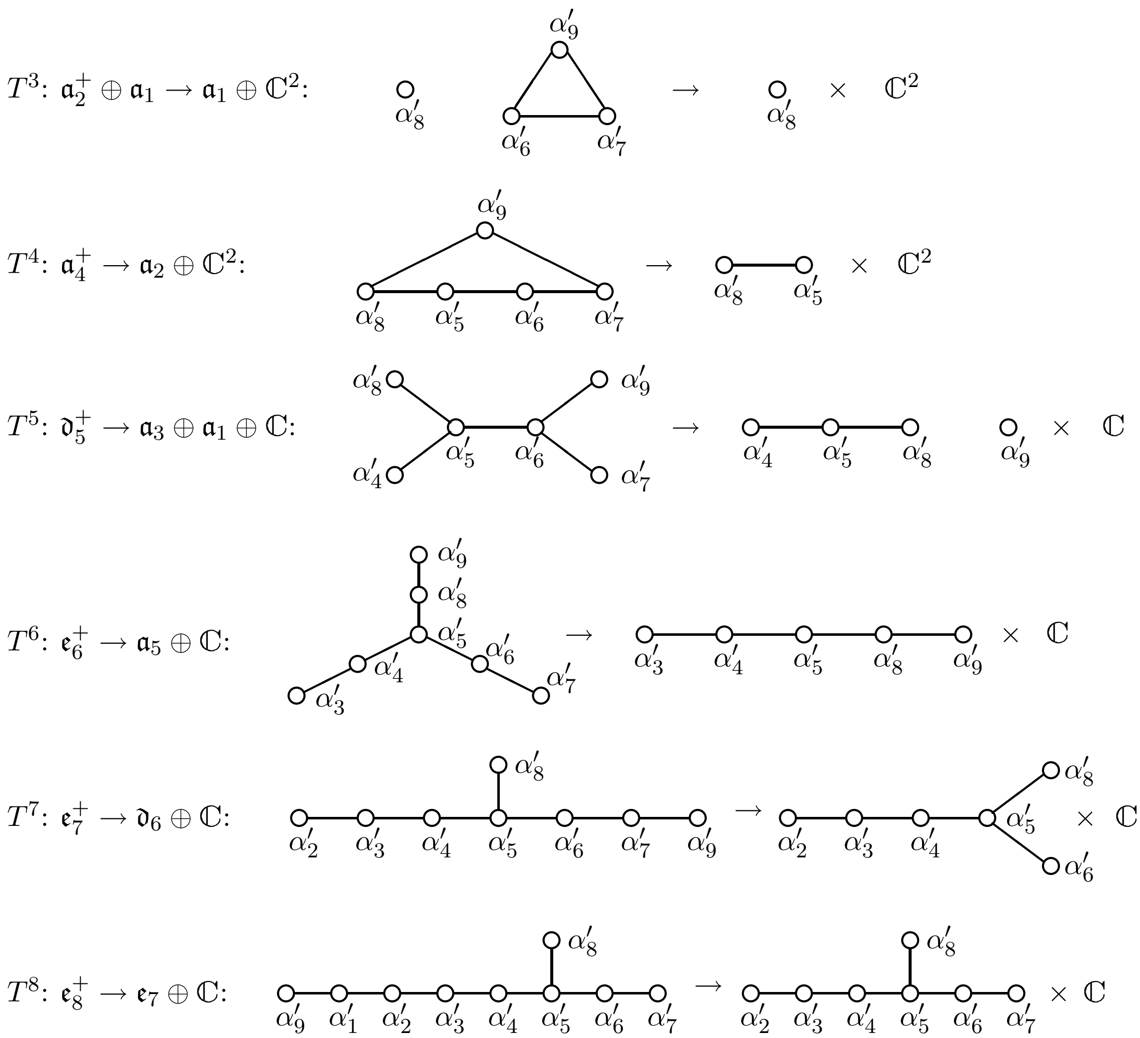


Figure 5.2: Diagrammatic method for $T^2/\mathbb{Z}_{n>2}$ orbifolds of M-theory

leading to $E'_9 = F_{\theta'_{E_8}} = \frac{i}{2}(H_7 - i(E_7 + F_7)) \in \mathfrak{g}^{(n-2)}$ with shift vector $\Lambda' = \Lambda'^1$, while $l_9^{E_8} = n-2$.

The results for $n = 2$ can be obtained directly by putting $l_7 = 0$, adding one more node to the diagrams instead of the abelian $\mathfrak{u}(1)$ factor. The results are summarized in Figure 5.3.

It was instructive to compare our method based on automorphisms induced by algebraic rotations and the standard classification of Lie algebra automorphisms based on shift vectors defining chief inner automorphisms. However, the mapping from one language to the other can be fairly obscure, in particular for orbifolds more complicated than the $T^2/\mathbb{Z}_n$ case treated above. In fact, in the $\mathfrak{a}_r$ serie of Lie algebras, for which the Coxeter labels are all equal, the necessary change of basis can be computed only once and trivially extended to larger algebras in the serie. In general, and in particular for exceptional algebras, one has to perform a different change of Cartan-Weyl basis whenever we consider the *same* orbifold in a larger U-duality symmetry algebra (or, geometrically speaking, when we compactify one more dimension).

Our method based on non-Cartan preserving automorphisms is thus more appropriate to treat a few particular orbifolds in a serie of algebras that are successively included one into the other, as is the case for the U-duality algebras of compactified supergravity theories. On the other hand, the method based on chief inner automorphisms is more amenable to classify all orbifolds of a unique algebra, for example all possible breakings of a given gauge group under

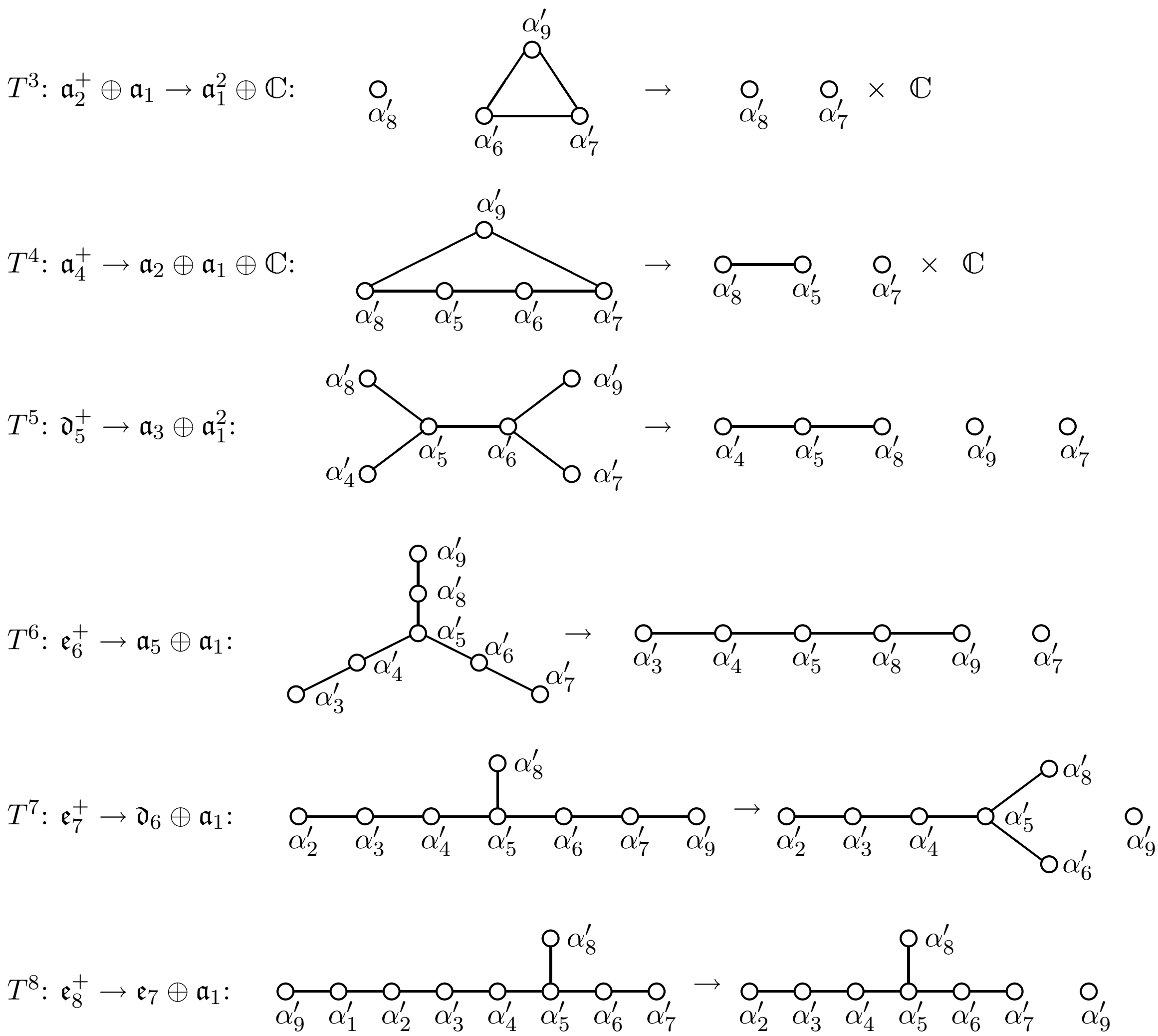


Figure 5.3: Diagrammatic method for $T^2/\mathbb{Z}_2$ orbifolds of M-theory

an orbifold action. For instance, the breakings of the $E_8 \times E_8$ gauge group of heterotic string theory have been treated this way by [163, 66]. It is also easier to generalize the method based on algebraic rotations to the infinite-dimensional case, since we can use the decomposition of $\mathfrak{e}_{10}$ in tensorial representations of $\mathfrak{sl}(10)$ and our intuition on the behaviour of tensorial indices under a physical rotation to identify non-trivial invariant objects.

We can draw a related conclusion from the explicit forms of the above basis transformation: when the orbifold is expressed in terms of the standard shift vector satisfying $(\Lambda|\theta) \leq n$, the geometric interpretation of the orbifold action gets blurred. More precisely, the directions in which the rotation is performed is determined above by the roots α'_i with coefficients $l_i = n-2$. For example, in $\mathfrak{e}_7$, our original Lorentz rotation by $\mathcal{K}_{9\,10}$ represented by α_7 appears in the standard basis as a gauge transformation generated by $\widetilde{\mathcal{Z}}_{456789}$. Similarly, in $\mathfrak{e}_8$, it seems that we are rotating in a direction corresponding to $(\widetilde{\mathcal{K}}_3)_{3456789\,10}$. Of course, mathematically, all conjugate Cartan-Weyl basis in a Lie algebra give rise to an isomorphic gradation of $\mathfrak{g}$, but the physical interpretation based on the decomposition of $\mathfrak{e}_r$ in tensorial representations of $\mathfrak{sl}(r)$ is obscured by the conjugation.

Indeed, our $T^q/\mathbb{Z}_2$ and $T^q/\mathbb{Z}_n$ orbifolds for $q = 2, 4$ all appear in the classification of $T^6/\mathbb{Z}_n$ orbifolds given in [163], where they are interpreted as $T^6/\mathbb{Z}_n$ orbifolds with particularly small breakings of the gauge group and degenerate shift vectors (in the sense of having lots of 0). It is however clear in our formalism that this degeneracy should actually be seen as having considered a rotation of null angle in certain directions.

5.8.2 Classes of shift vectors for $T^q/\mathbb{Z}_n$ orbifolds, for $q = 4, 6$

In the more complicated cases of $T^4/\mathbb{Z}_n$ and $T^6/\mathbb{Z}_n$ orbifolds, we will not give in detail the basis transformations necessary to obtain the standard shift vectors satisfying $(\Lambda'|\theta) \leq n$ for the whole serie of U-duality algebras. Rather, we will give the shift vectors in their universal form, which is valid for the whole serie of U-duality algebras. In particular, for $T^4/\mathbb{Z}_n$, the gradation of $\mathfrak{d}_5$ by eigensubspaces of $\mathcal{U}_4^{\mathbb{Z}_n}$ has been given in expressions (5.83), (5.84) and (5.85). A particularly natural choice of diagonal Cartan-Weyl basis for this decomposition is obtained by taking:

$$\begin{aligned}
E'_4 &= \tfrac{1}{\sqrt{2}}(E_4 + iE_{45})\,, & H'_4 &= \tfrac{1}{2}(2H_4 + H_5 - i(E_5 - F_5))\,,\\
E'_5 &= \tfrac{1}{2}(H_5 - i(E_5 + F_5))\,, & H'_5 &= i(E_5 - F_5)\,,\\
E'_6 &= \tfrac{1}{\sqrt{2}}(E_{56} - E_{67} + i(E_{567} + E_6))\,, & H'_6 &= \tfrac{1}{2}(H_5 + 2H_6 + H_7 - i(E_5 - F_5 + E_7 - F_7))\,,\\
E'_7 &= \tfrac{1}{\sqrt{2}}(H_7 - i(E_7 + F_7))\,, & H'_7 &= i(E_7 - F_7)\,,\\
E'_8 &= -\tfrac{1}{\sqrt{2}}(E_8 - iE_{58})\,, & H'_8 &= \tfrac{1}{2}(2H_8 + H_5 - i(E_5 - F_5))\,,
\end{aligned} \tag{5.133}$$

while F'_i is obtained from E'_i as above by conjugation and exchange of E and F. This leads to the following eigensubspace decomposition of $\mathfrak{d}_5$:

$$\begin{aligned}
\mathfrak{g}^{(0)} &= \ \mathrm{Span}\{H'_4; H'_5; H'_6; H'_7; H'_8; E'_6; E'_{567}; E'_{458}; E'_{4568}; E'_{45^2678}; E'_{45^26^278}; E' \leftrightarrow F'\}\,,\\
\mathfrak{g}^{(1)} &= \ \mathrm{Span}\{E'_4; E'_8; E'_{4567}; E'_{5678}; F'_{45}; F'_{58}; F'_{456}; F'_{568}\}\,,\\
\mathfrak{g}^{(2)} &= \ \mathrm{Span}\{E'_7; E'_{67}; E'_{45678}; F'_5; F'_{56}; F'_{45^268}\}\,.
\end{aligned} \tag{5.134}$$

The shift vector corresponding to this basis is given by:

$$\Lambda'^{\{4\}} = \Lambda'^4 + (n-2)\Lambda'^5 + 2\Lambda'^7 + \Lambda'^8\,,$$

which clearly reduces to $\Lambda'^{\{4\}} = \Lambda'^4 + \Lambda'^8$ in the case of $T^4/\mathbb{Z}_2$. By simply taking $E'_i = E_i$ for any additional roots that are unaffected by the orbifold action, this shift vector is valid in $\mathfrak{e}_r$, for $r = 6, .., 10$, as well.

For the case of $T^6/\mathbb{Z}_n$, we take the following Cartan-Weyl basis:

$$\begin{aligned}
E'_2 &= \tfrac{1}{\sqrt{2}}(E_2 + iE_{23})\,, & H'_2 &= \tfrac{1}{2}(2H_2 + H_3 - i(E_3 - F_3))\,,\\
E'_3 &= \tfrac{1}{2}(H_3 - i(E_3 + F_3))\,, & H'_3 &= i(E_3 - F_3)\,,\\
E'_4 &= \tfrac{1}{\sqrt{2}}(E_{34} + E_{45} - i(E_{345} - E_4))\,, & H'_4 &= \tfrac{1}{2}(H_3 + 2H_4 + H_5 + i(-E_3 + F_3 + E_5 - F_5))\,,\\
E'_5 &= \tfrac{1}{2}(H_5 + i(E_5 + F_5))\,, & H'_5 &= -i(E_5 - F_5)\,,\\
E'_6 &= \tfrac{1}{\sqrt{2}}(E_{56} - E_{67} - i(E_{567} + E_6))\,, & H'_6 &= \tfrac{1}{2}(H_5 + 2H_6 + H_7 + i(E_5 - F_5 + E_7 - F_7))\,,\\
E'_7 &= \tfrac{1}{\sqrt{2}}(H_7 + i(E_7 + F_7))\,, & H'_7 &= -i(E_7 - F_7)\,,\\
E'_8 &= -\tfrac{1}{\sqrt{2}}(E_8 + iE_{58})\,, & H'_8 &= \tfrac{1}{2}(2H_8 + H_5 + i(E_5 - F_5))\,,
\end{aligned} \tag{5.135}$$

that leads to the universal shift vector:

$$\Lambda'^{\{6\}} = \Lambda'^2 + (n-2)\Lambda'^3 + 2\Lambda'^5 + \Lambda'^6 + (n-4)\Lambda'^7 + (n-1)\Lambda'^8\,,$$

that is valid in $\mathfrak{e}_8$, $\mathfrak{e}_9$ and $\mathfrak{e}_{10}$, as well. It is obvious in this form that the degeneration of the coefficient l^7 when $n = 4$ leads to a larger invariant subalgebra with fewer abelian factors. On the other hand, as the invariant subalgebras for $T^6/\mathbb{Z}_4$ and $T^6/\mathbb{Z}_3$ both have no abelian factors, the coefficients of $\Lambda'^{\{6\}}$ does not allow to discriminate between them. Another fact worth noting is that setting $n = 2$ leads to $\Lambda'^{\{6\}} = \Lambda'^2 + \Lambda'^6 + \Lambda'^8$, corresponding to a $T^4/\mathbb{Z}_2$ orbifolds with respect to the nodes α_3 and α_5 and not to a $T^6/\mathbb{Z}_2$ orbifolds. This is natural since we chose the charge in the (x^9, x^{10})-plane to be $Q_3 = -2$, so that it reduces to the identity rotation for $n = 2$.

5.8.3 Roots of $\mathfrak{e}_{10}$ as physical class representatives

The universal shift vectors are mathematically interesting, but the original motivation to compute them was actually to give a physical interpretation of certain roots of $\mathfrak{e}_{10}$. Typically, our universal shift vectors Λ' are not roots, but we can use the self-duality of $Q(\mathfrak{e}_{10})$ and the periodicity modulo n of the orbifold action to replace Λ' by a root ξ generating the same orbifold action.

Self-duality of $Q(\mathfrak{e}_{10})$ relates the weight Λ' to a vector in the root lattice satisfying $(\Lambda'|\alpha') = (\tilde{\xi}|\alpha')\ \forall\alpha' \in \Delta(\mathfrak{g}, \mathfrak{h}')$. However, every root lattice vector is not a root. One should thus use the equivalence modulo n: $\Lambda' \equiv \Lambda' + n\sum_{i=-1}^{8} c_i\Lambda'^i = \overline{\Lambda}'$, for any 10-dimensional vector $\vec{c} \in \mathbb{Z}^{10}$, to find a weight $\overline{\Lambda}' = \sum_{i=-1}^{8} l_i\Lambda'^i$ such that:

$$\xi = \sum_{i,j=-1}^{8} (A^{-1})^{ij} l_j \alpha'_i$$

is a root of $\mathfrak{e}_{10}$. In fact, such a condition does not fix ξ uniquely either. However, it seems that there is a unique way to choose $\vec{c_q}$ so that $\xi^{[q,n]}$ is a root describing the orbifold $T^{10-q} \times T^q/\mathbb{Z}_n$ for all values of n.

From that point of view, we can see the shift vector as containing two parts: the *universal part*, that reflects the choices of orbifold directions and charges, and the *n-dependent part*, that defines the orbifold periodicity.

Concretely, it seems that $\vec{c_q}$ can always be chosen to be dual to a Weyl reflection of δ (at least for even orbifolds). In the case of $T^2/\mathbb{Z}_2$, for example, we had the universal part $\Lambda'^{\{2\}} = \Lambda'^6 - 2\Lambda'^7$, which is dual to $-\alpha'_7$. Adding $n(\Lambda'^7 - \Lambda'^8)$, i.e. the root $n\tilde{\delta}^{[2]} = n(\delta' + \sum_{i=-1}^{7} \alpha'_i)$, we obtain the desired form of shift vector in the physical basis:

$$\xi^{[2,n]} = (n, n, n, n, n, n, n, n, n-1, 1)\,.$$

From the tables of [115] it is easy to verify that this is a root of $\mathfrak{e}_{10}$ with $l = 3n$ for all values of $n \leq 6$, and it is very likely to be a root for any integer value of n. Note also that translating the results back in the original basis gives:

$$e^{i\frac{2\pi}{n}(\xi^{[2,n]}|\alpha')}\mathfrak{g}_{\alpha'} = e^{i2\pi(\alpha'|\tilde{\delta}^{[2]})}\,\mathrm{Ad}e^{\frac{2\pi}{n}(E_7-F_7)}\mathfrak{g}_{\alpha'}\,,$$

where the first factor does not contribute to the charge, so that the equivalence between the two descriptions, one in terms of shift vectors and the other of in terms of non-Cartan preserving inner automorphisms, is obvious.

For $T^4/\mathbb{Z}_n$, we similarly take $\overline{\Lambda}'^{\{4\}} = \Lambda'^4 - 2\Lambda'^5 + 2\Lambda'^7 + \Lambda'^8 + n(\Lambda'^5 - \Lambda'^6 - \Lambda'^8)$, which corresponds to

$$\begin{aligned}
\xi^{[4,n]} &= -\alpha'_5 + \alpha'_7 + n(\delta' + \alpha'_{-1} + \alpha'_0 + \alpha'_1 + \alpha'_2 + \alpha'_3 + \alpha'_4 + \alpha'_5) \\
&= (n, n, n, n, n, n, n-1, 1, n+1, n-1)\,.
\end{aligned}$$

Again, this is indeed a root $\forall n \leq 6$ and it can be checked to reduce to one of the 4 possible permutations proposed in [55] for $n = 2$. Furthermore, $H_{\xi^{[4,n]}} = nH_{\tilde{\delta}^{[4]}} - i(E_5 - F_5 - E_7 + F_7)$ as one would expect.

Finally, for $T^6/\mathbb{Z}_n$, one can check that $\overline{\Lambda}^{'\{6\}} = \Lambda'^2 - 2\Lambda'^3 + 2\Lambda'^5 + \Lambda'^6 - 4\Lambda'^7 - \Lambda'^8 + n(\Lambda'^7 - \Lambda'^8)$ has all desired properties. It is dual to

$$\begin{aligned}\xi^{[6,n]} &= -\alpha'_3 + \alpha'_5 - 2\alpha'_7 + n(\delta' + \alpha'_{-1} + \alpha'_0 + \alpha'_1 + \alpha'_2 + \alpha'_3 + \alpha'_4 + \alpha'_5 + \alpha'_6 + \alpha'_7) \\ &= (n, n, n, n, n, n-1, n+1, n+1, n-1, n-2, 2)\,,\end{aligned}$$

where the factor of -2 in front of α'_7 reminds us of the charge assignment $Q_3 = -2$. On the other hand, the - sign in front of α'_3 does not contradict our choice of $Q_1 = +1$, but is rather due to our Cartan-Weyl basis (5.135), in which H'_3 has a different conventional sign compared to H'_5 and H'_7. Accordingly, one obtains: $H_{\xi^{[6,n]}} = nH_{\tilde{\delta}^{[6]}} - i(E_3 - F_3 + E_5 - F_5 - 2(E_7 + F_7))$ as it should.

It is now easy to guess the general form of the shift vector for all $T^{10-q} \times T^q/\mathbb{Z}_n$ orbifolds, in which the orbifold projections are taken independently on each of the $(q/2)$ T^2 subtori (in other words, we exclude for example a $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold of T^6 for which one $\mathbb{Z}_3$ acts on the planes $\{x^5, x^6\}$ and $\{x^7, x^8\}$ and the other on the planes $\{x^7, x^8\}$ and $\{x^9, x^{10}\}$, since it contains two independent projections on the same T^2 subtorus).

By translating the tables of $\mathfrak{e}_{10}$ roots established by [115] in the physical basis, we can identify the roots which constitute class representatives of shift vectors (satisfying the conditions mentioned above) for orbifolds with various charge assignments, and build the classification represented in Tables 5.12 and 5.13. These listings deserve a few comments.

First of all, what we are really classifying are inner automorphisms of the type (5.62) with all different charges assignments (up to permutations of the shift vectors). Though some of these automorphisms allow to take a geometrical orbifold projection and descend to well-defined type IIA orbifolds, like the $T^4/\mathbb{Z}_n$ and the $T^6/\mathbb{Z}_n$ cases[14] we studied in Sections 5.5 and 5.6 for $n = 2, 3, 4, 6$, the Lefschetz fixed point formula would give a non-integer number of fixed points for most of the others. Clearly, such cases do not correspond to compactifications on geometrical orbifolds that can be made sense of in superstring theory (let alone preserve some supersymmetry). However, whether compactifications on such peculiar spaces makes sense in M-theory is, on the other hand, an open question. The invariant subalgebras and "untwisted" sectors can in any case be defined properly.

Second, we chose not to consider as different two shift vectors differing only by a permutation of $\tilde{\delta}$, but exhibiting the same universal part, for example $(3, 3, 3, 3, 3, 3, 3, 3, 2, 1)$ and $(3, 3, 3, 3, 3, 3, 3, 0, 2, 4)$.

Finally, looking at the Tables 5.12 and 5.13 in an horizontal way, one can identify series of shift vectors defining orbifold charges which appear as "subcharges" one of the others, when some Q_i's are set to zero. For example, starting from $T^8/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3$ for $q = 10$, one obtains successively $T^6/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3$, $T^4/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3$, $T^2/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3$ and $T^2/\mathbb{Z}_3$ for $q = 8, 6, 4, 2$, with shift vectors of monotonally decreasing squared lengths -8, -10, -12, -14 and -16.

Though, for evident typographical reasons, we were not able to accomodate all shift vectors related in this way on the same line, we have done so whenever possible to highlight the appearance of such families of class representatives. This explains the blank lines, whenever there was no such correspondence. An attentive study of Tables 5.12 and 5.13 shows that those families end up when a root of the serie reaches squared length 2. For example, going backwards and starting instead from $T^6/\mathbb{Z}'_3$ with a shift vector of null squared length for $q = 6$, one finds $T^2/\mathbb{Z}_6 \times T^6/\mathbb{Z}_3$ with a vector of squared length 2 for $q = 8$, but there is no $T^4/\mathbb{Z}_6 \times T^6/\mathbb{Z}_3$ for $q = 10$, since it would have to be generated by a vector of length 4, which is of course not a root.

[14]This kind of $T^6/\mathbb{Z}_n$ orbifold with charge assignment $(1, 1, -2)$ is denoted $T^4/\mathbb{Z}_n \times T^2/\mathbb{Z}_{n/2}$ in Tables 5.12 and 5.13 to distinguish it from the one with charge assignment $(1, -1, 1)$.

To extend this classification to orbifolds that are not induced by an automorphism of type (5.62), further computations are nevertheless necessary (to obtain the correct form of the universal parts). However, exactly the same methods can in principle be applied and we leave this matter for further research. When tables of roots of $\mathfrak{e}_{10}$ will be available up to higher levels in α_8, one could also study orbifolds for higher values of n. Of physical interest are perhaps values of n up to 12, which would in principle require knowledge of roots of levels up to 36.

A more speculative question is whether these orbifold-generating roots all have another physical interpretation, for example as solitonic M-theory objects with or without non-trivial fluxes, just as in [55]. A first look at the general shape of these roots in the physical basis seems to confirm this view, since the first $(10-q)$ n's remind of a $(10-q)$-brane transverse to the orbifolded torus, while the other components might be given an interpretation as fluxes through the orbifold. Indeed, both are expected to contribute to local anomaly cancellation at the orbifold fixed points. We do not have a general realization of this idea, yet, but we will describe a number of more concrete constructions in the following and discuss in particular all of the $\mathbb{Z}_2$ cases in detail, hinting at a possible interpretation of the general $\mathbb{Z}_n$ ones.

q=2

n	Orbifold	shift vector	Q_1	$\|\xi\|^2$	n	Orbifold	shift vector	Q_1	$\|\xi\|^2$
3	$T^2/\mathbb{Z}_3$	$(3,3,3,3,3,3,3,3,2,1)$	1	-4	5	$T^2/\mathbb{Z}_5$	$(5,5,5,5,5,5,5,5,4,1)$	1	-8
						$T^2/\mathbb{Z}_5'$	$(5,5,5,5,5,5,5,5,3,2)$	2	-12
4	$T^2/\mathbb{Z}_4$	$(4,4,4,4,4,4,4,4,3,1)$	1	-6	6	$T^2/\mathbb{Z}_6$	$(6,6,6,6,6,6,6,6,5,1)$	1	-10
	$T^2/\mathbb{Z}_2'$	$(4,4,4,4,4,4,4,4,2,2)$	2	-8		$T^2/\mathbb{Z}_3'$	$(6,6,6,6,6,6,6,6,4,2)$	2	-16
						$T^2/\mathbb{Z}_2''$	$(6,6,6,6,6,6,6,6,3,3)$	3	-18

q=4

n	Orbifold	shift vector	(Q_1,Q_2)	$\|\xi\|^2$
3	$T^4/\mathbb{Z}_3$	$(3,3,3,3,3,3,4,2,2,1)$	$(1,-1)$	-2
4	$T^4/\mathbb{Z}_4$	$(4,4,4,4,4,4,5,3,3,1)$	$(1,-1)$	-4
	$T^2/\mathbb{Z}_4 \times T^2/\mathbb{Z}_2$	$(4,4,4,4,4,4,5,3,2,2)$	$(1,-2)$	-6
	$T^4/\mathbb{Z}_2$	$(4,4,4,4,4,4,6,2,2,2)$	$(2,-2)$	0
	$T^2/\mathbb{Z}_2 \times T^2/\mathbb{Z}_4$	$(4,4,4,4,4,4,6,2,3,1)$	$(2,-1)$	2
5	$T^4/\mathbb{Z}_5$	$(5,5,5,5,5,5,6,4,4,1)$	$(1,-1)$	-6
	$T^4/\mathbb{Z}_5'$	$(5,5,5,5,5,5,6,4,3,2)$	$(1,-2)$	-10
	$T^4/\mathbb{Z}_5''$	$(5,5,5,5,5,5,7,3,4,1)$	$(2,-1)$	0
	$T^4/\mathbb{Z}_5'''$	$(5,5,5,5,5,5,7,3,3,2)$	$(2,-2)$	-2
6	$T^4/\mathbb{Z}_6$	$(6,6,6,6,6,6,7,5,5,1)$	$(1,-1)$	-8
	$T^2/\mathbb{Z}_3 \times T^2/\mathbb{Z}_6$	$(6,6,6,6,6,6,8,4,5,1)$	$(2,-1)$	2
	$T^2/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3$	$(6,6,6,6,6,6,7,5,4,2)$	$(1,-2)$	-14
	$T^4/\mathbb{Z}_3'$	$(6,6,6,6,6,6,8,4,4,2)$	$(2,-2)$	-8
	$T^2/\mathbb{Z}_2 \times T^2/\mathbb{Z}_3$	$(6,6,6,6,6,6,9,3,4,2)$	$(3,-2)$	2
	$T^2/\mathbb{Z}_6 \times T^2/\mathbb{Z}_2$	$(6,6,6,6,6,6,7,5,3,3)$	$(1,-3)$	-16
	$T^4/\mathbb{Z}_2'$	$(6,6,6,6,6,6,9,3,3,3)$	$(3,-3)$	0

q=6

n	Orbifold	shift vector	(Q_1,Q_2,Q_3)	$\|\xi\|^2$
3	$T^6/\mathbb{Z}_3$	$(3,3,3,3,2,4,4,2,2,1)$	$(-1,1,-1)$	0
4	$T^6/\mathbb{Z}_4$	$(4,4,4,4,3,5,5,3,3,1)$	$(-1,1,-1)$	-4
	$T^4/\mathbb{Z}_4 \times T^2/\mathbb{Z}_2$	$(4,4,4,4,3,5,5,3,2,2)$	$(1,1,-2)$	-4
	$T^2/\mathbb{Z}_4 \times T^4/\mathbb{Z}_2$	$(4,4,4,4,5,3,6,2,2,2)$	$(1,-2,2)$	2
5	$T^6/\mathbb{Z}_5$	$(5,5,5,5,4,6,6,4,4,1)$	$(-1,1,-1)$	-4
	$T^6/\mathbb{Z}_5'$	$(5,5,5,5,4,6,6,4,3,2)$	$(-1,1,-2)$	-8
	$T^6/\mathbb{Z}_5''$	$(5,5,5,5,4,6,7,3,4,1)$	$(-1,2,-1)$	2
	$T^6/\mathbb{Z}_5'''$	$(5,5,5,5,4,6,7,3,3,2)$	$(-1,2,-2)$	-2
6	$T^6/\mathbb{Z}_6$	$(6,6,6,6,5,7,7,5,5,1)$	$(-1,1,-1)$	-6
	$T^2/\mathbb{Z}_3 \times T^4/\mathbb{Z}_6$	$(6,6,6,6,4,8,7,5,5,1)$	$(-2,1,-1)$	2
	$T^4/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3$	$(6,6,6,6,5,7,7,5,4,2)$	$(-1,1,-2)$	-12
	$T^2/\mathbb{Z}_6 \times T^4/\mathbb{Z}_3$	$(6,6,6,6,5,7,8,4,4,2)$	$(-1,2,-2)$	-6
	$T^6/\mathbb{Z}_3'$	$(6,6,6,6,4,8,8,4,4,2)$	$(-2,2,-2)$	0
	$T^4/\mathbb{Z}_6 \times T^2/\mathbb{Z}_2$	$(6,6,6,6,5,7,7,5,3,3)$	$(-1,1,-3)$	-14
	$T^2/\mathbb{Z}_6 \times T^4/\mathbb{Z}_2$	$(6,6,6,6,5,7,9,3,3,3)$	$(-1,3,-3)$	2

Table 5.12: $\mathfrak{e}_{10}$ roots as class representatives of shift vectors for $\mathbb{Z}_n$ orbifolds

q=8

n	Orbifold	shift vector	$(Q_1, \ldots, Q_4)$	$\lvert\xi\rvert^2$
3	$T^8/\mathbb{Z}_3$	$(3,3,2,4,2,4,4,2,2,1)$	$(1,-1,1,-1)$	2
4	$T^8/\mathbb{Z}_4$	$(4,4,5,3,3,5,5,3,3,1)$	$(1,-1,1,-1)$	-2
	$T^6/\mathbb{Z}_4 \times T^2/\mathbb{Z}_2$	$(4,4,5,3,3,5,5,3,2,2)$	$(1,-1,1,-2)$	-2
5	$T^8/\mathbb{Z}_5$	$(5,5,6,4,4,6,6,4,4,1)$	$(1,-1,1,-1)$	-2
	$T^8/\mathbb{Z}_5'$	$(5,5,6,4,4,6,6,4,3,2)$	$(1,-1,1,-2)$	-6
	$T^8/\mathbb{Z}_5'''$	$(5,5,6,4,4,6,7,3,3,2)$	$(1,-1,2,-2)$	0
6	$T^8/\mathbb{Z}_6$	$(6,6,7,5,5,7,7,5,5,1)$	$(1,-1,1,-1)$	-4
	$T^2/\mathbb{Z}_3 \times T^6/\mathbb{Z}_6$	$(6,6,8,4,5,7,7,5,5,1)$	$(2,-1,1,-1)$	2
	$T^6/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3$	$(6,6,7,5,5,7,7,5,4,2)$	$(1,-1,1,-2)$	-10
	$T^4/\mathbb{Z}_6 \times T^4/\mathbb{Z}_3$	$(6,6,7,5,5,7,8,4,4,2)$	$(1,-1,2,-2)$	-4
	$T^2/\mathbb{Z}_6 \times T^6/\mathbb{Z}_3$	$(6,6,7,5,4,8,8,4,4,2)$	$(1,-2,2,-2)$	2
	$T^6/\mathbb{Z}_6 \times T^2/\mathbb{Z}_2$	$(6,6,7,5,5,7,7,5,3,3)$	$(1,-1,1,-3)$	-12
	$T^4/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3 \times T^2/\mathbb{Z}_2$	$(6,6,7,5,5,7,8,4,3,3)$	$(1,-1,2,-3)$	-6
	$T^2/\mathbb{Z}_6 \times T^4/\mathbb{Z}_3 \times T^2/\mathbb{Z}_2$	$(6,6,7,5,4,8,8,4,3,3)$	$(1,-2,2,-3)$	0

q=10

n	Orbifold	shift vector	$(Q_1, \ldots, Q_5)$	$\lvert\xi\rvert^2$
4	$T^{10}/\mathbb{Z}_4$	$(3,5,5,3,3,5,5,3,3,1)$	$(-1,1,-1,1,-1)$	2
	$T^8/\mathbb{Z}_4 \times T^2/\mathbb{Z}_2$	$(3,5,5,3,3,5,5,3,2,2)$	$(-1,1,-1,1,-2)$	0
5	$T^{10}/\mathbb{Z}_5$	$(4,6,6,4,4,6,6,4,4,1)$	$(-1,1,-1,1,-1)$	0
5	$T^{10}/\mathbb{Z}_5$	$(4,6,6,4,4,6,6,4,3,2)$	$(-1,1,-1,1,-2)$	-4
	$T^{10}/\mathbb{Z}_5'''$	$(4,6,6,4,4,6,7,3,3,2)$	$(-1,1,-1,2,-2)$	2
6	$T^{10}/\mathbb{Z}_6$	$(5,7,7,5,5,7,7,5,5,1)$	$(-1,1,-1,1,-1)$	-2
	$T^8/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3$	$(5,7,7,5,5,7,7,5,4,2)$	$(-1,1,-1,1,-2)$	-8
	$T^6/\mathbb{Z}_6 \times T^4/\mathbb{Z}_3$	$(5,7,7,5,5,7,8,4,4,2)$	$(-1,1,-1,2,-2)$	-2
	$T^8/\mathbb{Z}_6 \times T^2/\mathbb{Z}_2$	$(5,7,7,5,5,7,7,5,3,3)$	$(-1,1,-1,1,-3)$	-10
	$T^6/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3 \times T^2/\mathbb{Z}_2$	$(5,7,7,5,5,7,8,4,3,3)$	$(-1,1,-1,2,-3)$	-4
	$T^4/\mathbb{Z}_6 \times T^4/\mathbb{Z}_3 \times T^2/\mathbb{Z}_2$	$(5,7,7,5,4,8,8,4,3,3)$	$(-1,1,-2,2,-3)$	2

Table 5.13: $\mathfrak{e}_{10}$ roots as class representatives of shift vectors for $\mathbb{Z}_n$ orbifolds

5.9 $\mathbb{Z}_2$ orbifolds

The case of $\mathbb{Z}_2$-orbifolds is slightly degenerated and must be considered separately. In [56], the orbifold $T^{4m}/\mathbb{Z}_2$, $m = 1, 2$ and $T^{4m'+1}/\mathbb{Z}_2$, $m' = 0, 1, 2$ have been worked out, and the orbifold charges have been shown to be generated by a generic Minkowskian brane required [88, 249] for anomaly cancellation, living in the transverse space.

In this section, we will show how to treat all $T^q/\mathbb{Z}_2$ orbifolds for $q \in \{1, .., 9\}$. In Section 5.9.1, it will be shown how the algebraic results for invariant subalgebras in [55], which we henceforth refer to as $\mathbb{Z}_2$ orbifolds of the *first kind*, are recovered as a particular case in the more general framework of Section 5.8.

In Section 5.9.2, we investigate in detail the $q = 2, 3, 6, 7$ cases, or $\mathbb{Z}_2$ orbifolds of *the second kind*, which have not been considered in [55]. Let us stress that by *orbifolds of the second kind* we mean the purely algebraic implementation of the $\mathbb{Z}_2$ projection in the U-duality algebra. Then, we will extract from the construction of Section 5.8.3 the roots of $\mathfrak{e}_{10}$ defining the representatives of classes of shift vectors for these orbifolds of M-theory and give a tentative physical interpretation.

Concretely, let i, j, k be transverse spacelike coordinates and A, B, C coordinates on the orbifold, under a $\mathbb{Z}_2$-transformation, $11D$-supergravity and fields have charge assignment

$$\begin{aligned}
&(\text{all}) &:\quad & g_{ij} \to +g_{ij}\,, \qquad g_{iA} \to -g_{iA}\,, \qquad g_{AB} \to +g_{AB}\,,\\
&\begin{pmatrix}\text{odd}\\ \text{even}\end{pmatrix} &:\quad & C_{ijk} \to \mp C_{ijk}\,, \quad C_{ijA} \to \pm C_{ijA}\,, \quad C_{iAB} \to \mp C_{iAB}\,, \quad C_{ABC} \to \pm C_{ABC}\,,
\end{aligned}$$

odd and *even* referring to the dimension of the orbifold torus.

In contrast to the $\mathbb{Z}_{n>2}$ case, where the inner automorphisms generating the orbifold charges were pure $SO(r)$ rotations, the action of a $\mathbb{Z}_2$-orbifold can be regarded as an element of the larger $O(r) = \mathbb{Z}_2 \times SO(r)$. This distinctive feature of $\mathbb{Z}_2$-orbifold can be ascribed to the fact that while even orbifolds act as central symmetries and may be viewed as π-rotations, hence falling in $O^+(r)$ (positive determinant elements connected to the identity), odd orbifolds behave as mirror symmetries, and thus fall in $O^-(r)$. Concretely, negative determinant orthogonal transformations will contain, in the $\mathfrak{e}_{10}$ language, a rotation in the α_8 direction, namely $\text{Ad}(e^{\pi\,(E_8 - F_8)})$ or $\text{Ad}(e^{i\pi\, H_8})$, which, in this framework, behaves as a mirror symmetry.

The even case can be dealt with in a general fashion by applying the following theorem:

Theorem 5.9.1 *Let $T^q/\mathbb{Z}_2$ be a $\mathbb{Z}_2$ toroidal orbifold of $\mathfrak{e}_r$, for $q \in \{1, \ldots, 9\}$, $r \in \{q+1, \ldots, 10\}$. Let q be either $2m$ or $2m+1$. Given a set of (possibly non-simple) roots $\Delta_{\mathbb{Z}_2} = \{\beta_{(p)}\}_{p=1,..,m}$ satisfying $(\beta_{(p)}|\beta_{(l)}) = c_p \delta_{p,l}$, with $c_p \leqslant 2$ and provided the orbifold acts on the U-duality algebra $\mathfrak{g}^U$ with the operator $\mathcal{U}_{2m}^{\mathbb{Z}_2} \in G^U$ defined, according to expression (5.62), by*

$$\mathcal{U}_q^{\mathbb{Z}_2} = \prod_{p=1}^{m} \text{Ad}(e^{\pi\,(E_{\beta_{(p)}} - F_{\beta_{(p)}})})\,, \tag{5.136}$$

then, the orbifold action decomposes on the root-subspace $\mathfrak{g}^U_\alpha \subset \mathfrak{g}^U$ as

$$\mathcal{U}_q^{\mathbb{Z}_2} \cdot \mathfrak{g}^U_\alpha \equiv \prod_{p=1}^{m} \text{Ad}(e^{i\pi\, H_{\beta_{(p)}}}) \cdot \mathfrak{g}^U_\alpha = (-1)^{\sum_{p=1}^m (\beta_{(p)}|\alpha)}\, \mathfrak{g}^U_\alpha\,, \quad \forall \alpha \in \Delta(\mathfrak{g}^U)\,. \tag{5.137}$$

If the $\mathbb{Z}_2$-orbifold is restricted to extend in successive directions, starting from x^{10} downwards, it can be shown that for any root $\alpha = \sum_{j=-1}^{8} k^j \alpha_j \in \Delta(\mathfrak{g}^U)$, expression (5.137) assumes the simple form

$$\mathcal{U}_{2m}^{\mathbb{Z}_2} \cdot \mathfrak{g}^U_\alpha \equiv \text{Ad}(e^{i\pi\, \sum_{i=1}^m H_{9-2i}}) \cdot \mathfrak{g}^U_\alpha = \begin{cases} (-1)^{k^6}\, \mathfrak{g}^U_\alpha\,, & \text{for } m = 1 \\ (-1)^{k^{8-2m} + k^8}\, \mathfrak{g}^U_\alpha\,, & \text{for } 2 \leqslant m \leqslant 4 \end{cases}. \tag{5.138}$$

for even orbifold. For odd ones, we note the appearance of the mirror operator we mentioned above

$$\mathcal{U}^{\mathbb{Z}_2}_{2m+1} \cdot \mathfrak{g}^U_\alpha \equiv \mathrm{A}d\left(e^{i\pi\left(H_8+H_6+\sum_{i=1}^m H_{8-2i}\right)}\right) \cdot \mathfrak{g}^U_\alpha = (-1)^{k^{7-2m}}\,\mathfrak{g}^U_\alpha\,,\quad \text{for } m \geqslant 0\ . \tag{5.139}$$

Following Section 5.8.1, we are free to recast the orbifold charges resulting from expressions (5.138) and (5.139), by resorting to a shift vector $\xi^{[q,2]}$ such that:

$$\mathcal{U}^{\mathbb{Z}_2}_q \cdot \mathfrak{g}^U_\alpha = (-1)^{(\xi^{[q,2]}|\alpha)}\,\mathfrak{g}^U_\alpha\,,\quad \forall q = 1,\ldots,9\ .$$

The subalgebra invariant under $T^q/\mathbb{Z}_2$ is now reformulated as a KMA with root system

$$\Delta_{\mathrm{inv}} = \left\{\alpha \in \Delta(\mathfrak{g}^U)\,|(\xi^{[q,2]}|\alpha) = 0 \bmod 2\right\}\ . \tag{5.140}$$

This is definition of the $\mathbb{Z}_2$-charge used in [56, 55].

5.9.1 $\mathbb{Z}_2$ orbifolds of M-theory of the first kind

The orbifolds of M-theory with $q = 1, 4, 5, 8, 9$ have already been studied in [55], and a possible choice of shift vectors has been shown to be, in these cases, dual to prime isotropic roots, identified in [56] as Minkowskian branes. As such, they were interpreted as representatives of the 16 transverse M-branes stacked at the 2^q orbifold fixed points and required for anomaly cancellation in the corresponding M-theory orbifolds [249, 88].

In this section about $\mathbb{Z}_2$ orbifolds of the first kind, we will show how to rederive the results of [55] about shift vectors and invariant subalgebra, from the more general perspective we have developed in Section 5.8 by resorting to the Kac-Peterson formalism. After this cross check, we will generalize this construction to the $q = 2, 3, 6, 7$ cases, which have not been considered so far, and show how the roots $\xi_2^{[q,2]}$ are related to D-branes and involved in the cancellation of tadpoles due to O-planes of certain type 0B' orientifolds. For this purpose, we start by summarizing in Table 5.14 the shift vectors for the $q = 1, 4, 5, 8, 9$ cases found in [55], specifying in addition the $SL(10, \mathbb{R})$-representation they belong to. Next we will show how the results of Table 5.14 for

q	$\xi^{[q,2]}$	physical basis	Dynkin labels
1	$\alpha_{(-1)^2 0^4 1^6 2^8 3^{10} 4^{12} 5^{14} 6^9 7^6 8^7}$	$(2,2,2,2,2,2,2,2,4,1)$	$[200000001]$
4	$\alpha_{(-1)^2 0^4 1^6 2^8 3^{10} 4^{12} 5^{13} 6^8 7^5 8^6}$	$(2,2,2,2,2,2,1,1,3,1)$	$[100000100]$
5	$\alpha_{(-1)^2 0^4 1^6 2^8 3^{10} 4^{11} 5^{12} 6^8 7^4 8^5}$	$(2,2,2,2,2,1,1,1,1,1)$	$[000010000]$
8	$\alpha_{(-1)^2 0^4 1^5 2^6 3^7 4^8 5^9 6^6 7^3 8^4}$	$(2,2,1,1,1,1,1,1,1,1)$	$[010000000]$
9	δ	$(0,1,1,1,1,1,1,1,1,1)$	$[000000001]$

Table 5.14: Physical class representatives for $T^{10-q} \times T^q/\mathbb{Z}_2$ orbifolds of M-theory of the first kind

even orbifolds can be retrieved as special cases of the general solutions computed in Section 5.8.1.

The root of E_{10} relevant to the $q = 4$ orbifold can be determined as a special case of $T^4/\mathbb{Z}_n$ shift vectors, namely:

$$\xi^{[4,2]} = 2(\Lambda^5 - \Lambda^6 - \Lambda^8) + \Lambda^{\{4\}} = 2\tilde{\delta}^{[4]} - \alpha_5 + \alpha_7\,,$$

which coincides with the results of Table 5.14. This choice of weight is far from unique, but is the lowest height one corresponding to a root of E_{10} (given that $\Lambda^{\{4\}}$ is not a root). Likewise $\xi^{[8,2]}$ can in principle be deduced from the generic weight $\Lambda^{\{8\}}$ determining the $T^8/\mathbb{Z}_n$ charges.

Shift vectors for odd orbifolds of Table 5.14 can also be recast in a similar form, even though they do not generalize to $n > 2$. We can indeed rewrite:

$$\xi^{[1,2]} = 2(2\Lambda^7 - \Lambda^6) - \Lambda^7, \qquad \xi^{[5,2]} = 2(-\Lambda^8) + \Lambda^3, \qquad \xi^{[9,2]} = 2(-\Lambda^{-1}) + \Lambda^{-1}.$$

The last four E_{10} roots listed in Table 5.14 were identified in [56] as, respectively, Minkowskian Kaluza-Klein monopole (KK7M), M5-brane, M2-brane and Kaluza-Klein particle (KKp), with spatial extension in the transverse directions and have been presented in Table 5.1. The first root is the mysterious M-theory lift of the type IIA D8-brane, denoted as KK9M in this paper. In the language of Table 5.5, these roots correspond to the representation weights $(A \otimes \widetilde{K})_{(99)[1\cdots 9]}$, $D_{(10)[1\cdots 6\,10]}$, $B_{(1)[2\cdots 5]}$, $A_{(12)}$, $\widetilde{K}_{(2)[3\cdots 10]}$.

Furthermore, it has been shown in [249, 88] that the consistency of $\mathbb{Z}_2$ orbifolds of M-theory of the first kind requires the presence at the fixed points of appropriate solitonic configurations. For $T^{q=5,8}/\mathbb{Z}_2$, one needs respectively 16 M5-branes/M2-branes to ensure anomaly cancellation. In the case of $T^9/\mathbb{Z}_2$, 16 units of Kaluza-Klein momentum are needed, while Kaluza-Klein monopoles with a total Chern class of the KK gauge bundle amounting to 16 is required in the case of $T^4/\mathbb{Z}_2$.

For $q = 4, 5, 8, 9$, the transverse Minkowskian objects of Table 5.14 having all required properties were interpreted as generic representatives of these non-perturbative objects. However, their total multiplicity/charge cannot be inferred from the shift vectors. It was proposed in [55] that these numbers could be deduced from an algebraic point of view from the embedding of $\mathfrak{g}_{\rm inv}$ into a real form of the conjectured heterotic U-duality symmetry $\mathfrak{de}_{18}$. However, this idea seems to be difficult to generalize to the new examples treated in the present paper and we will not discuss it further.

For $q = 1$, the analysis is a bit more subtle, and needs to be carried out in type IIA language. To understand the significance of the shift vector in this case, it is convenient to reduce from M-theory on $T^8 \times S^1/\mathbb{Z}_2 \times S^1$ to type IIA theory on $T^8 \times S^1/\Omega I_1$, where I_1 is the space parity-operator acting on the S^1 as the original $\mathbb{Z}_2$ inversion, while Ω is the world-sheet parity operator. In this setup, the appropriate shift vector is $\xi_\sigma^{[1,2]} = (2,2,2,2,2,2,2,2,1,4)$, which can be interpreted as a KK9M of M-theory with mass:

$$M_{\rm KK9M} = M_P^{-9} V^{-1} e^{\langle \xi | H_m \rangle} = M_P^{12} R_1 \cdots R_8 R_{10}^3 \,.$$

Upon reduction to type IIA theory, we reexpress it in string units by setting

$$R_{10} = g_A M_s^{-1}\,, \quad \text{and} \quad M_P = g_A^{-1/3} M_s\,, \tag{5.141}$$

and take the limit $M_P R_{10} \to 0$:

$$M_{\rm KK9M} \to M_{\rm D8} = \frac{M_s^9}{g_A} R_1 \cdots R_8 \,.$$

The resulting mass is that of a D8-brane of type IIA theory. The appearance of this object reflects the need to align 8 D8-branes on each of the two O8 planes at both ends of the orbifold interval to cancel locally the -8 units of D8-brane charge carried by each O8 planes, a setup known as type I' theory.

The chain of invariant subalgebras $\mathfrak{g}_{\rm inv}$ in Table 5.15 is obtained by keeping only those root spaces of $\mathfrak{g}^U$ which have eigenvalue $+1$ under the action of $\mathcal{U}_{2m}^{\mathbb{Z}_2}$ (5.138) or $\mathcal{U}_{2m+1}^{\mathbb{Z}_2}$ (5.139). We can use the set of invariant roots (5.140) to build the Dynkin diagram of $\mathfrak{g}_{\rm inv}$, but this is not enough to determine root multiplicities in $D = 1$. In the hyperbolic case indeed, one will need to know the dimension of the root spaces $\mathfrak{g}_\alpha^U \subset \mathfrak{g}^U = \mathfrak{e}_{10|10}$ which are invariant under the actions (5.138) or (5.139) to determine the size of $\mathfrak{g}_{\rm inv}$. We will come back to this issue at the end of this section.

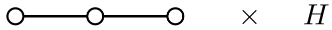
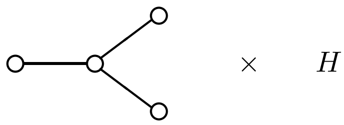
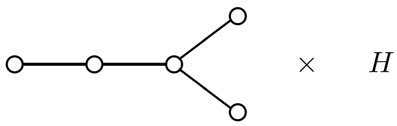
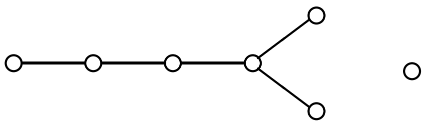
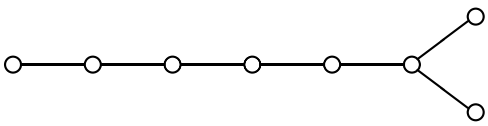
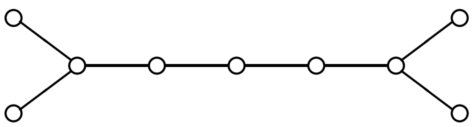
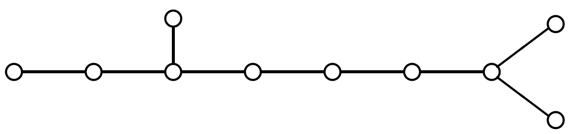
D	(Π_0, ϕ)	$\mathfrak{g}_{\text{inv}}$	$\sigma(\mathfrak{g}_{\text{inv}})$
8		$\mathfrak{so}(2,2) \oplus \mathfrak{so}(1,1)$	3
7		$\mathfrak{so}(3,3) \oplus \mathfrak{so}(1,1)$	4
6		$\mathfrak{so}(4,4) \oplus \mathfrak{so}(1,1)$	5
5		$\mathfrak{so}(5,5) \oplus \mathfrak{so}(1,1)$	6
4		$\mathfrak{so}(6,6) \oplus \mathfrak{sl}(2,\mathbb{R})$	7
3		$\mathfrak{so}(8,8)$	8
2		$\hat{\mathfrak{d}}_{8\|10}$	10
1		$\mathfrak{de}_{10\|10}$	10

Table 5.15: The split subalgebras $\mathfrak{g}_{\text{inv}}$ for $\mathbb{Z}_2$ orbifolds of the first kind

This construction of the root system leads, $\forall q = 1, 4, 5, 8, 9$, to a unique chain of invariant subalgebras, depicted in Table 5.15. Thus, we verify that the statement made in [55] for the hyperbolic case is true for all compactifications of type $T^{11-(D+q)} \times T^q/\mathbb{Z}_2$ with $q = 1, 4, 5, 8, 9$. There, this isomorphism was ascribed to the fact that, in $D = 1$, the shift vectors (5.14) are all prime isotropic and thus Weyl-equivalent to one another. The mathematical origin of this fact lies in the general method developed by Kac-Peterson explained in Section 5.8.1, which states that equivalence classes of shift vectors related by Weyl transformation and/or translation by n times any weight lattice vector lead to isomorphic fixed-point subalgebras.

Since a Weyl reflection in $\mathfrak{g}^U$ generates a U-duality transformation in M-theory and its low-energy supergravity, this isomorphism seems to indicate that all such orbifolds are dual in M-theory, as pointed out in [88, 220]. In fact, if one had chosen to reduce $\mathbb{Z}_2$ orbifolds of the first kind on a toroidal direction for q odd, and on an orbifolded direction for q even, one would have realized that they are all part of the serie of mutually T-dual orientifolds (a T-duality on x^i is denoted by $\mathcal{T}_i$):

$$\begin{aligned} &\text{type IIB on } T^9/\Omega \xrightarrow{\mathcal{T}_9} \text{type IIA on } T^9/\Omega I_1 \xrightarrow{\mathcal{T}_8} \text{type IIB on } T^9/(-1)^{F_L}\Omega I_2 \xrightarrow{\mathcal{T}_7} \\ &\quad \to \text{ type IIA on } T^9/(-1)^{F_L}\Omega I_3 \xrightarrow{\mathcal{T}_6} \text{type IIB on } T^9/\Omega I_4 \xrightarrow{\mathcal{T}_8} \dots \end{aligned} \tag{5.142}$$

where I_r denotes the inversion of the last r space-time coordinates, while Ω is as usual the world-sheet parity. The *space-time* left-moving fermions number $(-1)^{F_L}$ appear modulo 4 in these dualities.

The reality properties of $\mathfrak{g}_{\text{inv}}$ are easy to determine. Since the original balance between Weyl and Borel generators is preserved by the orbifold projection, the non-abelian part of $\mathfrak{g}_{\text{inv}}$ remains split. In $D = 8, \dots, 5$, the abelian $\mathfrak{so}(1,1)$ factor in $\mathfrak{g}_{\text{inv}}$ is generated by the non-compact element $H^{[q]}$ which also appears in $T^q/\mathbb{Z}_{n>2}$ orbifolds. For $q = 4$, it is, for instance, given by $H^{[4]} = H_8 - H_4$ in $D = 6$ and $2H_3 + 4H_4 + 3H_5 + 2H_6 + H_7$ in $D = 5$, as detailed in Section 5.5, and is enhanced, in $D = 4$, to the $\mathfrak{sl}(2,\mathbb{R})$ factor appearing in Table 5.15 when $H^{[4]}$ becomes the root $\gamma = \alpha_{23^24^35^36^278} \in \Delta_+(\mathfrak{e}_7)$. The procedure is similar for $q = 1, 5, 8, 9$, for different combinations $H^{[q]}$ and positive roots γ.

The root multiplicities in $\mathfrak{g}_{\text{inv}}$ is only relevant to the two cases $D = 2, 1$, for which the root multiplicities are inherited from $\mathfrak{e}_9$ and $\mathfrak{e}_{10}$. For $\mathfrak{g}^U = \mathfrak{e}_9$ we have $\mathfrak{g}_{\text{inv}} = \hat{\mathfrak{d}}_8$, since $\delta_{\mathfrak{g}_{\text{inv}}} = \delta$, and since δ and $\delta_{\hat{\mathfrak{d}}_8}$ both have multiplicity 8.

In $D = 1$, the story is different. In [55], it has been shown that $\mathfrak{g}_{\text{inv}}$ contains a subalgebra of type $\mathfrak{de}_{10}$. The authors have performed a low-level decomposition of both $\mathfrak{g}_{\text{inv}}$ and $\mathfrak{de}_{10}$. For a generic over-extended algebra $\mathfrak{g}^{\wedge\wedge}$, such a decomposition with respect to its null root $\delta_{G^{\wedge\wedge}}$ is given by $(\mathfrak{g}^{\wedge\wedge})_{[k]} \doteq \bigoplus_{(\alpha,\delta_{G^{\wedge\wedge}})=k} \mathfrak{g}^{\wedge\wedge}_\alpha$. In particular, they define: $(\mathfrak{g}_{\text{inv}})_{[k]} \doteq \mathfrak{g}_{\text{inv}} \cap (\mathfrak{e}_{10})_{[k]}$ and show that:

$$(\mathfrak{g}_{\text{inv}})_{[1]} \simeq (\mathfrak{de}_{10})_{[1]} \,, \qquad (\mathfrak{g}_{\text{inv}})_{[2]} \supset (\mathfrak{de}_{10})_{[2]} \,.$$

The first equality is a reformulation of $\mathfrak{g}_{\text{inv}} = \hat{\mathfrak{d}}_8$ for $\mathfrak{g}^U = \mathfrak{e}_9$. The second result comes from the fact that the orbifold projection selects certain preserved root subspaces without affecting their dimension. This feature is similar to what we have observed in the case of $\mathbb{Z}_n$ orbifolds, where the original root multiplicities are restored after modding out the Borcherds or KM algebras appearing in $D = 1$ by their centres and derivations.

5.9.2 $\mathbb{Z}_2$ orbifolds of M-theory of the second kind and orientifolds with magnetic fluxes

Let us first recall that by *orbifolds of the second kind* we mean the purely algebraic implementation of T^{10}/I_q, $q = 2, 3, 6, 7$, in the U-duality algebra. In this case, the connection to orbifolds of M-theory will be shown to be more subtle than in Section 5.9.1. Indeed, since the algebraic orbifolding procedure does not discriminate between two theories with the same bosonic untwisted sectors and different fermionic degrees of freedom, there are in principle several candidate orbifolds on the M-theory side to which these orbifolds of the second kind could be related.

The first (naive) candidate one can consider is to take M-theory directly on T^{10}/I_q, $q = 2, 3, 6, 7$. Then following the analysis of Section 5.9.1, a reduction of such orbifolds to type II string theory would result in a chain of dualities similar to expression (5.142), with the important difference that the $(-1)^{F_L}$ operator now appears in the opposite places. It is well known that the spectrum of such theories cannot be supersymmetric. Referring to the chain of dualities (5.142) with the required extra factor of the left-moving fermionic number operator, one observes that such theories do not come from a consistent truncation of type IIB string theory, since $(-1)^{F_L}\Omega$ is not a symmetry thereof, and all of them are therefore unstable.

A more promising candidate is M-theory on $T^q/\{(-1)^F S, \mathbb{Z}_2\}$, where $(-1)^F$ is now the total *space-time* fermion number and S represents a π shift in the M-theory direction. In contrast to the preceding case, these orbifolds are expected to be dual to orientifolds of type 0 theory, which are non-supersymmetric but are nonetheless believed to be stable, so that tadpole cancellation makes sense in such setups.

Let us work out, in these type 0 cases, a chain of dualities similar to expression (5.142). To start with, we review the argument stating that M-theory on $S^1/(-1)^F S$ is equivalent to the non-supersymmetric type 0A string theory in the small radius limit [27].

Considering the reduction of M-theory on $S^1 \times S^1/(-1)^F S$ to type IIA string theory on $S^1/(-1)^F S$, one can determine the twisted sector of this orbifold (with no fixed point) and perform a flip on $\{x^9, x^{10}\}$ to obtain the spectrum of M-theory on $S^1/(-1)^F S$. At the level of massless string states, all fermions are projected out from the untwisted sector and there appears a twisted sector that doubles the RR sector and adds a NSNS tachyon, leading to type 0A string theory in 10 dimensions. Interestingly, type 0 string theories have more types of $\mathbb{Z}_2$ symmetries and thus more consistent truncations. In $D = 10$, type 0A theory is symmetric under the action of Ω, while type 0B theory is symmetric under Ω, $\Omega(-1)^{f_L}$ and $\Omega(-1)^{F_L}$, where f_L and F_L are respectively the *world-sheet* and *space-time* left-moving fermion numbers. Furthermore, their compactified versions on T^{10} each belong to a serie of orientifold theories similar to (5.142). Among these four chains of theories, one turns out to be tachyon-free, the chain descending from type 0B string theory on $\Omega(-1)^{f_L}$. Let us concentrate on this family of orientifolds and show that the M-theory orbifolds of the second kind can all be seen to reduce to an orientifold from this serie in the small radius limit.

In order to see this, one can mimick the procedure used for (5.142) and reduce on a toroidal direction for q odd, and on an orbifolded direction for q even. The untwisted sectors of our orbifolds then turns out to correspond to those of an orientifold projection by $\Omega(-1)^{f_L} I_{(q)}$, resp. $\Omega(-1)^{f_R} I_{(q-1)}$, on type 0A string theory. A projection by $\Omega(-1)^{f_{L/R}} I_{(q)}$ has the following effects in type 0A string theory: it eliminates the tachyon and half of the doubled RR sector, the remaining half being distributed over the untwisted and twisted sectors of M-theory on $S^1/(-1)^F S$. Consequently, one expects to obtain theories that belong to the following chain of dual non-supersymmetric orientifolds:

$$\begin{aligned}\text{type 0B on } T^9/(-1)^{f_L}\Omega &\xrightarrow{\mathcal{T}_9} \text{type 0A on } T^9/(-1)^{f_R}\Omega I_1 \xrightarrow{\mathcal{T}_8} \text{type 0B on } T^9/(-1)^{f_R}\Omega I_2 \\ &\xrightarrow{\mathcal{T}_7} \text{ type 0A on } T^9/(-1)^{f_L}\Omega I_3 \xrightarrow{\mathcal{T}_6} \text{ type 0B on } T^9/(-1)^{f_L}\Omega I_4 \xrightarrow{\mathcal{T}_8} \ldots\end{aligned} \tag{5.143}$$

where $(-1)^{f_L}$ and $(-1)^{f_R}$ again appear modulo 4 in these dualities[15]. Complications might however arise at the twisted sector level when reducing to type 0A string theory on an orbifolded direction, since one should take into account a possible non-commutativity between the small radius limit and the orbifold limit. We will come back to this point later.

Instead, we first want to remind the reader that, as was shown in [46], type 0B string on $(-1)^{f_L}\Omega$ can be made into a consistent non-supersymmetric string theory by cancelling the tadpoles from the two RR 10-forms through the addition of 32 pairs of D9- and D9'-branes for a total $U(32)$ gauge symmetry. This setup is usually called type 0'. There is also a NSNS

[15]From the point of view of M-theory, these dualities sometimes exchange the untwisted and twisted sectors under $(-1)^F S$

dilaton tadpole, but this does not necessarily render the theory inconsistent. Rather, it leads to a non-trivial cosmological constant through the Fischler-Susskind mechanism [116, 117]. It was also shown in [46] that there is no force between the D9- and D9'-branes and that twisted sector open strings stretched between them lead to twisted massless fermions in the $\mathbf{496} \oplus \overline{\mathbf{496}}$ representation of $U(32)$. Even though the latter are chiral Majorana-Weyl fermions, it was shown in [215, 216] that a generalized Green-Schwarz mechanism ensures anomaly cancellation.

To characterize the twisted sectors of such orientifolds of type 0' string theory algebraicly, we will again use the equivalence classes of shift vectors that generate the orbifolds on the U-duality group. The simplest elements of these classes which are also roots give the set of real roots in Table 5.16.

q	$\tilde{\xi}^{[q,2]}$	physical basis	generator
2	α_7	$(0,0,0,0,0,0,0,0,1,-1)$	$\mathcal{K}_{[9\,10]}$
3	α_8	$(0,0,0,0,0,0,0,1,1,1)$	$\mathcal{Z}_{[89\,10]}$
6	$\alpha_{34^25^36^278^2}$	$(0,0,0,0,1,1,1,1,1,1)$	$\widetilde{\mathcal{Z}}_{[5\cdots10]}$
7	$\alpha_{1^22^33^44^55^66^47^28^3}$	$(0,0,2,1,1,1,1,1,1,1)$	$\widetilde{\mathcal{K}}_{(3)[3\cdots10]}$

Table 5.16: Universal shift vectors for $\mathbb{Z}_2$ orbifolds of the second kind

As is obvious from the second and third column, all such roots are in $\Delta_+(\mathfrak{e}_8)$ and correspond to instantons completely wrapping the orbifolded torus. Since they are purely $\mathfrak{e}_8$ roots, we do not expect them to convey information on the string theory twisted sectors. As such, this set of shift vectors does not lend itself to an interesting physical interpretation, but gives however certain algebraic informations. All these roots being in the same orbit of the Weyl subgroup of E_8, the resulting invariant subalgebras are again isomorphic (when existing) $\forall q = 2,3,6,7$. We list the invariant subalgebras for M-theory on $T^2/\{(-1)^F S, \mathbb{Z}_2\}$ together with their Dynkin diagrams in Table 5.17. The same invariant subalgebras appear for all values of q, but of course start to make sense only in lower dimensions.

The invariant subalgebras are not simple for $D \geq 3$ and all of them contain at least one $\mathfrak{sl}(2,\mathbb{R})$ factor with simple root $\tilde{\xi}^{[q,2]}$. When an abelian factor is present, it coincides with the non-compact Cartan element $H^{[q]}$ encountered in $T^q/\mathbb{Z}_{n>2}$ orbifolds. Furthermore, in contrast to the connected $\hat{\mathfrak{d}}_8$ diagram obtained for $q = 1,4,5,8,9$, the invariant subalgebra for $q = 2,3,6,7$ is given in $D = 2$ by an affine central product, as in the $T^{2,4,6}/\mathbb{Z}_{n>2}$ cases treated before. In $D = 1$, the invariant subalgebra is the following quotient of the KMA, whose Dynkin diagram is drawn in Table 5.17:

$$\mathfrak{g}_{\text{inv}} = {}^2\mathcal{KM}_{11|12}/\{\mathfrak{z}, d_1\}\,,$$

where $\mathfrak{z} = c_{\mathfrak{e}_7} - c_{\hat{\mathfrak{a}}_1}$. As in the $T^6/\mathbb{Z}_{3,4}$ cases, ${}^2\mathcal{KM}_{11}$ has a singular Cartan matrix with similar properties.

We will now show that certain equivalent shift vectors can be interpreted as configurations of D9 and D9'-branes cancelling R-R tadpoles in a type 0' string theory orientifold. This can be achieved by adding an appropriate weight lattice vector $\Lambda^{[q]}$ to $\tilde{\xi}^{[q,2]}$ that do not change the scalar products modulo 2. It should be chosen so that $\Lambda^{[q]} + \tilde{\xi}^{[q,2]}$ is a root, and gives insight on the possible M-theory lift of such constructions. More precisely, we want to convince the reader that certain choices of shift vectors generating M-theory orbifolds of the second kind can be seen as representing either magnetized D9-branes or their image D9'-branes in some type 0' theory with orientifold planes. Such branes carry fluxes that contribute to the overall D$(9-q)$-brane charge for even q and D$(10-q)$-brane charge for odd q, but not to the higher ones.

Let us first study the example of a $T^3/\mathbb{Z}_2 \times S^1/(-1)^F S$ orbifold of M-theory. Following

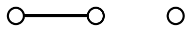
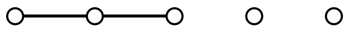
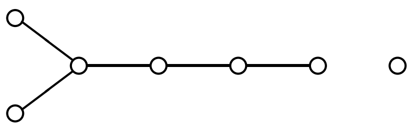
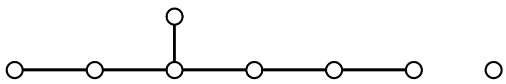
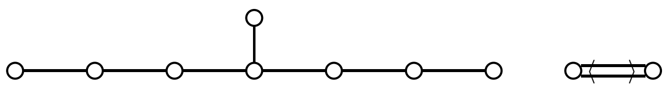
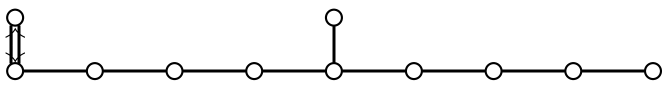
D	(Π_0, ϕ)	$\mathfrak{g}_{\text{inv}}$	$\sigma(\mathfrak{g}_{\text{inv}})$
8	 $\times$ H	$\mathfrak{sl}(2,\mathbb{R})^{\oplus 2} \oplus \mathfrak{so}(1,1)$	3
7	 $\times$ H	$\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ $\oplus\, \mathfrak{so}(1,1)$	4
6		$\mathfrak{sl}(4,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})^{\oplus 2}$	5
5		$\mathfrak{sl}(6,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$	6
4		$\mathfrak{so}(6,6) \oplus \mathfrak{sl}(2,\mathbb{R})$	7
3		$\mathfrak{e}_{7\|7} \oplus \mathfrak{sl}(2,\mathbb{R})$	8
2		$\hat{\mathfrak{e}}_{7\|9} \oplus \hat{\mathfrak{a}}_{1\|3}$	10
1		${}^2\mathcal{KM}_{11\|12}$	10

Table 5.17: The split subalgebras $\mathfrak{g}_{\text{inv}}$ for $\mathbb{Z}_2$ orbifolds of the second kind

the above construction, it should reduce in the limit $M_P R_{10} \to 0$ to type 0A string theory on $T^6 \times T^3/(-1)^{f_L}\Omega I_3$, which is T-dual to type 0B string theory on $T^7 \times T^2/(-1)^{f_R}\Omega I_2$. We summarize these dualities in the diagram below:

$$\begin{array}{l} \text{M-theory on } T^6 \times T^3/\mathbb{Z}_2 \times S^1/(-1)^F S \\ \qquad\qquad\quad \big\downarrow {\scriptstyle M_P R_{10} \to 0} \\ \text{type 0A on } T^6 \times (S^1 \times T^2)/(-1)^{f_L}\Omega I_3 \xrightarrow{\mathcal{T}_7} \text{type 0B on } T^6 \times S^1 \times T^2/(-1)^{f_R}\Omega I_2 \end{array}$$

In this last type 0B orientifold, there will be one orientifold plane carrying -4 units of D7-

and D7'-brane charge at each of the four orbifold fixed points. Suppose that we consider N pairs of magnetized D9- and D9'-branes carrying fluxes in the orbifolded plane (x'^8, x^9). This system induces two Chern-Simons couplings on the world-volume of the space-time filling branes:

$$\frac{M_s^{10}}{2(2\pi)^9}\int_{\mathbb{R}\times T^9} C_8 \wedge 2\pi\alpha' \mathrm{Tr}(F_2) = \frac{M_s^8}{(2\pi)^7}\int_{\mathbb{R}\times T^7} C_8 \cdot \frac{1}{2\pi}\int_{T^2/\mathbb{Z}_2} \mathrm{Tr}(F_2)\,,$$

and a similar expression involving C_8'. The quantized fluxes can then be chosen in such a way that the resulting total positive D7- and D7'-brane charges cancel the negative charges from the orientifold planes and ensure tadpole cancellation. Note that these charges are determined by the first Chern class c_1 of the $U(N)$ gauge bundle.

We can use an analogy with the supersymmetric case, where the system of O7-planes and magnetized D9-branes in a $T^8 \times T^2/(-1)^{F_L}\Omega I_2$ type IIB orientifold has a well-known T-dual equivalent [47, 191] built from D8-branes at angle with O8-planes in a $T^9 \times S^1/(-1)^{F_L}\Omega I_1$ type IIA orientifold, in which the flux is replaced by an angle χ in the following way:

$$2\pi\alpha' F_{89} = \frac{c_1}{N}\frac{\mathbb{I}_N}{M_s^2 R_8' R_9} \xrightarrow{\mathcal{T}_8} \cot(\chi) = \frac{c_1}{N}\frac{R_8}{R_9} \tag{5.144}$$

and where the type IIB radius[16] is $R_8' = 1/M_s^2 R_8$. In fact, the appearance of D7-brane charges in the absence of D9-brane ones on the type IIB side can be understood, in the dual setup, as the addition (resp. cancellation) of the charges due to the tilted D8-branes to those of their image branes with respect to the orientifold O8-plane. The image D8-brane indeed exhibits a different angle, being characterized by wrapping numbers $(c_1, -N)$ instead of (c_1, N) around the directions along the orbifold. In our non-supersymmetric case, however, one should keep in mind that the image brane of a D9-brane under $\Omega(-1)^{f_R}$ is a D9'-brane.

We will now show that the appropriate shift vector encodes not only explicit information on the presence of type 0' pairs of D9- and D9'-branes, but also on the tilting of their dual type 0A D8- and D8'-branes with respect to the O8-planes. One can then deduce the presence of fluxes from the angle χ.

To understand how this comes about, we note that both string theory D9- and D8-branes descend from the (conjectured) KK9M soliton of M-theory described by the $\mathfrak{e}_{10}$ roots that are permutations of $\xi = ((2)^6, 1, 2, 2, 4)$ in the following way:

$$M_{\mathrm{KK9M}} = M_P^{-9} V^{-1} e^{\langle \xi | H_R \rangle} = M_P^{12} R_1 \cdots R_6 R_8 R_9 R_{10}^3\,.$$

Following the chain of dualities (5.149), we successively obtain the D8- and D9-brane mass formulae:

$$M_P^{12} R_1 \cdots R_6 R_8 R_9 R_{10}^3 \xrightarrow{M_P R_{10}\to 0} M_{\mathrm{D8}} = \frac{M_s^9}{g_A} R_1 \cdots R_6 R_8 R_9 \xrightarrow{\mathcal{T}_7} M_{\mathrm{D9}} = \frac{M_s^{10}}{g_B} R_1 \cdots R_6 R_7' R_8 R_9 \tag{5.145}$$

Now, we select one definite shift vector $\tilde{\xi}^{[3,2]}$ from all equivalent ones, which has the particularity to correspond like ξ to a root of level 4. It is obtained from a permutation of the root $\tilde{\xi}^{[3,2]}$ that describes an orbifold in the directions $(x^7; x^8; x^9)$, namely $\tilde{\xi}_\sigma^{[3,2]} = ((0)^6, (1)^3, 0)$, as:

$$\xi^{[3,2]} = 2(\Lambda^6 - \Lambda^7 - \Lambda^8) - \tilde{\xi}_\sigma^{[3,2]} = ((2)^6, 3, 3, 1, 2)\,. \tag{5.146}$$

[16]Note that both the signs of the orientifold plane charges and this angle χ would also be sensitive to the presence of a quantized Kalb-Ramond background flux $\int B_{89}\, dx'^8 dx^9$, but we neglect this possibility here, since it would be the sign of a tilted geometry in the $(x^8; x^9)$ plane on the type A side (non-trivial complex structure of the torus).

Let us first blindly compute the ensuing mass formula, reduce it on x^{10} and T-dualize it on x^7:

$$M_P^{12} R_1 \cdots R_6 (R_7 R_8)^2 R_{10} \xrightarrow{M_P R_{10} \to 0} \frac{M_s^{11}}{g_A^3} R_1 \cdots R_6 (R_7 R_8)^2 \xrightarrow{\mathcal{T}_7} M_{\mathrm{D9}} = \frac{M_s^{10}}{g_B^3} R_1 \cdots R_7' R_8^2 \,. \tag{5.147}$$

On the type B side, R_8 and R_9 form the pair of orbifolded directions. Comparing with (5.145), we immediately see that we will have a D9-brane if: $R_9 \propto R_8/g_B^2$. As hinted above, we need to find an angle in the dual type 0A setup to identify the flux. In this perspective, we perform a further T-duality along x^8 that brings us to a $S^1/(-1)^{f_R}\Omega I_1$ orientifold of type 0A string theory in which the type 0B flux is mapped to an angle between the O8-plane, and the D8-brane obtained from (5.147) as

$$M_{\mathrm{D9}} = \frac{M_s^{10}}{g_B^3} R_1 \cdots R_7' R_8^2 \xrightarrow{\mathcal{T}_8} \frac{M_s^9}{g_A^3} R_1 \cdots R_6 R_7' R_8' \,. \tag{5.148}$$

This implies that there is a dual relation to $R_9 \propto R_8/g_B^2$ on the 0A side that has the same form:

$$R_9 \propto R_8/g_B^2 \xrightarrow{\mathcal{T}_8} R_9 \propto R_8'/g_A^2 \,.$$

Indeed, plugging back this dual relation in (5.148) clearly identifies the corresponding object with a D8-brane of type A string theory. Interestingly, (5.144) implies that there can be a non-right angle between the D8-brane and the O8-plane with $\cot(\chi) = \frac{c_1}{N}\frac{R_8}{R_9} \propto \frac{c_1}{N} g_A^2$. Unfortunately, our purely algebraic formalism does not allow us to see the individual values of c_1, N and the proportionality constant, but they must be physically chosen so that: $\frac{1}{2\pi}\int_{T^2/\mathbb{Z}_2} \mathrm{Tr}(F_{89}) dx^8 dx^9 = c_1 = 16$. This is similar to the case of [55], where the type of brane necessary for anomaly cancellation was obtained from the shift vector, but not their number.

Let us then study the case of a $T^2/\{(-1)^F S, \mathbb{Z}_2\}$ orbifold of M-theory (where the shift operator S only acts on x^{10}). We want to show that it gives an alternative M-theory lift of the same type 0' $T^2/(-1)^{f_R}\Omega I_2$ orientifold that we have just studied. Before we discuss the brane configuration, it is necessary to discuss the case of M-theory on $S^1/\{(-1)^F S, \mathbb{Z}_2\}$ to understand the effect of taking both orbifold projections on the same circle. We first remark that the orientifold group has four elements: $\{\mathbb{1}, (-1)^F S, I_1, (-1)^F I_1'\}$, where $I_1' = S I_1$. While I_1 is a reflexion of the coordinate x^{10} with respect to $x^{10} = 0$, I_1' is a reflection of x^{10} with respect to $x^{10} = \pi/2$. In particular, I_1 has two fixed points at $x^{10} = 0$ and π, while I_1' has two fixed points at $x^{10} = \pi/2$ and $3\pi/2$, and S has no fixed point. Consequently, the fundamental domain is an interval $[0, \pi/2]$ and there are three types of twisted sectors, the usual bosonic closed string twisted sector of $(-1)^F S$ that leads to a type 0 spectrum and two open strings twisted sectors sitting at each pair of fixed points. What is not known, however, is the precise resulting gauge symmetry and twisted spectrum. There is a dual picture of the same model, where one first uses the S symmetry to reduce the circle by half, and then considers the projection by I_1 which replaces the circle by the interval. This second picture resembles the non-supersymmetric heterotic orbifold of M-theory discussed in [112], except that these authors did not include a closed string twisted sector, which hopefully helps stabilizing the non-supersymmetric theory. We now conjecture that M-theory on $T^2/\{(-1)^F S, \mathbb{Z}_2\}$ is the strong coupling limit of the $S^1/(-1)^{f_R}\Omega I_1$ orientifold of type 0A string theory and is thus T-dual to the $T^3/\mathbb{Z}_2 \times S^1/(-1)^F S$ orbifold of M-theory through a double T-duality, modulo the appropriate breaking of gauge groups by Wilson lines.

Let us be more concrete. We need to reduce to type 0A string theory on an orbifolded direction, then T-dualize to type 0B on a normal toroidal direction to reach a type 0' $T^2/(-1)^{f_R}\Omega I_2$

orientifold, as in the following mapping:

$$\begin{array}{c} \text{M-theory on } T^8 \times T^2/\{(-1)^F S, \mathbb{Z}_2\} \\ \Big\downarrow M_P R_{10} \to 0 \\ \text{type 0A on } T^8 \times S^1/(-1)^{f_R}\Omega I_1 \xrightarrow{\mathcal{T}_8} \text{type 0B on } T^7 \times T^2/(-1)^{f_R}\Omega I_2 \end{array} \tag{5.149}$$

First, we have to select one definite shift vector from all equivalent ones. We take the one that has the particularity to descend from the more general $T^2/\mathbb{Z}_n$ serie of shift vectors of the form $n\tilde{\delta}^{[2]} - \alpha_7$, namely:

$$\xi^{[2,2]} = 2(\Lambda^7 - \Lambda^8) + \Lambda^{\{2\}} = 2\tilde{\delta}^{[2]} - \tilde{\xi}^{[2,2]} = ((2)^8, 1, 1)\,, \tag{5.150}$$

which lead to the mass formulae:

$$M_P^{-9} V^{-1} e^{\langle \xi^{[2,2]}, H_R \rangle} = M_P^9 R_1 \cdots R_8 \xrightarrow{M_P R_{10} \to 0} \frac{M_s^9}{g_A^3} R_1 \cdots R_8 \xrightarrow{\mathcal{T}_8} \frac{M_s^{10}}{g_B^3} R_1 \cdots R_7 R_8'^2\,. \tag{5.151}$$

We immediately see that we end up with the same objects as in (5.147) and (5.148) and the analysis of fluxes and angles is completely parallel. In a sense, the presence of these two different M-theory lifts of the same string orientifold reflects the equivalence between T-dualizing $S^1/\Omega\mathbb{Z}_2$ in the transverse space and T-dualizing $T^3/\Omega\mathbb{Z}_2$ along an orbifold direction. We will use a similar property later to relate $\xi^{[6,2]}$ and $\xi^{[7,2]}$.

We can now turn to the $T^6/\{(-1)^F S, \mathbb{Z}_2\}$ orbifold of M-theory. We will study this case along the same line as $T^2/\mathbb{Z}_2$, first reducing on an orbifolded direction to a type 0A theory orientifold, then T-dualizing along a transverse direction to a type 0B orientifold:

$$\begin{array}{c} \text{M-theory on } T^4 \times T^6/\{(-1)^F S, \mathbb{Z}_2\} \\ \Big\downarrow M_P R_{10} \to 0 \\ \text{type 0A on } T^4 \times T^5/(-1)^{f_R}\Omega I_5 \xrightarrow{\mathcal{T}_4} \text{type 0B on } T^3 \times T^6/(-1)^{f_R}\Omega I_6 \end{array} \tag{5.152}$$

We will again have a system of N pairs of magnetized D9- and D9'-branes, now contributing to cancel the -1/4 units of negative D3-brane charge carried by each of the 64 O3-planes. This can be achieved by the Chern-Simons coupling:

$$\frac{M_s^4}{(2\pi)^3} \int_{\mathbb{R}\times T^3} C_4 \cdot \frac{1}{(2\pi)^3} \int_{T^6/\mathbb{Z}_2} \mathrm{Tr}(F_2 \wedge F_2 \wedge F_2)\,.$$

In the case of a factorizable metric, we can separate $T^6/\mathbb{Z}_2$ into 3 $T^2/\mathbb{Z}_2$ sub-orbifolds, and only F_{56}, F_{78} and F_{49} yield non-trivial fluxes. Instead of c_1 and N, we now introduce for each pair of coordinates $(x^i; x^j)$ of the T^2's pairs of quantized numbers denoted by (m^a_{ij}, n^a_{ij}) [191]. The index a here numbers various stacks of N_a pairs of branes, with different fluxes. In the dual 0A picture, the m^a_{ij} and n^a_{ij}'s give wrapping numbers around the directions parallel, respectively perpendicular, to the O6-planes and a distinguishes between wrappings of branes around different homology cycles. With an appropriate normalization of cohomology bases on the homology cycles, one obtains:

$$\frac{1}{(2\pi)^3}\mathrm{Tr}\left(\int_{T^2/\mathbb{Z}_2} F_{56}\, dx^5 dx^6 \int_{T^2/\mathbb{Z}_2} F_{78}\, dx^7 dx^8 \int_{T^2/\mathbb{Z}_2} F_{49}\, dx'^4 dx^9\right) = \sum_a N_a m^a_{56} m^a_{78} m^a_{49} = 16$$

On the other hand, Chern-Simons couplings to higher forms such as C_5, C_7 and C_9 are determined by expressions which also include n^a_{ij} factors. For example, the D9-charge is related to

$\sum_a N_a n^a_{56} n^a_{78} n^a_{49}$. The wrapping numbers should then be chosen in a way that all those other total charges cancel. There are in principle several ways to achieve this, but it is not our main focus, so we will not give a specific example here (see [8] for concrete realizations in the supersymmetric case). Rather, following the $T^6/\mathbb{Z}_2$ case above, one wishes to study the magnetized D9-brane action given by our algebraic method, deduce from it that certain pairs of radii are related and then perform a triple T-duality along $(x^4; x^6; x^7)$ to exchange the fluxes against tilting angles between O6-planes and pairs of D6-branes and their image D6'-branes.

Keeping this framework in mind, we first recall the choice of shift vector that comes from the general $T^6/\mathbb{Z}_n$ orbifold serie. It is given by:

$$\xi^{[6,2]} = 2(\Lambda^7 - \Lambda^8) + \tilde{\xi}^{[6,2]} = 2\tilde{\delta}^{[6]} - \alpha_3 + \alpha_5 - \alpha_7 = ((2)^4, 1, 3, 3, 1, 1, 1)\,. \tag{5.153}$$

where $\tilde{\xi}^{[6,2]}$ differs from its expression $\xi^{[6,n]}$ for $n = 2$ given in Section 5.8.3 because the charge Q_3 is now -1 instead of -2. Let us again follow the dualities (5.152) to see how the D9-brane is expressed in this formalism:

$$\frac{e^{\langle \xi^{[6,2]}|H_R\rangle}}{M_P V} = M_P^9 R_1 \cdot\cdot R_4 (R_6 R_7)^2 \xrightarrow{M_P R_{10}\to 0} \frac{M_s^9}{g_A^3} R_1 \cdot\cdot R_4 (R_6 R_7)^2 \xrightarrow{\mathcal{T}_4} \frac{M_s^{10}}{g_B^3} R_1 \cdot\cdot R_3 (R_4' R_6 R_7)^2\,.$$

This can match the action of a D9-brane if $R_5 \propto R_6$, $R_7 \propto R_8$ and $R_9 \propto R_4'/g_B^2$. On the type A side, this again means that $R_9 \propto R_4/g_A^2$, and one verifies easily that $\mathcal{T}_4$ indeed maps the D9-brane to a D8-brane extended along all directions except x^4. This D8-brane is tilted with respect to the O4-plane in the $(x^4; x^9)$-plane by an angle $\cot(\chi^a_{49}) = \frac{m^a_{49} R_4}{n^a_{49} R_9}$ and still carries magnetic fluxes F_{56} and F_{78}. T-dualizing further along x^6 and x^7 leads to a D6-brane extended in the hypersurface along $(x^0; x^1; x^2; x^3; x^5; x^8; x^9)$ with mass:

$$\frac{M_s^7}{g_A} R_1 \cdots R_4 R_6 R_7 \sim \frac{M_s^7}{g_A^3} R_1 \cdots R_3 R_5 R_8 R_9\,.$$

Then, we can interpret this brane as one of the N_a D6-branes exhibiting two additional non-right angles with respect to the orientifold O6-plane, given by $\cot(\chi^a_{56}) = \frac{m^a_{56} R_6}{n^a_{56} R_5}$ and $\cot(\chi^a_{78}) = \frac{m^a_{78} R_7}{n^a_{78} R_8}$. It is of course understood in this discussion that the appropriate image Dp'-branes are always present.

Finally, we still wish to study $T^7/\{(-1)^F S, \mathbb{Z}_2\}$ orbifolds of M-theory. For this purpose, we use the permutation of $\tilde{\xi}^{[7,2]}$ describing an orbifold in $(x^3; \ldots; x^9)$ given by $\tilde{\xi}^{[7,2]}_\sigma = (0, 0, (1)^7, 2)$ in the following fashion:

$$\xi^{[7,2]} = 2(\Lambda^2 - \Lambda^3 + \Lambda^5 - 2\Lambda^8) + \tilde{\xi}^{[7,2]}_\sigma = (2, 2, 3, 3, 1, 3, 3, 1, 1, 2).$$

This time, we follow the successive mappings

$$\begin{array}{c} \text{M-theory on } T^2 \times T^8/(-1)^F S \\ \big\downarrow{\scriptstyle M_P R_{10}\to 0} \\ \text{type 0A on } \times T^2 \times T^7/(-1)^{f_L}\Omega I_7 \xrightarrow{\mathcal{T}_3} \text{type 0B on } T^3 \times T^6/(-1)^{f_R}\Omega I_6 \end{array}$$

leading to the mass formulae:

$$M_P^{12} R_1 R_2 (R_3 R_4 R_6 R_7)^2 R_{10} \xrightarrow{M_P R_{10}} \frac{M_s^{11}}{g_A^3} R_1 R_2 (R_3 R_4 R_6 R_7)^2 \xrightarrow{\mathcal{T}_3} \frac{M_s^{10}}{g_B^3} R_1 \cdots R_3' (R_4 R_6 R_7)^2\,.$$

and we obtain again the same type 0B $T^6/(-1)^{f_R}\Omega I_6$ orientifold as above, while tilting angles in the dual type IIA picture can again be obtained by $\mathcal{T}_{467}$.

q	$\xi^{[q,2]}$	physical basis	Dynkin label	$\|\Lambda\|^2$
2	$\alpha_{(-1)^2 0^4 1^6 2^8 3^{10} 4^{12} 5^{14} 6^{10} 7^5 8^6}$	$(2,2,2,2,2,2,2,2,1,1)$	[000000010]	-2
3	$\alpha_{(-1)^2 0^4 1^6 2^8 3^{10} 4^{12} 5^{15} 6^{11} 7^5 8^7}$	$(2,2,2,2,2,2,3,3,1,2)$	[010000001]	-2
6	$\alpha_{(-1)^2 0^4 1^6 2^8 3^9 4^{12} 5^{15} 6^{10} 7^5 8^6}$	$(2,2,2,2,1,3,3,1,1,1)$	[010001000]	2
7	$\alpha_{(-1)^2 0^4 1^7 2^{10} 3^{11} 4^{14} 5^{17} 6^{11} 7^5 8^7}$	$(2,2,3,3,1,3,3,1,1,2)$	[000100100]	2

Table 5.18: Physical class representatives for $T^{10-q} \times T^q/\mathbb{Z}_2$ orbifolds of M-theory of the second kind

Overall, we have a fairly homogeneous approach to these four different orbifolds of M-theory and it should not be too surprising that their untwisted sectors build the same algebra. We finally summarize the shift vectors we used for physical interpretation in Table 5.18. It is remarkable that these roots are found at level 6 and 7 in α_8, showing again that a knowledge of the $\mathfrak{e}_{10}$ root space at high levels is essential for the algebraic study of M-theory orbifolds.

Another fact worth mentioning is that our $\mathbb{Z}_2$ shift vectors either have norm 2 or -2, in contrast to the null shift vectors of Section 5.9.1. This lightlike characteristic has been proposed in [56, 55] to be a general algebraic property characterizing Minkowskian branes in M-theory. Similarly, these authors associated instantons with real roots of $\mathfrak{e}_{10}$, viewed as extensions of roots of $\mathfrak{e}_8$, that all have norm 2. However, we have just shown that Minkowskian objects can just as well have norm 2, or -2, and perhaps almost any. We suggest that the deciding factor is the threshold rather than the norm (at least for objects coupling to forms, forgetting for a while the exceptional case of Kaluza-Klein particles that have negative threshold, when they are instantonic and null threshold, when they are Minkowskian). Indeed, instantonic objects have threshold 0, while Minkowskian ones have threshold 1. This approach is compatible with the point of view of [87], as explained in Section 5.2.5, as well as with the results of this subsection. Some higher threshold roots also appear in Table 5.12 and 5.13, however, but we leave their interpretation for further investigation.

5.10 Shift vectors for $\mathbb{Z}_n$ orbifolds: an interpretative prospect

Now that we have an apparently coherent framework to treat $\mathbb{Z}_2$ M-theory orbifolds, it is tempting to try to generalize it to all $\mathbb{Z}_n$ orbifolds. To understand how this could be done, it is instructive to look at Tables 5.12 and 5.13. As mentioned at the end of Section 5.8.3, one notices that shift vectors for $T^q/\mathbb{Z}_n$ orbifolds can typically be grouped in series, for successive values of q and n. As an illustration, we give one such serie (i.e. relating orbifolds with all charges ± 1) in the following table:

$n\backslash q$	2	4	6	8	10
2	$((2)^8,1,1)$	$((2)^6,3,1,1,1)$	$((2)^4,1,3,3,1,1,1)$	$(2,2,3,1,1,3,3,1,1,1)$	/
3	$((3)^8,2,1)$	$((3)^6,4,2,2,1)$	$((3)^4,2,4,4,2,2,1)$	$(3,3,4,2,2,4,4,2,2,1)$	$(2,4,4,2,2,4,4,2,2,1)$
4	$((4)^8,3,1)$	$((4)^6,5,3,3,1)$	$((4)^4,3,5,5,3,3,1)$	$(4,4,5,3,3,5,5,3,3,1)$	$(3,5,5,3,3,5,5,3,3,1)$
5	$((5)^8,4,1)$	$((5)^6,6,4,4,1)$	$((5)^4,4,6,6,4,4,1)$	$(5,5,6,4,4,6,6,4,4,1)$	$(4,6,6,4,4,6,6,4,4,1)$
6	$((6)^8,5,1)$	$((6)^6,7,5,5,1)$	$((6)^4,5,7,7,5,5,1)$	$(6,6,7,5,5,7,7,5,5,1)$	$(5,7,7,5,5,7,7,5,5,1)$

From this table, it should be immediately apparent that typical shift vectors for $T^q/\mathbb{Z}_n$

orbifolds, with $q \in 2\mathbb{N}$ are given by (some permutation of):

$$\begin{aligned} \xi &= n\tilde{\delta} + \sum_{i=1}^{q/2} (-1)^{q/2-i}\, p_i\, \alpha_{7-q+2i} \\ &= ((n)^{10-q}, n+p_1, n-p_1, n-p_2, n+p_2, \ldots, n-p_{q/2}, p_{q/2}) \end{aligned}$$

and have a threshold bigger or equal to 1 since $1 \leq q_i \leq n-1$, $\forall i = 1, \ldots, q/2$. In analogy with the $\mathbb{Z}_2$ orientifold cases, it is tempting to think of the "average" value $((n)^9, 0)$ as spacetime-filling branes, and of the deviations $q_i\alpha_{7-q+2i}$ as fluxes in successive pairs of (orbifolded) dimensions. Of course, the fluxes are only directly interpretable as such after the reduction to string theory. In the $\mathbb{Z}_2$ examples, they appeared because an M-theory orbifold turns into a string theory orientifold with open strings twisted sectors exhibiting non-abelian Chan-Paton factors. This allowed us to invoke Chern-Simons couplings of the form:

$$\int C_{10-q} \cdot \int \mathrm{Tr}(F^{q/2}) \tag{5.154}$$

on the world-volume of the space-filling branes that participate to tadpoles cancellation at the orbifold fixed points. Geometrically speaking, the more orbifolded directions, the more non-trivial fluxes can be switched on, producing higher non-zero Chern numbers that reflect the increasingly complex topology in the presence of several conifold singularities at each fixed point. A further research direction is to determine which kind of flux could appear in which $\mathbb{Z}_n$ orbifolds.

In any case, one should not forget that the orbifolded directions in the string theory limit are not exactly the same as in the original M-theory orbifold, so that a bit of caution is required when trying to interpret the shift vector directly, without going through a chain of dualities leading to a better-known string theory soliton.

Our proposal is to regard the mass formulae associated to these shift vectors as M-theory lifts of the resulting string theory brane configurations, that are somehow necessary for the M-theory orbifolds to be well-defined, in a sense which remains to be understood.

It also remains unclear how the change of average value of the components of the shift vector from 2 to n determines the fact that we have a higher order orbifold. Intuitively, it should reflect the presence of more twisted sectors, but is a priori not related to the different number of fixed points.

All these questions are of course of primary interest to obtain non-trivial physical information from our algebraic toolkit and we will pursue them in forthcoming research projects. They will be addressed in future publications.

5.11 Conclusion

In this paper, we have aimed at developing a rigorous and general algebraic procedure to study orbifolds of supergravity theories using their U-duality symmetry. We were particularly interested in the $\mathfrak{e}_{11-D|11-D}$ serie of real split U-duality algebras for $D = 1, \ldots, 8$. Essentially, the procedure can be decomposed in the following successive steps. First, one constructs a finite order non-Cartan preserving inner automorphism describing the orbifold action in the complexified algebra $\mathfrak{e}_{11-D}$. This nth-order rotation automorphism reproduces the correct $\mathbb{Z}_n$-charges of the physical states of the theory, when using the "duality" mapping relating supergravity fields and directions in the coset $\mathfrak{e}_{11-D|11-D}/\mathfrak{k}(\mathfrak{e}_{11-D|11-D})$ (in the symmetric gauge). Next, one derives the complexified invariant subalgebra satisfied by the null charge sector and fixes its real properties by taking its fixed point subalgebra under the restricted conjugation. One then moves to an eigenbasis, on which the orbifold action takes the form of a Cartan-preserving (or chief) inner automorphism, and computes, in terms of weights, the classes of shift vectors reproducing the expected orbifold charges for all root spaces of $\mathfrak{e}_{11-D}$. In $D = 1$, one uses the invariance modulo n to show that every such class contains a root of $\mathfrak{e}_{10}$, which can be used as the class representative. In a number of cases, these roots can be identified with Minkowskian objects of M-theory or of the lower-dimensional string theories, and interpreted as brane configurations necessary for anomaly cancellation in the corresponding orbifold/orientifold setups.

In fact, for a given $T^q/\mathbb{Z}_n$ orbifold, the first two steps only have to be carried out explicitly once in $\mathfrak{e}_{q+1}$ for the compactification space $S^1 \times T^q/\mathbb{Z}_n$, and need not be repeated for all $T^p \times T^q/\mathbb{Z}_n$. Rather, one can deduce in which way the Dynkin diagram of the invariant subalgebra will get extended upon further compactifications. This is relatively straightforward until $D = 3$, but requires some more care in $D = 2, 1$, when the U-duality algebra becomes infinite-dimensional. In $\mathfrak{e}_{10}$, in particular, a complete determination of the root system of the invariant subalgebra requires in principle to look for all invariant generators. This could in theory be done, provided we know the full decomposition of $\mathfrak{e}_{10}$ in representations of $\mathfrak{sl}(10, \mathbb{R})$. However, one of the conclusions of our analysis is that once we understand the structure of $\mathfrak{g}_{\text{inv}}$ at low-level, its complete root system can be inferred from the general structure of Borcherds algebras.

By doing so, however, one realizes that there are three qualitatively different possible situations from which all cases can be inferred. The determining factor is the invariant subalgebra in $D = 3$. If this subalgebra of $\mathfrak{e}_{8|8}$ is simple, its extension in $\mathfrak{e}_{10|10}$ is hyperbolic and non-degenerate. This happens for $T^q/\mathbb{Z}_2$ for $q = 1, 4, 5, 8, 9$, as already shown in [55] by alternative methods. If it is on the other hand semi-simple, we obtain, in $D = 2$, what we called an affine central product. It denotes a product of the affinization of all simple factors present in $D = 3$, in which the respective centres and derivations of all factors are identified. Descending to $D = 1$, all affine factors reconnect through α_{-1} in a simple Dynkin diagram, leading to a degenerate hyperbolic Kac-Moody algebra, but without its natural centre(s) and derivation(s). This is the case for all remaining $\mathbb{Z}_2$ orbifolds, as well as for $T^6/\mathbb{Z}_n$ orbifolds with $n = 3, 4$. Finally, if an abelian factor is present in $\mathfrak{e}_8$, its affinization in $\mathfrak{e}_9$, $\hat{\mathfrak{u}}(1)$, turns into all multiples of an imaginary root in $\mathfrak{e}_{10}$, which also connects through α_{-1} to the main diagram, thus leading to a Borcherds algebra with one isotropic simple root. Although it was conceptually clear to mathematicians that Borcherds algebras can emerge as fixed-point subalgebras of Kac-Moody algebras under automorphisms, we found here several explicit constructions, demonstrating how this comes about in examples of a kind that does not seem to appear in the mathematical literature.

In the first case, the multiplicity of invariant roots is inherited from $\mathfrak{e}_{10}$, in the other two cases, however, great care should be taken in understanding how the original multiplicities split between different root spaces. In fact, the Borcherds/indefinite KM algebras appearing in these cases provide first examples of a splitting of multiplicities of the original KMA into multiplicities of several roots of its fixed point subalgebra. This is strictly speaking the case only for the algebras as specified by their Dynkin diagram, but one should keep in mind that the quotient by its possible derivations suppresses the operators that could differentiate between these roots, and recombines them into root spaces of the original dimension, albeit with a certain redistribution of the generators. In fact, it is likely that a computation of the root multiplicities by an appropriate Kac-Weyl formula for GKMA based on the root system of the Dynkin diagram would predict slightly smaller root spaces than those of the fixed-point subalgebra that are obtained from our method. However, it is not absolutely clear what is the right procedure to compute root multiplicities in GKMA. This is a still largely open question in pure mathematics, on which our method will hopefully shed some light.

Along the way, we also explicitly showed, in the $T^4/\mathbb{Z}_n$ case, how to go from our completely real basis for $\mathfrak{g}_{\text{inv}}$, described by a fixed-point subalgebra under the restricted conjugation, to the standard basis of its real form, obtained from the Cartan decomposition. This is especially interesting in the affine case, where we obtained the relation between the two affine parameters and their associate derivations.

Even though the present paper was focused on the breakings of U-duality symmetries, it is clear that, in another perspective, the same method can in principle be applied to obtain the known classification of (symmetric) breaking patterns of the $E_8 \times E_8$ gauge symmetry of heterotic string theory (or any other gauge symmetry) by orbifold projections. Indeed, our result in $D = 3$ for breakings of $\mathfrak{e}_8$ can be found in the tables of [163, 164], where they are derived from the Kac-Peterson method using chief inner automorphisms. Reciprocally, one might wonder why we did not use the Kac-Peterson method to study U-duality symmetry, too. It is certainly a beautiful and simple technique, very well suited to classify all possible non-isomorphic symmetry breakings of one group by various orbifold actions. However, calculating with $\mathbb{Z}_n$-rotation automorphisms instead of Cartan-preserving ones has a number of advantages when dealing with U-duality symmetries. In the Kac-Peterson method, one first fixes n, then lists all shift vectors satisfying the condition $(\Lambda, \theta_G) \leq n$ of Section 5.8, which allows to obtain all non-isomorphic breakings. In the end, however, one has sometimes to resort to different techniques to associate these breakings with a certain orbifold with determined dimension and charges.

Here, we adopt a quite opposite philosophy, by resorting to non-Cartan preserving inner automorphisms with a clear geometrical interpretation. In this perspective, one starts by fixing the dimension and charges of the orbifold and then computes the corresponding symmetry breaking, which allows to discriminate easily between a degenerate finite order rotation and an effective one. Only then do we reexpress this automorphism in an eigenbasis of the orbifold action, in which it takes the form of a chief inner automorphism, and compute the class of associated shift vectors. Doing so, we can unambiguously assign a particular class of shift vectors to a definite orbifold projection in space-time. Note that such shift vectors will typically not satisfy $(\Lambda, \theta_G) \leq n$, so that a further change of basis is required to relate them to their conjugate shift vector in the Kac-Peterson formalism (we have shown in Section 5.8 how to perform this change of basis explicitly). However, this process may obscure the number of orbifolded dimensions and the charge assignment on the Kac-Peterson side.

Furthermore, another reason for not resorting to the Kac-Peterson method is that we are not so much interested in all possible breakings of one particular group, say E_8, as in determining the fixed-point subalgebras for the whole E_r serie. Consequently, we can concentrate on the $T^q/\mathbb{Z}_n$ orbifold action in E_{q+1} and then extend the result to the whole serie without too many additional computations, since the orbifold rotation acts trivially on the additional compactified dimension and the natural geometrical interpretation of the $SL(r, \mathbb{R}) \in E_r$ generators has been preserved. On the other hand, the change of basis necessary to obtain a shift vector satisfying the Kac-Peterson condition can be completely different in E_r compared to E_{r-1}. Accordingly, starting from such a shift vector for E_{r-1}, there is no obvious way to obtain its extension describing the same orbifold in E_r. Finally, and much more important to us is the fact that there is no known way to extend the Kac-Peterson method to the infinite-dimensional case.

The above discussion has concentrated on the part of this work where the invariant subalgebras of $\mathfrak{e}_{11-D|11-D}$ under an nth-order inner automorphism were derived. In $D \geq 3$, these describe the residual U-duality symmetry and bosonic spectrum of supergravity theories compactified on orbifolds and map to the massless bosonic spectrum of the untwisted sector of orbifolded string theories. In such cases, these results have been known for a long time. They are however new in $D = 2, 1$, which was the main focus of this research project. In particular, the $D = 1$ case is very interesting, since the hyperbolic U-duality symmetry encountered there is expected to contain non-perturbative information, as well. Indeed, the specific class representatives of shift vectors we find correspond to higher level roots of $\mathfrak{e}_{10}$ which have no direct interpretation as supergravity fields. It is thus tempting to try to relate them to non-classical effects in M-theory which might give us information on the twisted sector of orbifolds/orientifolds of the descendant string theories.

Let us now discuss this more physical interpretative aspect inspired from the work of [55], where the shift vectors for a restricted class of $\mathbb{Z}_2$ orbifolds of M-theory were shown to reproduce the mass formulae of Minkowskian branes, which turned out to be the correct objects to be placed at each orbifold fixed point to ensure anomaly/tadpole cancellation. We have extended this analysis to incorporate other $\mathbb{Z}_2$ orbifolds of M-theory, which are non-supersymmetric and should be considered in bosonic M-theory. They have the particularity to break the infinite U-duality algebra to indefinite KMAs. These orbifolds reduce to $T^2/(-1)^{f_R}\Omega\mathbb{Z}_2$ and $T^6/(-1)^{f_R}\Omega\mathbb{Z}_2$ orientifolds of the type 0B string theory in which pairs of magnetized D9- and D9'-branes are used to cancel the O7- (resp. O3-)plane charges. They are part of a chain of dual orientifolds starting from type 0B string theory on $(-1)^{f_L}\Omega$, a tachyon-free theory believed to be well-defined, usually referred to as type 0' string theory. We have then shown that the $\mathfrak{e}_{10}$ roots playing the rôle of class representatives of shift vectors in these cases can be interpreted as such space-time filling D9-branes carrying the appropriate configuration of magnetic fluxes. This identification could in turn serve as a proposal for M-theory lifts of such type 0B orientifolds, as generated by certain exotic objects corresponding to $\mathfrak{e}_{10}$ roots that are not in $\mathfrak{e}_9$. Finally, these type IIB setups have an alternative reading in the T-dual type IIA pictures where the magnetic fluxes appear as tilting angles between O8- (resp. O6-)planes and D8- (resp. D6-)branes and their image branes, our analysis providing an algebraic characterization of this tilting angle.

As for $\mathbb{Z}_{n\geq 3}$ case, even though we have treated only a few examples explicitly, we have noticed that their associated shift vectors fall into series of roots of $\mathfrak{e}_{10}$, for successive values of q and n, with remarkable regularity. This has provided us with a facilitated procedure for constructing shift vectors for any $T^q/\mathbb{Z}_n$ orbifold which acts separately on each of the $(q/2)$ T^2 subtori. These roots of level $3n$ are classified in Tables 5.12 and 5.13. Despite the remarkably regular structure of such roots, it is not completely clear how to extract information on the correct anomaly/tadpole-cancelling brane configurations of the corresponding orbifolds. In particular, the components of the shift vectors transverse to the orbifold increase monotonously with n, so that their interpretation requires novel ideas. However, it is clearly of interest to generalize the identification of such brane constructions for $\mathbb{Z}_n$-shift vectors with $n > 2$, and to understand their possible relation to twisted sectors and/or fluxes present in the related string orbifolds. Hopefully, this can be done in a systematic manner, reproducing what is known about string theory orbifolds/orientifolds and leading to predictive results about less-known types of M-theory constructions.

Another future direction of research would consist in investigating more complicated orbifold setups in our algebraic framework, in which, for instance, several projections of various orders are acting on the same directions. This could possibly lead to new interesting classes of GKMAs. In general, however, not only one, but two or more shift vectors will be necessary to generate such orbifolds and should from a physical perspective be interpreted separately. This will hopefully open the door to working out the physical identification of yet a larger part of the $\mathfrak{e}_{10}$ root system, and constitute another step in the understanding of the precise relation between M-theory and $\mathfrak{e}_{10}$.

5.12 Appendix 1: Highest roots, weights and the Matrix R

i) The matrix R: herebelow, we give the expression of the matrix R used in Section 5.2.1 to define the root lattice metric g_ε (5.11) in the physical basis:

$$R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/3 & -2/3 & 1/3 \end{pmatrix} .$$

ii) Highest roots of the exceptional E_r chain: We list the highest roots of the finite Lie algebras of the chain $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \ldots \subset \mathfrak{a}_4 \subset \mathfrak{d}_5 \subset \mathfrak{e}_6 \subset \mathfrak{e}_7 \subset \mathfrak{e}_8$, appearing throughout this article:

$$\begin{aligned} \theta_{A_1} &= \alpha_8 , \\ \theta_{A_i} &= \alpha_{8-i} + .. + \alpha_7 , \quad i = 2, 3 \\ \theta_{A_4} &= \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 , \\ \theta_{D_5} &= \alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8 , \\ \theta_{E_6} &= \alpha_3 + 2\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + 2\alpha_8 , \\ \theta_{E_7} &= \alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + 2\alpha_8 , \\ \theta_{E_8} &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 . \end{aligned}$$

ii) Fundamental weights of $\mathfrak{e}_{10}$: The expression, on the set of simple roots, of the fundamental weights of $\mathfrak{e}_{10}$ defined by $(\Lambda^i|\alpha_j) = \delta^i_j$ for $i, j = -1, 0, 1, .., 8$ is obtained by inverting $\Lambda^i = (A(\mathfrak{e}_{10})^{-1})^{ij}\alpha_j$. In the physical basis, these weights have the particularly simple expression:

			$\lvert\Lambda\rvert^2$				$\lvert\Lambda\rvert^2$
$-\Lambda^{-1}$	=	$(0,1,1,1,1,1,1,1,1,1)$	0	$-\Lambda^4$	=	$(5,5,5,5,5,5,6,6,6,6)$	-30
$-\Lambda^0$	=	$(1,1,2,2,2,2,2,2,2,2)$	-2	$-\Lambda^5$	=	$(6,6,6,6,6,6,6,7,7,7)$	-42
$-\Lambda^1$	=	$(2,2,2,3,3,3,3,3,3,3)$	-6	$-\Lambda^6$	=	$(4,4,4,4,4,4,4,4,5,5)$	-18
$-\Lambda^2$	=	$(3,3,3,3,4,4,4,4,4,4)$	-12	$-\Lambda^7$	=	$(2,2,2,2,2,2,2,2,2,3)$	-4
$-\Lambda^3$	=	$(4,4,4,4,4,5,5,5,5,5)$	-20	$-\Lambda^8$	=	$(3,3,3,3,3,3,3,3,3,3)$	-10

For their expression in the root basis, see, for instance, [160]. It can be recast in the following recursion relations:

$$\begin{aligned} \Lambda^{-1} &= -\delta , & \Lambda^4 &= 2\Lambda^3 - \Lambda^2 - \alpha_3 , \\ \Lambda^0 &= -(\alpha_{-1} + 2\delta) , & \Lambda^5 &= 2\Lambda^4 - \Lambda^3 - \alpha_4 , \\ \Lambda^1 &= 2\Lambda^0 + \theta_{E_8} , & \Lambda^6 &= \Lambda^3 + \theta_{D_5} , \\ \Lambda^2 &= 2\Lambda^1 - \Lambda^0 - \alpha_1 , & \Lambda^7 &= 2\Lambda^6 - \Lambda^5 - \alpha_6 , \\ \Lambda^3 &= 2\Lambda^2 - \Lambda^1 - \alpha_2 , & \Lambda^8 &= \Lambda^2 + \theta_{E_6} . \end{aligned}$$

5.13 Appendix 2: The U-duality group for 11D supergravity

This appendix is meant as a complement to Section 5.2.4, and reviews the U-duality transformations for finite U-duality groups as presented in [203], which trivially extends to the hyperbolic case in $D = 1$.

It has been shown in [109, 205, 202], that the $E_{11-D|11-D}(\mathbb{Z})$ chain of U-duality group relevant for M-theory incorporates a generalized T-duality symmetry, which exchanges not only the radii of 10-dimensional IIA theory among themselves, but also the 11-dimensional radius R_{10} with any of them, leading to the transformation:

$$\mathcal{T}_{ijk} : R_i \to \frac{1}{M_P^3 R_j R_k}\,, \quad R_j \to \frac{1}{M_P^3 R_i R_k}\,, \quad R_k \to \frac{1}{M_P^3 R_i R_j}\,, \quad M_P^3 \to M_P^6 R_i R_j R_k\,. \tag{5.155}$$

for $i, j, k \in \{D, .., 10\}$. To get the whole Weyl group of E_{11-D}, one must supplement the transformation (5.155) with the permutation of all radii (belonging to the $SL(11-D, \mathbb{Z})$ modular group of the torus)

$$\mathcal{S}_{ij} : R_i \leftrightarrow R_j\,,$$

which is part of the permutation group $\mathcal{S}_{11-D}$ generated by the neighbour to neighbour permutations $\{S_{i,i+1}\}_{i=D,..,9}$. Then, taking the closure of the latter with the generator $\mathcal{T}_{89\,10}$ leads to the Weyl group:

$$W(E_{11-D}) = \mathbb{Z}_2 \overline{\times} \mathcal{S}_{11-D} \tag{5.156}$$

with $\mathbb{Z}_2 = \{\mathbb{1}, \mathcal{T}_{89\,10}\}$. This gives the whole set of Weyl generators in terms of their action on the M-theory radii.

If we compactify to IIA string theory by setting $M_P R_{10} \to 0$, then the generators

$$\mathcal{T}_{\hat{\imath}\hat{\jmath}10} : R_{\hat{\imath}} \to \frac{1}{M_s^2 R_{\hat{\jmath}}}\,, \quad R_{\hat{\jmath}} \to \frac{1}{M_s^2 R_{\hat{\imath}}}\,, \quad g_A \to \frac{g_A}{M_s^2 R_{\hat{\imath}} R_{\hat{\jmath}}}$$

for $\hat{\imath}, \hat{\jmath} \in \{D, .., 9\}$, represent a double T-duality symmetry mapping IIA string theory to itself. Likewise, the group of permutations is reduced to $\mathcal{S}_{10-D}$, generated by $\{S_{\hat{\imath},\hat{\imath}+1}\}_{\hat{\imath}=D,..,8}$, which belong to the $SL(10-D, \mathbb{Z})$ modular group of the IIA torus.

In $D = 1$, this setup naturally extends to the dilaton vector $H_R \in \mathfrak{h}(E_{10})$. The permutation group $\mathcal{S}_{10}$ acts as $H_R^i \to H_R^j$, for $i, j = 1, .., 10$, which corresponds to the dual Weyl transformation: $r_\alpha^\vee(H_R) = H_R - \langle H_R, \alpha\rangle \alpha^\vee$ for $\alpha = \alpha_{i-2} + \ldots + \alpha_{j-3} \in \Pi(A_9)$.

The $\mathbb{Z}_2$ factor in expression (5.156) on the other hand, corresponds to a Weyl reflection with respect to the electric coroot:

$$\begin{aligned} r_8^\vee(H_R) &= H_R - \langle H_R, \alpha_8\rangle \alpha_8^\vee \\ &= \left(H_R^1 + \frac{1}{3}\Delta H, H_R^2 + \frac{1}{3}\Delta H, .., H_R^7 + \frac{1}{3}\Delta H, H_R^8 - \frac{2}{3}\Delta H, H_R^9 - \frac{2}{3}\Delta H, H_R^{10} - \frac{2}{3}\Delta H\right)\,, \end{aligned}$$

with $\Delta H = H_R^8 + H_R^9 + H_R^{10}$.

On the generators of $\mathfrak{e}_{10}$, the Weyl group will act as $\sigma_\alpha = \exp\left[\frac{\pi i}{2}(E_\alpha + F_\alpha)\right]$ or alternatively as $\tilde{\sigma}_\alpha = \exp\left[\frac{\pi}{2}(E_\alpha - F_\alpha)\right]$, $\forall \alpha \in \Delta_+(E_{10})$, depending on the choice of real basis. In particular, a $\mathbb{Z}_4$ orbifold of M-theory can be represented in our language by a Weyl reflection, and is thus naturally incorporated in the U-duality group.

As mentioned in Section 5.2.4, from the point of view of its moduli space, the effect of acting with the subgroup $W(E_{11-D})$ of the U-duality group on the objects of M-theory on T^{10} will typically be to exchange instantons which shift fluxes, with instantons that induce topological changes. On the cosmological billiard, a Weyl transformation will then exchange the corresponding walls among themselves.

The rest of the U-duality group is given by the Borel generators. These act on the expectation values $\mathcal{C}_\alpha$, $\alpha \in \Delta_+(E_{11-D})$, appearing, in particular, as fluxes in expression (5.56). Picking, in a given basis, a root $\beta \in \Delta_+(E_{11-D})$, its corresponding Borel generator B_β will act on the (infinite) set $\{\mathcal{C}_\alpha\}_{\alpha\in\Delta_+(E_{11-D})}$ typically as [203, 124]:

$$B_\beta : \quad \mathcal{C}_\beta \to \mathcal{C}_\beta + 1 \qquad \mathcal{C}_\gamma \to \mathcal{C}_\gamma + \mathcal{C}_{\gamma-\beta}\,, \text{ if } \gamma - \beta \in \Delta_+(E_{11-D})\,. \tag{5.157}$$

If $\gamma - \beta \notin \Delta_+(E_{11-D})$, then $B_\beta : \mathcal{C}_\gamma \to \mathcal{C}_\gamma$. The first transformation in eqn.(5.157) is the M-theory spectral flow [203], generated by part of the Borel subalgebra of the arithmetic group $E_{11-D}(\mathbb{Z})$. Invariance under such a unity shift reflects the periodicity of the expectation values of the fields $\mathcal{A}^i{}_1$, C_3, $\widetilde{C}_6$ and $\widetilde{\mathcal{A}}^i{}_7$.

5.14 Appendix 3: Conventions and involutive automorphisms for the real form $\widehat{\mathfrak{so}}(8,6)$

i) *Conventions* for $\mathfrak{d}_7$: we recall the conventions used in Section 5.5.3 to label the basis of simple roots of the finite $\mathfrak{d}_7 \subset \hat{\mathfrak{d}}_7 \subset \mathfrak{g}_{\text{inv}}$ Lie algebra for the $T^5 \times T^4/\mathbb{Z}_{n>2}$ orbifold of M-theory:

$$\beta_1 \equiv \alpha_-\,, \quad \beta_2 \equiv \tilde{\alpha}\,, \quad \beta_3 \equiv \alpha_+\,, \quad \beta_4 \equiv \alpha_3\,, \quad \beta_5 \equiv \alpha_2\,, \quad \beta_6 \equiv \alpha_1\,, \quad \beta_7 \equiv \gamma\,. \tag{5.158}$$

The affine $\widehat{\mathfrak{d}}_r$ will be described by the following Dynkin diagram: The lexicographic order used

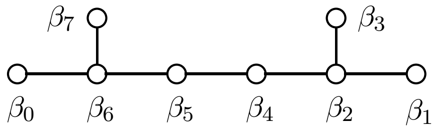


Figure 5.4: Dynkin diagram of $\hat{\mathfrak{d}}_7$ in the β-basis

in convention (5.158) is meant to naturally extend the $\mathfrak{a}_3 \subset \mathfrak{g}_{\text{inv}}$ subalgebra appearing for the $T^4/\mathbb{Z}_{n>2}$ orbifold in $D = 5$. In particular, we define $E_{\underline{i\cdots 4123}} \doteq [E_{\underline{i}}, \ldots [E_{\underline{4}}, E_{\alpha_-+\tilde{\alpha}+\alpha_+}]] \ldots]$ for $i \geqslant 4$.

For non-simple roots of level 2 in β_2, the corresponding ladder operator is defined by commuting two successive layers of simple root ladder operators, as, for instance, in:

$$E_{\underline{765^24^212^23}} \doteq [E_{\underline{5}}, [E_{\underline{4}}, [E_{\underline{2}}, E_{\underline{7654123}}]]]\,.$$

This implies in particular the useful relation

$$\mathcal{N}_{\beta_i,\beta_{i-1\cdots j^2(j-1)^2\cdots 4^212^23}} = \mathcal{N}_{\beta_j,\beta_{i\cdots j(j-1)^2\cdots 4^212^23}}\,,$$

which, combined with $\mathcal{N}_{-\alpha,-\beta} = -\mathcal{N}_{\alpha,\beta}$ and $\mathcal{N}_{\alpha,\gamma-\alpha} = -\mathcal{N}_{\alpha,-\gamma}$, induces

$$\begin{aligned}
\left[F_{\underline{i}}, E_{\underline{i\cdots(j+1)j^2\cdots 4^212^23}}\right] &= E_{\underline{i-1\cdots(j+1)j^2\cdots 4^212^23}}\,,\\
\left[F_{\underline{j}}, E_{\underline{i\cdots(j+1),j^2\cdots 4^212^23}}\right] &= E_{\underline{i\cdots j+1,j(j-1)^2\cdots 4^212^23}}\,.
\end{aligned}$$

ii) *The representation* Γ: the inner involutive automorphism written in the form (5.102) acts on elements of the algebra $\mathfrak{d}_7$ in the representation $\Gamma\{1,0,\ldots,0\}$ (see [71]) defined as follows. For general r, let the basis of simple roots $\mathfrak{d}_r$ characterized by the Dynkin diagram of Figure 5.4 be recast in terms of the orthogonal basis ε_i, $i = 1,..,r$

$$\begin{gathered}
\beta_1 = \varepsilon_{r-1} - \varepsilon_r\,, \qquad \beta_2 = \varepsilon_{r-2} - \varepsilon_{r-1}\,, \qquad \beta_3 = \varepsilon_{r-1} + \varepsilon_r\,,\\
\beta_i = \varepsilon_{r+1-i} - \varepsilon_{r+2-i}\,, \quad \forall i = 4,\ldots,r\,.
\end{gathered} \tag{5.159}$$

The remaining non-simple roots can be reexpressed as follows: for $1 \leqslant i < j \leqslant r-3$, we have

$$\begin{aligned}
\beta_{r+1-i} + \ldots + \beta_{r+1-j} &= \varepsilon_i - \varepsilon_{j+1}\,, \\
\beta_{r+1-i} + \ldots + \beta_4 + \beta_2 &= \varepsilon_i - \varepsilon_{r-1}\,,
\end{aligned}$$

$$\begin{array}{rclrcl}
\beta_{r+1-i} + \ldots + \beta_4 + \beta_2 + \beta_1 &=& \varepsilon_i - \varepsilon_r\,, & \beta_2 + \beta_1 &=& \varepsilon_{r-2} - \varepsilon_r\,, \\
\beta_{r+1-i} + \ldots + \beta_4 + \beta_3 + \beta_2 &=& \varepsilon_i + \varepsilon_r\,, & \beta_3 + \beta_2 &=& \varepsilon_{r-2} + \varepsilon_r\,, \\
\beta_{r+1-i} + \ldots + \beta_4 + \beta_3 + \beta_2 + \beta_1 &=& \varepsilon_i + \varepsilon_{r-1}\,, & \beta_3 + \beta_2 + \beta_2 &=& \varepsilon_{r-2} + \varepsilon_{r-1}\,,
\end{array} \tag{5.160}$$

while roots of level 2 in β_2 decompose as

$$\begin{array}{rcll}
\beta_{r+1-i} + \ldots + \beta_{r+1-j} + 2(\beta_{r-j} + \ldots + \beta_4 + \beta_2) + \beta_3 + \beta_1 &=& \varepsilon_i + \varepsilon_{j+1}\,, & 1 \leqslant i < j \leqslant r-4\,, \\
\beta_{r+1-i} + \ldots + \beta_4 + 2\beta_2 + \beta_3 + \beta_1 &=& \varepsilon_i + \varepsilon_{r-2}\,, & 1 \leqslant i \leqslant r-4\,.
\end{array}$$

Introducing the elementary matrices $\mathcal{E}_{i,j}$, with components $(\mathcal{E}_{i,j})_{kl} = \delta_{ik}\delta_{jl}$, the Cartan subalgebra of $\mathfrak{d}_r$ may be cast in the form

$$\begin{aligned}
\Gamma(H_{\underline{1}}) &= \frac{1}{\sqrt{r(r-1)}}\left(\mathcal{E}_{r-1,r-1} - \mathcal{E}_{r,r} + \mathcal{E}_{r+1,r+1} - \mathcal{E}_{r+2,r+2}\right)\,, \\
\Gamma(H_{\underline{2}}) &= \frac{1}{\sqrt{r(r-1)}}\left(\mathcal{E}_{r-2,r-2} - \mathcal{E}_{r-1,r-1} + \mathcal{E}_{r+2,r+2} - \mathcal{E}_{r+3,r+3}\right)\,, \\
\Gamma(H_{\underline{3}}) &= \frac{1}{\sqrt{r(r-1)}}\left(\mathcal{E}_{r-1,r-1} + \mathcal{E}_{r,r} - \mathcal{E}_{r+1,r+1} - \mathcal{E}_{r+2,r+2}\right)\,, \\
\Gamma(H_{\underline{i}}) &= \frac{1}{\sqrt{r(r-1)}}\left(\mathcal{E}_{r+1-i,r+1-i} - \mathcal{E}_{r+2-i,r+2-i} + \mathcal{E}_{r-1+i,r-1+i} - \mathcal{E}_{r+i,r+i}\right)\,, \quad \forall i = 4,\ldots,r\,.
\end{aligned}$$

The matrices representing the ladder operators of $\mathfrak{d}_r$, and solving in particular $[\Gamma(H_{\underline{i}}), \Gamma(E_{\underline{j}})] = A_{\underline{ji}}\Gamma(E_{\underline{j}})$, can be determined to be (see [69])

$$\begin{aligned}
\Gamma(E_{\varepsilon_i - \varepsilon_j}) &= \frac{1}{\sqrt{r(r-1)}}\left(\mathcal{E}_{i,j} + (-1)^{i+j+1}\mathcal{E}_{2r+1-j,2r+1-i}\right)\,, \\
\Gamma(E_{\varepsilon_i + \varepsilon_j}) &= \frac{1}{\sqrt{r(r-1)}}\left(\mathcal{E}_{i,2r+1-j} + (-1)^{i+j+1}\mathcal{E}_{j,2r+1-i}\right)\,.
\end{aligned} \tag{5.161}$$

Raising and lowering operators in the basis $\{\beta_i\}_{i=1,..,r}$ then readily follow from relations (5.159) and (5.160) and expressions (5.161).

Finally, this representation of $\mathfrak{d}_r$ preserves the metric $G_{D_r} = \begin{pmatrix} \mathbb{O} & g_{D_r}^\top \\ g_{D_r} & \mathbb{O} \end{pmatrix}$, where the off-diagonal blocs are given by g_{D_r} =offdiag$\{1,-1,1,-1,\ldots,(-1)^{r-1}\}$. It can be checked that indeed: $\Gamma(X)^\top G_{D_r} + G_{D_r}\Gamma(X) = 0$, for $X \in \mathfrak{d}_r$.

iii) Four involutive automorphisms for the real form $\mathfrak{so}(8,6)$: the set $\Delta_{(+1)}$ of roots generating the maximal compact subalgebra of the real form $\mathfrak{so}(8,6)$ appearing in Section 5.5.3 is determined for the four involutive automorphisms (5.112). Since $\dim \Delta_+(\mathfrak{d}_7) = 42$, and since all four cases have $\dim \Delta_{(+1)} = 18$, the corresponding involutive automorphisms all have signature $\sigma = 5$, and thus determine isomorphic real forms, equivalent to $\mathfrak{so}(8,6)$. This construction lifts to the affine extension $\hat{\mathfrak{d}}_7$ through the automorphism (5.102) building the Cartan decomposition (5.114) and (5.115).

Herebelow, we give the set of roots $\Delta_{(+1)}$ for the four cases (5.112) explicitly. We remind the reader that these four involutive automorphisms all have $e^{\beta'_2(\overline{H})} = +1$ and $e^{\beta'_{i\neq 2,4,6}(\overline{H})} = -1$. Moreover, the set $\Delta_{(+1)}$ generating the non-compact vector space $\mathfrak{p}$ (5.115) can be deduced from

$\Delta_{(-1)} = \Delta_+ \backslash \Delta_{(+1)}$, where Δ_+ is obtained from the system (5.159) and (5.160) by setting $r = 7$. In this case obviously $\dim \Delta_{(-1)} = 24$.

The first involutive automorphism defined by $e^{\beta'_4(\overline{H})} = e^{\beta'_6(\overline{H})} = +1$ has

$$\Delta_{(+1)} = \left\{\beta'_6, \beta'_4, \beta'_2, \beta'_{42}, \beta'_{765}, \beta'_{123}, \beta'_{7564}, \beta'_{5412}, \beta'_{5423}, \beta'_{4123}, \beta'_{76542}, \beta'_{65412}, \beta'_{65423}, \beta'_{412^23}, \beta'_{7654123}, \beta'_{765412^23}, \beta'_{7654^212^23}, \beta'_{65^24^212^23}\right\} . \quad (5.162)$$

The second, defined by $e^{\beta'_4(\overline{H})} = -e^{\beta'_6(\overline{H})} = +1$ has

$$\Delta_{(+1)} = \left\{\beta'_4, \beta'_2, \beta'_{76}, \beta'_{65}, \beta'_{42}, \beta'_{654}, \beta'_{123}, \beta'_{6542}, \beta'_{5412}, \beta'_{5423}, \beta'_{4123}, \beta'_{412^23}, \beta'_{765412}, \beta'_{765423}, b'_{654123}, \beta'_{65412^23}, \beta'_{654^212^23}, \beta'_{765^24^212^23}\right\} . \quad (5.163)$$

The third, defined by $e^{\beta'_4(\overline{H})} = -e^{\beta'_6(\overline{H})} = -1$ has

$$\Delta_{(+1)} = \left\{\beta'_6, \beta'_2, \beta'_{54}, \beta'_{765}, \beta'_{654}, \beta'_{542}, \beta'_{412}, \beta'_{423}, \beta'_{123}, \beta'_{6542}, \beta'_{54123}, \beta'_{765412}, \beta'_{765423}, \beta'_{654123}, \beta'_{5412^23}, \beta'_{65412^23}, \beta'_{7654^212^23}, \beta'_{65^24^212^23}\right\} . \quad (5.164)$$

The fourth, defined by $e^{\beta'_4(\overline{H})} = e^{\beta'_6(\overline{H})} = -1$ has

$$\Delta_{(+1)} = \left\{\beta'_2, \beta'_{76}, \beta'_{65}, \beta'_{54}, \beta'_{542}, \beta'_{412}, \beta'_{423}, \beta'_{123}, \beta'_{7654}, \beta'_{76542}, \beta'_{65412}, \beta'_{65423}, \beta'_{54123}, \beta'_{5412^23}, \beta'_{7654123}, \beta'_{765412^23}, \beta'_{654^212^23}, \beta'_{765^24^212^23}\right\} . \quad (5.165)$$

The four of them lead as expected to $\dim \Delta_{(+1)} = 18$.

Appendix A

Spinors in various dimensions: conventions and useful formulæ

In this appendix, we first give our conventions for Gamma-matrices in $(1+9)D$ and $(1+10)D$, whic hold throughout this thesis. Then, we present some general formulae (valid in all dimensions) involving antisymmetrized products of Gamma-matrices and spinor bilinear, and some useful properties of Weyl and Majorana spinors. We end this appendix by presenting a Fierz identity in $10D$ which is required to prove supersymmetry of the Yang-Mills action and of the BFSS matrix model.

A.1 Conventions for Dirac matrices in $10D$ and $11D$

Our conventions for Gamma-matrices in $(9+1)D$ and $(10+1)D$ are the following. Start from the Dirac matrices forming a $2^{[9/2]} = 16$ dimensional Majorana representation of $SO(9)$:

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij}\,\mathbb{1}_{16}$$

with γ^i real, $i = 1,..,9$. We can then construct a Majorana representation of $SO(9,1)$ by tensoring the above in the following way:

$$\Gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_{16} \\ -\mathbb{1}_{16} & 0 \end{pmatrix}, \qquad \Gamma^i = \begin{pmatrix} 0 & \gamma^i \\ \gamma^i & 0 \end{pmatrix}, \quad i = 1,..,9\,.$$

Fixing the Minkowskian metric to $\eta^{\mu\nu}$ =diag$\{-,+,..,+\}$ we have as expected $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_{32}$.

As usual in even dimensions, there is an additional linear independent matrix, the chirality matrix, which, in our choice of basis (suited to Majorana representation), is block diagonal and real:

$$\Gamma_* = \pm \prod_{\mu=0}^{9} \Gamma^\mu = \pm \begin{pmatrix} \mathbb{1}_{16} & 0 \\ 0 & \mathbb{1}_{16} \end{pmatrix}. \tag{A.1}$$

obviously $\{\Gamma_*, \Gamma^\mu\} = 0$. Identifying $\Gamma^{10} = \Gamma_*$, we obtain a $2^{[11/2]} = 32$ dimensional Majorana representation of the Clifford algebra of $SO(10,1)$. In particular, we can choose the charge conjugation matrix as $C = \Gamma^0$.

A.2 Product representations for spinors in various dimensions

In space-time dimensions $D = 2l{+}2$ and $2l{+}3$, the (complexe) the Dirac matrices have dimension $2^{[\frac{D}{2}]} = 2^{l+1}$ and realize the Clifford algebra:

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}\,\mathbb{I}_{2^{l+1}}\,. \tag{A.2}$$

For both odd and even dimensions, there are $2l+3$ such matrices, running from $\mu = 0,1,..,2l+2$ for odd dimensions, and from $\mu = 0,1,..,2l{+}1,*$, in even dimensions. In this case, the additional Gamma matrice is given by:

$$\Gamma_* \doteq \Gamma^{2l+2} = \pm i^{-l}\prod_{\mu=0}^{2l+1}\Gamma^\mu$$

Moreover, conjugation by the matrix Γ^0 is equivalent to hermitien conjugation of the Dirac matrices:

$$\Gamma^0\Gamma^\mu(\Gamma^0)^{-1} = -\Gamma^{\mu\dagger}\,, \qquad \Gamma^0\Gamma^{\mu_1..\mu_n}(\Gamma^0)^{-1} = (-1)^{\frac{n(n+1)}{2}}\Gamma^{\mu_1..\mu_n\dagger}\,.$$

The Clifford algebra is then spanned by the set $\{\Gamma^{\mu_1..\mu_n}\}_{n=0,..,2l+2}$, with $\Gamma^{\mu_1..\mu_n} \doteq \Gamma^{[\mu_1}\cdots\Gamma^{\mu_n]}$, which satisfy the orthogonality condition.

$$\frac{1}{32}\mathrm{Tr}\,(\Gamma^{\mu_1..\mu_n}\Gamma_{\lambda_1..\lambda_m}) = (-1)^{\frac{n(n-1)}{2}}n!\eta^{\mu_1}_{[\lambda_1}\cdots\eta^{\mu_n}_{\lambda_n]}\delta^n_m\,. \tag{A.3}$$

Introducing the charge conjugation matrix

$$C\Gamma^\mu C^{-1} = -\Gamma^{\mu\intercal}\,, \qquad \text{with } C^\intercal = -c_D C\,, \tag{A.4}$$

where c_l depends on the dimension of space-time. Thus, for $D = \{2,3,4\}$ mod 8, one has $c_D = -1$, while for $D = \{6,7,8\}$ mod 8, we get $c_D = 1$. In dimensions $D = 5$ mod 8, where no Majorana condition applies, as we will see later on, one uses the matrix C_2 instead, which satisfies $C_2\Gamma^\mu C_2^{-1} = \Gamma^{\mu\intercal}$ and $C_2^\intercal = -C_2$.

On antisymmetrized products of Dirac matrices, conjugation by C gives:

$$C\Gamma^{\mu_1..\mu_n}C^{-1} = (-1)^{\frac{n(n+1)}{2}}\Gamma^{\mu_1..\mu_n\intercal}\,.$$

Using the orthogonality relations (A.3), one can expand the product of two spinors on a complete set of independent tensors:

$$(\psi^\intercal C)^\alpha\lambda_\beta = \frac{1}{2^{l+1}}\sum_{n=0}^{2l+2}\frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\left(\psi^\intercal C\Gamma^{\mu_1..\mu_n}\lambda\right)\left(\Gamma_{\mu_1..\mu_n}\right)_\beta{}^\alpha\,. \tag{A.5}$$

In dimensions where the Majorana condition holds, *i.e.* for $l \in \{0,1,3\}$ mod 4, one can use the identity (A.17) below to replace $\psi^\intercal C\Gamma^{\mu_1..\mu_n}\lambda \to \pm\overline{\psi}\Gamma^{\mu_1..\mu_n}\lambda$ in the above expression.

The above expression corresponds to the decomposition in antisymmetric tensorial representations of $SO(2l+1,1)$ of the product of two 2^{l+1}-dimensional Dirac representations:

$$\mathbf{2}^{l+1}_{\text{Dirac}} \times \mathbf{2}^{l+1}_{\text{Dirac}} = [0]^2 + [1]^2 + .. + [l]^2 + [l+1] \tag{A.6}$$

The doubling of the antisymmetric representations $[k]$ for $k \leqslant l$, is due to the equivalence $[k] = [2l+2-k]$ under Hodge-dualization of the corresponding Gamma-matrices.

The Hodge-dual $(*\Gamma)^{\mu_1..\mu_n}$ is obtained by acting with the chirality matrix Γ_*. In a particular, we have in $D = 2l+2$:

$$\begin{aligned}(D-n)!\Gamma_*\Gamma^{\mu_1..\mu_n} &= \pm(-1)^{\frac{n(n+1)}{2}}i^{-l}\varepsilon^{\mu_1..\mu_n\nu_1..\nu_{D-n}}\Gamma_{\nu_1..\nu_{D-n}}\\ &= \pm(-1)^{\frac{n(n+1)}{2}}n!\varepsilon^{\mu_1..\mu_n\lambda_1..\lambda_{D-n}}(*\Gamma)_{\lambda_1..\lambda_{D-n}}\end{aligned} \tag{A.7}$$

where $\varepsilon_{012..D-1} = -\varepsilon^{012..D-1} = +1$ is the totally antisymmetric tensor in flat Minkowskian space satisfying $\varepsilon^{\mu_1..\mu_n\mu_{n+1}..\mu_D}\varepsilon_{\nu_1..\nu_n\mu_{n+1}..\mu_D} = -(D-n)!\delta^{[\mu_1}_{\nu_1}\cdots\delta^{\mu_n]}_{\nu_n}$. In particular, the sign of Γ_* (A.1) fixes, in even dimensions, the notion of self-duality or anti-self-duality for tensors such as $\psi^\intercal C\Gamma^{\mu_1..\mu_{l+1}}\lambda$.

A.3 Weyl spinors in various dimensions

In even $2l+2$ dimensions, the Dirac representation for spinors is reducible as a representation of the Lorentz group $SO(2l+1,1)$. The two irreducible 2^l Weyl representations are obtained from eigenvalue (±1) states (or right-/left-handed) under Γ_*. Thus one defines the following chirality operators

$$\mathcal{P}_\pm = \frac{1}{2}\left(\mathbb{1}\pm\Gamma_*\right)\,,$$

projecting the Dirac rep onto each of its two irreducible Weyl reps:

$$\mathcal{P}_\pm\psi_\pm = \psi_\pm\,,\qquad \mathcal{P}_\pm\psi_\mp = 0\,.$$

Using the properties of Γ_* one can verify that such operators meet all requirements to be projectors:

$$(\mathcal{P}_\pm)^2 = \mathbb{1}\,,\quad \mathcal{P}_\pm\mathcal{P}_\mp = 0\,,\quad \mathcal{P}_\pm{}^{-\mu} = {}^{-\mu}\mathcal{P}_\mp\,.$$

In $(9+1)D$, we usually identify $\mathcal{P}_\pm \equiv \frac{1}{2}\left(\mathbb{1}\pm\Gamma_{10}\right)$.

In particular since $\{\Gamma_*,\Gamma^\mu\}=0$, on can show that:

$$\mathcal{P}_\pm\Gamma^{\mu_1..\mu_n}\mathcal{P}_{\pm/\mp} = \Gamma^{\mu_1..\mu_n}\frac{1}{2}\left(\mathbb{1}\pm(-1)^n\Gamma_*\right)\mathcal{P}_{\pm/\mp} = \begin{cases}\Gamma^{\mu_1..\mu_n}\,, & \text{for } n \text{ even/odd}\\ 0\,, & \text{for } n \text{ odd/even}\end{cases}\,. \tag{A.8}$$

Using the properties of the charge conjugation matrix C (A.4) and $\Gamma_*^\intercal = \Gamma_*$ to show that in $D=2l+2$:

$$\Gamma_* C = \prod_{k=1}^{D-1}(-1)^{D-k}\,C\Gamma_* = (-1)^{l+1}C\Gamma_*\,, \tag{A.9}$$

then, the combination of eqns.(A.8) and (A.9) leads to

$$\psi_\pm^\intercal C\Gamma^{\mu_1..\mu_n}\lambda_{\pm/\mp} = \psi_\pm^\intercal C\Gamma^{\mu_1..\mu_n}\left[\frac{1}{2}\left(\mathbb{1}\pm(-1)^{n+l+1}\Gamma_*\right)\mathcal{P}_{\pm/\mp}\right]\lambda_{\pm/\mp}$$

implying that for l even and spinors of the same chirality, only bilinears with n odd survive, while for l odd, the converse is true. For two spinors of different chiralities, l even/odd kills all bilinear with n even/odd. This can be summarized in the well known decompositions of products of Weyl representations:

$$\mathbf{2}^l_\pm\times\mathbf{2}^l_\pm = \begin{cases}[1]+[3]+..+[l+1]_\pm\,, & \text{for } l \text{ even}\\ [0]+[2]+..+[l+1]_\pm\,, & \text{for } l \text{ odd}\end{cases}\,, \tag{A.10}$$

$$\mathbf{2}^l_\pm\times\mathbf{2}^l_\mp = \begin{cases}[0]+[2]+..+[l]\,, & \text{for } l \text{ even}\\ [1]+[3]+..+[l]\,, & \text{for } l \text{ odd}\end{cases}\,. \tag{A.11}$$

Furthermore, we can now exploit the properties of Γ_* and its appearance in the dualizing Gamma-matrices to rewrite the completeness relation (A.5) in a form where the action of Γ_* on the Weyl spinors is made manifest:

$$\begin{aligned}(\psi^\intercal C)^\alpha\lambda_\beta \;=\;& \frac{1}{2^{l+1}\cdot(l+1)!}\,(\psi^\intercal C\Gamma^{\mu_1..\mu_{l+1}}\lambda)\,(\Gamma_{\mu_1..\mu_{l+1}})_\beta{}^\alpha\,+ \\ &+\frac{1}{2^{l+1}}\sum_{n=0}^{2l+2}\frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\left[(\psi^\intercal C\Gamma^{\mu_1..\mu_n}\lambda)\,(\Gamma_{\mu_1..\mu_n})_\beta{}^\alpha + ((\Gamma_*\psi)^\intercal C\Gamma^{\mu_1..\mu_n}\lambda)\,(\Gamma_{\mu_1..\mu_n}\Gamma_*)_\beta{}^\alpha\right]\,.\end{aligned} \tag{A.12}$$

To derive this decomposition, we have used eqn.(A.7), (A.9) and the identity:
$\Gamma^{\mu_1..\mu_n} = -(\pm 1)(-1)^{\frac{n(n-1)}{2}} \frac{i^l}{n!} \epsilon^{\mu_1..\mu_n}{}_{\rho_1..\rho_{D-n}} \Gamma_* \Gamma^{\rho_1..\rho_{D-n}}$.

A.4 The Majorana condition

In even dimensions, the irreducible Weyl representations constructed in the preceding section are unique up to a change of basis. Then since the conjugate matrices $-(\Gamma^\mu)^*$ generate the same Clifford algebra (A.2) as the original Dirac matrices Γ^μ, both have to be related by a similarity transformation. Starting in even dimensions $D = 2l + 2$, and, in contrast to Section A.1, going to a basis where the Γ^0, Γ^1 and Γ^{2k}, $\forall k = 1, .., l$ are chosen real, while the Γ^{2k+1}, $\forall k = 1, .., l$ are chosen imaginary, complexe conjugation is achieved via the two following matrices:

$$B_1 = \prod_{k=1}^{l} \Gamma^{2k+1} , \quad B_1 = \Gamma_* B_2 \equiv (-1)^l \Gamma^3 \Gamma^5 \cdot .. \cdot \Gamma^{2l+1} \Gamma^{2l+2} = (\Gamma^0 \Gamma^1) \prod_{k=1}^{l} \Gamma^{2k} .$$

Since these matrices lead to the anticommutation relations:

$$B_1 \Gamma^\mu B_1^{-1} = (-1)^l \Gamma^{\mu *} , \quad B_2 \Gamma^\mu B_2^{-1} = (-1)^{l+1} \Gamma^{\mu *} , \quad B_i \Gamma_* B_i^{-1} = (-1)^l \Gamma_* , \ \forall i = 1, 2 . \quad \text{(A.13)}$$

they transform, in particular, $\Sigma_{\mu\nu} = -\frac{i}{2}\Gamma^{\mu\nu}$ into $B_i \Sigma_{\mu\nu} B_i^{-1} = -(\Sigma_{\mu\nu})^*$, $\forall i = 1, 2$, so that a Lorentz transformation acts on both the spinors ψ and $B_i^{-1}\psi^*$ in the same way. The Dirac representation is thus its own conjugate. In contrast, the chirality matrix Γ_* has its eigenvalues flipped under complexe conjugation (A.13) when l is odd, which tells us that the two irreducible Weyl reps are self-conjugate when l is even, and conjugate to each other when l is odd.

One can furthermore show, using the above relations, that:

$$B_1 B_1^* = (-1)^{\frac{l(l+1)}{2}} \mathbb{1} , \qquad B_2 B_2^* = (-1)^{\frac{l(l-1)}{2}} \mathbb{1} . \quad \text{(A.14)}$$

Then, self-conjugate spinors, or in other word spinors satisfying the *Majorana* condition:

$$\psi = B \psi^* \quad \text{(A.15)}$$

will occur for $l = 0 \bmod 4$, in which case $B = \{B_1, B_2\}$ (the two conditions are then equivalent, or *conjugate* by a similarity transformation), for $l = 1 \bmod 4$ with $B = B_2$, and for $l = 3 \bmod 4$ with $B = B_1$. When condition (A.15) is satisfied, spinors can be chosen real. In this case, there always is a basis where the Dirac matrices Γ^μ are all real, like in Section A.1. Obviously, we then have $B = \mathbb{1}$ and $C = \Gamma^0$, implying $(\Gamma^0)^\dagger = (\Gamma^0)^\intercal = -\Gamma^0$ and $(\Gamma^i)^\dagger = (\Gamma^i)^\intercal = \Gamma^i$. Alternatively, one can also always find a basis where $B = \Gamma_*$ and then $C = -\Gamma^1\Gamma^2 \cdots \Gamma^{2l+1}$. In particular, thes to choices can never be imposed in $D = 5, 6, 7 \bmod 8$.

Henceforth, we will denote by B either B_1 or B_2 in dimensions for which they satisfy $BB^* = \mathbb{1}$.Clearly, we have now

$$B\Gamma^\mu B^{-1} = \Gamma^{\mu *} , \qquad B\Gamma^{\mu_1..\mu_n} B^{-1} = \Gamma^{\mu_1..\mu_n *} .$$

Since $C\Gamma^0\Gamma^\mu(C\Gamma^0)^{-1} = \Gamma^{\mu *}$, then $C = B\Gamma^0$, so that:

$$C = \begin{cases} B_1\Gamma^0 , & \text{for } l = \{0, 3\} \bmod 4 \\ B_2\Gamma^0 , & \text{for } l = \{0, 1\} \bmod 4 \end{cases} .$$

As $C^\intercal = -c_D C$ (A.4), then $C^\intercal = (\Gamma^0)^{-1}B^{-1} = -C\Gamma^0 C^{-1} B^{-1}$, so that combining both expressions, we get:

$$B^\intercal = c_D B . \quad \text{(A.16)}$$

In general, the Majorana condition can be enforced on Weyl spinors when the difference between space dimensions s and time dimensions t satisfies: $s-t=0 \bmod 8$. In particular for $SO(2l+1,1)$ spinors, this holds for $l=4k$, where we have $BB^*=\mathbb{1}$ and each Weyl rep is conjugate to itself.

Finally, in odd dimensions $D=2l+3$, since the additional Dirac matrix is $\Gamma^{2l+2}=\Gamma_*$, the action of B on the latter will be compatible with its action on the rest of the Dirac matrices (A.13) only for $l=\{0,3\} \bmod 4$, and in both cases one can choose $B=B_1$. Then, the matrix $C'=B_1\Gamma^0$ acts as $C'\Gamma^\mu C'^{-1}=(-1)^{l+1}\Gamma^{\mu\intercal}$, behaving as the charge conjugation matrix C (A.4) for even l and as the matrix C_2 for odd l (see Section A.2).

A.5 Hermitian conjugation in $10D$

Thanks to the property (A.16), the Dirac conjugate and charge conjugate of Majorana spinor in dimensions $D=2l+2$ for $l=\{0,1,3\} \bmod 4$, are related to one another:

$$\overline{\psi}=\psi^\dagger\Gamma^0=c_D\psi^\intercal C\,. \tag{A.17}$$

Since in $(9+1)D$ we have $c_{10}=1$, bilinears of Majorana fermions can now equivalently be written as $\overline{\psi}\Gamma^{\mu_1..\mu_n}\lambda\equiv\psi^\intercal C\Gamma^{\mu_1..\mu_n}\lambda$. In particular, using the properties of the matrix C, one can show that:

$$\overline{\psi}\Gamma^{\mu_1..\mu_n}\lambda=(-1)^{\frac{n(n+1)}{2}}\bar{\lambda}\Gamma^{\mu_1..\mu_n}\psi\,, \tag{A.18}$$

which determines their hermiticty property of these tensors: $(\overline{\psi}\Gamma^{\mu_1..\mu_n}\lambda)^\dagger=-\bar{\lambda}\Gamma^{\mu_1..\mu_n}\epsilon$.

In addition, one can show, along a similar line, that

$$\psi^\dagger\Gamma^{i_1..i_n}\lambda=(-1)^{\frac{(n+2)(n+1)}{2}}\lambda^\dagger\Gamma^{i_1..i_n}\psi\,,\qquad \psi^\dagger\lambda=-\lambda^\dagger\psi \tag{A.19}$$

if the i_k are all space indices, and:

$$\psi^\dagger\Gamma^{\mu_1..\mu_n}\lambda=(-1)^{\frac{n(n-1)}{2}}\lambda^\dagger\Gamma^{\mu_1..\mu_n}\psi\,, \tag{A.20}$$

when one of the indices μ_k is equal to 0.

A.6 A ten-dimensional Fierz identity

In this section, we prove a Fierz identity for a triplet of right-handed Majorana spinors in $(9+1)D$ which we use in eqn.(1.49) to prove supersymmetry of the $10D$ SYM action, and in eqn.(1.72) to prove that of the BFSS matrix model. The expression in components of this identity is further needed in eqn.(1.87) to derive p-brane charges from the BFSS matrix model.

In order to prove that the antisymmetrized expression

$$(\overline{\psi}_+^{[I}\Gamma_\mu\psi_+^J)(\Gamma^\mu\psi_+^{K]})_\alpha=0 \tag{A.21}$$

vanishes, we rewrite it using the decomposition (A.12) as:

$$\begin{aligned}(\overline{\psi}_+^{[I})^\beta\,(\Gamma_\mu\psi_+^J)_\beta(\Gamma^\mu\psi_+^{K]})_\alpha &= \frac{1}{32\cdot 5!}(\Gamma_{\mu_1..\mu_5}\Gamma_\mu\psi_+^{[J})_\alpha\left(\overline{\psi}_+^I\Gamma^{\mu_1..\mu_5}\Gamma^\mu\psi_+^{K]}\right)\\ &\quad+\frac{1}{16}\sum_{n=0}^{4}\frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}(\Gamma_{\mu_1..\mu_n}\Gamma_\mu\psi_+^{[J})_\alpha\left(\overline{\psi}_+^I\Gamma^{\mu_1..\mu_n}\Gamma^\mu\psi_+^{K]}\right)\,.\end{aligned} \tag{A.22}$$

According to the decomposition (A.10) of product representations, since $l=4$ and the spinors are all of the same chirality, only bilinears for n odd survive in expression (A.22). Moreover, antisymmetry upon exchanging $I \leftrightarrow K$ and the property (A.18) on Majorana spinors only spares bilinears with $n = 1, 5$. Using in addition the commutation relations:

$$\Gamma^{\mu_1}\Gamma^{\mu} = \Gamma^{\mu_1\mu} + \eta^{\mu_1\mu}\mathbb{1}_{32}\,, \qquad \Gamma^{\mu_1\mu_2}\Gamma^{\mu} = \Gamma^{\mu_1\mu_2\mu} + 2\eta^{\mu[\mu_1}\Gamma^{\mu_2]}$$

the expansion (A.22) simplifies to:

$$\frac{1}{16}\left[(1-9)(\Gamma_\mu\psi_+^{[J})_\alpha\left(\overline{\psi}_+^I\Gamma^\mu\psi_+^{K]}\right) + \frac{1}{4!}(\Gamma_{\mu_1..\mu_4\mu}\psi_+^{[J})_\alpha\left(\overline{\psi}_+^I\Gamma^{\mu_1..\mu_4\mu}\psi_+^{K]}\right)\right]. \tag{A.23}$$

Concentrating on the first term in eqn.(A.23), we can replace $\psi_+^J \to \mathcal{P}_+\psi_+^J$. We then use antisymmetry of I, J and K and eqn.(A.18) to rewrite this term in components (eliminating ψ_+^J) as:

$$-\frac{1}{2\cdot 3}\epsilon_{JIK}\Big((\Gamma_\mu\mathcal{P}_+)_\alpha{}^\gamma\left(\overline{\psi}_+^I\Gamma^\mu\psi_+^K\right) + 2(\Gamma_\mu\psi_+^{[I})_\alpha\left(\overline{\psi}_+^{K]}\Gamma^\mu\mathcal{P}_+\right)^\gamma\Big) \tag{A.24}$$

Then, using the orthogonality property for $\frac{1}{16}\mathrm{Tr}(\Gamma^\mu\mathcal{P}_\pm\Gamma^\nu)$ (A.3), one can show that:

$$\Gamma^\mu\Gamma_\nu\Gamma_\mu = \Gamma^\mu\left(-\Gamma_\mu\Gamma_\nu + 2\mathbb{1}\eta_{\mu\nu}\right) = -8\Gamma_\nu = \frac{1}{2}\mathrm{Tr}\left(\Gamma_\mu\mathcal{P}_+\Gamma_\nu\right)\Gamma^\mu\,,$$

so that projecting (A.24) on the one-index basis tensor by contracting with $(\Gamma_\nu)_\gamma{}^\alpha$ we reobtain part of expression (A.21). In this case, we immediately see, using the identities (A.24) that:

$$-\frac{1}{2\cdot 3}\epsilon_{JIK}\Big(\mathrm{Tr}(\Gamma_\mu\mathcal{P}_+\Gamma_\nu)\left(\overline{\psi}_+^I\Gamma^\mu\psi_+^K\right) - 2\overline{\psi}_+^{[I}\mathcal{P}_-\Gamma^\mu\Gamma_\nu\Gamma_\mu\psi_+^{K]}\Big) = 0$$

since $\overline{\psi}_+\mathcal{P}_- = \overline{\mathcal{P}_+\psi_+} = \overline{\psi}_+$.

The second, five-index tensor part in eqn.(A.23) can be dealt with in the same fashion. In this case, it is even easier to show that this contribution vanishes, since $\mathrm{Tr}(\Gamma_{\mu_1..\mu_4\mu}\mathcal{P}_+\Gamma_\nu) = 0$ and $\Gamma^{\mu_1..\mu_4\mu}\Gamma_\nu\Gamma_{\mu_1..\mu_4\mu} = 0$. We have thus shown the identity (A.21)

We may now give the component expression for eqn.(A.21) which has been used in eqn.(1.87) of Section 1.4. By denoting $\{\alpha\beta\gamma\}$ the cyclic permutation of three indices and eliminating all spinorial components from eqn.(A.21), the latter can be recast into the form:

$$(\mathcal{P}_\pm\Gamma^0\Gamma^\mu)_\alpha{}^{\{\beta}(\mathcal{P}_\pm\Gamma^0\Gamma_\mu)^{\gamma\kappa\}} = (\mathcal{P}_\pm\Gamma^0\Gamma^i)_\alpha{}^{\{\beta}(\mathcal{P}_\pm\Gamma^0\Gamma_i)^{\gamma\kappa\}} - (\mathcal{P}_\pm)_\alpha{}^{\{\beta}(\mathcal{P}_\pm)^{\gamma\kappa\}}\,. \tag{A.25}$$

Multiplying by the expression $(\Gamma^{0k})_\lambda{}^\alpha$, we get

$$(\mathcal{P}_\pm\Gamma^{ki})_\lambda{}^{\{\beta}(\mathcal{P}_\pm\Gamma^0\Gamma_i)^{\gamma\kappa\}} - (\mathcal{P}_\pm\Gamma^0\Gamma^k)_\lambda{}^{\{\beta}(\mathcal{P}_\pm)^{\gamma\kappa\}} + (\mathcal{P}_\pm)_\lambda{}^{\{\beta}(\mathcal{P}_\pm\Gamma^0\Gamma^k)^{\gamma\kappa\}} = 0 \tag{A.26}$$

after having used $\Gamma^\mu\Gamma^\nu = \frac{1}{2}[\Gamma^\mu,\Gamma^\nu] + \frac{1}{2}\{\Gamma^\mu,\Gamma^\nu\} = \Gamma^{\mu\nu} + \eta^{\mu\nu}\mathbb{1}$.

One can also contract eqn.(A.25) on the left with $(\Gamma^{0k})_\gamma{}^\zeta$ and relabelling the indices, one arrives at a similar expression, with expressions in parenthesis interchanged. and $\left(\mathcal{P}_\pm\Gamma^0\Gamma_i\right)_\lambda{}^{(\beta}\left(\mathcal{P}_\pm\Gamma^{ki}\right)^{\gamma\kappa)}$

$$\begin{aligned}(\mathcal{P}_\pm\Gamma^0\Gamma_i)_\lambda{}^\beta(\mathcal{P}_\pm\Gamma^{ki})^{\gamma\kappa} - (\mathcal{P}_\pm\Gamma^0\Gamma_i)_\lambda{}^\kappa(\mathcal{P}_\pm\Gamma^{ki})^{\beta\gamma} - (\mathcal{P}_\pm\Gamma^0\Gamma_i)_\lambda{}^\gamma(\mathcal{P}_\pm\Gamma^{ki})^{\kappa\beta}&\\ -(\mathcal{P}_\pm)^{\{\beta}(\mathcal{P}_\pm\Gamma^0\Gamma^k)_\lambda{}^{\gamma\kappa\}} + (\mathcal{P}_\pm\Gamma^0\Gamma^k)^{\{\beta\gamma}(\mathcal{P}_\pm)_\lambda{}^{\kappa\}} &= 0\,.\end{aligned} \tag{A.27}$$

Adding expressions (A.26) and (A.27), we at the desired identity:

$$(\mathcal{P}_\pm\Gamma^{ki})_\lambda{}^\beta(\mathcal{P}_\pm\Gamma^0\Gamma_i)^{\gamma\kappa} + (\mathcal{P}_\pm\Gamma^{ki})^{\gamma\kappa}(\mathcal{P}_\pm\Gamma^0\Gamma_i)_\lambda{}^\beta + (\mathcal{P}_\pm\Gamma^{ki})_\lambda{}^\kappa(\mathcal{P}_\pm\Gamma^0\Gamma_i)^{\beta\gamma} - (\mathcal{P}_\pm\Gamma^{ki})^{\beta\gamma}(\mathcal{P}_\pm\Gamma^0\Gamma_i)_\lambda{}^\kappa = 0$$

In a basis where all Dirac matrices are real: $(\Gamma^0)^\intercal = -\Gamma^0$ and $(\Gamma^i)^\intercal = \Gamma^i$, $\forall i = 1,..,9$, so that the Γ^{0i} are symmetric, while the Γ^{ij} are antisymmetric. Then, the above expression simplifies to

$$(\mathcal{P}_\pm\Gamma^{ki})_\lambda{}^{(\beta}(\mathcal{P}_\pm\Gamma^0\Gamma_i)^{|\gamma|\kappa)} + (\mathcal{P}_\pm\Gamma^0\Gamma_i)_\lambda{}^{(\beta}(\mathcal{P}_\pm\Gamma^{ki})^{|\gamma|\kappa)} + (\mathcal{P}_\pm)_\lambda{}^\gamma(\mathcal{P}_\pm\Gamma^0\Gamma^k)^{\beta\kappa} - (\mathcal{P}_\pm\Gamma^0\Gamma^k)_\lambda{}^\gamma(\mathcal{P}_\pm)^{\beta\kappa} = 0\,.$$

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