

PHOTON-PHOTON SCATTERING CONTRIBUTION TO THE
SIXTH ORDER MAGNETIC MOMENTS OF THE MUON AND ELECTRON*

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ABSTRACT

We have calculated the three-photon exchange contribution to the sixth order anomalous magnetic moment of the leptons. Our result for the electron loop contribution to the muon moment $(18.4 \pm 1.1)(\alpha/\pi)^3$ brings the theoretical prediction into agreement with the CERN measurements within the one standard deviation experimental accuracy. The result for the electron loop contribution to the electron moment is $(0.36 \pm 0.04)(\alpha/\pi)^3$. The theoretical errors represent the accuracy of the required seven-dimensional numerical integrations.

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1. INTRODUCTION AND SUMMARY

The anomalous magnetic moments of the electron and muon have played central roles in the testing of the validity of quantum electrodynamics and the search for possible differences in the basic properties of the leptons. The increasing precision of present and projected measurements of the g-factor now promises a confrontation with the predictions of theory through sixth order in perturbation theory. In addition, the muon moment can provide a fundamental sum rule limit on the electromagnetic coupling to the entire spectrum of hadrons as well as a limit on the influence of weak interactions on the lepton field.¹

Unfortunately, the complete calculation of the sixth order radiative corrections to the lepton vertex — especially those graphs which cannot be obtained from insertions of second or fourth order corrections to the photon and fermion propagators — is horrendous. There are two central problems: (1) the reduction of matrix elements with three loop integrations to Feynman parametric form, and (2) the multidimensional integration of the resulting integrand.

In this paper we present a computation of the photon-photon scattering sub-diagram contribution to the sixth order magnetic moment of the electron and muon. In order to avoid computational errors in the reduction to parametric form we have carried out our calculation in two different ways: one follows the standard Landau techniques outlined in the book of Bjorken and Drell,² and the other is based on the method developed by Nakanishi³ and Kinoshita.⁴ We have calculated most integrands including all those that contribute to the $\ln(m_\mu/m_e)$ term by hand. In the end, all of the trace algebra and substitutions were performed automatically using REDUCE, an algebraic computation program developed by A. C. Hearn.⁵ For the practical solution to the second problem we have resorted to numerical integration using a novel program (originally developed by G. Sheppey at CERN⁶

and improved by one of us (AJD)) which on successive iterations improves the Riemann integration grid through a random variable sampling technique. In the rest of this section we present a comparison of theory and experiment and outline the remainder of the paper.

The most recent CERN measurement of the anomalous part of the muon g-factor gives⁷

$$a_{\text{exp}} = (116616 \pm 31) \times 10^{-8} . \quad (1.1)$$

The experimental error is about 7% of the $(\alpha/\pi)^2$ term in the theoretical prediction. Thus, for a serious confrontation of theory and experiment, the theoretical result must be improved to an accuracy of order 10^{-7} or better, which requires knowledge of the α^3 radiative corrections, hadronic corrections, and possibly corrections due to weak intermediate bosons.

The theoretical result for the muon g-factor which has been calculated previous to this work from standard quantum electrodynamics is

$$(1/2)(\alpha/\pi) + 0.76578 (\alpha/\pi)^2 + 3.00 (\alpha/\pi)^3 . \quad (1.2)$$

The fourth order term has been evaluated analytically up to and including terms of order $(\alpha/\pi)^2 (m_e/m_\mu)^2$.⁸ The last term consists of two parts, one is the contribution to the lepton vertex which involves only one type of lepton, and the other in which both leptons appear. An estimate of the first contribution based on the technique of sidewise dispersion relations gives $0.13 (\alpha/\pi)^3$.⁹ A term $[0.055 (\alpha/\pi)^3]$ not included in the above estimate was obtained recently by an analytic calculation of diagrams containing fourth order vacuum polarization due to muon pairs.¹⁰ (These mass-independent contributions are of course common to the electron g-factor.) The second part is obtained by insertion of electron loops of fourth and second order into the virtual photon lines of the second and fourth order

electromagnetic vertices of the muon.^{11, 12, 13, 14} This contribution can be written in the form

$$\left[\frac{2}{9} \left(\ln(m_\mu/m_e) \right)^2 - 1.114 \ln(m_\mu/m_e) + 2.44 \right] \left(\frac{\alpha}{\pi} \right)^3 = 2.82 \left(\frac{\alpha}{\pi} \right)^3. \quad (1.3)$$

It was found that the coefficients of the logarithmic terms can be obtained simply by algebraic manipulation of the renormalization constant Z_3 and the muon magnetic moment of the second and fourth orders.¹¹ Several terms contributing to the non-logarithmic terms in (1.3) have been calculated directly.^{11, 14} Although some non-logarithmic terms are still to be evaluated, they are at least estimated in Ref. 13.¹⁵ The error of this estimate will probably not exceed $\pm 0.5 (\alpha/\pi)^3$.¹⁶

The latest estimate of the contribution from strong interactions (vacuum polarization due to hadrons) to the muon g-factor, based on the Orsay colliding beam data for $e^+ + e^- \rightarrow \rho, \omega, \text{ and } \phi$ resonances, is¹⁷

$$a_{\text{hadrons}} = (6.5 \pm 0.5) \times 10^{-8}. \quad (1.4)$$

If one uses the value¹⁸

$$\alpha^{-1} = 137.03608 \pm 0.00026 \quad (1.5)$$

for the fine structure constant, one obtains from (1.2) and (1.4) the theoretical prediction

$$a = (116564 \pm 2) \times 10^{-8}, \quad (1.6)$$

which disagrees slightly (1.7 standard deviations) with the experimental value (1.1).

The error interval in (1.6) reflects the uncertainty in the strong interaction contribution $[0.5 \times 10^{-8}]$, in the value of $\alpha/2\pi [0.2 \times 10^{-8}]$, and in the sixth order correction (1.3) $[0.6 \times 10^{-8}]$. It does not take into account the uncertainty in the magnitude of the vacuum polarization contribution of higher mass hadrons.¹⁹ We have also not included possible weak interaction corrections to the muon moment,²⁰ which could be expected to be of order 1×10^{-8} .

Also not included in the above error estimate is the contribution from the sixth order diagrams containing photon-photon scattering subdiagrams (Fig. 1). Of course this is because it has not been successfully calculated or estimated thus far. Earlier attempts^{11, 13, 21} have been directed at finding out whether this contribution contains $\ln(m_\mu/m_e)$ terms or not. Unfortunately, it is not easy to detect the presence or absence of logarithmic terms without extensive calculations. In fact, on the basis of general consideration of the mass singularity,⁴ it can be shown that the individual diagrams of Fig. 1 may contribute to the logarithmic terms. On the other hand, these terms might cancel each other when contributions from all six diagrams are put together. Indeed several arguments have been put forward indicating such a cancellation.¹¹ However, since none of these arguments has been free from loopholes, we have been convinced that this question cannot be settled short of an all-out effort. Once we decided to settle the question of logarithmic terms by an extensive calculation, it was not much harder to evaluate the Feynman integrals for the graphs of Fig. 1 exactly.

The result of our calculation of the contribution from the three photon exchange diagrams turns out to be surprisingly large

$$\begin{aligned} \Delta a_{\text{photon-photon}} &= (18.4 \pm 1.1) (\alpha/\pi)^3 \\ &= (23.0 \pm 1.4) \times 10^{-8}. \end{aligned} \quad (1.7)$$

This leads us to a revised theoretical prediction

$$a_{\text{theory}} = (116587 \pm 3) \times 10^{-8} \quad (1.8)$$

and

$$\begin{aligned} a_{\text{exp}} - a_{\text{theory}} &= (29 \pm 34) \times 10^{-8} \\ &= (250 \pm 290) \text{ ppm.} \end{aligned} \quad (1.9)$$

Thus the addition of the photon-photon scattering contribution essentially eliminates the discrepancy mentioned above. The theoretical error in (1.8) includes the uncertainty due to the numerical integration of the contribution (1.7) $[1.4 \times 10^{-8}]$. This error could be reduced if necessary. We wish to emphasize that, with the inclusion of the photon-photon scattering contribution (1.7), all of the Feynman diagrams from quantum electrodynamics which contribute to the difference of the muon and electron magnetic moments through sixth order have been calculated or estimated.¹³

The largeness of the contribution (1.7) is closely related to a logarithmic dependence on the muon and electron mass ratio. In fact, in the limit of large m_μ/m_e the result (1.7) can be expressed in the form

$$\Delta a_{\text{photon-photon}} = \left[(6.4 \pm 0.1) \ln(m_\mu/m_e) + \text{const.} \right] (\alpha/\pi)^3. \quad (1.10)$$

Thus earlier arguments¹¹ indicating a cancellation among the diagrams of Fig. 1 for the logarithmic terms are disproved.

Since no approximations are made in the reduction of the Feynman integrals to parametric form, we can also obtain the photon-photon scattering contribution to the sixth order anomalous magnetic moment of the electron. Our result is

$$\begin{aligned} (\Delta a_e)_{\text{photon-photon}} &= (0.36 \pm 0.04) (\alpha/\pi)^3 \\ &= (0.45 \pm 0.05) \times 10^{-8}, \end{aligned} \quad (1.11)$$

where the error limits represent the uncertainty in the required numerical integrations in seven dimensions. For completeness, the mass independent contribution (1.11) must be added into the muon result (1.8).

Combining (1.11) with the previously calculated or estimated sixth order contributions given in Refs. 9 and 10, the theoretical prediction for the electron

moment is

$$a_e = (1/2)(\alpha/\pi) - 0.32848 (\alpha/\pi)^2 + 0.55 (\alpha/\pi)^3. \quad (1.12)$$

The last term is by no means the entire theoretical result for the sixth order coefficient, since second order vacuum polarization insertions into the fourth order vertex have not been calculated and, in addition, the reliability of the estimate of Ref. 9 is not certain. Note that the calculation of Mignaco and Remiddi¹⁰ corresponds to the contribution of three- and four-particle intermediate states in the sixth order Feynman diagrams containing fourth order vacuum polarization. The fact that this contribution is not so small casts some doubt on the validity of the two-particle approximation used in the dispersion-theoretical calculations.

The experimental value of the electron moment from the Michigan group is²²

$$\begin{aligned} (a_e)_{\text{exp}} &= (1159549 \pm 30) \times 10^{-9} \\ &= (1/2)(\alpha/\pi) - 0.32848 (\alpha/\pi)^2 - (7.0 \pm 2.4)(\alpha/\pi)^3, \end{aligned} \quad (1.13)$$

where we have used the value of α from (1.5) and the fourth order theoretical prediction to obtain an experimental determination of the sixth order coefficient. It will be interesting to see whether future experiments and further development of the theoretical result will confirm the indicated discrepancy of sign and magnitude of the sixth order coefficient.

In the next sections we discuss the calculation of the results (1.7) and (1.11). In Section 2 we introduce a method which enables us to extract the magnetic moment contribution of the diagrams of Fig. 1 automatically. This leads us to the introduction of the set of four modified Feynman diagrams shown in Fig. 2. There are, of course, many ways of introducing the Feynman parameters, and it is important to choose a method which gives as simple a result as possible, as well as exposing all the identities implicit in the formulas. Because of its simplicity

and versatility we shall use the double parametric representation of Feynman amplitudes introduced a few years ago.⁴ Its application to the diagrams of Fig. 2 is given in Section 3. In Section 4 we carry out the trace calculations and other simplifying operations and present the exact form of the Feynman integrals using "currents" as auxiliary variables, which is perhaps the most transparent and economical way of writing down these integrals. In Section 5 we discuss an alternate, more standard method which we have also used to derive the Feynman parametric integrals. The connections between the two reduction methods is discussed, and an important identity, readily utilized by REDUCE, to simplify numerator expressions with high powers of loop momenta is given. In Section 6 we study the behavior of the Feynman integrals in the limit where $\rho = (m_e/m_\mu)^2$ tends to zero. The method of numerical integration used to evaluate the integrals as well as the results of computation are discussed in Section 7. Some properties of the functions A_i and B_{ij} are described in Appendix A. In Appendix B we give the unsimplified output of REDUCE for graph IV. Some formulas needed in Section 6 are given in Appendix C.

2. EXTRACTION OF THE MAGNETIC MOMENT TERM

According to the Feynman-Dyson rules, we can write the contribution of the graphs of Fig. 1 in the form²³

$$\langle p'|S|p, \Delta \rangle = -i(2\pi)^4 \delta^4(p'-p-\Delta) \frac{1}{(2\pi)^{9/2}} \frac{m_\mu}{(2\Delta_0 p_0 p'_0)^{1/2}} eM, \quad (2.1)$$

where

$$M = \frac{e^2}{(2\pi)^8} \int d^4 p_1 d^4 p_3 p_1^{-2} p_2^{-2} p_3^{-2} \epsilon^\mu \Pi_{\kappa\rho\sigma\mu}(-p_1, p_2, p_3, -\Delta) \\ \times \bar{u}(p') \gamma^\kappa (\not{p}_4 - m_\mu)^{-1} \gamma^\rho (\not{p}_5 - m_\mu)^{-1} \gamma^\sigma u(p), \quad (2.2)$$

and $\Pi_{\kappa\rho\sigma\mu}$ is the polarization tensor of fourth rank representing the photon-photon scattering

$$\Pi_{\kappa\rho\sigma\mu}(-p_1, p_2, p_3, -\Delta) = \frac{-ie^4}{(2\pi)^4} \int d^4 p_6 \text{Tr} \left[\gamma_\kappa (\not{p}_6 - m_e)^{-1} \gamma_\rho (\not{p}_7 - m_e)^{-1} \gamma_\sigma (\not{p}_8 - m_e)^{-1} \gamma_\mu (\not{p}_9 - m_e)^{-1} \right. \\ \left. + \text{five other terms - regularization terms} \right]. \quad (2.3)$$

As usual, all momenta are restricted by the energy-momentum conservation law at each vertex. As is well known, individual terms of $\Pi_{\kappa\rho\sigma\mu}$ are logarithmically divergent for large p_6 , but the sum of all six terms is convergent and well defined if it is properly regularized. In the integral (2.2), each term may again diverge because of the photon-photon scattering subdiagrams. In addition each term may diverge logarithmically when all three momenta p_1 , p_3 , and p_6 go to infinity simultaneously. Nevertheless, it is expected that cancellation of ultraviolet divergences takes place, as in photon-photon scattering, and that there will be no real divergence problem as far as the magnetic moment term of (2.2) is concerned.

Although it is not difficult to show by direct calculation that this is in fact the case, it would be convenient if the formula (2.2) could be rewritten so that the cancellation of ultraviolet divergences is manifestly evident from the beginning. This can be achieved by making use of the identity

$$\Pi_{\kappa\rho\sigma\mu}(-p_1, p_2, p_3, -\Delta) = -\Delta^\nu \frac{\partial}{\partial \Delta^\mu} \Pi_{\kappa\rho\sigma\nu}(-p_1, p_2, p_3, -\Delta) , \quad (2.4)$$

which is easily obtained by differentiating the condition of gauge invariance

$$\Delta^\nu \Pi_{\kappa\rho\sigma\nu}(-p_1, p_2, p_3, -\Delta) = 0 \quad (2.5)$$

with respect to Δ^μ , regarding, e.g., Δ , p_1 , p_3 as independent variables.

Substituting (2.4) into (2.3) we obtain

$$M = \epsilon^\mu \Delta^\nu \bar{u}(p') M_{\mu\nu} u(p) \quad (2.6)$$

where

$$M_{\mu\nu} = -\frac{e^2}{(2\pi)^8} \int d^4 p_1 d^4 p_3 p_1^{-2} p_2^{-2} p_3^{-2} \times \left[\frac{\partial}{\partial \Delta^\mu} \Pi_{\kappa\rho\sigma\nu}(-p_1, p_2, p_3, -\Delta) \right] \gamma^\kappa (\not{p}_4 - m_\mu)^{-1} \gamma^\rho (\not{p}_5 - m_\mu)^{-1} \gamma^\sigma . \quad (2.7)$$

Now, when the differentiation with respect to Δ^μ is carried out explicitly in (2.7), M can be regarded as a sum of the modified Feynman diagrams shown in Fig. 2. Since each diagram of Fig. 2 contains an electron loop with five vertices, it is clear that no ultraviolet divergences arise from integrations over internal momenta. Thus no diagram of Fig. 2 requires subtraction or regularization any longer and each gives a well-defined and convergent contribution to the muon magnetic moment. This means that each term can be evaluated separately by a straightforward application of the techniques of Feynman parameterization.

Also since M in (2.6) is already proportional to Δ^ν , we can put $\Delta=0$ in $M_{\mu\nu}$ after differentiation to obtain the static magnetic moment.²⁴ This simplifies the calculation considerably.

In order to extract the magnetic moment term from the second rank tensor $\bar{u} M_{\mu\nu} u$, we note that, because of covariance under Lorentz transformations, it can be expressed in the form

$$\bar{u}(p') M_{\mu\nu} u(p) = \bar{u}(p') \left[A g_{\mu\nu} + B(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) + C P_\mu \gamma_\nu + D P_\nu \gamma_\mu + E P_\mu P_\nu \right] u(p) \quad (2.8)$$

where $P = (p+p')/2$ and we have omitted terms containing Δ in (2.8) according to our remark in the preceding paragraph.²⁴ Since $\Delta^\nu P_\nu = 0$, the D and E terms do not contribute to the magnetic moment. The C term does not contribute either since $\Delta^\nu \bar{u} \gamma_\nu u = 0$ by current conservation. The coefficient A must be equal to zero in order that (2.8) satisfy gauge invariance. Thus the only contribution to the magnetic moment arises from the B term and is equal to $\Delta a = -4 m_\mu B$. In order to project out the magnetic moment term in (2.6), we have only to multiply both sides of (2.8) by $\bar{u}(p) (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) u(p')$ and sum over initial and final positive energy spin states of the muon. Thus we obtain²⁵

$$\Delta a = -4 m_\mu B = \frac{1}{48 m_\mu} \lim_{p' \rightarrow p} \text{Tr} \left((\not{p}' + m_\mu) (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) (\not{p} + m_\mu) M_{\mu\nu} \right). \quad (2.9)$$

3. DOUBLE PARAMETRIC REPRESENTATION

In introducing Feynman parameters in (2.9), it is important to choose them so that the result can be expressed in as simple a form as possible. Otherwise problems of this complexity easily become unmanageable. We shall use the double parametric representation of Feynman amplitudes,⁴ probably one of the simplest systematic methods. Further simplification is achieved by a judicious choice of Feynman parameters common to all graphs of Fig. 2, enabling us to express the denominators of all integrands in identical form, and by introducing (as is shown in Section 4) auxiliary parameters, called "currents", which simplify the form of numerators enormously.

Let us first parameterize graph I whose lines are labeled as shown in Fig. 2. We shall write the propagator for the i^{th} internal boson line of mass m_i as

$$\left[(r_i + q_i)^2 - m_i^2 + i\epsilon \right]^{-1}, \quad (3.1)$$

where we have put $p_i = r_i + q_i$, r_i and q_i representing variable and fixed momenta. We choose r_i and q_i in such a way that they satisfy the separate "momentum conservation laws"

$$\sum \pm r_i = 0, \quad \sum \pm q_i + \text{external momenta} = 0 \quad (3.2)$$

for each vertex, where + or - is chosen according as $r_i + q_i$ is incoming or outgoing. Other than that they are left indeterminate for the moment. If the i^{th} line is a fermion line, the corresponding propagator is obtained by applying the operator

$$\pm \not{p}_i + m_i \quad \text{with} \quad D_i^\mu = \frac{1}{2} \int_{m_i^2}^{\infty} dm_i^2 \frac{\partial}{\partial q_{i\mu}} \quad (3.3)$$

on (3.1), where the sign + or - should be chosen according as q_i is in the direction of the arrow of the fermion line or not.

Noting that the D-operator (3.3) can be interchanged with the integration over the momenta r_1, r_3, r_6 in (2.7) because this integral is absolutely convergent, we can express the contribution of graph I to the anomalous magnetic moment of the muon (2.9) as follows:²⁶

$$\Delta a_I = \frac{1}{48m_\mu} \frac{2ie^6}{(2\pi)^{12}} F_0 F_I \int \frac{d^4 r_1 d^4 r_3 d^4 r_6}{\prod_{i=1}^{10} ((r_i + q_i)^2 - m_i^2 + i\epsilon)}, \quad (3.4)$$

where

$$F_0 = \text{Tr} \left[(\not{p} + m_\mu) (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) (\not{p}' + m_\mu) \gamma^k (\not{p}_4 + m_\mu) \gamma^\rho (\not{p}_5 + m_\mu) \gamma^\sigma \right] \quad (3.5)$$

and

$$F_I = \text{Tr} \left[\gamma_k (\not{p}_6 + m_e) \gamma_\rho (\not{p}_7 + m_e) \gamma_\sigma (\not{p}_8 + m_e) \gamma_\nu (\not{p}_9 + m_e) \gamma_\mu (\not{p}_{10} + m_e) \right]. \quad (3.6)$$

Before we carry out the r_i -integration, let us first collect all propagators whose integration momenta r_i are identical and can be expressed by a common variable r_α . The set of all such propagators will be called a chain α . For instance, the lines 1 and 4 have the same integration momentum $r_1 = r_4$, and will constitute the chain α . Making use of the Feynman parameters x_1 and x_4 with $x_1 + x_4 = 1$, we shall combine the corresponding propagators into the form²⁷

$$(n_\alpha - 1)! \int \frac{dx(\alpha)}{[(r_\alpha + q_\alpha)^2 - V_\alpha(x) + i\epsilon]^{n_\alpha}}, \quad (3.7)$$

where $n_\alpha = 2$ is the number of lines in the chain α and

$$dx(\alpha) = \delta(1 - x_1 - x_4) dx_1 dx_4, \quad (3.8)$$

$$V_\alpha(x) = x_1 m_1^2 + x_4 m_4^2 - x_1 x_4 (q_1 - q_4)^2.$$

Expressions of the form (3.7) can be written down in the same fashion for the chains $\beta = (3, 5)$ and $\gamma = (8, 9, 10)$. Remaining lines form chains by themselves: $\lambda = (7)$, $\mu = (6)$, $\nu = (2)$.

Now the integrand of (3.4) is a product of factors of the form (3.7), which can be combined into a single denominator using the formula

$$\prod_{i=1}^m \frac{(n_i-1)!}{a_i^{n_i}} = (n-1)! \int \frac{\delta(1-z_1 \dots -z_m) z_1^{n_1-1} dz_1 \dots z_m^{n_m-1} dz_m}{(z_1^{a_1} + z_2^{a_2} + \dots + z_m^{a_m})^n}, \quad (3.9)$$

where $n = n_1 + n_2 + \dots + n_m$. The resulting expression can be integrated easily with respect to the momenta r_α , etc., and the integral in (3.4) can be expressed in the double parametric form

$$3! i^3 \pi^6 \int \frac{dz}{U^2(z) [V(x, z) - i\epsilon]^4}, \quad (3.10)$$

where the discriminant $U(z)$ is a homogeneous polynomial of order 3 in z ,²⁸

$$V(x, z) = z_\alpha V_\alpha(x) + \dots + z_\nu V_\nu(x) + v(x, z), \quad (3.11)$$

$$\begin{aligned} -v(x, z)U(z) = & z_\beta z_\gamma z_\lambda (z_\alpha + z_\mu + z_\nu)(q_\beta - q_\gamma + q_\lambda)^2 \\ & + z_\alpha z_\gamma z_\mu (z_\beta + z_\lambda + z_\nu)(q_\alpha - q_\gamma + q_\mu)^2 \\ & + z_\alpha z_\beta z_\nu (z_\gamma + z_\lambda + z_\mu)(q_\alpha - q_\beta - q_\nu)^2 \\ & + z_\lambda z_\mu z_\nu (z_\alpha + z_\beta + z_\gamma)(-q_\lambda + q_\mu + q_\nu)^2 \\ & + z_\alpha z_\beta z_\lambda z_\mu (q_\alpha - q_\beta - q_\lambda + q_\mu)^2 \\ & + z_\alpha z_\gamma z_\lambda z_\nu (q_\alpha - q_\gamma + q_\lambda - q_\nu)^2 \\ & + z_\beta z_\gamma z_\mu z_\nu (q_\beta - q_\gamma + q_\mu + q_\nu)^2, \end{aligned} \quad (3.12)$$

and

$$dz \equiv \delta(1-z_\alpha \dots -z_\nu) dz(\alpha) \dots dz(\nu) ,$$

$$dz(\alpha) = z_\alpha^{n_\alpha-1} dz_\alpha dx(\alpha), \quad \text{etc.} \quad (3.13)$$

Substituting (3.10) into (3.4), we finally obtain

$$\Delta a_I = \frac{1}{256m_\mu} \left(\frac{\alpha}{\pi}\right)^3 F_0 F_I \int \frac{dz}{U^2(z) [V(x,z) - i\epsilon]^4} . \quad (3.14)$$

Advantages in adopting this parameterization are twofold: (1) The discriminant $U(z)$ is determined completely by the topological structure of the chain diagram and not by individual lines. Since all six graphs have the same chain structure, $U(z)$ is common to all graphs if we name the chains in an appropriate manner. (2) The denominator function $V(x,z)$ takes the most compact and explicit form for this parameterization. The formula for $v(x,z)$ will be much more lengthy than (3.12) for any other way of parameterization. In addition, if we introduce chains in the other graphs so that they have the same chain structure as graph I, for instance $\alpha = (1,4)$, $\beta = (3,5)$, $\gamma = (8,9)$, $\lambda = (7)$, $\mu = (6,10)$, $\nu = (2)$ for graph IV, we find that not only $U(z)$ but also $v(x,z)$ given by (3.12) (and hence $V(x,z)$) are identical with those of graph I; the only differences between different graphs being contained in the explicit expressions for q_α , $V_\alpha(x)$, and $dz(\alpha)$ given by (3.8) and (3.13). As is shown later, even these differences disappear in the end.

Thus, at least formally, the contributions Δa_{II} , Δa_{III} , Δa_{IV} to the muon magnetic moment from the remaining graphs can be expressed by the formula (3.14) if only we replace F_I by

$$F_{II} = \text{Tr} \left[\gamma_\mu (\not{D}_6 + m_e) \gamma_\rho (\not{D}_7 + m_e) \gamma_\sigma (\not{D}_8 + m_e) \gamma_\kappa (\not{D}_9 + m_e) \gamma_\nu (\not{D}_{10} + m_e) \right], \quad (3.15)$$

$$F_{III} = -\text{Tr} \left[\gamma_\kappa (\not{D}_6 + m_e) \gamma_\rho (\not{D}_7 + m_e) \gamma_\mu (\not{D}_9 + m_e) \gamma_\nu (\not{D}_{10} + m_e) \gamma_\sigma (\not{D}_8 + m_e) \right], \quad (3.16)$$

$$F_{IV} = \text{Tr} \left[\gamma_\kappa (\not{D}_{10} + m_e) \gamma_\mu (\not{D}_6 + m_e) \gamma_\rho (\not{D}_7 + m_e) \gamma_\sigma (\not{D}_8 + m_e) \gamma_\nu (\not{D}_9 + m_e) \right]. \quad (3.17)$$

The factor -1 in F_{III} arises because Δ' flows in the opposite direction around the electron loop relative to the other three graphs.

4. TRACES AND D-OPERATIONS

Our next task is to perform the trace calculation and determine the effect of D-operations explicitly.

The trace calculation for graph I is simplified considerably if one notes that F_I can be written as

$$F_I = \left(-D_9 D_{10} + m_e^2 \right) F'_I + \text{the remainder} , \quad (4.1)$$

with

$$F'_I = \text{Tr} \left[\gamma_\kappa (\not{D}_6 + m_e) \gamma_\rho (\not{D}_7 + m_e) \gamma_\sigma \gamma_\mu (\not{D}_8 - m_e) \gamma_\nu \right] , \quad (4.2)$$

where the remainder consists of terms which are either symmetric in μ and ν or proportional to $D_8 - D_9$ and $D_9 - D_{10}$ and thus give vanishing contribution to Δa_I . Furthermore we have²⁹

$$\left(-D_9 D_{10} + m_e^2 \right) \int \frac{dz}{U^2 V^4} = \frac{1}{3} \int \frac{dz'}{\left[U^2 V^3 \right]_{x_{10}=0}} , \quad (4.3)$$

where dz' has the same form as dz defined by (3.13) except that $dz(\gamma)$ is replaced by

$$dz'(\gamma) = z_\gamma dz_\gamma dx_8 dx_9 \delta(1-x_8-x_9) . \quad (4.4)$$

Using (4.1) and (4.3) we can therefore simplify (3.14) to

$$\Delta a_I = \frac{1}{768m_\mu} \left(\frac{\alpha}{\pi} \right)^3 F_0 F'_I \int \frac{dz'}{\left[U^2 V^3 \right]_{x_{10}=0}} . \quad (4.5)$$

In the same fashion Δa_{II} and Δa_{III} can be expressed in the form (4.5) if we replace F'_I by

$$F'_{II} = \text{Tr} \left[\gamma_\kappa \gamma_\mu (\not{D}_6 - m_e) \gamma_\nu \gamma_\rho (\not{D}_7 + m_e) \gamma_\sigma (\not{D}_8 + m_e) \right] \quad (4.6)$$

and

$$F'_{\text{III}} = -\text{Tr} \left[\gamma_{\kappa} (\not{D}_6 + m_e) \gamma_{\rho} \gamma_{\nu} (\not{D}_7 - m_e) \gamma_{\mu} \gamma_{\sigma} (\not{D}_8 + m_e) \right], \quad (4.7)$$

and interpret dz' somewhat differently. In the case of Δ_{IV} , we do not obtain too much simplification. But we write F_{IV} as

$$\begin{aligned} F_{\text{IV}} = F'_{\text{IV}} = & \text{Tr} \left[\gamma_{\kappa} \gamma_{\mu} \gamma_{\rho} \not{D}_7 \gamma_{\sigma} \gamma_{\nu} \right] \left(-D_6 D_{10} + m_e^2 \right) \left(-D_8 D_9 + m_e^2 \right) \\ & + 2 D_{8\nu} \text{Tr} \left[\gamma_{\kappa} \gamma_{\mu} \gamma_{\rho} (\not{D}_7 + m_e) \gamma_{\sigma} (\not{D}_9 + m_e) \right] \left(-D_6 D_{10} + m_e^2 \right) \\ & + 2 D_{6\mu} \text{Tr} \left[\gamma_{\kappa} (\not{D}_{10} + m_e) \gamma_{\rho} (\not{D}_7 + m_e) \gamma_{\sigma} \gamma_{\nu} \right] \left(-D_8 D_9 + m_e^2 \right) \\ & + 4 D_{6\mu} D_{8\nu} \text{Tr} \left[\gamma_{\kappa} (\not{D}_{10} + m_e) \gamma_{\rho} (\not{D}_7 + m_e) \gamma_{\sigma} (\not{D}_9 + m_e) \right] \\ & + \text{vanishing terms}, \end{aligned} \quad (4.8)$$

and apply formulas similar to (4.3).

It is also convenient to write F_0 coming from the muon lines as

$$\begin{aligned} F_0 = & 2m_{\mu} \text{Tr} \left[p^{\kappa} \gamma^{\rho} \not{D}_3 \gamma^{\sigma} + p^{\sigma} \gamma^{\kappa} \not{D}_1 \gamma^{\rho} \right] \left\{ \not{D} (\gamma^{\mu} \gamma^{\nu} - \gamma^{\nu} \gamma^{\mu}) + (\gamma^{\mu} \gamma^{\nu} - \gamma^{\nu} \gamma^{\mu}) \not{D} \right\} \\ & + m_{\mu} \text{Tr} \left[\gamma^{\kappa} \not{D}_1 \gamma^{\rho} \not{D}_3 \gamma^{\sigma} \left\{ \not{D} (\gamma^{\mu} \gamma^{\nu} - \gamma^{\nu} \gamma^{\mu}) + (\gamma^{\mu} \gamma^{\nu} - \gamma^{\nu} \gamma^{\mu}) \not{D} \right\} \right]. \end{aligned} \quad (4.9)$$

This is obtained using the identities resulting from the mass shell condition $p^2 = m_{\mu}^2$:

$$\begin{aligned} (\not{D} + m_{\mu}) \gamma^{\kappa} (\not{D}_4 + m_{\mu}) &= (\not{D} + m_{\mu}) (\gamma^{\kappa} \not{D}_1 + 2p^{\kappa}), \\ (\not{D}_5 + m_{\mu}) \gamma^{\sigma} (\not{D} + m_{\mu}) &= (\not{D}_3 \gamma^{\sigma} + 2p^{\sigma}) (\not{D} + m_{\mu}), \end{aligned} \quad (4.10)$$

where we have made use of the equations

$$\begin{aligned} D_1 &= D_4 - p, \\ D_3 &= D_5 - p, \end{aligned} \quad (4.11)$$

which follow from (A.3).

In order to carry out the D-operation explicitly, it is convenient to introduce the functions

$$Q_i^\mu = -\frac{1}{2x_{i\alpha} z_\alpha} \frac{\partial V}{\partial q_{i\mu}}, \quad (4.12)$$

$$g_{\mu\nu} B_{ij} = \frac{U}{2x_{i\alpha} z_\alpha x_{j\beta} z_\beta} \frac{\partial^2 V}{\partial q_i^\mu \partial q_j^\nu}, \quad (4.13)$$

where the lines i and j belong to the chains α and β , respectively. Then it is easy to see that the result of applying D-operators on $1/V^n$, n being a sufficiently large positive integer, can be expressed in terms of Q's and B's as follows:

$$\begin{aligned} D_{i\mu} \frac{1}{V^n} &= \frac{Q_{i\mu}}{V^n}, \\ D_{i\mu} D_{j\nu} \frac{1}{V^n} &= \frac{Q_{i\mu} Q_{j\nu}}{V^n} - \frac{1}{2(n-1)} \frac{g_{\mu\nu} B_{ij}}{UV^{n-1}}, \\ D_{i\mu} D_{j\nu} D_{k\lambda} \frac{1}{V^n} &= \frac{Q_{i\mu} Q_{j\nu} Q_{k\lambda}}{V^n} - \frac{1}{2(n-1)} \frac{Q_{i\mu} g_{\nu\lambda} B_{jk} + Q_{j\nu} g_{\mu\lambda} B_{ik} + Q_{k\lambda} g_{\mu\nu} B_{ij}}{UV^{n-1}}, \\ &\dots \dots \dots \end{aligned} \quad (4.14)$$

As is shown in Appendix A, the quantity Q_i satisfies the "Kirchhoff's laws."² Thus it can be regarded as the "current" running through the i^{th} line for given "external currents" p, p' and given "resistances" (i.e., given values of Feynman parameters) of internal lines.

In our problem in which we put $p' = p$ eventually, all internal currents $Q_{i\mu}$ become proportional to p_μ and thus the proportionality coefficient A_i defined by

$$Q_{i\mu} = \frac{A_i}{U} p_\mu \quad (4.15)$$

itself may be regarded as a current satisfying the Kirchoff's laws. The functions A_i and B_{ij} are homogeneous polynomials of $z_\alpha, z_\beta, \dots, z_\nu$. Their explicit form and properties are discussed in Appendix A.

We can now carry out the trace calculations and D-operations and write down the integrals for $\Delta a_{I'}, \dots, \Delta a_{IV}$ explicitly in terms of A_i 's and B_{ij} 's. We shall write them as follows:

$$\Delta a_{I'} = \left(\frac{\alpha}{\pi}\right)^3 \left[M_{I'a} + M_{I'b} + M_{I'c} + M_{I'd} \right], \quad I' = I, II, III, IV, \quad (4.16)$$

where $M_{I'a}$ and $M_{I'c}$ are obtained from the first term of (4.9) and $M_{I'b}$ and $M_{I'd}$ from the second term of (4.9). Also $M_{I'a}$ and $M_{I'b}$ are terms which arise from the m_e -independent part of the electron trace $F_{I'}$, and $M_{I'c}$ and $M_{I'd}$ represent the remainders.

In these integrals it is trivial to carry out the x_9 and x_{10} integrations. After this is done, differences among $V(x, z)$ of different graphs disappear completely, as was mentioned in Section 3. We shall now introduce the new z variables as follows:

$$\begin{aligned} z_1 &= z_\alpha x_1, & z_2 &= z_\nu, & z_3 &= z_\beta x_3, & z_4 &= z_\alpha x_4, \\ z_5 &= z_\beta x_5, & z_6 &= z_\mu, & z_7 &= z_\lambda, & z_8 &= z_\gamma. \end{aligned} \quad (4.17)$$

In terms of these variables $U(z)$ can be written as

$$\begin{aligned} U(z) &= (z_1+z_4) (z_3+z_5) (z_6+z_7+z_8) \\ &+ (z_1+z_4) \{ z_2 (z_6+z_7+z_8) + z_7 (z_6+z_8) \} \\ &+ (z_3+z_5) \{ z_2 (z_6+z_7+z_8) + z_6 (z_7+z_8) \} \\ &+ z_8 (z_2 z_6 + z_2 z_7 + z_6 z_7) . \end{aligned} \quad (4.18)$$

We shall also introduce the notation

$$dz_0 \equiv \delta \left(1 - \sum_{i=1}^8 z_i \right) \prod_{i=1}^8 dz_i \quad (4.19)$$

and

$$W = m_{\mu}^{-2} UV, \quad (4.20)$$

where V is given by (3.11) and (3.12).

Since we have carried out all D-operations, we can now put $p' = p$ and $\Delta=0$ in the denominator V . Then, after one puts the photon mass equal to zero, the function W takes the very simple form

$$W = az_4^2 + bz_5^2 + c(z_4+z_5)^2 + \rho(z_6+z_7+z_8) U, \quad (4.21)$$

where

$$\rho = \left(\frac{m_e}{m_{\mu}} \right)^2 \quad (4.22)$$

and

$$\begin{aligned} a &= B_{47} - B_{46} = (z_3+z_5)(z_6+z_7+z_8) + z_7z_8, \\ b &= B_{56} - B_{57} = (z_1+z_4)(z_6+z_7+z_8) + z_6z_8, \\ c &= B_{48} - B_{47} = B_{58} - B_{56} = z_2(z_6+z_7+z_8) + z_6z_7, \end{aligned} \quad (4.23)$$

B_{ij} being given in Appendix A.

We are now ready to write down our integrals:

$$\begin{aligned} M_{1a} &= -4 \int \frac{dz_0}{U^3 W^3} z_8 A_1 A_6 A_7 A_8 \\ &+ \int \frac{dz_0}{U^3 W^2} z_8 \left[3A_7 A_8 B_{46} + A_6 (A_8 B_{47} + A_7 B_{48}) + 2A_1 (A_7 B_{68} + A_8 B_{67}) \right] \\ &- \int \frac{dz_0}{U^3 W} z_8 \left[2B_{47} B_{68} + B_{48} B_{67} \right], \end{aligned} \quad (4.24)$$

$$\begin{aligned}
M_{Ib} = & -3 \int \frac{dz_0}{U^4 W^3} z_8 A_1 A_3 A_6 A_7 A_8 \\
& + \frac{1}{2} \int \frac{dz_0}{U^4 W^2} z_8 \left[A_6 A_7 A_8 B_{45} + 2A_1 A_6 (A_7 B_{58} + 8A_8 B_{57}) + A_1 A_3 (A_8 B_{67} + 4A_7 B_{68}) \right] \\
& - \frac{1}{2} \int \frac{dz_0}{U^4 W} z_8 \left[2A_1 (B_{56} B_{78} + 5B_{57} B_{68}) + 3A_8 (B_{45} B_{67} + 5B_{46} B_{57}) \right. \\
& \left. - 2A_6 (B_{47} B_{58} - 4B_{48} B_{57}) \right], \tag{4.25}
\end{aligned}$$

$$\begin{aligned}
M_{Ic} = & 4\rho \int \frac{dz_0}{UW^3} z_8 A_1 (A_1 + A_7) \\
& - \rho \int \frac{dz_0}{UW^2} z_8 (B_{48} + 3B_{47} - 3B_{46}), \tag{4.26}
\end{aligned}$$

$$\begin{aligned}
M_{Id} = & \rho \int \frac{dz_0}{U^2 W^3} z_8 A_1 A_3 (A_6 + A_7 + A_8) \\
& + \frac{\rho}{2} \int \frac{dz_0}{U^2 W^2} z_8 \left[(2A_7 - 3A_8) B_{45} - 2A_1 (5B_{57} - B_{56}) \right], \tag{4.27}
\end{aligned}$$

$$\begin{aligned}
M_{IIa} = & 2 \int \frac{dz_0}{U^3 W^3} z_6 (A_7 - A_6) A_6 A_7 A_8 \\
& + \frac{1}{2} \int \frac{dz_0}{U^3 W^2} z_6 \left[A_7 A_8 (B_{56} - B_{46}) + 3A_6 A_8 (B_{57} - B_{47}) + A_6 A_7 (B_{58} - 3B_{48}) \right. \\
& \left. - 2A_1 (A_8 B_{67} + A_7 B_{68} - A_6 B_{78}) + 2A_3 (A_8 B_{67} + A_6 B_{78}) \right] \\
& - \frac{1}{2} \int \frac{dz_0}{U^3 W} z_6 \left[B_{56} B_{78} + 2B_{58} B_{67} + B_{46} B_{78} - 2B_{47} B_{68} - 2B_{48} B_{67} \right], \tag{4.28}
\end{aligned}$$

$$\begin{aligned}
M_{\text{IIb}} &= \int \frac{dz_0}{U^4 W^3} z_6^{A_1 A_3 A_6 A_7 A_8} \\
&+ \frac{1}{2} \int \frac{dz_0}{U^4 W^2} z_6 \left[A_6 A_7 A_8 B_{45} - A_3 A_8 (2A_6 B_{47} + A_7 B_{46}) \right. \\
&\quad \left. - A_1 (A_7 A_8 B_{56} + 2A_6 A_8 B_{57} + 2A_6 A_7 B_{58}) + A_1 A_3 (A_6 B_{78} - 2A_7 B_{68}) \right] \\
&+ \frac{1}{2} \int \frac{dz_0}{U^4 W} z_6 \left[3A_6 (B_{47} B_{58} - B_{45} B_{78}) + A_7 (2B_{46} B_{58} + B_{48} B_{56}) + A_8 (B_{46} B_{57} + 2B_{47} B_{56}) \right. \\
&\quad \left. + A_1 (5B_{57} B_{68} + B_{58} B_{67}) + A_3 (B_{47} B_{68} - B_{48} B_{67}) \right], \tag{4.29}
\end{aligned}$$

$$\begin{aligned}
M_{\text{IIc}} &= 2\rho \int \frac{dz_0}{UW^3} z_6 \left[A_1 (A_6 - A_7 - A_8) + A_3 (A_6 - A_7 + A_8) \right] \\
&\quad - \frac{\rho}{2} \int \frac{dz_0}{UW^2} z_6 \left[B_{46} - B_{47} - B_{48} + B_{56} - 3B_{57} + 3B_{58} \right], \tag{4.30}
\end{aligned}$$

$$\begin{aligned}
M_{\text{IId}} &= \rho \int \frac{dz_0}{U^2 W^3} z_6^{A_1 A_3} (A_6 - 3A_7 + A_8) \\
&\quad - \frac{\rho}{2} \int \frac{dz_0}{U^2 W^2} z_6 \left[(3A_6 - A_7 - A_8) B_{45} - A_1 (5B_{57} - B_{58}) - 3A_3 (B_{47} - B_{48}) \right], \tag{4.31}
\end{aligned}$$

$$\Delta a_{\text{III}} = \Delta a_{\text{II}}, \tag{4.32}$$

$$\begin{aligned}
M_{\text{IVa}} &= \int \frac{dz_0}{U^3 W^2} (B_{67} - z_2 z_8) (z_6^{A_1 A_6} + z_7^{A_3 A_7}) \\
&\quad - 3 \int \frac{dz_0}{U^3 W} (B_{67} - z_2 z_8) (z_6^{B_{46}} + z_7^{B_{57}}), \tag{4.33}
\end{aligned}$$

$$\begin{aligned}
M_{IVb} = & - \int \frac{dz_0}{U^4 W^2} \left[z_8 A_1 A_8 (B_{67} A_3 + z_2 z_8 A_7) + (z_7 A_7)^2 \{ (z_1 + z_4) A_1 + z_6 A_6 \} \right] \\
& + 3 \int \frac{dz_0}{U^4 W^2} \left[z_8 B_{48} (A_8 B_{67} - A_7 B_{68}) - z_6 z_7 B_{67} \{ (z_1 + z_4) A_1 - z_6 A_6 \} \right], \quad (4.34)
\end{aligned}$$

$$M_{IVc} = -2\rho \int \frac{dz_0}{UW^2} z_6 (B_{46} + z_7 z_8), \quad (4.35)$$

$$M_{IVd} = \rho \int \frac{dz_0}{UW^2} z_4 z_7 (z_6^{-z_8}). \quad (4.36)$$

We have obtained all terms except M_{IVb} by hand calculation. The complete integrand was obtained with the help of REDUCE.⁵ Also some of the formulas given in Appendix A have been used to simplify the integrands. For the benefit of those readers who wish to check our calculations, we shall give in Appendix B the integrals M_{IVa}, \dots, M_{IVd} in unsimplified form.

5. AN ALTERNATIVE METHOD

As an alternative and check of the calculation presented in Section 3 and Section 4, we have also performed the reduction of the formula (2.2) to the parametric form (formulas (4.24) – (4.36)) using the standard techniques outlined in Section 18.4 of Ref. 2. The required extension of these techniques to the graphs of Fig. 1 is discussed here.

After the traces and index contractions are performed to project out Δa as in (2.9), and after some simplification with respect to the lines 9 and 10, our integrals can be reduced to the basic form

$$I = \int \frac{d^4 \ell_1 d^4 \ell_2 d^4 \ell_3 F(p_1, \dots, p_8)}{\prod_{j=1}^8 (p_j^2 - m_j^2)}, \quad (5.1)$$

where F is a polynomial in p_1, \dots, p_8 and the denominator may be multiplied by another factor of $p_j^2 - m_j^2$, $j = 6, 7, \text{ or } 8$, depending on which graph of Fig. 2 we are considering. Our labeling of loop momenta ℓ_1, ℓ_2, ℓ_3 is shown in Fig. 3. In accordance with the prescription of Ref. 2, we shall write p_j as

$$p_j = k_j + \ell_j = k_j + \sum_{r=1}^3 \eta_{jr} \ell_r, \quad (5.2)$$

where η_{jr} is the projection ($\pm 1, 0$) of p_j along ℓ_r . The k_j can be any choice of fixed momenta (independent of ℓ_r) such that four momentum conservation is satisfied at the six vertices of Fig. 3.

Next we introduce Feynman parameters z_1, \dots, z_8 , and rewrite (5.1) as

$$I = 7! \int \delta\left(1 - \sum_{k=1}^8 z_k\right) dz_1 \dots dz_8 \int \frac{d^4 \ell_1 d^4 \ell_2 d^4 \ell_3 F(p)}{\left[\sum_{j=1}^8 z_j (p_j^2 - m_j^2)\right]^8}. \quad (5.3)$$

If we choose the k_j such that

$$\sum_{j=1}^8 z_j^{k_j} \eta_{jr} = 0, \quad r = 1, 2, 3, \quad (5.4)$$

then the denominator in (5.3) has no $k \cdot l$ cross terms:

$$\sum_{j=1}^8 z_j \left(p_j^2 - m_j^2 \right) = -D + \sum_{r,r'=1}^3 U_{rr'} \ell_r \cdot \ell_{r'} \quad (5.5)$$

where

$$D = \sum_{j=1}^8 z_j \left(m_j^2 - k_j^2 \right) \quad (5.6)$$

and

$$U_{rr'} = \sum_{j=1}^8 z_j \eta_{jr} \eta_{jr'} \quad (5.7)$$

The fixed momenta k_1, \dots, k_8 are subject to (6-1) linearly independent equations (momentum conservation at each vertex, or Kirchhoff's first law) and three equations (5.4) (Kirchhoff's second law). Thus these momenta are completely and uniquely determined as functions of external momenta and Feynman parameters. Since Q_j 's defined by (4.12) also satisfy the same set of equations as is shown in Appendix A, and since the solution is unique, k_j must be identical with Q_j . Note that, although q_j defined by (3.1) and k_j defined by (5.2) look quite similar, they are in fact entirely different. The former does not satisfy Kirchhoff's second law while the latter does. The former is a constant vector independent of Feynman parameter z while the latter is a function of z .

Although k_j is identical with Q_j and is thus given explicitly in Appendix A, it will be instructive to see how they may be determined directly by the Kirchhoff's

laws. Let us first write down the second law (5.4) explicitly

$$\begin{aligned}
 z_1 k_1 + z_2 k_2 + z_4 k_4 - z_6 k_6 &= 0 , \\
 z_2 k_2 - z_3 k_3 - z_5 k_5 + z_7 k_7 &= 0 , \\
 z_6 k_6 + z_7 k_7 + z_8 k_8 &= 0 .
 \end{aligned}
 \tag{5.8}$$

Making use of the first law (four-momentum conservation)

$$\begin{aligned}
 k_4 &= k_1 + p, & k_5 &= k_3 + p , \\
 k_2 &= k_4 - k_5 = k_1 - k_3 , \\
 k_6 &= k_8 - k_1, & k_7 &= k_8 - k_3 ,
 \end{aligned}
 \tag{5.9}$$

we may rewrite (5.8) as

$$\begin{aligned}
 (z_1 + z_2 + z_4 + z_6) k_1 - z_2 k_3 - z_6 k_8 &= -z_4 p , \\
 -z_2 k_1 + (z_2 + z_3 + z_5 + z_7) k_3 - z_7 k_8 &= -z_5 p , \\
 -z_6 k_1 - z_7 k_3 + (z_6 + z_7 + z_8) k_8 &= 0 .
 \end{aligned}
 \tag{5.10}$$

The solution k_1 is given by

$$\begin{aligned}
 k_1 &= \frac{p}{U} \begin{vmatrix} -z_4 & -z_2 & -z_6 \\ -z_5 & z_2 + z_3 + z_5 + z_7 & -z_7 \\ 0 & -z_7 & z_6 + z_7 + z_8 \end{vmatrix} \\
 &= \frac{p}{U} A_1 = Q_1 ,
 \end{aligned}
 \tag{5.11}$$

where A_1 is given by (A.7) and U by

$$U(z) = \det(U_{rr}) , \tag{5.12}$$

or, more explicitly, by (4.18). Other k_j 's can be determined in the same fashion, confirming

$$k_j = (A_j/U)p = Q_j, \quad j = 1, 2, \dots, 8. \quad (5.13)$$

Substituting these k_j 's in D of (5.6), we also obtain

$$D = V = m_\mu^2 W/U, \quad (5.14)$$

where V is given by (3.11) and W is defined by (4.21).

Now, when the substitution (5.2) is made in the integral (5.1), and averaging over the direction of p is made, the integrand $F(p_1, \dots, p_8)$ becomes a polynomial in l_r . We are now ready to carry out the integration over $l_1, l_2,$ and l_3 . The basic integration over loop momenta is

$$7! \int \frac{d^4 l_1 d^4 l_2 d^4 l_3}{\left[-D + \sum_{r,r'=1}^3 U_{rr'} l_r \cdot l_{r'}\right]^8} = i^3 \pi^6 \frac{1}{U^2 D^2}. \quad (5.15)$$

Integrands of (5.1) containing extra denominator factor $p_k^2 - m_k^2$ can be integrated using parametric differentiation of (5.15) with respect to m_k^2 . Similarly, integrands of (5.1) which contain numerator factors $l_j \cdot l_k$ can be integrated³⁰ using parametric differentiation with respect to the $U_{rr'}$. For example,

$$\begin{aligned} & 8! \int \frac{(l_j \cdot l_k) d^4 l_1 d^4 l_2 d^4 l_3}{\left[-D + \sum_{r,r'} U_{rr'} l_r \cdot l_{r'}\right]^9} \\ &= - \sum_{s,s'=1}^3 \eta_{js} \eta_{ks'} \frac{\partial}{\partial \bar{U}_{ss'}} 7! \int \frac{d^4 l_1 d^4 l_2 d^4 l_3}{\left[-D + \sum_{r,r'} U_{rr'} l_r \cdot l_{r'}\right]^8} \\ &= - \frac{i^3 \pi^6}{D^2} \sum_{s,s'} \eta_{js} \eta_{ks'} \frac{\partial}{\partial \bar{U}_{ss'}} \left(\frac{1}{U^2}\right) = 2i^3 \pi^6 \frac{B_{jk}}{U^3 D^2}. \end{aligned} \quad (5.16)$$

We have defined

$$\bar{U}_{ss'} = \begin{cases} U_{ss'} & \text{for } s = s' , \\ 2U_{ss'} & \text{for } s \neq s' , \end{cases} \quad (5.17)$$

and

$$B_{jk} = \sum_{s,s'} \eta_{js} \eta_{ks'} B_{ss'} = \sum_{s,s'} \eta_{js} \eta_{ks'} \frac{\partial U}{\partial \bar{U}_{ss'}} . \quad (5.18)$$

Notice that $B_{ss'}$ is the signed cofactor of $U_{ss'}$ in U . Consequently $B_{ss'}/U$ is the inverse of the matrix $U_{ss'}$. Again the calculation of B_{jk} given here agrees with that of Appendix A in terms of z_1, \dots, z_8 .

In addition to quadratic terms, numerators with up to six powers of loop momenta ℓ_r appear in the computation of graph IV. An important identity for reducing the required higher order derivatives is

$$U \frac{\partial^2 U}{\partial \bar{U}_{ab} \partial \bar{U}_{cd}} = B_{ab} B_{cd} - \frac{1}{2} B_{ac} B_{bd} - \frac{1}{2} B_{ad} B_{bc} , \quad (5.19)$$

which holds for symmetric matrices $U_{rr'} = U_{r'r}$. To prove this, let us first assume that $U_{rr'}$ is not symmetric and all its elements are independent, and show that

$$U \frac{\partial^2 U}{\partial U_{ab} \partial U_{cd}} = B_{ab} B_{cd} - B_{ad} B_{bc} , \quad B_{ab} \equiv \frac{\partial U}{\partial U_{ab}} , \quad (5.20)$$

holds for such a U . We start from the identity

$$\sum_{j=1}^3 U_{ij} B_{cj} = \delta_{ic} U . \quad (5.21)$$

Differentiating both sides with respect to U_{ab} we obtain

$$\delta_{ai} B_{cb} + \sum_j U_{ij} \frac{\partial^2 U}{\partial U_{ab} \partial U_{cj}} = \delta_{ic} B_{ab} \quad (5.22)$$

Multiplication by B_{id} on both sides and summation over i then yields (5.20). The proof of (5.19) for the symmetric case is the same except

$$\frac{\partial U_{ij}}{\partial \bar{U}_{ab}} = \frac{1}{2} \delta_{ia} \delta_{jb} + \frac{1}{2} \delta_{ib} \delta_{ja} \quad (5.23)$$

We note that Eqs. (5.19) and (5.20) hold for matrices of any finite dimension $n \geq 2$. We also note that (5.19) is equivalent to (A.26).

As a consequence of (5.19), we readily find

$$\begin{aligned} & 9! \int \frac{d^4 \ell_1 d^4 \ell_2 d^4 \ell_3 (\ell_a \cdot \ell_b)(\ell_c \cdot \ell_d)}{\left[-D + \sum U_{rr'} \ell_r \cdot \ell_{r'}\right]^{10}} \\ &= \frac{i^3 \pi^6}{D^2} \sum_{ss'} \sum_{rr'} \eta_{as} \eta_{bs'} \eta_{cr} \eta_{dr'} \frac{\partial^2}{\partial \bar{U}_{ss'} \partial \bar{U}_{rr'}} \left(\frac{1}{U^2}\right) \\ &= \frac{i^3 \pi^6}{D^2 U^4} \left[4B_{ab} B_{cd} + B_{ad} B_{bc} + B_{ac} B_{bd}\right] \end{aligned} \quad (5.24)$$

and

$$\begin{aligned}
& 9! \int \frac{d^4 l_1 d^4 l_2 d^4 l_3 (l_a \cdot l_b)(l_c \cdot l_d)(l_e \cdot l_f)}{[-D + \sum_{rr'} U_{rr'} l_r \cdot l_{r'}]^{10}} \\
&= \frac{i^3 \pi^6}{D} \sum_{ss'} \sum_{rr'} \sum_{tt'} \eta_{as} \eta_{bs'} \eta_{cr} \eta_{dr'} \eta_{et} \eta_{ft'} \cdot \frac{\partial^3}{\partial \bar{U}_{ss'} \partial \bar{U}_{rr'} \partial \bar{U}_{tt'}} \left(\frac{1}{U^2} \right) \\
&= \frac{-i^3 \pi^6}{U^5 D} \left[8B_{ab} B_{cd} B_{ef} + 2B_{ab} B_{cf} B_{ed} + 2B_{ab} B_{ce} B_{df} \right. \\
&\quad + 2B_{ac} B_{bd} B_{ef} + 2B_{ad} B_{bc} B_{ef} + 2B_{ae} B_{bf} B_{cd} + 2B_{af} B_{be} B_{cd} \\
&\quad + \frac{1}{2} B_{ad} B_{be} B_{cf} + \frac{1}{2} B_{ae} B_{bd} B_{cf} + \frac{1}{2} B_{ac} B_{bf} B_{de} \\
&\quad + \frac{1}{2} B_{af} B_{bc} B_{de} + \frac{1}{2} B_{ad} B_{bf} B_{ce} + \frac{1}{2} B_{af} B_{bd} B_{ce} \\
&\quad \left. + \frac{1}{2} B_{ac} B_{be} B_{df} + \frac{1}{2} B_{ae} B_{bc} B_{df} \right]. \tag{5.25}
\end{aligned}$$

In our calculation of Δa for the graphs of Fig. 2, REDUCE, after it performed the traces and index contractions, made substitutions including (5.24) and (5.25) to complete the reduction to parametric form. The result agrees exactly with Eqs. (4.24) - (4.36). The final form given for M_{IV} is obtained after algebraic reduction using the Kirchhoff's laws given in Appendix A.

6. LOGARITHMIC TERMS

We shall now study the behavior of Δa in the limit where $\rho = (m_e/m_\mu)^2$ tends to zero. For this purpose we note that the denominator function W is positive everywhere within the domain of integration as is seen from (4.21) and (4.23). Therefore any singularity which the integrals (4.24) - (4.36) may have at $\rho=0$ can come only from the domain of integration in the neighborhood of the boundary defined by

$$z_4 = 0, \quad z_5 = 0 \quad . \quad (6.1)$$

According to the general analysis of mass singularity,⁴ this singularity at $z_4=z_5=0$ is associated with the vanishing of photon and electron masses. The formula (4.21) also shows that W vanishes at

$$z_6 = z_7 = z_8 = 0 \quad . \quad (6.2)$$

However this takes place because U vanishes there and not because V vanishes. Thus it is associated with the singularity at large virtual momentum of the electron loop²⁹ and does not lead to any singularity at $\rho=0$.

We shall therefore examine the behavior of our integrals in the neighborhood of $z_4=z_5=0$. It is then easy to see by counting the power of z_4 and z_5 in the numerators and denominators that the integrals $M_{I'b}$, $M_{I'c}$, $M_{I'd}$, $I' = I, II, III, IV$, are all convergent as $\rho \rightarrow 0$, and only the integrals $M_{I'a}$, $I' = I, II, III, IV$, may have a logarithmic singularity in ρ . In order to determine the coefficients of $\ln \rho$ in these integrals, we may carry out the integration with respect to z_4 and z_5 over a small domain in the neighborhood of $z_4=z_5=0$. For this purpose let us consider the integral

$$\int_{0 \leq z_4 + z_5 \leq K} \frac{dz_4 dz_5}{W} \quad , \quad (6.3)$$

where W is given by (4.21) and K is a small fixed positive number satisfying $\rho \ll K \ll 1$ such that the terms of order $(z_4+z_5)^3$ in W can be ignored. The integration in (6.3) can be easily performed giving

$$\int_{0 \leq z_4+z_5 \leq K} \frac{dz_4 dz_5}{W} = -\frac{1}{2} (\ln \rho) G + \text{nonlogarithmic terms}, \quad (6.4)$$

where

$$\begin{aligned} G &= G(a_0, b_0, c_0) \\ &= \frac{1}{\sqrt{\Delta_0}} \left(\tan^{-1} \frac{a_0}{\sqrt{\Delta_0}} + \tan^{-1} \frac{b_0}{\sqrt{\Delta_0}} \right), \end{aligned} \quad (6.5)$$

$$\begin{aligned} \Delta_0 &= a_0 b_0 + a_0 c_0 + b_0 c_0 \\ &= (z_6+z_7+z_8) U_0, \end{aligned} \quad (6.6)$$

and a_0, b_0, c_0 are a, b, c defined by (4.23) evaluated at $z_4=z_5=0$. Similarly U_0 is U of (4.18) evaluated at $z_4=z_5=0$. Differentiating both sides of (6.4) with respect to $a_0, b_0,$ and c_0 , we can obtain further relations of the type (6.4). They are given in Appendix C.

We may now express the A_i 's in terms of z_4, z_5 and the B_{ij} 's as given by (A.16) in the integrals (4.24), (4.28), and (4.33), and carry out the z_4 and z_5 integrations with the help of formulas (6.4), (C.1), ..., (C.5). After a straightforward but lengthy calculation we find that

$$M_{Ia} + M_{IIa} + M_{IIIa} = \frac{\pi}{2} (\ln \rho) \int dz''_0 \frac{z_8(z_1 z_7 - z_8(z_2+z_7))}{U_0 \Delta_0^{3/2}} + \dots, \quad (6.7)$$

and

$$M_{IVa} = \frac{5\pi}{2} (\ln \rho) \int dz_0'' \frac{z_1 z_3 z_6 B_{46}}{U_0^3 \Delta_0^{1/2}} + \dots, \quad (6.8)$$

where

$$dz_0'' \equiv \delta(1-z_1-z_2-z_3-z_6-z_7-z_8) \\ \times dz_1 dz_2 dz_3 dz_6 dz_7 dz_8. \quad (6.9)$$

In deriving (6.7) and (6.8), we have made extensive use of the "Kirchhoff's laws" discussed in Appendix A, the identity

$$\tan^{-1} \frac{a_0}{\sqrt{\Delta_0}} + \tan^{-1} \frac{b_0}{\sqrt{\Delta_0}} + \tan^{-1} \frac{c_0}{\sqrt{\Delta_0}} = \frac{\pi}{2}, \quad (6.10)$$

as well as the symmetry of the integrals in (6.7) and (6.8) under the transformations

- i) $z_1 \longleftrightarrow z_3$, $z_6 \longleftrightarrow z_7$, z_2, z_8 unchanged,
- ii) $z_2 \longleftrightarrow z_3$, $z_6 \longleftrightarrow z_8$, z_1, z_7 unchanged,
- iii) $z_1 \longleftrightarrow z_2$, $z_7 \longleftrightarrow z_8$, z_3, z_6 unchanged.

Clearly the leading term of (6.8) is positive definite since B_{46} is negative everywhere. On the other hand, the leading term of (6.7) has both positive and negative contributions and its sign cannot be determined by inspection. However, in view of the fact that the pairs (z_1, z_7) and (z_2, z_8) are more or less equivalent according to the symmetry iii), it is plausible that (6.7) is also positive. This is in fact confirmed in Section 7 by numerical integration.

7. NUMERICAL INTEGRATION

It is obviously beyond our capability to evaluate the seven-fold integrals (4.24) - (4.36) analytically. We have therefore resorted to the method of numerical integration. This was greatly facilitated by the availability of a multiple integration program written by G. Sheppey,⁶ which could be readily modified to suit our need.

We are primarily interested in the values of Δa at $\rho = (1/207)^2$ (muon moment) and $\rho=1$ (electron moment). However, in view of the results of Section 6, we are also interested in examining numerically the functional dependence of Δa on ρ .

Sheppey's program is essentially a simple Riemann summation combined with a sampling technique which produces an efficient grid by successive approximations. At the start of the iteration process the domain of integration is divided up into a number of hypercubes by the user's specification of the number and size of the integration intervals along each axis. (Initial specification can be somewhat arbitrary because on successive iterations the program will automatically readjust all interval sizes based on the relative errors it associates with each.) Two points x_n^1 and x_n^2 are selected at random within each hypercube n as points for the evaluation of the integrands — rather than choosing the central value. The arithmetic average of the two values is used for the Riemann sum estimate

$$I = \sum_n \frac{1}{2} \left[f(x_n^1) + f(x_n^2) \right] \Delta V_n , \quad (7.1)$$

where ΔV_n is the n -th hypercube volume. A variance for each cube is defined as the square of one-half the difference between the random estimates of the integrand value. The associated error for the Riemann sum is the square root of the

sum of all variances multiplied by 1.82 to give a 91% confidence level:

$$\sigma^2 = (1.82)^2 \sum_n \left| \frac{f(x_n^1) - f(x_n^2)}{2} \right|^2 (\Delta V_n)^2 \quad (7.2)$$

Upon completion of such an iteration, those cubes which are found to contain the greatest relative variance are reduced in size along each edge in proportion to that dimension's contribution to the error, and the process then cycles through another iteration based on this new set of intervals.

The successive iterated values of the integral I_i and error σ_i , $i=1, 2, \dots, N$, are accumulated under the assumption that they are normally distributed (verified independently by histograms). Weights (W_i) for each iteration are calculated as

$$W_i = (I_i/\sigma_i)^2 \quad (7.3)$$

which gives the most probable (weighted mean) value of the integral:

$$\bar{I} = \frac{\sum_{i=1}^N I_i W_i}{\sum_{i=1}^N W_i} \quad (7.4)$$

with a standard deviation

$$\bar{\sigma} = \frac{\bar{I}}{\left(\sum_{i=1}^N W_i \right)^{1/2}} \quad (7.5)$$

The result of each iteration I_i was found in practice to overlap with \bar{I} within the error σ_i more than 90% of the time.

The integration package was tested on many multidimensional integrals, some of which were five dimensional parametric forms similar to the function analyzed in this paper, but with known analytic solution. Full confidence in the utility of the program was obtained before it was applied to the problem at hand.

In applying this program to our problem, we instructed REDUCE to punch out the result of trace calculation and D-operations described in Section 4 in a FORTRAN-compatible form so that it can be directly fed into the integration program. The integrand takes the form of a ratio of polynomials times a theta function and is well-behaved everywhere in the domain of integration for $\rho > 0$.

In Fig. 4 we show the anomalous magnetic moment contribution Δa arising from the graphs of Fig. 1 divided by $(\alpha/\pi)^3$ as a function of $X = \log_{10} (\rho^{-1}) = \log_{10} (m_\mu/m_e)^2$. The error bars indicate a better than 91% confidence interval. Typical points required 10 minutes of computation time on the SLAC IBM 360/91, after an initial 30 minutes had been used to obtain a distribution of the 50,000 hypercubes which would be approximately valid for all X.

The result (1.7) for the special case $\rho = (1/207)^2$ represents the result of more extensive effort and was obtained after about 30 iterations (about 1 per minute) with up to 90,000 hypercubes. Results consistent with (1.7) were also obtained with grids constrained to have a minimum of five points per axis. But results with smallest σ_i were those on which no such constraints were imposed. A typical result for an iteration is (apart from the factor $(\alpha/\pi)^3$)

$$I_i = 17.7, \quad \sigma_i = 2.1 \quad (7.6)$$

with 2, 4, 2, 22, 25, 2, 6 intervals along the seven axes, respectively.³¹ We have found that largest number of points are concentrated along the z_4 and z_5 axes, which is not surprising in view of consideration of Section 6.

A convergent value for $\rho=1$ proved much more difficult to obtain, partly because the integrand is not peaked in any particular regions of the variable space and partly because the theta function constraint on the variables occurs where the integrand is not small. The latter problem could be avoided and eliminated by a

change of variables

$$\begin{aligned}
 z_7 &= (1-z_1-z_2-\dots-z_6) \alpha_7 , \\
 z_6 &= (1-z_1-z_2-\dots-z_5) \alpha_6 , \\
 &\vdots \\
 z_1 &= \alpha_1 ,
 \end{aligned}
 \tag{7.7}$$

which turns the integral into the form

$$\int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \dots \int_0^1 d\alpha_7 f(z)(1-z_1)(1-z_1-z_2)\dots(1-z_1-z_2-\dots-z_6) .$$

(7.8)

It was also found convenient to switch z_2 and z_8 . Using this form and 600,000 hypercubes, the integral gave consistent results with small error in a one-hour run (5 iterations) on the IBM 360/91. The two best individual iterations gave

$$\begin{aligned}
 I_i &= 0.34 , & \sigma_i &= 0.06 , \\
 I_i &= 0.37 , & \sigma_i &= 0.09 .
 \end{aligned}
 \tag{7.9}$$

The cumulative result is given by (1.11). All other runs including those without the change of variables (7.7) overlapped with this result. On the best run the grid chosen had 583,200 cubes consisting of 12, 45, 4, 6, 3, 3, 5 intervals along each of the axes α_1 through α_7 .³²

As was shown in Section 6, the analytic dependence of the photon-photon scattering contribution on $\rho = (m_e/m_\mu)^2$ for small ρ is of the form

$$\Delta a(\rho) = \left(\frac{\alpha}{\pi}\right)^3 \left[C_1 \ln \rho + C_2 \right] , \quad \rho \ll 1 .$$

(7.10)

The coefficient C_1 as given by (6.7) and (6.8) was numerically integrated over a five dimensional space, giving

$$C_1 = - 3.19 \pm 0.04 \quad (7.11)$$

and the result (1.10). As a consistency check we have also integrated (6.7) and (6.8) separately. They gave approximately equal contributions and their sum agreed with the above result. The result (7.11) for C_1 is not inconsistent with a linear fit to the points of Fig. 4 for small ρ .

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APPENDIX A

KIRCHHOFF'S LAWS FOR A_i 'S AND B_{ij} 'S

The quantity Q_i^μ defined by (4.12) satisfies the "Kirchhoff's laws", namely, the sum of "currents Q_i^μ " entering any vertex v is conserved

$$\sum_v \pm Q_i^\mu = - \sum_v \pm (\text{external currents}) , \quad (\text{A.1})$$

where + or - is chosen according as Q_i is incoming or outgoing, and the sum of "voltage drops" around any closed loop C is zero

$$\sum_C \eta_{iC} x_{iZ}^\alpha Q_i^\mu = 0 , \quad (\text{A.2})$$

where the Feynman parameter x_{iZ}^α is regarded as the "resistance" of the line i of chain α , and η_{iC} is the projection (+1, -1) of q_i along C .

The first law (A.1) follows from

$$\left(\sum_v \pm D_i^\mu \right) \int \frac{d^4 r_1 d^4 r_3 d^4 r_6}{\prod((r_j + q_j)^2 - m_j^2)} = \int \frac{\left(\sum_v \pm (r_i^\mu + q_i^\mu) \right) d^4 r_1 d^4 r_3 d^4 r_6}{\prod((r_j + q_j)^2 - m_j^2)} , \quad (\text{A.3})$$

where q_i are fixed momenta satisfying (3.2). The second law (A.2) is a consequence of the fact that $V(x, z)$ is invariant under the simultaneous transformation of all $q_{i(C)}$

$$q_{i(C)} \longrightarrow q_{i(C)} + \eta_{iC} q^C , \quad (\text{A.4})$$

where $q_{i(C)}$ represents q_i belonging to the closed loop C and q^C is an arbitrary constant 4-vector common to all lines of the loop C .⁴ This invariance leads us to

$$\left(\sum_C \eta_{iC} \frac{\partial V}{\partial q_{i(C)}} \right) q^C = 0 , \quad (\text{A.5})$$

which is equivalent to (A.2) as is easily seen from the definition (4.12).

In our problem in which the only external current is p^μ (since $\Delta=0$), all internal currents are proportional to p^μ . Thus the proportionality coefficients A_i defined by

$$Q_i^\mu = \frac{A_i}{U} p^\mu \quad (\text{A. 6})$$

may themselves be regarded as currents satisfying the Kirchhoff's laws. In this Appendix we shall write down the explicit forms of A_i and B_{ij} and show that these functions of z in fact satisfy the Kirchhoff's laws.

Let us first calculate A_i 's for graph I from the definitions (4.12) and (A.6). Although they are functions of $x_1 z_\alpha, \dots, x_{10} z_\gamma$, it is sufficient for our purpose to write them down for the case

$$x_9 = x_{10} = 0 \quad (\text{A. 7})$$

Then, in terms of the new z variables defined by (4.17), they can be written down as follows:

$$\begin{aligned} A_1 &= -z_4 \left((z_3+z_5)(z_6+z_7+z_8) + z_7 z_8 \right) - (z_4+z_5) \left(z_2(z_6+z_7+z_8) + z_6 z_7 \right), \\ A_2 &= -z_4 \left((z_3+z_5)(z_6+z_7+z_8) + z_7 z_8 \right) + z_5 \left((z_1+z_4)(z_6+z_7+z_8) + z_6 z_8 \right), \\ A_3 &= -z_5 \left((z_1+z_4)(z_6+z_7+z_8) + z_6 z_8 \right) - (z_4+z_5) \left(z_2(z_6+z_7+z_8) + z_6 z_7 \right), \\ A_6 &= z_4 \left(z_8(z_2+z_7) + (z_7+z_8)(z_3+z_5) \right) + z_5 \left(z_2 z_8 - z_7(z_1+z_4) \right), \\ A_7 &= z_5 \left(z_8(z_2+z_6) + (z_6+z_8)(z_1+z_4) \right) + z_4 \left(z_2 z_8 - z_6(z_3+z_5) \right), \\ A_8 &= -z_4 \left(z_6(z_2+z_3+z_5+z_7) + z_2 z_7 \right) - z_5 \left(z_7(z_1+z_2+z_4+z_6) + z_2 z_6 \right). \end{aligned} \quad (\text{A. 8})$$

We have not written down A_4 , A_5 , A_9 , and A_{10} explicitly since they are easily obtained from others making use of the current conservation at external vertices (i.e., vertices to which at least one external line is attached):

$$\begin{aligned} A_4 - A_1 &= U, \\ A_5 - A_3 &= U, \\ A_9 &= A_{10} = A_8. \end{aligned} \quad (\text{A. 9})$$

It is now easy to check the current conservation at internal vertices:

$$\begin{aligned}
 A_1 + A_6 - A_{10} &= 0 \quad , \\
 A_2 + A_6 - A_7 &= 0 \quad , \\
 A_2 - A_4 + A_5 &= 0 \quad , \\
 A_3 + A_7 - A_8 &= 0 \quad .
 \end{aligned}
 \tag{A.10}$$

Similarly the second Kirchhoff's law can be checked:

$$\begin{aligned}
 z_6 A_6 + z_7 A_7 + z_8 A_8 &= 0 \quad , \\
 z_1 A_1 + z_2 A_2 + z_4 A_4 - z_6 A_6 &= 0 \quad , \\
 z_2 A_2 - z_3 A_3 - z_5 A_5 + z_7 A_7 &= 0 \quad .
 \end{aligned}
 \tag{A.11}$$

Next we shall examine B_{ij} defined by (4.13). Since B_{ij} is symmetric in i and j , there are 55 B_{ij} 's altogether. However they are related to each other by various identities. For instance, it is obvious from the definition (4.13) that

$$\begin{aligned}
 B_{1i} &= B_{4i} && \text{if } i \neq 1 \text{ or } 4 \quad , \\
 B_{3i} &= B_{5i} && \text{if } i \neq 3 \text{ or } 5 \quad , \\
 B_{8i} &= B_{9i} = B_{10i} && \text{if } i \neq 8, 9, \text{ or } 10 \quad , \\
 B_{89} &= B_{8,10} = B_{9,10} \quad .
 \end{aligned}
 \tag{A.12}$$

From (4.13), (3.11), and (3.12) we see that B_{ij} are second derivatives of V , which is quadratic in the q_j , with respect to the q_j . Since the derivatives are taken before the q_j are fixed, B_{ij} cannot depend on the external momenta or their routing, but only on the topological structure of the graph. In order to find further properties of B_{ij} let us note that Q_i^μ can be expressed as a linear combination of B_{ij}

$$U Q_i^\mu = - \sum_{j=1}^{10} q_j^\mu x_j z \alpha_j B_{ij}, \quad i = 1, 2, \dots, 10 \quad ,
 \tag{A.13}$$

where α_j is the chain to which the line j belongs. The q_j are arbitrary constant momenta subject only to the 4-momentum conservation law (3.2); we need not restrict ourselves to Q_i^μ proportional to p^μ . This equation follows from definitions (4.12), (4.13) and the fact that $V(x, z)$ is quadratic in q_j . If we choose

$$q_4 = q_5 = p, \quad \text{all other } q\text{'s} = 0 \quad (\text{A.14})$$

in (A.13), which is consistent with (3.2), we obtain

$$U Q_i^\mu = (-z_4 B_{4i} - z_5 B_{5i}) p^\mu \quad (\text{A.15})$$

Since $\Delta=0$ for this choice of q_j , we may use (A.6) to get

$$A_i = -z_4 B_{4i} - z_5 B_{5i} \quad (\text{A.16})$$

Substituting (A.16) into (A.9) or (A.15) into (A.1) for external vertices, and taking (A.12) into account, we get equations involving diagonal B_{ij}

$$\begin{aligned} B_{14} - B_{44} &= U/z_4 \quad , \\ B_{35} - B_{55} &= U/z_5 \quad . \end{aligned} \quad (\text{A.17})$$

Relaxing condition (A.7) and choosing different q_i and external momenta consistent with (3.2), we get equations similar to (A.15). Substituting these into (A.1) for external vertices and using (A.12) we get the general relation

$$B_{ii} - B_{ij} = U/z_{\alpha_i} x_i \quad , \quad (\text{A.18})$$

where lines i and j belong to the same chain α_i .

Similarly, substitution of (A.16) into (A.10) and of expressions like (A.15) into (A.1) for internal vertices yields

$$\begin{aligned} B_{1i} + B_{6i} - B_{10i} &= 0 \quad , \\ B_{2i} + B_{6i} - B_{7i} &= 0 \quad , \\ B_{2i} - B_{4i} + B_{5i} &= 0 \quad , \\ B_{3i} + B_{7i} - B_{8i} &= 0 \quad , \end{aligned} \quad (\text{A.19})$$

if none of the B_{ij} are diagonal. Otherwise we need a slight modification. For instance

$$B_{22} + B_{26} - B_{27} = -U/z_2 \quad . \quad (\text{A. 20})$$

These relations may be regarded as Kirchhoff's first law for B_{ij} .

In order to obtain the second law for B_{ij} , we substitute (A.16) into (A.11) and expressions like (A.15) into (A.2). Again using $x_9 = x_{10} = 0$, we find

$$\begin{aligned} z_6 B_{6i} + z_7 B_{7i} + z_8 B_{8i} &= 0 \quad , \\ z_1 B_{1i} + z_2 B_{2i} + z_4 B_{4i} - z_6 B_{6i} &= 0 \quad , \\ z_2 B_{2i} - z_3 B_{3i} - z_5 B_{5i} + z_7 B_{7i} &= 0 \quad . \end{aligned} \quad (\text{A. 21})$$

Finally we give some B_{ij} explicitly in terms of new z variables defined by (4.17) which corresponds to putting $x_9 = x_{10} = 0$:

$$\begin{aligned} B_{45} &= z_2(z_6 + z_7 + z_8) + z_6 z_7 \quad , \\ B_{46} &= -z_8(z_2 + z_3 + z_5 + z_7) - z_7(z_3 + z_5) \quad , \\ B_{47} &= z_6(z_3 + z_5) - z_2 z_8 \quad , \\ B_{48} &= z_6(z_2 + z_3 + z_5 + z_7) + z_2 z_7 \quad , \\ B_{56} &= z_7(z_1 + z_4) - z_2 z_8 \quad , \\ B_{57} &= -z_8(z_1 + z_2 + z_4 + z_6) - z_6(z_1 + z_4) \quad , \\ B_{58} &= z_7(z_1 + z_2 + z_4 + z_6) + z_2 z_6 \quad , \\ B_{67} &= z_2(z_1 + z_3 + z_4 + z_5 + z_8) + (z_1 + z_4)(z_3 + z_5) \quad , \\ B_{68} &= (z_1 + z_4)(z_2 + z_3 + z_5 + z_7) + z_2(z_3 + z_5) \quad , \\ B_{78} &= (z_3 + z_5)(z_1 + z_2 + z_4 + z_6) + z_2(z_1 + z_4) \quad . \end{aligned} \quad (\text{A. 22})$$

Other formulas of great use can be derived from the observation that, aside from the term $\sum_{\alpha} z_{\alpha} \left(\sum_i^{\alpha} x_i m_i^2 \right)$, the denominator function $V(x, z)$ can be regarded as the "power" burned up in the network.² This leads us to a set of equations²⁹

$$-Q_i^2 + m_i^2 = \partial V / \partial z_i, \quad i = 1, 2, \dots, 8, \quad (\text{A.23})$$

where Q_i is defined by (4.12). With the help of (A.6) and (4.20) this can be transformed into

$$A_i^2 = W(\partial U / \partial z_i) - U(\partial W / \partial z_i) + \rho_i U^2, \quad (\text{A.24})$$

where

$$\rho_i = \begin{cases} 0 & \text{for } i = 1, 2, 3, \\ 1 & \text{for } i = 4, 5, \\ \rho & \text{for } i = 6, 7, 8. \end{cases} \quad (\text{A.25})$$

Suppose we express W and A_i in (A.24) in terms of B_{ij} and z_4 and z_5 . Then, comparing the coefficients of z_4 and z_5 in the resulting expression and using the Kirchhoff's law (A.19) repeatedly, we can write down an enormous number of formulas quadratic in B_{ij} 's. Some of them are shown below:

$$\begin{aligned} B_{46}B_{57} - B_{47}B_{56} &= z_8 U, \\ B_{48}B_{56} - B_{46}B_{58} &= z_7 U, \\ B_{47}B_{58} - B_{48}B_{57} &= z_6 U, \\ B_{45}B_{67} - B_{47}B_{56} &= B_{48}B_{67} - B_{47}B_{68} = B_{58}B_{67} - B_{56}B_{78} \\ &= B_{67}B'_{88} - B_{68}B_{78} = B_{48}B_{78} - B_{47}B'_{88} = B_{58}B_{68} - B_{56}B'_{88} \\ &= B_{45}B'_{88} - B_{48}B_{58} = z_2 U, \\ B'_{66}B'_{88} - B_{68}B_{68} &= B_{48}B_{68} - B_{46}B'_{88} = (z_2 + z_3 + z_5 + z_7) U, \\ B'_{77}B'_{88} - B_{78}B_{78} &= B_{58}B_{78} - B_{57}B'_{88} = (z_1 + z_2 + z_4 + z_6) U, \\ B'_{66}B_{78} - B_{67}B_{68} &= (z_3 + z_5) U, \\ B'_{66}B_{58} - B_{56}B_{68} &= (z_2 + z_7) U, \end{aligned} \quad (\text{A.26})$$

where B'_{ii} is the polynomial part of B_{ii} which may also be written as $\partial U / \partial z_i$.

We note that (A.26) is equivalent to the result (5.19).

Formulas such as (A.11), (A.21), and (A.26) are extremely useful in simplifying the numerators of Feynman integrals. Although the usefulness of Kirchhoff's laws for the study of analytic properties of Feynman integrals (which are derived from the properties of denominators of the integrands) is well-known, it appears that their use in the simplification of Feynman integrals has not been emphasized thus far.

It will be obvious that the above results are also applicable to other graphs of Fig. 2 if we interpret A_9 , A_{10} , B_{9i} , and $B_{10,i}$ in an appropriate fashion.

APPENDIX B

UNSIMPLIFIED INTEGRANDS FOR GRAPH IV

For the benefit of readers who wish to check our calculations, we give here the unsimplified output of REDUCE for graph IV. Simplification of these integrands with the help of the identities given in Appendix A leads us to the formulas (4.33) - (4.36). As is defined in (A.26), B'_{ii} is the polynomial part of B_{ii} and is equal to $\partial U / \partial z_i$.

$$\begin{aligned}
 M_{IVa} = & \int \frac{dz_0}{U^4 W^2} z_6 z_8 \left[A_1 A_6 (B_{68} B_{78} - B_{67} B'_{88}) + A_3 A_7 (B'_{66} B'_{88} - B_{68}^2) \right] \\
 & + \int \frac{dz_0}{U^3 W^2} \left[z_6 (A_1 A_6 B_{67} - A_3 A_7 B'_{66}) - z_8 A_3 A_7 B'_{88} \right] \\
 & + \int \frac{dz_0}{U^2 W^2} A_3 A_7 \\
 & + \frac{1}{2} \int \frac{dz_0}{U^4 W} z_6 z_8 \left[B_{48} (B_{67} B_{68} - B'_{66} B_{78}) + B_{47} (B'_{66} B'_{88} - B_{68}^2) + 5B_{46} (B_{67} B'_{88} - B_{68} B_{78}) \right. \\
 & \quad \left. + 3B_{57} (B_{68}^2 - B'_{66} B'_{88}) + 3B_{56} (B_{68} B_{78} - B_{67} B'_{88}) + 3B_{58} (B_{67} B_{68} - B'_{66} B_{78}) \right] \\
 & + \frac{1}{2} \int \frac{dz_0}{U^3 W} \left[z_6 (3B_{57} B'_{66} + 3B_{56} B_{67} - B_{47} B'_{66} - 5B_{46} B_{67}) \right. \\
 & \quad \left. + z_8 (3B_{57} B'_{88} + 3B_{58} B_{78} - B_{47} B'_{88} + B_{48} B_{78}) \right] \\
 & + \frac{1}{2} \int \frac{dz_0}{U^2 W} \left[B_{47} - 3B_{57} \right], \tag{B.1}
 \end{aligned}$$

$$\begin{aligned}
M_{IVb} = & \frac{1}{2} \int \frac{dz_0}{U^5 W^2} z_6 z_8 \left[A_1 A_3 \left\{ A_6 (B_{68} B_{78} - B_{67} B'_{88}) + A_7 (B'_{66} B'_{88} - B_{68}^2) + A_8 (B_{78} B'_{66} - B_{67} B_{68}) \right\} \right. \\
& + A_1 \left\{ A_6 A_7 (B_{58} B_{68} - B_{56} B'_{88}) + A_6 A_8 (B_{58} B_{67} - B_{56} B_{78}) + A_7 A_8 (B_{56} B_{68} - B_{58} B'_{66}) \right\} \\
& + A_3 \left\{ A_6 A_7 (B_{46} B'_{88} - B_{48} B_{68}) + A_6 A_8 (B_{48} B_{67} - B_{46} B_{78}) + A_7 A_8 (B_{46} B_{68} - B_{48} B'_{66}) \right\} \\
& \left. + A_6 A_7 A_8 (B_{48} B_{56} - B_{46} B_{58}) \right] \\
& + \frac{1}{2} \int \frac{dz_0}{U^4 W^2} z_6 \left[A_1 A_3 (A_6 B_{67} - A_7 B'_{66}) + A_6 A_7 (A_1 B_{56} - A_3 B_{46}) \right] \\
& + \frac{1}{2} \int \frac{dz_0}{U^4 W^2} z_8 \left[-A_1 A_3 (A_7 B'_{88} + A_8 B_{78}) + A_7 A_8 (A_3 B_{48} + A_1 B_{58}) \right] \\
& + \frac{1}{2} \int \frac{dz_0}{U^3 W^2} A_1 A_3 A_7 \\
& + \frac{1}{2} \int \frac{dz_0}{U^5 W} z_6 z_8 \left[3A_1 (B_{56} B_{67} B'_{88} - B_{58} B'_{66} B_{78}) \right. \\
& + A_3 \left\{ 2B_{47} (B_{68}^2 - B'_{66} B'_{88}) - B_{46} (B_{67} B'_{88} + 2B_{68} B_{78}) + B_{48} (4B_{67} B_{68} - B'_{66} B_{78}) \right\} \\
& + A_6 \left\{ B_{45} (B_{67} B'_{88} - B_{68} B_{78}) + 3B_{46} B_{58} B_{78} \right. \\
& \quad \left. + 2B_{47} (B_{56} B'_{88} - B_{58} B_{68}) + B_{48} (B_{56} B_{78} - 4B_{58} B_{67}) \right\} \\
& + A_7 \left\{ B_{45} (B_{68}^2 - B'_{66} B'_{88}) - B_{46} (2B_{56} B'_{88} + B_{58} B_{68}) + B_{48} (4B_{58} B'_{66} - B_{56} B_{68}) \right\} \\
& + A_8 \left\{ B_{45} (B'_{66} B_{78} - B_{67} B_{68}) + B_{46} (2B_{56} B_{78} + B_{58} B_{67}) \right. \\
& \quad \left. + 2B_{47} (B'_{66} B_{58} - B_{56} B_{68}) - 3B_{48} B_{56} B_{67} \right\} \\
& \left. \right]
\end{aligned}$$

+

$$\begin{aligned}
& + \frac{1}{2} \int \frac{dz_0}{U^4 W} z_6 \left[-3A_1 B_{56} B_{67} + A_3 (2B_{47} B'_{66} + B_{46} B_{67}) \right. \\
& \quad \left. - A_6 (2B_{47} B_{56} + B_{45} B_{67}) + A_7 (2B_{46} B_{56} + B_{45} B'_{66}) \right] \\
& + \frac{1}{2} \int \frac{dz_0}{U^4 W} z_8 \left[3A_1 B_{58} B_{78} + A_3 (2B_{47} B'_{88} + B_{48} B_{78}) \right. \\
& \quad \left. + A_7 (B_{45} B'_{88} - 4B_{48} B_{58}) - A_8 (B_{45} B_{78} + 2B_{47} B_{58}) \right] \\
& + \frac{1}{2} \int \frac{dz_0}{U^3 W} \left[-2A_3 B_{47} - A_7 B_{45} \right] , \tag{B.2}
\end{aligned}$$

$$\begin{aligned}
M_{IVc} &= \rho \int \frac{dz_0}{U^2 W^2} z_6 z_8 \left[(B_{48} B_{67} - B_{46} B_{78}) + (B_{56} B_{68} - B_{58} B'_{66}) \right. \\
& \quad \left. + (B_{46} B'_{88} - B_{48} B_{68}) + (B_{58} B_{68} - B_{56} B'_{88}) \right] \\
& + \rho \int \frac{dz_0}{UW^2} \left[z_8 B_{58} + z_6 (B_{56} - B_{46}) \right] , \tag{B.3}
\end{aligned}$$

$$\begin{aligned}
M_{IVd} &= \frac{\rho}{2} \int \frac{dz_0}{U^3 W^2} z_6 z_8 \left[(A_6 + A_7 + A_8) (B_{46} B_{58} - B_{48} B_{56}) \right. \\
& \quad \left. + A_1 \left\{ (B_{56} B'_{88} - B_{58} B_{68}) + (B_{56} B_{68} - B_{58} B'_{66}) + (B_{56} B_{78} - B_{58} B_{67}) \right\} \right. \\
& \quad \left. + A_3 \left\{ (B_{48} B_{68} - B_{46} B'_{88}) + (B_{48} B'_{66} - B_{46} B_{68}) + (B_{48} B_{67} - B_{46} B_{78}) \right\} \right] \\
& + \frac{\rho}{2} \int \frac{dz_0}{U^2 W^2} \left[z_6 (A_3 B_{46} - A_1 B_{56}) + z_8 (A_1 B_{58} - A_3 B_{48}) \right] . \tag{B.4}
\end{aligned}$$

APPENDIX C

FORMULAS DERIVED FROM (6.4)

Differentiating both sides of (6.4) with respect to a , b , and c , where we have omitted the suffix 0 in a_0 , etc., we obtain further relations:

$$\int \frac{z_4^2 dz_4 dz_5}{W^2} = \frac{\ln \rho}{4\Delta_0} \left(\frac{c}{a+c} - (b+c) G \right) + \dots, \quad (\text{C.1})$$

$$\int \frac{z_4 z_5 dz_4 dz_5}{W^2} = \frac{\ln \rho}{4\Delta_0} (-1+cG) + \dots, \quad (\text{C.2})$$

$$\int \frac{z_4^4 dz_4 dz_5}{W^3} = \frac{\ln \rho}{16\Delta_0^2} \left(\frac{c(5\Delta_0 + 3c^2)}{(a+c)^2} - 3(b+c)^2 G \right) + \dots, \quad (\text{C.3})$$

$$\int \frac{z_4^3 z_5 dz_4 dz_5}{W^3} = \frac{\ln \rho}{16\Delta_0^2} \left(-\frac{2\Delta_0 + 3c^2}{a+c} + 3c(b+c) G \right) + \dots, \quad (\text{C.4})$$

$$\int \frac{z_4^2 z_5^2 dz_4 dz_5}{W^3} = \frac{\ln \rho}{16\Delta_0^2} \left(3c - (\Delta_0 + 3c^2) G \right) + \dots. \quad (\text{C.5})$$

The domain of integration is $0 \leq z_4 + z_5 \leq K$ in all cases. Similar formulas for the integrals

$$\int \frac{z_5^2 dz_4 dz_5}{W^2}, \quad \int \frac{z_4 z_5^3 dz_4 dz_5}{W^3}, \quad \int \frac{z_5^4 dz_4 dz_5}{W^3}$$

are obtained from (C.1), (C.4), and (C.3) by interchanging a and b .

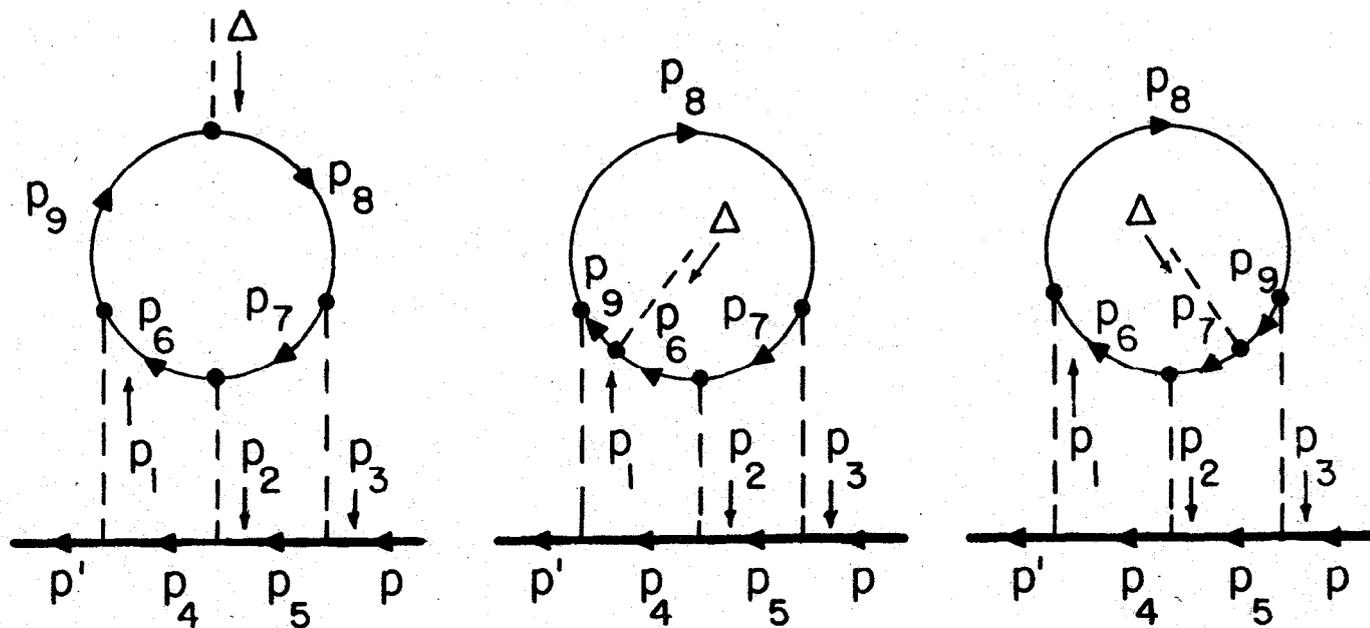
REFERENCES AND FOOTNOTES

1. For a review and references, see F. J. M. Farley, The First Meeting of the European Physical Society at Florence (1969); S. J. Brodsky, 1969 International Conference on Electron and Photon Interactions at High Energies, Daresbury. See also Ref. 18.
2. J. D. Bjorken and S. D. Drell, Relativistic Quantum Fields, (McGraw-Hill, New York, 1965); Sec. 18.4.
3. N. Nakanishi, *Progr. Theoret. Phys.* 17, 401 (1957).
4. T. Kinoshita, *J. Math. Phys.* 3, 650 (1962).
5. A. C. Hearn, Stanford University Report No. ITP-247 (unpublished) and A. C. Hearn, "Reduce—A User-Oriented Interactive Systems for Algebraic Simplification," in Interactive Systems for Experimental Applied Mathematics, Ed. by M. Klerer and J. Reinfelds (Academic Press, New York, 1968).
6. We wish to thank Dr. G. R. Henry for bringing this program to our attention.
7. J. Bailey, W. Bartl, G. Von Bochmann, R. C. A. Brown, F. J. M. Farley, H. Jöstlein, E. Picasso, and R. W. Williams, *Phys. Letters* 28B, 287 (1968).
8. H. H. Elend, *Phys. Letters* 20, 682 (1966); erratum *ibid.* 21, 720 (1966).
See also H. Suura and E. H. Wichmann, *Phys. Rev.* 105, 1930 (1957);
A. Petermann, *Phys. Rev.* 105, 1931 (1957); A. Petermann, *Fortschr. Phys.* 6, 505 (1958).
9. S. D. Drell and H. R. Pagels, *Phys. Rev.* 140, B397 (1965); R. G. Parsons, *Phys. Rev.* 168, 1562 (1968).
10. J. A. Mignaco and E. Remiddi, *Nuovo Cimento* 60A, 519 (1969).
11. T. Kinoshita, *Nuovo Cimento* 51B, 140 (1967).
12. S. D. Drell and J. S. Trefil, unpublished. For a discussion of their preliminary result, see S. D. Drell, Proceedings of the XIII International Conference on High Energy Physics (Berkeley, 1966), p. 93, and S. D. Drell "Precision

Tests of Quantum Electrodynamics at Low Energies," in Particle Interactions at High Energies, Scottish University Summer School, Edited by T. W. Preist and L. L. J. Vick (Oliver and Boyd, Edinburgh and London, 1966).

13. T. Kinoshita, Cargese Lectures in Physics, Vol. 2 (Gordon and Breach, New York, 1968); p. 209.
14. B. E. Lautrup and E. deRafael, Phys. Rev. 174, 1835 (1968).
15. For more discussion on this point, see Ref. 7 of our preliminary report: J. Aldins, T. Kinoshita, S. J. Brodsky, and A. J. Dufner, Phys. Rev. Letters 23, 441 (1969).
16. An additional support of this estimate is given by the calculation of B. E. Lautrup and E. deRafael, CERN preprint, TH 1042 (1969).
17. M. Gourdin and E. deRafael, Nuclear Phys. B10, 667 (1969). See also T. Kinoshita and R. J. Oakes, Phys. Letters 25B, 143 (1967), and J. E. Bowcock, Z. Physik 211, 400 (1968).
18. This is the value of α^{-1} derived by the adjustment of fundamental constants using no QED data, given by B. N. Taylor, W. H. Parker, and D. N. Langenberg, Rev. Mod. Phys. 41, 375 (1969). See also W. H. Parker, B. N. Taylor, and D. N. Langenberg, Phys. Rev. Letters 18, 287 (1967). Recent fine structure measurements in H and D and the hyperfine splitting in H yield values of α^{-1} consistent with (1.5).
19. H. Terazawa, Progr. Theoret. Phys. 39, 1326 (1968); J. S. Bell and E. deRafael, Nucl. Phys. B11, 611 (1969).
20. R. A. Shaffer, Phys. Rev. 135, B187 (1964); S. J. Brodsky and J. D. Sullivan, Phys. Rev. 156, 1644 (1967); T. Burnett and M. J. Levinc, Phys. Letters 24B, 467 (1967).
21. H. Terazawa, Progr. Theoret. Phys. 38, 863 (1967).

22. D. T. Wilkinson and H. R. Crane, Phys. Rev. 130, 852 (1963); A. Rich, Phys. Rev. Letters 20, 967 (1967) and erratum, *ibid.* 20, 1221 (1968); G. R. Henry and J. E. Silver, Phys. Rev. 180, 1262 (1969).
23. Our metric and conventions are the same as that of Ref. 2. The Born current corresponds to $M = \bar{u}(p') \epsilon^\mu \gamma_\mu u(p)$, $p' = p + \Delta$.
24. From Eq. (2.4) alone it is not possible to exclude the possibility that $\frac{\partial}{\partial \Delta^\mu} \Pi_{\kappa\rho\sigma\nu}$ and hence $M_{\mu\nu}$ have a mild singularity (less singular than Δ^{-1}) in the neighborhood of $\Delta = 0$. However the analysis of mass singularity discussed in Ref. 4 shows that no such singularity is possible insofar as $m_e \neq 0$. This is why we can put $\Delta = 0$ in $M_{\mu\nu}$.
25. Alternately, one can use general projection operators for the muon form factors $F_1(q^2)$, $F_2(q^2)$ and obtain (2.9) as a special case for $F_2(0)$. See S. J. Brodsky and J. D. Sullivan, Ref. 20.
26. This is actually the sum of contributions from two diagrams, one is that of graph I of Fig. 2 and the other is that of a graph in which the direction of the arrow of the electron loop is reversed. Both contribute equally to (3.4).
27. For a general treatment of double parametric representation, see Ref. 4.
28. See Ref. 4, formula (2.18), for an explicit form of this $U(z)$. A form more convenient for our purpose is given later by formula (4.18).
29. N. Nakanishi, Supplement, Progr. Theoret. Phys. 18, 1 (1961).
30. Note that explicit dependence on the direction of the external four vectors can always be removed by tensor methods as in (2.8) and (2.9).
31. The variable z_8 was eliminated beforehand using the δ function.
32. The variable z_2 was eliminated using the δ function, and the variables $\alpha_1, \dots, \alpha_7$ were defined by (7.7) in which z_2 was replaced by z_8 .



1513A4

Fig. 1

Feynman diagrams containing subdiagrams of photon-photon scattering type. The heavy, thin, and dotted lines represent the muon, electron, and photon, respectively. There are three more diagrams obtained by reversing the direction of the electron loop.

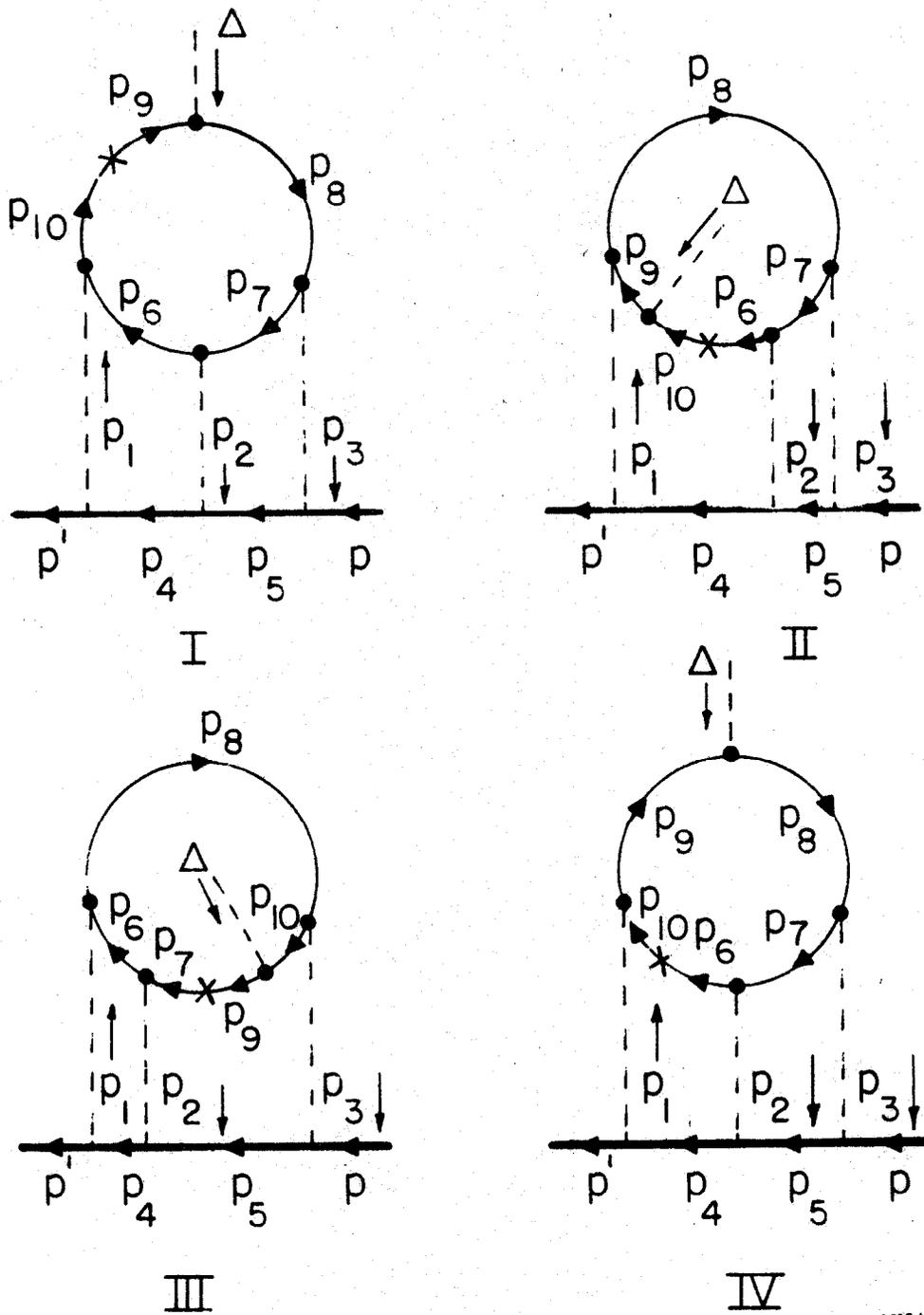


Fig. 2

1513A2

Feynman diagrams obtained from those of Fig. 1 by differentiation with respect to Δ . The crosses represent differentiation vertices. The external momenta are routed so that Δ always flows through the middle photon line.

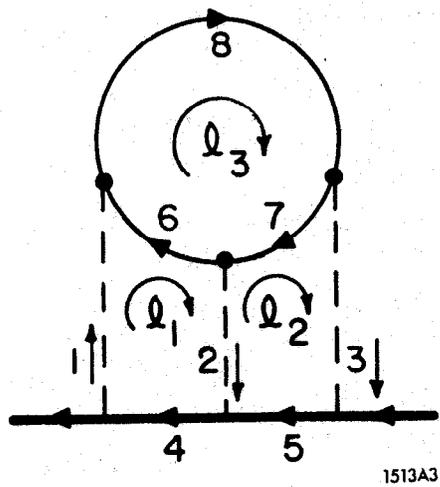
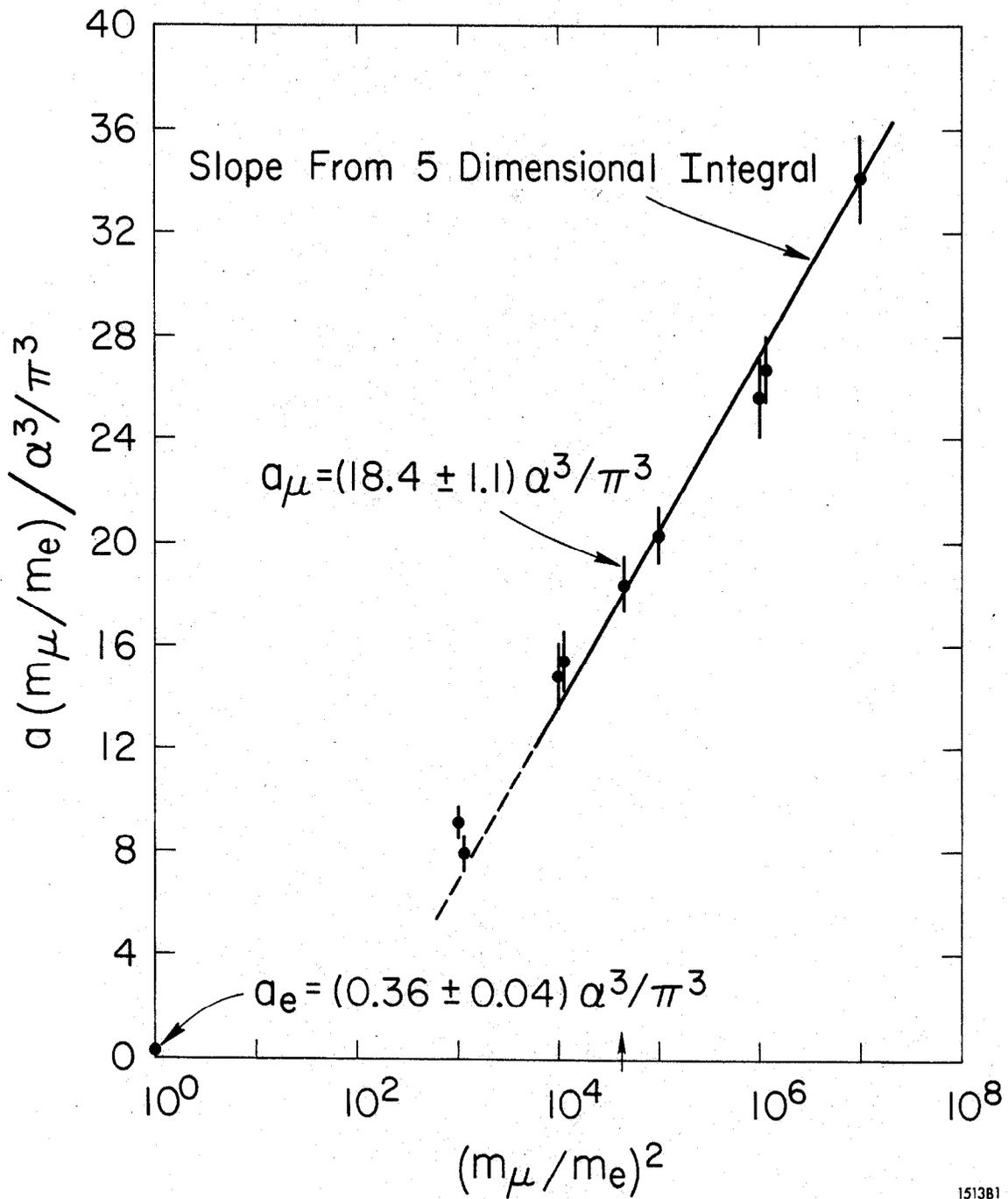


Fig. 3

A diagram indicating the labeling of loop momenta l_1 , l_2 , l_3 .



151381

Fig. 4

The anomalous magnetic moment contribution Δa as a function of

$$X = \log_{10}(\rho^{-1}) = \log_{10}(m_\mu/m_e)^2.$$