16th International Symposium on Particles Strings and Cosmology (PASCOS 2010) IOP Publishing
Journal of Physics: Conference Series **259** (2010) 012050 doi:10.1088/1742-6596/259/1/012050

Black Holes in 5D Hořava Lifshitz Theory of Gravity

George Koutsoumbas, Eleftherios Papantonopoulos, Pavlos Pasipoularides, Minas Tsoukalas

Department of Physics, National Technical University of Athens, Zografou Campus GR 157 73, Athens, Greece.

E-mail: kutsubas@central.ntua.gr, lpapa@central.ntua.gr, paul@central.ntua.gr, minasts@central.ntua.gr,

Abstract. In this talk we discuss black hole solutions in five dimensions in the context of Hořava-Lifshtz gravity. There are several classes of solutions which are characterized by an AdS_5, dS_5 or flat large distance asymptotic behaviour, plus the standard $1/r^2$ tail of the usual five-dimensional Schwarzschild black holes. In addition there are solutions with an unconventional short or large distance behaviour, and for a special range of the coupling parameters solutions which coincide with black hole solutions of conventional relativistic five-dimensional Gauss-Bonnet gravity.

1. Introduction

A lot of attention has been addressed lately to Hořava-Lifshtz model of gravity [1],[2]. The model is based on an anisotropy between space and time coordinates, which is expressed via the scalings $t \to b^z t$ and $x \to bx$, where z is a dynamical critical exponent. In the particular model the four-dimensional diffeomorphism invariance of general relativity is sacrificed in order to achieve power-counting renormalizability. Instead we have foliation preserving diffeomorphisms

$$x^i \to x'^i = x'^i(x^i, t), \quad t \to t' = t'(t). \tag{1.1}$$

The action contains a kinetic plus a potential term, which both respect a restricted $(3 + 1)$ diffeomorphism invariance. Due to (1.1) we can add higher order spatial derivative terms, which improve the UV behavior, rendering the model power-counting renormalizable. However the absence of full diffeomorphism invariance introduces an additional scalar mode which can lead to strong coupling problems or instabilities [5], [6], [7], [8], [9].

Originally the potential term, [1] was formed under the so called "detailed balance principle". An alternative way for constructing an action is to include all possible operators which are compatible with the renormalizability of the model. This implies that all operators with dimension less or equal to six are allowed in the action (for the exact form of the action see [3],[4]). There are two versions which are known as projectable and non-projectable. In the projectable version the lapse function N^2 depends only on the time coordinate, while in the non–projectable version N^2 is a function of both the space and time coordinates.

Issues connected with broken Lorentz invariance were studied in the spherically symmetric solutions of 4D HL gravity. In the case of detailed balance, such spherically symmetric solutions

© 2010 IOP Publishing Ltd

were found [10], but they exhibited an unconventional large distance asymptotic behaviour. The correct Schwarzchild-flat asymptotic behaviour can be recovered if the detailed balance action is modified in the IR by a term proportional to Ricci scalar, and the cosmological constant term is considered to be zero [11]. A similar study, in the case of non-vanishing cosmological constant, has also been carried out [12]. A generalization to topological black holes was obtained in [13]. Finally, a systematic study of static spherically symmetric solutions of 4D HL gravity was presented in [14],[17] where the most general spherically symmetric solution for $\lambda \neq 1$ and general coupling parameters was obtained.

In this talk we present a full study of spherically symmetric solutions in the non-projectable version of the five-dimensional Horava-Lifshitz gravity, for $z = 4$, [16]. For the construction of the 5D action we do not use the "detailed balance principle", but we include all the terms which are compatible with the renormalizability of the model. In particular, we can include all spatial curvature terms with dimension less than or equal to eight. However, the large number of possible terms, which are allowed in the action, leads to equation of motion of great complexity. For this reason we restrict our study only to terms of up to second order in the curvature. Also, we suppose that in the IR limit 5D the HL gravity reduces to the 5D General Relativity plus a bulk cosmological constant. A class of spherically symmetric solutions of the 5D HL gravity has been considered previously [15], but only for a very specific choice of the couplings.

2. Action and equations of motion

The 5D action of the model is constructed from a kinetic plus a potential term according to the equation

$$S = \frac{1}{16\pi G_5 c}\int dt d^d x \sqrt{|g|} N \left\{\mathcal{L}_K + \mathcal{L}_V\right\} \tag{2.1}$$

in which d ($D = d+1 = 5$) is the spatial dimension and G_5 is the five dimensional Newton constant. The kinetic part in the above Lagrangian of Eq. (2.1) can be expressed via the extrinsic curvature as:

$$\mathcal{L}_K = (K^{ij}K_{ij} - \lambda K^2), \quad K_{ij} = \frac{1}{2N}\left\{-\partial_t g_{ij} + \nabla_i N_j + \nabla_j N_i\right\}, \quad i,j = 1,2,3,4 \tag{2.2}$$

For the construction of the potential term we will follow the approach, according to which the potential term is constructed by including all possible renormalizable operators, that have dimension smaller or equal to eight, hence we write

$$\mathcal{L}_V = \mathcal{L}_R + \mathcal{L}_{R^2} + \mathcal{L}_{R^3} + \mathcal{L}_{\Delta R^2} + \mathcal{L}_{R^4} + \mathcal{L}_{\Delta R^3} + \mathcal{L}_{\Delta^2 R^2} \ . \tag{2.3}$$

where the symbol Δ is defined as $\Delta = \partial^i \partial_i$ ($i = 1,2,3,4$).

The dimensions of the various terms in the Lagrangian read

$$[R] = 2, \ [R^2] = 4, \ [R^3] = [\Delta R^2] = 6, [R^4] = [\Delta R^3] = [\Delta^2 R^2] = 8 \ . \tag{2.4}$$

In this work we are mainly interested in the lowest order operator $\mathcal{L}_R$ and the operator $\mathcal{L}_{R^2}$, which contains contributions of second order in the curvature:

$$\mathcal{L}_R = \eta_{0a} + \eta_{1a} R, \quad \mathcal{L}_{R^2} = \eta_{2a} R^2 + \eta_{2b} R^{ij} R_{ij} + \eta_{2c} R^{ijkl} R_{ijkl} \ . \tag{2.5}$$

Note that in the case of three spatial dimensions the term $R^{ijkl}R_{ijkl}$ is absent, as the Weyl tensor in three dimensions automatically vanishes. However, in four spatial dimensions this term cannot be omitted from the action.

The first term $\mathcal{L}_R$ is necessary in order to recover 5D general relativity with a cosmological constant in the IR limit. The second term $\mathcal{L}_{R^2}$, includes all possible quadratic terms in curvature, and becomes important in the short distance regime of the theory. Moreover, η_{0a} plays the role of the cosmological constant, while η_{1a}, η_{2a}, η_{2b}, and η_{2c} are dimensionful coupling constants with dimensions

$$[\eta_{1a}] = 6, \quad [\eta_{2a}] = [\eta_{2b}] = [\eta_{2c}] = 4 \ . \tag{2.6}$$

In the present analysis we ignore higher order Lagrangian terms, of dimension six and eight.

If this model is to make sense, it is necessary that the 5D general relativity (with a cosmological constant in our case) is recovered in the IR limit. Although there is no theoretical proof for this difficult question, we will assume that the renormalization group flow towards the IR leads the parameter λ to the value one ($\lambda = 1$), hence 5D general relativity is recovered. Also, to obtain the Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G_5} \int dx^0 d^4x \sqrt{g} N \left(\tilde{K}_{ij} \tilde{K}^{ij} - \tilde{K}^2 + R + \eta_{0a} \right) \ , \tag{2.7}$$

we have to set $\eta_{1a} = c^2$, and

$$\tilde{K}_{ij} = \frac{1}{2N} \left\{ -\partial_0 g_{ij} + \nabla_i \left(\frac{N_j}{c} \right) + \nabla_j \left(\frac{N_i}{c} \right) \right\} \ . \tag{2.8}$$

where the time-like coordinate x_0 is defined as $x_0 = ct$.

We will present 5D spherically symmetric solutions of the Horava-type gravity model. We use the following ansatz for the metric

$$ds^2 = -N(r)^2 dt^2 + f^{-1}(r) \ dr^2 + r^2 d\Omega_k^2 \ , \tag{2.9}$$

in which r is a radius coordinate that corresponds to the extra dimension, and $d\Omega_k^2$ is the metric of a 3D maximally symmetric space, where k is the spatial curvature of 3D hypersurfaces and for $k = 1, -1, 0$ we have a sphere, hyperboloid or 3D torus topology correspondingly. In what follows it is convenient to perform the transformation

$$f(r) = k + r^2 Z(r) \ . \tag{2.10}$$

Then the action of the model to second order in curvature terms is

$$S = \frac{1}{16\pi G_5} \int dt d^d x \sqrt{|g|} N \left(K^{ij} K_{ij} - \lambda K^2 + \eta_{0a} + \eta_{1a} R + \eta_{2a} R^2 + \eta_{2b} R^{ij} R_{ij} + \eta_{2c} R^{ijkl} R_{ijkl} \right) \tag{2.11}$$

can be put into the form

$$S \left[N(r), Z(r), \frac{dZ(r)}{dr} \right] = \int_0^{+\infty} dr \ L \left[N(r), Z(r), \frac{dZ(r)}{dr} \right] \ , \tag{2.12}$$

after we integrate out the angular coordinates, where

$$L \left[N, Z, \frac{dZ}{dr} \right] \sim r^3 \sqrt{\frac{N^2}{f}} \left(P \left(r \frac{dZ}{dr} \right)^2 + M(Z) \left(r \frac{dZ}{dr} \right) + Q(Z) \right) \tag{2.13}$$

and

$$\begin{aligned} P &= 3(3\eta_{2a} + \eta_{2b} + \eta_{2c}) \ , \\ M(Z) &= 6(12\eta_{2a} + 3\eta_{2b} + 2\eta_{2c}) Z - 3\eta_{1a} \ , \\ Q(Z) &= 12(12\eta_{2a} + 3\eta_{2b} + 2\eta_{2c}) Z^2 - 12\eta_{1a} Z + \eta_{0a} \ . \end{aligned} \tag{2.14}$$

The solutions come from the Euler-Lagrange equations of the aforementioned Lagrangian.

3. Solutions

3.1. No quadratic terms, $\eta = 0$ and $\varrho = 0$
If we set $\eta = \varrho = 0$ we find the simple solution

$$f(r) = k + r^2 Z = k + \frac{\eta_{0a}}{12} r^2 + \frac{\tilde{C}_\mu}{3r^2} \ , \tag{3.1}$$

where $\tilde{C}_\mu$ is a constant of integration and $N(r)^2 = f(r)$. If we set

$$\Lambda_{eff} = -\eta_{0a} \ , \quad \mu = -\frac{\tilde{C}_\mu}{3} \ , \tag{3.2}$$

the above equation takes the well-known form

$$f(r) = k - \frac{\Lambda_{eff}}{12} r^2 - \frac{\mu}{r^2} \ , \tag{3.3}$$

which is the standard AdS_5 (for $\Lambda_{eff} < 0$) or dS_5 (for $\Lambda_{eff} > 0$) or asymptotically flat (for $\Lambda_{eff} = 0$) Schwarzschild black hole solution of 5D general relativity with a cosmological constant.

3.2. $\eta = 0$ and $\varrho \neq 0$
If $\eta = 0$ then $P = 0$, then the equation for Z reads as

$$r \frac{dZ}{dr} = -\frac{\eta_{0a} - 12Z + 12\varrho Z^2}{-3 + 6\varrho Z} \ . \tag{3.4}$$

Integration of this equation yields:

$$3\varrho Z^2 - 3Z + \frac{\eta_{0a}}{4} + \frac{\tilde{C}_\mu}{r^4} = 0 \ , \tag{3.5}$$

where $\tilde{C}_\mu$ is an integration constant which is related to the mass of the black hole. The algebraic equation (3.5) gives two solutions

$$Z(r) = \frac{1}{2\varrho} + \sigma \frac{\sqrt{3(3 - \varrho\eta_{0a})r^4 - 12\varrho\tilde{C}_\mu}}{6\varrho \ r^2} \ , \tag{3.6}$$

where σ is a sign ($\sigma = \pm 1$), so for the function $f(r) = k + r^2 Z$ we obtain

$$f(r) = k + \frac{r^2}{2\varrho} \left[1 + \sigma \sqrt{\left(1 - \frac{\varrho\eta_{0a}}{3} \right) - \frac{4\varrho\tilde{C}_\mu}{3r^4}} \right] \ . \tag{3.7}$$

In what follows we will assume that $\varrho\tilde{C}_\mu < 0$, because for $\varrho\tilde{C}_\mu > 0$ the range of radius r has a lower bound ($r > r_{min}$).

The Euler-Lagrange equations for $N(r)$ yield

$$N(r)^2 = f(r) \ . \tag{3.8}$$

This solution is in direct connection with the solutions for black holes in Gauss-Bonnet theory which take the following form

$$f(r) = k + \frac{r^2}{4\hat{a}} \left[1 + \sigma \sqrt{1 - 8\hat{a}n^2 + 8\frac{a\mu_g}{r^4}} \right] , \ \sigma = \pm 1 \ , \tag{3.9}$$

in which μ_g is a constant of integration which is related with the mass of the black hole, and the parameter $n^2 = -2\Lambda$ corresponds to a negative bulk cosmological constant.

If we replace

$$\hat{a} \to \frac{\varrho}{2},\ n^2 \to \frac{\eta_{0a}}{12},\ \mu_g \to -\frac{C_\mu}{3} \tag{3.10}$$

in equation (3.9), we recover the black hole solution of Eq. (3.7), for the specific case $\eta = 0$ and $\varrho \neq 0$ of the previous section.

Note, that the condition $\eta = 3\eta_{2a}+\eta_{2b}+\eta_{2c} = 0$ is satisfied in the case of GB coefficient $\eta_{2a} = \hat{a}$, $\eta_{2b} = -4\hat{a}$ and $\eta_{2c} = \hat{a}$, but there are other different combinations of the coupling parameters $\eta_{2a},\ \eta_{2b},\ \eta_{2c}$ which give $\eta = 0$ and $\varrho \neq 0$. This is a very interesting result which merits further investigation. Note also that the relation $1 - \frac{\varrho\eta_{0a}}{3} = 0$ corresponds to the Chern-Simons limit of GB gravity.

3.3. $\varrho = 0$ and $\eta \neq 0$

Substituting $\varrho = 0$ and $\eta \neq 0$ we find for the function $f(r) = k + r^2 Z(r)$

$$f(r) = k + \frac{\eta_{0a}}{12} r^2 + \frac{r^2}{16\eta}\left(W_L^2\left(\frac{\tilde{C}_\mu}{r^4}\right) + 2W_L\left(\frac{\tilde{C}_\mu}{r^4}\right)\right) . \tag{3.11}$$

where $W_L(x)$ is the Lambert function, which is defined as the real solution of the equation $e^{W_L(x)}W_L(x) = x$. The large r asymptotic behaviour of Eq. (3.11) is found to be

$$f(r) = k + \frac{\eta_{0a}}{12} r^2 + \frac{\tilde{C}_\mu}{8\eta r^2} + O\left(\frac{1}{r^6}\right) . \tag{3.12}$$

The function $N(r)^2$ can be expressed in the following closed form

$$N(r)^2 = f(r)\tilde{N}(Z(r))^2 = \frac{\tilde{C}_\mu^2 f(r)}{r^8\left(W_L^2\left(\frac{\tilde{C}_\mu}{r^4}\right) + W_L\left(\frac{\tilde{C}_\mu}{r^4}\right)\right)^2} . \tag{3.13}$$

In the large r regime we find, from the above equation, that:

$$N(r)^2 = f(r)\ \left(1 + \frac{\tilde{C}_\mu^2}{r^8} + O\left(\frac{\tilde{C}_\mu^3}{r^{12}}\right)\right) , \tag{3.14}$$

hence in the large distance limit we recover the standard asymptotic behavior $N(r)^2 \simeq f(r)$.

3.4. Static solutions in the generic case ($\eta \neq 0$ and $\varrho \neq 0$)

For the generic case where $\eta \neq 0$ and $\varrho \neq 0$ we will expose the main features of the solutions. For more details can be found in [16]. First we define the following constants

$$A \equiv -3\eta_{0a}\eta + \frac{9\eta}{\varrho} ,\quad B \equiv \varrho(\varrho - 4\eta) . \tag{3.15}$$

Case (i): $B \geq 0$ and $|\sqrt{B}/\varrho| < 1$, $A > 0$

In this case we have in general two branches of solutions for function $f(r)$ where both of them have AdS_5, dS_5 or flat large distance behavior. In the case of $k = 0$ one of the solutions exhibits a naked singularity. On the other hand we can see that the large distance behavior of $N(r)^2$, is

identical with that of $f(r)$. In the case where $A < 0$, the radius r must have an upper bound and the function $f(r)$ has no large distance limit. This case lacks of physical interest.

Case (ii): $B \geq 0$ and $|\sqrt{B}/\varrho| > 1$, $A > 0$

Here we have only one branch for the function $f(r)$. However there is an even more significant difference, as this class of solutions does not have regular AdS_5, dS_5 or flat asymptotic behavior. Furthermore we see that $N(r)^2$ vanishes for large values of r. For $A < 0$ the solutions behave similarly.

Case (iii): $B > 0$ and $A = 0$

Setting $A = 0$, we obtain for the functions $f(r)$ and $\hat{N}(r)$:

$$f(r) = k + \frac{r^2}{2\varrho} - \frac{C_0}{3}\, r^{2\frac{\sqrt{B}+\varrho}{\sqrt{B}-\varrho}} \tag{3.16}$$

and

$$\hat{N}(r) = \frac{\bar{C}_N}{C_0^2\sqrt{B}|\varrho+\sqrt{B}|} r^{-4\frac{\sqrt{B}+\varrho}{\sqrt{B}-\varrho}} \ . \tag{3.17}$$

As we see the above solutions exhibit an unconventional asymptotic behaviour which is not of the type AdS_5, dS_5 or flat 5D Schwarzschild form.

Case (iv): $B < 0$ and $A > 0$

Here if $B < 0$ then A must always be positive. Function $f(r)$ has several branches of solutions. In contrast with the previous case ($B > 0$) the range of radius r terminates at a lower non-zero bound. Solutions for function $f(r)$ have normal AdS_5, dS_5 large distance behavior. Again there are solutions where naked singularities occur.

Case (v): $B = 0$

Finally for $B = 0$ we have the following solutions

$$f(r) = k + \left(\frac{1}{2\varrho} + \frac{\sigma\sqrt{A}}{3\varrho}\right) r^2 \pm \frac{\tilde{C}_\mu}{3\varrho r^2} \ . \tag{3.18}$$

This has the standard form of a AdS_5, dS_5 or flat solution. For $N(r)^2$ we get,

$$N(r)^2 = \frac{1}{f(r)} \ . \tag{3.19}$$

4. Conclusions

We presented static spherically symmetric solutions in the framework of the 5D Horava-Lifshitz gravity. We considered an action consisting of terms up to second order in the curvature and we solve the theory with a non-projectable spherically symmetric ansatz for the metric. The black hole spectrum we found is controlled by three parameters η, ϱ and η_{0a}, where η_{0a} is a cosmological constant.

More specifically, there are three main sets of solutions: the two special cases ($\eta = 0$, $\varrho \neq 0$), and ($\eta \neq 0$, $\varrho = 0$) and the generic case ($\eta \neq 0$, $\varrho \neq 0$). In all cases we obtained analytic black hole solutions which have the standard AdS_5, dS_5 of flat asymptotic behaviour, plus the well-known $1/r^2$ tail. However, we also obtained solutions with an unconventional short and large distance asymptotic behaviour. Also, in many cases we obtained solutions with a naked singularity.

We also found static solutions which, after a proper identification of coupling parameters, coincide with static black hole solutions of relativistic gravity theories with quadratic curvature

correction terms. One class of these solutions consists of the Schwarzschild-AdS black hole solutions of five-dimensional Lanczos-Lovelock gravity theories. Another class of solutions contains the well-known Gauss-Bonnet black hole solutions. The interesting result we obtained in our investigation is that the non-relativistic solutions of the HL gravity corresponding to the Gauss-Bonnet solutions can be obtained for various combinations of the coupling parameters η and ϱ and not just the standard Gauss-Bonnet combination. This may be attributed to the fact that the HL static solutions are insensitive to the coupling parameter λ, so they hold even if $\lambda \neq 1$ (the value which signals the breaking of Lorentz invariance).

Acknowledgments

M.T. would like to thank the organizers of PASCOS 2010 for giving him the opportunity to give this talk.

References

[1] Horava P 2009, *JHEP* **0903** 020 (*Preprint* 0812.4287 [hep-th])
[2] Horava P 2009, *Phys. Rev.* D **79**, 084008 (*Preprint* 0901.3775 [hep-th])
[3] Sotiriou T P, Visser M and Weinfurtner S 2009, *JHEP* **0910**, 033 (*Preprint*0905.2798 [hep-th])
[4] Sotiriou T P, Visser M and Weinfurtner S 2009, *Phys. Rev. Lett.* **102** 251601 (*Preprint* 0904.4464 [hep-th])
[5] Charmousis C, Niz G, Padilla A and Saffin P M 2009, *JHEP* **0908** 070 (*Preprint* 0905.2579 [hep-th])
[6] Blas D, Pujolas O and Sibiryakov S 2010, *Phys. Rev. Lett.* **104** 181302 (*Preprint* 0909.3525 [hep-th])
[7] Papazoglou A and Sotiriou T P 2010, *Phys. Lett.* B **685** 197 (*Preprint* 0911.1299 [hep-th])
[8] Kimpton I and Padilla A 2010, *JHEP* **1007**, 014 (*Preprint* 1003.5666 [hep-th])
[9] Padilla A 2010, (*Preprint* 1009.4074 [hep-th])
[10] Lu H, Mei J and Pope C N 2009, *Phys. Rev. Lett.* **103**, 091301 (*Preprint* 0904.1595 [hep-th])
[11] Kehagias A and Sfetsos K 2009, *Phys. Lett.* B **678** 123 (*Preprint* 0905.0477 [hep-th])
[12] Park M i 2009, *JHEP* **0909** 123 [arXiv:0905.4480 [hep-th]].
[13] R. G. Cai, L. M. Cao and N. Ohta, Phys. Rev. D **80** (2009) 024003 (*Preprint* 0904.3670 [hep-th])
[14] Kiritsis E and Kofinas G 2010, *JHEP* **1001** 122 (*Preprint* 0910.5487 [hep-th])
[15] Cai R G, Liu Y and Sun Y W 2009, *JHEP* **0906**, 010 (*Preprint* 0904.4104 [hep-th]) [arXiv:0904.4104 [hep-th]].
[16] Koutsoumbas G, Papantonopoulos E, Pasipoularides P and Tsoukalas M 2010, *Phys. Rev.* D **81** 124014 (*Preprint* 1004.2289 [hep-th])
[17] Koutsoumbas G and Pasipoularides P 2010, *Phys. Rev.* D **82** 044046 (*Preprint* 1006.3199 [hep-th])