

Near horizon geometry, Brick wall model and the Entropy of a scalar field in the Reissner-Nordstrom black hole backgrounds

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Abstract.

In this article we will find the entropy of a scalar field in the Reissner-Nordstrom black hole backgrounds using the brick wall model of 't Hooft. We will use the semi-classical WKB approximation. We will consider the modes which are globally stationary so that the WKB quantization rule used in the brick wall model remains valid. In the Schwarzschild black hole this consideration had led to a new expression of the entropy different from the conventional expression which is inversely divergent in the brick wall cut-off parameter and in terms of a proper distance cut-off parameter, is proportional to the area of the event horizon. The new expression of the scalar field entropy obtained in this article is logarithmically divergent in the brick wall cut-off parameter and is not proportional to the area of the black hole event horizon. For the extremal Reissner-Nordstrom black hole background the entropy of the scalar field is again divergent in the brick wall cut-off parameter and vanishes if the temperature of the Hawking radiation and the black hole is taken to be zero. We will next consider the entropy for a thin shell of matter field surrounding the black hole horizon. When expressed in terms of a covariant cut-off parameter, the entropy of a thin shell of matter field surrounding the horizon in the non-extreme Reissner-Nordstrom black hole background is given by an expression proportional to the area of the black hole horizon. We will briefly explain the significance of this result.

1. Introduction

The thermodynamical aspects of the black holes was first established by Bekenstein [1]. He obtained an expression for the entropy of the black holes. This expression was proportional to the area of the horizons of the black holes. Hawking obtained the exact expression for the entropy by considering the behaviour of matter fields in the black hole backgrounds [2]. The entropy of a black hole, considered as a thermodynamical system, was found out to be $\frac{A}{4}$. Here A is the horizon surface area. The thermodynamical properties of a minimally coupled matter fields in black hole backgrounds was discussed by 't Hooft [3]. The black hole is assumed to be in thermal equilibrium with surrounding matter. For a static black hole the rate of particles radiated by the black hole should be equal to the rate of absorption of matter by the black hole. 't Hooft [3] assumed that the matter field wave function is vanishing near the horizon as well as at infinity, i.e, for a large value of the radial coordinate. Since then a lot of works had been done on the entropy of matter fields in the black hole backgrounds using the brick-wall model [4,5,6,7,8,9,10,11,12 13, 14].

In a previous article [15] we had discussed a few aspects of the entropy of a scalar field in the Schwarzschild black hole background using the brick-wall model of 't Hooft [3] with a proper consideration of the centrifugal potential. In the brick wall model of 't Hooft the semi-classical

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WKB quantization rule is applied to the radial part of the matter field solution for the purpose of the counting of the states. We considered the fact that the modes for which the WKB quantization rule, used in the brick wall model, remains to be valid should be stationary throughout the range of the radial variable. We found that, with this consideration, the entropy of a scalar field is different than that obtained in [3]. The expression of entropy obtained by 't Hooft is inversely divergent in the brick wall cut-off parameter and in terms of a proper distance cut-off parameter, is proportional to the area of the event horizon. The new expression of the scalar field entropy obtained in [15] is logarithmically divergent in the brick wall cut-off parameter and is not proportional to the area of the black hole event horizon. We also note that the term logarithmically divergent in the brick wall cut-off parameter is independent of the mass of the black hole although the leading order infrared divergent term is dependent on the mass of the black hole.

In the present article we will find the entropy of a scalar field in the Reissner-Nordstrom and the extremal Reissner-Nordstrom black hole background. We will use the brick wall model of t' Hooft together with the counting of the states followed in [15]. The expression for the entropy of a massless scalar field, obtained in this article, in the non-extreme Reissner-Nordstrom black hole background is similar to that of the Schwarzschild black hole and is logarithmically divergent in the brick wall cut-off parameter. For the extremal Reissner-Nordstrom black hole background the entropy of the scalar field is again divergent in the brick wall cut-off parameter. In this case the divergent part contains a term inversely dependent on the brick wall cut-off parameter and a term logarithmically dependent on the same. The matter field entropy vanishes if the temperature of the Hawking radiation and the black hole is taken to be zero.

In section:IV we will consider the entropy of a thin shell of massless scalar field surrounding the black hole event horizon [3]. The thickness of the thin shell is taken to be the same as the brick wall cut-off parameter. In the flat space-time the entropy of a thin shell of scalar field, obtained by using the WKB approximation, is proportional to the volume of the thin shell in the limit the thickness is small compared to the radius of the inner surface. The internal energy and the entropy vanish when the thickness vanishes. In the black hole backgrounds we will find that the entropy of a thin shell of massless scalar field surrounding the horizon following the procedure of section:III. We will find that, in terms of a proper distance cut-off parameter, the scalar field entropy for a thin shell surrounding the black hole horizon is proportional to the area of the black hole event horizon. We will find that the free energy and the entropy are inversely dependent on the coordinate cut-off parameter and is more strongly divergent in comparison to the earlier case of a scalar field occupying the complete manifold. This is expected because for a thin shell of matter field surrounding the horizon, the upper limit of the angular quantum number is restricted by eqn (16) and when the thickness of the matter shell is small compared to the radius of the horizon, a greater number of modes contribute to the free energy and the entropy. This is associated with the vanishing of the centrifugal potential at the horizon. Similar situation will remain valid with the Schwarzschild black hole [24]. It is remarkable that the entropy is given by a universal expression *proportional to the area of the horizon*. The expressions of the matter field entropy obtained in this section are important in the context of explaining the black hole entropy in terms of the near horizon states [20] and also for the entanglement entropy approach to explain the black hole entropy [21].

2. Brick-wall model and the Entropy of a scalar field in the Reissner-Nordstrom black hole background

The radial part of the wave equation of a massive scalar field in the Schwarzschild black hole background is given by the following expression:

$$\left(1 - \frac{2M}{r}\right)^{-1} E^2 \psi(r) + \frac{1}{r^2} \partial_r [r(r - 2M) \partial_r \psi(r)] - \left[\frac{l(l+1)}{r^2} + m^2\right] \psi(r) = 0 \quad (1)$$

Here l is the angular momentum quantum number and E is the energy.

We follow the procedure of t' Hooft to calculate the entropy of a scalar field using the brick wall model and the WKB quantization rule [3]. We consider massless fields but the discussions can also

be extended for massive fields. The boundary conditions on the scalar field are the following:

$$\psi(2M + h), \psi(L) = 0 \quad (2)$$

where $h \ll 2M$ and $L \gg 2M$. Here h is the brick-wall cut-off parameter. We consider the solutions of the wave-equation obtained by the WKB approximation. The solutions are assumed to be of the form $\rho(r)\exp[iS(r)]$. The solutions are stationary w.r.t the radial variable throughout the spatial manifold and the amplitude is assumed to be a slowly varying function of the radial coordinate r . These solutions are also important for the analysis of the Hawking radiation.

We now consider the free-energy calculation using a proper density of states. Let us first consider the WKB quantization condition together with the brick wall boundary condition. We have the following condition:

$$\pi n = \int_{r_1}^L dr k(r, l, E) \quad (3)$$

Here $r_1 = 2M + h$ and $k(r, l, E)$ is the radial wave number. We assume that for a given set of values for E, l, r_1 and L there exist one solution of the above integral equation. The upper value of l is restricted by the condition that $k(r)$ should remain real throughout the complete region of integration. We have,

$$k(r) = \frac{1}{V(r)} \sqrt{E^2 - \frac{V(r)}{r^2}} l(l + 1) \quad (4)$$

Here we are considering a massless scalar field. At $r = 2M + h$ with $h \rightarrow 0$, l can be varied freely. However for a state for which $k(r, E, l)$ is real throughout the complete region from $r = 2M + h$ to $r = L$ the upper limit of l is restricted by the maximum value of $\frac{V(r)}{r^2}$ which occurs at $r = 3M$ for the Schwarzschild black hole. The upper limit of l is given by the following expression,

$$l(l + 1) = 27E^2 M^2 \quad (5)$$

The expression for the free energy of a massless scalar field is given by the following expression [17]:

$$\pi\beta F = -\beta \int_0^\infty \frac{dE}{[\exp(\beta E) - 1]} \int_{2M+h}^L \frac{dr}{V(r)} \int dl (2l + 1) \sqrt{E^2 - \frac{V(r)}{r^2}} [l(l + 1)] \quad (6)$$

Here $V(r) = 1 - \frac{2M}{r}$. The range of E integration is from zero to infinity although one should be considering the back reaction to the metric. The range of r integration is from $2M + h$ to L where $L \gg 2M$. The l integration in eqn.(6) for the free energy introduces a factor $\frac{r^2}{V(r)}$ in the radial integral. We can take the upper limit of l integration to be $l(l + 1) = s(27E^2 M^2)$, where s can be taken to be arbitrarily close to one but not one so that $k(r)$ is real for all values of r . The product $s \frac{V(r)}{r^2} (27E^2 M^2)$ is less than E^2 for all values of r . We first do the l -integration. The free energy is given by the following expression:

$$\pi\beta F = -\frac{2}{3}\beta \int \frac{dE}{[\exp(\beta E) - 1]} \int \frac{r^2 dr}{V^2(r)} [E^3 - \{E^2 - s(27E^2 M^2) \frac{V(r)}{r^2}\}^{\frac{3}{2}}] \quad (7)$$

We can do a binomial expansion in s to find out the leading divergent term in the brick-wall cut-off parameter. The maximum value of the term $27M^2 \frac{V(r)}{r^2}$ is one. We thereafter do the radial integration and the energy integration respectively. It can easily be seen the term associated with the linear order term in s is the most divergent term in the brick-wall cut-off parameter h . The entropy is given by $S = \beta^2 \frac{dF}{d\beta}$. The entropy associated with the linear order term in ' s ' is given by,

$$S = \frac{27sI}{128\pi^4} \left[\frac{L}{M} + 2\ln\left(\frac{L}{h}\right) \right] \quad (8)$$

Where $I = \int_0^\infty \frac{x^3 dx}{(\exp x - 1)} = \frac{\pi^4}{15}$. The divergent part of the entropy associated with the brick-wall cut-off parameter is logarithmically divergent in the cut-off parameter. The divergent behaviour is expected as the constant-time foliations intersect at the horizon and can be related with infinite red shift. We have assumed $h \ll 2M$ and $L \gg 2M$. Here β is given by the usual expression $\beta = 8\pi M$ and h is the brick-wall cut-off parameter. We find that with this expression of β the divergent part of the entropy in terms of the brick-wall cut-off parameter is independent of the horizon surface area. We also note that the term logarithmically divergent in the brick wall cut-off parameter is independent of the mass of the black hole although the leading order infrared divergent term is dependent on the mass of the black hole. The higher order terms in s are not divergent in h and vanish in the limit $L \rightarrow \infty$. We will later discuss the situation with the massive fields.

In passing we note that in the original reference, [3], the upper limit of the l integration in equation (6) was taken so that k is zero. This makes the upper limit dependent on r and many solutions do not have a part stationary throughout the complete range of r and we can not apply the WKB quantization rule for those modes. The scalar field entropy is given by the following expression:

$$S = \frac{8\pi^3}{45h} 2M \left(\frac{2M}{\beta}\right)^3 \quad (9)$$

Unlike eqn(8) this expression is inversely dependent on h . In terms of a coordinate invariant cut-off, given by eqn (16), this expression is proportional to the area of the horizon and is independent of the mass or charge of a particular black hole.

We now consider the entropy of a massless scalar field in the non-extreme Reissner-Nordstrom black hole background. The procedure remains the same and only the metric function $V(r) = (1 - \frac{2M}{r})$ is replaced by the new metric function $V(r) = (1 - \frac{2M}{r} + \frac{Q^2}{r^2})$, where Q is the charge of the black hole and we have $M > |Q|$. The black hole horizon is at $r = r_+ = M + \sqrt{(M^2 - Q^2)}$. In this case finding out the maximum value of $\frac{V(r)}{r^2}$ is little cumbersome and this occurs at,

$$\tilde{r} = \frac{1}{2}[3M + \sqrt{(9M^2 - 8Q^2)}] \quad (10)$$

which is outside the event horizon. Proceeding in the same way as above we obtain the following expression for the most significant part of the scalar field entropy:

$$S = \frac{4\pi^3 s}{15\beta^3} [L + r_+ \ln\left(\frac{L}{r_+}\right) + \frac{r_+^2}{r_+ - r_-} \ln\left(\frac{r_+}{h}\right)] \left[\frac{\tilde{r}^2}{V(\tilde{r})}\right] \quad (11)$$

Where $r_- = M - \sqrt{(M^2 - Q^2)}$. We thus find that the divergent part of the scalar field entropy in the non-extreme Reissner-Nordstrom black hole background is similar to that of the Schwarzschild black hole and in particular the near horizon contribution is again logarithmically divergent in the brick wall cut-off parameter. The expression reduces to that in the Schwarzschild black hole in the limit $Q = 0$. In the next section we will find that the expression of the entropy of a thin shell of matter field surrounding the horizon is given by an expression proportional to the horizon surface area with the proportionality constant being the same for both the Schwarzschild black hole and the non-extreme Reissner-Nordstrom black hole.

We now consider the extremal Reissner-Nordstrom black hole in detail. In this case we have,

$$V(r) = \left(1 - \frac{M}{r}\right)^2 \quad (12)$$

The maximum value of $\frac{V(r)}{r^2}$ occurs at $r = 2M$ and is given by $\frac{1}{16M^2}$. The divergent part of the free energy, both ultraviolet and infrared, is given by,

$$F = -\frac{16\pi^3}{15} \frac{M^2}{\beta^4} \left[\left(\frac{M^2}{h}\right) + M \ln\left(\frac{L}{h}\right) + L\right] \quad (13)$$

Here ' s ' is taken to be one. Hence the divergent part of the massless scalar field entropy is given by the following expression:

$$S = \frac{64\pi^3}{15} \frac{M^2}{\beta^3} \left[\left(\frac{M^2}{h} \right) + M \ln \left(\frac{L}{h} \right) + L \right] \quad (14)$$

The temperature of the Hawking radiation and hence that of the extremal Reissner-Nordstrom black hole is zero and hence both the free-energy and the entropy of the scalar field vanish. However it is well-known from the imaginary-time Euclidean sector analysis [18,19] that the temperature can be taken to be arbitrary as there is no conical singularity to avoid in the corresponding Euclidean manifold. In this case the scalar field entropy is divergent. The topology of the extremal Reissner-Nordstrom black hole is different from that of the non-extreme black holes and the corresponding expressions for the scalar field entropies are also different.

3. Entropy of a thin shell of scalar field surrounding the black hole event horizon

We now consider a thin shell of matter field surrounding the outer horizon in the non-extreme Reissner-Nordstrom black hole background. The boundary conditions on the scalar field are the following:

$$\psi(r_+ + h), \psi(r_+ + 2h) = 0 \quad (15)$$

We again assume $h \ll r_+$. We can proceed similar to the preceding section. Within the region of interest the maximum of $\frac{V(r)}{r^2}$ occurs at $r = r_+ + 2h$ and to apply the WKB quantization rule, the angular quantum number, l , should be less than the value obtained from the following equation:

$$N(N + 1) = \frac{E^2(r_+)^4}{2h(r_+ - r_-)} \quad (16)$$

As before we consider the upper limit of the l -integration to be given by sN with $s < 1$. With this choice, $k(r)$ is real throughout the region of interest. In this case the free energy calculation is a little cumbersome. Unlike the previous cases, we will have near horizon contributions from the higher order terms in ' s '. These terms are inversely dependent on the brick wall parameter and we have,

$$F = - \left[2sl \ln(2) - \frac{2}{3} \sum (-1)^p \frac{(\frac{3}{2}!)^{s^p}}{(\frac{3}{2}-p)! p!} \frac{(2^{p-1}-1)2^{2-p}}{(p-1)} \right] \left[\frac{\pi^3}{60\beta^4 h} \frac{(r_+)^6}{(r_+ - r_-)^2} \right] \quad (17)$$

Where $p \geq 2$, is an integer, and $h \ll r_+$. In this case the integral for the free energy, for $s < 1$, picks divergent contribution only from the near horizon region and can be associated with the brick wall cut-off h . With the h -dependent part explicitly factored out, the remaining terms within the third parentheses is finite and can be shown to be greater than $\ln[4\exp(3/2)]$ by a small value.

The brick wall cut-off parameter is a radial parameter and we can replace it by the covariant cut-off parameter ϵ given by the following expression:

$$\epsilon = \int_{r_+}^{(r_+ + h)} \frac{dr}{\sqrt{V(r)}} \quad (18)$$

For $h \ll r_+$ we have,

$$\epsilon^2 = \left(\frac{4r_+^2}{r_+ - r_-} \right) h \quad (19)$$

In terms of this covariant cut-off parameter we have the following expression for the scalar field entropy,

$$S = \eta \frac{4\pi^3}{15\beta^3} \left[\frac{r_+^8}{(r_+ - r_-)^3} \right] \frac{1}{\epsilon^2} \quad (20)$$

Where ' η ' is the numerical factor within the third parentheses of the equation (17). ' β ' is the inverse of the Hawking temperature and is given by, $\beta = \frac{4\pi r_+^2}{r_+ - r_-}$. With this value of ' β ', the scalar field entropy becomes proportional to the area of the event horizon and is given by the following expression:

$$S = \frac{\eta}{960\pi} \frac{A}{\epsilon^2} \quad (21)$$

Where ' A ' is the area of the black hole event horizon. Thus we have a universal expression for the matter field entropy which is proportional to the area of the event horizon. If we now equate this expression to the Bekenstein-Hawking entropy [3], we find that the value of the covariant cut-off parameter becomes independent of the mass of the black hole and is given by the following expression:

$$\epsilon = \sqrt{\frac{\eta}{240\pi}} \quad (22)$$

This expression is different from the corresponding expression obtained by 't Hooft,

$$\epsilon = \sqrt{\frac{1}{90\pi}} \quad (23)$$

Similar situation will also remain valid for the Schwarzschild black hole background [24]. Thus the value of ' ϵ ' given by eqn(22) is a universal constant, independent of the black hole parameters. It is same for both the non-extreme Reissner-Nordstrom black hole and the Schwarzschild black hole. The value is expressed in terms of the Planck length. The situation for the extreme Reissner-Nordstrom black hole is not much illuminating and the scalar field entropy vanishes when Hawking temperature is taken to be zero.

4. conclusion

To conclude, we have considered the entropy of a scalar field in the the Reissner-Nordstrom black hole backgrounds. We have used the brick wall model of 't Hooft. The scalar field is vanishing near the horizon as well as far away from the horizon. The density of states is calculated using the WKB approximation. We found that the entropy associated with the solutions which are stationary throughout the region of interest so that the WKB quantization rule is valid is different from that obtained by 't Hooft. The expression for the entropy is divergent in the brick wall cut-off parameter and this ill-behavedness is associated with the intersection of the constant-time foliations at the horizon which is a fixed point set of the time-like Killing vector. The scalar field entropy is not proportional to the area of the black hole event horizon. It will be interesting to compare the discussions in this article with the calculations of the matter field entropy made in the imaginary-time Euclidean manifold [19]. The Euclidean sector topology of the extremal Reissner-Nordstrom black hole is different from that of the non-extreme Reissner-Nordstrom black hole and leads to a vanishing entropy for the black hole [17]. The entropy of the matter field also vanishes if we choose the temperature to be zero. We also considered the entropy of a thin shell of scalar field surrounding the black hole horizon. We have obtained an expression for the matter field entropy which is proportional to the horizon surface area. This is valid for both the Schwarzschild and the non-extreme Reissner-Nordstrom black holes and the proportionality constants are the same. The expressions of the thin shell matter field entropy obtained in the article are important in the context of explaining the black hole entropy in terms of the near horizon states [20] and also for the entanglement entropy approach to explain the black hole entropy [21,22]. The fact that the solutions of the matter field may not contain a global oscillatory term for all values of ' l ' is also

important to discuss the scattering and absorption of matter fields in black hole backgrounds. These discussions may also be important to consider backscattering and to find out the luminosity of Hawking radiation [19]. The logarithmically divergent expression of the matter field entropy may be significant for the optical metric analysis of the black hole thermodynamics [13].

References

- [1] Bekenstein J 1973 *Phys. Rev. D* **7** 2333
- [2] Hawking S W 1975 *Commun. Math. Phys.* **43** 199
- [3] 't Hooft G 1985 *Nucl. Phys. B* **256** 727
- [4] Susskind L and Uglum J 1994 *Phys. Rev. D* **50** 2700
- [5] Ghosh A and Mitra P 1995 *Phys. Lett. B* **357** 295
- [6] Ichinose I and Satoh Y 1995 *Nucl. Phys. B* **447** 34
- [7] Solodukhin S N 1995 *Phys. Rev. D* **51** 68
- [8] Cai R G and Zhang Y Z 1996 *Mod. Phys. Lett. A* **11**
- [9] Cognola G 1998 *Phys. Rev. D* **57** 6292
- [10] Ghosh K 2003 *Phys. Rev. D* **60** 124027
- [11] Ghosh K 2012 *A Few Comments on Classical Electrodynamics: Supplement IX* **V153**
<http://www.arxiv.org/physics/0605061>
- [12] Bombelli L, Kaul R K, Lee J and Sorkin R 1986 *Phys. Rev. D* **34** 373
- [13] de Alwis S P and Ohta N 1995 *Phys. Rev. D* **52** 3529
- [14] Setare M R 2006 *Int. J. Mod. Phys. A* **21** 6183-6190
- [15] Ghosh K 2009 *Nucl. Phys. B* **814** 212-216
- [16] Gibbons G W and Hawking S W 1979 *Commun. Math. Phys.* **66** 291
- [17] Tolman R C 1979 *The principles of Statistical Mechanics* (Dover Publications, Inc. New York)
- [18] Hawking S W, Horowitz G T and Ross S F 1995 *Phys. Rev. D* **51** 4302
- [19] Hawking S W 1977 *Commun. Math. Phys.* **56** 133
- [20] Birmingham D, Gupta K S and Sen S 2001 *Phys. Lett. B* **505** 191
- [21] Srednicki M 1993 *Phys. Rev. Lett.* **71** 666
- [22] Birrel N D and Davies P C W 1982 *Quantum Fields in Curved Space* (Cambridge University Press)
- [23] Ghosh K 1999 *Phys. Rev. D* **60** 104003
- [24] Ghosh K 2012 *Communicated*