New Journal of Physics
The open-access journal for physics

Inverse bremsstrahlung heating beyond the first Born approximation for dense plasmas in laser fields

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New Journal of Physics **14** (2012) 065010 (21pp)
Received 26 January 2012
Published 25 June 2012
Online at http://www.njp.org/
doi:10.1088/1367-2630/14/6/065010

Abstract. Inverse bremsstrahlung (IB) heating, an important process in the laser–matter interaction, involves two different kinds of interaction—the interaction of the electrons with the external laser field and the electron–ion interaction. This makes analytical approaches very difficult. In a quantum perturbative approach to the IB heating rate in strong laser fields, usually the first Born approximation with respect to the electron–ion potential is considered, whereas the influence of the electric field is taken exactly in the Volkov wave functions. In this paper, a perturbative treatment is presented adopting a screened electron–ion interaction potential. As a new result, we derive the momentum-dependent, angle-averaged heating rate in the first Born approximation. Numerical results are discussed for a broad range of field strengths, and the conditions for the applicability of a linear approximation for the heating rate are analyzed in detail. Going a step further in the perturbation series, we consider the transition amplitude in the second Born approximation, which enables us to calculate the heating rate up to the third order of the interaction strength.

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Contents

1. Introduction

Inverse bremsstrahlung (IB) heating is one of the most important heating processes governing the interaction of laser radiation with matter. In dense plasmas produced by laser excitation, the free electrons exchange photons with the laser field permanently. When they are accelerated in the Coulomb field of the ions, the electrons emit photons, a phenomenon commonly known as bremsstrahlung. In the presence of the external field, the electrons absorb laser photons as well. In the time average, the absorption exceeds the emission, and the plasma is heated due to IB. Since the IB heating results from electron–ion collisions in the plasma, the electron–ion heating rate or the electron–ion collision frequency, which is also related to the dynamic conductivity, are important quantities commonly discussed in this context.

There exists a variety of works considering plasmas in the presence of a linearly polarized laser field. Considering classical plasmas, Dawson and Oberman [1] calculated the high-frequency conductivity for fully ionized plasmas on the basis of the Vlasov–Poisson equations in the framework of classical dielectric theory. Their work, in which collisional absorption of the lowest order with respect to the laser field strength was taken into account, was later extended by Decker *et al* [2] to the nonlinear case of arbitrarily strong fields. Rae and Burnett [3] found similar results accounting for collective plasma effects but using the Mermin dielectric function. Calculating the momentum loss of single electrons to stationary ions, the cycle-averaged absorption has been investigated classically by Pert [4] and by Mulser *et al* [5, 6] within the ballistic model. Shvets and Fisch [7] used a similar approach whose results—after averaging with respect to a Maxwell distribution—were found to match the results of Decker *et al* [2].

Expressions for the electron–ion collision frequency for laser fields in the nonlinear regime were derived from classical kinetic equations by Silin [8] using a non-Markovian Landau collision term and by Klimontovich [9] using a Lenard–Balescu collision term. The creation of a non-Maxwellian distribution of electron momenta during the IB heating was predicted by Langdon [10] and investigated, e.g., by Jones and Lee [11] and later by Hilse *et al* [12].

A first quantum mechanical treatment of IB in first order with respect to the electron–ion interaction was due to Rand [13]. In seminal approaches performed by Bunkin and Fedorov [14] and Kroll and Watson [15], transition rates and cross sections for the elementary absorption and emission processes were derived in the first Born approximation. Shima and Yatom [16] used these expressions taking statistical averages with distribution functions for different plasma conditions. A relativistic generalization of the transition rates was given by Schlessinger and Wright [17]. Brehme [18] and Daniele *et al* [19] investigated in detail the cross sections following the works [14, 15].

In the linear regime, a semiclassical approach was due to Cauble and Rozmus [20] by using quantum pair potentials within a memory function kinetic formalism. A quantum statistical T-matrix approach in the linear response regime was used by Reinholz *et al* [21] and Wierling *et al* [22]. Krainov [23, 24] derived expressions for the absorption rate of a single electron in the limiting cases of small and large electron momenta for small laser fields on the basis of the quantum mechanical differential cross section for spontaneous IB.

A rigorous quantum kinetic approach to plasmas in strong laser fields was presented by Kremp *et al* [25] using nonequilibrium Green's function techniques. Based on these results, Bornath *et al* [26] derived quantum statistical expressions for the electron-ion collision frequency for high-frequency laser fields. Similar results were obtained by Kull and Plagne [27] starting from the quantum Vlasov equation.

Nonlinear IB for solid-density quantum plasmas including correlations in the electronic and ionic subsystems were investigated in [12, 28–30]. In an approach adopting a connection between IB absorption and the stopping power for ions, a generalized expression for the electron–ion collision frequency was obtained by Grinenko and Gericke [31].

We wish to mention that in the references mentioned so far, mainly linearly polarized laser fields were investigated. Analytical expressions for circularly polarized fields were derived, e.g., in [7, 32, 33].

Several authors used classical molecular dynamics (MD) simulations to study IB heating [12, 34–36]. In order to investigate the dependence of individual collision events on certain quantitites such as the impact parameter in more detail, the numerical solution to the classical equations of motion for the scattering of electrons on isolated ions in the presence of a laser field was obtained by Fraiman *et al* [37]. This approach was extended by Brantov *et al* [38] to derive the IB heating rate for weakly coupled, low-density plasmas, and recently by Moll *et al* [39] for dense clusters in intense laser fields. In these classical simulations, it was shown that rescattering events of an electron to the same ion play an important role and that 'straight line' electron trajectories can be modified by the laser field significantly. These simulations are numerically quite expensive and do not provide analytical results.

The analytical results for the IB heating rate listed above differ in certain points. Some are obtained from kinetic equations and some from averaged probabilities for elementary absorption or emission processes. Some results were derived classically for weakly coupled plasmas, while with other expressions also degenerate quantum systems can be described or collective effects are included. Besides the results from the T-matrix approach [21, 22] that are only valid in the linear response limit, all analytical expressions follow from perturbative treatments where perturbation theory is performed with respect to the electron–ion interaction potential.

It is the aim of this paper to take with analytical methods a step beyond the first-order perturbation theory. To achieve this, we present in section 2 a derivation of the IB heating rate for electrons with momentum $\mathbf{p}$ interacting with an external electric field in terms of scattering quantities. Then we derive and discuss the momentum-dependent, angle-averaged heating rate in the first Born approximation. In section 3, we go beyond this and derive—on the basis of the transition amplitude in the second Born approximation—the heating rate containing additional correction terms in the third order of the electron–ion interaction. A discussion of numerical results is given for weak fields. Finally, we draw some conclusions.

2. The inverse bremsstrahlung heating rate in the first Born approximation

2.1. General derivation of the heating rate

We consider the scattering of an electron on an atomic ion in the presence of a linearly polarized laser field. The transition rate for the scattering process is defined as

$$w_{\mathbf{p}_\mathrm{i}\to\mathbf{p}_\mathrm{f}} = \lim_{t\to\infty} \frac{1}{t}\,|T_\mathrm{fi}|^2 \tag{1}$$

with the transition amplitude

$$T_\mathrm{fi} = \langle \Psi_\mathrm{f}^-(t)|\Psi_\mathrm{i}^+(t)\rangle. \tag{2}$$

The functions $\Psi_\mathrm{i}^+(t)$ and $\Psi_\mathrm{f}^-(t)$ represent wave functions of the electron with initial momentum $\mathbf{p}_\mathrm{i}$ and final momentum $\mathbf{p}_\mathrm{f}$, respectively. Both wave functions $\Psi_\mathrm{i}^+(\mathbf{r},t)$ and $\Psi_\mathrm{f}^-(\mathbf{r},t)$ are solutions to the Schrödinger equation (throughout this paper, we use Hartree atomic units with $m_\mathrm{e} = e = \hbar = 1$)

$$\mathrm{i}\frac{\partial\Psi}{\partial t} = \left(\hat{H}^{(V)} + U\right)\Psi. \tag{3}$$

Here U is the electron–ion interaction potential. The Hamilton operator $\hat{H}^{(V)}$ is determined by the operator of the electron momentum, $\hat{\mathbf{p}}$, and the vector potential of the laser field, $\mathbf{A}$, according to

$$\hat{H}^{(V)} = \frac{1}{2}\left(\hat{\mathbf{p}} + \frac{1}{c}\mathbf{A}(t)\right)^2. \tag{4}$$

The solutions to the unperturbed Schrödinger equation (that means for $U = 0$) are the Volkov wave functions $\Psi_\mathbf{p}^V$, which will be given explicitly in section 2.2. The wave functions in (2) obey the boundary conditions

$$\begin{aligned} \Psi_\mathrm{i}^+(\mathbf{r},t) &\underset{t\to-\infty}{\longrightarrow} \Psi_{\mathbf{p}_\mathrm{i}}^V(\mathbf{r},t),\\ \Psi_\mathrm{f}^-(\mathbf{r},t) &\underset{t\to+\infty}{\longrightarrow} \Psi_{\mathbf{p}_\mathrm{f}}^V(\mathbf{r},t). \end{aligned} \tag{5}$$

According to (1), we need to calculate the transition amplitude

$$T_\mathrm{fi}^+ = \lim_{t\to\infty}\langle \Psi_\mathrm{f}^-(t)|\Psi_\mathrm{i}^+(t)\rangle. \tag{6}$$

We consider here processes in which photons are absorbed or emitted by the electron, which means processes with $|\mathbf{p}_\mathrm{i}| \neq |\mathbf{p}_\mathrm{f}|$. The Volkov wave functions represent a full orthogonal set of wave functions. Hence, we can subtract from the right-hand side of (6) the Kronecker symbol $\langle\Psi_{\mathbf{p}_\mathrm{f}}^V(t)|\Psi_{\mathbf{p}_\mathrm{i}}^V(t)\rangle = \delta_{\mathbf{p}_\mathrm{f}\mathbf{p}_\mathrm{i}}$ and with (5) we can write

$$\begin{aligned} T_\mathrm{fi}^+ &= \lim_{t\to\infty}\langle\Psi_{\mathbf{p}_\mathrm{f}}^V(t)|\Psi_\mathrm{i}^+(t)\rangle - \lim_{t\to-\infty}\langle\Psi_{\mathbf{p}_\mathrm{f}}^V(t)|\Psi_\mathrm{i}^+(t)\rangle\\ &= \int_{-\infty}^{+\infty}\mathrm{d}t\,\frac{\partial}{\partial t}\langle\Psi_{\mathbf{p}_\mathrm{f}}^V(t)|\Psi_\mathrm{i}^+(t)\rangle. \end{aligned} \tag{7}$$

By using the expansion of the wave function $\Psi_{\rm i}^{+}(\mathbf{r}, t)$ in terms of the Volkov wave functions,

$$\Psi_{\rm i}^{+}(\mathbf{r}, t) = \sum_{\mathbf{p}''} a_{\mathbf{p}_{\rm i}\to\mathbf{p}''} \Psi^{V}_{\mathbf{p}''}(\mathbf{r}, t), \tag{8}$$

we obtain

$$T_{\rm fi}^{+} = \int_{-\infty}^{+\infty} \mathrm{d}t \, \frac{\partial a_{\mathbf{p}_{\rm i}\to\mathbf{p}_{\rm f}}}{\partial t}. \tag{9}$$

Below we will see that the electron's final energy is given as

$$\frac{p_{\rm f}^2}{2} = \frac{p_n^2}{2} = \frac{p_{\rm i}^2}{2} + n\omega, \tag{10}$$

which means during the scattering event the electron can absorb or emit a total number of n photons. Due to this, the total transition rate (1) can be written as

$$w_{\mathbf{p}_{\rm i}\to\mathbf{p}_{\rm f}} = \sum_{n=-\infty}^{\infty} w_{\mathbf{p}_n}, \tag{11}$$

where $w_{\mathbf{p}_n}$ is a shorthand notation for $w_{\mathbf{p}_{\rm i}\to\mathbf{p}_n}$. The rate W_n for the absorption or emission of n photons in the plasma is given by the individual rates $w_{\mathbf{p}_n}$ multiplied by the number $N_{\rm i}$ of ions in the volume V and by the density of final states $V\mathrm{d}^3p_n/(2\pi)^3$, and by integrating with respect to the respective final momentum $\mathbf{p}_n$

$$W_n = N_{\rm i}\frac{V}{(2\pi)^3}\int \mathrm{d}^3p_n\, w_{\mathbf{p}_n} = N_{\rm i}\frac{V}{(2\pi)^3}\int \mathrm{d}\Omega_{\mathbf{p}_n}\int \mathrm{d}p_n\, p_n^2\, w_{\mathbf{p}_n}. \tag{12}$$

This rate depends on the initial momentum $\mathbf{p}_{\rm i}$ of the electron as well as on the angle between $\mathbf{p}_{\rm i}$ and the electrical field strength vector $\mathbf{F}$. The angle-averaged rate will be denoted by $\langle W_n\rangle$. Then the IB heating rate is given as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \sum_{n\geqslant n_{\rm min}}^{\infty} n\omega\,\langle W_n\rangle. \tag{13}$$

In (13), terms with $n > 0$ correspond to the absorption of photons while terms with $n < 0$ correspond to photon emission. The number of photons that can be absorbed by the electron is not limited from above; therefore $n < \infty$. The number of photons that can be emitted by an electron having initially the energy $p_{\rm i}^2/2$ is limited due to energy conservation; therefore $n_{\rm min} \geqslant -p_{\rm i}^2/(2\omega)$.

In the following, we will evaluate the expressions (11)–(13) within a perturbation theory with respect to the electron–ion potential.

2.2. Heating rate in the first Born approximation for arbitrary field strengths

In order to derive the IB heating rate in the first Born approximation, we start from the Schrödinger equation (3) and consider for the electron–ion interaction the Debye potential

$$U(r) = -\frac{Z}{r}\exp(-\kappa r). \tag{14}$$

Here Z is the mean charge of an atomic ion in the plasma. Screening due to the surrounding plasma medium is taken into account in (14) via the inverse screening length κ. By setting

$U = 0$ in (3), one obtains as solutions of the unperturbed Schrödinger equation the Volkov wave functions

$$\Psi_{\mathbf{p}}^{V}(\mathbf{r},t) = \frac{1}{\sqrt{V}} \exp\left(\mathrm{i}\mathbf{p}\cdot\mathbf{r} - \frac{\mathrm{i}}{2}\int_{-\infty}^{t} \mathrm{d}t' \left(\mathbf{p} + \frac{1}{c}\mathbf{A}(t') \right)^2 \right) \tag{15}$$

with V being a normalization volume. Typically, the laser wavelength is much greater than the atomic scale in the plasma such as the Bohr radius. Then the dipole approximation can be used in which the vector potential is related to the electric field strength $\mathbf{F}$ of the linearly polarized laser field according to

$$\mathbf{F}(t) = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}(t), \qquad \mathbf{F}(t) = \mathbf{F}_0 \sin(\omega t), \tag{16}$$

where ω is the laser frequency and $\mathbf{F}_0$ is the field strength amplitude. We assume that the electromagnetic field adiabatically disappears at $t \to -\infty$. Then the Volkov wave functions can be written as

$$\Psi_{\mathbf{p}}^{V}(\mathbf{r},t) = \frac{1}{\sqrt{V}} \exp\left(\mathrm{i}\mathbf{p}\cdot\mathbf{r} - \frac{\mathrm{i}}{2}p^2 t - \frac{\mathrm{i}}{\omega^2}\mathbf{p}\cdot\mathbf{F}_0 \sin(\omega t) - \frac{\mathrm{i}}{2c^2}\int_{-\infty}^{t} \mathrm{d}t' \mathbf{A}^2(t') \right). \tag{17}$$

Here the Volkov wave functions are given in the velocity gauge. If instead the length gauge was used, the exponent of the Volkov wave functions would have a different form. However, for the calculation of the free–free matrix element below, the choice of the gauge is insignificant. This is in contrast to, e.g., bound-free matrix elements in the Keldysh–Faisal–Reiss theory where the expression for the ionization rate is gauge dependent [40].

According to (8), the solution to the full Schrödinger equation (3) for an electron having initial momentum $\mathbf{p}$ is given as an expansion in terms of the Volkov wave functions. From (3) and (8), it follows that the transition amplitudes $a_{\mathbf{p}\to\mathbf{p}'}(t)$ obey the equation

$$\mathrm{i}\frac{\partial a_{\mathbf{p}\to\mathbf{p}'}(t)}{\partial t} = \sum_{\mathbf{p}''} a_{\mathbf{p}\to\mathbf{p}''}(t)\, \langle \Psi_{\mathbf{p}'}^{V}(\mathbf{r},t)|U(r)|\Psi_{\mathbf{p}''}^{V}(\mathbf{r},t)\rangle. \tag{18}$$

The initial state of the electron before the scattering process corresponds to the Volkov wave function of an electron with momentum $\mathbf{p}$; therefore in the zeroth approximation the amplitudes are given as

$$a_{\mathbf{p}\to\mathbf{p}''}^{(0)}(t) = \begin{cases} 1 & \text{if } \mathbf{p}'' = \mathbf{p}, \\ 0 & \text{if } \mathbf{p}'' \neq \mathbf{p} \end{cases} \tag{19}$$

and the transition amplitude in the first order, $a_{\mathbf{p}\to\mathbf{p}'}^{(1)}(t)$, for the transition $\mathbf{p} \to \mathbf{p}'$ is determined from

$$\mathrm{i}\frac{\partial a_{\mathbf{p}\to\mathbf{p}'}^{(1)}(t)}{\partial t} = \langle \Psi_{\mathbf{p}'}^{V}(\mathbf{r},t)|U(r)|\Psi_{\mathbf{p}}^{V}(\mathbf{r},t)\rangle. \tag{20}$$

Inserting the Volkov wave functions (17) and using the Jacobi–Anger expansion [41]

$$\mathrm{e}^{\mathrm{i}z\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(z)\,\mathrm{e}^{\mathrm{i}n\theta}, \tag{21}$$

to expand the exponent of the matrix element into a series of Bessel functions J_n, one obtains

$$\mathrm{i}\frac{\partial a_{\mathbf{p}\to\mathbf{p}'}^{(1)}(t)}{\partial t} = \frac{U_{\mathbf{p}'\mathbf{p}}}{V}\sum_{n=-\infty}^{\infty} J_n\left(\frac{(\mathbf{p}-\mathbf{p}')\cdot\mathbf{F}_0}{\omega^2}\right) \exp\left[\mathrm{i}\left(\frac{p'^2}{2} - \frac{p^2}{2} - n\omega\right)t\right], \tag{22}$$

which contains the Fourier component of the electron–ion interaction potential

$$U_{\mathbf{p}'\mathbf{p}} = \int \mathrm{d}^3r\, U(r)\, \exp[\mathrm{i}(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}] = -\frac{4\pi Z}{|\mathbf{p}-\mathbf{p}'|^2+\kappa^2}. \tag{23}$$

With (1), (9) and (22), the transition rate for the scattering of electrons with initial momentum $\mathbf{p}$ and final momentum $\mathbf{p}'$ is in the first Born approximation given as

$$\begin{aligned} w^{(1)}_{\mathbf{p}\to\mathbf{p}'} &= \frac{1}{\tau}\left|\int_{-\infty}^{\infty}\mathrm{d}t\,\frac{\partial}{\partial t}a^{(1)}_{\mathbf{p}\to\mathbf{p}'}(t)\right|^2 \\ &= \frac{4\pi^2}{\tau}\frac{|U_{\mathbf{p}'\mathbf{p}}|^2}{V^2}\sum_{n,n'=-\infty}^{\infty} J_n\left(\frac{(\mathbf{p}-\mathbf{p}')\cdot\mathbf{F}_0}{\omega^2}\right) J_{n'}\left(\frac{(\mathbf{p}-\mathbf{p}')\cdot\mathbf{F}_0}{\omega^2}\right) \\ &\quad\times\delta\left(\frac{p'^2}{2}-\frac{p^2}{2}-n\omega\right)\delta\left(\frac{p'^2}{2}-\frac{p^2}{2}-n'\omega\right), \end{aligned} \tag{24}$$

where $\tau\to\infty$ is the collision time. Only terms with $n=n'$ give a contribution in (24) because it holds that $\delta(x)\delta(y)=0$ if $x\neq y$. Now we use

$$\delta^2\left(\frac{p'^2}{2}-\frac{p^2}{2}-n\omega\right) = \delta\left(\frac{p'^2}{2}-\frac{p^2}{2}-n\omega\right)\delta(0) = \delta\left(\frac{p'^2}{2}-\frac{p^2}{2}-n\omega\right)\frac{\tau}{2\pi}. \tag{25}$$

Then the transition rate is derived as

$$w^{(1)}_{\mathbf{p}\to\mathbf{p}'} = \sum_{n=-\infty}^{\infty} w^{(1)}_{\mathbf{p}_n} \tag{26}$$

with

$$w^{(1)}_{\mathbf{p}_n} = \frac{2\pi}{V^2}|U_{\mathbf{p}_n\mathbf{p}}|^2 J_n^2\left(\frac{(\mathbf{p}-\mathbf{p}_n)\cdot\mathbf{F}_0}{\omega^2}\right)\delta\left(\frac{p_n^2}{2}-\frac{p^2}{2}-n\omega\right). \tag{27}$$

In (27), energy conservation for the absorption ($n>0$) or emission ($n<0$) of $|n|$ laser photons is provided by the relation $p_n^2/2 = p^2/2+n\omega$. The rate for the absorption of n photons in first-order perturbation theory with respect to the potential U is derived according to (12). By introducing the number density of the ions $n_\mathrm{i}=N_\mathrm{i}/V$, this rate can be written as

$$W^{(1)}_n = n_\mathrm{i}\,p\int \mathrm{d}\Omega_{\mathbf{p}_n}\frac{\mathrm{d}\sigma^{(1)}_n}{\mathrm{d}\Omega_{\mathbf{p}_n}} \tag{28}$$

containing the incident electron flux $n_\mathrm{i}p$ and the differential cross section for the absorption of n photons in the first Born approximation

$$\frac{\mathrm{d}\sigma^{(1)}_n}{\mathrm{d}\Omega_{\mathbf{p}_n}} = \frac{\mathrm{d}\sigma^{(1)}(\mathbf{p}\to\mathbf{p}_n)}{\mathrm{d}\Omega_{\mathbf{p}_n}} J_n^2\left(\frac{(\mathbf{p}-\mathbf{p}_n)\cdot\mathbf{F}_0}{\omega^2}\right), \qquad \frac{p_n^2}{2} = \frac{p^2}{2}+n\omega, \tag{29}$$

where

$$\frac{\mathrm{d}\sigma^{(1)}(\mathbf{p}\to\mathbf{p}_n)}{\mathrm{d}\Omega_{\mathbf{p}_n}} = \frac{1}{4\pi^2}\frac{p_n}{p}|U_{\mathbf{p}_n\mathbf{p}}|^2 \tag{30}$$

is the cross section for inelastic scattering of an electron on an atomic ion. In the limit $\omega\to 0$, equation (30) matches the elastic scattering cross section in the first Born approximation. In their 'low-frequency approximation', Kroll and Watson [15] stated that for small ω, equation (29) is an exact relation even if the cross section in the first Born approximation (30) is replaced by the exact cross section. However, by considering semiclassical results of Bersons [42] it was pointed out by Fedorov [43] that this statement is not correct and that relation (29) holds only in the first Born approximation.

In the case of the Debye potential, the expression (23) for $U_{\mathbf{p}_n\mathbf{p}}$ can be inserted into (28) and the transfer momentum $\mathbf{q}$ can be introduced according to

$$\mathbf{q}^2 = |\mathbf{p} - \mathbf{p}_n|^2 = p^2 + p_n^2 - 2pp_n \cos\chi, \tag{31}$$

where χ is the angle between $\mathbf{p}$ and $\mathbf{p}_n$ yielding

$$\int \mathrm{d}\Omega_{\mathbf{p}_n} (\cdots) = 2\pi \int_0^\pi \mathrm{d}\chi \, \sin\chi \, (\cdots) = \frac{2\pi}{pp_n} \int_{|p-p_n|}^{p+p_n} \mathrm{d}q \, q \, (\cdots). \tag{32}$$

By introducing the angle θ between $\mathbf{q}$ and the electric field vector $\mathbf{F}$, the rate (28) can be averaged with respect to the field direction and one obtains

$$\langle W_n^{(1)} \rangle = \frac{Z\omega_p^2}{p} \int_{|p-p_n|}^{p+p_n} \frac{q\,\mathrm{d}q}{(q^2+\kappa^2)^2} \int_0^\pi \mathrm{d}\theta \, \sin\theta \, J_n^2\left(\frac{qF_0\cos\theta}{\omega^2}\right) \tag{33}$$

in which $\omega_p = (4\pi Z n_\mathrm{i})^{1/2}$ is the plasma frequency. With the substitution $x = \cos\theta$, the IB heating rate in the first Born approximation for arbitrary field strengths in the non-relativistic regime is determined as

$$\begin{aligned}\frac{\mathrm{d}E^{(1)}}{\mathrm{d}t} &= \sum_{n\geqslant n_\mathrm{min}}^{\infty} n\hbar\omega \, \langle W_n^{(1)} \rangle \\ &= \frac{2Z\omega_p^2 \omega m_\mathrm{e}^2 e^2 \hbar}{p} \sum_{n\geqslant n_\mathrm{min}}^{\infty} n \int_{|p-p_n|}^{p+p_n} \frac{q\,\mathrm{d}q}{(q^2+\hbar^2\kappa^2)^2} \int_0^1 \mathrm{d}x \, J_n^2\left(\frac{eqF_0x}{m_\mathrm{e}\hbar\omega^2}\right).\end{aligned} \tag{34}$$

Here the quantities e, m_e and $\hbar$ were written explicitly. For a given initial momentum p of the electrons, the value $n_\mathrm{min} < 0$ in (34) is defined from the relation $p_n = (p^2 + 2m_\mathrm{e} n\hbar\omega)^{1/2}$. The heating rate (34) has a nonlinear dependence on the electric field strength which is contained in the Bessel functions. The terms in the sum with respect to n that formally originates from the Jacobi–Anger expansion (21) can now be interpreted as the rates for the respective absorption or emission processes of $|n|$ laser photons.

In the numerical evaluation of the heating rate (34), the absorption and emission rates (33) have to be calculated individually for each photon number n. We want to address the issue of how many terms have to be considered in the numerical computation. From the upper integration limit in (33), $p + p_n$, we obtain for the maximum value of the transfer momentum $q_\mathrm{max} > 2p$. The asymptotic expression for the Bessel functions in the case $n \gg 1$ is [41]

$$J_n(z) \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{z\exp(1)}{2n}\right)^n. \tag{35}$$

For large n, this expression decreases rapidly if $n > 1.36z$. Hence, by roughly estimating $n \sim z$ and setting $q > 2p$, we obtain (in atomic units) for the photon number to be considered the estimate

$$n \approx \frac{2pF_0}{\omega^2}. \tag{36}$$

This means that with increasing field strength $\mathbf{F}_0$, the electron can absorb a larger number of photons. Formally, the number of photons to be considered increases not only with increasing field strength, but also with increasing initial electron momentum p. At this point, the balance between photon absorption and photon emission comes into play. If one wants to calculate only the total absorption rate, i.e. all terms with $n > 0$, or only the total emission rate, i.e. all terms with $n < 0$, the number of terms to be considered is estimated according to (36). However, because the probability of absorption is only a little larger than the probability of emission in the plasma, the total absorption rate and the total emission rate cancel out each other to a large extent.

A comment shall be made concerning the sign of the heating rate. Based on the Kroll–Watson cross sections (30), Brantov *et al* [38] discussed for the Coulomb case ($\kappa \to 0$) the momentum-dependent heating rate in the special cases that the initial electron momentum $\mathbf{p}$ is either parallel or perpendicular to the electric field vector $\mathbf{F}_0$. For such situations, the sign of the heating rate depends on the magnitude of $|\mathbf{p}|$ and on the field direction in both the classical [38, 39] and the quantum case [43]. We wish to point out that in this paper, we performed an angular average with respect to the field direction. As can be seen from the numerical results in section 2.4, the resulting heating rate (34) is positive for any fixed momentum p which holds also in the classical case.

2.3. Heating rate for weak laser fields

To lowest order with respect to the field strength, the IB heating rate (34) appears to be proportional to F_0^2. This is called the linear regime because the heating rate increases linearly with the laser intensity $I_0 = cF_0^2/(8\pi)$. For arbitrary values of the laser field strength, the heating rate has a nonlinear dependence on F_0 via the Bessel functions. The linear dependence results if the argument of the Bessel functions becomes much smaller than unity. Then it is sufficient to consider only the absorption and emission of single photons, i.e. only terms with $n = \pm 1$ in (34) and to approximate $J_1(z) \approx z/2$. This will define a condition under which the field can be considered to be weak; see below. Returning to the atomic system of units, the heating rate becomes

$$\frac{\mathrm{d}E^{(1)}}{\mathrm{d}t} = \frac{Z\omega_p^2 F_0^2}{6p\omega^3} \left\{ \int_{|p-p_1|}^{p+p_1} \frac{q^3\,\mathrm{d}q}{(q^2+\kappa^2)^2} - \int_{|p-p_{-1}|}^{p+p_{-1}} \frac{q^3\,\mathrm{d}q}{(q^2+\kappa^2)^2} \right\}. \tag{37}$$

By expanding the integration limits according to

$$p + p_{\pm 1} \approx 2p \pm \frac{\omega}{p} - \frac{\omega^2}{2p^3} \pm \frac{\omega^3}{2p^5}, \qquad p - p_{\pm 1} \approx \mp \frac{\omega}{p} + \frac{\omega^2}{2p^3} \mp \frac{\omega^3}{2p^5}, \tag{38}$$

one finally obtains in lowest order with respect to ω/p^2:

$$\frac{\mathrm{d}E^{(1)}}{\mathrm{d}t} = \frac{Z\omega_p^2 F_0^2 p}{6\omega^2} \left\{ \frac{1}{(p^2+\kappa^2/4)^2} + \frac{\omega^4}{p^4(\omega^2+\kappa^2 p^2)^2} \right\}. \tag{39}$$

In some special cases, the linear expression (39) can be simplified further. If $\kappa \ll \omega/p \ll p$, it yields

$$\frac{\mathrm{d}E^{(1)}}{\mathrm{d}t} = \frac{Z\omega_p^2 F_0^2}{3\omega^2 p^3}. \tag{40}$$

This formula was found by Marcuse already in 1962 by applying a double perturbation theory both with respect to the electron–ion interaction and to the laser field strength in the case of the

Coulomb potential [43, 44]. Otherwise, for $\omega/p \ll \kappa \ll p$, one finds that

$$\frac{\mathrm{d}E^{(1)}}{\mathrm{d}t} = \frac{Z\omega_p^2 F_0^2}{6\omega^2 p^3}. \tag{41}$$

Finally, in the case of large screening, $\omega/p \ll p \ll \kappa$, the heating rate becomes

$$\frac{\mathrm{d}E^{(1)}}{\mathrm{d}t} = \frac{8Z\omega_p^2 F_0^2 p}{3\omega^2 \kappa^4}. \tag{42}$$

We obtained the linear approximations (39)–(42) for the heating rate under the assumption that the argument of the Bessel functions in (34) is much smaller than unity. From the maximum value of the transfer momentum, $q_{\max} \approx 2p$, we can estimate that this condition is fulfilled only for very small values of the field strength, namely

$$F_0 \ll \frac{m_e \hbar \omega^2}{ep}. \tag{43}$$

Fedorov [43], who considered the non-averaged heating rate, pointed out that for special collision geometries the condition for a linear behavior is fulfilled for larger field strengths than those determined by (43), in particular if one considers small-angle scattering with $q \approx \omega/p$. Such a condition can be found also in the present case of the angle-averaged heating rate. The numerical results (see figure 1) show a linear behavior of the properly normalized heating rate going far beyond the range given by (43). The condition for this will be derived in the following for the case $\kappa \to 0$, however, it is also valid for $\kappa \neq 0$; for numerical results, see section 2.4. We consider a special form of Neumann's addition theorem for the Bessel functions [41]

$$\sum_{n=-\infty}^{\infty} J_n^2(z)\, \mathrm{e}^{in\phi} = J_0\left(2z \,\sin\frac{\phi}{2}\right), \tag{44}$$

yielding for $\phi = 0$ the result

$$\sum_{n=-\infty}^{\infty} J_n^2(z) = 1. \tag{45}$$

By subtracting (45) from (44), one obtains

$$\sum_{n=-\infty}^{\infty} J_n^2(z)\, \left(\mathrm{e}^{in\phi} - 1\right) = J_0\left(2z \,\sin\frac{\phi}{2}\right) - 1. \tag{46}$$

For $\phi \to 0$, it follows from (46) by a Taylor expansion of the exponent and of the Bessel function J_0 that

$$-\frac{\phi^2}{2} \sum_{n=-\infty}^{\infty} n^2 J_n^2(z) = -\frac{z^2\phi^2}{4} \tag{47}$$

and therefore

$$\frac{1}{2} \sum_{n=-\infty}^{\infty} n^2 J_n^2(z) = \sum_{n=1}^{\infty} n^2 J_n^2(z) = \frac{z^2}{4}. \tag{48}$$

The nonlinear heating rate (34) can be written as

$$\frac{\mathrm{d}E^{(1)}}{\mathrm{d}t} = \frac{2Z\omega_p^2\omega}{p} \sum_{n=1}^{\infty} n \left\{ \int_{|p-p_n|}^{p+p_n} \mathrm{d}q\, f(q) - \int_{|p-p_{-n}|}^{p+p_{-n}} \mathrm{d}q\, f(q) \right\}, \tag{49}$$

where $f(q)$ is the same integrand as in (34). In contrast to section 2.3, we now take into account not only single-photon absorption and emission, but multiphoton processes. By expanding the integration limits under the condition that $n\omega \ll p^2$ and by rearranging the integrals, the expression in the curly brackets of (49) becomes

$$\{\ldots\} \approx \int_{2p-n\omega/p}^{2p+n\omega/p} \mathrm{d}q \; f(q) + \int_{\frac{n\omega}{p}-\frac{n^2\omega^2}{2p^3}}^{\frac{n\omega}{p}+\frac{n^2\omega^2}{2p^3}} \mathrm{d}q \; f(q). \tag{50}$$

In (50), the first term describes backscattering of electrons with the transfer momentum $q \approx 2p$, whereas the second term indicates small-angle scattering with $q \approx n\omega/p$. We can approximate these integrals by using

$$\int_{Q-\delta q}^{Q+\delta q} \mathrm{d}q \; f(q) \approx 2\delta q \; f(Q) \tag{51}$$

and in the limit $\kappa \to 0$, the heating rate becomes

$$\frac{\mathrm{d}E^{(1)}}{\mathrm{d}t} = \frac{2Z\omega_p^2\omega}{p} \sum_{n=1}^{\infty} \left\{ \frac{n^2\omega}{4p^4} \int_0^1 \mathrm{d}x \; J_n^2\left(\frac{2pF_0x}{\omega^2}\right) + \frac{1}{\omega} \int_0^1 \mathrm{d}x \; J_n^2\left(\frac{nF_0x}{p\omega}\right) \right\}. \tag{52}$$

By using for the first term the relation (48) and for the second term the relation [41]

$$\sum_{n=1}^{\infty} J_n^2(nz) = \frac{1}{2}\left(\frac{1}{\sqrt{1-z^2}} - 1\right), \tag{53}$$

we obtain

$$\frac{\mathrm{d}E^{(1)}}{\mathrm{d}t} = \frac{2Z\omega_p^2\omega}{p} \left\{ \frac{F_0^2}{12p^2\omega^3} + \frac{p}{2F_0}\left(\arcsin\left(\frac{F_0}{p\omega}\right) - \frac{F_0}{p\omega}\right) \right\}. \tag{54}$$

Expression (54) was derived under the condition $n\omega \ll p^2$. The maximum value of n that gives a significant contribution can be estimated according to (36). Hence, the condition under which the linear approximation (54) for the heating rate can be used is

$$F_0 \ll \omega p. \tag{55}$$

The condition (55) is the well-known classical condition for small laser fields, namely that the oscillatory quiver velocity $v_{\mathrm{os}} = eF_0/(m_{\mathrm{e}}\omega)$ of the electrons is much smaller than the electron velocity $v = p/m_{\mathrm{e}}$. In the opposite case, non-perturbative effects such as multiple scattering become important. With the condition (55), (54) can be simplified further, and one obtains in lowest order with respect to the parameter $F_0/(\omega p)$

$$\frac{\mathrm{d}E^{(1)}}{\mathrm{d}t} = \frac{2Z\omega_p^2\omega}{p} \left\{ \frac{F_0^2}{12p^2\omega^3} + \frac{F_0^2}{12p^2\omega^3} \right\} = \frac{Z\omega_p^2F_0^2}{3\omega^2p^3}, \tag{56}$$

which is the same as the Marcuse result (40). It should be noted that according to (50) and (56), half of this result originates from the scattering at large angles (backscattering), while the other half is due to small-angle scattering.

We want to stress that according to (43), the argument of the Bessel functions is much smaller than unity only for very small fields $F_0 \ll \omega^2/p$. Only for such small fields, multiphoton absorption and emission do not occur and the heating is solely determined by single-photon processes. Nevertheless, the more general condition for the applicability of the linear approximation for the angle-averaged heating rate is $F_0 \ll \omega p$, which is fulfilled even for considerably larger field strengths when multiphoton absorption and emission processes are involved. Apparently, as long as (55) is fulfilled, the multiphoton processes cancel out each other in such a way that it seems as though the plasma heating is determined only by single-photon absorption and emission.

2.4. Numerical results in the first Born approximation

By substituting the transfer momentum q in (34) by $y = qp/\omega$, the nonlinear heating rate can be written as

$$\frac{\mathrm{d}E^{(1)}}{\mathrm{d}t} = \frac{2Z\omega_p^2 p}{\omega} \sum_{n \geqslant n_{\min}}^{\infty} n \int_{|p^2 - pp_n|/\omega}^{(p^2+pp_n)/\omega} \frac{y\,\mathrm{d}y}{\left(y^2 + \left(\frac{\kappa p}{\omega}\right)^2\right)^2} \int_0^1 \mathrm{d}x\, J_n^2\left(xy\,\frac{p_{\mathrm{os}}}{p}\right). \tag{57}$$

We will discuss the heating rate as a function of the dimensionless parameter p_{os}/p appearing in the Bessel functions where $p_{\mathrm{os}} = F_0/\omega$ is the quiver momentum. Throughout this paper, we will use $n_{\mathrm{i}} = 0.0024\,\mathrm{au} = 1.6 \times 10^{22}\,\mathrm{cm}^{-3}$ as a typical value for dense plasmas. In general, the inverse screening length κ is derived from the electron temperature T_{e} and the chemical potential of the electrons μ_{e} according to [45]

$$\kappa^2 = \frac{8\pi}{k_{\mathrm{B}} T_{\mathrm{e}} \Lambda_{\mathrm{e}}^3} I_{-1/2}\left(\frac{\mu_{\mathrm{e}}}{k_{\mathrm{B}} T_{\mathrm{e}}}\right), \quad \Lambda_{\mathrm{e}} = \left(\frac{2\pi}{k_{\mathrm{B}} T_{\mathrm{e}}}\right)^{1/2}, \tag{58}$$

where $I_{-1/2}(z)$ is a Fermi integral and only screening due to the electrons was taken into account. In the classical limit, (58) reduces to $\kappa^2 = 4\pi n_{\mathrm{i}} Z/(k_{\mathrm{B}} T_{\mathrm{e}})$. In the following, we will treat κ as a free parameter where 1 au corresponds to $1 a_{\mathrm{B}}^{-1}$.

In figure 1, the heating rate normalized to the product $\omega\, U_{\mathrm{pond}}$ is shown as a function of p_{os}/p where $U_{\mathrm{pond}} = F_0^2/(4\omega^2)$ is the ponderomotive energy of the electrons. The upper curves show results for $p = 2\,\mathrm{au}$ and the lower curves indicate results for $p = 4\,\mathrm{au}$, corresponding to energies $E = p^2/2 = 54.4\,\mathrm{eV}$ and $E = 217.6\,\mathrm{eV}$, respectively. A weak plasma screening ($\kappa = 0.03$ au, full dark curves) and a stronger screening ($\kappa = 0.3$ au, dashed dark curves) are considered for $\omega = 0.0553$ au, which corresponds to infrared laser radiation with the wavelength $\lambda = 825\,\mathrm{nm}$. The dashed bright curves show results from the linear expression (39) in the case of $\kappa = 0.3$ au. For small values of p_{os}/p, condition (55) is fulfilled, and the normalized heating rate is constant. It shows a maximum at $p_{\mathrm{os}}/p \sim 1$ and decreases with increasing field strength for large values of p_{os}/p. The linear expression (39) is a good approximation up to $p_{\mathrm{os}}/p \sim 0.5$, compare the bright and the full dashed lines. For the larger momentum p (lower curves), the heating rate yields lower values than for the smaller initial momentum (upper curves). This means that slower electrons can absorb more energy from the laser field than fast electrons. A larger plasma screening yields a reduction of the heating rate (compare the dashed and the full dark curves).

The dependence of the heating rate on the screening parameter κ is shown in figure 2 for the electron momentum $p = 3\,\mathrm{au}$ and for $\omega = 0.2\,\mathrm{au}$ corresponding to laser radiation with $\lambda = 275\,\mathrm{nm}$. A strong laser field ($p_{\mathrm{os}}/p = 5$, full bright curve) and a small field ($p_{\mathrm{os}}/p = 0.1$, full dark curve) are considered where the dashed dark curves show results from the limiting cases (40)–(42) for weak fields. For small values of κ, the normalized heating rate is constant, and decreases with increasing plasma screening. This is because the screening reduces the

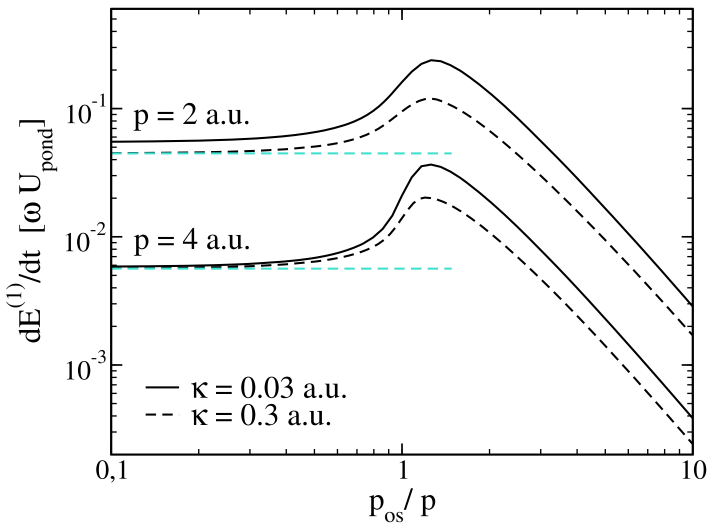

Figure 1. Heating rate normalized to ωU_{pond} as a function of the parameter p_{os}/p for $Z = 1$, $\omega = 0.0553$ au and for different values of the momentum p in the case of weak screening ($\kappa = 0.03$ au, full curves) and stronger screening ($\kappa = 0.3$ au, dashed curves). The bright dashed lines show the results from the linear expression (39) for $\kappa = 0.3$ au.

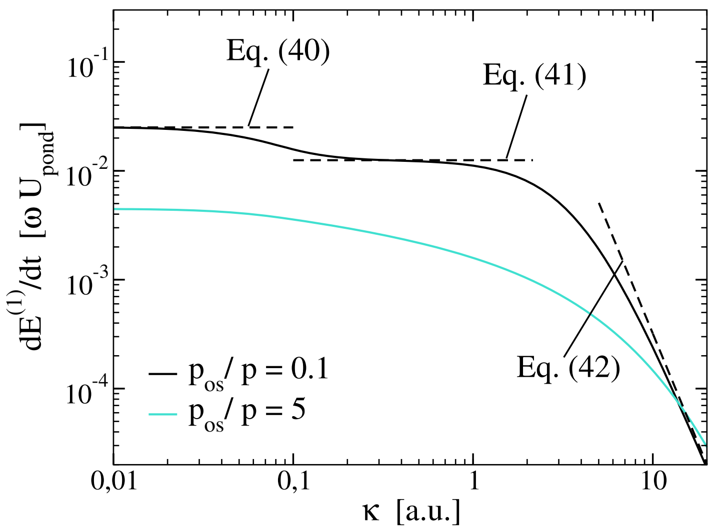

Figure 2. Heating rate normalized to ωU_{pond} as a function of the screening parameter κ for $p = 3$ au, $\omega = 0.2$ au and $Z = 1$. The dashed lines show the results from the limiting cases (40)–(42) for small fields.

electron–ion interaction and therefore the deflection of the electrons which is responsible for IB. In the case of the small laser field (dark curve), the normalized heating rate shows a plateau for intermediate values of κ where the limiting case (41) is reached. This plateau is absent in the strong field case (bright curve). The reason for this is high-order multiphoton processes occurring for strong fields, which yield a 'smoothening' of the heating rate as a function of κ.

3. The second Born approximation

In order to go beyond the calculations of section 2.2, we now consider the transition amplitude in the second Born approximation. According to (18), this amplitude is given as

$$
\begin{aligned}
\mathrm{i}\frac{\partial a^{(2)}_{\mathbf{p}\to\mathbf{p}'}(t)}{\partial t} &= \sum_{\mathbf{p}''} a^{(1)}_{\mathbf{p}\to\mathbf{p}''}(t)\, \langle \Psi^{V}_{\mathbf{p}'}(\mathbf{r},t)|U(r)|\Psi^{V}_{\mathbf{p}''}(\mathbf{r},t)\rangle \\
&= -\sum_{\mathbf{p}''} \frac{U_{\mathbf{p}'\mathbf{p}''}U_{\mathbf{p}''\mathbf{p}}}{V^2} \sum_{m,m'=-\infty}^{\infty} J_m[\mathbf{p},\mathbf{p}'']J_{m'}[\mathbf{p}'',\mathbf{p}'] \frac{\exp\left[\mathrm{i}\left(\frac{p'^2-p^2}{2}-(m+m')\omega\right)t\right]}{\frac{p''^2}{2}-\frac{p^2}{2}-m\omega-\mathrm{i}\lambda}.
\end{aligned}
\tag{59}
$$

Here the matrix element was given from (20) and (22), $a^{(1)}_{\mathbf{p}\to\mathbf{p}''}(t)$ was obtained from the integration of (22) with respect to the time and $\lambda \to +0$ is a small quantity to account for the poles of (59). In order to shorten the notation we introduced the abbreviation

$$
J_n[\mathbf{p}_1,\mathbf{p}_2] := J_n\left(\frac{(\mathbf{p_1}-\mathbf{p_2})\cdot\mathbf{F}_0}{\omega^2}\right), \tag{60}
$$

where $\mathbf{p}_1$ and $\mathbf{p}_2$ are arbitrary momenta and $\mathbf{F}_0$ is the electric field strength amplitude. The transition rate is derived according to

$$
w^{(2)}_{\mathbf{p}\to\mathbf{p}'} = \frac{1}{\tau}\left|\int_{-\infty}^{\infty}\mathrm{d}t\left(\frac{\partial}{\partial t}a^{(1)}_{\mathbf{p}\to\mathbf{p}'}(t)+\frac{\partial}{\partial t}a^{(2)}_{\mathbf{p}\to\mathbf{p}'}(t)\right)\right|^2. \tag{61}
$$

By taking into account both transition amplitudes (22) and (59) and setting $n = m + m'$, we obtain for the transition rate in the second Born approximation

$$
w^{(2)}_{\mathbf{p}\to\mathbf{p}'} = \sum_{n=-\infty}^{\infty} w^{(2)}_{\mathbf{p}_n} \tag{62}
$$

with

$$
\begin{aligned}
w^{(2)}_{\mathbf{p}_n} = \frac{2\pi}{V^2}&\left|-U_{\mathbf{p}_n\mathbf{p}}\,J_n[\mathbf{p},\mathbf{p}_n] + \frac{1}{V}\sum_{\mathbf{p}''}\sum_{m=-\infty}^{\infty}\frac{U_{\mathbf{p}_n\mathbf{p}''}U_{\mathbf{p}''\mathbf{p}}\,J_m[\mathbf{p},\mathbf{p}'']\,J_{n-m}[\mathbf{p}'',\mathbf{p}_n]}{\frac{p''^2}{2}-\frac{p^2}{2}-m\omega-\mathrm{i}\lambda}\right|^2 \\
&\times\delta\left(\frac{p_n^2}{2}-\frac{p^2}{2}-n\omega\right).
\end{aligned}
\tag{63}
$$

In this expression, energy conservation is guaranteed via the relation $p_n^2/2 = p^2/2 + n\omega$. By replacing

$$
\frac{1}{V}\sum_{\mathbf{p}''} \to \frac{1}{(2\pi)^3}\int_0^{\infty}\mathrm{d}p''\,p''^2\int\mathrm{d}\Omega_{\mathbf{p}''}, \tag{64}
$$

one finally obtains the rate for the absorption ($n > 0$) or emission ($n < 0$) of $|n|$ photons

$$W_n^{(2)} = \frac{Z\omega_p^2 p_n}{\pi} \int \mathrm{d}\Omega_{\mathbf{p}_n} \left| \frac{J_n\left(\frac{(\mathbf{p}-\mathbf{p}_n)\cdot\mathbf{F}_0}{\omega^2}\right)}{|\mathbf{p}-\mathbf{p}_n|^2+\kappa^2} + \frac{Z}{2\pi^2}\int_0^\infty \mathrm{d}p''\, p''^2 \sum_{m=-\infty}^{\infty} \int \mathrm{d}\Omega_{\mathbf{p}''} \right.$$
$$\left. \times \frac{J_m\left(\frac{(\mathbf{p}-\mathbf{p}'')\cdot\mathbf{F}_0}{\omega^2}\right) J_{n-m}\left(\frac{(\mathbf{p}''-\mathbf{p}_n)\cdot\mathbf{F}_0}{\omega^2}\right)}{\left(\frac{p''^2}{2}-\frac{p^2}{2}-m\omega-i\lambda\right)\left(|\mathbf{p}_n-\mathbf{p}''|^2+\kappa^2\right)\left(|\mathbf{p}-\mathbf{p}''|^2+\kappa^2\right)} \right|^2. \qquad (65)$$

The absorption rate (65) contains not only the initial and the final momentum $\mathbf{p}$ and $\mathbf{p}_n$, but also an intermediate momentum $\mathbf{p}''$. The heating rate including terms from the transition amplitudes in both the first and the second Born approximation is given as

$$\frac{\mathrm{d}E^{(2)}}{\mathrm{d}t} = \sum_{n\geqslant n_{\min}}^{\infty} n\omega\,\langle W_n^{(2)}\rangle. \qquad (66)$$

Here, $\langle\cdots\rangle$ denotes averaging of (65) with respect to the direction of the electric field vector of the laser. In order to obtain the heating rate (66), six integrations and two summations with respect to Bessel functions have to be performed. Unfortunately, in the case when the condition (55) for low fields is not fulfilled, the number of terms to be considered in the summations estimated from (36) is quite large.

In the case of low laser fields, (65) can be simplified significantly (for the derivation, see the appendix) and the heating rate is given as

$$\frac{\mathrm{d}E^{(2)}}{\mathrm{d}t} = \omega\left(\langle W_1^{(2)}\rangle - \langle W_{-1}^{(2)}\rangle\right) \qquad (67)$$

with

$$\langle W_1^{(2)}\rangle = \frac{n_{\mathrm{i}}Z^2 p_1}{2}\int_0^\pi \mathrm{d}\Theta_1\,\sin\Theta_1 \frac{4\pi}{3}\left(|A|^2+|B|^2\right) = \langle W_{-1}^{(2)}\rangle\,|_{\omega\leftrightarrow-\omega,\,p_1\leftrightarrow p_{-1}}, \qquad (68)$$

where the quantities A and B are lengthy expressions given as (A.19) and (A.20) in the appendix. For the calculation of the heating rate in the case of low fields, (67), only two integrations have to be carried out numerically.

Care must be taken concerning the correct perturbation order. The Born series represents an expansion of the exact transition amplitude with respect to the interaction potential U, hence with respect to Z. The transition rate is derived from the absolute square of the sum of transition amplitudes according to (61). The heating rate in the first Born approximation, (34), is of the order of Z^2. In contrast to this, the heating rate (66) contains terms up to the order Z^4. However, in the sense of a perturbation theory with respect to the interaction potential, the second Born approximation is not 'complete' because some terms of the order Z^4 are missing. These terms are contained in the third Born approximation term, which is too cumbersome to calculate. Therefore, in order to stay within a correct perturbation theory with respect to Z, we will in the following consider only the terms up to the order Z^3 in the heating rate.

In figure 3, the heating rate in the first Born approximation, corresponding to the order Z^2, is compared to the heating rate containing the additional correction terms of the order Z^3. Again the numerical results are normalized to $\omega\, U_{\mathrm{pond}}$. For the ion charge, the values $Z = 1$ and $Z = 2$ are considered. In the case of very large electron momenta, the results of the orders Z^2 and Z^3 almost coincide. This means that the correction terms of the order Z^3 do not significantly change

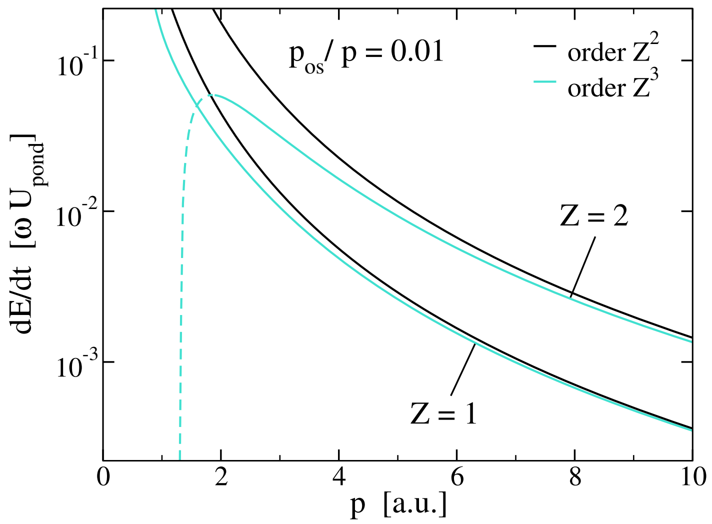

Figure 3. Heating rate normalized to ωU_{pond} as a function of the electron momentum p for $\omega = 0.0553$ au, $\kappa = 0.03$ au and $p_{\text{os}}/p = 0.01$. For $Z = 1$ and $Z = 2$, results are shown containing terms of the order Z^2 and terms up to Z^3.

the results in the first Born approximation as it should be expected. With decreasing p, the Z^3-correction terms lead to a reduction of the heating rate. For small momenta, the perturbation theory starts to lose its validity and correction terms of higher order become important. For values of p that are too small, the model breaks down. Because the terms in the perturbation series scale in orders of Z, the breakdown is more apparent for $Z = 2$ than it is for $Z = 1$; see the dashed line in figure 3.

4. Conclusion

There have been several attempts to derive analytical expressions for the IB heating rate for both classical and quantum plasmas. Most approaches are based on either the first Born approximation or its classical analogue, the 'straight line path' approximation. These perturbative calculations take into account only single scattering events of an electron to the same ion with scattering mostly into small angles. Classical simulations including multiple scattering are time-consuming and cannot provide analytical expressions for the heating rate.

In this paper, a momentum-dependent expression for the heating rate has been derived in the first Born approximation for the Debye potential taking into account plasma screening. In the calculations, an angular averaging with respect to the field direction was performed. For small laser fields, simple analytical results were obtained. The conditions for the applicability of a linear approximation for the angle-averaged heating rate have been analyzed in detail. The linear approximation for this quantity is valid even for field strengths for which the total absorption rate as well as the total emission rate contain high-order multiphoton processes. In the heating rate—which results from the difference of absorption and emission—these multiphoton terms cancel out to a large extent. The heating rate as a function of the Debye screening parameter κ shows a plateau for small fields which is absent for strong laser fields.

In the double perturbation theory both with respect to the laser field and the electron–ion interaction, we considered—as a further new result—the transition amplitude in the second Born approximation. In the weak field case, it has been evaluated numerically. A consistent perturbative correction to the heating rate in the first Born approximation could be obtained in the order Z^3 where Z is the ion charge. These correction terms yield a reduction of the heating rate compared to the first Born approximation. Due to the correction terms, the scope of applicability of the perturbative approach could be extended to the region of smaller electron momenta, hence smaller electron energies.

Acknowledgment

This work was supported by the Deutsche Forschungsgemeinschaft within Sonderforschungsbereich 652.

Appendix. Absorption rate in the second Born approximation for small fields

In the case of arbitrary laser field strengths, the absorption rate including terms from the transition amplitude in the second Born approximation is given as (65) in section 3. In order to consider low fields, we take into account only the terms with $(n=1, m=0)$ and $(n=1, m=1)$. By using $J_1(z) \approx z/2$ and by taking the average with respect to the direction of $\mathbf{F}$ (denoted as $\langle \cdots \rangle$), we obtain

$$\langle W_1^{(2)} \rangle = \frac{n_i Z^2 p_1}{2\omega^4} \int_0^\pi \mathrm{d}\Theta \sin\Theta \int_0^\pi \mathrm{d}\Theta_1 \sin\Theta_1 \int_0^{2\pi} \mathrm{d}\phi_1 \times \left| \frac{p F_0 \cos\Theta - p_1 F_0 \cos\theta_1}{p^2 + p_1^2 + \kappa^2 - 2 p p_1 \cos\Theta_1} + \frac{Z}{\pi^2} I \right|^2, \tag{A.1}$$

where

$$I = \int_0^\infty \mathrm{d}p'' p''^2 \int_0^\pi \mathrm{d}\Theta' \sin\Theta' \int_0^{2\pi} \mathrm{d}\phi' \frac{1}{(|\mathbf{p}_1 - \mathbf{p}''|^2 + \kappa^2)(|\mathbf{p} - \mathbf{p}''|^2 + \kappa^2)} \times \left\{ \frac{p'' F_0 \cos\theta' - p_1 F_0 \cos\theta_1}{p''^2 - p^2 - i\lambda} + \frac{p F_0 \cos\Theta - p'' F_0 \cos\theta'}{p''^2 - p^2 - 2\omega - i\lambda} \right\}. \tag{A.2}$$

The angle Θ' in (A.1) and (A.2) is defined below in (A.6), whereas the notation of the other angles is taken from figure A.1.

We introduce the abbreviations

$$a = |\mathbf{p}_1 - \mathbf{p}''|^2 + \kappa^2, \quad b = |\mathbf{p} - \mathbf{p}''|^2 + \kappa^2 \tag{A.3}$$

and

$$\mathbf{P} = \frac{1-t}{2}\mathbf{p} + \frac{1+t}{2}\mathbf{p}_1, \quad z^2 = \kappa^2 + p_1^2 \frac{1+t}{2} + p^2 \frac{1-t}{2}. \tag{A.4}$$

By using the 'Feynman trick'

$$\frac{1}{ab} = \frac{1}{2} \int_{-1}^{1} \mathrm{d}t \left[a\frac{1+t}{2} + b\frac{1-t}{2} \right]^{-2}, \tag{A.5}$$

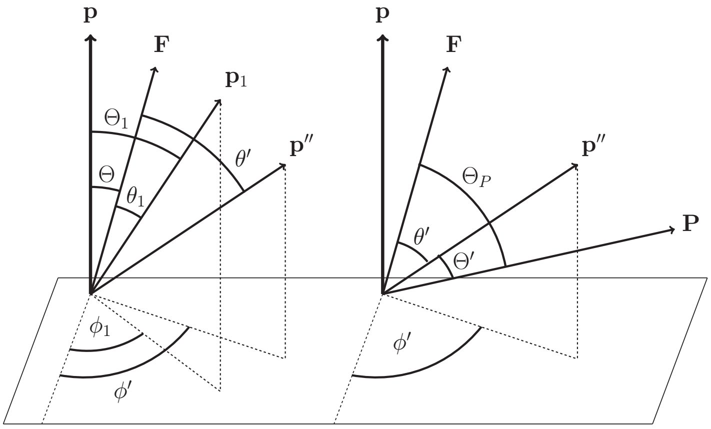

Figure A.1. Definition of the angles from the momenta and the field strength vector. The initial electron momentum $\mathbf{p}$ has been fixed and the angles ϕ_1 and ϕ' are obtained from the projections of $\mathbf{p}_1$ and $\mathbf{p}''$ on the plane perpendicular to $\mathbf{p}$.

we obtain

$$\frac{1}{ab} = \frac{1}{2}\int_{-1}^{1} \frac{\mathrm{d}t}{(z^2 + p''^2 - 2p''P\cos\Theta')^2} = -\frac{1}{2}\int_{-1}^{1} \mathrm{d}t \frac{1}{2z}\frac{\partial}{\partial z}\frac{1}{z^2 + p''^2 - 2p''P\cos\Theta'}, \tag{A.6}$$

where Θ' is the angle between $\mathbf{p}''$ and $\mathbf{P}$. Hence, (A.2) becomes

$$I = -\frac{F_0}{4}\int_0^{\pi} \mathrm{d}\Theta' \sin\Theta' \int_0^{2\pi} \mathrm{d}\phi' \int_{-1}^{1} \mathrm{d}t \frac{1}{z}\frac{\partial}{\partial z}\int_0^{\infty} \mathrm{d}p''\, p''^2 \frac{1}{z^2 + p''^2 - 2p''P\cos\Theta'}$$

$$\times \left\{ \frac{p''\cos\theta' - p_1\cos\theta_1}{p''^2 - p^2 - \mathrm{i}\lambda} + \frac{p\cos\Theta - p''\cos\theta'}{p''^2 - p_1^2 - \mathrm{i}\lambda} \right\}. \tag{A.7}$$

Here the angles are connected according to

$$\begin{aligned} \cos\theta' &= \cos\Theta'\cos\Theta_P + \sin\Theta'\sin\Theta_P\cos\phi', \\ \cos\theta_1 &= \cos\Theta\cos\Theta_1 + \sin\Theta\sin\Theta_1\cos\phi_1 \end{aligned} \tag{A.8}$$

and $\cos\Theta_P$ is given via

$$P\cos\Theta_P = \frac{1-t}{2}p\cos\Theta + \frac{1+t}{2}p_1(\cos\Theta\cos\Theta_1 + \sin\Theta\sin\Theta_1\cos\phi_1). \tag{A.9}$$

By inserting the first equation of (A.8) into (A.7), the integration with respect to ϕ' can be carried out. Substituting $p'' \to -p''$ and $\Theta' \to \pi - \Theta'$ allows us to change the integration limits according to

$$\int_0^{\infty} \mathrm{d}p''\, p''^2(\cdots) \to \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}p''\, p''^2(\cdots). \tag{A.10}$$

For the further simplification of (A.7), we consider expressions of the form

$$T_n(\xi) = \int_{-\infty}^{\infty} \mathrm{d}p'' p''^2 \frac{(p'')^{n-1}}{(p''^2 - \xi^2 - \mathrm{i}\lambda)(z^2 + p''^2 - 2p''P\cos\Theta')} \tag{A.11}$$

with $n = 1, 2$ and $\xi = p$ or $\xi = p_1$, respectively. By closing the contour with a semicircle in the upper half plane, these expressions are evaluated with residue theory and yield

$$T_n(\xi) = \frac{\mathrm{i}\pi\xi^n}{z^2+\xi^2-2\xi P\cos\Theta'} + \frac{\pi(P\cos\Theta'+\mathrm{i}W)^{n+1}}{W[(P\cos\Theta'+\mathrm{i}W)^2-\xi^2-\mathrm{i}\lambda]}, \tag{A.12}$$

where $W = \sqrt{z^2 - P^2\cos^2\Theta'}$. Now we introduce the abbreviation $\Delta = \sqrt{z^2 - P^2}$ and make use of

$$\frac{1}{z}\frac{\partial}{\partial z}(\cdots) = \frac{1}{\Delta}\frac{\partial}{\partial\Delta}(\cdots) \tag{A.13}$$

to consider expressions of the form

$$R_n(\xi) = \frac{1}{2\Delta}\frac{\partial}{\partial\Delta}\int_0^\pi \mathrm{d}\Theta' \sin\Theta'[\cos\Theta']^{n-1}T_n(\xi). \tag{A.14}$$

Then (A.7) can be written as

$$I = -\frac{F_0\pi}{2}\int_{-1}^{1}\mathrm{d}t\,\{[R_2(p)-R_2(p_1)]\cos\Theta_P - R_1(p)p_1\cos\theta_1 + R_1(p_1)p\cos\Theta\}. \tag{A.15}$$

In (A.15), the curly brackets can be evaluated explicitly to yield

$$\begin{aligned}\{\cdots\} = {} & \frac{\pi\cos\Theta_P}{P\Delta}\left[\frac{z^2-\mathrm{i}p\Delta}{(p+\mathrm{i}\Delta)^2-P^2} - \frac{z^2-\mathrm{i}p_1\Delta}{(p_1+\mathrm{i}\Delta)^2-P^2}\right] \\ & + \frac{\mathrm{i}\pi\cos\Theta_P}{2P^2}\left[\ln\left(\frac{p+P+\mathrm{i}\Delta}{p-P+\mathrm{i}\Delta}\right) - \ln\left(\frac{p_1+P+\mathrm{i}\Delta}{p_1-P+\mathrm{i}\Delta}\right)\right] \\ & + \frac{\pi}{\Delta}\left[\frac{p\cos\Theta}{(p_1+\mathrm{i}\Delta)^2-P^2} - \frac{p_1\cos\theta_1}{(p+\mathrm{i}\Delta)^2-P^2}\right].\end{aligned} \tag{A.16}$$

With the connection between the angles, (A.8) and (A.9), the absorption rate (A.1) can be written in the form

$$\begin{aligned}\langle W_1^{(2)}\rangle &= \frac{n_\mathrm{i}Z^2p_1}{2}\int_0^\pi \mathrm{d}\Theta_1\sin\Theta_1\int_0^\pi\mathrm{d}\Theta\sin\Theta\int_0^{2\pi}\mathrm{d}\phi_1|A\cos\Theta + B\sin\Theta\cos\phi_1|^2 \\ &= \frac{n_\mathrm{i}Z^2p_1}{2}\int_0^\pi\mathrm{d}\Theta_1\sin\Theta_1\frac{4\pi}{3}(|A|^2+|B|^2),\end{aligned} \tag{A.17}$$

where the quantities A and B (given below) are independent of Θ and ϕ_1. By introducing the final abbreviations

$$\Psi_1 = \frac{1-t}{2}p + \frac{1+t}{2}p_1\cos\Theta_1, \qquad \Psi_2 = \frac{1+t}{2}p_1\sin\Theta_1, \qquad \Omega_\pm = \kappa^2 \pm \omega + \omega t, \tag{A.18}$$

the real and imaginary parts of the quantities A and B in (A.17) are given as

$$\begin{aligned}\begin{pmatrix}\mathrm{Re}A\\ \mathrm{Re}B\end{pmatrix} = {} & \frac{\binom{1}{0}pF_0 - p_1F_0\binom{\cos\Theta_1}{\sin\Theta_1}}{\omega^2(2p^2+2\omega+\kappa^2-2pp_1\cos\Theta_1)} - \frac{F_0Z}{2\omega^2}\int_{-1}^{1}\mathrm{d}t\frac{1}{P^2\Delta} \\ & \times\left\{\frac{\binom{\Psi_1}{\Psi_2}(2p^2\Delta^2-z^2\Omega_+) + P^2p_1\binom{\cos\Theta_1}{\sin\Theta_1}\Omega_+}{\Omega_+^2+4p^2\Delta^2} - \frac{\binom{\Psi_1}{\Psi_2}(2p_1^2\Delta^2-z^2\Omega_-) + \binom{1}{0}P^2p\Omega_-}{\Omega_-^2+4p_1^2\Delta^2}\right. \\ & \left. - \frac{\binom{\Psi_1}{\Psi_2}\Delta}{2P}\left[\mathrm{atan}\left(\frac{2P\Delta}{\Omega_+-2\Delta^2}\right) - \mathrm{atan}\left(\frac{2P\Delta}{\Omega_--2\Delta^2}\right)\right]\right\},\end{aligned} \tag{A.19}$$

$$\begin{pmatrix}\mathrm{Im}A\\ \mathrm{Im}B\end{pmatrix} = -\frac{F_0 Z}{2\omega^2}\int_{-1}^{1}\mathrm{d}t\,\frac{1}{P^2}\left\{\frac{\binom{\Psi_1}{\Psi_2}}{4P}\ln\left[\frac{(p^2+z^2+2pP)(p_1^2+z^2-2p_1P)}{(p^2+z^2-2pP)(p_1^2+z^2+2p_1P)}\right]\right.$$
$$\left.+\frac{\binom{\Psi_1}{\Psi_2}p(\Omega_+-2z^2)+2pp_1P^2\binom{\cos\Theta_1}{\sin\Theta_1}}{\Omega_+^2+4p^2\Delta^2}-\frac{\binom{\Psi_1}{\Psi_2}p_1(\Omega_--2z^2)+\binom{1}{0}2pp_1P^2}{\Omega_-^2+4p_1^2\Delta^2}\right\}. \quad \text{(A.20)}$$

In the brackets $\binom{\cdots}{\cdots}$ of (A.19) and (A.20), the upper value belongs to the quantity A, whereas the lower value belongs to B. The emission rate $\langle W_{-1}^{(2)}\rangle$ is derived from (A.17) after the substitution of $\omega \to -\omega$ and $p_1 \to p_{-1}$ into A and B.

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