Anomalous dimension of spin-1/2 baryons in many-flavor QCD

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The anomalous dimension of spin-$1/2$ baryon operators in QCD is derived at leading $1/N_f$ order using the minimal subtraction scheme. A residual ambiguity, originating from the presence of evanescent operators in dimensional regularization, is parametrized by a function of the renormalized coupling. Our result is shown to agree with previous two- and three-loop calculations performed in two different renormalization schemes.

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I. INTRODUCTION

Scaling functions in QCD depend on $\alpha = g^2/4\pi$ and $1/N_f$. An inspection of the five-loop β function [1] (see also [2]), five-loop mass anomalous dimension γ_m [3], and three-loop anomalous dimension of spin-$1/2$ baryons $\gamma_\pm$ [4] reveals that these may be re-organized in the $\overline{\rm MS}$ scheme as an expansion in α and $\sim N_f/10$ with coefficients of order unity or smaller. From this empirical observation we may conclude that ordinary perturbation theory should be reliable when α is sufficiently smaller than unity, whereas a large N_f calculation should provide a reasonably accurate estimate of the exact, nonperturbative result for $N_f \gtrsim 10$.

Nevertheless, investigations of QCD in the limit of large number N_f of massless flavors are quite useful in practice. At the very least, they serve as nontrivial consistency checks for high-order calculations in ordinary perturbation theory. Furthermore, they represent an interesting laboratory for the study of dualities, see e.g. [5] and references therein.

Yet, because even the first nontrivial order effectively re-sums an entire αN_f series, large N_f calculations offer a unique, and systematically improvable probe of the nonperturbative regime of gauge theories (see [6] for a review of some concrete applications). Therefore, despite the fact that for realistic numbers of massless flavors this approach is not fully justifiable,[1] one can still hope that some quantity of physical interest be approximated reasonably well by the first few orders in $1/N_f$ even when $N_f \lesssim 10$. Of course the QCD beta function cannot be such an example, since it changes abruptly with N_c/N_f. Other renormalization group (RG) functions, such as the anomalous dimension of the quark bilinear or of baryons, might be more promising candidates.

An analysis of large N_f QED was initiated in [8,9] with the calculation of the anomalous dimension of the mass operator. The leading-order result straightforwardly generalizes to large N_f QCD. The beta function at next to leading order was calculated for QED in [9] and for QCD in [5]. Here we wish to derive an intrinsically non-Abelian quantity that has no counterpart in QED: the anomalous dimension of baryons. This is currently known up to three loops within standard perturbation theory [4].

II. SPIN-1/2 OPERATORS IN QCD

We adopt a Weyl spinor notation, where all fermions are left-handed: ψ is a $\mathbf{3}$ of $SU(3)$ color and a fundamental of $SU(N_f)_L$ whereas $\tilde\psi$ is a $\bar{\mathbf{3}}$ of $SU(3)$ color and the anti-fundamental of $SU(N_f)_R$.

Using Fierz transformations it is easy to show that, in exactly $d = 4$ dimensions, the lowest dimensional

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[1]The authors of [7] suggested the replacement $N_f \to -3\beta_0/2$, with $\beta_0 = 11 - 2N_f/3$ the coefficient of the QCD one-loop beta function, as a way to improve large N_f computations to smaller N_f. This "naive non-Abelianization" effectively includes additional gluonic loops; however, it is an unsystematic (and gauge-dependent) truncation of the series and it is hard to judge its reliability. A more convincing way to quantify the impact of gluonic loops would be to calculate subleading $1/N_f$ corrections.

operators interpolating spin-1/2 baryons appear in two Lorentz structures:

$$[B_+]^{ijk}_\alpha = \psi^{\{i}_\alpha(\psi^{j\}}\psi^k) \quad [B_-]^{i\tilde{j}\tilde{k}}_\alpha = \psi^i_\alpha(\tilde{\psi}^{\tilde{j}}\tilde{\psi}^{\tilde{k}})^*, \qquad (1)$$

plus their conjugates. In this notation $(\psi_1\psi_2) = \psi_1^t\epsilon\psi_2$—with $\alpha, \beta, \cdots$ Lorentz indices and ϵ the fully antisymmetric 2 by 2 matrix—, contractions of color is understood, and i, j, k, $\tilde{i}$, $\tilde{j}$, $\tilde{k}$ are flavor indices. The latter will often be suppressed for brevity, unless necessary to avoid ambiguities. We defined $\psi^{\{i}\psi^{j\}} \equiv \psi^i\psi^j + \psi^j\psi^i$.

The operators (1) are in different representations of the flavor group $SU(N_f)_L \times SU(N_f)_R$. B_+ transforms as the $\frac{1}{3}\mathbf{N_f}(\mathbf{N_f}^2 - 1)$-dimensional representation of $SU(N_f)_L$, that has mixed symmetry properties, while B_- is a $(\mathbf{N_f}, [\mathbf{N_f} \otimes \mathbf{N_f}]_{\rm antisym})$.

In the chiral limit, and still in $d = 4$ dimensions, $B_\pm$ do not mix under RG within any mass-independent renormalization scheme:

$$\mu\frac{d}{d\mu}\begin{pmatrix} B^r_+ \\ B^r_- \end{pmatrix} = -\begin{pmatrix} \gamma_+ & \\ & \gamma_- \end{pmatrix}\begin{pmatrix} B^r_+ \\ B^r_- \end{pmatrix}. \qquad (2)$$

Here and in the following $B_\pm$ ($B^r_\pm$) denote the bare (renormalized) composite operators. By Parity conservation, similar relations hold for $\tilde{B}_+ = \tilde{\psi}(\tilde{\psi}\tilde{\psi})$ and $\tilde{B}_- = \tilde{\psi}(\psi\psi)^*$, that have anomalous dimensions γ_+, γ_- respectively.[2]

Unfortunately, in multiloop calculations a mixing between $B_\pm$ and additional operators turns out to be unavoidable in all regularization schemes. While this is obvious in a generic mass-dependent scheme, it also extend to dimensional regularization, that is the scheme adopted here and in virtually all multiloop calculations. The point is that dimensional regularization violates the assumption $d = 4$. This introduces a mixing with evanescent operators with Lorentz structures $\Gamma\psi(\psi\Gamma\psi)$. We will see below that with an appropriate qualification one can still make sense of (2). However, the presence of evanescent operators will nevertheless impact our calculation of $\gamma_\pm$.

A. Definition of the renormalization scheme

Diagrams are regulated via dimensional regularization with $d = 4 - \varepsilon$ throughout the paper. Furthermore, we assume that the 2-dimensional matrices $\bar{\sigma}^\mu$, σ^μ and the anti-symmetric tensor $\epsilon^{\alpha\beta,\dot{\alpha}\dot{\beta}}$ are defined in d dimensions. We use the notation of [10].

Now, consider the following correlators of the bare operators:

$$\begin{aligned} \langle B_+\rangle_{\dot{\alpha}\dot{\beta}\dot{\gamma}\delta} &\equiv \langle\psi^\dagger(p_1)^i_{\dot{\alpha}a}\psi^\dagger(p_2)^j_{\dot{\beta}b}\psi^\dagger(p_3)^k_{\dot{\gamma}c} \\ &\quad\times [B_+(-p_1-p_2-p_3)]^{ijk}_\delta\rangle \\ \langle B_-\rangle_{\dot{\alpha}\beta\gamma\delta} &\equiv \langle\psi^\dagger(p_1)^i_{\dot{\alpha}a}\tilde{\psi}(p_2)^{\tilde{j}}_{\beta b}\tilde{\psi}(p_3)^{\tilde{k}}_{\gamma c} \\ &\quad\times [B_-(-p_1-p_2-p_3)]^{i\tilde{j}\tilde{k}}_\delta\rangle, \end{aligned} \qquad (4)$$

where the repeated flavor indices are not summed. At leading order in $1/N_f$, we find

$$\langle B\rangle^{(1)} = \mathcal{D}_{00}(\varepsilon, p)\langle B\rangle^{(0)} + \mathcal{D}_{01}(\varepsilon, p)\langle T\rangle^{(0)}. \qquad (5)$$

The divergent part of $\langle B\rangle^{(1)}$ contains terms proportional to the tree correlator $\langle B\rangle^{(0)}$, as well as to nontrivial spinor structures that we call $\langle T\rangle^{(0)}$. Up to a contraction with the antisymmetric $\epsilon^{\alpha\beta,\dot{\alpha}\dot{\beta}}$ tensor, as defined in the parenthesis of (1), the associated tensor structures are $\langle B\rangle^{(0)} = 3(1\otimes 1\otimes 1)$ and $\langle T\rangle^{(0)} = 1\otimes\Gamma_{\mu\nu}\otimes\Gamma^{\mu\nu} + \Gamma_{\mu\nu}\otimes 1\otimes\Gamma^{\mu\nu} + \Gamma_{\mu\nu}\otimes\Gamma^{\mu\nu}\otimes 1$ (see Eqs. (11) and (12) for the explicit form). The latter structure reduces to $3(1\otimes 1\otimes 1)$ in $d = 4$ dimensions, i.e. $T\to B$ as $\varepsilon\to 0$. However, for $4 - d = \varepsilon \neq 0$, the Gamma matrices are not complete, T is independent, and B is not multiplicatively renormalized. In order to have a set of operators closed under RG, we must extend (2) by introducing the operators associated to T, or more conveniently a linear combination E_1 of T, B that vanishes as $\varepsilon \to 0$. Such a combination E_1 is usually referred to as an evanescent operator. The latter then mixes with other evanescent operators involving a higher number of Gamma matrices and so on. The bottom line is that in dimensional regularization B mixes with an infinite number of evanescent operators $E_{a=1,2,3,\ldots}$, invalidating (2). This complication is well appreciated in the context of four-fermion operators, see e.g. [11,12] for earlier literature, and [13] for a lucid discussion.

Denoting the complete operator basis by O_A ($= B, E_1, E_2, E_3, \ldots$), the bare and renormalized operators are related via

$$O_A = Z_{AB}O^r_B, \qquad (6)$$

[2] In the literature a different operator basis has often been adopted. A connection with the latter is straightforwardly obtained introducing a 4-component Dirac fermion $\Psi = (\Psi_L, \Psi_R)^t$ with $\Psi_L = \psi$, $\Psi_R = \epsilon\tilde{\psi}^*$, and defining $B_1 \equiv \Psi(\Psi^t C\Psi)$, $B_2 = \gamma^5\Psi(\Psi^t C\gamma^5\Psi)$, where $C = i\gamma^0\gamma^2$. The latter basis is commonly used in lattice QCD simulations, since $B_{1,2}$ have the right Parity properties to interpolate a nucleon. (B_1 vanishes in the nonrelativistic limit and therefore B_2 is usually preferred.) From the relations $(B_2 \pm B_1)_L = -2B_\pm$, $(B_2 \pm B_1)_R = +2\epsilon\tilde{B}_\pm$, and (2), we see that

$$\mu\frac{d}{d\mu}\begin{pmatrix} B^r_1 \\ B^r_2 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} \gamma_+ + \gamma_- & \gamma_+ - \gamma_- \\ \gamma_+ - \gamma_- & \gamma_+ + \gamma_- \end{pmatrix}\begin{pmatrix} B^r_1 \\ B^r_2 \end{pmatrix}. \qquad (3)$$

Again, this expression is exact in the limit of unbroken chiral symmetry and Parity, $d = 4$, and for any mass-independent scheme. The constraint (3) is consistently satisfied by the three-loop calculation of [4].

with $Z_{AB}=Z_{AB}^{\rm conn}Z_\psi^{3/2}$. By construction, the bare evanescent operators have vanishing tree-level matrix elements with $\varepsilon\to 0$. On the other hand, the renormalized operators E_a^r may contribute at loop level, but their matrix elements are not independent. In fact, in exactly 4-dimensions there exist finite functions f_a of the renormalized coupling such that $\langle E_a^r\rangle = f_a\langle B^r\rangle$, see e.g. [14]. The functions f_a are scheme-dependent. One can always choose a prescription where $f_a=0$. [12] Such a scheme is especially useful when matching with a more fundamental theory at some high scale. The authors of [12,13] also found that $\gamma_{a0}=0$ in this particular case, so the running of the phenomenologically relevant parameters is simply controlled by the 00 component of the infinite-dimensional anomalous dimension matrix $\gamma = Z^{-1}\mu dZ/d\mu$, which itself receives contributions from loops involving evanescent operators starting at the second order in the perturbative expansion (see below). The conclusion is that, with the particular choice $f_a=0$, Eq. (2) is correct. In a generic scheme with $f_a\neq 0$ the evanescent operators also contribute to the matching, but it is easy to see that the scaling of Green's functions with an insertion of B^r is controlled by [11,12]

$$\gamma = \gamma_{00} + \gamma_{0a} f_a. \tag{7}$$

Therefore we may say that the scaling laws are still governed by (2) even in the general case $f_a\neq 0$. At leading order in $1/N_f$ the mixing with the evanescent operators, parametrized by the $\gamma_{0a}f_a$ term above, is not relevant since f_a and γ_{AB} first arise at $\mathcal{O}(1/N_f)$, i.e. we have $\gamma=\gamma_{00}+\mathcal{O}(1/N_f^2)$. Irrespective of f_a we can thus write:

$$\gamma_\pm = \mu\frac{d}{d\mu}\left(\delta Z^{\rm conn}_{00\pm}+\frac{3}{2}\delta Z_\psi\right)+\mathcal{O}(1/N_f^2), \tag{8}$$

where $\delta Z_{AB}=Z_{AB}-\delta_{AB}=\mathcal{O}(1/N_f)$. We conclude that, at this order, loops involving evanescent operators do not affect γ_{00}. Mixing with the evanescent operators starts at the second order in the perturbative expansion.

Yet, there is an additional subtle way in which the evanescent operators impact physical processes, which holds at any order in $1/N_f$ *and for any* f_a. Indeed, the very definition of bare E_a is not unique, and the choice we make ultimately affects the matrix elements of the renormalized physical operators $B_\pm^r$ [13]. In fact, in complete generality, we can *define*

$$E_1 = T - s(\varepsilon)B, \tag{9}$$

where $s(\varepsilon)=1+\sum_{n=1}s_n\varepsilon^n$ is an arbitrary function with $s(0)=1$, and still satisfy the constraint $T\to B$ (as $\varepsilon\to 0$). As a result, B^r also depends on s in general. Specifically, employing for instance a minimal subtraction scheme, we may go back to (5) and take:

$$\langle B_\pm^r\rangle \equiv {\rm finite}(1+\mathcal{D}_{00}+s_\pm(\varepsilon)\mathcal{D}_{01})\langle B\rangle + \mathcal{O}(1/N_f^2). \tag{10}$$

We thus see that $B_\pm^r$, and as a consequence also their anomalous dimensions, depend in general on a scheme-dependent function $s(\varepsilon)$.The matrix elements of $B_\pm^r$ are already affected at one-loop order in ordinary perturbation theory. This follows from (10) and the fact that, at one loop, one has $\mathcal{D}_{01}\propto 1/\varepsilon$. On the other hand, we will see that the dependance of $\gamma_\pm$ on $s_\pm$ starts at two-loop. The implication of $s_\pm$ on the anomalous dimensions $\gamma_\pm$ is analyzed in Sec. II B, whereas that on the matrix elements of $B_\pm^r$ is discussed in Sec. II D. Note that a renormalization scheme is uniquely defined only once $s_\pm$ is given.

B. Anomalous dimension of the physical operators $B_\pm^r$

Having introduced our general renormalization prescription (10) we can now derive an explicit expression for (8).

At leading $1/N_f$ order, the diagrams that contribute to $\gamma_\pm$ are the same as a one-loop analysis, with the gluon propagator resumming all fermion bubbles, see (A1). Within dimensional regularization, and using the formulas (A5) and (A6) from the Appendix, the divergent parts of the *connected* diagrams contributing to $\langle B_\pm\rangle$ read (compare to (5)):

$$\begin{aligned}
{\rm div}\langle B_-\rangle^{\rm conn}_{\dot\alpha\beta\dot\gamma\delta} &= i(P_1)_{\alpha\dot\alpha}i(P_2)_{\beta\dot\beta}i(P_3)_{\gamma\dot\gamma}\epsilon_{abc}\epsilon^{\dot\sigma\dot\rho}\\
&\quad\times{\rm div}\left[T^-_{\alpha\dot\beta\dot\gamma\delta\dot\sigma\dot\rho}\sum_{n=0}^\infty I_n+\xi T'^+_{\alpha\dot\beta\dot\gamma\delta\dot\sigma\dot\rho}I_0\right]\\
{\rm div}\langle B_+\rangle^{\rm conn}_{\dot\alpha\dot\beta\dot\gamma\delta} &= i(P_1)_{\alpha\dot\alpha}i(P_2)_{\beta\dot\beta}i(P_3)_{\gamma\dot\gamma}\epsilon_{abc}\epsilon^{\sigma\rho}\\
&\quad\times{\rm div}\left[T^+_{\alpha\beta\gamma\delta\sigma\rho}\sum_{n=0}^\infty I_n+\xi T'^+_{\alpha\beta\gamma\delta\sigma\rho}I_0+(\rho\leftrightarrow\sigma)\right],
\end{aligned}\tag{11}$$

where $P_i=\not p_i/p_i^2$ is the tree-level fermion propagator, and ξ the gauge parameter. The terms $\rho\leftrightarrow\sigma$ in the second line arise from the symmetrization of the ij indices in the definition of B_+ (see (1)). In the above expressions we introduced the tensorial structures

$$\begin{aligned}
T^-_{\alpha\dot\beta\dot\gamma\delta\dot\sigma\dot\rho} &= -[\delta_{\delta\alpha}(\bar\Gamma^{\mu\nu})_{\dot\beta\dot\sigma}(\bar\Gamma^{\mu\nu})_{\dot\gamma\dot\rho}-(\Gamma^{\mu\nu})_{\delta\alpha}\delta_{\dot\beta\dot\sigma}(\bar\Gamma^{\mu\nu})_{\dot\gamma\dot\rho}\\
&\quad-(\Gamma^{\mu\nu})_{\delta\alpha}(\bar\Gamma^{\mu\nu})_{\dot\beta\dot\sigma}\delta_{\dot\gamma\dot\rho}]\\
T'^-_{\alpha\beta\gamma\delta\sigma\rho} &= +3[\delta_{\delta\alpha}\delta_{\dot\beta\dot\sigma}\delta_{\dot\gamma\dot\rho}]\\
T^+_{\alpha\beta\gamma\delta\sigma\rho} &= -[\delta_{\delta\alpha}(\Gamma^{\mu\nu})_{\sigma\beta}(\Gamma_{\mu\nu})_{\rho\gamma}+(\Gamma^{\mu\nu})_{\delta\alpha}\delta_{\sigma\beta}(\Gamma_{\mu\nu})_{\rho\gamma}\\
&\quad+(\Gamma^{\mu\nu})_{\delta\alpha}(\Gamma_{\mu\nu})_{\sigma\beta}\delta_{\rho\gamma}]\\
T'^+_{\alpha\beta\gamma\delta\sigma\rho} &= +3[\delta_{\delta\alpha}\delta_{\sigma\beta}\delta_{\rho\gamma}],
\end{aligned}\tag{12}$$

where the d dimensional anti-symmetric tensors $\Gamma^{\mu\nu},\bar\Gamma^{\mu\nu}$ are defined via $\sigma^\mu\bar\sigma^\nu=g^{\mu\nu}-2i\Gamma^{\mu\nu}$ and $\bar\sigma^\mu\sigma^\nu=g^{\mu\nu}-2i\bar\Gamma^{\mu\nu}$, whereas

$$I_n = -\frac{2}{3} i g^2 \frac{4}{d} \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{\ell^4}{(\ell^2-\Delta)^4} [\Pi(\ell)]^n$$
$$= -\frac{1}{N_f}\left(-\frac{\lambda}{\varepsilon Z_A}\right)^{n+1} \frac{\bar{\Pi}^n}{n+1}\left(\frac{\mu^2}{\Delta}\right)^{(n+1)\frac{\varepsilon}{2}}$$
$$\times \frac{(1-\frac{\varepsilon}{6})\Gamma(1+(n+1)\frac{\varepsilon}{2})}{(1+n\frac{\varepsilon}{2})(1+n\frac{\varepsilon}{4})(1+n\frac{\varepsilon}{6})\Gamma(1+n\frac{\varepsilon}{2})}. \quad (13)$$

with the factor of 2/3 due to the group theory identity $T^A_{a'a}[T^A_{bb'}]^* \epsilon^{a'b'c} = -\frac{2}{3}\epsilon^{abc}$, obtained for fermions in the fundamental representation ($T_R = 1/2$).

The quantity Δ depends on Lorentz-scalar combinations of the $p^\mu_{1,2,3}$ and therefore on the corresponding Feynman diagram. However, the results of the Appendix (see also [6]) imply that Δ does appear in the divergent parts, as it must be in our regularization scheme. Because our main focus is the evaluation of the anomalous dimensions, Δ can therefore be ignored. It is still worth emphasizing that, as opposed to $\gamma_\pm$, the momentum-dependent finite terms are generically affected by renormalon poles [15].

As argued below (5), we are free to write $T^\pm$ as

$$T^\pm = 3[s_\pm(\varepsilon)] 1\otimes 1\otimes 1 + E_1^\pm \quad (14)$$

for any nonsingular $s_\pm$ satisfying $s_\pm(0) = 1$. Here $T^\pm$ are explicitly given in (12) whereas $E_1^\pm$ is the tensor structure associated to the evanescent operators at this order. From our prescription (10) follows that $\delta Z_{00} = \mathrm{div}(\mathcal{D}_{00} + s\mathcal{D}_{01})$, and we have:

$$\delta Z^{\rm conn}_{00,\pm} = \mathrm{div}\left[3 s_\pm \sum_{n=0}^\infty I_n + 3\xi I_0\right] + \mathcal{O}(1/N_f^2)$$
$$= -\sum_{n=1}^\infty \frac{1}{n}\mathrm{div}\left[\left(\frac{\lambda}{\varepsilon}\right)^n G_0^B(\varepsilon) s_\pm(\varepsilon)\right]$$
$$+ \frac{3\xi^r\lambda}{N_f}\frac{1}{\varepsilon} + \mathcal{O}(1/N_f^2), \quad (15)$$

with

$$G_0^B(\varepsilon) = -\frac{3}{N_f}\frac{(1-\varepsilon)(1-\frac{\varepsilon}{3})\Gamma(1-\varepsilon)}{(1-\frac{\varepsilon}{2})^2(1-\frac{\varepsilon}{4})\Gamma(1+\frac{\varepsilon}{2})\Gamma^3(1-\frac{\varepsilon}{2})}. \quad (16)$$

In deriving the second line of (15) we note that, once the expression of I_n given in (13) is used, the first line of (15) has precisely the structure of (A5); its divergent part is thus obtained as shown in (A6). Note that $\mathcal{D}_{00} + s\mathcal{D}_{01}$ in general depends on the external momenta, whereas $\delta Z^{\rm conn}_{00,\pm}$ do not.

Regarding the quark wave-function term in (8), note that Z_ψ can be calculated by replacing the tree gluon propagator with (A1) in the familiar one-loop diagram. As usual we define $Z_\psi = 1 + \delta Z_\psi = 1 + \mathrm{div}(\bar{\Sigma}) + \mathcal{O}(1/N_f^2)$, where $\Sigma(q) \equiv \not{q}\bar{\Sigma}(q^2)$ is the 1-particle irreducible fermion 2-point function. Using the general formulas of the Appendix we find (see also [6])

$$Z_\psi = 1 - \frac{2\xi^r\lambda}{N_f}\frac{1}{\varepsilon} - \sum_{n=1}^\infty \frac{1}{n}\mathrm{div}\left[\left(\frac{\lambda}{\varepsilon}\right)^n G_0^\psi(\varepsilon)\right] + \mathcal{O}(1/N_f^2),$$
$$G_0^\psi(\varepsilon) = -\frac{3}{2N_f}\varepsilon\frac{(1-\varepsilon)(1-\frac{\varepsilon}{3})^2\Gamma(1-\varepsilon)}{(1-\frac{\varepsilon}{2})^2(1-\frac{\varepsilon}{4})\Gamma(1+\frac{\varepsilon}{2})\Gamma^3(1-\frac{\varepsilon}{2})}. \quad (17)$$

This quantity was computed previously by other authors (see for instance [16] for a calculation in the $\xi = 0$ gauge). Its determination does not present any subtlety associated to evanescent operators.

Plugging (15) and (17) into (8) and using (A7) we arrive at our main result:

$$\gamma_\pm(\lambda) = \lambda\left[G_0^B(\lambda)s_\pm(\lambda) + \frac{3}{2}G_0^\psi(\lambda)\right]$$
$$= -\frac{3}{N_f}\lambda\frac{(1-\lambda)(1-\frac{\lambda}{3})^2\Gamma(1-\lambda)}{(1-\frac{\lambda}{2})^2(1-\frac{\lambda}{4})\Gamma(1+\frac{\lambda}{2})\Gamma^3(1-\frac{\lambda}{2})}$$
$$\times\left(\frac{3}{4}\lambda + \frac{s_\pm(\lambda)}{1-\frac{\lambda}{3}}\right) + \mathcal{O}(1/N_f^2), \quad (18)$$

where $\lambda = \alpha_r N_f/3\pi$ and $s_\pm(\lambda) = 1 + s_1^\pm\lambda + \cdots$. Consistently, $\gamma_\pm$ do not depend on ξ^r. This is because, in any mass-independent scheme (the $\overline{\rm MS}$ scheme is adopted here), the anomalous dimension of gauge invariant operators cannot depend on the gauge parameter. In our case, once this is verified at one-loop, the result trivially extends to all terms at first order in $1/N_f$ because the longitudinal component of the gluon propagator is not renormalized; see (A1).

An important property of (18) is that it explicitly depends on the scheme-dependent functions $s_\pm$. This dependence starts at two-loop order, i.e. $\mathcal{O}(\lambda^2)$. Indeed, in a one-loop approximation, and using the gauge $\xi = 0$ for simplicity, the divergent term in (15) simply reads $\delta Z^{\rm conn}_{00,\pm} = +(3\lambda/N_f)\mathrm{div}[s_\pm(\varepsilon)/\varepsilon] = +(3\lambda/N_f)1/\varepsilon$, and in particular $\mathrm{div}\langle B\rangle^{(1)} \propto 1\otimes 1\otimes 1$, and the anomalous dimension does not depend on our definition of the evanescent operators. Starting at two loops, the $1/\epsilon$ powers become large enough to win over the positive ones in $s_\pm$ and contribute to the counterterm $\delta Z^{\rm conn}_{00,\pm}$. The computation carried out in this paper, while being technically analogous to a one-loop calculation, is intrinsically multiloop and forces us to deal with the ambiguity associated to the presence of the evanescent operators. In the following, we will compare the scheme dependence of (18) with the one obtained in previous multiloop calculations.

C. Two schemes for $s_\pm$

We would like to compare (18) to results obtained using ordinary perturbation theory. We consider the two- and three-loop calculations performed by [14,4]. We find that these are associated, respectively, to

$$s_\pm = 1 \ [14], \qquad \frac{d(d-1)}{12} \ [4]. \tag{19}$$

The fact that $s_\pm = 1$ in [14] follows immediately from (11) and the subtraction scheme introduced in that reference. Using (11) we see that the anomalous dimension of the general operator $O = \psi_\alpha \psi_\beta \psi_\gamma$ in the scheme of [14] is:

$$\gamma_O = -\frac{3}{N_f}\lambda \frac{(1-\lambda)(1-\frac{\lambda}{3})^2 \Gamma(1-\lambda)}{(1-\frac{\lambda}{2})^2(1-\frac{\lambda}{4})\Gamma(1+\frac{\lambda}{2})\Gamma^3(1-\frac{\lambda}{2})} \times \left(\frac{3}{4}\lambda \mathbb{C}_0 + \frac{1}{3(1-\frac{\lambda}{3})}\mathbb{C}_2\right) + \mathcal{O}(1/N_f^2), \tag{20}$$

where

$$\mathbb{C}_0 = 1 \otimes 1 \otimes 1$$
$$\mathbb{C}_2 = 1 \otimes \Gamma_{\mu\nu} \otimes \Gamma^{\mu\nu} + \Gamma_{\mu\nu} \otimes 1 \otimes \Gamma^{\mu\nu} + \Gamma_{\mu\nu} \otimes \Gamma^{\mu\nu} \otimes 1.$$

The 4-dimensional scalings of the 3-quark operators of spin $1/2, 3/2$ are obtained from (20) replacing $\mathbb{C}_2 = +3, -3$, respectively, and $\mathbb{C}_0 = 1$ in both of them (there is a factor of $1/2$ difference in our definition of $\Gamma^{\mu\nu}$ compared to [14]). An analog expression holds for $\tilde{O} = \psi_\alpha (\tilde{\psi}_{\dot\beta} \tilde{\psi}_{\dot\gamma})^*$.

To see why $s_\pm = d(d-1)/12$ characterizes the formalism of [4] is a bit more complicated. Rather than repeating the calculation using the operator basis defined there, we can get to (19) observing that in the basis used by Ref. [4] only the spinor structure $1 \otimes 1 \otimes 1$ appears at order $1/N_f$, so the expression corresponding to (14) simply reads $T^\pm_{\rm ref} = 3 s^{\rm ref}_\pm(\epsilon)(1 \otimes 1 \otimes 1)$. Obviously, if we contract the Lorentz indices in $\langle B_\pm \rangle$ with the external momenta as

$$p_1^\mu \bar\sigma_\mu^{\dot\alpha\delta} \epsilon^{\dot\beta\dot\gamma} \langle B_+ \rangle_{\dot\alpha\dot\beta\dot\gamma\delta}, \qquad p_1^\mu \bar\sigma_\mu^{\dot\alpha\delta} \epsilon^{\beta\gamma} \langle B_- \rangle_{\dot\alpha\beta\gamma\delta}, \tag{21}$$

the resulting expressions are multiplicatively renormalized as well. What is less obvious is that the contractions (21) are always multiplicatively renormalized, irrespective of the definition of the evanescent operators. This follows from the fact that traces of the Gamma matrices can be simplified in any dimension using $\{\sigma^\mu, \bar\sigma^\nu\} = 2g^{\mu\nu}$. (As a matter of fact, all tensor structures in (11) become products of identities when contracted with $\bar{\not{p}}_1 \otimes \epsilon$). These contractions, being Lorentz scalar combinations of the external momenta, do not depend on our basis of fermionic operators. Hence, making the identification contraction$(T^\pm)$ = contraction$(T^\pm_{\rm ref})$, we obtain $s^{\rm ref}_\pm = d(d-1)/12$, as anticipated. That this factor appears in the comparison between the two schemes was already observed in [4].

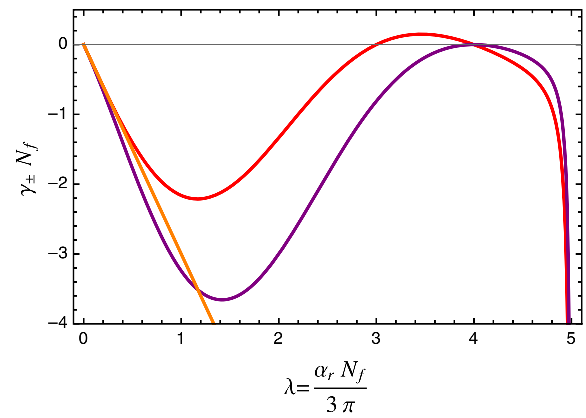

FIG. 1. Anomalous dimension of baryons in the $\overline{\rm MS}$ scheme at leading order in $1/N_f$ and all orders in the coupling $\lambda = \alpha_r N_f/3\pi$ for different definitions of evanescent operators, or equivalently $s_\pm$: in purple $s_\pm = 1$ [14], in red $s_\pm = d(d-1)/12$ [4], and in orange $s_\pm$ such that $\gamma_\pm$ equals the one-loop result.

Expanding $\gamma_\pm$ in powers of $a = \alpha_r/4\pi$ we get:

$$\gamma_\pm^{(s_\pm = \frac{d(d-1)}{12})} = -4a - \frac{4}{9}N_f a^2 + \frac{260}{81}N_f^2 a^3 + \frac{4}{81}(51 - 48\zeta_3)N_f^3 a^4 + \mathcal{O}(N_f^4 a^5),$$
$$\gamma_\pm^{(s_\pm = 1)} = -4a - \frac{32}{9}N_f a^2 + \frac{112}{27}N_f^2 a^3 + \frac{64}{81}(5 - 3\zeta_3)N_f^3 a^4 + \mathcal{O}(N_f^4 a^5), \tag{22}$$

with $\zeta_3 = 1.20206\ldots$. The first few terms consistently agree with the calculation of [14,4].[3] The full expression (18) is shown in Fig. 1. For generic $s_\pm$ it has a simple pole at $\lambda = 5$, that also sets the radius of convergence of the λ series.

D. Independence of observables on $s_\pm$

The scheme-dependent function $s_\pm$ appearing in the definition of the bare evanescent operators affects the matrix elements of the renormalized physical operator, see (10), as well as its running, see (18). The former effect already appears at one-loop order in ordinary perturbation theory, as discussed below (10), whereas the latter starts at two loops, as emphasized below (18). It is now important to verify the *$s_\pm$-independence* of physical observables.

From (10) and recalling that $s = 1 + \sum_{n=1}^\infty s_n \epsilon^n$, we have

[3] For the latter, this is true up to an overall factor of -2 arising from a different definition of $\gamma_\pm$. Our conventions conform with those adopted in the one-loop analysis of [17] and the two-loop calculation of [18] ($N_f = 3$) and [19] (general N_f), as well as [14].

$$\frac{d}{ds_n}\ln\langle B^r\rangle = \frac{d}{ds_n}\mathrm{finite}(s\mathcal{D}_{01}) + \mathcal{O}(1/N_f^2)$$
$$= -\sum_{k=0}^{\infty}[G_0^B]_k\frac{\lambda^{n+k}}{n+k} + \mathcal{O}(1/N_f^2), \quad (23)$$

with $G_0^B(\varepsilon) = \sum_{k=0}[G_0^B]_k\varepsilon^k$. Here we used the fact that, because the derivative brings positive powers of ε, only the *divergent* part of $\mathcal{D}_{01}$ contributes to this expression. Note that the right-hand side is $\mathcal{O}(\lambda)$, in agreement with the discussion below (10).

A more convenient way to write the above law is

$$-\frac{d}{ds_n}\ln\langle B^r\rangle = \sum_{k=0}^{\infty}[G_0^B]_k\frac{\lambda^{n+k}}{n+k} = \int_0^\lambda d\lambda G_0^B(\lambda)\lambda^{n-1}$$
$$= \frac{d}{ds_n}\int_0^\lambda d\lambda\frac{\gamma}{\lambda^2}. \quad (24)$$

Importantly, (24) is momentum independent. This is crucial to ensure scheme independence of physical quantities, because it allows the dependence of $\langle B^r\rangle$ on $s_\pm$ to be canceled by that in appropriate Wilson coefficients. To show how this cancellation works, consider an effective field theory defined by the following effective ($d = 4-\varepsilon$ dimensional) Hamiltonian:

$$\mathcal{H} = J\left[C^rB^r + \sum_{a=1}C_a^rE_a^r\right]. \quad (25)$$

Here J is a fermionic current that sources a baryon with the appropriate flavor quantum number. Assume for definiteness we are interested in processes with a single insertion of $\mathcal{H}$, say $\langle J|\mathcal{H}|\psi\psi\psi\rangle$. At leading order in $1/N_f$ we have $\langle J|\mathcal{H}|\psi\psi\psi\rangle \propto C^r\langle B^r\rangle$.

Recall now that, by RG invariance of the physical observables, we have $\ln C^r(\mu)/C^r(\mu') = \int_{\lambda'}^{\lambda} d\lambda\gamma/\lambda^2$, where we used $\mu d\lambda/d\mu = \lambda^2$. Because the scheme dependence starts at higher orders in the loop expansion, by IR-freedom of our large N_f theory, $\frac{d}{ds_n}C^r(\mu')$ approaches zero as $\mu' \to 0$. From this follows that

$$\frac{d}{ds_n}\ln C^r(\mu) = \frac{d}{ds_n}\int_0^\lambda d\lambda\frac{\gamma}{\lambda^2}. \quad (26)$$

Combining the latter result and (24) we get $\frac{d}{ds_n}\langle J|\mathcal{H}|\psi\psi\psi\rangle = 0$, which is what we wanted to prove to $\mathcal{O}(1/N_f)$ and all orders in α_rN_f. In ordinary perturbation theory the independence of our physical quantity on $s_\pm$ might seem a bit surprising: the scheme dependence of γ starts at two loops, but this translates into a change in the Wilson coefficients of order α_r, see Eq. (26), that can thus cancel the one-loop effect in (23).

We end with a comment on the conformal window of many-flavor QCD. It is well known that the QCD beta function has zeros at $N_f^c \le N_f \le 16$, for some unknown number N_f^c. In a large N_f expansion one finds $N_f^c = 7$ [5] (see also [20]). At these IR fixed points, critical exponents like $\gamma_\pm$ become physical, scheme-independent quantities. However, the scheme independence of (18) might seem surprising given that in the $\overline{\mathrm{MS}}$ scheme the renormalized coupling does not carry any information about the evanescent operators. This puzzle is solved observing that the defining condition $\beta(\lambda_*) = 0$ requires cancellations between terms of different order in the $1/N_f$ expansion; it then becomes possible for such terms to conspire so as to remove any s_n dependence from $\gamma_\pm(\lambda_*)$. From this observation we learn two lessons. First, the next-to-leading terms in the large N_f expansion of $\gamma_\pm$ must also depend on the s_n's of (18). This is necessary for the above cancellation to take place. Second, varying $\gamma_\pm$ as a function of $s_\pm$, see Fig. 1, should give us a rough measure of the size of the next to leading corrections.

III. DISCUSSION

The anomalous dimension of the QCD spin-1/2 baryons, at $\mathcal{O}(1/N_f)$ and all orders in $\lambda = \alpha_rN_f/3\pi < 5$, can be written in the minimal subtraction scheme as [see Eq. (18)],

$$\gamma_\pm(\lambda) = \frac{1}{2}\gamma_m(\lambda)\left(\frac{3}{4}\lambda + \frac{s_\pm(\lambda)}{1-\frac{\lambda}{3}}\right) + \mathcal{O}(1/N_f^2), \quad (27)$$

where γ_m is the anomalous dimension of the mass operator, first calculated in [9]. (In our notation the scaling dimension of the quark bilinear is $3+\gamma_m$ while that of baryons $4.5+\gamma_\pm$.)

Here $s_\pm(\lambda)$ are scheme-dependent polynomials of the renormalized coupling satisfying $s_\pm(0) = 1$. The residual scheme dependence they entail stems from an ambiguity in the definition of evanescent operators that must be introduced in dimensional regularization. We showed that the two schemes adopted in [4,14] correspond to $s_\pm = 1$, $(1-\lambda/3)(1-\lambda/4)$, respectively. The appearance of $s_\pm$ is due to the intrinsic multiloop nature of the present calculation. In this sense our analysis is qualitatively different from a one-loop calculation.

At leading $1/N_f$ order the evanescent operators do not mix with the physical operators under RG evolution, such that (2) remains approximately valid. However, their very definition impacts the anomalous dimension as well as the matrix elements of the renormalized physical operators $B_\pm^r$. In any observable, such dependence is exactly compensated by that of appropriate Wilson coefficients. We explicitly saw how this works to all orders in α_rN_f in a simple example.

The anomalous dimensions $\gamma_\pm$ at large N_f can be used as nontrivial checks of multiloop calculations in real QCD. They also find a phenomenological application in scenarios beyond the Standard Model, for example in the calculation of the proton decay rate [21] or in models with exotic QCD-like dynamics in the conformal window [22]. Unfortunately, Eq. (18) can only provide *qualitative information* about the actual size of the anomalous dimension of baryons in the conformal window. The problem becomes intrinsically nonperturbative in the interesting region $N_f < 13$, and our tool cannot offer quantitatively trustable estimates. On the one hand, for $13 \le N_f \le 16$, perturbation theory appears to be reliable and even the one-loop result seems reasonably accurate.[4] For example, at the zero of the five-loop beta function with $N_f = 13$, $\alpha_* = 0.406$, the numerical value of (18) lies within a few percent of the one-, two-, and three-loop expressions [4,14]. On the other hand, when $N_f < 13$ ordinary perturbation theory becomes unreliable, as testified by the fact that the IR fixed point found at two, three, and four loops *disappears* at five loops. Similarly, the residual scheme dependence in (18), argued to be of order $3/N_f$ in the previous section, becomes uncomfortably large, signaling a loss of perturbative control.

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APPENDIX: LARGE N_f AT LEADING ORDER

We collect here a few useful formulas for large-N_f calculations (see [6] for a more comprehensive review).

At leading order in $1/N_f$, QCD correlators have a very simple structure. The dominant diagrams may be simply derived by replacing the bare gluon propagator in a one-loop analysis with an improved bare quantity,

$$\langle A_\mu A_\nu\rangle = -\frac{i}{q^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\sum_{n=0}^{\infty}\Pi^n - \frac{i}{q^2}\xi\frac{q_\mu q_\nu}{q^2}, \tag{A1}$$

where

$$\begin{aligned}\Pi &= -\frac{\lambda}{\varepsilon Z_A}\left(-\frac{\mu^2}{q^2}\right)^{\varepsilon/2}\bar\Pi(\varepsilon),\\ \bar\Pi(\varepsilon) &\equiv \frac{(1-\frac{\varepsilon}{2})^2\Gamma(1+\frac{\varepsilon}{2})\Gamma^2(1-\frac{\varepsilon}{2})}{(1-\varepsilon)(1-\frac{\varepsilon}{2})(1-\frac{\varepsilon}{3})\Gamma(1-\varepsilon)}.\end{aligned} \tag{A2}$$

In the previous expression, we introduced the renormalized coupling and the gluon wave-function,

$$\lambda = T_R\frac{2\alpha_r N_f}{3\pi}, \qquad Z_A = 1 - \frac{\lambda}{\varepsilon}, \tag{A3}$$

with $\mathrm{tr}(T^aT^b) = T_R\delta^{ab}$ for a given fermion color representation with generators T^a ($T_R = 1/2$ for the fundamental representation, whereas $T_R = 1$ in QED). The renormalized coupling $\alpha_r = \mu^{-\varepsilon}Z_A g^2(4\pi)^{-1+\varepsilon/2}$ (g is the bare coupling) satisfies

$$\mu\frac{d\lambda}{d\mu} = -\varepsilon\lambda Z_A = (\lambda-\varepsilon)\lambda. \tag{A4}$$

Equation (A1) includes all fermion bubbles and therefore resums the entire λ series. At the order we are working the relation between the bare and renormalized vector fields is $A_\mu = \sqrt{Z_A}A_\mu^r$, whereas the gauge parameter is renormalized according to $\xi = Z_A\xi^r$.

All $1/N_f$ diagrams are found to be of the form

$$C = \sum_{n=1}^{\infty}\frac{1}{n}\left(-\frac{\lambda}{\varepsilon Z_A}\right)^n G(\varepsilon, n\varepsilon), \tag{A5}$$

with $G(\varepsilon, n\varepsilon) = \sum_{k=0}^{\infty}G_k(\varepsilon)(n\varepsilon)^k$. This key relation, which first appeared in the calculation of the anomalous dimension of the fermion mass operator [9], allows us to systematically analyze diagrams at the leading $1/N_f$ order.

Once our diagrams are put in the standard form (A5), the extraction of the anomalous dimension in a minimal subtraction scheme becomes a trivial task. Indeed, expanding in powers of the renormalized coupling (recall (A3)) using the negative binomial series we find that the divergent part in perturbation theory is given by

$$\begin{aligned}\mathrm{div}(C) &= -\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{div}\left[\left(\frac{\lambda}{\varepsilon}\right)^n G_0(\varepsilon)\right]\\ &= -\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\frac{\lambda^n}{n}\frac{[G_0]_m}{\varepsilon^{n-m}},\end{aligned} \tag{A6}$$

where we wrote $G_0(\varepsilon) = \sum_{m=0}^{\infty}[G_0]_m\varepsilon^m$ as yet another formal series. Employing (A4) and rearranging the sum, we arrive at a very useful compact expression [6,9,15]:

$$\begin{aligned}\mu\frac{d}{d\mu}\mathrm{div}(C) &= \sum_{n=1}^{\infty}\sum_{m=0}^{n-1}[G_0]_m\left[\frac{\lambda^n}{\varepsilon^{n-m-1}} - \frac{\lambda^{n+1}}{\varepsilon^{n-m}}\right]\\ &= \sum_{m=0}^{\infty}[G_0]_m\sum_{n=m+1}^{\infty}\left[\frac{\lambda^n}{\varepsilon^{n-m-1}} - \frac{\lambda^{n+1}}{\varepsilon^{n-m}}\right]\\ &= \sum_{m=0}^{\infty}[G_0]_m\lambda^{m+1} = \lambda G_0(\lambda).\end{aligned} \tag{A7}$$

Equation (A7) is a manifestation of the fact that it is the coefficient of $1/\varepsilon$ in the correlator C that contains information on the RG evolution [24].

[4] This is why [23] finds, not surprisingly, small anomalous dimensions in this region.

As a final comment, we observe that Eqs. (A5) and (A7) show that the dependence on ε of the regulated Feynman diagrams secretly encodes the perturbative series in λ. We may say that the very existence of a reliable perturbative expansion in λ justifies our series in ε. Physically, the relation between the two expansions stems from the fact that in $d = 4 - \varepsilon$ dimensions, large N_f QCD has a nontrivial IR fixed point at $\lambda^* = \varepsilon + \mathcal{O}(1/N_f)$, see (A4). Because anomalous dimensions of gauge-invariant operators are functions solely of the renormalized coupling, the critical exponents $\gamma(\varepsilon)$ in $d = 4 - \varepsilon$ dimensions can be mapped onto $\gamma(\lambda)$, up to subleading $1/N_f$ corrections. This approach has been applied to many-flavor QCD in [25].

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