STOCHASTIC DE SITTER (INFLATIONARY) STAGE IN THE EARLY UNIVERSE

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Abstract

The dynamics of a large-scale quasi-homogeneous scalar field producing the de Sitter (inflationary) stage in the early universe is strongly affected by small-scale quantum fluctuations of the same scalar field and, in this way, becomes stochastic. The evolution of the corresponding large-scale space-time metric follows that of the scalar field and is stochastic also. The Fokker-Planck equation for the evolution of the large-scale scalar field is obtained and solved for an arbitrary scalar field potential. The average duration of the de-Sitter stage in the new inflationary scenario is calculated (only partial results on this problem were known earlier). Applications of the developed formalism to the chaotic inflationary scenario and to quantum inflation are considered. In these cases, the main unsolved problem lies in initial pre-inflationary conditions.

1. Introduction

In the models of the early universe with an initial or intermediate metastable de Sitter (inflationary) stage with an effective cosmological constant produced both by quantum gravitational corrections to the Einstein equations |1| and by a scalar field |2-4|, of extreme importance is the exit from this stage that depends on the way of decay of the effective cosmological constant because it determines the spectrum and amplitude of metric perturbations for the subsequent evolution. These perturbations break the homogeneity and isotropy achieved earlier at the inflationary stage and can, in the worst case, destroy all the advantages of inflation. Two ways of decay of the effective cosmological constant are possible : via (quasi) homogeneous classical instability and via inhomogeneous quantum fluctuations. In the first case, the amplitude of perturbations of the de Sitter space-time in the modes which preserve (exactly or approximately) the isotropy and homogeneity of the 3-space in some frame of reference is much more than the amplitude of other, inhomogeneous perturbations. Thus, we have a classical (quasi)-homogeneous perturbation from the very beginning and the subsequent evolution is deterministic ; the duration of the de Sitter stage is totally determined by the initial amplitude of this perturbation. This type of decay takes place, for example, in the author's model |1| for the case of the closed 3-space section if the spatial dimension of this section was of the order of H^{-1} at the beginning of the de Sitter stage (in the paper, we put $\hbar = c = 1$; $a(t)$ is the scale factor of the Friedmann-Robertson-Walker isotropic cosmological model ; $H = \dot{a}/a$).

The existence of a quasi-homogeneous classical scalar field is also assumed in the "chaotic" inflationary scenario |5| (for the inclusion of the R^2 term where R is the Ricci scalar, see |6|). Here, the term "chaotic" simply means the unspecified dependence of the metric and the scalar field on space coordinates though this dependence is weak enough, so that the spatial derivatives of all variables are much less than the temporal ones.

In the second case, we have no large (quasi) homogeneous perturbation at the beginning of the de Sitter stage. This possibility was first pointed in |7| in connection with the model |1|. But, in fact, this situation is more typical for the models where the de Sitter stage arises from the initially radiation-dominated, "hot" universe in the course of a non-equilibrium, close to the II order phase transition (for example, the "new" inflationary scenario). Here, nevertheless, a large quasi-homogeneous "classical" perturbation with characteristic wavelengths $\gg H^{-1}$ can arise during the de Sitter stage from small-scale quantum perturbations. In other words, "classical order" appears from "quantum chaos". In spite of being effectively classical, the evolution of this large-scale perturbation and the space-time metric as a whole is essentially stochastic. The duration of the de Sitter stage also becomes a stochastic quantity in this case.

This is just the process we are interested in. It belongs to the class of the so-called "synenergetic" problems which arise in different branches of science and attract much interest at the present time. We shall consider the new inflationary scenario where the role of the abovementioned perturbation is played by the non-zero large-scale scalar field Φ . It is assumed that $\Phi = 0$ (or sufficiently small) at the beginning of the de Sitter stage. We shall obtain the Fokker-Planck equation for the evolution of the probability distribution of Φ (Sec.2) and calculate the average duration of the de Sitter stage in the new inflationary scenario in Sec.3 (only partial results on this problem or order-of-magnitude estimates were obtained earlier |8-10|). After that, we shall turn to the chaotic inflationary scenario (Sec.4) and discuss the modern state of the problem of the "creation" of the universe briefly (Sec.5).

2. Evolution of a scalar field in the new inflationary scenario.

The de Sitter stage in the new inflationary scenario is assumed to be produced by the vacuum energy of some scalar field with the Lagrangian density

$$L = \frac{1}{2} \Phi_{,i} \Phi^{,k} - V(\Phi) \tag{1}$$

where the vacuum effective potential $V(\Phi)$ has the following properties :

$$\begin{aligned} &V(\Phi_0) = 0 \\ &V(\Phi)_{\Phi \to 0} = V_0 + \frac{1}{2} M^2 \Phi^2 - \frac{1}{3} \nu \Phi^3 - \frac{1}{4} \lambda \Phi^4 , \end{aligned} \tag{2}$$

M^2 can have both signes. $\Phi = \Phi_0$ is the flat space-time (true vacuum). $\Phi = 0$ is the false vacuum. We include the term in Φ^3 to describe the case of the so-called "primordial" inflation |11| simultaneously. At the non-zero temperature T, the potential V acquires the additional thermal term which is either small or, with the sufficient accuracy, has the form $\frac{1}{2} B T^2 \Phi^2$, $B \ll 1$, $T \propto a^{-1}$.
At the de Sitter stage, $H = H_0 = \text{const}$, $a = a_0 \exp(H_0 t)$, where $H_0^2 = 8\pi G V_0/3$ (the spatial curvature is negligable). In order to have enough long de Sitter stage and enough small perturbations at the subsequent stages, the following conditions should be fulfilled :

$$|M^2| \lesssim H_0^2/20 \quad ; \quad \nu/H_0 \lesssim 10^{-6} \quad ; \quad \lambda \lesssim 10^{-12} . \tag{3}$$

The Coleman-Weinberg potential does not evidently meet these requirements, so it is usually assumed now that Φ is some weakly interacting scalar field, in particular, it should be the singlet with respect to SU(5) or any other grand unification group. In such a way, the spirit, though not the letter, of the "new" inflationary scenario is maintained.
The de Sitter stage begins when $T^4 \sim V_0$. It can be divided into two successive periods: "hot" and "cold" (vacuum). During the hot period, the temperature $T \gg H_0$ and quantum-gravitational effects caused by the space-time curvature are unimportant. The duration of this period is rather short ; in dimensionless units,

$$H_0 \Delta t_h \sim \ln (V_0^{1/4}/H_0) \sim \ln (G^{-1/2} V_0^{-1/4}) , \tag{4}$$

that is of the order of 10 typically. After that, the cold (vacuum) period begins where $T \ll H_0$ and, in fact, temperature effects can be neglected (except only for the calculation of the initial dispersion of Φ ; see Eq.(13) below). This period is the most interesting because quantum-gravitational effects connected with the space-time curvature play the decisive role here (we denote its beginning by t_0).

To obtain quantitatively (not only qualitatively) correct results one should not use such quantities as $<\Phi>$ or $<\Phi^2>$ (the approaches based on these quantities have been correctly criticized in [12, 13]). Instead of this, we represent the quantum scalar field Φ (the Heisenberg operator) in the form :

$$\Phi = \bar{\Phi}(t,\vec{r}) + \frac{1}{(2\pi)^{3/2}} \int d^3k.\ \Theta(k - \varepsilon a(t) H_0) \cdot$$

$$\cdot [\hat{a}_k \varphi_k(t) \exp(-i\vec{k}\vec{r}) + \hat{a}_k^+ \varphi_k^*(t) \exp(i\vec{k}\vec{r})] + \delta\varphi. \quad (5)$$

$$k = |\vec{k}| \quad , \quad \Theta(z) = \begin{cases} 1 \ , & z > 0 \\ 0 \ , & z < 0 \end{cases} \qquad \varepsilon = \text{const.} \ll 1 .$$

Here, $\bar{\Phi}(t, \vec{r})$ contains only long wavelength modes with $k \ll H_0 a(t)$, $\delta\varphi$ is the small correction that can be neglected in the leading order in small parameters $|M^2|/H_0^2$, ν/H_0, λ and the second integral term in Eq. (5) satisfies the free massless scalar wave equation in the de Sitter background : $\Box\varphi = 0$. Thus,

$$\varphi_k = H_0 (2k)^{-1/2} \left(\eta - \frac{i}{k}\right) \exp(-ik\eta) \ ; \ \eta = \int \frac{dt}{a(t)} = -[a(t)H_0]^{-1} \quad (6)$$

and $\hat{a}_k^+$ and $\hat{a}_k$ are the usual creation and annihilation Bose-operators. The auxiliary small parameter ε is introduced to refine the derivation, it will not appear in all final equations. In fact, it cannot be arbitrarily small ; the immediate comparison of different terms in Eq. (5) suggests that $\varepsilon \gg |M|/H_0$ but more refined treatment consisting in the substitution of the solution (6) by the solution of the free massive wave equation $\Box\varphi + M^2\varphi = 0$ in the de Sitter background (that does not change Eq. (8) below in the leading approximation in $|M^2|/H_0^2$) shows that the significantly weaker condition $|\ln\varepsilon| \ll \max(H_0^2/M^2, H_0/\nu, \lambda^{-1})$ is sufficient. It can be also seen immediately that the account of the abovementioned thermal correction to $V(\Phi)$ results in the substitution

$$k \rightarrow (k^2 + k_0^2)^{1/2} \quad , \qquad k_0^2 = BT^2a^2 = \text{const.} \quad (7)$$

in Eq. (6). This gives an effective infrared cut-off that can be important in some problems.

The scalar field Φ satisfies the operator equation of motion $\Box\Phi + dV/d\Phi = 0$

exactly. Using (5, 6) and the conditions of "slow rolling" (3), one obtains the following equation for $\bar{\Phi}$ in the leading order :

$$\dot{\bar{\Phi}}(t,\vec{r}) = -\frac{1}{3H_0}\frac{dV(\bar{\Phi})}{d\bar{\Phi}} + f(t,\vec{r}) ;$$

$$f(t,\vec{r}) = \frac{\varepsilon a H_0^2}{(2\pi)^{3/2}}\int d^3k .\ \delta(k-\varepsilon a H_0)\cdot\frac{(-i)H_0}{\sqrt{2}\,k^{3/2}}\left[\hat{a}_k \exp(-i\vec{k}\vec{r}) - \hat{a}_k^{+}\exp(i\vec{k}\vec{r})\right]. \tag{8}$$

That is the main point : the large-scale scalar field $\bar{\Phi}$ changes not only due to the classical force d $V(\bar{\Phi})/$d $\bar{\Phi}$ but also due to the flow of initially small-scale quantum fluctuations across the de Sitter horizon k = a(t)H_0 in the process of expansion. Moreover, the evolution of inhomogeneous fluctuations is linear inside the de Sitter horizon and even in some region outside it ; on the other hand, the evolution of $\bar{\Phi}$ is non-linear but here the spatial and second time derivatives of $\bar{\Phi}$ are small. Below, we shall omit the bar above $\bar{\Phi}$, so Φ will mean the large-scale field only. Two important consequences follow from Eq. (8). Firstly, there are no spatial derivatives in Eq (8) at all . This means that the evolution of Φ can be studied locally, in the "point" (this "point" has, in fact, spatial dimension $\sim H_0^{-1}$). The temporal evolution of Φ is slow as compared to H_0^{-1} (if the inflation exists at all), so our time "differential" dt can be also chosen $\sim H_0^{-1}$; only the processes with characteristic times $\tau \gg H_0^{-1}$ will be considered. Secondly, though Φ and f have a complicated operator structure, it can be immediately seen that all terms in Eq. (8) commute with each other because $\hat{a}_k$ and $\hat{a}_k^+$ appear only in one combination for each possible $\vec{k}$! Thus, we can consider Φ and f as classical, c-number quantities. But they are certainly stochastic, simply because we can not ascribe any definite numerical value to the combination $[\hat{a}_k \exp(-i\vec{k}\vec{r}) - \hat{a}_k^{+}\exp(i\vec{k}\vec{r})]$. As a result, the peculiar properties of the de Sitter space-time — the existence of the horizon and the appearance of the large "friction" term $3H_0\dot{\Phi}$ in the wave equation — simplify the problem of a non-equilibrium phase transition greatly and make its solution possible, in contrast to the case of the flat space-time.

It is clear now that Eq. (8) can be considered as the Langevin equation for Φ(t) with the stochastic force f(t). The calculation of the correlation function for f(t) is straighforward and gives ($\vec{r}$ is the same throughout) :

$$\langle f(t_1)\, f(t_2)\rangle = H_0^3\,(4\pi^2)^{-1}\,\delta(t_1 - t_2) \tag{9}$$

Thus f(t) has the properties of white noise. This appears to be the case because different moments of time correspond to different k because of the δ -function in the definition of f, and $\hat{a}_k$ and $\hat{a}_k^+$ with different $\vec{k}$ commute. For spatially separated points,

$$\langle f(t_1, \vec{r}_1)\, f(t_2, \vec{r}_2) \rangle = H_0^3 (4\pi^2)^{-1} \delta(t_1 - t_2) \cdot \frac{\sin\left(\varepsilon a H_0 |\vec{r}_1 - \vec{r}_2|\right)}{\varepsilon a H_0 |\vec{r}_1 - \vec{r}_2|} \quad (10)$$

We are interested in the average values $\langle F(\hat{\Phi})\rangle$ where F is an arbitrary function. For that case, one can introduce the normalized probability distribution $\rho(\Phi, t)$ for the classical stochastic quantity Φ

$$\left(\int_{-\infty}^{\infty} \rho(\Phi, t)\, d\Phi = 1 \right) \quad \text{, so that}$$

$$\langle F(\hat{\Phi}) \rangle = \int_{-\infty}^{\infty} \rho(\Phi, t)\, F(\Phi)\, d\Phi \quad (11)$$

By the standard procedure, the Fokker-Planck (or, better to say, Einstein-Smoluchowski) equation for ρ follows from (8) and (9) :

$$\frac{\partial \rho}{\partial t} = \frac{H_0^3}{8\pi^2} \frac{\partial^2 \rho}{\partial \Phi^2} + \frac{1}{3H_0} \frac{\partial}{\partial \Phi} \left(\frac{dV}{d\Phi} \rho \right). \quad (12)$$

This equation has to be supplemented by some initial condition for ρ at $t = t_0$. It should be noted also that Eq. (12) is applicable at the stage of "slow rolling" ($|\dot{\Phi}| \ll H_0 \Phi$) only. When this condition ceases to be valid (that takes place at $\Phi \sim \min(H_0 \lambda^{-1/2}, H_0^2 \nu^{-1})$), the second time derivative of Φ comes into play (though spatial derivatives are still unimportant), the de Sitter stage ends and Φ reaches its flat space-time equilibrium value Φ_0 during the time interval less than H_0^{-1}. After that, a number of oscillations around Φ_0 is possible. Thus, strictly speaking, we can use Eq. (12) if only $|\Phi| \ll \min(H_0 \lambda^{-1/2}, H_0^2 \nu^{-1})$. But just because we are not interested in time intervals $\Delta t \sim H_0^{-1}$ when calculating such quantities, as e.g., the average duration of the de Sitter stage, we can safely substitute $V(\Phi)$ in Eqs. (8,12) by its expansion for $|\Phi| \ll \Phi_0$ (the second line in Eq. (2)) and use Eqs. (8, 12) for arbitrary Φ. Then the (stochastic) moment of time t_s when the de Sitter stage ends coincides with the sufficient accuracy ($\Delta t \sim H_0^{-1}$) with the moment when $|\Phi|$ reaches infinity according to Eq. (8) (the stochastic force f(t) becomes unimportant at the last stage of evolution).

A note should be added about time reversibility. The microscopic evolution of the total scalar field operator $\hat{\Phi}$ is, certainly, time-reversible, so the apparent, diffusion-like irreversibility of the evolution of $\rho(\Phi, t)$ is due to, as usually, "coarse-graining" that takes place continuously in the process of neglecting more and more information contained in separate modes with different $\vec{k}$.

3. Average duration of the de Sitter stage in the new inflationary scenario.

Now we have to introduce the initial condition for ρ at the beginning of the "cold" part of the inflation : $\rho = \rho_0(\Phi)$ at $t = t_0$. The simplest possible choice would be $\rho_0(\Phi) = \delta(\Phi)$. In fact, the situation is more complicated and depends on the initial conditions at the Planckian moment $t_p = G^{\frac{1}{2}}$. If one assumes thermal equilibrium before the de Sitter stage, then the contribution of thermal quanta of the scalar field Φ with the rest mass $m^2(T) \ll T^2$ $(B \ll 1)$ to $\rho_0(\Phi)$ is gaussian with the dispersion

$$\langle \Phi_T^2 \rangle = \frac{1}{\pi^2} \int_0^\infty k^2 dk\, |\varphi_k|^2 \left[\exp\left(\sqrt{k^2+k_0^2}/aT\right) - 1\right]^{-1} \quad (13)$$

At $T \ll H_0$, the main contribution to the integral is due to the region $k \sim k_0 \ll aT$, $|k\eta| \ll 1$. Using (6,7), we obtain [10,14] :

$$\langle \Phi_T^2 \rangle = \frac{H_0^2 a T}{2\pi^2} \int_0^\infty \frac{k^2 dk}{(k^2+k_0^2)^2} = H_0^2 / 8\pi\sqrt{B} \gg H_0^2 \quad (14)$$

This expression is valid if the modes with $k \sim k_0$ are inside the horizon at the beginning of the de Sitter stage that requires $B \gg G V_0^{\frac{1}{2}} \sim H_0/M_p$, where $M_p = G^{-\frac{1}{2}}$ is the Planck mass. In the opposite case, the modes with $k < k_1 = H_0\, a(t = H_0^{-1})$ are never inside the horizon. For these modes, $\varphi_k \simeq$ const. and, in fact, nothing definite can be said about their occupation numbers. The probability distribution needs not be gaussian either, but it is independent of time (we do not include the term $R\Phi^2/12$ into the Lagrangian (1) because then the fine-tuning between M^2 and H_0^2 is necessary for the inflation to occur). In this case, the reasonable lower limit on the initial dispersion can be obtained by integrating from k_1 to ∞ in Eqs. (13,14) that gives

$$\langle \Phi^2(t = t_0) \rangle \gtrsim H_0 V_0^{1/4} \sim H_0^{3/2} M_p^{1/2} \gg H_0^2 \quad , (15)$$

if thermal equilibrium is assumed in the whole region inside the horizon at the beginning of the de Sitter stage.

Thus, the initial dispersion of Φ , in general, exceeds H_0^2 significantly. Nevertheless, it appears (see below) that if

$$\langle \Phi^2(t = t_0) \rangle \ll \min.\left(H_0^2/\lambda^{1/2} ,\ H_0^{8/3}/\nu^{2/3} \right), \quad (16)$$

then the initial dispersion can be neglected because its effect on the average duration of the de Sitter stage proves to be small. Therefore, there exists a set of possible (though not necessary) initial conditions at $t = t_p$ for which we can use the initial condition $\rho_0(\Phi) = \delta(\Phi)$ at $t = t_0$.

Note that, if the last term in Eq.(12) can be neglected (that takes place in the bening of the "cold" period of inflation), then Eq. (12)is the usual diffusion equation. Thus, the initially gaussian distribution $\rho(\Phi)$ remains gaussian in the course of time evolution and its dispersion changes as

$$\langle \Phi^2 \rangle = \langle \Phi^2 (t=t_0) \rangle + \frac{H_0^3}{4\pi^2} (t-t_0) \ . \tag{17}$$

This is just the result obtained in [9,10,15]. In the presence of the quadratic potential $V = M^2\Phi^2/2$, the distribution remains gaussian and the dispersion can be obtained from the "one-loop" equation |10|

$$\frac{d}{dt} \langle \Phi^2 \rangle = - \frac{2M^2}{3H_0} \langle \Phi^2 \rangle + \frac{H_0^3}{4\pi^2} \ . \tag{18}$$

In this case, Eq.(20) below reduces to that of the harmonic oscillator and can be solved analytically.
In the general case, the solution of Eq.(12) is :

$$\rho(\Phi, t) = \exp\left(-\frac{4\pi^2 V(\Phi)}{3H_0^4}\right) \sum_n c_n \Psi_n(\Phi) \exp\left(-E_n \frac{H_0^3}{4\pi^2}(t-t_0)\right), \tag{19}$$

where $\Psi_n(\Phi)$is the complete orthonormal set of eigenfunctions of the Schrodinger equation

$$\frac{1}{2}\frac{d^2\Psi_n}{d\Phi^2} + (E_n - W(\Phi))\Psi_n = 0 \ ;$$

$$|\Psi_n(\pm\infty)| = 0 \ ; \tag{20}$$

$$W(\Phi) = \frac{8\pi^4}{9H_0^8}\left(\frac{dV}{d\Phi}\right)^2 - \frac{2\pi^2}{3H_0^4}\frac{d^2V}{d\Phi^2} = \frac{1}{2}(v'^2 - v'') \ ;$$

$$v(\Phi) = 4\pi^2 V(\Phi)/3H_0^4 \ .$$

It was explained at the end of Sec.2 that we may set $V(\infty) = -|V(-\infty)| = -\infty$. Therefore, $W(\pm\infty) = \infty$ and Eq.(20) has the discrete spectrum of eigenvalues only. For $V(\Phi)$ given in Eq.(2), it is the equation of the anharmonic (or doubly anharmonic) oscillator. The coefficients c_n are obtained from the initial condition for $\rho(\Phi, t)$ at $t = t_0$:

$$c_n = \int_{-\infty}^{\infty} d\Phi \, \rho_0(\Phi) \exp(v(\Phi)) \Psi_n(\Phi) \ . \tag{21}$$

The behaviour of $\rho(\Phi,t)$ at large times is, as usually, determined by the lowest energy level E_0. E_0 is strictly positive that follows from the "supersymmetric" form of the potential $W(\Phi)$.

In practice, we are more interested not in $\rho(\phi,t)$ itself but in $w(t_s)$ - the probability distribution for the stochastic moment t_s when the de Sitter stage ends. $w(t_s)$ can be obtained from $\rho(\phi,t)$ by the following way. Let the rolling of the scalar field to both sides is possible : $V(\pm\infty) = -\infty$. The integral $\int d\phi\left(\frac{dV}{d\phi}\right)^{-1}$ converges at $|\phi|\to\infty$ that means that $|\phi|$ approaches infinity in finite time. For $|\phi|\to\infty$, the evolution of Φ becomes deterministic ; both the stochastic force in Eq.(8) and the second derivative with respect to Φ in Eq.(12) can be neglected. Then the solution of Eq.(12) for $\Phi\to\pm\infty$ is, correspondingly,

$$\rho = \left(\frac{dV}{d\Phi}\right)^{-1} g\left(t + 3H_0\int_{\pm\infty}^{\Phi} d\Phi\left(\frac{dV}{d\Phi}\right)^{-1}\right), \quad (22)$$

where g is some unknown function that has to be determined from the previous evolution. The form of the solution represents the fact that the probability is transported without changing along the classical paths

$$t + 3H_0\int_{\pm\infty}^{\Phi} d\Phi\left(\frac{dV}{d\Phi}\right)^{-1} = \mathrm{const.} = t_s \ . \quad (23)$$

Therefore, one can introduce $w(t_s)\propto g(t_s)$. The exact coefficient of proportionality is determined by the condition of probability conservation

$$w(t_s) = \rho(\Phi,t)\ \left|\left(\partial\Phi/\partial t_s\right)_t\right| , \quad (24)$$

along the path (23). If we do not make difference between rolling down to the left and to the right sides, then the resulting expression for $w(t_s)$ is

$$w(t_s) = \frac{1}{3H_0}\left(\lim_{\Phi\to+\infty} + \lim_{\Phi\to-\infty}\right)\left|\frac{dV(\Phi)}{d\Phi}\right|\rho(\Phi,t_s). \quad (25)$$

If the rolling of the scalar field is possible to the right side only ($V(-\infty) = \infty$, $V(\infty) = -\infty$; e.g., when $\lambda = 0$ in Eq.(2)), the second limit in Eq.(25) has to be omitted. The distribution $w(t_s)$ is certainly non-gaussian. Its behaviour for large t_s is exponential and is determined by the lowest energy level E_0. Though $w(t_s)$ cannot be computed analitically, it is remarkable that the closed explicit expressions for all moments $\langle(H_0(t_s-t_0))^n\rangle$ with integer n can be obtained in the form of successive integrals. The approach used here is similar to the Stratonovich's "first time passage" method.

Let us consider a set of the functions

$$Q_n(\Phi) = \int_{t_0}^{\infty} (t-t_0)^n \, \rho(\Phi,t) \, dt \; ; \quad n = 0,1,2\ldots \tag{26}$$

Then

$$\frac{1}{3H_0}\left(\lim_{\Phi\to+\infty} + \lim_{\Phi\to-\infty}\right)\left|\frac{dV}{d\Phi}\right| Q_n(\Phi) = \int_{t_0}^{\infty}(t-t_0)^n w(t)\,dt = \langle (t_s-t_0)^n \rangle \tag{27}$$

Integrating both sides of Eq.(12) over t from $t = t_0$ to $t = \infty$, we obtain the ordinary differential equation

$$\frac{H_0^3}{8\pi^2} Q_0'' + \frac{1}{3H_0}\left(\frac{dV}{d\Phi} Q_0\right)' = -\rho_0(\Phi) . \tag{28}$$

Its solution, subjected to the boundary conditions $Q_0(\pm\infty) = 0$ (because $\rho(\pm\infty,t)=0$), is

$$Q_0 = \frac{8\pi^2}{H_0^3} e^{-2v(\Phi)} \int_{\Phi}^{\infty} e^{2v(\Phi_1)} d\Phi_1 \left(\int_{-\infty}^{\Phi_1} \rho_0(\Phi_2)\, d\Phi_2 - C \right) ;$$

$$C = \frac{\int_{-\infty}^{\infty} e^{2v(\Phi)} d\Phi \int_{-\infty}^{\Phi} \rho_0(\Phi_1)\, d\Phi_1}{\int_{-\infty}^{\infty} e^{2v(\Phi)}\, d\Phi} = \text{const.} ; \quad 0 < C < 1 . \tag{29}$$

If the rolling is possible to the right (left) side only, then C=0 (C=1). For the symmetric case $V(-\Phi) = V(\Phi)$ and $\rho_0(-\Phi) = \rho_0(\Phi)$, $C = \frac{1}{2}$. Now,

$$\frac{1}{3H_0}\left(\lim_{\Phi\to+\infty} + \lim_{\Phi\to-\infty}\right)\left|\frac{dV}{d\Phi}\right| Q_0(\Phi) = (1-C) + C = 1 = \int_{t_0}^{\infty} w(t)\,dt \tag{30}$$

Thus, the probability $w(t_s)$ introduced according to Eq.(25) is properly normalized. By multiplying both sides of Eq.(12) by $(t-t_0)^n$ and integrating over t from t_0 to $t = \infty$, the recurrence relation between Q_n can be found. It has the form $(n \geq 1)$:

$$\frac{H_0^3}{8\pi^2} Q_n'' + \frac{1}{3H_0}\left(\frac{dV}{d\Phi} Q_n\right)' = -n\, Q_{n-1} . \tag{31}$$

The boundary conditions are $Q_n(\pm\infty) = 0$ for all n. Then

$$Q_n = \frac{8\pi^2 n}{H_0^3} e^{-2v(\Phi)} \int_{\Phi}^{\infty} e^{2v(\Phi_1)} d\Phi_1 \left(\int_{-\infty}^{\Phi_1} Q_{n-1}(\Phi_2)\, d\Phi_2 - C_n \right) ;$$

$$C_m = \frac{\int_{-\infty}^{\infty} e^{2v(\Phi)}\, d\Phi \int_{-\infty}^{\Phi} Q_{m-1}(\Phi_1)\, d\Phi_1}{\int_{-\infty}^{\infty} e^{2v(\Phi)}\, d\Phi} = \text{const.} \tag{32}$$

Using Eq.(27), we obtain

$$\langle H_0^n (t_s - t_0)^n \rangle = n H_0^n \int_{-\infty}^{\infty} Q_{n-1}(\Phi)\, d\Phi \tag{33}$$

In particular, the average dimensionless duration of the de Sitter stage is equal to

$$\langle H_0 \Delta t \rangle = H_0 \Delta t_h + \langle H_0 (t_s - t_0) \rangle = H_0 \Delta t_h + H_0 \int_{-\infty}^{\infty} Q_0(\Phi)\, d\Phi \,, \tag{34}$$

where Δt_h is given in Eq.(4) and Q_0 is presented in Eq.(29).
Let us now consider several particular cases. Let $\nu = 0$ in Eq.(2) (that corresponds to the original picture of the "new" inflation) and $\rho_0(\Phi) = \delta(\Phi)$. Then Eq.(34) simplifies ($C = \frac{1}{2}$) :

$$\langle H_0 \Delta t \rangle = H_0 \Delta t_h + \frac{8\pi^2}{H_0^2} \int_0^{\infty} d\Phi\, e^{-2v(\Phi)} \int_{\Phi}^{\infty} e^{2v(\Phi_1)} d\Phi_1 \,; \tag{35}$$

$$v(\Phi) = \frac{4\pi^2 V(\Phi)}{3 H_0^4} = \frac{4\pi^2}{3 H_0^4} \left(\frac{M^2 \Phi^2}{2} - \frac{\lambda \Phi^4}{4} \right)$$

(the constant term in the potential may be omitted because it cancels in Eq.(35)). After some manipulation, the expression (35) can be represented in the form containing only one integration :

$$\langle H_0 \Delta t \rangle = H_0 \Delta t_h + \frac{\pi\sqrt{3}}{\sqrt{\lambda}} \left[\sqrt{\frac{\pi}{2}} \int_0^1 \frac{dx}{\sqrt{x(1-x^2)}} \exp\left(\frac{\alpha^2 x}{2}\right) + \right.$$

$$\left. + \alpha \int_0^1 \frac{dx}{\sqrt{1-x^2}}\, \Phi\left(1, \frac{3}{2}, \frac{\alpha^2 x}{2}\right) \right] , \quad \alpha = \frac{2\pi M^2}{\sqrt{3\lambda}\, H_0^2} , \tag{36}$$

where Φ is the confluent hypergeometric function.
Three more particular cases are of special interest.
1) $M^2 < 0$; $\lambda^{\frac{1}{2}} H_0^2 \ll |M^2| \ll H_0^2$; $|\alpha| \gg 1$.
Then

$$\langle H_0 \Delta t \rangle = H_0 \Delta t_h + \frac{3 H_0^2}{2|M^2|} \left(\ln \frac{16\pi^2 M^4}{3\lambda H_0^4} + \gamma \right) , \tag{37}$$

where $\gamma = 0.577$ ... is the Euler constant. In this case, one-loop approximation which consists in the substitution of $\langle\Phi^4\rangle$ by $3(\langle\Phi^2\rangle)^2$ in the equation for $\langle\Phi^2\rangle$ gives the result which is correct with the logarithmic accuracy :

$$\langle H_0 \Delta t \rangle_{\text{one-Loop}} = H_0 \Delta t_{\hbar} + \frac{3\ H^2}{2\,|M^2|} \ln \frac{8\pi^2 M^4}{9\lambda H_0^4} \cdot \tag{38}$$

However, more accurate approach was developed in |10| for this case which gave the right answer. It consists in the observation that in this case the stochastic force f(t) in Eq.(8) is important then and only then when the classical force $(-dV(\Phi)/d\Phi)$ can be neglected and vice versa. Thus, Eq.(8) can be integrated directly that gives the following result for the stochastic quantity t_s itself |10| :

$$H_0(t_s - t_0) = \frac{3H_0^2}{2\,|M^2|} \ln \frac{|M^2|}{\lambda \Phi_1^2} , \tag{39}$$

where Φ_1 is a gaussian stochastic quantity with zero average and the dispersion

$$\langle \Phi_1^2 \rangle = 3\,H_0^4 / 8\pi^2 |M^2| \tag{40}$$

(the thermal contribution to $\langle\Phi_1^2\rangle$ is neglected here for simplicity). After averaging $\ln \Phi_1$ in Eq.(39) over the gaussian distribution, just the correct result (37) appears.

2) $|M^2| \ll \lambda^{1/2} H_0^2$; $|\alpha| \ll 1$.

For this case, only one-loop |10| or order-of-magnitude |9| estimates were known earlier. It follows from Eq.(36) that

$$\langle H_0 \Delta t \rangle = H_0 \Delta t_{\hbar} + \frac{\pi\sqrt{3}}{4} \Gamma^2(1/4)\lambda^{-1/2} \simeq H_0 \Delta t_{\hbar} + 17.88\,\lambda^{-1/2} \tag{41}$$

One-loop approximation gives the numerical coefficient in the second term equal to $\pi^2/\sqrt{2} \simeq 6.98$ that is 2.56 times less.

It is intructive to consider the case of a many-component scalar field Φ_a with the symmetry group O(N) and see how the one-loop approximation becomes exact in the limit $N \to \infty$. Let $\Phi = (\Phi_a \Phi^a)^{1/2}$. The strightforward application of the developed approach shows that the corresponding generalization of Eq.(12) to the $N \neq 1$ case is :

$$\frac{\partial \rho}{\partial t} = \frac{H_0^3}{8\pi^2 \Phi^{N-1}} \frac{\partial}{\partial \Phi}\left(\Phi^{N-1} \frac{\partial \rho}{\partial \Phi}\right) + \frac{1}{3H_0 \Phi^{N-1}} \frac{\partial}{\partial \Phi}\left(\frac{dV}{d\Phi} \Phi^{N-1} \rho\right) \tag{42}$$

$$S_N \int_0^\infty \Phi^{N-1} \rho(\Phi, t)\, d\Phi = 1 \ ; \qquad S_N = N\pi^{N/2} / \Gamma(1 + \tfrac{N}{2}),$$

where S_N is the area of the N-dimensional sphere (O(N)-symmetrical initial condition for ρ is also assumed). If $\rho(\Phi,t) = \delta(\Phi)$ at $t = t_0$, then, instead of Eq.(35), the following expression for the average duration of the de Sitter stage results :

$$\langle H_0 \Delta t \rangle = H_0 \Delta t_h + \frac{8\pi^2}{H_0^2}\int_0^\infty d\Phi \, . \, \Phi^{N-1} e^{-2v(\Phi)} \int_\Phi^\infty d\Phi_1 \cdot \Phi_1^{1-N} e^{2v(\Phi_1)} \quad (43)$$

For $V(\Phi) = V_0 - \lambda \Phi^4 / 4N$,

$$\langle H_0 (t_s - t_0) \rangle = \frac{\pi^2 \sqrt{3}}{\sqrt{2\lambda}} \cdot \frac{\sqrt{N}\,\Gamma(N/4)}{2\,\Gamma(1/2 + N/4)} ;$$

$$\langle H_0 (t_s - t_0) \rangle_{one\text{-}Loop} = \frac{\pi^2 \sqrt{3}}{\sqrt{2\lambda}} \cdot \sqrt{\frac{N}{N+2}} . \quad (44)$$

Thus, both expressions tend to the same limit $\pi^2\sqrt{3}/\sqrt{2\lambda} \simeq 12.09\,\lambda^{-1/2}$ at $N \to \infty$ (but from different sides).

Now we return to the N = 1 case and calculate the dispersion of the quantity $H_0(t_s - t_0)$. By the use of Eqs.(32,33), we obtain

$$\langle H_0^2 (t_s - t_0)^2 \rangle = \frac{96\pi^2}{\lambda}\int_0^\infty e^{u^4/2} du \int_u^\infty e^{-x^4/2} dx \int_0^x e^{y^4/2} dy \int_y^\infty e^{-z^4/2} dz ;$$

$$\delta^2 = \langle H_0^2 (t_s - t_0)^2 \rangle - (\langle H_0 (t_s - t_0) \rangle)^2 = \quad (45)$$

$$= \frac{12\pi^2}{\lambda}\left(\frac{\Gamma^2(1/4)}{64} - \int_0^1 \frac{F^2(\arccos t, 1/\sqrt{2})}{1 - t^4} dt \right);$$

$\delta \simeq 0.6408 \langle H_0 (t_s - t_0) \rangle$,

where $F(\varphi,k)$ is the elliptic integral of the first kind. Also interesting is to calculate the change in the result (41) due to the spreading of the initial condition at $t = t_0$ (the "thermal" correction). If $\rho_0(\Phi)$ is the gaussian distribution with the zero average and the dispersion Φ_T^2 (see Eqs.(14,15), then by applying Eq.(29) with C = ½ the following result can be found :

$$\langle H_0 (t_s - t_0) \rangle = 17.88\,\lambda^{-1/2} - 4\pi^2 \Phi_T^2 H_0^{-2} ;$$

$$\sqrt{\lambda}\,\Phi_T^2 \ll H_0^2 . \quad (46)$$

Thus, if the condition (16) is satisfied, then the thermal correction is small ; in the opposite case, the inflationary stage is very short.

3) $\lambda^{1/2} H_0^2 \ll M^2 \ll H_0^2$; $\alpha \gg 1$.

In this case, the result (36) simplifies to the form :

$$\langle H_0 \Delta t \rangle = H_0 \Delta t_h + \frac{3\pi H_0^2}{\sqrt{2}\, M^2} \exp\left(\frac{2\pi^2 M^4}{3\lambda H_0^4}\right) . \quad (47)$$

The exponent just coincides with the result obtained by Hawking and Moss |8| with the help of the de Sitter instanton. Thus, our approach reproduces the instanton results without using instantons at all. Moreover, we have obtained a little more - the coefficient of the exponential, that corresponds to the summation of all one-loop diagrams on the instanton background in the standard functional integral approach. The corresponding probability distribution $w(t_s)$ is determined by the lowest energy level E_0 of Eq.(20) with the excellent accuracy and, thus, is purely exponential :

$$w(t_s) = \frac{E_0 H_0^3}{4\pi^2} \exp\left(-E_0 . \frac{H_0^3 (t_s - t_0)}{4\pi^2}\right) ;$$

$$\frac{E_0 H_0^2}{4\pi^2} = \frac{\sqrt{2} M^2}{3\pi H_0^2} \exp\left(-\frac{2\pi^2 M^4}{3\lambda H_0^4}\right) = \langle H_0 (t_s - t_0) \rangle^{-1} , \tag{48}$$

It is clear in our approach that the transition of the scalar field through the potential barrier takes place only locally, that is, in the volume $\sim H_0^{-3}$ (in fact, somewhat larger), but not in the whole 3-space. This fact can be also understood in the functional integral approach if one rewrites the de Sitter instanton in the static, "thermal" form :

$$dS^2 = (1 - H_0^2 r^2)\, d\tau^2 + (1 - H_0^2 r^2)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \tag{49}$$

$$\Phi = \Phi_{max} = M \lambda^{-1/2} ,$$

where τ is periodic with the period $2\pi H_0^{-1}$. Then the instanton tells us that Φ has reached the top of the potential barrier inside the horizon ($r < H_0^{-1}$) but gives us no information about the behaviour of Φ outside the horizon.

That is enough for the case of the "new" inflation. Now we shall turn to the so-called "primordial" inflation |11| where it is assumed that $\nu \neq 0$, $\lambda = 0$ and present the most interesting results briefly. In this case, the average duration of the de Sitter stage is given by Eqs.(34,29) with C = 0. Two limiting cases are the most important and representative.

1) $|M^2| \ll H_0^{4/3} \nu^{2/3}$.

Then

$$\langle H_0 (t_s - t_0) \rangle = 4 \left(\frac{\pi}{3}\right)^{2/3} \Gamma^2(1/3) \left(\frac{H_0}{\nu}\right)^{2/3} \simeq 29.60 \, (H_0/\nu)^{2/3} . \tag{50}$$

2) $H_0^{4/3} \nu^{2/3} \ll M^2 \ll H_0^2$.

In this case,

$$\langle H_0 (t_s - t_0) \rangle = \frac{6\pi H_0^2}{M^2} \exp\left(\frac{4\pi^2 M^6}{9 H_0^4 \nu^2}\right) \tag{51}$$

Again, the exponent is just the action for the Hawking-Moss instanton which is equal to the difference between the actions for the de Sitter instantons (49) with $\Phi = \Phi_{max} = M^2/\nu$ and $\Phi = \Phi_{min} = 0$. The third case $M^2<0$, $H_0^{4/3}\nu^{2/3} \ll |M^2| \ll H_0^2$ reduces, in fact, to the second one after shifting the scalar field : $\Phi = \Phi_1 - |M^2|/\nu$.
The quantitative results presented in the Sec. 2,3 were first published by the author in the shorter form in Russian in |16,17|. Two points should be emphasized, however.
Firstly, though the quantity $\ln(a(t_s)/a(t_0)) = H_0(t_s-t_0)$ has the well-defined probability distribution $w(t_s)$, the quantity $a(t_s)/a(t_0)$ does not, because $E_0H_0^2 \ll 1$ in all cases. Thus, it seems that the quantity $\ln a(t)$ is more suitable for the description of the stochastic inflation than the scale factor $a(t)$ itself.
Secondly, the calculated duration of the de Sitter stage gives us the typical size of causally connected regions. However, only a minor last part of this inflation produces regions those remain approximately homogeneous and isotropic in the course of subsequent evolution. This follows from the fact that after the inflation, the space-time metric at scales much larger than the cosmological post-inflationary particle horizon has the following simple structure in the proper ("ultra-synchronous") gauge. |10|

$$ds^2 = dt^2 - \exp(h(\vec{r}))\, a^2(t)(dx^2 + dy^2 + dz^2) \; ; \qquad (52)$$
$$h(\vec{r}) = 2\ln(a(t_s(\vec{r}))/a(t_0)),$$

where $h(\vec{r})$ is not assumed to be small and $a(t)$ is the scale factor for the strictly isotropic and homogeneous solution. The quantity $h(\vec{r})$ is essentially stochastic, its rms value is of the order of its average (see, e.g., Eq.(45)). Thus, the metric (52) becomes anisotropic and inhomogeneous in the course of the after-inflationary expansion when spatial gradients of $h(\vec{r})$ (omitted in Eq.(52) in the leading approximation) come into play. This situation illustrates the well-known fact that "general" inflation produces neither isotropy nor homogeneity of the present-day universe and, therefore, cannot "explain" them without further assumptions. Nevertheless, if the conditions (3) are fulfilled, then the last, "useful" part of inflation does produce sufficiently large regions with the degree of homogeneity and isotropy that matches the observations. It is important that during this part of inflation the stochastic force $f(t)$ in Eq.(8) becomes small as compared to the classical force $(-dV(\Phi)/(d\Phi)$. Then, for regions those are not too large, $h(\vec{r})$ can be represented in the form which was used in |10,18-21| :

$$h(\vec{r}) = \text{const} + \delta h(\vec{r}) \; ; \quad \delta h(\vec{r}) = -2H_0\, \delta\Phi(t,\vec{r})/\dot{\Phi} \; , \qquad (53)$$

where $\delta\Phi$ is the small fluctuation of $\Phi(t)$ produced by $f(t,\vec{r})$. Here $\delta h(\vec{r})$ is really small.
The duration Δt_1 of this "useful" part of inflation (when $|\delta h|<1$) is easily estimated using the expression for perturbations (53) :

$$
\begin{aligned}
H_0 \Delta t_1 &\sim \lambda^{-1/3} \quad ; \quad |M^2| \ll \lambda^{1/3} H_0^2 \ , \ \nu = 0 \ ; \\
&\sim \frac{H_0^2}{|M^2|} \ln \frac{|M|^6}{\lambda H_0^6} \quad ; \quad |M^2| \gg \lambda^{1/3} H_0^2 \ , \ \nu = 0 \ ; \\
&\sim \left(\frac{H_0}{\nu}\right)^{1/2} \quad ; \quad |M^2| \ll \nu^{1/2} H_0^{3/2} \ , \ \lambda = 0 \ ; \\
&\sim \frac{H_0^2}{|M^2|} \ln \frac{M^4}{\nu H_0^3} \quad ; \quad |M^2| \gg \nu^{1/2} H_0^{3/2} \ , \ \lambda = 0 \ .
\end{aligned}
\tag{54}
$$

$H_0 \Delta t_1$ contains no exponentially large multipliers. If λ or ν are fixed, then Δt_1 is maximal and the amplitude of perturbations at the given present-day scale is minimal when $|M^2| \ll \lambda^{1/3} H_0^2$ or $|M^2| \ll \nu^{1/2} H_0^{3/2}$; the upper limits on λ and ν presented in Eq.(3), strictly speaking, refer just to these cases. If M does not satisfy these conditions, the duration of the "useful" part of inflation diminishes ; however, the numerical restrictions on λ and ν remain practically unchanged due to the first condition in Eq.(3). It should be pointed also that the case $M^2>0$ presents no more advantages than the case $M^2<0$.

4. Evolution of the scalar field in the chaotic inflationary scenario.

In the chaotic inflationary scenario, it is assumed that the initial value of the quasi-homogeneous scalar field Φ is non-zero and, in fact, large ; typically, $|\Phi| > M_P$ at $t=t_P$. The potential $V(\Phi)$ can be a rather arbitrary function ; the only condition is that it should grow less faster than exp(const.$|\Phi|$) for $|\Phi| \to \infty$. Typical examples are $V(\Phi) = \lambda \Phi^4/4$ |5| and even $V(\Phi) = M^2 \Phi^2/2$ with $M^2>0$ (the dynamics of the latter model was studied in |22-26|). Here, the quantity $H = \dot{a}/a$ cannot be constant in general, but if $|\dot{H}| \ll H^2$, then the expansion of the universe is quasi-exponential. Thus, the notion of the quasi-de Sitter stage with the slow varying H arises. The scalar field should also change slowly during this stage : $|\dot{\Phi}| \ll H\Phi$. Then, $H^2 = 8\pi GV(\Phi)$.

We can now repeat the derivation of Eqs.(8,12) (Sec.2) for this case. Because of the dependence of H on t, the quantity $\ln a(t) = \int H(t)dt$ appears to be more proper and fundamental independent variable than the time t. Eq.(6) retains its form with the change : $H_0 \to H$. It is straightforward to obtain the following equation for the large-scale scalar field :

$$
\frac{\partial \Phi}{\partial \ln a} = -\frac{1}{3H^2} \frac{dV}{d\Phi} + \frac{f}{H} \ ; \tag{55}
$$

$$\langle f(\ln a)\, f(\ln a_1) \rangle = H^4 (4\pi^2)^{-1} \delta(\ln a - \ln a_1)$$

Then the corresponding Fokker-Planck equation takes the form (H^2 can be expressed through $V(\Phi)$) :

$$\frac{\partial \rho}{\partial \ln a} = \frac{G}{3\pi} \frac{\partial^2}{\partial \Phi^2} (V\rho) + \frac{1}{8\pi G} \frac{\partial}{\partial \Phi} \left(\frac{d \ln V}{d\Phi} \rho \right) . \qquad (56)$$

It is worthwhile to note that this equation has just the form one would expect to follow from quantum cosmology because it is no longer depends on such classical quantities as t or H, but contains only fundamental variables $\ln$ a and Φ which remain in quantum case.

Now, the problem of the initial condition for $\rho(\Phi, \ln a)$ arises. In the studies of classical chaotic inflation, it is usually assumed that $\Phi = \Phi_0$ at $t=t_p$ that corresponds to $\rho_0(\Phi) \propto \delta(\Phi - \Phi_0)$ for some $\ln a_0$. But such a condition contradicts the whole spirit of quantum cosmology. A natural idea is to consider stationary solutions (e.g., independent of $\ln a$) of Eq.(56). They can be thought of as being in "equilibrium with space-time foam" which may arise at planckian curvatures. At first, we introduce the notion of the probability flux $j(\Phi, \ln a)$ by rewriting Eq.(56) in the form

$$\frac{\partial \rho}{\partial \ln a} = - \frac{\partial \dot{j}}{\partial \Phi} ;$$

$$- \dot{j} = \frac{G}{3\pi} \frac{\partial}{\partial \Phi} (V\rho) + \frac{1}{8\pi G} \frac{d \ln V}{d\Phi} \rho . \qquad (57)$$

Then, two types of stationary solutions arise : with no flux and with a constant flux j_0 :

$$\rho = \text{const.}\, V^{-1} \exp(3/8\, G^2 V) - 3\pi j_0 (GV)^{-1} \exp(3/8\, G^2 V) \int^{\Phi} d\Phi_1 \exp(-3/8\, G^2 V(\Phi_1)). \qquad (58)$$

The first solution (with j = 0) is just the envelope of the Hartle-Hawking time-symmetric wave function |27| in the classically permitted region ($a^2 \gg (8\pi GV)^{-1}$) ; the exponent is the action for the de Sitter instanton with Φ = const (with the correct sign). Moreover, we have obtained the coefficient of the exponent, so the solution appears to be normalizable. It is easy to verify that the average value of Φ calculated with the use of this solution practically coincides with Φ_s —the value of Φ for which $|\dot{H}| \sim H^2$ and the de Sitter stage ends ($\Phi_s \sim M_p$ if $V(\Phi) = \lambda_n \Phi^n/n$). This does not mean that the dimension of the universe after inflation is small (because all $\ln a$ are equally probable for stationary solutions) but suggests that the "useful" part of inflation is typically very small (if exists at all) in this case. It is possible to obtain the "useful" part of inflation that is long enough, but with the very small probability $\sim \exp(-3/8G^2V(\Phi_s)) \sim \exp(-10^{10})$.

It is interesting that the second solution with $j \neq 0$ does not, in fact, contain any exponential at all. For $G^2V(\Phi) \ll 1$ that corresponds to curvatures much less than the planckian one, its form for $j_0<0$ is :

$$\rho_2 \simeq |j_0| \cdot 8\pi G V \Big/ \frac{dV}{d\Phi} \, . \tag{59}$$

In this case, the stochastic force is unimportant. Thus, we have only two possibilities: either the stationary solution contains the instanton contribution exp(-S) (where S is the action for the instanton, S<0) or the solution is non-exponential. We have not obtained the solution proportional to exp(S) = exp(-|S|) which was advocated by several authors (including the author of this paper) some time ago |28-30|. It seems that the latter solution describing the process of "quantum creation" of the universe via quantum tunneling to the de Sitter stage, though possible formally, has a very small probability also (with the same order of magnitude as above). This conclusion is similar to that obtained by Rubakov |31| though we suppose that his terminology of "catastrophic particle creation in the process of quantum tunneling" is inadequate ; in fact, no real particle creation takes place at the de Sitter instanton solution.

5. Conclusions and discussion.

We introduced and elaborated the approach consisting in taking into account the change in a large-scale scalar field due to the continuous flow of small-scale quantum perturbations of the same scalar field across the de Sitter horizon during the de Sitter (inflationary) stage. That gave us the possibility to find the explicit expressions for the average duration of the de Sitter stage (and for any higher moment if necessary) in the case when the initial probability distribution of the scalar field before the beginning of de Sitter stage was known. Certainly, the method used in the paper (as any other mathematical method) cannot solve the problem of initial pre-inflationary conditions ; new physical hypothesises (or "principles") are necessary for this purpose.

What can be said now about the possibility of "spontaneous quantum creation of the universe" which was so extensively discussed in |32-35| ? To make the terminology more precise, the author proposed some time ago |36| (see also |30|) to call the "quantum creation of the universe" the situation when we have a solution for the wave function of the universe with a non-zero probability flux emerging from the region of small values of a (or, equivalently, large values of space-time curvature). This proposal can be used in our stochastic approach also. Then the first stationary solution of Eq.(56) (the first term in Eq.(58)) corresponds to the time-symmetric universe which has no beginning and was not created. This coincides with the Hawking's interpretation of the Hartle-Hawking wave function in quantum cosmology. In the case of our first solution, we encounter the serious difficulty connected (as was explained in Sec. 4) with the very small probability of having the large duration of the "useful" part of inflation.

The second stationary solution with the non-zero probability flux does correspond to the "creation" of the universe but this creation has very little in common with the picture that was introduced in |32-35|. In particular, no quantum tunneling takes place, and the evolution of the metric and the scalar field remains classical up to the planckian curvatures. This type of creation was called the "classical creation" in |28| but it should be clear that the "classical creation" is not a new concept but simply the paraphrase of the standard classical picture of a singularity as a boundary of the space-time through which the space-time cannot be continued ; the only difference is that now this boundary is assumed to have a finite thickness .

The difficulties with the second solution are connected with our impossibility at the present time to prove the very existence of such a solution (in other words, to prove the possibility of the quantum change of topology) and to say something definite about the value of j_0, if it is non-zero. Thus, the problem of the possibility of the quantum creation of the universe remains open.

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