

## A geometric approach to non-perturbative quantum mechanics

CODESIDO SANCHEZ, Santiago

### Abstract

This work explores the connection between spectral theory and topological strings. A concrete example (the  $Y(3,0)$  geometry) of a conjectured exact relation between both based on mirror symmetry (TS/ST correspondence) is analysed in detail, to find a complete agreement. This is used as motivation to apply string theory tools, and in particular the refined holomorphic anomaly, to other Schrödinger-type spectral problems. With it, we efficiently compute their all-orders WKB expansion. We also upgrade the refined holomorphic anomaly to include non-perturbative corrections to the WKB series. This is used to retrieve the transseries generated by previously known exact quantization conditions for quantum mechanical problems. Via resurgence, it will allow us to reproduce the large order behaviour of the WKB coefficients, for both the Schrödinger problems and the quantum mirror curves of the TS/ST correspondence.

### Reference

CODESIDO SANCHEZ, Santiago. *A geometric approach to non-perturbative quantum mechanics*. Thèse de doctorat : Univ. Genève, 2018, no. Sc. 5167

DOI : [10.13097/archive-ouverte/unige:102512](https://doi.org/10.13097/archive-ouverte/unige:102512)

URN : [urn:nbn:ch:unige-1025125](https://nbn-resolving.org/urn:nbn:ch:unige-1025125)

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UNIVERSITÉ  
DE GENÈVE

# A geometric approach to non-perturbative quantum mechanics

THÈSE

présentée à la Faculté des Sciences de l'Université de Genève  
pour obtenir le grade de  
Docteur ès sciences, mention interdisciplinaire

par

Santiago Codesido Sánchez

de

Santiago de Compostela (Espagne)

Thèse N° 5167



**UNIVERSITÉ  
DE GENÈVE**

**FACULTÉ DES SCIENCES**

**DOCTORAT ÈS SCIENCES, MENTION INTERDISCIPLINAIRE**

**Thèse de Monsieur Santiago CODESIDO SANCHEZ**

intitulée :

**«A Geometric Approach to  
Non-perturbative Quantum Mechanics»**

La Faculté des sciences, sur le préavis de Monsieur M. MARIÑO BEIRAS, professeur ordinaire et directeur de thèse (Section de mathématiques), Monsieur J. SONNER, professeur associé (Département de physique théorique), Monsieur R. KASHAEV, professeur associé (Section de mathématiques), Monsieur W. LERCHE, professeur (Département de physique théorique, Organisation européenne pour la recherche nucléaire - CERN, Genève, Suisse), autorise l'impression de la présente thèse, sans exprimer d'opinion sur les propositions qui y sont énoncées.

Genève, le 19 janvier 2018

**Thèse - 5167 -**

**Le Doyen**

## Abstract

This work explores the connection between spectral theory and topological strings. A concrete example (the  $Y^{3,0}$  geometry) of a conjectured exact relation between both based on mirror symmetry (TS/ST correspondence) is analysed in detail, to find a complete agreement. This is used as motivation to apply string theory tools, and in particular the refined holomorphic anomaly, to other Schrödinger-type spectral problems. With it, we efficiently compute their all-orders WKB expansion. We also upgrade the refined holomorphic anomaly to include non-perturbative corrections to the WKB series. This is used to retrieve the transseries generated by previously known exact quantization conditions for quantum mechanical problems. Via resurgence, it will allow us to reproduce the large order behaviour of the WKB coefficients, for both the Schrödinger problems and the quantum mirror curves of the TS/ST correspondence.

## Acknowledgements

First and foremost I would like to thank my supervisor, Marcos Mariño. Not to mention the support and advice, this work would not even exist without him in the first place. I could not have been luckier than to end up writing a thesis about the very topic I spent my undergraduate years wondering about.

Then there are, of course, my colleagues in the project. First, Alba Grassi, the best guiding hand one could ask for to start a PhD. And Szabolcs Zakany, who has discussed with me once and then again any kind of little detail of a computation, of a malfunctioning piece of code, of a not-very-well explained section in a paper; who has been a second couple of eyes in every revision of my work –including this very thesis– and a fundamental source of feedback. Thank you both.

I extend these thanks to my collaborators Ricardo Schiappa and Jie Gu; and to all those with whom I have had the many discussions that helped with the project: Massimiliano Ronzani, Davide Racco, Alexander Glazman, and Yasuyuki Hatsuda.

For the time when I was still learning my way around physics, I am also grateful to José Edelstein. I arrived to where I am thanks to him, and his lectures were surely the reason for which I took the decision of heading into theoretical physics. On the other side of the very low fence dividing our fields, my thanks to Jose A. Oubiña and F. Adrián F. Tojo, for making this physicist remember during those crucial years that mathematics are not just for decoration.

And a special thanks to Sandra and Carlos, for all their help and encouragement.

## Resumé en français

Cette thèse explore la connexion entre la théorie spectrale et la théorie des cordes topologiques. Un exemple concret (la géométrie  $Y^{3,0}$ ) d’une relation exacte (appelée correspondance TS/ST) conjecturée entre les deux, venant de l’idée de *mirror symmetry* est analysé en détail, et on trouve un très bon accord. Ceci est utilisé comme la motivation pour appliquer des outils de théorie des cordes – en particulier la *refined holomorphic anomaly* – à d’autres problèmes spectraux du type Schrödinger. Avec cela, nous calculons leur expansion WKB.

Aussi, nous modifions la *holomorphic anomaly* pour inclure des corrections non-perturbatives dans la série WKB. Ceci est utilisé pour récupérer la transseries obtenue en utilisant des conditions de quantification exactes déjà connues pour des problèmes de mécanique quantique. À l’aide de la *resurgence*, il nous permettra également de reproduire le comportement à grand ordre des coefficients WKB, tant pour les problèmes type Schrödinger que les *quantum mirror curves* de la correspondance TS/ST.

Dans la théorie spectrale, c’est la mécanique quantique qui nous concerne. Du point de vue mathématique, elle est un sous-ensemble de la théorie spectrale qui utilise des algèbres d’opérateur très concrètes. Notamment, l’algèbre de Heisenberg  $[x, p] = i\hbar$ . Une des questions les plus directes que l’on peut alors demander, c’est quelles sont les énergies du système correspondant – c’est-à-dire, les valeurs propres de l’opérateur Hamiltonien correspondant,  $H(x, p)$ . Pour des exemples très choisis, tout étudiant sait obtenir des résultats exacts. Pour le reste, on recourt normalement aux approximations. Mathématiquement, les réponses qu’on cherche – comme les niveaux d’énergie – sont transformées en expansions en série de puissances autour d’un paramètre petit. C’est-à-dire, en *perturbations* d’un problème connu. La mécanique quantique a naturellement un tel petit paramètre, la constante de Planck  $\hbar$ . Le problème déjà connu dans ce cas est, tout simplement, son homologue dans la mécanique classique. Par exemple, une fois qu’on sait la trajectoire d’un système à une particule classique, on peut calculer une approximation à ses niveaux d’énergie dans la mécanique quantique avec la bien connue méthode de Wentzel-Kramers-Brillouin.

Bien sûr, ces problèmes ont été déjà bien étudiés. L’approximation WKB peut être étendue à un algorithme récursif qui produit toute la série de puissances en  $\hbar$  déterminant la fonction d’onde. Malheureusement, trouver les niveaux d’énergie implique l’évaluation des intégrales de plus en plus compliquées. Celles-ci ne peuvent pas être calculés d’une façon simple et algorithmique, par opposition aux fonctions d’onde.

Alors, de nouvelles perspectives permettent d’obtenir de nouveaux résultats et dans notre cas, la nouvelle perspective est la théorie des cordes. À l’origine née d’une description de la phénoménologie des hadrons, elle est devenu plus tard une candidate pour une “théorie de tout” – une fois qu’il a été réalisé que toutes les versions connues des théories de cordes accueillent une particule obéissant les mêmes lois que le graviton. Ceci était un accomplissement notable, pour autant qu’il donnait au

moins une théorie quantique de la gravité *cohérente*. Cependant, nous ne sommes pas concernés par la question de savoir à quel point – ou même si – les théories des cordes peuvent être réalistes du point de vue de la phénoménologie. Enfin, elles sont des théories d’objets géométriques dont la dynamique est fondamentalement liée aux propriétés des espaces dans lesquels elles vivent.

Le développement des outils nécessaires pour obtenir des prédictions concrètes a eu comme résultat tout type de découvertes mathématiques. Cette interaction entre la physique théorique et les mathématiques n’a rien de nouveau, c’est sûr. Mais la théorie des cordes a été particulièrement utile à cause de son interaction avec la géométrie algébrique, entre autres. D’intérêt particulier pour nous est la théorie de cordes topologique, et sa connexion aux problèmes spectraux.

Le but principal de cette thèse est d’éclairer, au moins un peu, la structure non-perturbative de problèmes dans la mécanique quantique. Pour le faire, on montrera en détail comment on peut construire – et résoudre – des problèmes spectraux en termes des informations géométriques contenues dans la théorie de cordes topologique. Finalement, on appliquera les outils de la théorie de cordes à la mécanique quantique. Avec ceci on donnera une façon systématique d’examiner leur secteur non-perturbatif.

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# Chapter 1

## Introduction

As anyone who has ever faced a physics problem knows, there are two fundamental steps involved in obtaining its answers. The first is of course coming up with a model, say Newton's laws of dynamics, so that we can turn our questions into precise mathematical statements – a simple set of differential equations. The second is being actually able to answer those questions. Crucially, this is not only a matter of whether we can prove that the answer exists – but rather of having a feasible way to produce the result.

Time and time again, new ways of solving old problems have come from unexpected paths of research. The gain is sometimes in mere computational efficiency, sometimes in the form of a new insight. To continue with the example, Hamilton's formalism is in a sense just a repainting of Newton's laws. Yet, a repainting crucial to –among others– the development of quantum mechanics.

It is precisely quantum mechanics itself that concerns us. In a narrow sense, it is a subset of spectral theory working on very concrete operator algebras. Notably, the Heisenberg algebra  $[x, p] = i\hbar$ . One of the most direct questions that can then be asked is which are the energies of a system – what are the eigenvalues of the corresponding Hamiltonian operator  $H(x, p)$ . Writing down the eigenproblem is the aforementioned first step. As for the second, in a few selected examples every student knows how to derive exact results in a closed form. With the rest, we normally resort to approximations. Mathematically, the answers we look for –such as energy levels– are transformed into power series expansions around some small parameter. That is, *perturbations* of a known problem. Quantum mechanics naturally has one such small parameter, Planck's constant  $\hbar$ . The “known problem” is merely its classical mechanics counterpart. For instance, once we know the trajectory of a classical one-particle system, we can compute a first approximation to its quantum mechanical energy levels via the well known Wentzel-Kramers-Brillouin approximation.

These perturbative solutions are extremely useful –after all, most of the successful predictions of Quantum Field Theory are perturbations around free fields– but come with a couple of caveats. First, in many cases the power series is not convergent for any value of the perturbation parameter. Without a way to make sense of this, there is an inherent limit to the precision of any statement we could

in principle make. Then, even if we accept just an approximate answer, we need a systematic way of finding corrections. And finally, supposing there is one, simply calculating a few such corrections can be a computationally impracticable task.

By all means, these problems have been studied extensively, in particular in the context of quantum mechanics. The WKB approximation can be extended to a recursive algorithm that generates the quantum mechanical wavefunction as an  $\hbar$  power series. Unfortunately, determining the energy levels involves the evaluation of increasingly complicated period integrals. These cannot be computed in a simple, algorithmic way.

The practical difficulty of calculations aside, this all-orders-in- $\hbar$  WKB series does not converge for any  $\hbar$ . It can still be understood through regularization techniques, notably Borel resummation. This is, in essence, a particular way of completing the perturbative power series. There is certainly not a unique way of achieving this completion. For example, two clearly different functions of  $\hbar$  may share the same power series around zero – and differ in a non-analytic term like  $\exp(-1/\hbar)$ . Such a term is, in fact, the hallmark of what we will call *non-perturbative* effects. The variety of different completions immediately raises the question: which one is *correct*? In some concrete quantum mechanical problems, one can find exact conditions that the eigenvalues must fulfil, which in turn constrains the non-perturbative completion to the power series.

As stated before, new results come from new perspectives, and in our case, the new perspective is string theory. Originally born from a “stringy” description of hadron phenomenology, it later became a candidate for a “theory of everything” – once it was realized that all known versions of string theory host a particle obeying the same laws as the graviton. This was a notable achievement, making it at least a consistent quantum theory of gravity. However, we are not concerned with how close –or even if– string theories are to be phenomenologically realistic. They are theories of geometrical objects, whose dynamics are fundamentally linked to the properties of the spaces in which they live. Just developing the tools needed to obtain concrete predictions led to fruitful mathematical discoveries. This interplay between mathematics and theoretical physics is nothing new, for sure. The need to solve Newton’s equations of motion advanced the study of differential equations, and a better understanding of analysis led to the Euler-Lagrange reformulation; Riemannian geometry set up the framework for general relativity, who in turn sparked more interest in its mathematical foundations. String theory has been particularly fruitful in its interaction with algebraic geometry – although not only. Of particular interest to us is topological string theory, and its surprising connection to spectral problems. Starting from there, we will explain in detail how one can build – and solve – these problems in terms of the geometrical information contained in topological string theories.

Ultimately, it is the main objective of this thesis to shed some light on the non-perturbative structure of quantum mechanics. Much has been written already about it, and surely much more remains. We will therefore mostly focus on a concrete aspect, the WKB coefficients and their relation to quantization conditions.

We will apply the tools of topological string theory to them, and we will provide a new, systematic way of looking into their non-perturbative sector.

## 1.1 A brief review of the foundations

### 1.1.1 The all-orders WKB method

Our starting point is the Schrödinger equation for a one-dimensional particle in a potential  $V(x)$

$$-\frac{\hbar^2}{2}\psi''(x) + V(x)\psi(x) = E\psi(x). \quad (1.1)$$

For simplicity, let us suppose that the potential has only two turning points – like the harmonic oscillator. We require the wavefunction  $\psi(x)$  to be square normalizable so that it can be understood as a probability density. As usual, if the potential is binding, the equation becomes a discrete eigenvalue problem for the energy  $E$ . We can get a first idea of what are the eigenvalues by recasting the problem in terms of the classical momentum,  $p(x)$ ,

$$\hbar^2\psi''(x) + p^2(x; E)\psi(x) = 0, \quad p(x; E) = \sqrt{2(E - V(x))}. \quad (1.2)$$

Then, the  $n$ -th energy level is approximated by the Bohr-Sommerfeld condition

$$\oint_A p(x; E_n) dx \sim 2\pi\hbar n \quad (1.3)$$

where  $A$  is a trajectory in phase space between the turning points of the potential. From a physical viewpoint, we are *quantizing* the volume enclosed by a trajectory in phase space. This is what is usually called the *old* quantum theory, that was based on more or less the intuition that *something* had to be quantized, and the action had the right units of Planck's constant  $\hbar$ .

A systematic approach to this idea is the given by the all-orders WKB method of Dunham [1]. First, we set a WKB ansatz for the wavefunction,

$$\psi(x; E, \hbar) = \exp \left[ \frac{i}{\hbar} \int^x Q(x'; E, \hbar) dx' \right]. \quad (1.4)$$

The Schrödinger equation for  $\psi(x)$  becomes a Riccati equation for  $Q(x)$ ,

$$Q(x; E, \hbar)^2 - i\hbar \frac{\partial Q(x; E, \hbar)}{\partial x} = p(x; E)^2, \quad (1.5)$$

which can be solved with a power series in  $\hbar$ . In fact,  $Q$  can be rewritten as

$$Q(x; E, \hbar) = P(x; E, \hbar) + \frac{i\hbar}{2} \frac{\partial}{\partial x} \log P(x; E, \hbar) \quad (1.6)$$

where  $P$  is a power series in just  $\hbar^2$ ,

$$P(x; E, \hbar) = \sum_{n=0}^{\infty} P_n(x; E) \hbar^{2n} = p(x; E) + O(\hbar^2). \quad (1.7)$$

With it the wavefunction can be recast into the familiar form

$$\psi(x; E, \hbar) = \frac{1}{\sqrt{P(x; E, \hbar)}} \exp \left[ \frac{i}{\hbar} \int^x P(x'; E, \hbar) dx' \right]. \quad (1.8)$$

Now we ask what happens to this ansatz in the different parts of the classically allowed region. Recall we assumed a binding potential – growing as  $x \mapsto \pm\infty$  – with a single such allowed region. The leading order part  $p(x; E)$  becomes imaginary beyond the classical turning points  $x_{\pm}$ , defined by

$$V(x_{\pm}) = E. \quad (1.9)$$

This means that the wavefunction past the turning point will decrease exponentially – like we would expect from the classical intuition. For instance, after the turning point on the right side of the real line,

$$\psi(x > x_+) \simeq \alpha \exp \left( -\frac{1}{\hbar} \int_{x_+}^x \sqrt{2(V(x') - E)} dx' + O(\hbar^0) \right). \quad (1.10)$$

In the classically allowed region,  $p(x; E)$  is real and the wavefunction oscillating,

$$\begin{aligned} \psi(x_- < x < x_+) \simeq & \beta_+ \exp \left( \frac{i}{\hbar} \int_{x_-}^x \sqrt{2(E - V(x'))} dx' + O(\hbar^0) \right) + \\ & + \beta_- \exp \left( -\frac{i}{\hbar} \int_{x_-}^x \sqrt{2(E - V(x'))} dx' + O(\hbar^0) \right). \end{aligned} \quad (1.11)$$

Notice that the Riccati equation admits in general two solutions of differing sign – but after the turning point, the requirement of vanishing at infinity directly selects just one of them. The quantization condition follows from imposing square-normalizability on the *whole* wavefunction. For that, we would need to “glue”  $\alpha$  with  $\beta_{\pm}$ . Unfortunately, near the transition point, the WKB wavefunction diverges at every order, which is quite a problem if we want to glue the wavefunctions there. However, we know this must be an artefact of the approximation – the full wavefunction itself is just as well behaved as the potential it lives in. In fact, let us expand the potential as  $V(x) = E + \alpha(x - x_+) + O((x - x_+)^2)$ . If we just keep the first order approximation, directly from the Schrödinger equation, we get

$$\psi(x \sim x_+) \propto \text{Ai} \left[ \left( \frac{2\alpha}{\hbar^2} \right)^{1/3} (x - x_+) \right], \quad (1.12)$$

with Ai the Airy function. There is also another independent solution with the second Airy function, Bi. Importantly, the Airy function shows oscillations at one side and an exponentially suppressed behaviour at the other, just like the two solutions at each side of the turning point<sup>1</sup>. The idea of the uniform WKB method (see for

<sup>1</sup>In fact, as we will see in figure 3.1 when reviewing resurgence, the asymptotic expansions of the Airy function for each side break at the turning point, just like the WKB ansatz.

instance [2–4]) is to use the Airy function as an intermediary between the classically allowed and forbidden regions to do the matching. There are of course other ways of obtaining the spectrum of Schrödinger operators, see [5–7].

To understand why this matching leads to quantization, think that we could have as well started by the wavefunction on the forbidden region in the negative axis,

$$\psi(x < x_-) \simeq \gamma \exp \left( +\frac{1}{\hbar} \int_{x_-}^x \sqrt{2(V(x') - E)} dx' + O(\hbar^0) \right). \quad (1.13)$$

If we repeat the matching procedure twice – from negative forbidden region to classically allowed to positive forbidden – in general we will not find (1.10), but a combination of that and the diverging solution.

The concrete matching of the left and right exponential solutions will be determined by what happens in the middle. The WKB wavefunction is written in such a way that

$$\int^x Q(x') dx \quad (1.14)$$

captures precisely the change of the phase of the wavefunction. Then, the relation between the coefficients of the exponential solutions can be written as a function of the phase that the wavefunction acquires by traversing the classical region. This phase is given by (half) the period integral of  $Q(x)$

$$\int_{x_-}^{x_+} Q(x) dx. \quad (1.15)$$

By a careful analysis order by order in  $\hbar^2$ , one can see that the exponentially suppressed solutions at each side can be matched through the classical region precisely when

$$\oint_A P(x; E_n, \hbar) dx = 2\pi\hbar \left( n + \frac{1}{2} \right), \quad (1.16)$$

where  $A$  is, like in (1.3), a trajectory between the two turning points. This generalizes the semiclassical condition to something that includes, a priori, all the  $\hbar$  orders. The function  $P(x) dx$  plays the role of a *quantum* differential on phase space, in the sense that  $p(x) dx$  was the symplectic differential of classical mechanics. Importantly, the  $\hbar^2$  corrections to this quantum differential can be computed with the Riccati equation in an algebraic, recursive way.

### 1.1.2 Beyond perturbation

Unfortunately for the computation, and fortunately for the richness of the theory, the practical story is not so simple. The  $\hbar$  expansion of the period integral gives well defined coefficients at every order,

$$t_n := \oint_A P_n(x; E) dx. \quad (1.17)$$

Even if computing  $P_n$  is rather straightforward, the integrals themselves are a different matter. Nevertheless, one could evaluate them, say, numerically. The real problem comes when one looks at what happens to these terms when  $n$  is large. It is well known that the coefficients of series in standard perturbation theory diverge factorially [8]. In the WKB case, the situation is slightly different, but not better. As we will see later on, the WKB coefficients diverge double-factorially<sup>2</sup>,

$$t_n \sim (2n)! \quad (1.18)$$

Not only does this grow too fast to converge everywhere – it does not even converge for *any* value of  $\hbar$ ! But of course, as physicists we do not let ourselves be defeated by such small details.

Consider the following function, defined for positive  $x$ ,

$$f(x) = \int_0^\infty \frac{e^{-t}}{1+tx} dt. \quad (1.19)$$

This is a perfectly well defined integral, and it does not look very threatening. The function solves the following problem

$$x^2 g'(x) + (x+1)g(x) = 1, \quad x \in [0, \infty). \quad (1.20)$$

This is very nice, but as we have already mentioned, many times we can only solve equations like (1.20) by approximate solutions, such as power series. We can see what it looks like for  $f(x)$  by either plugging a series ansatz into (1.20) or directly taking the  $n$ -th derivative of the definition.

$$\frac{1}{n!} \partial_x^n f(x) \Big|_{x=0} = \int_0^\infty e^{-t} (-t)^n dx = (-1)^n n! \quad (1.21)$$

So the Taylor expansion of  $f(x)$  goes like

$$f(x) \simeq 1 - x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + \dots = \sum_{n=0}^{\infty} (-1)^n n! x^n, \quad (1.22)$$

where the sum sign is purely formal, since the series has zero radius of convergence! And yet, we know it all comes from a perfectly well defined integral. We call this formal series an *asymptotic* expansion (hence the  $\simeq$ , instead of equality): even if the first few terms give a reasonable approximation to the actual function, there is an order from where it eventually diverges.

This is the idea lying behind Borel resummation. A detailed review of this and other methods for making sense of divergent series can be found in [10]. Let us give a sketch by supposing we have a power series ansatz for some problem, and it turns out that the coefficients diverge at large  $n$  like  $n!$ ,

$$p(x) = \sum_{n=0}^{\infty} c_n x^n, \quad c_n \sim n! \quad (1.23)$$

---

<sup>2</sup>This is a behaviour that many readers will recognize from string theory amplitudes [9].

where the sum is again formal. Then we define a Borel transform

$$\hat{p}(x) := \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n, \quad (1.24)$$

which now has a finite radius of convergence – so it is an actual function, as opposed to an asymptotic series. The problem of extending it for all  $x$  is the traditional one of analytic continuation, the same way that

$$\sum_{n=0}^{\infty} x^n \quad (1.25)$$

has radius of convergence 1 but it is just the series around 0 of the meromorphic function  $1/(1-x)$ . Finally, we define the Borel sum of  $p(x)$  as the function

$$(\mathcal{B}p)(x) := \int_0^{\infty} \hat{p}(tx) e^{-t} dt. \quad (1.26)$$

There are some subtleties in the analytic continuation of  $\hat{p}$ , say the possible presence of poles along the positive real axis. We will have to worry about this when the time comes. But barring anything of the sort, for now let us be optimistic and just say that this is a well defined integral. And, indeed, its Taylor expansion is the formal power series  $p(x)$ . Intuitively, we are taking the  $n!$  divergence out of  $c_n$ , and rewriting it as an integral via the gamma function definition,

$$p(x) = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n \left( \int_0^{\infty} t^n e^{-t} dt \right). \quad (1.27)$$

And then, not because we are allowed to but by defining  $(\mathcal{B}p)$  that way, we switch the sum with the integral.

One should now wonder: is this the only way? The answer is no: consider a more general version of  $f(x)$ , given by

$$f(x) = \int_0^{\infty} \frac{e^{-t}}{1-tx/A} dt, \quad \begin{cases} x \in [0, \infty), \\ A \in \mathbb{C}, A \notin [0, \infty). \end{cases} \quad (1.28)$$

$A$  is usually called the action, a name originating in instanton physics<sup>3</sup>. The reason we do not want it right now to be a positive real number is, as mentioned, to avoid poles along the integration path. Now  $f(x)$  solves the equation

$$x^2 g'(x) + (x-A)g(x) + A = 0. \quad (1.29)$$

However, this is not the only solution. The general case is

$$g(x) = f(x) + \frac{c}{x} e^{-A/x} \quad (1.30)$$

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<sup>3</sup>Just take a look at the quantum mechanical propagator,  $\int \mathcal{D}x e^{iS[x]/\hbar}$ . When Wick rotated, it has saddle points at the extrema of the action called *instanton* configurations. They contribute as  $\exp(-S/\hbar)$ .

for arbitrary  $c$ . If the real part of  $A$  is greater than zero, then

$$\lim_{x \rightarrow 0^+} e^{-A/x} = 0 \quad (1.31)$$

and similarly for all of its derivatives. Then, all the different  $g(x)$  have exactly the same<sup>4</sup> power series. The  $\exp(-A/x)$  term is responsible for what we call *non-perturbative* effects, because it cannot be seen (directly) in the *perturbative* power series. We say directly, because the two are not completely unrelated. Indeed, the asymptotic expansion of  $f(x)$  gives

$$f(x) \simeq \sum_{n=0}^{\infty} n! \left(\frac{x}{A}\right)^n. \quad (1.32)$$

There is a manifest relation between the size of the exponentially small term and the large order behaviour, through the value of  $A$ . This connection does not stop there, and it is just the tip of the iceberg of the phenomenon known as *resurgence* [11]. In this thesis we will make heavy use of it to test our results.

Summing up, when we can only compute perturbations as a power series, a resummation recipe such as the Borel sum will give us a *particular* solution of our original problem – be it (1.20) or something more complex like a partition function. In any case we expect to have a whole family of functions reproducing the original series, of the form

$$\begin{aligned} f(x) = & c_0 + c_1 x + c_2 x^2 + \dots + \\ & + e^{-A_1/x} \left( c_0^{(1)} + c_1^{(1)} x + c_2^{(1)} x^2 + \dots \right) + \\ & + e^{-A_2/x} \left( c_0^{(2)} + c_1^{(2)} x + c_2^{(2)} x^2 + \dots \right) + \dots \end{aligned} \quad (1.33)$$

This is a multiple expansion in two parameters, the small  $x$  and the small(er)  $\exp(-A/x)$ , called a *transseries*. For a comprehensive explanation of Borel resummation, transseries and related phenomena relevant for this thesis, the reader can refer to [12].

As said earlier, one of the main purposes of this work is to tackle the asymptotic series of WKB and ultimately provide an algorithmic way to compute whatever is missing from the non-perturbative sector. To do that, we will need to build a transseries framework to extend WKB. Now, nobody wants to reinvent the wheel. So we have taken much inspiration from previous efforts at understanding this kind of situation.

And if someone has had a hard time dealing with its own perturbative nature –providing us a great deal of tools in doing so– well, that is string theory.

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<sup>4</sup>For those with technical fears about the definition of a power series without a left derivative: one might as well send  $x \mapsto y^2$ . Then  $\exp(-A/y^2)$  becomes an infinitely differentiable function over the real line, with all derivatives at  $y = 0$  vanishing. All  $g(y)$  still have the same power series around  $y = 0$ .

### 1.1.3 From strings to spectral theory

So, what *is* string theory?

Just as classical mechanics is concerned with the position  $x$  of a particle, living in some space  $X$ , at a time  $t$ ,

$$x(t) \in X, \quad (1.34)$$

string theory “simply” broadens its concern to one dimensional, extended objects. We will give a very broad overview of what this implies, but for more details entry-level discussions can be found in [13, 14]. At fixed  $t = t_0$ , one can see these extended objects as a collection of points  $x(t_0, s)$  parametrized by a real number  $s$ . The trajectory of this extended object is then

$$x(t, s) \in X. \quad (1.35)$$

From a geometrical point of view, this is the embedding of a surface  $\Sigma$  in some ambient space  $X$ . In analogy to the *worldline* described by a point particle, we call it the *worldsheet*. It gets its (classical) dynamics from extremizing an action, just as with Hamilton’s principle. The minimal dynamic, coordinate independent action is

$$S[\Sigma, h] = \int_{\Sigma} dt ds \sqrt{h} G_{\mu\nu} h^{ab} \partial_a x^\mu(t, s) \partial_b x^\nu(t, s). \quad (1.36)$$

Here,  $G_{\mu\nu}$  is the metric of the ambient space  $X$ ,  $h^{ab}$  is the metric on the worldsheet itself, and  $h = \det(h_{ab})$ . The latter is what describes the *shape* of the worldsheet, and will be as much of a dynamical ingredient as the position  $x$  itself. We can actually see this as a theory –called *sigma model*– of fields  $x^\mu \in X$  living in a two dimensional space  $\Sigma$ , coupled to a two dimensional gravity through the metric  $h$ .

Just like in the particle case, we can promote the theory to a quantum version through the path integral. If we want to compute the probability to change from one string configuration to a different one later on, we should integrate over all possible worldsheets that connect them to get the corresponding amplitude. In a purely formal sense, we want

$$Z = \int_{\mathcal{M}} \int_{\mathcal{X}} \mathcal{D}x \mathcal{D}h e^{\frac{i}{\hbar} S[\Sigma(x), h]} \quad (1.37)$$

where  $\mathcal{X}$  and  $\mathcal{M}$  represent all possible configurations of respectively the position and metric of  $\Sigma$  – all possible surfaces – given our chosen boundaries. Admittedly, this is not very well defined. Not even in the sense the average mathematician will say the particle path integral is not well defined, just because the measure  $\mathcal{D}x$  over all positions is difficult to formalize. It is the much worse sense of trying to define a path integral for a metric  $h$  in  $\mathcal{M}$ , which in those terms sounds much like the problem of quantizing gravity.

However, it is not quite that case – the action for  $h$  is much simpler than the Einstein-Hilbert one. First, let us bring some ideas from QFT to help us. There, we tend to end up doing a perturbative expansion in some coupling  $g$ . Not to

say one cannot do exact computations in QFT – or even do a completely different kind of approximation by working on the lattice, but perturbation theory is the most common angle of attack. With strings, what is seen from afar as a particle is a single tube, something that from a distance resembles a line. We can see interactions –vertices in Feynman diagrams– as the joining and splitting of these tubes. In the QFT expansion, a diagram with  $n$  loops will be preceded by a power of the (hopefully) small coupling  $g^n$ . In strings, the loops are “handles” in the worldsheet, in the sense that the torus has a hole. Geometrically, this is the genus  $g$  of the surface  $\Sigma_g$ . Therefore, if string interactions have some coupling  $g_s$ , a sensible theory would give us some power of  $g_s$  in front of a contribution involving  $\Sigma_g$ . As we shall see now, this hierarchy of surfaces in terms of their genus gives actually computable contributions to the partition function  $Z$ . Unfortunately and unlike in QFT, such a perturbative coupling expansion is the best “bare” string theory can do, without a better definition for  $Z$ .

Also following QFT, we take a Wick rotation of  $t$  into imaginary time  $i\tau = t$ . We also reorganize the partition function into the free energy  $F = \log Z$ , so that  $F$  contains only the connected components of the diagrams contributing to  $Z$ . Then, we define  $F$  to be a formal series in the coupling  $g_s$ ,

$$F = \sum_{g=0}^{\infty} g_s^{2g-2} F_g \quad (1.38)$$

where the genus  $g$  contribution is given by

$$F_g = \int_{\mathcal{M}_g} \int_{\mathcal{X}_g} \mathcal{D}x \mathcal{D}h e^{-\frac{1}{\hbar} S[\Sigma_g, h]}. \quad (1.39)$$

$\mathcal{X}_g$  is still the space of all position configurations of the worldsheet  $\Sigma_g$ . Essentially, that part of the integral is a QFT living on  $\Sigma_g$ . The integral over all metrics can now be understood as follows. The action (1.36) is invariant under conformal transformations  $h_{ab}(t, s) \mapsto \Omega(t, s) h_{ab}(t, s)$ . This means that physically inequivalent configurations must be given by conformally inequivalent metrics – which is the same, for Riemann surfaces, as inequivalent complex structures. But the space of complex structures is a well known object: the moduli space  $\mathcal{M}_g$  of surfaces of genus  $g$ . This is a  $3g - 3$  complex dimensional manifold, integrals over which have been more than studied by mathematicians.

Actually, the conformal invariance of (1.36) is obvious... at the *classical* level. Quantum anomalies can, and do, spoil the symmetry in general scenarios. Of course one can insist in using (1.39) as a definition for the theory regardless of symmetry, but if we really want be able to interpret it as the integral over all possible metrics  $h$  we need to make sure that conformal invariance is preserved at the quantum level. One of the constraints to avoid these anomalies is that, at first order in the string coupling, the background has to be Ricci flat. In other words, it has to satisfy Einstein’s vacuum equation. This is probably the single most important source of interest in string theory, since it makes it a consistent quantum theory

that includes in its classical limit Einstein's gravity, who has otherwise refused to be quantized<sup>5</sup>. This has proved incredibly useful in trying to understand what *some* quantum theory of gravity might look like. Unfortunately for phenomenologists, string theory implies gravity, and much more. Including all kinds of other particles. As it turns out, another constraint coming from anomaly cancellation is the number of fields of the sigma model, that depends on whether we include purely bosonic or also anticommuting ones. From the point of view of the particles propagating in the target space  $X$ , we are determining the *dimension* of spacetime. And in superstring theories, which include fermions via worldsheet supersymmetry, the requirement is for the dimension of spacetime to be 10. This is not a problem, except for the fact that we only observe 4.

This led to the idea of *compactification*, that essentially says that the extra 6 dimensions are compact and small enough so that they cannot be probed by point particles – only “stringy” phenomena can feel them, for instance by wrapping around in non trivial ways. It is not very different from the extra dimension added to Kaluza-Klein theory. As in Kaluza-Klein, the shape of these compact dimensions would have a fundamental impact on the macroscopic theory. One should consider the effective theory obtained by integrating out the extra dimensions – which would hopefully look somehow like the Standard Model. The way in which strings are allowed to interact inside the compactified space will fundamentally depend on its geometry, and will give rise to different couplings in the effective theory. For this reason, much focus has been put on studying string theory on six dimensional compactifications. Since they should be Ricci flat, a very interesting subclass are the so called Calabi-Yau manifolds, who on top of flatness have a trivial canonical bundle. The original reason is that this condition enforces  $\mathcal{N} = 1$  supersymmetry on the target space effective theory, which at the time was seen as a desirable property by phenomenologists. From the mathematician's point of view, they are interesting simply because of how much can be actually computed by having a nice Kähler structure.

Quite some time has passed in the search for compactifications that give rise to known physics in the point particle regime. Without much luck, for the moment. However, independently of how much trust one places in string theory as a theory of everything, the undeniable fact is that in its context physicists and mathematicians, drawing inspiration from either side, have made remarkable advances in understanding the geometry and topology of Calabi-Yau manifolds and other related objects. The standard geometry of manifolds studies distances between their *points*. When one tries to see how *extended* objects such as strings live in them, what is under study is the space of *loops* and higher embeddings into the manifold. In this sense string theory gives rise to a notion of “stringy” or “extended” geometry [15]. This is not, by far, the only area of mathematics that has profited from the developments

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<sup>5</sup>One should at least acknowledge here Loop Quantum Gravity for trying, although it is not clear how much of a theory of *gravity* it is as it has not been yet proven that general relativity is its classical limit.

of the last decades in string theory, but it is the one we will be mostly interested in for this thesis.

An important class of theories with a very direct geometrical interpretation are topological string theories, reviewed in [16]. The idea is to start with the  $\mathcal{N} = (2, 2)$  supersymmetric sigma model on the worldsheet Riemann surface. This a theory for the (bosonic) position fields  $x_\mu$  together with the necessary fermionic ones to meet the supersymmetry requirements. Then, one uses the  $U(1)$  R-symmetry between the supersymmetry generators to mix them with the spacetime Lorentz symmetry generators, in order to produce a theory that is topological. It is so in the following sense. With the new mix of supersymmetry and Lorentz generators, one builds a fermionic BRST symmetry  $Q$  such that the stress-energy tensor of the theory can be written, for some fermionic object  $b_{\mu\nu}$ , as

$$T_{\mu\nu} = \{Q, b_{\mu\nu}\}, \quad (1.40)$$

where  $\{\cdot, \cdot\}$  is the anticommutator. Since the stress-energy is the variation of the action w.r.t. the spacetime metric,

$$\frac{\delta S}{\delta G_{\mu\nu}} = T^{\mu\nu}, \quad (1.41)$$

this means that the variation of the action as a response variations in the ambient metric is  $Q$  exact. The whole point of BRST symmetries is establishing equivalence classes of physically indistinguishable states, which means the contribution of the action to the path integral is independent of the *spacetime* metric. The most important consequence of this is that the path integral *localizes* on its saddle points. Since we are free to choose whatever representative of each equivalence class of spacetime embeddings, we can in particular take them towards the semiclassical limit, which are the extrema of the action. Understanding the action as a definition for the “size” of the worldsheet, the path integral over spacetime becomes just a sum over, in a certain sense, “extremal surfaces”. Two contributions will be inequivalent, and contribute separately to the path integral, if they cannot be continuously deformed into the other: if they are *topologically* different. In the end, what the path integral computes is in how many different ways can one wrap the Riemann surface onto the target space. Because of this, topological string theory has been very successful in explicitly computing enumerative invariants of Calabi-Yau geometries, where traditional methods could not.

Going back to the “twist” used to turn the theory topological, one has the freedom to choose the sign of the  $U(1)$  R-charge used. Physically, this choice should be irrelevant. Geometrically, the computed quantities turn out to be completely different. The two resulting twisted topological theories are usually called A and B model. The former is sensitive to the Kähler structure of the Calabi-Yau, which roughly speaking tells us about its *size*. The latter, on the other hand, depends on the moduli of the complex structure, which is related to the *shape*. A simple example of this structures is the torus built as a quotient of the complex plane by



can be parametrized by period integrals of the non-vanishing holomorphic 3-form guaranteed to exist by the Calabi-Yau condition. Here is where one makes contact with spectral problems of Heisenberg operators, since as we saw in the first part of the introduction, their semiclassical quantization is given in terms of periods. There is a certain choice of coordinates in some mirror manifold  $\widehat{X}$  that allow us to naturally upgrade them to Heisenberg operators  $[\mathbf{x}, \mathbf{p}] = i\hbar$ . This operator is what we therefore call a *quantum mirror curve*. Remarkably, it can be shown [18] that the “stringy” corrections to those complex structure periods in the so-called Nekrasov-Shatashvili limit are precisely the WKB corrections to the semiclassical volume of the associated operator. Via mirror symmetry, we have a wealth of information on the structure of those corrections from purely geometrical calculations in the A model on the manifold  $X$ , such as [19].

## 1.2 Outline and related publications

This thesis is divided as follows. Most of the content of chapter 2, including plots and derivations, has been developed in

- [20] S. Codesido, J. Gu and M. Marino, *Operators and higher genus mirror curves*, *JHEP* **02** (2017) 092, [[arXiv:1609.00708](#)],

which in turn builds on the results of

- [21] S. Codesido, A. Grassi and M. Marino, *Spectral Theory and Mirror Curves of Higher Genus*, *Annales Henri Poincaré* **18** (2017) 559–622, [[arXiv:1507.02096](#)],
- [22] S. Codesido, A. Grassi and M. Marino, *Exact results in  $\mathcal{N} = 8$  Chern-Simons-matter theories and quantum geometry*, *JHEP* **07** (2015) 011, [[arXiv:1409.1799](#)].

In that chapter a conjecture [23] relating topological string and the spectral determinant of a certain class of Heisenberg type difference operators and their perturbations will be presented. We will analyse in full detail a concrete example, the  $Y^{3,0}$  geometry, that is convoluted enough to show the full power of the generalized conjecture, that includes computing spectral traces of some perturbations to the operators. We will show how we can recover analytically the large  $\hbar$  expansion of the traces, and for some concrete values of  $\hbar$  we will compute them exactly. The interested reader should refer to [23] for the original statement or [24] for a review.

In chapter 3 we will consider a Schrödinger problem very closely related to topological strings, the  $\cosh x$  or modified Mathieu potential, based on the work in

- [25] S. Codesido, M. Marino and R. Schiappa, *Non-Perturbative Quantum Mechanics from Non-Perturbative Strings*, [[arXiv:1712.02603](#)].

First we will review in depth the perturbative holomorphic anomaly of [26], or rather its refinement [27]. For our purposes, it will serve as an extremely efficient tool to compute the WKB periods of the spectral problem. Then, following the spirit

of [28], we will show how this can actually be used to retrieve non-perturbative information about the system. We will test it through several concrete predictions. More importantly, we will build a new framework to obtain non-perturbative WKB corrections, just from the holomorphic anomaly.

Chapter 4 will examine four applications of the holomorphic anomaly to spectral problems, based on [25] and

- [29] S. Codesido and M. Marino, *Holomorphic Anomaly and Quantum Mechanics*, [arXiv:1612.07687](#).

For the first two, we argue that the holomorphic anomaly that applies to the modified Mathieu potential due to its relation with topological strings should actually be a feature of the WKB recursion even when unrelated to stringy scenarios. With comprehensive computations we show that this is the case for two genus one spectral curves in quantum mechanics. Then, we apply the non-perturbative framework to study the large order behaviour of both the WKB coefficients and the energy levels of the difference operators introduced in chapter 2. We will also discuss how the transseries relates to the Borel resummation of their asymptotic expansion, as a follow-up to the results of [30, 31].

Finally, the article

- [32] S. Codesido, *On the resummation of the Lee-Yang edge singularity coupled to gravity*, [arXiv:1712.02752](#),

was written in the context of this thesis to consider the implications of [30] in a “simpler” string scenario. One might find in it a quick introduction, via example, to the numerical details of Borel resummation and asymptotic expansions.

The chapters try to be as self contained as possible. The reader interested in topological string theory and mirror curves will find chapter 2 most relevant. The reader who has been drawn here by the implications for quantum mechanics can skip directly to chapter 3, and for just an understanding of how to extract non-perturbative corrections to WKB periods from the refined holomorphic anomaly section 3.3 should be enough.



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## Chapter 2

# Spectral theory and topological strings

Let us start then by understanding where the whole idea of applying stringy methods to spectral problems came from. In [23], a detailed –yet conjectural– correspondence was proposed between topological strings on toric Calabi–Yau (CY) manifolds, and the spectral theory of certain quantum-mechanical operators on the real line (see [24] for a review). For this reason it is referred to as the Topological String/Spectral Theory (TS/ST) correspondence. The operators arise by the quantization of the mirror curve to a toric CY, as suggested originally in [33]. This correspondence builds upon previous work on quantization and mirror symmetry [18, 34–37], and on the exact solution of the ABJM matrix model [38–44] (reviewed in [45, 46].) It leads to exact expressions for the spectral traces and the Fredholm determinants of these operators, in terms of BPS invariants of the CY. Conversely, the genus expansion of the topological string free energy arises as an asymptotic expansion of their spectral traces, in a certain ‘t Hooft-like regime. In this way, the correspondence provides a non-perturbative completion of the topological string free energy. Although the general correspondence of [23] is still conjectural, it has passed many tests. In the last two years, techniques have been developed to calculate the corresponding quantities in spectral theory and shown to be in perfect agreement with the predictions of the conjecture [47–51]. Other aspects of the correspondence have been discussed in [52–55].

The conjectural correspondence of [23] was formulated for mirror curves of genus one. In this chapter we will study in detail examples of the correspondence for higher genus  $g_\Sigma$  curves. The framework that we will later develop for quantum mechanical problems will also apply only to simpler genus one curves. Indeed, one of the objectives of this section is to give an overview of the tools necessary to deal with higher genus curves, that could be used to extend the quantum mechanical framework in the same direction. The original generalization for mirror curves of toric CY was proposed in [21]. In this case, the quantization of the curve leads naturally to  $g_\Sigma$  different operators, and one can define a generalized Fredholm determinant which encodes their spectral properties. In [21], the higher genus version of the correspon-

dence was verified in detail in the example of the resolved  $\mathbb{C}^3/\mathbb{Z}_5$  orbifold, arguably the simplest toric CY with a genus two mirror curve. Here we will focus on a more complicated example.

After briefly reviewing the spectral theory of the operators associated to toric CY's, we will study the CY geometry that leads to the  $SU(3)$  relativistic periodic Toda lattice (recently studied in [56]). The geometry has an extra mass parameter, which was absent in the example analyzed in [21]. mirror curves. The conjectural formulae of [21] pass the tests with flying colors. Along the way, we will present the exact integral kernel for the inverse of a four-term operator which is relevant for the relativistic Toda case, generalizing the results in [47, 49].

## 2.1 The higher genus TS/ST correspondence

### 2.1.1 Building operators from mirror curves

First we will review the construction of the  $g_\Sigma$  operators corresponding to the mirror curve of a toric CY, their spectral theory, and the connection to the topological string theory compactified on the toric CY. We mainly follow [21, 23, 48, 49]. For the interested reader, more on mirror symmetry on toric CYs can be found in [17, 57–59].

The toric CY threefolds which we are interested in can be described as symplectic quotients,

$$X = \mathbb{C}^{k+3} // G, \quad (2.1.1)$$

where  $G = U(1)^k$ . The action of each  $U(1)$  on the coordinates of  $\mathbb{C}^{k+3}$  is specified through of a matrix of charges  $Q_i^\alpha$ ,  $i = 0, \dots, k+2$ ,  $\alpha = 1, \dots, k$ , so that a rotation  $\phi$  by the  $\alpha$ -th component of  $G$  sends

$$u_i \mapsto e^{iQ_i^\alpha \phi} u_i, \quad \{u_i\} \in \mathbb{C}^{k+3}. \quad (2.1.2)$$

The charges must satisfy the condition,

$$\sum_{i=0}^{k+2} Q_i^\alpha = 0, \quad \alpha = 1, \dots, k. \quad (2.1.3)$$

which implies the CY condition on the quotient. The interest of the construction is that their mirrors can be written in terms of  $3+k$  complex coordinates  $Y^i \in \mathbb{C}^*$ ,  $i = 0, \dots, k+2$ , which satisfy the constraint

$$\sum_{i=0}^{k+2} Q_i^\alpha Y^i = 0, \quad \alpha = 1, \dots, k. \quad (2.1.4)$$

The mirror CY manifold  $\widehat{X}$  is then given by

$$w^+ w^- = W_X, \quad (2.1.5)$$

where

$$W_X = \sum_{i=0}^{k+2} x_i e^{Y_i}, \quad (2.1.6)$$

and the  $x_i$  parametrize its complex structure. It is possible to solve the constraints (2.1.4), modulo a global translation, in terms of two variables which we will denote by  $x$ ,  $y$ , and we then obtain a function  $W_X(e^x, e^y)$  from (2.1.6). The equation

$$W_X(e^x, e^y) = 0 \quad (2.1.7)$$

defines a Riemann surface  $\Sigma$  embedded in  $\mathbb{C}^* \times \mathbb{C}^*$ , which we will call the *mirror curve* to the toric CY threefold  $X$ . We note that there is a group of reparametrization symmetries of the mirror curve given by [60],

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow G \begin{pmatrix} x \\ y \end{pmatrix}, \quad G \in SL(2, \mathbb{Z}). \quad (2.1.8)$$

The moduli space of the mirror curve can be parametrized by the  $k+3$  coefficients  $x_i$  of its equation (2.1.6), among which three can be set to 1 by the  $\mathbb{C}^*$  scaling acting on  $e^x, e^y$  and an overall  $\mathbb{C}^*$  scaling. Equivalently, one can use the Batyrev coordinates

$$z_\alpha = \prod_{i=0}^{k+2} x_i^{Q_i^\alpha}, \quad \alpha = 1, \dots, k, \quad (2.1.9)$$

which are invariant under the  $\mathbb{C}^*$  actions. In order to write down the mirror curves, it is also useful to introduce a two-dimensional Newton polygon  $\mathcal{N}$ . The points of this polygon are given by

$$\nu^{(i)} = (\nu_1^{(i)}, \nu_2^{(i)}), \quad (2.1.10)$$

in such a way that the extended vectors

$$\bar{\nu}^{(i)} = \left(1, \nu_1^{(i)}, \nu_2^{(i)}\right), \quad i = 0, \dots, k+2, \quad (2.1.11)$$

satisfy the relations

$$\sum_{i=0}^{k+2} Q_i^\alpha \bar{\nu}^{(i)} = 0. \quad (2.1.12)$$

This Newton polygon is nothing else but the support of the 3d toric fan of the toric Calabi-Yau threefold on a hyperplane located at  $(1, *, *)$ . The function on the l.h.s. of (2.1.7) is then given by the Newton polynomial of the polygon  $\mathcal{N}$ ,

$$W_X(e^x, e^y) = \sum_{i=0}^{k+2} x_i \exp\left(\nu_1^{(i)} x + \nu_2^{(i)} y\right). \quad (2.1.13)$$

Clearly, there are many sets of vectors satisfying the relations (2.1.12), but they lead to curves differing in a reparametrization or a global translation, which are therefore equivalent. It can be seen that the genus of  $\Sigma$ ,  $g_\Sigma$ , is given by the number of inner

points of  $\mathcal{N}$ . We also notice that among the coefficients  $x_i$  of the mirror curve,  $g_\Sigma$  of them are “true” moduli of the geometry, corresponding to the inner points of  $\mathcal{N}$ , while  $r_\Sigma$  of them are the so-called “mass parameters”, corresponding to the points on the boundary of the polygon (this distinction has been emphasized in [61, 62]). In order to distinguish them, we will denote the former by  $\kappa_i$ ,  $i = 1, \dots, g_\Sigma$  and the latter by  $\xi_j$ ,  $j = 1, \dots, r_\Sigma$ . It is obvious that we can translate the Newton polygon in such a way that the inner point associated to a given  $\kappa_i$  is the origin. In this way we obtain what we will call the canonical forms of the mirror curve

$$\mathcal{O}_i(x, y) + \kappa_i = 0, \quad i = 1, \dots, g_\Sigma, \quad (2.1.14)$$

where  $\mathcal{O}_i(x, y)$  is a polynomial in  $e^x$ ,  $e^y$ . Note that, for  $g_\Sigma = 1$ , there is a single canonical form. Different canonical forms are related by reparametrizations of the form (2.1.8) and by overall translations, which lead to overall monomials, so we will write

$$\mathcal{O}_i + \kappa_i = \mathcal{P}_{ij}(\mathcal{O}_j + \kappa_j), \quad i, j = 1, \dots, g_\Sigma, \quad (2.1.15)$$

where  $\mathcal{P}_{ij}$  is of the form  $e^{\lambda x + \mu y}$ ,  $\lambda, \mu \in \mathbb{Z}$ . Equivalently, we can write

$$\mathcal{O}_i = \mathcal{O}_i^{(0)} + \sum_{j \neq i} \kappa_j \mathcal{P}_{ij}. \quad (2.1.16)$$

The functions  $\mathcal{O}_i(x, y)$  appearing in the canonical forms of the mirror curves can be quantized [21, 23]. To do this, we simply promote the variables  $x, y$  to Heisenberg operators  $\mathbf{x}, \mathbf{y}$  satisfying

$$[\mathbf{x}, \mathbf{y}] = i\hbar, \quad (2.1.17)$$

and we use the Weyl quantization prescription,

$$e^{\alpha x + \beta y} \mapsto e^{\alpha \mathbf{x} + \beta \mathbf{y}}. \quad (2.1.18)$$

In this way we obtain  $g_\Sigma$  different operators, which we will denote by  $\mathbf{O}_i$ ,  $i = 1, \dots, g_\Sigma$ . These operators are self-adjoint. The equation of the mirror curve itself is promoted to an operator

$$\mathbf{W}_{X,i} \equiv \mathbf{O}_i + \kappa_i, \quad (2.1.19)$$

which we call the *quantum mirror curve*. Different canonical forms are related by the quantum version of (2.1.15),

$$\mathbf{O}_i + \kappa_i = \mathbf{P}_{ij}^{1/2}(\mathbf{O}_j + \kappa_j) \mathbf{P}_{ij}^{1/2}, \quad i, j = 1, \dots, g_\Sigma, \quad (2.1.20)$$

where  $\mathbf{P}_{ij}$  is the operator corresponding to the monomial  $\mathcal{P}_{ij}$ . We will also denote by  $\mathbf{O}_i^{(0)}$  the operator associated to the function  $\mathcal{O}_i^{(0)}$  in (2.1.16). This can be regarded as an “unperturbed” operator, while the moduli  $\kappa_j$  encode different perturbations of it. We also define the inverse operators,

$$\rho_i = \mathbf{O}_i^{-1}, \quad \rho_i^{(0)} = \left(\mathbf{O}_i^{(0)}\right)^{-1}, \quad i = 1, \dots, g_\Sigma. \quad (2.1.21)$$

It is easy to see that [21]

$$P_{ij} = P_{ji}^{-1}, \quad i \neq j, \quad (2.1.22)$$

and

$$P_{ik} = P_{ij}^{1/2} P_{jk} P_{ij}^{1/2}, \quad i \neq k. \quad (2.1.23)$$

We want to study now the spectral theory of the operators  $O_i$ ,  $i = 1, \dots, g_\Sigma$ . The appropriate object to consider turns out to be the generalized spectral determinant introduced in [21]. Let us consider the following operators,

$$A_{jl} = \rho_j^{(0)} P_{jl}, \quad j, l = 1, \dots, g_\Sigma. \quad (2.1.24)$$

Let us suppose that these operators are of trace class (this turns out to be the case in all known examples, provided some positivity conditions on the mass parameters are satisfied). Then, the generalized spectral determinant associated to the CY  $X$  is given by

$$\Xi_X(\boldsymbol{\kappa}; \hbar) = \det(1 + \kappa_1 A_{j1} + \dots + \kappa_{g_\Sigma} A_{jg_\Sigma}). \quad (2.1.25)$$

Due to the trace class property of the operators  $A_{jl}$ , this quantity is well-defined, and its definition does not depend on the choice of the index  $j$ , due to the similarity transformation

$$A_{il} = P_{ij}^{-1/2} A_{jl} P_{ij}^{1/2}. \quad (2.1.26)$$

As shown in [63], (2.1.25) is an entire function of the moduli  $\kappa_1, \dots, \kappa_{g_\Sigma}$ . In particular, it can be expanded around the origin  $\boldsymbol{\kappa} = 0$ , as follows,

$$\Xi_X(\boldsymbol{\kappa}; \hbar) = \sum_{N_1 \geq 0} \dots \sum_{N_{g_\Sigma} \geq 0} Z_X(\mathbf{N}; \hbar) \kappa_1^{N_1} \dots \kappa_{g_\Sigma}^{N_{g_\Sigma}}, \quad (2.1.27)$$

with the convention that

$$Z_X(0, \dots, 0; \hbar) = 1. \quad (2.1.28)$$

This expansion defines the (generalized) fermionic spectral traces  $Z_X(\mathbf{N}; \hbar)$  of the toric CY  $X$ . Both  $\Xi_X(\boldsymbol{\kappa}; \hbar)$  and  $Z_X(\mathbf{N}; \hbar)$  depend in addition on the mass parameters, gathered in a vector  $\boldsymbol{\xi}$ . When needed, we will indicate this dependence explicitly and write  $\Xi_X(\boldsymbol{\kappa}; \boldsymbol{\xi}, \hbar)$ ,  $Z_X(\mathbf{N}; \boldsymbol{\xi}, \hbar)$ . As shown in [21], one can use classical results in Fredholm theory to obtain determinant expressions for these fermionic traces. Let us consider the kernels  $A_{jl}(x_m, x_n)$  of the operators defined in (2.1.24), and let us construct the following matrix:

$$R_j(x_m, x_n) = A_{jl}(x_m, x_n) \quad \text{if} \quad \sum_{s=1}^{l-1} N_s < m \leq \sum_{s=1}^l N_s. \quad (2.1.29)$$

Then, we have that <sup>1</sup>

$$Z_X(\mathbf{N}; \hbar) = \frac{1}{N_1! \dots N_{g_\Sigma}!} \int \det_{m,n} (R_j(x_m, x_n)) d^N x, \quad (2.1.30)$$

<sup>1</sup>The determinant of the matrix  $R_j(x_m, x_n)$  is independent of the label  $j$ , just like the Fredholm determinant.

where

$$N = \sum_{s=1}^{g_\Sigma} N_s. \quad (2.1.31)$$

In the case  $g_\Sigma = 2$ , this formula becomes

$$Z_X(N_1, N_2; \hbar) = \frac{1}{N_1! N_2!} \int \det \begin{pmatrix} A_{j1}(x_1, x_1) & \cdots & A_{j1}(x_1, x_N) \\ \vdots & & \vdots \\ A_{j1}(x_{N_1}, x_1) & \cdots & A_{j1}(x_{N_1}, x_N) \\ A_{j2}(x_{N_1+1}, x_1) & \cdots & A_{j2}(x_{N_1+1}, x_N) \\ \vdots & & \vdots \\ A_{j2}(x_N, x_1) & \cdots & A_{j2}(x_N, x_N) \end{pmatrix} dx_1 \cdots dx_N. \quad (2.1.32)$$

One finds, for example

$$\begin{aligned} Z_X(1, 1; \hbar) &= \text{Tr } \mathbf{A}_{j1} \text{Tr } \mathbf{A}_{j2} - \text{Tr } (\mathbf{A}_{j1} \mathbf{A}_{j2}) \\ &= \int dx_1 dx_2 (A_{j1}(x_1, x_1) A_{j2}(x_2, x_2) - A_{j1}(x_1, x_2) A_{j2}(x_2, x_1)), \end{aligned} \quad (2.1.33)$$

as well as

$$\begin{aligned} Z_X(2, 1; \hbar) &= \text{Tr } (\mathbf{A}_{j1}^2 \mathbf{A}_{j2}) - \frac{1}{2} \text{Tr } (\mathbf{A}_{j1}^2) \text{Tr } \mathbf{A}_{j2} + \\ &\quad + \frac{1}{2} (\text{Tr } \mathbf{A}_{j1})^2 \text{Tr } \mathbf{A}_{j2} - \text{Tr } \mathbf{A}_{j1} \text{Tr } (\mathbf{A}_{j1} \mathbf{A}_{j2}), \\ Z_X(1, 2; \hbar) &= \text{Tr } (\mathbf{A}_{j1} \mathbf{A}_{j2}^2) - \frac{1}{2} \text{Tr } \mathbf{A}_{j1} \text{Tr } (\mathbf{A}_{j2}^2) + \\ &\quad + \frac{1}{2} \text{Tr } \mathbf{A}_{j1} (\text{Tr } \mathbf{A}_{j2})^2 - \text{Tr } (\mathbf{A}_{j1} \mathbf{A}_{j2}) \text{Tr } \mathbf{A}_{j2}. \end{aligned} \quad (2.1.34)$$

Alternatively, one can also derive them for arbitrary  $g_\Sigma$  from Liouville's formula for the determinant,

$$\begin{aligned} \det(1 + \mathbf{O}) &= \exp \left\{ \text{Tr } \log(1 + \mathbf{O}) \right\} = \\ &= \exp \left\{ \text{Tr } (\mathbf{O}) - \frac{\text{Tr } (\mathbf{O}^2)}{2} + \frac{\text{Tr } (\mathbf{O}^3)}{3} + \dots \right\} = \\ &= 1 + \text{Tr } (\mathbf{O}) + \frac{\text{Tr } (\mathbf{O})^2 - \text{Tr } (\mathbf{O}^2)}{2} + \\ &\quad + \frac{\text{Tr } (\mathbf{O})^3 - 3 \text{Tr } (\mathbf{O}) \text{Tr } (\mathbf{O}^2) + 2 \text{Tr } (\mathbf{O}^3)}{6} + \dots \end{aligned} \quad (2.1.35)$$

which naturally gives the small  $\kappa_i$  expansion that defines the fermionic traces in (2.1.27). In our genus two case the expressions are rather simple. With

$$\mathbf{O} = \kappa_1 \mathbf{A}_{j1} + \kappa_2 \mathbf{A}_{j2} \quad (2.1.36)$$

we can extract for instance  $Z_X(1, 1; \hbar)$  from the  $\kappa_1\kappa_2$  coefficient from the expansion of the determinant, since only a finite number of terms contribute to a fixed order in  $\mathcal{O}$ ,

$$\begin{aligned} \frac{\mathrm{Tr}(\mathcal{O})^2 - \mathrm{Tr}(\mathcal{O}^2)}{2} &= \kappa_1\kappa_2 \left( \mathrm{Tr}(A_{j_1}) \mathrm{Tr}(A_{j_2}) - \mathrm{Tr}(A_{j_1}A_{j_2}) \right) + \\ &+ \kappa_1^2 \frac{\mathrm{Tr}(A_{j_1})^2 - \mathrm{Tr}(A_{j_1}^2)}{2} + \kappa_2^2 \frac{\mathrm{Tr}(A_{j_2})^2 - \mathrm{Tr}(A_{j_2}^2)}{2}. \end{aligned} \quad (2.1.37)$$

The  $\kappa_1^n\kappa_2^m$  coefficient is precisely the fermionic trace  $Z_X(n, m; \hbar)$ . For arbitrary genus, one would have

$$Z(\mathbf{N}; \hbar) = \exp \left\{ \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{m} \mathrm{Tr} \left[ \left( \sum_{i=1}^{g_\Sigma} \kappa_i A_{j_i} \right)^m \right] \right\} \Big|_{\kappa_1^{N_1} \dots \kappa_{g_\Sigma}^{N_{g_\Sigma}}}. \quad (2.1.38)$$

The generalized spectral determinant encodes the spectral properties of all the operators  $\mathcal{O}_i$  in a single strike. Indeed, one has [21],

$$\det(1 + \kappa_i \rho_i) = \frac{\Xi_X(\boldsymbol{\kappa}; \hbar)}{\Xi_X(\kappa_1, \dots, \kappa_{i-1}, 0, \kappa_{i+1}, \dots, \kappa_{g_\Sigma}; \hbar)}, \quad i = 1, \dots, g_\Sigma. \quad (2.1.39)$$

In addition,

$$\det(1 + \kappa_i \rho_i^{(0)}) = \Xi_X(0, \dots, 0, \kappa_i, 0, \dots, 0; \hbar), \quad i = 1, \dots, g_\Sigma, \quad (2.1.40)$$

i.e. the generalized spectral determinant specializes to the spectral determinants of the unperturbed operators appearing in the different canonical forms of the mirror curve.

The standard spectral determinant of a single trace-class operator determines the spectrum of eigenvalues, through its zeros. The generalized spectral determinant (2.1.25) vanishes in a codimension one submanifold of the moduli space. It follows from (2.1.39) that this submanifold contains all the information about the spectrum of the operators  $\rho_i$  appearing in the quantization of the mirror curve, as a function of the moduli  $\kappa_j$ ,  $j \neq i$ .

### 2.1.2 The view from spectral theory

Before moving on to why are we even building operators out of geometrical objects like mirror curves, we will show some explicit results on the spectral theory side. After all, it would be pointless to try to find a relation between both if we cannot even say something just from looking at the operators.

The simplest form in which they appear is the three-term operator

$$\mathcal{O}_{m,n} = e^x + e^y + e^{-mx-ny}, \quad m, n \in \mathbb{Z}_{>0}. \quad (2.1.41)$$

This is the bare minimum that produces a genus one curve that has at least one modulus. From the point of the physical intuition, the  $e^x$  and  $e^{-x}$  produce a confining

potential so that we can have a trace class operator to begin with. Mass parameters (that is, extra parameters that do not contribute to the genus of the curve) appear as perturbations by another exponential term. These operators were introduced and studied in [47]. It turns out that the integral kernel of their inverses  $\rho_{m,n} = \mathbf{O}_{m,n}^{-1}$  can be explicitly computed in terms of Faddeev's quantum dilogarithm. This makes it possible to calculate the standard traces

$$\mathrm{Tr} \rho_{m,n}^\ell, \quad \ell = 1, 2, \dots, \quad (2.1.42)$$

in terms of integrals over the real line. These integrals can be computed by using the techniques of [64], or by using the recursive methods developed in [41, 50, 65, 66]. Once the standard spectral traces (2.1.42) have been computed, the fermionic spectral traces follow the combinatorics shown in the previous section. Mixed traces, as those appearing in (2.1.33), (2.1.34), can be also obtained in terms of integrals.

Finding the kernel for an exponential operator with an arbitrary number of terms is not straightforward. Constraining ourselves to what we will need, we will study four-term operators that arise as perturbations of (2.1.41). This concrete operator reads,

$$\mathbf{O}_{m,n,\xi} = e^x + e^y + e^{-mx-ny} + \xi e^{-(1+m)x-(n-1)y}, \quad m, n \in \mathbb{Z}_{>0}. \quad (2.1.43)$$

First, as in [47], we introduce the Heisenberg operators  $\mathbf{q}$  and  $\mathbf{p}$  satisfying the normalized commutation relations

$$[\mathbf{p}, \mathbf{q}] = \frac{1}{2\pi i}. \quad (2.1.44)$$

They are related to the operators  $x, y$  appearing in (2.1.43) by

$$x \equiv 2\pi b \frac{(n+1)\mathbf{p} + n\mathbf{q}}{m+n+1}, \quad y \equiv -2\pi b \frac{m\mathbf{p} + (m+1)\mathbf{q}}{m+n+1}, \quad (2.1.45)$$

so that

$$\hbar = \frac{2\pi b^2}{m+n+1}. \quad (2.1.46)$$

Let us also rescale the mass parameter by introducing  $\zeta$  as

$$\xi = e^{2\pi b \zeta}. \quad (2.1.47)$$

We will now obtain an explicit expression for the integral kernel of

$$\rho_{m,n,\xi} = \mathbf{O}_{m,n,\xi}^{-1}, \quad (2.1.48)$$

based on similar derivations in [47] and [49]. We decompose

$$\begin{aligned} e^{-y/2} \mathbf{O}_{m,n,\xi} e^{-y/2} &= e^{x-y} + 1 + e^{-mx-(n+1)y} + \xi r e^{-(1+m)x-ny} \\ &= e^{2\pi b(\mathbf{p}+\mathbf{q})} + 1 + e^{2\pi b\mathbf{q}} + \xi e^{-2\pi b\mathbf{p}} \\ &= e^{-\pi b\mathbf{p}} \left( e^{2\pi b(2\mathbf{p}+\mathbf{q})} + e^{2\pi b\mathbf{p}} + e^{2\pi b\mathbf{q}} + \xi \right) e^{-\pi b\mathbf{p}}. \end{aligned} \quad (2.1.49)$$

We now use Faddeev's quantum dilogarithm  $\Phi_b(x)$  [67–69] (our conventions for this function are as in [47]). It has the following integral definition,

$$\Phi_b(x) = \exp \left( \int_{\mathbb{R}+i\epsilon} \frac{e^{-2i x z}}{4 \sinh(z\mathbf{b}) \sinh(z\mathbf{b}^{-1})} \frac{dz}{z} \right). \quad (2.1.50)$$

It satisfies the following identity (a similar identity was already used in [47]):

$$\Phi_b(\mathbf{p}) \Phi_b^*(\mathbf{q}) e^{2\pi\mathbf{b}\mathbf{p}} \Phi_b(\mathbf{q}) \Phi_b^*(\mathbf{p}) = e^{2\pi\mathbf{b}(2\mathbf{p}+\mathbf{q})} + e^{2\pi\mathbf{b}\mathbf{p}} + e^{2\pi\mathbf{b}(\mathbf{p}+\mathbf{q})}. \quad (2.1.51)$$

Its behaviour under complex conjugation is given by,

$$\Phi_b^*(z) = \frac{1}{\Phi_b(z^*)}. \quad (2.1.52)$$

Let us denote

$$\widehat{\mathcal{O}}_{m,n,\xi} = e^{\pi\mathbf{b}\mathbf{p}-y/2} \mathcal{O}_{m,n,\xi} e^{\pi\mathbf{b}\mathbf{p}-y/2}. \quad (2.1.53)$$

We now recall that Faddeev's quantum dilogarithm satisfies the following difference equation,

$$\frac{\Phi_b(x + c_b + i\mathbf{b})}{\Phi_b(x + c_b)} = \frac{1}{1 - qe^{2\pi\mathbf{b}x}}, \quad (2.1.54)$$

where

$$q = e^{2\pi i \mathbf{b}^2}, \quad c_b = i \frac{\mathbf{b} + \mathbf{b}^{-1}}{2}. \quad (2.1.55)$$

By using this equation, we obtain

$$\begin{aligned} \Phi_b(\mathbf{q}) \Phi_b^*(\mathbf{p}) \widehat{\mathcal{O}}_{m,n,\xi} \Phi_b(\mathbf{p}) \Phi_b^*(\mathbf{q}) &= \xi + e^{2\pi\mathbf{b}\mathbf{p}} = \xi (1 + e^{2\pi\mathbf{b}(\mathbf{p}-\zeta)}) \\ &= \xi \frac{\Phi_b(\mathbf{p} - \zeta - i\mathbf{b}/2)}{\Phi_b(\mathbf{p} - \zeta + i\mathbf{b}/2)}, \end{aligned} \quad (2.1.56)$$

By taking the inverse, we find,

$$\Phi_b(\mathbf{q}) \Phi_b^*(\mathbf{p}) e^{-\pi\mathbf{b}\mathbf{p}+y/2} \xi \mathcal{O}_{m,n,\xi}^{-1} e^{-\pi\mathbf{b}\mathbf{p}+y/2} \Phi_b(\mathbf{p}) \Phi_b^*(\mathbf{q}) = \frac{\Phi_b(\mathbf{p} - \zeta + i\mathbf{b}/2)}{\Phi_b(\mathbf{p} - \zeta - i\mathbf{b}/2)}, \quad (2.1.57)$$

and we can write,

$$\xi \mathcal{O}_{m,n,\xi}^{-1} = e^{\pi\mathbf{b}\mathbf{p}-y/2} \Phi_b(\mathbf{p}) \Phi_b^*(\mathbf{q}) \frac{1}{\Phi_b(\mathbf{p} - \zeta - i\mathbf{b}/2)} \Phi_b(\mathbf{p} - \zeta + i\mathbf{b}/2) \Phi_b(\mathbf{q}) \Phi_b^*(\mathbf{p}) e^{\pi\mathbf{b}\mathbf{p}-y/2}. \quad (2.1.58)$$

We now use the quantum pentagon identity [68]

$$\Phi_b(\mathbf{p} - p_0) \Phi_b(\mathbf{q}) = \Phi_b(\mathbf{q}) \Phi_b(\mathbf{p} - p_0 + \mathbf{q}) \Phi_b(\mathbf{p} - p_0), \quad (2.1.59)$$

where  $p_0 = \zeta - i\mathbf{b}/2$ , to obtain

$$\xi \mathcal{O}_{m,n,\xi}^{-1} = e^{\pi\mathbf{b}\mathbf{p}-y/2} \frac{\Phi_b(\mathbf{p})}{\Phi_b(\mathbf{p} - p_0^*)} \frac{1}{1 + e^{2\pi\mathbf{b}(\mathbf{p}+\mathbf{q}-\zeta)}} \frac{\Phi_b(\mathbf{p} - p_0)}{\Phi_b(\mathbf{p})} e^{\pi\mathbf{b}\mathbf{p}-y/2}. \quad (2.1.60)$$

Let us now introduce the parameters [47]

$$a = \frac{\mathbf{b}m}{2(m+n+1)}, \quad c = \frac{\mathbf{b}}{2(m+n+1)}, \quad h = a + c - nc, \quad (2.1.61)$$

as well as the function

$$\Psi_{a,c}(x) \equiv \frac{e^{2\pi ax}}{\Phi_{\mathbf{b}}(x - i(a+c))}. \quad (2.1.62)$$

We will also relabel

$$\mathbf{p} + \mathbf{q} - \zeta \rightarrow \mathbf{q}. \quad (2.1.63)$$

Then, we have

$$\rho_{m,n,\xi} = \mathbf{O}_{m,n,\xi}^{-1} = \Psi_{a,c}^*(\mathbf{p}) \Psi_{nc,0}(\mathbf{p} - \zeta) |\Psi_{a+c,cn}(\mathbf{q})|^2 \Psi_{a,c}(\mathbf{p}) \Psi_{nc,0}^*(\mathbf{p} - \zeta), \quad (2.1.64)$$

where we used again the property (2.1.54). In particular, its kernel can be written, in the momentum representation

$$\mathbf{p}|x) = x|x), \quad (2.1.65)$$

as

$$(x|\rho_{m,n,\xi}|y) = \rho_{m,n,\xi}(x, y) = \frac{\overline{f(x)}f(y)}{2\mathbf{b} \cosh\left(\pi \frac{x-y+ih}{\mathbf{b}}\right)}, \quad (2.1.66)$$

where

$$f(x) = \Psi_{a,c}(x) \Psi_{nc,0}^*(x - \zeta) = \frac{\Phi_{\mathbf{b}}(x - \zeta + ic)}{\Phi_{\mathbf{b}}(x - i(a+c))} e^{2\pi(a+nc)x} e^{-2\pi c\zeta}. \quad (2.1.67)$$

It can be easily checked, by using the properties of Faddeev's quantum dilogarithm, that as  $\xi \rightarrow 0$  (which corresponds to  $\zeta \rightarrow -\infty$ ), we recover the kernel of  $\mathbf{O}_{m,n}$  determined in [47].

Using the results (2.1.66), (2.1.67), it is possible to compute explicitly the spectral traces of  $\rho_{m,n,\xi}$ . Let us consider a simple illustrative example,  $m = n = 1$ . This is actually the operator related to the topological string on local  $\mathbb{P}^2$  [23]. Let us focus on the so called maximally supersymmetric case  $\hbar = 2\pi$ , which corresponds to  $\mathbf{b} = \sqrt{3}$ . The diagonal integral kernel is given by

$$\begin{aligned} \rho_{1,1,\xi}(x, x) &= \frac{1}{2\mathbf{b} \cos(\pi/6)} e^{4\pi x/\mathbf{b} - 2\pi\zeta/\mathbf{b}} \frac{\Phi_{\mathbf{b}}(x + i\mathbf{b}/3) \Phi_{\mathbf{b}}(x - \zeta + i\mathbf{b}/6)}{\Phi_{\mathbf{b}}(x - i\mathbf{b}/3) \Phi_{\mathbf{b}}(x - \zeta - i\mathbf{b}/6)} \\ &= \frac{1}{2\mathbf{b} \cos(\pi/6)} \frac{e^{4\pi x/\mathbf{b} - 2\pi\zeta/\mathbf{b}}}{(1 + e^{2\pi x/\mathbf{b}} + e^{4\pi x/\mathbf{b}})(1 + e^{2\pi(x-\zeta)/\mathbf{b}})}. \end{aligned} \quad (2.1.68)$$

This can be integrated, to obtain

$$\mathrm{Tr} \rho_{1,1,\xi} = \frac{\pi (2 - \xi^{1/3}) + \sqrt{3} \xi^{1/3} \log(\xi)}{18\pi (\xi^{2/3} - \xi^{1/3} + 1)}. \quad (2.1.69)$$

Similarly, one finds<sup>2</sup>

$$\begin{aligned} \mathrm{Tr}\rho_{1,1,\xi}^2 = & -\frac{6\sqrt{3}(1+\xi) + \pi(-4 + 4\xi^{1/3} - 13\xi^{2/3} + 6\xi)}{108\pi(1+\xi^{1/3})(1-\xi^{1/3} + \xi^{2/3})^2} \\ & + \frac{\xi^{2/3}\log\xi}{18\sqrt{3}\pi(1-\xi^{1/3} + \xi^{2/3})^2} + \frac{\xi^{2/3}(\log\xi)^2}{36\pi^2(1+\xi^{1/3})(1-\xi^{1/3} + \xi^{2/3})^2}. \end{aligned} \quad (2.1.70)$$

These traces are functions of  $\xi$  with a branch cut at  $\xi = 0$ . Their limit when  $\xi \rightarrow 0$  exists and gives back the traces for the operator  $\rho_{1,1} \equiv \rho_{1,1,0}$  calculated in for example [47]. The theory for  $\xi = 1$  is particularly simple and we find

$$\mathrm{Tr}\rho_{1,1,\xi=1} = \frac{1}{18}, \quad \mathrm{Tr}\rho_{1,1,\xi=1}^2 = \frac{7}{216} - \frac{1}{6\sqrt{3}\pi}. \quad (2.1.71)$$

### 2.1.3 Introducing topological strings

What we just did was build a certain class of operators out of the same information we build toric CY – namely, the charge vectors  $Q_i^\alpha$ . That could have been done with any geometric object, of course. What we will do now is formulate a connection between both, using a *physical* theory living on the CY, topological strings. Indeed, the main conjectural result of [21,23] is an explicit formula expressing the generalized spectral determinant and spectral traces of one of these operators in terms of enumerative invariants of the corresponding CY  $X$ . This set of invariants is precisely what is captured by topological string theory.

To state this result, we need some basic geometric ingredients. As discussed in the previous section, the CY  $X$  has  $g_\Sigma$  moduli denoted by  $\kappa_i$ ,  $i = 1, \dots, g_\Sigma$ . We will introduce the associated “chemical potentials”  $\mu_i$  by

$$\kappa_i = e^{\mu_i}, \quad i = 1, \dots, g_\Sigma. \quad (2.1.72)$$

The CY  $X$  also has  $r_\Sigma$  mass parameters,  $\xi_j$ ,  $j = 1, \dots, r_\Sigma$ . Let  $n_\Sigma \equiv g_\Sigma + r_\Sigma$ . The Batyrev coordinates  $z_i$  introduced in (2.1.9) can be written as

$$-\log z_i = \sum_{j=1}^{g_\Sigma} C_{ij}\mu_j + \sum_{k=1}^{r_\Sigma} \alpha_{ik} \log \xi_k, \quad i = 1, \dots, n_\Sigma. \quad (2.1.73)$$

The coefficients  $C_{ij}$  determine a  $n_\Sigma \times g_\Sigma$  matrix which can be read off from the toric data of  $X$ . One can choose the Batyrev coordinates in such a way that, for  $i = 1, \dots, g_\Sigma$ , the  $z_i$ 's correspond to true moduli, while for  $i = g_\Sigma + 1, \dots, g_\Sigma + r_\Sigma$ , they correspond to mass parameters. For such a choice, the non-vanishing coefficients

$$C_{ij}, \quad i, j = 1, \dots, g_\Sigma, \quad (2.1.74)$$

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<sup>2</sup>We would like to thank Szabolcs Zakany for adapting the techniques of [50] to check this result.

form an invertible matrix, which agrees (up to an overall sign) with the charge matrix  $C_{ij}$  appearing in [70]. We also recall that the standard mirror map expresses the Kähler moduli  $t_i$  of the CY in terms of the Batyrev coordinates  $z_i$

$$-t_i = \log z_i + \tilde{\Pi}_i(\mathbf{z}), \quad i = 1 \dots, n_\Sigma, \quad (2.1.75)$$

where  $\tilde{\Pi}_i(\mathbf{z})$  is a power series in  $z_i$  with finite radius of convergence. Together with (2.1.73), this implies that

$$t_i = \sum_{j=1}^{g_\Sigma} C_{ij} \mu_j + \sum_{k=1}^{r_\Sigma} \alpha_{ik} \log \xi_k + \mathcal{O}(e^{-\mu}). \quad (2.1.76)$$

By using the quantized mirror curve, one can promote the classical mirror map to a quantum mirror map  $t_i(\hbar)$  depending on  $\hbar$  [18] (see [62, 71] for examples.)

$$-t_i(\hbar) = \log z_i + \tilde{\Pi}_i(\mathbf{z}; \hbar), \quad i = 1 \dots, n_\Sigma. \quad (2.1.77)$$

This quantum mirror map will play an important role in our construction. In addition to the quantum mirror map, we need the following enumerative ingredients. First of all, we need the conventional genus  $g$  free energies  $F_g(t)$  of  $X$ ,  $g \geq 0$ , in the so-called large radius frame (LRF), which encode the information about the Gromov–Witten invariants of  $X$ . They have the structure<sup>3</sup>

$$\begin{aligned} F_0(t) &= \frac{1}{6} \sum_{i,j,k=1}^{n_\Sigma} a_{ijk} t_i t_j t_k + 4\pi^2 \sum_{i=1}^{n_\Sigma} b_i^{NS} t_i + \sum_d N_0^d e^{-d \cdot t}, \\ F_1(t) &= \sum_{i=1}^{n_\Sigma} b_i t_i + \sum_d N_1^d e^{-d \cdot t}, \\ F_g(t) &= C_g + \sum_d N_g^d e^{-d \cdot t}, \quad g \geq 2. \end{aligned} \quad (2.1.78)$$

In these formulae,  $N_g^d$  are the Gromov–Witten invariants of  $X$  at genus  $g$  and multi-degree  $d$ , which in a intuitive way compute the number of inequivalent wrappings in the cohomology class of  $d$  that can be put into the target space  $X$ . The interest of topological strings lies in the fact that these free energies can actually be computed for certain classes of CY manifolds, for instance with the topological vertex [72] or as we will see later, with the holomorphic anomaly [26]. The coefficients  $a_{ijk}$ ,  $b_i$  are cubic and linear couplings characterizing the perturbative genus zero and genus one free energies, while  $C_g$  is the so-called constant map contribution [26]. This genus zero term can be computed directly from the classical periods on the curve, as we shall soon see. The constants  $b_i^{NS}$ , which can be obtained from the refined holomorphic anomaly equation [27, 73], usually appear in the linear term of

<sup>3</sup>The formula of  $F_0(t)$  differs from the one in the topological string literature by the linear term, which usually doesn't play a role in non-compact CY models. The addition of this term makes the formulae in the rest of the paper more compact.

$F^{NS}(t, \hbar)$  (see below, (2.1.85)). The total free energy of the topological string is the formal series,

$$F^{WS}(t, g_s) = \sum_{g \geq 0} g_s^{2g-2} F_g(t) = F^{(p)}(t, g_s) + \sum_{g \geq 0} \sum_d N_g^d e^{-d \cdot t} g_s^{2g-2}, \quad (2.1.79)$$

where

$$F^{(p)}(t, g_s) = \frac{1}{6g_s^2} \sum_{i,j,k=1}^{n_\Sigma} a_{ijk} t_i t_j t_k + \sum_{i=1}^{n_\Sigma} \left( b_i + \frac{4\pi^2}{g_s^2} b_i^{NS} \right) t_i + \sum_{g \geq 2} C_g g_s^{2g-2} \quad (2.1.80)$$

and  $g_s$  is the topological string coupling constant.

As found in [74], the sum over Gromov–Witten invariants in (2.1.79) can be resummed order by order in  $\exp(-t_i)$ , at all orders in  $g_s$ . This resummation involves the Gopakumar–Vafa (GV) invariants  $n_g^d$  of  $X$ , and it has the structure

$$F^{GV}(t, g_s) = \sum_{g \geq 0} \sum_d \sum_{w=1}^{\infty} \frac{1}{w} n_g^d \left( 2 \sin \frac{w g_s}{2} \right)^{2g-2} e^{-w d \cdot t}. \quad (2.1.81)$$

Note that, as formal power series, we have

$$F^{WS}(t, g_s) = F^{(p)}(t, g_s) + F^{GV}(t, g_s). \quad (2.1.82)$$

In the case of toric CYs, the Gopakumar–Vafa invariants are special cases of the refined BPS invariants [19, 75, 76]. These refined invariants depend on the degrees  $d$  and on two non-negative half-integers or “spins”,  $j_L, j_R$ . We will denote them by  $N_{j_L, j_R}^d$ . They mirror the refinement from de Rham cohomology to Dolbeault cohomology, in that they count curves taking into account not only their degree but also their holomorphic/antiholomorphic dependence. For toric CY, the physical theory is sensible to this refinement and we can obtain different free energies from it. They can also be computed explicitly, be it with the (refined) vertex formalism [19] or the (refined) holomorphic anomaly [75]. We then define the NS free energy as

$$F_{NS}(t, \hbar) = F_{NS}^{pert}(t, \hbar) + F_{NS}^{inst}(t, \hbar), \quad (2.1.83)$$

where

$$F_{NS}^{pert}(t, \hbar) = \frac{1}{6\hbar} \sum_{i,j,k=1}^{n_\Sigma} a_{ijk} t_i t_j t_k + \left( \hbar + \frac{4\pi^2}{\hbar} \right) \sum_{i=1}^{n_\Sigma} b_i^{NS} t_i, \quad (2.1.84)$$

and

$$F_{NS}^{inst}(t, \hbar) = \sum_{j_L, j_R} \sum_{w, d} N_{j_L, j_R}^d \frac{\sin \frac{\hbar w}{2} (2j_L + 1) \sin \frac{\hbar w}{2} (2j_R + 1)}{2w^2 \sin^3 \frac{\hbar w}{2}} e^{-w d \cdot t}. \quad (2.1.85)$$

In this equation, the coefficients  $a_{ijk}$  are the same ones that appear in (2.1.78). By expanding (2.1.83) in powers of  $\hbar$ , we find the NS free energies at order  $n$ ,

$$F^{NS}(t, \hbar) = \sum_{n=0}^{\infty} F_n^{NS}(t) \hbar^{2n-1}. \quad (2.1.86)$$

The first term in this series,  $F_0^{NS}(t)$ , is equal to  $F_0(t)$ , the standard genus zero free energy.

Following [44], we now define the modified grand potential of the CY  $X$ . It is the sum of two functions. The first one is

$$\begin{aligned} J_X^{WKB}(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar) &= \sum_{i=1}^{n_\Sigma} \frac{t_i(\hbar)}{2\pi} \frac{\partial F^{NS}(t(\hbar), \hbar)}{\partial t_i} + \frac{\hbar^2}{2\pi} \frac{\partial}{\partial \hbar} \left( \frac{F^{NS}(t(\hbar), \hbar)}{\hbar} \right) + \\ &+ \frac{2\pi}{\hbar} \sum_{i=1}^{n_\Sigma} (b_i + b_i^{NS}) t_i(\hbar) + A(\boldsymbol{\xi}, \hbar). \end{aligned} \quad (2.1.87)$$

We note that the function  $A(\boldsymbol{\xi}, \hbar)$  is only known in a closed form in some simple geometries. The second function is the “worldsheet” modified grand potential, which is obtained from the generating functional (2.1.81),

$$J_X^{WS}(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar) = F^{GV} \left( \frac{2\pi}{\hbar} t(\hbar) + \pi i \mathbf{B}, \frac{4\pi^2}{\hbar} \right). \quad (2.1.88)$$

It involves a constant integer vector  $\mathbf{B}$  (sometimes called B-field<sup>4</sup>) which depends on the geometry under consideration. This vector satisfies the following requirement: for all  $d$ ,  $j_L$  and  $j_R$  such that  $N_{j_L, j_R}^d$  is non-vanishing, we must have

$$(-1)^{2j_L + 2j_R + 1} = (-1)^{\mathbf{B} \cdot d}. \quad (2.1.89)$$

We note that the characterization above only defines  $B$  up to  $(2\mathbb{Z})^{n_\Sigma}$ . A different choice of  $B$  does not change  $J_X^{WS}(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar)$ . The total, modified grand potential is the sum of the above two functions,

$$J_X(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar) = J_X^{WKB}(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar) + J_X^{WS}(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar), \quad (2.1.90)$$

and it was introduced in [44]. Note that if we define

$$F_{top}(\mathbf{t}, g_s) = \frac{1}{6g_s^2} \sum_{i,j,k=1}^{n_\Sigma} a_{ijk} t_i t_j t_k + \sum_{i=1}^{n_\Sigma} \left( b_i + \frac{4\pi^2}{g_s^2} b_i^{NS} \right) t_i + F^{GV}(\mathbf{t}, g_s), \quad (2.1.91)$$

which differs from  $F(\mathbf{t}, g_s)$  by the terms proportional to  $C_g$ ,  $J_X(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar)$  can be written in a slightly more compact form, i.e.<sup>5</sup>

$$\begin{aligned} J(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar) &= \sum_{i=1}^{n_\Sigma} \frac{t_i}{2\pi} \frac{\partial}{\partial t_i} F_{NS}^{inst}(\mathbf{t}(\hbar), \hbar) + \frac{\hbar^2}{2\pi} \frac{\partial}{\partial \hbar} \left( \frac{F_{NS}^{inst}(\mathbf{t}(\hbar), \hbar)}{\hbar} \right) + A(\boldsymbol{\xi}, \hbar) \\ &+ \widehat{F}_{top} \left( \frac{2\pi}{\hbar} \mathbf{t}(\hbar), \frac{4\pi^2}{\hbar} \right), \end{aligned} \quad (2.1.92)$$

<sup>4</sup>The naming is a bit unfortunate, since it is not quite the B-field mentioned in the introduction. There should be no risk of confusion, as we will only use the integer-vector meaning in our topological string construction.

<sup>5</sup>Note that  $F_{top}(\mathbf{t}, g_s)$  is  $F^{WS}(\mathbf{t}, g_s)$  with the contributions from the constant maps removed. The latter can be understood as absorbed in  $A(\boldsymbol{\xi}, \hbar)$ .

where all but the constant  $A(\boldsymbol{\xi}, \hbar)$  are hidden in (refined) topological string free energies. Here we introduce the notation  $\widehat{f}(\mathbf{t})$ , meaning that  $\mathbf{t}$  is shifted by  $\pi i \mathbf{B}$  in the terms of order  $\exp(-t_i)$  (the instanton contributions). In particular,

$$\widehat{F}_{top}(\mathbf{t}, g_s) = \frac{1}{6g_s^2} \sum_{i,j,k=1}^{n_\Sigma} a_{ijk} t_i t_j t_k + \sum_{i=1}^{n_\Sigma} \left( b_i + \frac{4\pi^2}{g_s^2} b_i^{NS} \right) t_i + F^{GV}(\mathbf{t} + \pi i \mathbf{B}, g_s). \quad (2.1.93)$$

Finally, notice that (2.1.92) involves (2.1.81) and (2.1.85). They both have arbitrarily high order poles for any  $\hbar$  or  $g_s$  that is a rational multiple of  $\pi$ . One can, in that case, expand their expressions around the chosen value of  $\hbar$  or  $g_s = 4\pi^2/\hbar$ , which will make the poles at every order manifest. When building (2.1.92), *all the poles from  $F_{NS}$  will cancel out with all the poles from  $\widehat{F}_{top}$* . This pole cancellation mechanism was introduced in the context of ABJM theory [42], and guarantees that (2.1.92) is a well defined function.

### 2.1.4 The conjecture

Let  $X$  be a toric Calabi-Yau,  $J_X$  the generating function of its standard and refined BPS invariants as defined in (2.1.92), and  $\Xi_X$  the generalized spectral determinant of the operators associated to the mirror curve to  $X$  as defined in (2.1.25). Then, we state the conjecture

$$\Xi_X(e^\mu, e^\xi; \hbar) = \sum_{n \in \mathbb{Z}^{g_\Sigma}} \exp(J_X(\boldsymbol{\mu} + 2\pi i n, \boldsymbol{\xi}, \hbar)). \quad (2.1.94)$$

The right hand side of (2.1.94) defines a quantum-deformed (or generalized) Riemann theta function by

$$\Xi_X(\boldsymbol{\kappa}; \hbar) = \exp(J_X(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar)) \Theta_X(\boldsymbol{\kappa}; \hbar). \quad (2.1.95)$$

It can be computed as an expansion around the large radius point of moduli space. In the so-called ‘‘maximally supersymmetric case’’  $\hbar = 2\pi$ , it can be written down in closed form in terms of a conventional theta function. There is an equivalent form of the conjecture which gives an integral formula for the fermionic spectral traces:

$$Z_X(\mathbf{N}; \hbar) = \frac{1}{(2\pi i)^{g_\Sigma}} \int_{-i\infty}^{i\infty} d\mu_1 \cdots \int_{-i\infty}^{i\infty} d\mu_{g_\Sigma} \exp \left\{ J_X(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar) - \sum_{i=1}^{g_\Sigma} N_i \mu_i \right\}. \quad (2.1.96)$$

In practice, the contour integration along the imaginary axis can be deformed to a contour where the integral is convergent. For example, in the genus one case the integration contour is the one defining the Airy function (see [23, 43]).

An important consequence of the representation (2.1.96) is the existence of a 't Hooft-like limit in which one can extract the genus expansion of the conventional topological string. The 't Hooft limit is defined by

$$\hbar \rightarrow \infty, \quad N_i \rightarrow \infty, \quad \frac{N_i}{\hbar} = \lambda_i \text{ fixed}, \quad i = 1, \dots, g_\Sigma. \quad (2.1.97)$$

In this 't Hooft limit, the integral in (2.1.96) can be evaluated in the saddle-point approximation, and in order to have a non-trivial result, we have to consider the modified grand potential in the limit

$$\hbar \rightarrow \infty, \quad \mu_i \rightarrow \infty, \quad \frac{\mu_i}{\hbar} = \zeta_i \text{ fixed}, \quad i = 1, \dots, g_\Sigma. \quad (2.1.98)$$

In this limit, the quantum mirror map appearing in the modified grand potential becomes trivial. We will assume that the mass parameters  $\xi$  scale in such a way that

$$\log \hat{\xi}_j = \frac{2\pi}{\hbar} \log \xi_j, \quad j = 1, \dots, r_\Sigma, \quad (2.1.99)$$

are fixed as  $\hbar \rightarrow \infty$ . In the regime (2.1.98), the modified grand potential has an asymptotic genus expansion of the form,

$$J_X^{\text{'t Hooft}}(\zeta, \hat{\xi}, \hbar) = \sum_{g=0}^{\infty} J_g^X(\zeta, \hat{\xi}) \hbar^{2-2g}, \quad (2.1.100)$$

where

$$\begin{aligned} J_0^X(\zeta, \hat{\xi}) &= \frac{1}{16\pi^4} \widehat{F}_0(T) + A_0(\hat{\xi}) \\ J_1^X(\zeta, \hat{\xi}) &= A_1(\hat{\xi}) + \widehat{F}_1(T), \\ J_g^X(\zeta, \hat{\xi}) &= A_g(\hat{\xi}) + (4\pi^2)^{2g-2} \left( \widehat{F}_g(T) - C_g \right), \quad g \geq 2. \end{aligned} \quad (2.1.101)$$

In these equations, we have introduced the rescaled Kähler parameter

$$T = \frac{2\pi}{\hbar} t. \quad (2.1.102)$$

The arguments  $\zeta$  and  $\hat{\xi}$  of the modified grand potential are related to the rescaled Kähler parameters  $T$  by

$$T_i - \sum_{j=1}^{r_\Sigma} \alpha_{ij} \log \hat{\xi}_j = 2\pi \sum_{j=1}^{g_\Sigma} C_{ij} \zeta_j, \quad i = 1, \dots, g_\Sigma + r_\Sigma. \quad (2.1.103)$$

We have assumed that the function  $A(\xi, \hbar)$  has the expansion

$$A(\xi, \hbar) = \sum_{g=0}^{\infty} A_g(\hat{\xi}) \hbar^{2-2g}. \quad (2.1.104)$$

The saddle point of the integral (2.1.96) as  $\hbar \rightarrow \infty$  is then given by

$$\lambda_i = \sum_{j=1}^{n_\Sigma} \frac{C_{ji}}{8\pi^3} \left( \frac{\partial \widehat{F}_0}{\partial T_j} + 4\pi^2 b_j^{NS} \right), \quad i = 1, \dots, g_\Sigma. \quad (2.1.105)$$

It follows from this equation that the 't Hooft parameters are flat coordinates on the moduli space. The frame defined by these coordinates will be called the maximal conifold frame (MCF). The submanifold in moduli space defined by

$$\lambda_i = 0, \quad i = 1, \dots, g_\Sigma, \quad (2.1.106)$$

has dimension  $r_\Sigma$  (the number of mass parameters of the toric CY), and we will call it the maximal conifold locus (MCL). It is a submanifold of the conifold locus of the CY  $X$ . By evaluating the integral (2.1.96) in the saddle-point approximation, we find that the fermionic spectral traces have the following asymptotic expansion in the 't Hooft limit:

$$\log Z(\mathbf{N}; \hbar) \sim \sum_{g \geq 0} \mathcal{F}_g(\boldsymbol{\lambda}) \hbar^{2-2g}. \quad (2.1.107)$$

The leading function in this expansion is given by a Legendre transform,

$$\mathcal{F}_0(\boldsymbol{\lambda}) = J_0^X(\boldsymbol{\zeta}, \boldsymbol{\xi}) - \boldsymbol{\lambda} \cdot \boldsymbol{\zeta}. \quad (2.1.108)$$

If we choose the Batyrev coordinates in such a way that the first  $g_\Sigma$  correspond to true moduli, and the remaining  $r_\Sigma$  correspond to mass parameters, we find

$$\frac{\partial \mathcal{F}_0}{\partial \lambda_i} = -\zeta_i = -\sum_{j=1}^{g_\Sigma} \frac{C_{ij}^{-1}}{2\pi} \left( T_j - \sum_{k=1}^{r_\Sigma} \alpha_{jk} \log \xi_k \right), \quad i = 1, \dots, g_\Sigma, \quad (2.1.109)$$

where  $C^{-1}$  denotes the inverse of the truncated matrix (2.1.74). In view of the results of [77], the higher order corrections  $\mathcal{F}_g(\boldsymbol{\lambda})$  can be computed in a very simple way: the integral (2.1.96) implements a symplectic transformation from the LRF to the MCF. The functions  $\mathcal{F}_g(\boldsymbol{\lambda})$  are precisely the genus  $g$  free energies of the topological string in the MCF.

## 2.2 A concrete example: $Y^{3,0}$ geometry

We will now present a detailed study of a genus two geometry with a mass parameter. In a sense, this showcases all the power of the conjecture in a minimal way: more mass parameters make calculations worse, but there is no additional enlightenment we would achieve. Higher genus in the mirror curve makes calculations *much* worse, and just amounts to a different bound in the sums over the moduli.

We will first focus on how to recover the large  $\hbar$  traces of the operator from the topological string side by using the saddle point expansion of (2.1.96). For this we will need to compute explicitly some  $\mathcal{F}_g$ , and of course some traces of the operator. Afterwards, we will make sense of (2.1.96) in a completely direct way, for fixed, finite  $\hbar$ . We will do this by reinterpreting the formula as a quickly convergent series of Airy integrals. But before that, of course, let us give some details about the geometry and the operators we wish to study.

### 2.2.1 Curve and operator

The generic  $Y^{N,q}$  geometry<sup>6</sup> has been studied in some detail in [78]. The mirror curve is given by

$$a_1 e^p + a_2 e^{-p+(N-q)x} + \sum_{i=0}^N b_i e^{ix} = 0, \quad (2.2.110)$$

where  $p, x$  are variables. As said, to make it maximally interesting while minimally complicated, we will focus on  $N = 3, q = 0$ . Then, the mirror curve has the form

$$a_1 e^p + a_2 e^{-p+3x} + b_3 e^{3x} + b_2 e^{2x} + b_1 e^x + b_0 = 0. \quad (2.2.111)$$

The corresponding charge vectors are

$$Q^1 = (0, 0, 1, -2, 1, 0), \quad Q^2 = (0, 0, 0, 1, -2, 1), \quad Q^3 = (1, 1, -1, 0, 0, -1), \quad (2.2.112)$$

and the Batyrev coordinates are

$$z_1 = \frac{b_3 b_1}{b_2^2}, \quad z_2 = \frac{b_0 b_2}{b_1^2}, \quad z_3 = \frac{a_1 a_2}{b_3 b_0}. \quad (2.2.113)$$

The canonical forms of the mirror curve (2.2.111) are

$$\begin{aligned} b_3 e^x + e^y + e^{-x-y} + b_0 e^{-2x} + b_1 e^{-x} + b_2 &= 0, \\ b_0 e^u + e^y + e^{-u-y} + b_3 e^{-2u} + b_2 e^{-u} + b_1 &= 0, \end{aligned} \quad (2.2.114)$$

where we set  $a_1 = a_2 = 1$ . To obtain the first curve, start from (2.2.111) and set

$$p = y + 2x. \quad (2.2.115)$$

After multiplying the resulting equation by  $e^{-2x}$ , we obtain precisely the first curve in (2.2.114). To obtain the second curve, we multiply (2.2.111) by  $e^{-x}$  and we perform the symplectic transformation from  $(p, x)$  to  $(y, u)$

$$y = 2x - p, \quad u = -x. \quad (2.2.116)$$

Note that both curves are identical after exchanging  $b_1 \leftrightarrow b_2$  and  $b_0 \leftrightarrow b_3$ . This need not be the case for general geometries, of course (see [20] for an analysis of  $\mathbb{C}^3/\mathbb{Z}_6$  where it is clearly not the case). We regard  $b_1, b_2$  as moduli of the curve, and  $b_3, b_0$  as parameters. We can further set one of  $b_0, b_3$  to one by using the remaining  $\mathbb{C}^*$  rescaling freedom, but have we refrained from doing it to make the symmetry between the two canonical forms in (2.2.114) apparent.

The operators  $\mathcal{O}_i^{(0)}$ ,  $i = 1, 2$ , obtained by quantization of the canonical forms are

$$\begin{aligned} \mathcal{O}_1^{(0)} &= b_3 e^x + e^y + e^{-x-y} + b_0 e^{-2x}, \\ \mathcal{O}_2^{(0)} &= b_0 e^u + e^y + e^{-u-y} + b_3 e^{-2u}. \end{aligned} \quad (2.2.117)$$

---

<sup>6</sup>To be precise, we are talking about the resolution of the cone over the  $Y^{N,q}$  singularity.

Both of them can be regarded as perturbations of the operator  $\mathcal{O}_{1,1}$  introduced in (2.1.41), up to an overall normalization. In the first case, we change the normalization of  $x, y$  in such a way that

$$e^x \rightarrow b_3^{-2/3} e^x, \quad e^y \rightarrow b_3^{1/3} e^y, \quad (2.2.118)$$

and then we find

$$b_3 e^x + e^y + e^{-x-y} + b_0 e^{-2x} + b_1 e^{-x} \rightarrow b_3^{1/3} \left( e^x + e^y + e^{-x-y} + b_3 b_0 e^{-2x} + b_1 b_3^{1/3} e^{-x} \right), \quad (2.2.119)$$

Note that, when  $b_1 = 0$ , the operator inside the parentheses is precisely the operator  $\mathcal{O}_{1,1,\xi}$  introduced in (2.1.43), where

$$\xi = b_3 b_0 = \frac{1}{z_3}. \quad (2.2.120)$$

In the second operator the normalization is fixed by requiring

$$e^u \rightarrow b_0^{-2/3} e^u, \quad e^y \rightarrow b_0^{1/3} e^y, \quad (2.2.121)$$

and then we find

$$b_0 e^u + e^y + e^{-u-y} + b_3 e^{-2u} + b_2 e^{-u} \rightarrow b_0^{1/3} \left( e^u + e^y + e^{-u-y} + b_3 b_0 e^{-2u} + b_2 b_0^{1/3} e^{-u} \right), \quad (2.2.122)$$

Note in particular that

$$\begin{aligned} \mathrm{Tr} \left( \rho_1^{(0)} \right)^N &= b_3^{-N/3} \mathrm{Tr} \rho_{1,1,\xi}^N, \\ \mathrm{Tr} \left( \rho_2^{(0)} \right)^N &= b_0^{-N/3} \mathrm{Tr} \rho_{1,1,\xi}^N. \end{aligned} \quad (2.2.123)$$

In the following, we will set

$$\kappa_1 = b_2, \quad \kappa_2 = b_1. \quad (2.2.124)$$

Note that the truncated matrix appearing in (2.1.74) is in this case the Cartan matrix of  $SU(3)$ ,

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad (2.2.125)$$

with inverse

$$C^{-1} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \quad (2.2.126)$$

## 2.2.2 Topological string ingredients

First of all, we will compute the basic building blocks from the topological string side. The prepotential  $F_0$  can be computed from period integrals along the 2  $g_\Sigma$  independent cycles of the CY,

$$\Pi_{A,i} = t_i, \quad \Pi_{B,i} = \frac{\partial F_0}{\partial t_i}, \quad i : 1, \dots, g_\Sigma. \quad (2.2.127)$$

The A periods are precisely the classical mirror maps relating the Kähler parameters of the CY to the moduli of the mirror curve. These period integrals are not computed directly, in any case. Instead, we use the fact that they are annihilated by Picard-Fuchs (PF) operators [17],

$$L_\alpha \Pi_i = 0 \quad (2.2.128)$$

with

$$L_\alpha = \prod_{Q_i^\alpha > 0} \partial_{b_i}^{Q_i^\alpha} - \prod_{Q_i^\alpha < 0} \partial_{b_i}^{-Q_i^\alpha}, \quad (2.2.129)$$

where  $b_i$  are the parameters of the mirror curve, and  $Q_i^\alpha$  the charges that define the CY. It is better to rewrite them in terms of the invariant moduli,

$$z_\alpha = \prod_i b_i^{Q_i^\alpha} \quad (2.2.130)$$

given in (2.2.113). In this coordinates the PF system can always be rewritten in terms of logarithmic derivatives

$$\theta_\alpha = z_\alpha \partial_{z_\alpha}. \quad (2.2.131)$$

Explicitly, we rearrange the derivatives to find

$$\begin{aligned} \theta_{12} &:= 2\theta_2 - \theta_1, \quad \theta_{21} := 2\theta_1 - \theta_2, \\ L_1 &= \theta_{21}(\theta_3 - \theta_1) - z_1 \theta_{12}(1 + \theta_{12}) := \theta_{21} \theta_3 - P_1, \\ L_2 &= L_1 \quad (1 \leftrightarrow 2), \\ L_3 &= \theta_3^2 - z_3(\theta_3 - \theta_1)(\theta_3 - \theta_2). \end{aligned} \quad (2.2.132)$$

One also needs boundary conditions, of course. At large radius (small  $z_\alpha$ ), these are provided by the fact that the Kähler parameters go as

$$t_i \simeq -\log z_i + O(z), \quad (2.2.133)$$

while the B period is fixed by the structure (2.1.78), so that

$$\frac{\partial F_0}{\partial t_i} \simeq \frac{1}{2} \sum_{j,k} a_{ijk} t_i t_j t_k + O(z). \quad (2.2.134)$$

There is an extra operator that follows from the sum of the charge vectors,

$$L_0 = \theta_3^2 - z_1 z_2 z_3 \theta_{12} \theta_{21}, \quad (2.2.135)$$

and that is relevant to completely determine the solutions of the system at some degenerate points of moduli space.

We know, however, that  $z_3$  is just a mass parameter, a deformation of the  $z_3 = 0$  case. It should appear in a simple –i.e. algebraic– way in the periods. In fact, it is direct to see that  $\log z_3$  is an element of the kernel of the operators, so the corresponding large radius Kähler parameter is trivial. Still, the situation is not so

obvious for the B period or for other points in moduli space. Ideally, we would like to rewrite the PF system without  $z_3$  derivatives. With that in mind, let us first isolate the crossed  $\theta_{1,2}\theta_3$  terms on the  $L_{1,2}$  operators.

$$0 = (L_1 + L_2)\Pi = [(\theta_1 + \theta_2)\theta_3 - (P_1 + P_2)]\Pi \implies (\theta_1 + \theta_2)\theta_3\Pi = (P_1 + P_2)\Pi, \quad (2.2.136)$$

where  $\Pi$  is any of the periods. Now we can relate this to the  $L_3$  operator,

$$0 = L_3\Pi = [(1 - z_3)\theta_3^2 - z_3\theta_1\theta_2]\Pi + z_3(\theta_1 + \theta_2)\theta_3\Pi, \quad (2.2.137)$$

$$\theta_3^2\Pi = \frac{z_3}{1 - z_3}(P_1 + P_2 - \theta_1\theta_2)\Pi. \quad (2.2.138)$$

We have expressed the  $z_3$  derivatives in terms of  $\theta_{1,2}$ . In  $L_{1,2}$  we have single derivatives of  $z_3$ , so we rather look at

$$\theta_3 L_1 = \theta_{21}(\theta_3^2) - \theta_1(\theta_{21}\theta_3) - z_1(1 + \theta_{12}\theta_3)(\theta_{12}\theta_3). \quad (2.2.139)$$

Now by the definition of  $P_{1,2}$  in (2.2.132)

$$0 = L_1\Pi = \theta_{21}\theta_3\Pi - P_1\Pi \implies \theta_{21}\theta_3\Pi = P_1\Pi, \quad (2.2.140)$$

and

$$\begin{aligned} 0 &= (z_3 - 1)\theta_3 L_1\Pi = \\ &= [z_3\theta_{21}(P_1 + P_2 - \theta_1\theta_2) + (1 - z_3)(\theta_1 P_1 + z_1(1 + \theta_{12})P_2)]\Pi. \end{aligned} \quad (2.2.141)$$

We get new operators

$$\begin{aligned} \tilde{L}_1 &= z_3\theta_{21}(P_1 + P_2 - \theta_1\theta_2) + (1 - z_3)[\theta_1 P_1 + z_1(1 + \theta_{12})P_2], \\ \tilde{L}_2 &= \tilde{L}_1 \quad (1 \leftrightarrow 2). \end{aligned} \quad (2.2.142)$$

As for  $L_0$ , we can look at  $(1 - z_3)/z_3 L_0$  from which we get

$$\tilde{L}_0 = P_1 + P_2 - \theta_1\theta_2 - z_1 z_2 (z_3 - 1)\theta_{21}\theta_{12}. \quad (2.2.143)$$

The periods calculated with this PF system  $\{\tilde{L}_0, \tilde{L}_1, \tilde{L}_2\}$  at large radius in  $z_{1,2}$  are exact in  $z_3$ .

We can actually get more information about the  $z_3$  structure. Consider

$$\frac{L_3(z_1^{n_1} z_2^{n_2} f(z_3))}{z_1^{n_1} z_2^{n_2}} = (z_3^2 - z_3^3) f''(z_3) + [z_3 + z_3^2(n_1 + n_2 - 1)] f'(z_3) - n_1 n_2 z_3 f(z_3). \quad (2.2.144)$$

If we equate this to zero, it is in fact a hypergeometric equation whose regular solution around  $z_3 = 0$  is

$$f(z_3) = C \cdot {}_2F_1(-n_1, -n_2, 1; z_3). \quad (2.2.145)$$

This is, for any nonnegative  $n_1, n_2$ , a polynomial in  $z_3$  - which also evaluates trivially when  $z_3 = 0$ ,

$${}_2F_1(-n_1, -n_2, 1; 0) = 1. \quad (2.2.146)$$

Consider now the A-period solutions, which around the large radius should be of the form

$$\Pi_{A,i} = -\log z_i + \sum_{n_1, n_2} f_{n_1, n_2}(z_3) z_1^{n_1} z_2^{n_2}. \quad (2.2.147)$$

Since  $L_3(\Pi_{A,i}) = 0$ , the coefficients  $f_{n_1, n_2}(z_3)$  are also solutions of the hypergeometric equation, and we have

$$\Pi_{A,i} = -\log z_i + \sum_{n_1, n_2} C_{n_1, n_2} \cdot {}_2F_1(-n_1, -n_2, 1; z_3) \cdot z_1^{n_1} z_2^{n_2}. \quad (2.2.148)$$

Recall that the hypergeometric function is 1 when  $z_3 = 0$ , so that

$$\Pi_{A,i}|_{z_3=0} = -\log z_i + \sum_{n_1, n_2} C_{n_1, n_2} z_1^{n_1} z_2^{n_2}. \quad (2.2.149)$$

This means we can find the exact coefficients  $f_{n_1, n_2}(z_3)$  of  $z_1, z_2$  (which are polynomials in  $z_3$ ) by solving the PF in the  $z_3 = 0$  case, and taking the  $C_{n_1, n_2}$  from its solution - a much more efficient procedure than directly solving  $\{\tilde{L}_0, \tilde{L}_1, \tilde{L}_2\}$  in the general  $z_3$  case. The same idea applies of course to find the regular part of the B-periods.

One ingredient we will need for the comparison with spectral theory is the B-field appearing in (2.1.88). In this geometry it is given by [56]

$$\mathbf{B} = (0, 0, 1). \quad (2.2.150)$$

This means that in the instanton part of the periods, computed as a power series like above (but not in the log terms), we have to change  $z_3 \rightarrow -z_3$ . Finally, to simplify the complicated algebraic manipulations, in most of what follows we will set  $z_3 = 1$ . That is, *after* inserting the B field. The whole analysis can be repeated without (added) difficulty for other values of  $z_3$ , but the general-value formulas are too cumbersome for a comprehensible presentation. The main difference after all lies in whether  $z_3$  is zero or not, which is the difference between (2.2.117) being relatively simple three-term operators like (2.1.41) or the more interesting perturbed version in (2.1.43).

### 2.2.2.1 Large radius periods

Now on to some concrete computations – do recall that we have inserted the B field. By solving (2.2.132), we get A periods leading as

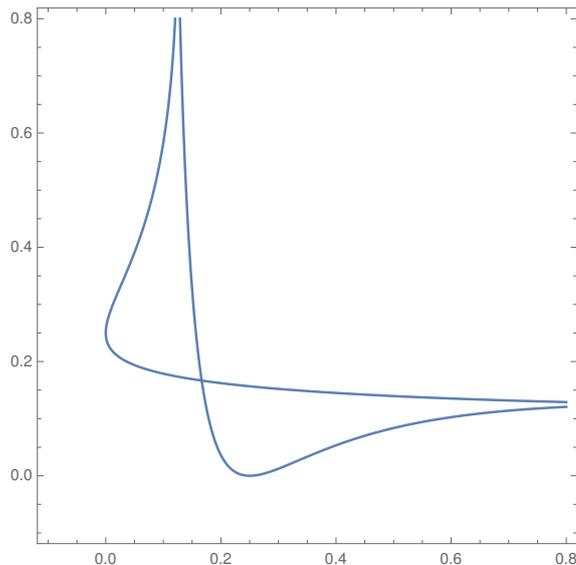
$$\begin{aligned}
t_1 &= -\log(z_1) + (-2z_1 + z_2) + \left(-3z_1^2 + \frac{3z_2^2}{2}\right) + \\
&+ \left(-\frac{20z_1^3}{3} + 2z_1^2z_2 - z_1z_2^2 + \frac{10z_2^3}{3}\right) + \\
&+ \left(-\frac{35z_1^4}{2} + 8z_1^3z_2 - 4z_1z_2^3 + \frac{35z_2^4}{4} + 4z_1^2z_2z_3 - 2z_1z_2^2z_3\right) + \\
&+ \left(-\frac{252z_1^5}{5} + 30z_1^4z_2 - 15z_1z_2^4 + \frac{126z_2^5}{5} + 24z_1^3z_2z_3 - 12z_1z_2^3z_3\right) + O(z)^6, \\
t_2 &= t_1(z_1 \leftrightarrow z_2), \\
t_3 &= -\log(z_3),
\end{aligned} \tag{2.2.151}$$

and B periods as

$$\begin{aligned}
\frac{\partial F_0}{\partial t_1} &= \left(\log^2(z_1) + \log(z_1)\log(z_2) + \frac{1}{2}\log^2(z_2) + \frac{2}{3}\log(z_1)\log(z_3) + \frac{1}{3}\log(z_2)\log(z_3)\right) + \\
&+ (2z_1 + 3\log(z_1)z_1 + \log(z_2)z_1 + \log(z_3)z_1 + \log(z_2)z_2) + \\
&+ \left(7z_1^2 + \frac{9}{2}\log(z_1)z_1^2 + \frac{3}{2}\log(z_2)z_1^2 + \frac{3}{2}\log(z_3)z_1^2 - z_1z_2 + z_2^2 + \frac{3}{2}\log(z_2)z_2^2\right) + O(z)^3, \\
\frac{\partial F_0}{\partial t_2} &= \frac{\partial F_0}{\partial t_1}(z_1 \leftrightarrow z_2).
\end{aligned} \tag{2.2.152}$$

We have solved these equations explicitly as a power series in  $z_3$  with (2.2.132) to show that neither the A periods nor the B periods involve pure functions of  $z_3$ , other than the leading log of the mass period itself. This is important because the reduced,  $z_3$ -exact set of PF equations in (2.2.142) cannot determine a pure  $z_3$  part: any function of only  $z_3$  will be annihilated by both  $\tilde{L}_1$  and  $\tilde{L}_2$ , so we are free to add it to the periods. Thankfully, the solution of the full equations shows us that the  $z_3$  power series is zero, so that we only have to care about the log  $z_3$  leading piece. This pure  $z_3$  part is, or would be, relevant if it was there, and that is the case for the  $\mathbb{C}^3/\mathbb{Z}_6$  geometry in [20]. For instance, it appears in a non-trivial way in the theta function defined by (2.1.95).

Simplicity is also the reason we choose  $z_3 = 1$  (which is really  $z_3 = -1$  before the B-field), in order to remove the log  $z_3$  terms too. Notice, however, that the periods are still a non-trivial function of  $z_3$  in the higher orders of  $z_1$  and  $z_2$ . But now we can use (2.2.142) to get the  $z_3$ -exact solutions for the periods, without worrying about the integration constant.



**Figure 2.1:** Discriminant locus of of  $Y^{3,0}$ ,  $z_3 = 1$ , plotted in  $z_1$  and  $z_2$ .

### 2.2.2.2 Conifold periods

As we saw in the introduction to this section, one of the main consequences of the integral form of the conjecture (2.1.96) is that the traces of the operators should be directly related to the free energies in the maximal conifold frame (MCF) (2.1.107). In the conifold, the geometry develops a singularity. We can see where this happens extracting the discriminant of the mirror curve (B-field included),

$$\begin{aligned}
\Delta(z_1, z_2, z_3) = & 729z_1^4z_2^4 + 729z_1^4z_3^3z_2^4 + 2187z_1^4z_3^2z_2^4 + 2187z_1^4z_3z_2^4 - 972z_1^3z_2^3 + \\
& + 216z_1^2z_2^3 + 1215z_1^3z_3^2z_2^3 - 27z_1^2z_3^2z_2^3 + 243z_1^3z_3z_2^3 - 540z_1^2z_3z_2^3 + \\
& + 216z_1^3z_2^2 + 270z_1^2z_2^2 - 27z_1^3z_3^2z_2^2 - 144z_1z_2^2 - 540z_1^3z_3z_2^2 + \\
& + 513z_1^2z_3z_2^2 - 36z_1z_3z_2^2 + 16z_2^2 - 144z_1^2z_2 + 68z_1z_2 - \\
& - 36z_1^2z_3z_2 + z_1z_3z_2 - 8z_2 + 16z_1^2 - 8z_1 + 1.
\end{aligned} \tag{2.2.153}$$

It vanishes over the locus shown in figure 2.1. In the MCF the A period vanishes at this conifold locus, and the B periods become singular along it. Together they give the MCF prepotential  $\mathcal{F}_0$ .

To find this, first we look for a linear change of variables such that the discriminant becomes factorized at first order (i.e., the variables that describe the tangent lines to the conifold locus at the MCL). We find

$$\begin{aligned}
z_1 &= \frac{1}{2} (3 - \sqrt{5}) u_1 + \frac{1}{2} (-3 - \sqrt{5}) u_2 + \frac{1}{6}, \\
z_2 &= \frac{1}{2} (-3 - \sqrt{5}) u_1 + \frac{1}{2} (3 - \sqrt{5}) u_2 + \frac{1}{6}.
\end{aligned} \tag{2.2.154}$$

Introducing this change in the discriminant, we get indeed

$$\Delta = -\frac{1125}{8}u_1u_2 + \left(-\frac{135}{2}\sqrt{5}u_1^3 + \frac{405}{2}\sqrt{5}u_1^2u_2 + \frac{405}{2}\sqrt{5}u_1u_2^2 - \frac{135}{2}\sqrt{5}u_2^3\right) + O(u)^4. \quad (2.2.155)$$

We know that the A periods should vanish around the conifold, so they must be of the form

$$\sum_{n_1, n_2=0}^{\infty} A_{n_1, n_2} u_1^{n_1} u_2^{n_2}. \quad (2.2.156)$$

The change of variables from  $z_i$  to  $u_i$  is straightforward, and with the PF equations (2.2.142) we find three solutions: two linearly leading in each of  $u_1$  and  $u_2$ , and one whose leading order is quadratic in both. We can constrain the two periods (that must be a combination of these three solutions) by the requirement that they should be vanishing over the discriminant locus. In other words, if we rewrite the discriminant in terms of the A periods via the mirror map, it should factor out both periods at every order. We just need to enforce the condition at next to leading order, and the rest must follow.

We can see that this is obtained for a period leading as

$$A_{1,0} = 1, \quad A_{0,1} = 0, \quad A_{0,2} = \frac{12}{5\sqrt{5}} \quad (2.2.157)$$

which is enough to determine the solution

$$\begin{aligned} \sigma_1 = u_1 + & \left(\frac{112u_1^2}{5\sqrt{5}} - \frac{12u_1u_2}{5\sqrt{5}} + \frac{12u_2^2}{5\sqrt{5}}\right) + \left(\frac{8984u_1^3}{75} - \frac{1212}{25}u_1^2u_2 + \frac{588}{25}u_1u_2^2 + \frac{228u_2^3}{25}\right) + \\ & + \left(\frac{3812896u_1^4}{1125\sqrt{5}} - \frac{221888u_1^3u_2}{125\sqrt{5}} + \frac{1824u_1^2u_2^2}{5\sqrt{5}} + \frac{38112u_1u_2^3}{125\sqrt{5}} + \frac{32544u_2^4}{125\sqrt{5}}\right) + O(u)^5. \end{aligned} \quad (2.2.158)$$

The other linear solution is just obtained by the exchange of the variables,

$$\sigma_2 = \sigma_1(u_1 \leftrightarrow u_2). \quad (2.2.159)$$

The mirror map follows from inverting the series,

$$\begin{aligned} u_1 = \sigma_1 + & \left(-\frac{112\sigma_1^2}{5\sqrt{5}} + \frac{12\sigma_1\sigma_2}{5\sqrt{5}} - \frac{12\sigma_2^2}{5\sqrt{5}}\right) + \left(\frac{29912\sigma_1^3}{375} + \frac{492}{25}\sigma_1^2\sigma_2 - \frac{348}{25}\sigma_1\sigma_2^2 + \frac{1404\sigma_2^3}{125}\right) + \\ & + \left(-\frac{6903584\sigma_1^4}{5625\sqrt{5}} - \frac{843072\sigma_1^3\sigma_2}{625\sqrt{5}} + \frac{370896\sigma_1^2\sigma_2^2}{625\sqrt{5}} + \frac{18128\sigma_1\sigma_2^3}{625\sqrt{5}} - \frac{117376\sigma_2^4}{625\sqrt{5}}\right) + O(\sigma)^5, \\ u_2 = u_1(\sigma_1 \leftrightarrow \sigma_2). \end{aligned} \quad (2.2.160)$$

Introducing these periods into the discriminant we can see that it indeed vanishes whenever  $\sigma_1$  or  $\sigma_2$  are zero.

$$\begin{aligned} \Delta = -\frac{1125}{8}\sigma_1\sigma_2 \left[ 1 + \left(-\frac{136\sigma_1}{5\sqrt{5}} - \frac{136\sigma_2}{5\sqrt{5}}\right) + \right. \\ \left. + \left(\frac{19328\sigma_1^2}{375} + \frac{55876\sigma_1\sigma_2}{125} + \frac{19328\sigma_2^2}{375}\right) + O(\sigma)^3 \right]. \end{aligned} \quad (2.2.161)$$

So  $\sigma_{1,2}$  correspond to the A periods (up to a scale) in the conifold frame.

As for the B periods, they must be singular at the conifold, the branch cuts of which are given as we have just seen by either  $\sigma_1 = 0$  or  $\sigma_2 = 0$ . Then we look for B periods leading as

$$\sigma_i \log \sigma_i + \sum_{n_1, n_2=0}^{\infty} \tilde{B}_{n_1, n_2} \sigma_1^{n_1} \sigma_2^{n_2}. \quad (2.2.162)$$

Just as with the A period, we find a solution leading like  $u \log u$ , but we also find one that leads quadratically. In this case, we can fix it by requiring that the B periods must be the derivatives of a single function of  $\sigma_1, \sigma_2$  (i.e. the conifold prepotential  $\mathcal{F}_0$ ) at the first non-trivial order, as we did with the A periods. Notice that the  $\log \sigma_i$  term will actually generate negative powers of  $u_i$  upon expansion. We also fix to zero the coefficients leading as the A periods. The solutions should be of the form

$$\log u_1 \sum_{n_1, n_2=0}^{\infty} C_{n_1, n_2} u_1^{n_1} \left( \frac{u_2}{u_1} \right)^{n_2} + \sum_{n_1, n_2=0}^{\infty} B_{n_1, n_2} u_1^{n_1} \left( \frac{u_2}{u_1} \right)^{n_2} \quad (2.2.163)$$

Our requirements are met for

$$C_{0,1} = 0, C_{1,0} = 1, B_{1,0} = 0, B_{1,1} = 0, B_{2,0} = \frac{96}{5\sqrt{5}}, \quad (2.2.164)$$

After introducing the mirror map, all negative powers of  $u_i$  should cancel with those coming from the log expansion. Indeed, we get

$$\begin{aligned} s_1 &= \log(\sigma_1) \sigma_1 + \left( -\frac{16\sigma_1^2}{5\sqrt{5}} + \frac{6\sigma_1\sigma_2}{\sqrt{5}} + \frac{3\sigma_2^2}{\sqrt{5}} \right) + \left( \frac{3464\sigma_1^3}{1125} - \frac{2046}{125}\sigma_1^2\sigma_2 - \frac{318}{125}\sigma_1\sigma_2^2 - \frac{682\sigma_2^3}{125} \right) + \\ &+ \left( -\frac{102848\sigma_1^4}{3375\sqrt{5}} + \frac{495232\sigma_1^3\sigma_2}{1875\sqrt{5}} - \frac{7872\sigma_1^2\sigma_2^2}{625\sqrt{5}} - \frac{5248\sigma_1\sigma_2^3}{625\sqrt{5}} + \frac{123808\sigma_2^4}{1875\sqrt{5}} \right) + O(\sigma)^5, \\ s_2 &= s_1(\sigma_1 \leftrightarrow \sigma_2). \end{aligned} \quad (2.2.165)$$

The B periods must lead like these functions, although there could be a linear  $\sigma_i$  term, so we write

$$\frac{\partial \mathcal{F}_0}{\partial \sigma_1} = s_1 + \left( \alpha + \frac{1}{2} \right) \sigma_1 + \beta \sigma_2, \quad \frac{\partial \mathcal{F}_0}{\partial \sigma_2} = s_2 + \left( \alpha + \frac{1}{2} \right) \sigma_2 + \beta \sigma_1, \quad (2.2.166)$$

we integrate and find

$$\begin{aligned} \mathcal{F}_0 &= \frac{\alpha\sigma_1^2}{2} + \frac{1}{2} \log(\sigma_1) \sigma_1^2 + \beta \sigma_1 \sigma_2 + \frac{\alpha\sigma_2^2}{2} + \frac{1}{2} \log(\sigma_2) \sigma_2^2 + \\ &+ \left( -\frac{16\sigma_1^3}{15\sqrt{5}} + \frac{3\sigma_1^2\sigma_2}{\sqrt{5}} + \frac{3\sigma_1\sigma_2^2}{\sqrt{5}} - \frac{16\sigma_2^3}{15\sqrt{5}} \right) + \\ &+ \left( \frac{866\sigma_1^4}{1125} - \frac{682}{125}\sigma_1^3\sigma_2 - \frac{159}{125}\sigma_1^2\sigma_2^2 - \frac{682}{125}\sigma_1\sigma_2^3 + \frac{866\sigma_2^4}{1125} \right) + \\ &+ \left( -\frac{102848\sigma_1^5}{16875\sqrt{5}} + \frac{123808\sigma_1^4\sigma_2}{1875\sqrt{5}} - \frac{2624\sigma_1^3\sigma_2^2}{625\sqrt{5}} - \right. \\ &\quad \left. - \frac{2624\sigma_1^2\sigma_2^3}{625\sqrt{5}} + \frac{123808\sigma_1\sigma_2^4}{1875\sqrt{5}} - \frac{102848\sigma_2^5}{16875\sqrt{5}} \right) + O(\sigma)^6. \end{aligned} \quad (2.2.167)$$

Note that we still have the freedom of rescaling  $\sigma_i$  (and also  $s_i$  as to keep the same leading form for  $F_0$ ).

The first correction to the prepotential has a well known structure, see [26] for the general case or [78] for our particular geometry. Its form is

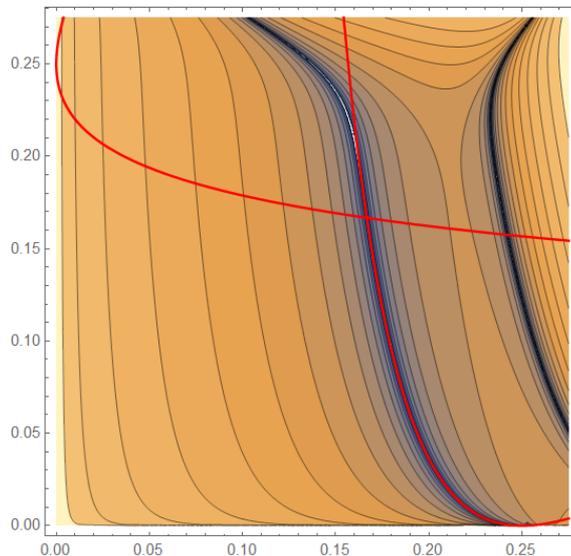
$$\mathcal{F}_1 = -\frac{1}{12} \log(\Delta z_1^a z_2^b) - \frac{1}{2} \log \det \frac{\partial \sigma_i}{\partial z_j}. \quad (2.2.168)$$

with  $a = b = 1/8$  for our geometry. We can use the mirror map (2.2.154) and (2.2.160) to expand it as a function of  $\sigma_1, \sigma_2$ , discarding the constant term which is related to the scale of  $\sigma_i$  through  $\log \Delta$ .

$$\begin{aligned} \mathcal{F}_1 = & -\frac{1}{12} \log(\sigma_1 \sigma_2) + \left( \frac{16\sigma_1}{15\sqrt{5}} + \frac{16\sigma_2}{15\sqrt{5}} \right) + \left( -\frac{146\sigma_1^2}{1125} + \frac{533\sigma_1\sigma_2}{125} - \frac{146\sigma_2^2}{1125} \right) + \\ & + \left( -\frac{9472\sigma_1^3}{16875\sqrt{5}} - \frac{48256\sigma_1^2\sigma_2}{1875\sqrt{5}} - \frac{48256\sigma_1\sigma_2^2}{1875\sqrt{5}} - \frac{9472\sigma_2^3}{16875\sqrt{5}} \right) + \\ & + \left( \frac{291664\sigma_1^4}{46875} + \frac{1548844\sigma_1^3\sigma_2}{46875} - \frac{6703678\sigma_1^2\sigma_2^2}{46875} + \frac{1548844\sigma_1\sigma_2^3}{46875} + \frac{291664\sigma_2^4}{46875} \right) + O(\sigma)^5. \end{aligned} \quad (2.2.169)$$

Further corrections can be found with the holomorphic anomaly of [26]. But that is a whole other story, one we will discuss in deep detail when we come back to quantum mechanics. For the moment, to illustrate the TS/ST conjecture it will suffice with  $\mathcal{F}_1$ .

### 2.2.2.3 Matching with large radius



**Figure 2.2:**  $\log |\lambda_1|$  for  $Y^{3,0}$ ,  $z_3 = 1$ , plotted in  $z_1$  and  $z_2$ .

We mentioned that  $\sigma_i$  were only defined by the PF system up to a multiplicative constant. There is in fact a way of fixing this. The conjecture gives a very precise

relation (2.1.105) between the large radius periods and the parameters of the 't Hooft expansion of the operator traces (2.1.97). Since these parameters are vanishing over the conifold locus, they should be the A periods. From the large radius perspective, they were given by the saddle point of the Airy integral,

$$\lambda_i = \sum_j \frac{C_{ij}}{8\pi^3} \left( \frac{\partial F_0}{\partial t_j} + 4\pi^2 b_j^{NS} \right). \quad (2.2.170)$$

For this geometry, recall we have

$$C_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad (2.2.171)$$

and

$$b^{NS} = \left( -\frac{1}{6}, -\frac{1}{6} \right). \quad (2.2.172)$$

One can evaluate them as a function of the moduli  $z_i$  by computing (2.2.152) to a high order. In figure 2.2 we plot the logarithm of the absolute value for  $\lambda_1$ , with darker regions indicating more negative values – that is,  $\lambda_1$  closer to zero. Superimposed is the conifold locus itself, and we see that indeed a “valley” forms along  $\Delta = 0$ . Because there are only two couples of independent periods in moduli space,  $\lambda_i$  must be a linear combination of the  $\sigma_i$ . Indeed, by direct numerical comparison, one can find that

$$\sigma_i(u_1, u_2) = \frac{\pi^2}{3\sqrt{3}} \lambda_i(z_1, z_2), \quad i = 1, 2. \quad (2.2.173)$$

Then we can rescale the prepotential to get

$$\begin{aligned} \mathcal{F}_0(\lambda_1, \lambda_2) &= \frac{1}{2} \lambda_1^2 \log(\lambda_1) + \frac{1}{2} \lambda_2^2 \log(\lambda_2) + \left( \frac{\alpha \lambda_1^2}{2} + \frac{\alpha \lambda_2^2}{2} + \beta \lambda_1 \lambda_2 \right) - \gamma(\lambda_1 + \lambda_2) \\ &+ \left( -\frac{16\pi^2 \lambda_1^3}{45\sqrt{15}} + \frac{\pi^2 \lambda_1^2 \lambda_2}{\sqrt{15}} + \frac{\pi^2 \lambda_1 \lambda_2^2}{\sqrt{15}} - \frac{16\pi^2 \lambda_2^3}{45\sqrt{15}} \right) \\ &+ \left( \frac{866\pi^4 \lambda_1^4}{30375} - \frac{682\pi^4 \lambda_1^3 \lambda_2}{3375} - \frac{53\pi^4 \lambda_1^2 \lambda_2^2}{1125} + \dots \right) + \\ &+ \left( -\frac{102848\pi^6 \lambda_1^5}{1366875\sqrt{15}} + \frac{222616\pi^6 \lambda_1^4 \lambda_2}{151875\sqrt{15}} - \frac{2624\pi^6 \lambda_1^3 \lambda_2^2}{50625\sqrt{15}} + \dots \right) + \mathcal{O}(\lambda_i^6), \end{aligned} \quad (2.2.174)$$

and for the one loop correction

$$\begin{aligned} \mathcal{F}_1(\lambda_1, \lambda_2) &= -\frac{1}{12} \log(\lambda_1 \lambda_2) + \left( \frac{16\pi^2 \lambda_1}{45\sqrt{15}} + \frac{16\pi^2 \lambda_2}{45\sqrt{15}} \right) + \left( -\frac{146\pi^4 \lambda_1^2}{30375} + \frac{533\pi^4 \lambda_1 \lambda_2}{3375} - \frac{146\pi^4 \lambda_2^2}{30375} \right) \\ &+ \left( -\frac{9472\pi^6 \lambda_1^3}{1366875\sqrt{15}} - \frac{48256\pi^6 \lambda_1^2 \lambda_2}{151875\sqrt{15}} - \frac{48256\pi^6 \lambda_1 \lambda_2^2}{151875\sqrt{15}} - \frac{9472\pi^6 \lambda_2^3}{1366875\sqrt{15}} \right) + \\ &+ \left( \frac{291664\pi^8 \lambda_1^4}{34171875} + \frac{1548844\pi^8 \lambda_1^3 \lambda_2}{34171875} - \frac{6703678\pi^8 \lambda_1^2 \lambda_2^2}{34171875} + \dots \right) + \mathcal{O}(\lambda_i^5) \end{aligned} \quad (2.2.175)$$

### 2.2.3 Computing large $\hbar$ traces

So we would like to see how all of the previous computations can reproduce the standard results from spectral theory. As we saw from (2.1.107), one of the most direct consequences of the conjecture is that the topological string amplitudes  $\mathcal{F}_g$  in the MCF can be obtained from the 't Hooft limit (2.1.97) of the fermionic spectral traces (2.1.27). This is a double scaling limit that takes  $\hbar \rightarrow \infty$ . Therefore, we have an asymptotic expansion as in (2.1.107). When as in our case the genus of the mirror curve is two, we can write

$$\log Z(N_1, N_2; \hbar) = \mathcal{F}(\lambda_1, \lambda_2; \hbar) = \sum_{g=0}^{\infty} \hbar^{2-2g} \mathcal{F}_g(\lambda_1, \lambda_2) , \quad (2.2.176)$$

As we saw explicitly with the computations in the previous sections, the genus  $g$  topological string free energy can be in turn expanded around the MCL  $\lambda_1 = \lambda_2 = 0$ , and one finds

$$\mathcal{F}_g(\lambda_1, \lambda_2) = \mathcal{F}_g^{\log}(\lambda_1, \lambda_2) + \sum_{\substack{i,j=0 \\ (i,j) \neq (0,0)}}^{\infty} \mathcal{F}_{g;i,j} \lambda_1^i \lambda_2^j , \quad (2.2.177)$$

where  $\mathcal{F}_g^{\log}(\lambda_1, \lambda_2)$  contains  $\log \lambda_1, \log \lambda_2$ , and is zero for  $g \geq 1$ . The remaining part is a power series in  $\lambda_1, \lambda_2$ . The numbers  $\mathcal{F}_{g;i,j}$  can be regarded as ‘‘conifold’’ Gromov-Witten invariants. In order to extract these invariants from the fermionic traces, we replace  $\lambda_i$  by  $N_i/\hbar$ . One gets

$$\log Z(N_1, N_2; \hbar) = \mathcal{F}^{\log}(N_1/\hbar, N_2/\hbar) \hbar^{-2} + \sum_{n \geq -1} \hbar^{-n} \sum_{\substack{i+j \leq n+2 \\ i,j \geq 0, (i,j) \neq (0,0) \\ i+j \equiv n \pmod{2}}} \mathcal{F}_{\frac{n-i-j}{2}+1;i,j} N_1^i N_2^j . \quad (2.2.178)$$

Note that since  $\mathcal{F}^{\log}(\lambda_1, \lambda_2) \sim \log(\lambda_i) \lambda_i^2$ , the first term  $\mathcal{F}^{\log}(N_1/\hbar, N_2/\hbar) \hbar^{-2}$  is of the order  $\mathcal{O}(\hbar^0)$ . We find out that in the power series part of  $\log Z(N_1, N_2; \hbar)$ , only a finite number of MCF Gromov-Witten invariants contributes at each order  $\hbar^{-n}$ . Therefore a simple linear algebra calculation allows us to extract the numbers  $\mathcal{F}_{g;i,j}$  from the large  $\hbar$  expansion of  $\log Z(N_1, N_2; \hbar)$ , evaluated with different values of  $N_1, N_2$ . For instance,

$$\begin{aligned} \mathcal{F}_{0;3,0} &= \frac{1}{6} (-2 \log Z(1, 0) + \log Z(2, 0)) \Big|_{\hbar^{-1}} , \\ \mathcal{F}_{1;1,0} &= \frac{1}{6} (8 \log Z(1, 0) - \log Z(2, 0)) \Big|_{\hbar^{-1}} , \\ \mathcal{F}_{0;2,1} &= \frac{1}{2} (\log Z(0, 1) + 2 \log Z(1, 0) - 2 \log Z(1, 1) - \log Z(2, 0) + \log Z(2, 1)) \Big|_{\hbar^{-1}} . \end{aligned} \quad (2.2.179)$$

In the above expressions, we suppress ‘‘ $\hbar$ ’’ in  $Z(N_1, N_2; \hbar)$  to simplify the notation. We use  $|_{\hbar^{-n}}$  to denote the contribution of the power series at the order  $\hbar^{-n}$ . Finally, the fermionic traces are just simple algebraic combinations of standard traces, given by formulas as the ones shown in (2.1.33).

Time to compute them. Take the first curve in (2.2.117) with mass parameter set to one,  $b_0 = b_3 = 1$ , and the perturbation to zero,  $b_1 = 0$ . Its operator version reads

$$(e^{\mathbf{x}} + e^{\mathbf{y}} + e^{-\mathbf{x}-\mathbf{y}} + e^{-2\mathbf{x}}) \Psi = -b_2 \Psi. \quad (2.2.180)$$

The kernel of the operator

$$\rho_{m,n,\xi} = [e^{m\mathbf{u}} + e^{n\mathbf{v}} + e^{-m\mathbf{u}-n\mathbf{v}} + \xi e^{-(1+m)\mathbf{u}-(n-1)\mathbf{v}}]^{-1} \quad (2.2.181)$$

is given in (2.1.66). Since the 't Hooft expansion is a large  $\hbar$  expansion (or a large  $\mathbf{b}$  expansion) we need to get the power series of Faddeev's dilogarithm. However, the kernel features shifts of its argument involving  $\mathbf{b}$  itself, which would need to be resummed in the general expansion. A careful computation shows that

$$\log \Phi_{\mathbf{b}}(x + \delta \mathbf{b}) = \frac{\mathbf{b}^2}{2\pi i} \sum_{k=0}^{\infty} \frac{1}{\mathbf{b}^k} \sum_{j=0}^{k/4} \text{Li}_{2+2j-k}(-e^{2\pi\delta}) \frac{(2\pi)^{k-2j} (-1)^j B_{2j}(\frac{1}{2})}{(k-4j)!(2j)!} x^{k-4j}, \quad (2.2.182)$$

where  $B_n(x)$  is the  $n$ th Bernoulli polynomial.

With such a formula it is algorithmic to expand the kernel as a power series in  $1/\mathbf{b}$ . Recall we are using the  $m = n = 1$  case, with  $\xi = 1$  (corresponding to  $b_0 = b_3 = z_3 = 1$  in the geometry). For simplicity we will rename

$$\rho := \rho_{1,1,1}. \quad (2.2.183)$$

The idea is to use the  $1/\mathbf{b}$  expansion as a saddle point expansion, so that the kernel is evaluated around a Gaussian approximation. We can take advantage of the  $\delta$  parameter in (2.2.182) to force the  $\mathbf{b}$  coefficient of  $\Phi_{\mathbf{b}}$  to vanish. We can do this since a constant shift in all the  $x$  variables is irrelevant for the traces. It is achieved by

$$e^{2\pi\delta} = \frac{1}{2} (1 + \sqrt{5}). \quad (2.2.184)$$

For the first trace we just need the diagonal kernel,

$$\rho(x, x) = e^{\alpha \mathbf{b}^2} e^{-\beta x^2} \sum_{k=0}^{\infty} \frac{\gamma_k(x)}{\mathbf{b}^k} \quad (2.2.185)$$

where  $\gamma_k(x)$  is a polynomial in  $x$  of degree  $3k$ ,

$$\begin{aligned} \alpha &= -\frac{i\text{Li}_2\left(-\frac{1}{2}\sqrt[3]{-1}(1+\sqrt{5})\right)}{2\pi} + \frac{i\text{Li}_2\left(\frac{1}{2}\sqrt[3]{-1}(1+\sqrt{5})\right)}{2\pi} \\ &\quad - \frac{i\text{Li}_2\left(-\frac{1}{2}(-1)^{2/3}(1+\sqrt{5})\right)}{2\pi} + \frac{i\text{Li}_2\left(\frac{1}{2}(-1)^{2/3}(1+\sqrt{5})\right)}{2\pi} + \frac{2}{3}\text{csch}^{-1}(2), \\ \beta &= \frac{\sqrt{15}\pi}{2}, \\ \gamma_1(x) &= \frac{1}{2}\sqrt{3}\pi^2 x^3, \\ \gamma_2(x) &= \frac{1}{24}\left(9\pi^4 x^6 + 2\sqrt{15}\pi^3 x^4 - \sqrt{15}\pi\right), \\ \gamma_3(x) &= \frac{1}{80}\pi^2 x \left(\pi\left(\pi x^2\left(5\pi x^2\left(\sqrt{3}\pi x^2 + 2\sqrt{5}\right) - 44\sqrt{3}\right) - 5\sqrt{5}\right) x^2 + 10\sqrt{3}\right), \\ &\quad \dots \end{aligned} \tag{2.2.186}$$

The integral

$$\int \rho(x, x) dx \tag{2.2.187}$$

is then at every order in  $1/b$  just a gaussian moment integral. In particular, every odd order vanishes, since it always involves odd powers of  $x$  in  $\gamma_i(x)$  - which is what we expect in order to have a  $1/\hbar = 1/b^2$  expansion.

We obtain

$$\begin{aligned} \text{Tr}(\rho) &= \int \rho_{1,1,1}(x, x) dx = \\ &= o(b^0) + \frac{4\pi^2}{75\sqrt{3}b^4} - \frac{116\pi^3}{3375\sqrt{5}b^6} + \frac{452\pi^4}{28125\sqrt{3}b^8} + O(b^{-9}), \\ \text{Tr}(\rho^2) &= \int \rho_{1,1,1}(x, y)\rho_{1,1,1}(y, x) dx dy = \\ &= o(b^0) - \frac{8\pi}{9\sqrt{15}b^2} + \frac{152\pi^2}{675b^4} - \frac{12616\pi^3}{10125\sqrt{15}b^6} + \frac{1575688\pi^4}{2278125b^8} + O(b^{-9}), \\ \text{Tr}(\rho^3) &= \int \rho_{1,1,1}(x, y)\rho_{1,1,1}(y, z)\rho_{1,1,1}(z, x) dx dy dz = \\ &= o(b^0) - \frac{4\pi}{9\sqrt{5}b^2} + \frac{76\pi^2}{225\sqrt{3}b^4} - \frac{16444\pi^3}{30375\sqrt{5}b^6} + \frac{66956\pi^4}{84375\sqrt{3}b^8} + O(b^{-9}). \end{aligned} \tag{2.2.188}$$

The fermionic  $Z(N, 0)$  traces are recovered via (2.1.38),

$$\log Z(1, 0) = o(\hbar^0) + \frac{16\pi^4}{675\hbar^2} - \frac{928\pi^6}{30375\sqrt{15}\hbar^3} + \frac{6592\pi^8}{2278125\hbar^4} + O(\hbar^{-5}), \tag{2.2.189}$$

$$\log Z(2, 0) = o(\hbar^0) - \frac{32\pi^2}{15\sqrt{15}\hbar} + \frac{4424\pi^4}{10125\hbar^2} - \frac{1075264\pi^6}{455625\sqrt{15}\hbar^3} + \frac{122231744\pi^8}{102515625\hbar^4} + O(\hbar^{-5}). \tag{2.2.190}$$

$$\log Z(3, 0) = o(\hbar^0) - \frac{128\pi^2}{15\sqrt{15}\hbar} + \frac{7648\pi^4}{3375\hbar^2} - \frac{309088\pi^6}{16875\sqrt{15}\hbar^3} + \frac{464070848\pi^8}{34171875\hbar^4} + O(\hbar^{-5}). \quad (2.2.191)$$

Then we can use (2.2.178) to recover the expansion of the conifold free energy  $\mathcal{F}$ . For instance, at order  $1/\hbar$ ,

$$\begin{aligned} \mathcal{F}_{0;3,0} + \mathcal{F}_{1;1,0} &= 0, \\ 8\mathcal{F}_{0;3,0} + 2\mathcal{F}_{1;1,0} &= -\frac{32\pi^2}{15\sqrt{15}}, \end{aligned} \quad (2.2.192)$$

and so on for  $1/\hbar^n$ . Higher orders can be obtained without any conceptual complication (although the process becomes burdensome) to find similar equations. The first few terms are

$$\begin{aligned} \mathcal{F}_{0;3,0} &= -\frac{16\pi^2}{45\sqrt{15}}, & \mathcal{F}_{1;1,0} &= \frac{16\pi^2}{45\sqrt{15}}, \\ \mathcal{F}_{0;4,0} &= \frac{866\pi^4}{30375}, & \mathcal{F}_{1;2,0} &= -\frac{146\pi^4}{30375}, \\ \mathcal{F}_{0;5,0} &= -\frac{102848\pi^6}{1366875\sqrt{15}}, & \mathcal{F}_{1;3,0} &= -\frac{9472\pi^6}{1366875\sqrt{15}}, \\ \mathcal{F}_{0;6,0} &= \frac{5527264\pi^8}{307546875}, & \mathcal{F}_{1;4,0} &= \frac{291664\pi^8}{34171875}. \end{aligned} \quad (2.2.193)$$

These are the coefficients we obtained in (2.2.174) and (2.2.175) for  $\mathcal{F}_0(\lambda_1, \lambda_2 = 0)$  and  $\mathcal{F}_1(\lambda_1, \lambda_2 = 0)$ .

Just notice what this means. **The period integrals over the CY know about the large  $\hbar$  expansion of the spectral traces.** It is a very distinct prediction: other than the original physical insight of the Fermi gas formulation of ABJM theory [42], there is was no previous reason why these two different fields of mathematics should be related at all.

It is true that in taking the 't Hooft limit we have neglected the NS free energy in (2.1.92). But still this is a very non-trivial test of the conjecture. The generating function (2.1.92) is built out of two basic blocks, the NS and the standard free energies. That the NS free energy generates the small  $\hbar$  (or WKB) expansion of related spectral problems is a well known fact [34], and should not be surprising for us here. What we do when taking the 't Hooft expansion is a computation at *large*  $\hbar$ , where the NS sector is exponentially suppressed and but instead the standard topological string provides relevant contributions. For those who might still have doubts, that the interplay between both leads to a well defined spectral determinant at finite  $\hbar$  will be verified in the next section, with the Airy function method.

For the moment, we can extract a bit more from this analysis. The conjecture gives other information than operator traces. After all, there is nothing special about the  $\lambda_2 = 0$  slice from the point of view of the geometry. It is for that reason that we introduced the generalized determinant (2.1.25). We will now use it to look at mixed traces of two operators. The operator we were working with can be rewritten as

$$e^{\mathbf{x}} + e^{\mathbf{y}} + e^{-\mathbf{x}-\mathbf{y}} + e^{-2\mathbf{x}} + b_1 e^{-\mathbf{x}} = \mathbf{A}_{11} + b_1 \mathbf{P}_{11}. \quad (2.2.194)$$

For simplicity we were denoting  $\rho = \mathbf{A}_{11}$ . The generalized determinant reads

$$\Xi(b_i) = \det(1 + b_2\rho + b_1\rho e^{-\mathbf{x}}) := \det(1 + b_2\rho + b_1\tilde{\rho}). \quad (2.2.195)$$

The kernel for  $\tilde{\rho}$  comes next. First, let us reexpress the operator  $\mathbf{x}$  in terms of the kernel coordinates (2.1.45),

$$\mathbf{x} = 2\pi\mathbf{b}\frac{\mathbf{p} + \mathbf{q}}{3}, \quad (2.2.196)$$

With this we can obtain the action of  $\mathbf{P}_{11}$  over a vector in the position representation,

$$\hat{x}|x\rangle = x|x\rangle, \quad (|x\rangle)^\dagger = \langle x|, \quad (2.2.197)$$

$$\begin{aligned} \int dx f(x) \langle x| e^{-\mathbf{x}} &= \int dx f(x) \langle x| e^{-2\pi\mathbf{b}\frac{\mathbf{p}+\mathbf{q}}{3}} = e^{-\frac{\pi i}{9}\mathbf{b}^2} \int dx f(x) \langle x| e^{-\frac{2\pi\mathbf{b}\mathbf{q}}{3}} e^{-\frac{2\pi\mathbf{b}\mathbf{p}}{3}} = \\ &= e^{-\frac{\pi i}{9}\mathbf{b}^2} \int dx f\left(x + \frac{i\mathbf{b}}{3}\right) e^{-\frac{2\pi\mathbf{b}x}{3}} \langle x|. \end{aligned} \quad (2.2.198)$$

This means that the kernel for  $\tilde{\rho}$  is just a shifted version of the  $\rho$  kernel in (2.1.66). It explicitly reads

$$\tilde{\rho}(x, y) = e^{-\frac{\pi i}{9}\mathbf{b}^2} e^{-\frac{2\pi\mathbf{b}y}{3}} \rho\left(x, y + \frac{i\mathbf{b}}{3}\right). \quad (2.2.199)$$

As opposed to the expression we had for  $\rho$ , this is not so manifestly self-adjoint. However, the topological string side was perfectly symmetrical in  $\lambda_1$  and  $\lambda_2$ . This means that both kernels actually encode the same spectral information. Let us compute the traces for a finite value of  $\hbar$  such as  $2\pi$ , corresponding to  $\mathbf{b} = \sqrt{3}$ . First we look at the normal, unperturbed  $\rho$ . Using the quasi-periodicity of Faddeev's dilogarithm,

$$\frac{\Phi_{\mathbf{b}}(x + i\mathbf{b}^{-1})}{\Phi_{\mathbf{b}}(x)} = \frac{1}{1 - q_- e^{2\pi\mathbf{b}^{-1}(x - c_{\mathbf{b}})}}, \quad (2.2.200)$$

with

$$q_- = e^{2\pi i\mathbf{b}^{-2}}, \quad c_{\mathbf{b}} = \frac{i(\mathbf{b} + \mathbf{b}^{-1})}{2}, \quad (2.2.201)$$

we see that

$$\begin{aligned} \rho_{1,1,1}(x, y)|_{\hbar=2\pi} &= -\frac{e^{\frac{2\pi(x+y)}{\sqrt{3}}} \operatorname{sech}\left(\frac{\pi(x-y+\frac{i}{2\sqrt{3}})}{\sqrt{3}}\right)}{2\sqrt{3}\left(e^{\frac{2\pi x}{\sqrt{3}}} + 1\right)\left((-1)^{1/3}e^{\frac{2\pi x}{\sqrt{3}}} + 1\right)\left((-1)^{2/3}e^{\frac{2\pi y}{\sqrt{3}}} - 1\right)} \times \\ &\times \frac{\Phi_{\mathbf{b}}(x)\Phi_{\mathbf{b}}\left(y + \frac{i}{2\sqrt{3}}\right)}{\Phi_{\mathbf{b}}(y)\Phi_{\mathbf{b}}\left(x + \frac{i}{2\sqrt{3}}\right)}, \end{aligned} \quad (2.2.202)$$

which in the diagonal case gives  $\text{Tr}(\rho) = 1/18$ , as we found in (2.1.71). Let us do the same derivation for  $\tilde{\rho}$ , with (2.2.200) and (2.2.199). We get the general kernel

$$\begin{aligned} \tilde{\rho}(x, y)|_{\hbar=2\pi} = & - \frac{(-1)^{2/3} e^{\frac{2}{3}\pi(\sqrt{3}x+i)} \text{sech}\left(\frac{\pi(x-y-\frac{i}{2\sqrt{3}})}{\sqrt{3}}\right)}{2\sqrt{3} \left(e^{\frac{2\pi x}{\sqrt{3}}} + 1\right) \left((-1)^{1/3} e^{\frac{2\pi x}{\sqrt{3}}} + 1\right) \left((-1)^{2/3} e^{\frac{2\pi y}{\sqrt{3}}} + 1\right)} \times \\ & \times \frac{\Phi_{\mathbf{b}}(x) \Phi_{\mathbf{b}}\left(y + \frac{i}{2\sqrt{3}}\right)}{\Phi_{\mathbf{b}}(y) \Phi_{\mathbf{b}}\left(x + \frac{i}{2\sqrt{3}}\right)}, \end{aligned} \quad (2.2.203)$$

and its diagonal slice

$$\tilde{\rho}(x, x)|_{\hbar=2\pi} = - \frac{(\sqrt{3} - i) \left(\tanh\left(\frac{\pi x}{\sqrt{3}}\right) - 1\right)}{12 \left(\sqrt{3} + 2i \sinh\left(\frac{2\pi x}{\sqrt{3}}\right)\right)}. \quad (2.2.204)$$

Even if it looks a bit more threatening, its integral is also 1/18. Again, this is what one expects from the  $\lambda_1, \lambda_2$  symmetry on the topological string side.

In fact one can write any combination of these kernels in a manifestly real form with a just a bit of manipulation. Take  $\rho_j \in \{\rho, \tilde{\rho}\}$  and define the shift

$$\delta_j = \begin{cases} 0 & \text{iff } \rho_j = \rho, \\ \frac{i\mathbf{b}}{6} & \text{iff } \rho_j = \tilde{\rho}. \end{cases} \quad (2.2.205)$$

First notice how the interaction term that appears in the denominator of the kernel is self-adjoint,

$$I(x, y) = 2\mathbf{b} \cosh\left(\pi \frac{x - y + i\hbar}{\mathbf{b}}\right) = \bar{I}(y, x). \quad (2.2.206)$$

Then, a generic trace can be written as

$$\begin{aligned} \text{Tr}(\rho_1 \dots \rho_n) &= \int \prod_{j=1}^n dx_j \frac{\bar{f}(x_{j+1}) f(x_j + 2\delta_j)}{I(x_{j+1}, x_j + 2\delta_j)} e^{4\pi i \delta_j^2} e^{4\pi i x_j \delta_j} = \\ &= \int \prod_{j=1}^n dx_j \frac{\bar{f}(x_j - \delta_j) e^{4\pi i x_j \delta_j} f(x_j + \delta_j)}{I(x_{j+1} - \delta_{j+1}, x_j + \delta_j)} = \\ &= \int \prod_{j=1}^n dx_j \frac{\overline{f(x_j + \delta_j) e^{4\pi i x_j \delta_j} f(x_j + \delta_j)}}{\left[ I(x_{j+1} - \delta_{j+1}, x_j + \delta_j) I(x_j - \delta_j, x_{j+1} + \delta_{j+1}) \right]^{1/2}} = \\ &= \int \prod_{j=1}^n dx_j \frac{e^{4\pi i x_j \delta_j} |f(x_j + \delta_j)|^2}{|I(x_{j+1} - \delta_{j+1}, x_j + \delta_j)|}, \end{aligned} \quad (2.2.207)$$

where  $x_{j+1} := x_1$ . This makes explicit that the traces, even when mixing both operators, must be real. It is also useful for computing the higher order traces, since the integral is again split into relatively simple interaction terms and ‘‘potential’’

terms that factorize over the  $x_i$ . This is important to make an efficient computation since the most intricate details (and in particular, the dilogarithm) are located in those terms.

We now perform the  $1/\mathbf{b}$  expansion of the crossed terms. For the  $\tilde{\rho}$  terms, we must include the imaginary shift  $\delta_j$  in the quantum dilogarithm. As in the  $\rho$  case, we want the  $1/\mathbf{b}$  series to be a saddle-point expansion around a Gaussian point. A real shift is then needed to cancel the  $\mathbf{b}$  coefficient of  $\Phi_{\mathbf{b}}$  that is linear in  $p$ . We achieve this by a shift (2.2.182)

$$e^{2\pi\delta} = \frac{1}{2} \left( -1 + \sqrt{5} \right) e^{2\pi\frac{i}{6}}. \quad (2.2.208)$$

The traces of  $\tilde{\rho}$  are computed in exactly the same way as we did for (2.2.185),

$$\begin{aligned} \mathrm{Tr}(\tilde{\rho}) &= o(\mathbf{b}^0) + \frac{4\pi^2}{75\sqrt{3}\mathbf{b}^4} - \frac{116\pi^3}{3375\sqrt{5}\mathbf{b}^6} + \frac{452\pi^4}{28125\sqrt{3}\mathbf{b}^8} + O(\mathbf{b}^{-9}), \\ \mathrm{Tr}(\tilde{\rho}^2) &= o(\mathbf{b}^0) - \frac{8\pi}{9\sqrt{15}\mathbf{b}^2} + \frac{152\pi^2}{675\mathbf{b}^4} - \frac{12616\pi^3}{10125\sqrt{15}\mathbf{b}^6} + \frac{1575688\pi^4}{2278125\mathbf{b}^8} + O(\mathbf{b}^{-9}), \end{aligned} \quad (2.2.209)$$

and as the symmetry on the topological string side told us, of course they are the same results as for  $\rho$ . The interesting part are the mixed traces

$$\begin{aligned} \mathrm{Tr}(\tilde{\rho}\rho) &= o(\mathbf{b}^0) - \frac{2\pi}{5\sqrt{15}\mathbf{b}^2} + \frac{26\pi^2}{375\mathbf{b}^4} - \frac{1468\pi^3}{5625\sqrt{15}\mathbf{b}^6} + \frac{47242\pi^4}{421875\mathbf{b}^8} + O(\mathbf{b}^{-9}), \\ \mathrm{Tr}(\tilde{\rho}\rho^2) &= o(\mathbf{b}^0) - \frac{58\pi}{225\sqrt{5}\mathbf{b}^2} + \frac{1138\pi^2}{5625\sqrt{3}\mathbf{b}^4} - \frac{17968\pi^3}{50625\sqrt{5}\mathbf{b}^6} + \frac{10393358\pi^4}{18984375\sqrt{3}\mathbf{b}^8} + O(\mathbf{b}^{-9}), \\ \mathrm{Tr}(\tilde{\rho}^2\rho) &= o(\mathbf{b}^0) - \frac{58\pi}{225\sqrt{5}\mathbf{b}^2} + \frac{1138\pi^2}{5625\sqrt{3}\mathbf{b}^4} - \frac{17968\pi^3}{50625\sqrt{5}\mathbf{b}^6} + \frac{10393358\pi^4}{18984375\sqrt{3}\mathbf{b}^8} + O(\mathbf{b}^{-9}). \end{aligned} \quad (2.2.210)$$

From the general Liouville formula (2.1.38),

$$\log Z(1, 1) = o(\hbar^0) + \frac{2\pi^2}{\sqrt{15}\hbar} - \frac{166\pi^4}{675\hbar^2} + \frac{25216\pi^6}{30375\sqrt{15}\hbar^3} - \frac{591676\pi^8}{2278125\hbar^4} + O(\hbar^{-5}), \quad (2.2.211)$$

$$\log Z(2, 1) = \log Z(1, 2) = o(\hbar^0) + \frac{58\pi^2}{15\sqrt{15}\hbar} - \frac{14506\pi^4}{10125\hbar^2} + \frac{4444448\pi^6}{455625\sqrt{15}\hbar^3} + O(\hbar^{-4}). \quad (2.2.212)$$

In the same way we used (2.2.178) to extract the  $\mathcal{F}_{g;i,0}$ , we get a series of linear equations on the mixed terms. For instance, from the first  $\hbar$  order of  $Z(1, 1)$  we get

$$0 = \mathcal{F}_{0;0,3} + \mathcal{F}_{0;1,2} + \mathcal{F}_{0;2,1} + \mathcal{F}_{0;3,0} + \mathcal{F}_{1;0,1} + \mathcal{F}_{1;1,0} - \frac{2\pi^2}{\sqrt{15}}. \quad (2.2.213)$$

We already have the values of  $\mathcal{F}_{0;0,i}$  and  $\mathcal{F}_{1;0,i}$  in (2.2.193). Therefore the first non-trivial mixed term we have in  $\mathcal{F}_0$  is  $\mathcal{F}_{0;1,2}$  and  $\mathcal{F}_{0;2,1}$ . We need another equation, so we turn our attention to  $Z(1, 2)$  (equivalently,  $Z(2, 1)$ ) who gives

$$0 = 8\mathcal{F}_{0;0,3} + 4\mathcal{F}_{0;1,2} + 2\mathcal{F}_{0;2,1} + \mathcal{F}_{0;3,0} + 2\mathcal{F}_{1;0,1} + \mathcal{F}_{1;1,0} - \frac{58\pi^2}{15\sqrt{15}} \quad (2.2.214)$$

to get

$$\mathcal{F}_{0;1,2} = \mathcal{F}_{0;2,1} = \frac{\pi^2}{\sqrt{15}}. \quad (2.2.215)$$

This is precisely the coefficient  $\lambda_1^2 \lambda_2$  of  $\mathcal{F}_0$  in (2.2.174). Higher  $Z(N_1, N_2)$  give more terms in  $\mathcal{F}_0$  and  $\mathcal{F}_1$ . Again, the moral is that the topological string theory contains the traces of the mixing of the operator with its perturbation.

## 2.2.4 Computing exact $\hbar$ traces

These are very strong analytical tests of (2.1.96), and in particular of the not at all trivial non-perturbative completion for the asymptotic WKB part of the spectral determinant. But as we already mentioned, in taking the  $\hbar \rightarrow \infty$  limit, we are effectively discarding the  $F_{NS}$  terms in (2.1.92), who are just the ones responsible for the WKB expansion. This was because we are also rescaling the Kähler parameters, following (2.1.102). Recall from (2.1.85) that the NS free energy is written as series in

$$e^{-t} = \exp\left(-\frac{T\hbar}{2\pi}\right), \quad (2.2.216)$$

where we keep  $T$  constant for the 't Hooft limit. In particular, this means that all contributions coming from  $F_{NS}$  are exponentially small in  $\hbar \rightarrow \infty$ , and they cannot be seen in the  $1/\hbar$  expansions we have considered in the previous section.

But there is nothing about (2.1.96) that makes it intrinsically perturbative. The goal of this section is to evaluate it at finite values of  $\hbar$ , to obtain *exact* results. For that, we need the ingredients of (2.1.92) as an exact function of  $q$ . There are several ways one can go about this. First, notice that (2.1.81) and (2.1.85) are already resummed in  $\hbar$ , with  $g_s = 4\pi^2/\hbar$ . One can compute the Gopakumar-Vafa  $n_g^d$  and the refined  $N_{JL, JR}^d$  by expanding those free energies in a perturbative series and the matching order by order with the perturbative computation offered by the holomorphic anomaly. Even if it requires quite a bit of setup, this is eventually an efficient way if one wishes to write (2.1.81) up to a high order in  $e^{-t}$ .

Another option is to use the refined topological vertex of [19], which gives both  $F_{NS}$  and  $F^{GV}$  directly as exact functions of  $\hbar$  and  $g_s$ . However, as opposed to the holomorphic anomaly, the topological vertex produces a series that is perturbative in  $e^{-t}$ . For instance, the standard free energy for this geometry is

$$\begin{aligned} F^{GV}(t_i, g_s) = & \frac{2q^2(Q_1 + Q_2)}{(q^2 - 1)^2} + \frac{q^4 Q_1^2 + q^4 Q_2^2 + 2(q^3 + q)^2 Q_1 Q_2}{(q^4 - 1)^2} + \\ & + \frac{2q^6(Q_1^3 + Q_2^3)}{3(q^6 - 1)^2} - \frac{q^2 Q_1 Q_2 Q_3}{(q^2 - 1)^2} + O(Q^4), \quad q = e^{ig_s/2}, \end{aligned} \quad (2.2.217)$$

while in the NS limit

$$F_{NS}^{inst}(t_i, \hbar) = \frac{(q^2 + 1)(Q_1 + Q_2)}{q^2 - 1} + \frac{(q^4 + 1)Q_1^2 + (q^4 + 1)Q_2^2 + 4(q^2 + 1)^2 Q_1 Q_2}{4(q^4 - 1)} + \\ + \frac{(q^6 + 1)(Q_1^3 + Q_2^3) - 9(q^5 + q^3 + q)Q_1 Q_2 Q_3}{9(q^6 - 1)} + O(Q^4), \quad q = e^{i\hbar/2}, \quad (2.2.218)$$

with

$$Q_i = e^{-t_i}. \quad (2.2.219)$$

As a remainder, when writing (2.1.92), one must evaluate  $F^{GV}$  with  $\hbar \mapsto 4\pi^2/\hbar$  and  $t \mapsto 2\pi/t$ .

The topological vertex has provided the free energies as functions of the Kähler parameters. On the other hand, both the spectral problem and (2.1.96) are written in terms of the moduli of the mirror curve, one being related to the other via (2.1.73). The A periods we computed in (2.2.151) give the classical (order  $\hbar^0$ ) relation between  $t_i$  and  $z_i$ . The quantum corrections  $\mathbf{t}(\hbar)$  needed for (2.1.92) can be obtained as an exact function of  $\hbar$  with the method of [18] mentioned when we set up the conjecture. We will show an explicit example of its use. Let us write the operator corresponding to the first parametrization in (2.2.114), with  $b_3 = 1$ , acting on a wavefunction  $\Psi(x)$ ,

$$X\Psi(X) + \Psi(q^2 X) + \frac{1}{qX}\Psi(q^{-2}X) + \frac{b_0}{X^2}\Psi(X) + \frac{b_1}{X}\Psi(X) + b_2\Psi(X) = 0, \quad (2.2.220)$$

where we have introduced  $X = e^x$  for convenience, and  $q$  is still  $\exp(i\hbar/2)$ . Notice that

$$e^{x+p} = q^{-1} e^x e^p \quad (2.2.221)$$

from the BakerCampbellHausdorff formula. Define the quotient of the wavefunction with its displacement as

$$V(X) = \gamma \frac{\Psi(q^2 X)}{\Psi(X)}. \quad (2.2.222)$$

With a bit of rescaling, setting  $\gamma = (z_1^2 z_2 z_3)^{1/3}$ , the equation can be written in terms of the moduli

$$V(X) + \frac{z_2 z_3 z_1}{qX V(q^{-2}X)} + \frac{z_2}{X^2} + Xz_1 + \frac{1}{X} + 1. \quad (2.2.223)$$

One can now find the function  $V(X)$  as a power series in any of the moduli. For instance, define

$$r(X) = 1 + \frac{1}{X} + Xz_1, \quad s(X) = \frac{1}{X^2}, \quad t(X) = \frac{q z_1 z_3}{X}. \quad (2.2.224)$$

In this way the equation is manifestly recursive,

$$V(X) + r(X) = z_2 \left[ s(X) + \frac{t(X)}{V(q^{-2}X)} \right]. \quad (2.2.225)$$

The leading order of  $V(X)$  is clearly  $-r(X)$ , and we can use the ansatz

$$V(X) = -r(X) \exp\left(\sum_{n=1}^{\infty} v_n(X) z_2^n\right). \quad (2.2.226)$$

Every  $v_n(X)$  is simply an algebraic combination of the previous ones. For instance,

$$v_1(X) = \frac{s(X) - \frac{t(X)}{r(q^{-2}X)}}{r(X)} \quad (2.2.227)$$

and so on. Then, following [18], the wavefunction near infinity should behave as

$$\Psi(x) \propto \exp\left(\frac{\Pi_A(z_i, \hbar)}{\hbar} x\right) \quad (2.2.228)$$

where  $\Pi_A(z_i, \hbar)$  is a *quantum* period relating the Kähler parameters and the moduli at finite  $\hbar$ . In other words, using (2.2.222) for the quotient of wavefunctions,

$$\Pi_A(z_i, \hbar) = \text{Res} \frac{\log(\gamma^{-1} V(X))}{X} \Big|_{X=0}. \quad (2.2.229)$$

We chose our ansatz to give directly the expansion in  $z_2$  of the period, so

$$\begin{aligned} \Pi_A(z_i, \hbar) = & -\frac{1}{3} \log(z_1^2 z_2 z_3) \text{Res} \frac{1}{X} \Big|_{X=0} + \text{Res} \frac{\log(-r(X))}{X} \Big|_{X=0} + \\ & + \sum_{n=1}^{\infty} \text{Res} \frac{v_n(X)}{X} \Big|_{X=0} z_2^n, \end{aligned} \quad (2.2.230)$$

and finally

$$\Pi_A(z_i, \hbar) = -\frac{1}{3} \log(z_1^2 z_2 z_3) - z_2 + \frac{1}{2} z_2^2 \left( \frac{2z_1(q^2 z_3 + q + z_3)}{q} - 3 \right) + O(z^3). \quad (2.2.231)$$

From the symmetry of the problem, it is obvious that there is another period

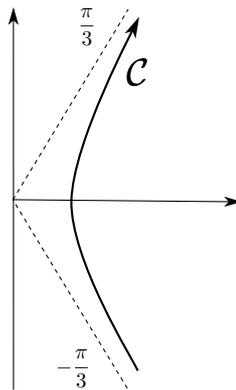
$$\Pi'_A = \Pi_A(z_1 \leftrightarrow z_2), \quad (2.2.232)$$

as well as the trivial mass parameter period  $t_3 = -\log z_3$ . Finally, to get the right quantum mirror maps, we just need to ensure that they are linear combinations leading as prescribed in (2.1.77). For instance, re-expanding in  $z_1$  and  $z_2$ ,

$$\begin{aligned} t_1(z_i, \hbar) = & 2 \Pi'_A - \Pi_A + \frac{1}{3} \log z_3 = -\log z_1 - 2z_1 + z_2 + \frac{3z_2^2}{2} - 3z_1^2 + \\ & + \frac{(q^2 + 1) z_1 (2z_1 - z_2) z_2 z_3}{q} + \frac{1}{3} (10z_2^3 - 20z_1^3 + 6z_2 z_1^2 - 3z_2^2 z_1) + O(z)^3. \end{aligned} \quad (2.2.233)$$

which is, order by order, exact in  $z_3$  and  $q$ . Of course, it just reduces to the classical period (2.2.151) when  $q = 1$ .

With this we are ready to evaluate (2.1.96). On the one hand, since we have the integral kernels of the operators  $A_1, A_2$  associated to the  $Y^{3,0}$  geometry, we can directly compute the fermionic traces  $Z(N_1, N_2; \hbar)$ . In particular, the kernels  $A_1(x, y) = \rho(x, y)$  and  $A_2(x, y)$  given in (2.2.199) are tremendously simplified when  $\hbar$  is a rational multiple of  $2\pi$ . On the other, as pointed in [21, 23], (2.1.96) gives a convenient way to compute  $Z(N_1, N_2; \hbar)$  in terms of Airy functions. Let us review the Airy function method and generalize it slightly.



**Figure 2.3:** The integration contour  $\mathcal{C}$  used in the computation of  $Z(\mathbf{N}; \boldsymbol{\xi}, \hbar)$ .

We first separate  $J(\boldsymbol{\mu}; \boldsymbol{\xi}, \hbar)$  into the perturbative part  $J^{(p)}$ , which is a cubic polynomial in  $\mu$ , and the non-perturbative part  $J^{(np)}$ , which is a power series in  $e^{-\mu}$ . The formula (2.1.96) can be separated as

$$Z(\mathbf{N}; \boldsymbol{\xi}, \hbar) = \int_{\mathcal{C}} e^{J^{(p)} + J^{(np)} - \sum_{i=1}^{g_{\Sigma}} N_i \mu_i} \prod_{i=1}^{g_{\Sigma}} \frac{d\mu_i}{2\pi i}. \quad (2.2.234)$$

Here the integration path  $\mathcal{C}$  for each  $d\mu_i$  has been deformed so that it asymptotes to  $e^{\pm i\pi/3}\infty$ , as seen in Fig. 2.3. Let us ignore the non-perturbative contributions for the moment, and let us define

$$Z^{(p)}(\mathbf{N}; \boldsymbol{\xi}, \hbar) = \int_{\mathcal{C}} e^{J^{(p)} - \sum_{i=1}^{g_{\Sigma}} N_i \mu_i} \prod_{i=1}^{g_{\Sigma}} d\mu_i. \quad (2.2.235)$$

When  $g_{\Sigma} = 1$ , this is nothing else but an Airy function, since the coefficients of the cubic terms are triple intersection numbers of the geometry and must therefore be real and positive. In the case of  $g_{\Sigma} = 2$ , we can always find a linear combination of  $\mu_{1,2}$

$$\nu_i = \sum_{j=1}^2 K_{ij} \mu_j, \quad i = 1, 2 \quad (2.2.236)$$

such that  $J^{(p)}$  has the following form

$$J^{(p)} = \sum_{i=1}^2 \left( \frac{C_i}{3} \nu_i^3 + D_i \nu_i^2 + B_i \nu_i \right) + \zeta \nu_1 \nu_2 + A, \quad (2.2.237)$$

where the coefficient  $C_i$  is proportional to  $\hbar^{-1}$  and positive, while  $D_i, B_i, A$  are functions of the mass parameters  $\xi_j$  and  $\hbar$  (the coefficient  $\zeta$  should not be confused with other occurrences of the same symbol in this paper, in (2.1.98) and (2.1.47)). The Jacobian must not vanish, i.e.

$$\left( \frac{\partial \mu_i}{\partial \nu_j} \right) \neq 0 \quad (2.2.238)$$

for the integral measure in (2.2.234) to be still meaningful after the coordinate transformation. In addition, we define  $M_i$  by

$$\sum_{i=1}^2 \mu_i N_i = \sum_{i=1}^2 \nu_i M_i. \quad (2.2.239)$$

If  $\zeta = 0$ , the quadratic mixing term is absent, and  $Z^{(p)}$  splits to a product of Airy functions

$$\begin{aligned} Z^{(p)}(\mathbf{N}; \boldsymbol{\xi}, \hbar) &= \left( \frac{\partial \mu_i}{\partial \nu_j} \right) e^A \prod_i (C_i)^{-1/3} \exp \left( D_i C_i^{-1} (M_i - B_i + \frac{2}{3} D_i^2 C_i^{-1}) \right) \\ &\quad \times \text{Ai} \left( (C_i)^{-1/3} (M_i - B_i + D_i^2 C_i^{-1}) \right). \end{aligned} \quad (2.2.240)$$

This is the scenario discussed for instance in [21]. Furthermore, the non-perturbative term can be expanded as a power series in the moduli  $z_i$ ,

$$e^{J^{(np)}} = \sum_{k_i \geq 0} P_{\mathbf{k}}(\boldsymbol{\mu}; \boldsymbol{\xi}, \hbar) \prod_{i=1}^{n_\Sigma} z_i^{k_i}, \quad (2.2.241)$$

where  $P_{\mathbf{k}}(\boldsymbol{\mu}; \boldsymbol{\xi}, \hbar)$  is a polynomial in  $\mu_i$ . Since by (2.1.73)

$$z_i = \exp \left\{ - \sum_j C_{ij} \mu_j + \text{mass parameters} \right\}, \quad (2.2.242)$$

the powers of the moduli will effectively act as a shift in the

$$\exp \left\{ - \sum_i N_i \mu_i \right\} \quad (2.2.243)$$

term of the Airy integral. Finally, the polynomial terms in  $\mu$ ,  $P_{\mathbf{k}}$ , can be generated by taking derivatives w.r.t. the  $N_i$ . In short,

$$Z(\mathbf{N}; \boldsymbol{\xi}, \hbar) = \sum_{k_i \geq 0} \left( \prod_{\ell} \xi_{\ell}^{-\sum_i k_i \alpha_{i\ell}} \right) P_{\mathbf{k}}(-\partial_{\mathbf{N}}; \boldsymbol{\xi}, \hbar) \circ Z^{(p)}(\mathbf{N} + \mathbf{k} \cdot C; \boldsymbol{\xi}, \hbar). \quad (2.2.244)$$

Here  $P_{\mathbf{k}}(-\partial_{\mathbf{N}}; \boldsymbol{\xi}, \hbar)$  is the differential operator obtained by replacing variable  $\mu_i$  by  $-\partial_{N_i}$  in the polynomial  $P_{\mathbf{k}}(\boldsymbol{\mu}; \boldsymbol{\xi}, \hbar)$ , and we use  $\circ$  to denote its action on  $Z^{(p)}(\{N_i + \sum_j k_j C_{ji}\}; \boldsymbol{\xi}, \hbar)$ .

If however the mixing term  $\zeta\nu_1\nu_2$  is present,  $Z^{(p)}(\mathbf{N}; \boldsymbol{\xi}, \hbar)$  cannot be split to a product of conventional Airy functions. Instead we need to expand  $\exp(\zeta\nu_1\nu_2)$ , and replace each term in the expansion by a differential operator that acts on the  $Z^{(p)}$  with the mixing term removed, which now can be written as a product of conventional Airy functions, just like what one has done for the terms in  $J^{(np)}$ . In compact form, one finds

$$\begin{aligned} Z^{(p)}(\mathbf{N}; \boldsymbol{\xi}, \hbar) &= \\ &= \left( \frac{\partial \mu_i}{\partial \nu_j} \right) e^A \exp(\zeta \partial_{M_1} \partial_{M_2}) \circ \left( \prod_i (C_i)^{-1/3} \exp \left\{ (D_i C_i^{-1} (M_i - B_i + \frac{2}{3} D_i^2 C_i^{-1})) \right\} \times \right. \\ &\quad \left. \times \text{Ai} \left\{ (C_i)^{-1/3} (M_i - B_i + D_i^2 C_i^{-1}) \right\} \right). \end{aligned} \tag{2.2.245}$$

The formula for computing  $Z(\mathbf{N}; \boldsymbol{\xi}, \hbar)$  in terms of derivatives of  $Z^{(p)}(\mathbf{N}; \boldsymbol{\xi}, \hbar)$  is the same. In actual computations, both the expansion of  $\exp(\zeta\nu_1\nu_2)$  and the expansion of  $e^{J^{(np)}}$  have to be truncated, which greatly constrains the precision of the calculations one can reach.

Fortunately for the geometry  $Y^{3,0}$ , when  $\xi = 1$ , which is the only situation that we will consider here, the quadratic mixing term is absent (again, see [20] to see how one would deal with it for general mass parameter). The perturbative grand potential  $J^{(p)}$  is

$$J^{(p)}(\boldsymbol{\mu}; \hbar) = \frac{1}{12\pi\hbar} (8\mu_1^3 - 3\mu_1^2\mu_2 - 3\mu_1\mu_2^2 + 8\mu_2^3) + \left( \frac{\pi}{3\hbar} - \frac{\hbar}{12\pi} \right) (\mu_1 + \mu_2), \tag{2.2.246}$$

and it splits with the following change of variables

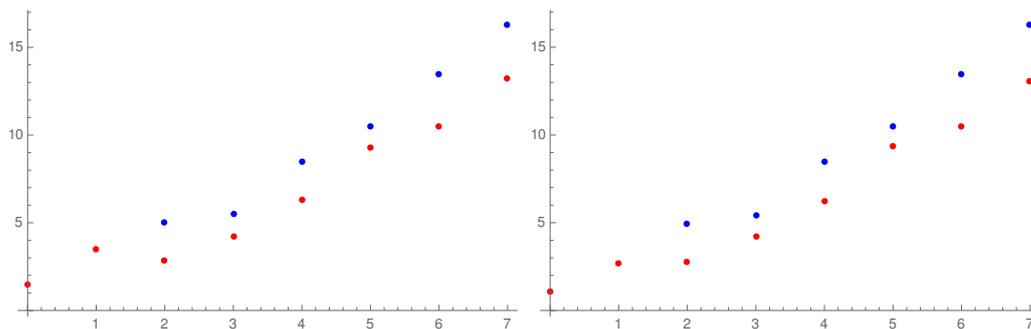
$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{7-3\sqrt{5}}{2} \\ \frac{7-3\sqrt{5}}{2} & 1 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}. \tag{2.2.247}$$

We will consider two particular cases. The first is the maximal supersymmetric case where  $\hbar = 2\pi$ . The first fermionic trace  $Z(1, 0)$  is identified with the trace of  $\rho_{1,1,\xi}$ , which is given by (2.1.69). When  $\xi = 1$ , one has from (2.1.71),

$$Z(1, 0; \xi = 1, \hbar = 2\pi) = \frac{1}{18}, \tag{2.2.248}$$

and by using the relation (2.1.38) between the fermionic traces and the spectral traces, we find in addition,

$$Z(2, 0; \xi = 1, \hbar = 2\pi) = \frac{108 - 19\sqrt{3}\pi}{1296\sqrt{3}\pi}. \tag{2.2.249}$$



**Figure 2.4:** Airy expansion vs. exact spectral traces. Relative matching digits of  $Z(1,0)$  [left], and  $Z(2,0)$  [right] associated to  $Y^{3,0}$  with  $\xi = 1$ , plotted against the order of  $e^{-\mu}$  instanton corrections included in the Airy expansion. The blue and the red dots correspond to  $\hbar = 2\pi$  and  $\hbar = 4\pi/3$  respectively.

We also use the Airy function method to compute these fermionic traces, including up to seven orders of instanton corrections in the modified grand potential  $J(\boldsymbol{\mu}; \xi = 1, \hbar = 2\pi)$ . To measure the degree of agreement between the results of the Airy function method and those using the integral kernel, we define the relative matching degree between two numbers  $x$  and  $y$  to be

$$-\log_{10} \left| \frac{x - y}{y} \right|. \quad (2.2.250)$$

Roughly speaking it gives the number of identical digits between  $x$  and  $y$ . We plot in Figs. 2.4 in blue dots the relative matching degrees between the Airy function results and the analytic results against the order of instanton corrections used in the former method. One finds very good agreement, and it improves consistently when more instanton contributions are included.

The second case is  $\hbar = 4\pi/3$  so that  $\mathbf{b} = \sqrt{2}$ . The integral kernel also simplifies. We find

$$\begin{aligned} \rho_{1,1,\xi=1}(x, x) &= \frac{1}{2\mathbf{b} \cos(\pi/6)} e^{\frac{4}{3}\pi\mathbf{b}x} \frac{\Phi_{\mathbf{b}}(x + \frac{i}{3}\mathbf{b}^{-1}) \Phi_{\mathbf{b}}(x + \frac{2i}{3}\mathbf{b}^{-1})}{\Phi_{\mathbf{b}}(x - \frac{i}{3}\mathbf{b}^{-1}) \Phi_{\mathbf{b}}(x - \frac{2i}{3}\mathbf{b}^{-1})} \\ &= \frac{1}{2\mathbf{b} \cos(\pi/6)} \frac{e^{\frac{4}{3}\pi\mathbf{b}x}}{1 + \sqrt{3}e^{2\pi\mathbf{b}^{-1}x} + e^{4\pi\mathbf{b}^{-1}x}}, \end{aligned} \quad (2.2.251)$$

so that the trace is

$$Z(1,0; \xi = 1, \hbar = 4\pi/3) = \text{Tr} \rho_{1,1,\xi=1} \left( \hbar = \frac{4}{3}\pi \right) = \frac{\cos \frac{\pi}{9} - \sqrt{3} \sin \frac{\pi}{9}}{3}. \quad (2.2.252)$$

It is also possible to compute the double spectral trace by numerical integration, from which one gets the second fermionic trace, and it reads

$$Z(2,0; \xi = 1, \hbar = 4\pi/3) = 0.003565431804217254350 \dots \quad (2.2.253)$$

Similarly these fermionic traces can be computed by the Airy function method. We give the plots of the relative matching degrees in Figs 2.4 in red dots, with the order of instanton corrections used in the Airy function method increased up to seven. Once again one finds very good agreement between the two results.

To sum up, the conjectural formula (2.1.96) allows us to reproduce the exact spectral information of certain difference operators from related topological string theories. That is, not only it matches the results perturbatively in  $\hbar$ , as known from WKB, or in  $1/\hbar$ , as we saw in the previous section. It also does so for finite, concrete values. This is remarkable, since the  $\hbar$  expansions of spectral traces are only asymptotic in  $\hbar$ . So are  $F_{NS}$  and  $F_{top}$  from (2.1.92) in their respective parameters,  $\hbar$  and  $g_s$ . Even when using a resummed form, such as the ones in (2.2.218) and (2.2.217) the problems are manifest because both expressions have poles over a dense set of points in the  $\hbar$  real line – namely, rational multiples of  $\pi$ . It is only through the combination of *both* that a sensible answer can be formulated. That is, it was key to include *non-perturbative* terms in the form of

$$\widehat{F}_{top} \left( \frac{2\pi t}{\hbar}, \frac{4\pi^2}{\hbar} \right) = \sum_n c_n(\hbar) e^{-\frac{2\pi t}{\hbar}}. \quad (2.2.254)$$

There is one last interesting thing to say about (2.1.96). From the spectral theory point of view, the  $\mathbf{N}$  parameters are evidently non-negative integers. However, there is nothing keeping us from evaluating the integral for arbitrary complex values of  $\mathbf{N}$ , where it is still a perfectly well behaved function [22]. That is, the Airy integral not only reproduces the spectral theory traces, but also serves as a natural analytic continuation for them.



## Chapter 3

# The refined holomorphic anomaly

The main lesson of the previous chapter is that refined topological string theory encodes *all* the spectral information of a certain class of operators. This is not a completely new story, of course. For example, the Nekrasov–Shatashvili limit of  $\mathcal{N} = 2$  gauge theories is equivalent to the all-orders WKB quantization of certain quantum integrable systems. By using the Nekrasov instanton partition function [79], one can then obtain new, exact quantization conditions for these systems. Studies of this connections between quantum mechanics and refined topological strings/gauge theory have been made in for example [18, 35, 80–85]. In particular the all-orders WKB coefficients of the quantum mechanical problems are directly given by quantum periods on mirror curve of the topological string, where the refined string coupling is identified with the quantum mechanical  $\hbar$ .

One of the most powerful tools to recover the genus expansion of the topological string theory (or equivalently, the  $\hbar$  expansion of the WKB coefficients for the related systems) is the so-called holomorphic anomaly. The original holomorphic anomaly equations were first introduced in [26] to describe the non-holomorphic dependence of standard topological string amplitudes on the Kähler parameters. It turns out that these equations constrain these amplitudes to a large extent. In some local CY geometries, the holomorphic anomaly equations, combined with modularity and appropriate boundary conditions, can be used to compute recursively (and efficiently) the topological string free energies at all genera [86, 87]. The equations of [26] can be extended to refined topological string theory and to  $\mathcal{N} = 2$  gauge theories [27, 73]. As we noted above, the refined NS limit of certain such theories gives the WKB expansion for certain quantum mechanical problems.

We know very well that there is a whole set of interesting information in spectral problems that cannot be accessed from perturbation theory. The archetypal example is of course quantum mechanical tunnelling through a barrier. Closer to the string side, in the previous chapter we saw the key importance of including non-perturbative terms in order to have a well defined spectral theory. It is then natural to wonder if we could recover anything of the sort in quantum mechanics with the tools we already have for stringy theories. Yet, by the very perturbative nature of string theory, the holomorphic anomaly was constructed as a recursion that returns

higher  $\hbar$  corrections, order by order.

However, the anomaly itself is a consequence of a very basic requirement on the string amplitudes [26]. For the simplified genus one case, their classical part, the prepotential (related to the leading WKB coefficient) is given by the two independent periods on the associated curve. They form a vectorial representation of the modular group  $SL(2, \mathbb{Z})$  – we can exchange them, or add them together, without affecting the underlying geometry. Then the holomorphic anomaly follows from demanding the amplitudes to be invariant under such modular transformations.

There is nothing in that statement that is inherently perturbative. If we extend our perturbative ansatz to include non-perturbative *transseries* reflecting exponentially small corrections, it would be logical to demand from them as well to remain modular invariant. In fact, the standard anomaly recursion governing the conventional topological string has been already generalized non-perturbatively in [28]. This has led for example to a very precise determination of the large order behavior of the genus expansion for topological string theories on toric CY such as we studied in the previous chapter. It has also been used in [31] to provide a semiclassical decoding of the conjecture (2.1.96).

Following those ideas, in this chapter we will develop a *transseries* formalism to apply the holomorphic anomaly to quantum mechanical problems. We will do so by constructing in parallel an explicit demonstration of the method’s power when applied to a spectral problem. This example will be the modified Mathieu potential, that has been well studied in [56]. The choice is motivated by its direct connection to Seiberg-Witten theory [79]. The perturbative WKB corrections to the periods of the (modified) Mathieu potential are in fact obtained from the Nekrasov-Shatashvili limit of the refined SW free energies [34, 79, 88]. These NS free energies can in turn be computed with a refined version of the holomorphic anomaly of [26], introduced in [27]. In choosing this example we will constrain ourselves to curves with genus one for simplicity, although the machinery of the holomorphic anomaly has been generalized to higher genus curves, see [70].

Let us insist that it is well established that the *perturbative* WKB periods of the Mathieu potential can be computed with the holomorphic anomaly, through its connection to SW theory. Our own goal here is twofold: to go *beyond* perturbation, and to prepare the ground for applying the holomorphic anomaly to quantum mechanical problems *unrelated* to stringy scenarios.

Another word of warning: this is not at all to say this is the first work to consider the non-perturbative structure of quantum mechanical spectra, not even in the context of the Mathieu potential. For instance, the resummation of the asymptotic series of the Mathieu energies as a function of the *energy level* was taken care of in [83]. That kind of expansion can be obtained, for instance, from standard perturbation theory around a harmonic oscillator. What we present here as new results is concretely the non-perturbative and large order study of the WKB asymptotic series. This is a different story from the one in [83] – the WKB coefficients are functions of the *energy* itself, and have a radically distinct divergent behaviour.

The chapter is structured as follows. First, we will show how to obtain the per-

turbative WKB periods in an explicit way. Then, we will write the general framework for computing their non-perturbative corrections. We will use it to perform several checks of resurgent behaviour. And finally, we will see how we can recover a previously known exact quantization condition purely *within* our framework.

### 3.1 The modified Mathieu potential

The modified Mathieu equation describes a one-dimensional, quantum mechanical particle in a  $\cosh(x)$  potential. The classical Hamiltonian is

$$H(p, x) = p^2 + 2 \cosh(x), \quad (3.1.1)$$

and the corresponding Schrödinger equation is

$$\left( -\hbar^2 \frac{d^2}{dx^2} + 2 \cosh(x) - E \right) \psi(x) = 0. \quad (3.1.2)$$

The spectral problem associated to the modified Mathieu equation leads to an infinite number of discrete energy levels, labelled by an integer  $n = 0, 1, 2, \dots$ . These energy levels can be determined by using the all-orders WKB method. First, recall the WKB wavefunction ansatz,

$$\psi(x) = \exp \left[ \frac{i}{\hbar} \int^x Q(x') dx' \right]. \quad (3.1.3)$$

that gives the Riccati equation

$$Q^2(x) - i\hbar \frac{\partial Q(x)}{\partial x} = p^2(x), \quad (3.1.4)$$

The WKB computation becomes perturbative when we take the  $Q$  function as a series,

$$Q(x) = \sum_{k=0}^{\infty} Q_k(x) \hbar^k. \quad (3.1.5)$$

Then, the Riccati equation turns into an algebraic recursion

$$\begin{aligned} Q_0(x) &= p(x), \\ Q_{n+1}(x) &= \frac{1}{2Q_0(x)} \left( i \frac{\partial Q_n(x)}{\partial x} - \sum_{k=1}^n Q_k(x) Q_{n+1-k}(x) \right). \end{aligned} \quad (3.1.6)$$

One can see that in fact  $Q$  splits into a total derivative and a function of  $\hbar^2$ ,

$$Q(x) = P(x) + \frac{i\hbar}{2} \frac{\partial}{\partial x} \log P(x). \quad (3.1.7)$$

Indeed,  $P(x)$  satisfies, from the Riccati equation,

$$P(x)^2 - p(x)^2 = \hbar^2 \frac{2 P(x) P''(x) - 3 P'(x)^2}{4 P(x)^2}, \quad (3.1.8)$$

which is clearly worse from the point of view of resolution, but shows explicitly that  $P(x)$  can be written a series in  $\hbar^2$ ,

$$P(x) = \sum_{n=0}^{\infty} P_n(x) \hbar^{2n}. \quad (3.1.9)$$

The first couple of terms for modified Mathieu are

$$\begin{aligned} P_0(x) &= Q_0(x) = p(x) = \sqrt{E - 2 \cosh x}, \\ P_1(x) &= Q_2(x) = \frac{9 - 4E \cosh(x) - \cosh(2x)}{16 (E - 2 \cosh x)^{5/2}}. \end{aligned} \quad (3.1.10)$$

The classical part of the phase space volume can be computed as the integral,

$$\begin{aligned} \text{vol}_0(E) &= \oint_{\text{turn. points}} P_0(x) dx = 4 \int_0^{x_+} \sqrt{E - 2 \cosh x} dx = \\ &= 8\sqrt{E+2} \left[ \mathbf{K} \left( \frac{E-2}{E+2} \right) - \mathbf{E} \left( \frac{E-2}{E+2} \right) \right], \end{aligned} \quad (3.1.11)$$

where  $\mathbf{K}$ ,  $\mathbf{E}$  denote the complete elliptic integrals of the first and the second kind, as a function of the squared elliptic modulus  $k^2$ . The integral is taken around the classically allowed trajectory that goes between the turning points given by  $V(x_{\pm}) = E$ ,

$$x_{\pm} = \pm \cosh^{-1} \left( \frac{E}{2} \right). \quad (3.1.12)$$

This classical volume determines the approximate Bohr-Sommerfeld quantization,

$$\text{vol}_0(E_n^{\text{classic}}) \approx 2\pi\hbar n \quad (3.1.13)$$

by quantizing the phase space volume in units of  $\hbar$ . As we motivated in the introduction, the *all-orders* WKB quantization condition promotes it to a *quantum* volume,

$$\text{vol}(E_n, \hbar) = \oint_{\text{turn. points}} P(x) dx = 2\pi\hbar \left( n + \frac{1}{2} \right), \quad (3.1.14)$$

The  $1/2$  factor comes from the integral of total derivative in  $Q(x)$  around the classical cycle, since  $\log P(x)$  has a branch cut and gains a  $2\pi i$  factor after a complete turn.

The all-orders WKB coefficients that follow from (3.1.6) give an asymptotic expansion for the quantum volume appearing in (3.1.14). It is possible to calculate the very first orders of this expansion by using brute force integration. One finds, for example, at the next-to-leading order,

$$\text{vol}_1(E) = \oint_{\text{turn. points}} dx P_1(x) = \frac{2 \mathbf{K} \left( \frac{2E}{E+2} - 1 \right) - E \mathbf{E} \left( \frac{2E}{E+2} - 1 \right)}{6(E-2)\sqrt{2+E}}, \quad (3.1.15)$$

where the all-orders *perturbative* WKB volume is expanded as

$$\text{vol}(E, \hbar) = \text{vol}_0(E) + \hbar^2 \text{vol}_1(E) + O(\hbar^4). \quad (3.1.16)$$

Now, the Riemann curve given by

$$H(x, p) = 0 \tag{3.1.17}$$

being genus one, it has another independent period. By the translation symmetry of the problem, it is associated to a cycle that goes around the imaginary axis, and is given by

$$\begin{aligned} \mathbf{a}(E, \hbar) &= \frac{1}{2\pi i} \int_{-\pi i}^{\pi i} dx \left( \sqrt{E - 2 \cosh x} + \sum_{n=1}^{\infty} \hbar^{2n} P_n(x) \right) = \\ &= \sum_{n=0}^{\infty} \mathbf{a}_n(E) \hbar^{2n}. \end{aligned} \tag{3.1.18}$$

The classical limit of this period will be called  $\mathbf{a}_0(E)$ , and can also be integrated explicitly,

$$\mathbf{a}_0(E) = \frac{2\sqrt{E}}{\pi} \sqrt{1 + \frac{2}{E}} \mathbf{E} \left( \frac{4}{2 + E} \right). \tag{3.1.19}$$

Just like in topological string theory, we can use this to define a prepotential  $\mathcal{F}_0^{(0)}$ ,

$$\frac{\partial \mathcal{F}_0^{(0)}(\mathbf{a}_0)}{\partial \mathbf{a}_0} = \text{vol}_0(E(\mathbf{a}_0)). \tag{3.1.20}$$

The naming of the periods has been chosen in accordance to SW theory. Indeed, one can expand and invert  $\mathbf{a}_0(E)$  and insert it in this definition. Integrating, the prepotential<sup>1</sup> at large  $\mathbf{a}_0$  is

$$\mathcal{F}_0^{(0)}(\mathbf{a}_0) = 4\mathbf{a}_0^2 \log(2\mathbf{a}_0) - 6\mathbf{a}_0^2 - \frac{1}{2\mathbf{a}_0^2} - \frac{5}{64\mathbf{a}_0^6} - \frac{3}{64\mathbf{a}_0^6} + \mathcal{O}(\mathbf{a}_0^{-10}) \tag{3.1.21}$$

which is that of SW [88]. With the change of variables

$$p^2 \mapsto \frac{2p^2}{1-x^2} - \frac{1-x^2}{x}, \quad x \mapsto \log x \tag{3.1.22}$$

one can see that the curve  $H(x, p) = 0$  can in fact be written in the SW form

$$p^2 = (x - u)(x^2 - 1), \tag{3.1.23}$$

after identifying

$$E = 2u. \tag{3.1.24}$$

One might wonder why are we even interested in this other period, from the point of view of WKB. We will find several answers to that question – it actively participates in the large order behaviour of the volumes, and it generates the non-perturbative corrections in the exact quantization condition – but the most straightforward is that both periods are related by the Riemann bilinear relations. It will

<sup>1</sup>The (0) superscript means that it is part of the *perturbative* sector in  $\hbar$  – a notation whose purpose will be clear when we deal with transseries corrections.

be explained more carefully when dealing with other quantum mechanical problems, but the fact is that knowledge of the free energies allows us to recover the energy as a function of the quantum volume – which, by the WKB condition, should be basically the energy level times  $2\pi\hbar$ . Save for the fact that the series is asymptotic, this allows us to write  $E_n$  as a function of the level  $n$ . In the case of Seiberg-Witten, this connection is the famous Matone relation [89], in its full quantum version [90],

$$E - 2 = \frac{\hbar}{16} + \frac{\hbar^3}{4} \partial_{\hbar} \left[ \frac{\mathcal{F}_D}{\hbar^2} \right], \quad (3.1.25)$$

where  $\mathcal{F}_D$  is the free energy obtained when we integrate  $\mathbf{a}$  w.r.t. the quantum volume,

$$\frac{\partial \mathcal{F}_{D,0}^{(0)}}{\partial \text{vol}_0} = -\mathbf{a}_0. \quad (3.1.26)$$

For the moment, do not worry much more about it – we will come back to it later, when dealing with the quantization condition.

Written in this language, we can easily get the differential equation the periods must satisfy, i.e. the Picard-Fuchs equation. There is a rather systematic way of finding them by ansatz. Consider the following differential operator

$$L = \alpha_2(E) \partial_{EE} + \alpha_1(E) \partial_E + 1. \quad (3.1.27)$$

The classical periods are integrals of the classical momentum,

$$p(x) = \sqrt{E - 2 \cosh x}. \quad (3.1.28)$$

The operator  $L$  will annihilate a period if it produces something that vanishes when integrated over a cycle – that is, a total derivative. In order to match the powers of  $e^x$ , we look for

$$L p(x) = \partial_x \left[ \frac{\beta_{-1}(E) e^{-x} + \beta_0(E) + \beta_1(E) e^x}{p(x)} \right]. \quad (3.1.29)$$

This is solved by

$$\begin{aligned} \alpha_1 &= 0, \quad \alpha_2 = 4(E^2 - 4), \\ \beta_{-1} &= 2, \quad \beta_0 = 0, \quad \beta_1 = 2. \end{aligned} \quad (3.1.30)$$

If  $\Pi(E)$  is a period over some cycle,

$$L \Pi(E) = \oint L p(x) dx = \oint \partial_x (\dots) dx = 0. \quad (3.1.31)$$

In short, the Picard-Fuchs equation for (3.1.1) is

$$\left[ 16 \left( \left( \frac{E}{2} \right)^2 - 1 \right) \partial_{EE} + 1 \right] \Pi(E) = 0. \quad (3.1.32)$$

It is easy to see that both (3.1.19) and (3.1.11) satisfy this equation. One can derive similar equations for higher  $\hbar$  corrections to the periods in a similar way, although it quickly becomes a computationally difficult task. Interestingly, at higher orders one will need higher derivatives for the PF operator. This is because the higher order corrections are derivatives of the classical periods. For instance, with the expressions we already have in (3.1.11) and (3.1.15),

$$\text{vol}_1(E) = \left( -\frac{1}{24} \partial_E + \frac{E}{48(E^2 - 4)} \right) \text{vol}_0(E), \quad (3.1.33)$$

and so on for  $\text{vol}_n$ . Since both periods are integrals of the same underlying differential and satisfy the same Picard-Fuchs equations, we also have

$$\mathbf{a}_1(E) = \left( -\frac{1}{24} \partial_E + \frac{E}{48(E^2 - 4)} \right) \mathbf{a}_0(E). \quad (3.1.34)$$

## 3.2 The perturbative holomorphic anomaly

### 3.2.1 Free energies and recursion

In a similar fashion to the prepotential, we can define an all-orders free energy  $\mathcal{F}^{(0)}$  by

$$\frac{\partial \mathcal{F}^{(0)}(\mathbf{a}, \hbar)}{\partial \mathbf{a}} = \text{vol}(E(\mathbf{a}, \hbar), \hbar) = \sum_{n=0}^{\infty} \mathcal{F}_n^{(0)}(\mathbf{a}) \hbar^{2n}. \quad (3.2.35)$$

Notice also that we have defined it as a function of the *quantum* period  $\mathbf{a}$ . In particular, the leading term is just the prepotential evaluated at  $\mathbf{a}(E, \hbar)$  (as opposed to  $\mathbf{a}_0$ ). This means that if we want to recover the quantum volume, we would need both the  $\mathcal{F}_n^{(0)}(\mathbf{a}, \hbar)$  and the quantum  $\mathbf{a}(E, \hbar)$  period,

$$\begin{aligned} \text{vol}(E, \hbar) &= \left. \frac{\partial \mathcal{F}^{(0)}(\mathbf{a}, \hbar)}{\partial \mathbf{a}} \right|_{\mathbf{a}=\mathbf{a}(E, \hbar)} = \partial_{\mathbf{a}_0} \mathcal{F}_0^{(0)}(\mathbf{a}_0(E)) + \\ &+ \hbar^2 \left[ \partial_{\mathbf{a}_0} \mathcal{F}_1^{(0)}(\mathbf{a}_0(E)) + \mathbf{a}_1(E) \partial_{\mathbf{a}_0 \mathbf{a}_0} \mathcal{F}_0^{(0)}(\mathbf{a}_0(E)) \right] + O(\hbar^4). \end{aligned} \quad (3.2.36)$$

who are both  $\hbar$  expansions. For convenience, we define a “corrected” energy  $\xi(\mathbf{a})$ , by

$$\xi(\mathbf{a}(E, \hbar)) = E. \quad (3.2.37)$$

This means we can write the *quantum*  $\mathbf{a}$  period as

$$\mathbf{a}(E, \hbar) = \mathbf{a}_0(\xi) = \frac{2\sqrt{\xi}}{\pi} \sqrt{1 + \frac{2}{\xi}} \mathbf{E} \left( \frac{4}{2 + \xi} \right). \quad (3.2.38)$$

and the prepotential as a function of the quantum  $\mathbf{a}$ ,

$$\partial_{\mathbf{a}} \mathcal{F}_0^{(0)}(\mathbf{a}) \Big|_{\mathbf{a}=\mathbf{a}(E, \hbar)} = 8\sqrt{\xi + 2} \left[ \mathbf{K} \left( \frac{\xi - 2}{\xi + 2} \right) - \mathbf{E} \left( \frac{\xi - 2}{\xi + 2} \right) \right]. \quad (3.2.39)$$

In this language, the first correction is given by [27, 73]

$$\mathcal{F}_1^{(0)}(\mathbf{a}) = -\frac{1}{24} \log \Delta(\xi). \quad (3.2.40)$$

This is a general feature of the first WKB free energy, as we will see when we study boundary conditions for the problem.  $\Delta$  is the discriminant of the curve  $H(x, p) = 0$  or equivalently (3.1.23),

$$\Delta(E) = \left(\frac{E}{2}\right)^2 - 1, \quad (3.2.41)$$

which can be read from finding the singular points of the PF equation (3.1.32). We can also check that this reproduces the first correction to the quantum volume by using (3.2.36). One can get  $\mathbf{a}_1(E)$  from (3.1.34), and  $\mathcal{F}_0^{(0)}(E)$  from (3.1.11). Then, the  $\hbar^2$  term of the quantum volume in (3.2.36) is

$$\begin{aligned} & \partial_{\mathbf{a}_0} \mathcal{F}_1^{(0)}(\mathbf{a}_0(E)) + \mathbf{a}_1(E) \partial_{\mathbf{a}_0 \mathbf{a}_0} \mathcal{F}_0^{(0)}(\mathbf{a}_0(E)) = \\ &= \frac{\partial_E \mathcal{F}_1^{(0)}(\mathbf{a}_0(E))}{\partial_E \mathbf{a}_0(E)} + \mathbf{a}_1(E) \frac{\partial_E \partial_{\mathbf{a}_0} \mathcal{F}_0^{(0)}(\mathbf{a}_0(E))}{\partial_E \mathbf{a}_0(E)} = \\ &= \frac{\pi E}{12(2-E)\sqrt{E+2} \mathbf{K}\left(\frac{4}{E+2}\right)} + \\ & \quad + \frac{\mathbf{K}\left(\frac{E-2}{E+2}\right) \left[ E \mathbf{E}\left(\frac{4}{E+2}\right) - (E-2) \mathbf{K}\left(\frac{4}{E+2}\right) \right]}{6(E-2)\sqrt{E+2} \mathbf{K}\left(\frac{4}{E+2}\right)} = \\ &= \frac{2 \mathbf{K}\left(\frac{2E}{E+2} - 1\right) - E \mathbf{E}\left(\frac{2E}{E+2} - 1\right)}{6(E-2)\sqrt{2+E}} = \text{vol}_1(E), \end{aligned} \quad (3.2.42)$$

precisely what we computed directly from the period integral in (3.1.15). In order to simplify the last line we used Legendre's relation

$$\mathbf{K}(k^2) \mathbf{E}(1-k^2) + \mathbf{K}(1-k^2) \mathbf{E}(k^2) - \mathbf{K}(k^2) \mathbf{K}(1-k^2) = \frac{\pi}{2}. \quad (3.2.43)$$

No doubt it is much more convenient for any concrete evaluation to first compute the quantum period  $\mathbf{a}(E)$ , and then insert it into the  $\mathcal{F}_n^{(0)}(\mathbf{a})$ . This also makes sense only if computing  $\mathcal{F}_n(\mathbf{a})$  is actually more efficient than doing the integrals for  $\text{vol}_n(E)$ . Which is exactly what the holomorphic anomaly does.

The algorithm to compute the actual spectrum would go as follows. One computes the dual free energy  $\mathcal{F}_D(\text{vol}, \hbar)$  as a function of the quantum volume  $\text{vol}(E, \hbar)$  – with the holomorphic anomaly. Then, one retrieves the physical energy  $E$  as a function of  $\text{vol}$  with the Matone relation (3.1.25), or its equivalents in quantum mechanics. Finally, the all-orders WKB condition (3.1.14) identifies  $\text{vol}$  with the energy level, and we obtain  $E_n$  as an asymptotic expansion in  $\hbar$  with coefficients in  $n$ , as was found by direct means in [83], where the resummation of this asymptotic series was also taken care of.

In order to introduce the holomorphic anomaly itself, we will need some ingredients from the geometry of the curve. We will do a brief review of some well known

aspects of the theory of modular forms, to shed light on the path to the the anomaly and to establish a bit of notation. First, the modular parameter of the curve is given by the derivative of one period w.r.t. the other,

$$\tau(\mathbf{a}_0) = \frac{\beta}{2\pi i} \frac{\partial^2 \mathcal{F}_0^{(0)}(\mathbf{a}_0)}{\partial \mathbf{a}_0^2}. \quad (3.2.44)$$

The modular parameter  $\tau$  parametrizes, as the energy varies, all the inequivalent genus one curves  $H(x, p) = 0$  in the sense of algebraic geometry. For higher genus curves, one would need a matrix  $\tau_{ij}$  containing every couple of period derivatives [70].

Actually, we are interested in the free energies as a function of the whole quantum period  $\mathbf{a}$ , and will evaluate the previous equation at  $\tau(\mathbf{a})$ , so that instead of the energy  $E$  we parametrize the curves by the corrected  $\xi$ . For simplicity, from now on we will just write  $\mathcal{F}_n^{(0)}(\mathbf{a}) = \mathcal{F}_n^{(0)}$ . The  $\beta$  coefficient takes into account the normalization for  $\mathcal{F}_0^{(0)}$ . To find  $\beta$  for our geometry, just take derivatives of the classical periods (3.1.11) and (3.1.19),

$$\frac{\partial^2 \mathcal{F}_0^{(0)}}{\partial \mathbf{a}^2} = \frac{\partial_\xi \partial_{\mathbf{a}} \mathcal{F}_0^{(0)}}{\partial_\xi \mathbf{a}} = 4\pi \frac{\mathbf{K}\left(\frac{\xi-2}{\xi+2}\right)}{\mathbf{K}\left(\frac{4}{\xi+2}\right)}. \quad (3.2.45)$$

Compare this with the relation between  $\tau$  and the argument of the elliptic functions,

$$2\pi i \tau = -2\pi \frac{\mathbf{K}(1-k^2)}{\mathbf{K}(k^2)} \iff k^2 = \frac{\vartheta_2^4(q)}{\vartheta_3^2(q)}, \quad (3.2.46)$$

where the Jacobi elliptic functions  $\vartheta_i(q)$  are given in terms of the nome  $q$  by

$$\begin{aligned} \vartheta_2(q) &= 2q^{1/4} \sum_{n=0}^{\infty} q^{n(n+1)}, \\ \vartheta_3(q) &= 1 + 2 \sum_{n=0}^{\infty} q^{n^2}, \\ \vartheta_4(q) &= 1 + 2 \sum_{n=0}^{\infty} (-1)^n q^{n^2}, \\ q &= e^{i\pi\tau}. \end{aligned} \quad (3.2.47)$$

The two immediate consequences are that for the Mathieu potential

$$\beta = -\frac{1}{2}, \quad (3.2.48)$$

and the corrected energy is

$$\xi = 4 \frac{\vartheta_3^4(q)}{\vartheta_2^4(q)} - 2. \quad (3.2.49)$$

Notice that we can invert this relation to express the periods as a function<sup>2</sup> of  $\tau$

$$\mathbf{a}(\tau), \quad \partial_{\mathbf{a}} \mathcal{F}_0^{(0)}(\tau). \quad (3.2.50)$$

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<sup>2</sup>This is a slight abuse of notation, but it should generally be clear that  $\mathcal{F}(\tau)$  means  $\mathcal{F}(\mathbf{a}(\tau))$ .

Recall that we defined them as

$$\partial_{\mathbf{a}}\mathcal{F}_0^{(0)} = \oint_A dx P(x, \hbar), \quad \mathbf{a} = \frac{1}{2\pi i} \oint_B dx P(x, \hbar). \quad (3.2.51)$$

It is straightforward to see that an exchange of periods in (3.2.44), changing their respective normalization constants, sends

$$\tau \mapsto -\frac{1}{\tau}, \quad (3.2.52)$$

called S transformation. It is a *duality* operation on the periods. There is also a T transformation

$$\tau \mapsto \tau + 1 \quad (3.2.53)$$

who corresponds to adding (with the  $2\pi i/\beta$  factor from the normalization) the  $\mathbf{a}$  period to the  $\partial_{\mathbf{a}}\mathcal{F}_0^{(0)}$  period. Together they generate the  $SL(2, \mathbb{Z})$  modular group, whose general element is the transformation

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \quad (3.2.54)$$

where

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1, \quad a, b, c, d \in \mathbb{Z}. \quad (3.2.55)$$

The first quantum correction to the free energy,  $\mathcal{F}_1^{(0)}$ , has naturally weight zero when this group acts on the periods. With the S-dual nome,

$$q_D = \exp\left(i\pi \frac{-1}{\tau}\right) \quad (3.2.56)$$

the Jacobi thetas transform in the following way.

$$\vartheta_2^4(q_D) = -\tau^2 \vartheta_4^4(q), \quad \vartheta_3^4(q_D) = -\tau^2 \vartheta_3^4(q), \quad \vartheta_4^4(q_D) = -\tau^2 \vartheta_2^4(q). \quad (3.2.57)$$

Together with the identity

$$\vartheta_2^4(q) + \vartheta_4^4(q) = \vartheta_3^4(q) \quad (3.2.58)$$

and the relation to the corrected energy (3.2.49), we can see that the first free energy (3.2.40) is indeed invariant under modular transformations of the periods. However, the higher corrections are not, and in general have highly non-trivial transformation rules. Notably, as we found in (3.1.33), they are generated by derivatives of the lower order corrections, which *do* have a simple behaviour under S. This phenomenon is actually nothing new. In the theory of modular forms, it is well known that their derivatives are not modular themselves, but rather they are what is called *quasi-modular*. The best example of this are the Eisenstein series, defined for integer  $k$  as

$$E_{2k}(\tau) = 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^{2n}}{1-q^{2n}}, \quad (3.2.59)$$

where  $q$  is the nome

$$q = e^{i\pi\tau} \quad (3.2.60)$$

and  $\zeta(s)$  Riemann's theta function. It is well known that their derivatives w.r.t. the modular parameter are given in terms of themselves. For instance, one can be convinced by comparing the  $q$  series that

$$\frac{1}{2\pi i} \frac{\partial E_2}{\partial \tau} = \frac{E_2^2 - E_4}{12}. \quad (3.2.61)$$

or

$$\frac{1}{2\pi i} \frac{\partial E_8}{\partial \tau} = \frac{2(E_4^2 E_2 + E_6 E_4)}{3}. \quad (3.2.62)$$

They are all invariant under the T transform,

$$E_{2k}(\tau + 1) = E_{2k}(\tau). \quad (3.2.63)$$

In addition, for  $k \geq 1$  they are modular forms of weight  $2k$ . A modular form is an holomorphic function of the upper half complex plane, invariant under T, and transforming as follows under S

$$E_{2k}\left(-\frac{1}{\tau}\right) = \tau^{2k} E_{2k}(\tau). \quad (3.2.64)$$

The weight is the power with which  $\tau$  appears after the T transformation, and it gives a natural grading to the ring of modular forms.

Now, this is not quite the case for  $E_2$ , who gets an extra term upon transformation,

$$E_2\left(-\frac{1}{\tau}\right) = \tau^2 E_2(\tau) + \frac{6\tau}{i\pi}. \quad (3.2.65)$$

For this reason, it is called a *quasi-modular form*. One can have a modular form by just finding some object of weight 2 that compensates this behaviour. Interestingly, the imaginary part of  $\tau$  fulfils the role,

$$\frac{6}{i\pi(\tau - \bar{\tau})} \Big|_{\substack{\tau \mapsto -1/\tau \\ \bar{\tau} \mapsto -1/\bar{\tau}}} = \frac{6\tau\bar{\tau}}{i\pi(\tau - \bar{\tau})} = \tau^2 \frac{6}{i\pi(\tau - \bar{\tau})} - \frac{6\tau}{i\pi} \quad (3.2.66)$$

and it is obviously invariant under  $\tau \mapsto \tau + 1$ ,  $\bar{\tau} \mapsto \bar{\tau} + 1$ . This means that if we extend ourselves to non-holomorphic functions,

$$\hat{E}_2(\tau, \bar{\tau}) = E_2(\tau) + \frac{6}{i\pi(\tau - \bar{\tau})} \quad (3.2.67)$$

satisfies

$$\begin{aligned} \hat{E}_2(\tau + 1, \bar{\tau} + 1) &= \hat{E}_2(\tau, \bar{\tau}), \\ \hat{E}_2(-1/\tau, -1/\bar{\tau}) &= \tau^2 \hat{E}_2(\tau, \bar{\tau}). \end{aligned} \quad (3.2.68)$$

For this, it is called an *almost holomorphic modular form*, since it now transforms appropriately under the modular group, and the lack of holomorphicity is contained

in a relatively simple function of the imaginary part of  $\tau$ . A general almost holomorphic modular form of weight  $k$  transforms with a factor  $\tau^k$  under the modular group and can be written as a polynomial in  $(\text{Im}\tau)^{-1}$  with holomorphic coefficients. Quasi-modular forms are simply the holomorphic part of almost holomorphic forms.

The (apparent) downside of including the non-holomorphic term in  $\hat{E}_2$  is that derivatives w.r.t.  $\tau$  become a mess. To fix that, define the Maass derivative, a graded derivation defined by

$$f_k(-1/\tau) = \tau^k f_k(\tau) \implies D_\tau f_k = \left( \frac{1}{2\pi i} \frac{\partial}{\partial \tau} + \frac{k}{2\pi i(\tau - \bar{\tau})} \right) f_k. \quad (3.2.69)$$

In particular, from the  $\tau$  derivative of  $E_2$ , it follows that

$$D_\tau \hat{E}_2 = \frac{\hat{E}_2^2 - E_4}{12}, \quad (3.2.70)$$

and similarly

$$D_\tau E_8 = \frac{2(E_4^2 \hat{E}_2 + E_6 E_4)}{3}. \quad (3.2.71)$$

Indeed,  $D_\tau$  closes over the ring of almost holomorphic modular forms.

The fundamental lesson we learn here is that lack of modularity can be “cured” by a non-holomorphic dependence. The insight of [26] is to apply this to the free energies. The idea is to build a set of modular invariant  $F_n^{(0)}$  such that

$$\lim_{\bar{\tau} \rightarrow i\infty} F_n^{(0)}(\tau, \bar{\tau}) = \mathcal{F}_n^{(0)}(\tau). \quad (3.2.72)$$

Just like with  $\hat{E}_2$ , we recover the holomorphic (*quasi-modular*) part by sending the complex conjugate of the modular parameter to the point at infinity. And just like with  $\hat{E}_2$ , we expect the non-holomorphic dependence to be heavily constrained if modularity is to be respected. **This constraint is precisely the (refined) holomorphic anomaly.**

As said in the introduction to this chapter, the WKB periods of the Mathieu potential follow from the Nekrasov-Shatashvili free energies of Seiberg-Witten theory [79]. The holomorphic anomaly constraining refined (and in particular, NS) free energies was given in [27]. We will now write it down for our problem, and show how it can be used to recover them recursively.

We will need some more geometric ingredients. First, the second derivative of the prepotential – the modular parameter – defines a Kähler metric in the  $\tau$  moduli space through

$$G_{\text{a}\bar{\text{a}}} = \partial_{\text{a}\bar{\text{a}}} \mathcal{F}_0^{(0)} + \partial_{\bar{\text{a}}\bar{\text{a}}} \bar{\mathcal{F}}_0^{(0)} = \frac{2\pi i}{\beta} (\tau - \bar{\tau}). \quad (3.2.73)$$

Second, the derivative of the modular parameter w.r.t. the period gives the equivalent to a Yuwaka coupling, which will be a key element in the construction,

$$Y_{\text{aaa}} = \frac{\partial^3 \mathcal{F}_0^{(0)}}{\partial \text{a}^3}. \quad (3.2.74)$$

Then, the non-holomorphic part of the modular invariant, perturbative, WKB free energies must satisfy

$$\frac{\partial F_n^{(0)}}{\partial \bar{\mathbf{a}}} = \frac{1}{2} \bar{Y}_{\bar{\mathbf{a}}}^{\mathbf{a}\mathbf{a}} \sum_{r=1}^{n-1} \frac{\partial F_r^{(0)}}{\partial \mathbf{a}} \frac{\partial F_{n-r}^{(0)}}{\partial \mathbf{a}}, \quad n \geq 1, \quad (3.2.75)$$

where we treat  $\mathbf{a}$  and  $\bar{\mathbf{a}}$  as independent variables.

We have also raised indices in the conjugate of the Yukawa with the Kähler metric,

$$\bar{Y}_{\bar{\mathbf{a}}}^{\mathbf{a}\mathbf{a}} = G^{\mathbf{a}\bar{\mathbf{a}}} G^{\bar{\mathbf{a}}\mathbf{a}} \bar{Y}_{\bar{\mathbf{a}}\bar{\mathbf{a}}}. \quad (3.2.76)$$

The mixed-index Yukawa is not a very convenient object to work with, and neither are the period derivatives. We would like to recast the holomorphic anomaly in the language of modular forms, where derivatives satisfy simple algebraic relations like the ones shown for the Eisenstein series. Define the *propagator*  $S^{\mathbf{a}\mathbf{a}}$ , a non-holomorphic function that must satisfy

$$\partial_{\bar{\mathbf{a}}} S^{\mathbf{a}\mathbf{a}} = \bar{Y}_{\bar{\mathbf{a}}}^{\mathbf{a}\mathbf{a}}. \quad (3.2.77)$$

This allows us to use the propagator as the new non-holomorphic “coordinate”,

$$F_n^{(0)}(\mathbf{a}, \bar{\mathbf{a}}) \rightarrow F_n^{(0)}(\mathbf{a}, S^{\mathbf{a}\mathbf{a}}(\mathbf{a}, \bar{\mathbf{a}})) \quad (3.2.78)$$

and by the chain rule

$$\partial_{S^{\mathbf{a}\mathbf{a}}} F_n^{(0)}(\mathbf{a}, S^{\mathbf{a}\mathbf{a}}) = \frac{1}{2} \sum_{r=1}^{n-1} \partial_{\mathbf{a}} F_r^{(0)}(\mathbf{a}, S^{\mathbf{a}\mathbf{a}}) \partial_{\mathbf{a}} F_{n-r}^{(0)}(\mathbf{a}, S^{\mathbf{a}\mathbf{a}}), \quad n \geq 1. \quad (3.2.79)$$

The requirement on  $S^{\mathbf{a}\mathbf{a}}$  leaves an arbitrary holomorphic part undetermined. But we did say we wanted to rewrite things with modular forms, and we should use that freedom for this purpose. We can choose it so, very explicitly, as

$$S^{\mathbf{a}\mathbf{a}} = -\frac{\beta}{12} \hat{E}_2. \quad (3.2.80)$$

To see that this satisfies the requirement for the propagator, just take the derivative

$$\partial_{\bar{\mathbf{a}}} S^{\mathbf{a}\mathbf{a}} = -\frac{\beta}{12} \partial_{\bar{\mathbf{a}}} \left( E_2(\tau) + \frac{6}{i\pi(\tau - \bar{\tau})} \right) = -\frac{\beta}{2\pi i} \frac{\partial_{\bar{\mathbf{a}}} \bar{\tau}}{(\tau - \bar{\tau})^2}. \quad (3.2.81)$$

The derivative of the modular parameter should be recast in terms of the prepotential,

$$\bar{\tau} = -\frac{\beta}{2\pi i} \partial_{\bar{\mathbf{a}}\bar{\mathbf{a}}} \bar{\mathcal{F}}_0^{(0)} \implies \partial_{\bar{\mathbf{a}}} \bar{\tau} = -\frac{\beta}{2\pi i} \bar{Y}_{\bar{\mathbf{a}}\bar{\mathbf{a}}\bar{\mathbf{a}}}, \quad (3.2.82)$$

and

$$\partial_{\bar{\mathbf{a}}} S^{\mathbf{a}\mathbf{a}} = \left( \frac{\beta}{2\pi i} \frac{1}{\tau - \bar{\tau}} \right)^2 \bar{Y}_{\bar{\mathbf{a}}\bar{\mathbf{a}}\bar{\mathbf{a}}} = G^{\mathbf{a}\bar{\mathbf{a}}} G^{\bar{\mathbf{a}}\mathbf{a}} \bar{Y}_{\bar{\mathbf{a}}\bar{\mathbf{a}}\bar{\mathbf{a}}} \quad (3.2.83)$$

just like we wanted.

To transform the derivative w.r.t. the period into a derivative w.r.t. the modular parameter, we can also use the chain rule,

$$\partial_{\mathbf{a}} = \frac{\partial \tau}{\partial \mathbf{a}} \partial_{\tau} = \beta Y_{\mathbf{aaa}} \frac{\partial \tau}{2\pi i}. \quad (3.2.84)$$

The  $F_n^{(0)}$  are built to be modular invariant, so they have weight zero. Meaning that on them

$$\frac{1}{2\pi i} \partial_{\tau} F_n^{(0)} = D_{\tau} F_n^{(0)}. \quad (3.2.85)$$

However, since we will build the free energies out of (almost holomorphic) modular forms, it is more convenient to already use the latter. A modular-form ready version of the holomorphic anomaly follows,

$$\partial_{\hat{E}_2} F_n^{(0)} = -\frac{\beta^3}{24} Y^2 \sum_{r=1}^{n-1} D_{\tau} F_r^{(0)} D_{\tau} F_{n-r}^{(0)}, \quad (3.2.86)$$

and we just write  $S = S^{\mathbf{aa}}$ ,  $Y = Y_{\mathbf{aaa}}$  for simplicity. Remember that for the particular case of the Mathieu potential,  $\beta = -1/2$ .

### 3.2.2 Modular generators

As it is well known, the differential ring of almost holomorphic modular forms requires only three generators. This is key for efficiently solving (3.2.86), turning a differential problem into an algebraic one. We choose the following generators, defined in terms of the Jacobi elliptic functions introduced in (3.2.47) and the Eisenstein series (3.2.59)

$$\begin{aligned} K_2(\tau) &= \vartheta_3^4(q) + \vartheta_4^4(q), \\ L_2(\tau) &= \vartheta_2^4(q), \\ \hat{E}_2(\tau) &= E_2(\tau) - \frac{3}{\pi \operatorname{Im} \tau}. \end{aligned} \quad (3.2.87)$$

Actually,  $K_2$  and  $L_2$  are not quite modular forms. They are T invariant, but they get exchanged under the S transform. This is necessary to properly represent the periods, which *also* form a vectorial representation of the modular group, rather than being invariant. Concretely, from the S transform on elliptic thetas (3.2.57),

$$\begin{aligned} K_2(-1/\tau) &= -\tau^2 \frac{K_2(\tau) + 3L_2(\tau)}{2}, \\ L_2(-1/\tau) &= -\tau^2 \frac{K_2(\tau) - L_2(\tau)}{2}. \end{aligned} \quad (3.2.88)$$

On the other hand, the Eisenstein series is an actual (almost holomorphic/quasi) modular form (3.2.65), (3.2.68). As promised, differentiation w.r.t. the modular

parameter closes over the three generators,

$$\begin{aligned}\frac{1}{2\pi i}\partial_\tau K_2 &= \frac{1}{6}E_2K_2 + \frac{1}{12}(3L_2^2 - K_2^2), \\ \frac{1}{2\pi i}\partial_\tau L_2 &= \frac{1}{6}E_2L_2 + \frac{1}{6}K_2L_2, \\ \frac{1}{2\pi i}\partial_\tau E_2 &= \frac{1}{12}E_2E_2 - \frac{1}{48}(K_2^2 + 3L_2^2).\end{aligned}\tag{3.2.89}$$

which one can easily check upon expansion in  $q$ . Equivalently, using the Maass derivative

$$\begin{aligned}D_\tau K_2 &= \frac{1}{6}\hat{E}_2K_2 + \frac{1}{12}(3L_2^2 - K_2^2), \\ D_\tau L_2 &= \frac{1}{6}\hat{E}_2L_2 + \frac{1}{6}K_2L_2, \\ D_\tau \hat{E}_2 &= \frac{1}{12}\hat{E}_2\hat{E}_2 - \frac{1}{48}(K_2^2 + 3L_2^2),\end{aligned}\tag{3.2.90}$$

now in terms of  $\hat{E}_2$ . Effectively, if we use these three generators to express the  $F_n^{(0)}$ , the recursion (3.2.86) becomes algebraic. An important observation that follows from the algebra is that  $\partial_{\hat{E}_2}$  and  $D_\tau$  do not commute in general. From (3.2.90),

$$\begin{aligned}\partial_{\hat{E}_2}D_\tau &= \frac{1}{6}\left[K_2\partial_{K_2} + L_2\partial_{L_2} + \hat{E}_2\partial_{\hat{E}_2}\right] + D_\tau\partial_{\hat{E}_2} = \\ &= \frac{k}{12} + D_\tau\partial_{\hat{E}_2}.\end{aligned}\tag{3.2.91}$$

The first object to rewrite are the periods (3.1.11) and (3.1.19). In (3.2.49) we found the expression for the corrected energy, that can be written with our generators as

$$\xi(\tau) = 2\frac{K_2(\tau)}{L_2(\tau)},\tag{3.2.92}$$

and the discriminant is

$$\Delta(\tau) = \left(\frac{\xi(\tau)}{2}\right)^2 - 1 = \frac{K_2(\tau)^2}{L_2(\tau)^2} - 1.\tag{3.2.93}$$

The elliptic integrals can also be restated in terms of Jacobi thetas,

$$\mathbf{K}\left(\frac{4}{\xi+2}\right) = \mathbf{K}(k^2) = \frac{\pi}{2}\vartheta_3^2(q) = \frac{\pi}{\sqrt{8}}\sqrt{K_2 + L_2},\tag{3.2.94}$$

and  $\mathbf{K}(1-k^2)$  follows from (3.2.46). The second integral is in fact related to the Eisenstein series,

$$E_2(\tau) = \frac{6}{\pi}\mathbf{E}(k^2)\vartheta_3^2(q) - \vartheta_3^4(q) - \vartheta_4^4(q) \implies \mathbf{E}(k^2) = \frac{\pi}{3\sqrt{2}}\frac{E_2 + K_2}{\sqrt{K_2 + L_2}}.\tag{3.2.95}$$

Finally,  $\mathbf{E}(1 - k^2)$  can be obtained from Legendre's relation (3.2.43). The A period is given as a function of  $\tau$  by

$$\mathbf{a}(\tau) = \frac{2 E_2(\tau) + K_2(\tau)}{3 \sqrt{L_2(\tau)}}. \quad (3.2.96)$$

We are overloading the notation here, compared to  $\mathbf{a}(E, \hbar)$  in (3.1.18). Indeed, when we say  $\mathbf{a}(\tau)$  the  $(E, \hbar)$  dependence is contained in  $\tau$  via the corrected energy  $\xi$  (3.2.92). As the reader will see, as long as we use  $\tau$  or  $(E, \hbar)$  as variables there will be no risk of confusion. Respectively, the volume – from now on, B period – as a function of  $\tau$  is

$$\partial_{\mathbf{a}} \mathcal{F}_0^{(0)}(\tau) = -\frac{8 i \pi \tau [E_2(\tau) + K_2(\tau)] + 6}{3 \sqrt{L_2(\tau)}}. \quad (3.2.97)$$

When we write  $\mathcal{F}_0^{(0)}(\tau)$  we mean, of course,  $\mathcal{F}_0^{(0)}(\mathbf{a}(\tau))$ , but we will commonly use the shorthand here too for simplicity. It is convenient to define a *dual* variable

$$\mathbf{a}_D(\tau_D) = \frac{2}{3} \frac{2 E_2(\tau_D) - K_2(\tau_D) - 3 L_2(\tau_D)}{\sqrt{2 K_2(\tau_D) - 2 L_2(\tau_D)}} \quad (3.2.98)$$

where  $\tau_D = -1/\tau$ , in terms of which the B period is

$$\partial_{\mathbf{a}} \mathcal{F}_0^{(0)}(\tau) = -4\pi \mathbf{a}_D(-1/\tau) = -\frac{8 i \pi \tau [E_2(\tau) + K_2(\tau)] + 6}{3 \sqrt{L_2(\tau)}}. \quad (3.2.99)$$

The derivative w.r.t. the A period can also be written in terms of modular generators by using (3.2.84),

$$\partial_{\mathbf{a}} = \frac{1}{\partial_{\tau} \mathbf{a}} \partial_{\tau} = \frac{8 \sqrt{L_2}}{L_2^2 - K_2^2} \frac{\partial_{\tau}}{2\pi i}, \quad (3.2.100)$$

If we apply this to the B period, we get, as we should from (3.2.44),

$$\partial_{\mathbf{a}\mathbf{a}} \mathcal{F}_0^{(0)} = -4\pi i \tau, \quad (3.2.101)$$

Equation (3.2.44), which for  $\beta = -1/2$  reads

$$\partial_{\mathbf{a}} = -\frac{Y}{2} \frac{\partial_{\tau}}{2\pi i}. \quad (3.2.102)$$

tells us the Yukawa,

$$Y = \partial_{\mathbf{a}\mathbf{a}\mathbf{a}} \mathcal{F}_0^{(0)} = \frac{16 \sqrt{L_2}}{K_2^2 - L_2^2}. \quad (3.2.103)$$

### 3.2.3 Frames and dual free energy

The  $\mathbf{a}$  period (3.2.96) has a very similar structure to the  $\mathbf{a}_D$  variable (3.2.98) that gives the B period. In fact, define

$$a(\tau, \bar{\tau}) = \frac{2 \hat{E}_2(\tau, \bar{\tau}) + K_2(\tau)}{3 \sqrt{L_2(\tau)}}, \quad (3.2.104)$$

as the modular completion of the period  $\mathbf{a}$ . In the holomorphic limit,

$$\lim_{\bar{\tau} \rightarrow i\infty} a(\tau, \bar{\tau}) = \mathbf{a}(\tau). \quad (3.2.105)$$

Since  $a$  has weight one, we can use the S transform to compute the holomorphic limit in the dual variable,

$$\lim_{\bar{\tau}_D \rightarrow i\infty} a(\tau, \bar{\tau}) = \lim_{\bar{\tau}_D \rightarrow i\infty} a\left(\frac{-1}{\tau_D}, \frac{-1}{\bar{\tau}_D}\right) = \frac{1}{i\tau} \mathbf{a}_D(\tau_D) = -\frac{1}{4\pi i \tau} \partial_{\mathbf{a}} \mathcal{F}_0^{(0)}(\tau) \quad (3.2.106)$$

So, up to a  $\tau$  factor, we obtain the two periods as different limits of the same modular form  $a$ . This prompts us to define two different *frames* for holomorphic limits, in the following way. Let  $f_k$  be an almost holomorphic form of weight  $k$ , then define two applications from the ring of almost holomorphic modular forms to the ring of quasi-modular forms,

$$\begin{aligned} f_k(\tau, \bar{\tau}) &\mapsto [f_k]_e(\tau) = \lim_{\bar{\tau} \rightarrow \infty} f_k(\tau, \bar{\tau}), \\ f_k(\tau, \bar{\tau}) &\mapsto [f_k]_m(\tau) = \tau^k \lim_{\bar{\tau}_D \rightarrow \infty} f_k(\tau, \bar{\tau}). \end{aligned} \quad (3.2.107)$$

We call them *electric*,  $[\cdot]_e$ , and *magnetic*,  $[\cdot]_m$ , frames – following the naming of SW theory. Concretely, the frames for  $a$  are

$$\begin{aligned} [a]_e &= \mathbf{a}(\tau), \\ [a]_m &= -i \mathbf{a}_D(\tau_D). \end{aligned} \quad (3.2.108)$$

The magnetic frame can be easily implemented in terms of the generators via a *magnetic map*  $M$ , defined by its action on them,

$$\begin{aligned} M[K_2(\tau)] &= \frac{-K_2(\tau) - 3L_2(\tau)}{2}, \\ M[L_2(\tau)] &= \frac{-K_2(\tau) + L_2(\tau)}{2}, \\ M[\hat{E}_2(\tau, \bar{\tau})] &= \hat{E}_2(\tau, \bar{\tau}) \end{aligned} \quad (3.2.109)$$

and by requiring it to be distributive on the ring of modular forms. This is essentially the effect S of the transform, without the  $\tau^2$ . The magnetic map is simply the application

$$f_k(\tau, \bar{\tau}) \mapsto [f_k]_m(\tau_D) = \lim_{\bar{\tau}_D \rightarrow i\infty} M[f_k(\tau_D, \bar{\tau}_D)]. \quad (3.2.110)$$

Notice that the difference between frames is not just the choice of  $\tau$  or  $\tau_D$  as a variable. That is true of  $K_2$  and  $L_2$ , but the behaviour of  $\hat{E}_2$  is completely different,

$$\begin{aligned} \lim_{\bar{\tau} \rightarrow i\infty} \hat{E}_2(\tau, \bar{\tau}) &= E_2(\tau), \\ \lim_{\bar{\tau}_D \rightarrow i\infty} \hat{E}_2(\tau, \bar{\tau}) &= E_2(\tau) + \frac{6}{i\pi\tau}. \end{aligned} \quad (3.2.111)$$

Most of the time we will use objects of weight zero, such as the free energies. For them, there is no difference (3.2.107) between the magnetic frame and taking the

holomorphic limit after an S transform. Things get interesting when we consider derivatives. First, we define a *modular period derivative* by analogy with (3.2.84),

$$D_a = \beta Y D_\tau. \quad (3.2.112)$$

Applying (3.2.90) to the definition of  $a$ ,

$$D_\tau a = -\frac{2}{Y}, \quad (3.2.113)$$

and (3.2.112) is, with  $\beta = -1/2$ ,

$$D_a = \frac{D_\tau}{D_\tau a}. \quad (3.2.114)$$

So in fact,  $D_a$  is a derivation on the ring of modular forms satisfying  $D_a a = 1$ , which makes it a derivative w.r.t.  $a$ . Since the holomorphic limits of  $a$  are  $\mathbf{a}$  and  $\mathbf{a}_D$ , this has an interesting consequence,

$$\begin{aligned} [D_a f]_e &= \frac{\partial [f]_e}{\partial \mathbf{a}}, \\ [D_a f]_m &= i \frac{\partial [f]_m}{\partial \mathbf{a}_D}. \end{aligned} \quad (3.2.115)$$

This is quite convenient, since the two frames use in a natural way  $\mathbf{a}$  and  $\mathbf{a}_D$  as their variables. Consider that we would rather use  $\tau$  for the electric frame, since in it  $\hat{E}_2$  becomes  $E_2(\tau)$ , a power series in  $q$ . In the magnetic frame, however,  $\hat{E}_2$  goes to  $E_2(\tau_D)$ . If we insist on using  $\tau$  as a variable, the transformation term gives us factors of  $1/\log q$  which are not very convenient for expansions. On the other hand,  $E_2(\tau_D)$  is again a power series in  $q_D$ . The connection to  $\mathbf{a}$  and  $\mathbf{a}_D$  comes from their own expansion. From (3.2.96) and (3.2.98), we find

$$\mathbf{a}(\tau) = \frac{1}{\sqrt{q}} \left( \frac{1}{2} + 3q^2 - \frac{21q^4}{2} + 33q^6 + \dots \right), \quad (3.2.116)$$

$$\mathbf{a}_D(\tau_D) = -16q_D - 96q_D^2 - 384q_D^3 + \dots \quad (3.2.117)$$

so that we get nice power series expansions for the respective limits of  $\hat{E}_2$

$$\begin{aligned} E_2(\tau) &= 1 - \frac{3}{2\mathbf{a}^4} - \frac{81}{32\mathbf{a}^8} + O(\mathbf{a}^{-12}), & \mathbf{a} &= \mathbf{a}(\tau), \\ E_2(\tau_D) &= 1 - \frac{3\mathbf{a}_D^2}{32} - \frac{9\mathbf{a}_D^3}{128} + O(\mathbf{a}_D^4), & \mathbf{a}_D &= \mathbf{a}_D(\tau_D). \end{aligned} \quad (3.2.118)$$

Consider an almost holomorphic completion of (3.2.97), and let us name it  $\Phi$ ,

$$D_a \Phi = -4 \left( i\pi\tau a + \frac{4}{\sqrt{L_2}} \right). \quad (3.2.119)$$

In the electric frame,

$$\begin{aligned} [D_a \Phi]_e &= 8\mathbf{a} (\log(2\mathbf{a}) - 1) + \frac{1}{\mathbf{a}^3} + \frac{15}{32\mathbf{a}^7} + \\ &+ \frac{15}{32\mathbf{a}^{11}} + \frac{10283}{16384\mathbf{a}^{15}} + \frac{40239}{40960\mathbf{a}^{19}} + \dots = \frac{\partial \mathcal{F}_0^{(0)}}{\partial \mathbf{a}}. \end{aligned} \quad (3.2.120)$$

We would not expect less, since the holomorphic limit of  $a$  is  $\mathbf{a}$ . What happens in the magnetic frame?

$$\begin{aligned} -i[D_a \Phi]_m &= 16 + 4\mathbf{a}_D \left( \log\left(\frac{\mathbf{a}_D}{16}\right) - 1 \right) + \\ &+ \frac{3\mathbf{a}_D^2}{4} + \frac{5\mathbf{a}_D^3}{32} + \frac{55\mathbf{a}_D^4}{1024} + \frac{189\mathbf{a}_D^5}{8192} + \frac{3689\mathbf{a}_D^6}{327680} + \dots \end{aligned} \quad (3.2.121)$$

This also looks like a prepotential, but where the variable is  $\mathbf{a}_D$ . Indeed, this is what we call the *dual prepotential*, defined by a Legendre transform of  $\mathcal{F}_0^{(0)}$ ,

$$\mathcal{F}_{D,0}^{(0)}(\tau_D) = \mathcal{F}_0^{(0)}(\tau) + 4\pi\mathbf{a}_D(\tau_D) \mathbf{a}(\tau) \quad (3.2.122)$$

so that

$$\frac{\partial \mathcal{F}_{D,0}^{(0)}(\tau_D)}{\partial \mathbf{a}_D} = \left[ 4\pi\mathbf{a}_D(\tau_D) + \frac{\partial \mathcal{F}_0^{(0)}(\tau)}{\partial \mathbf{a}(\tau)} \right] \frac{\partial \mathbf{a}(\tau)}{\partial \mathbf{a}_D(\tau_D)} + 4\pi\mathbf{a}(\tau) = 4\pi\mathbf{a}(\tau), \quad (3.2.123)$$

basically exchanging the function of each period. By writing  $\mathbf{a}$  as a function of  $\tau_D$ ,

$$\mathbf{a}(\tau_D) = \frac{\sqrt{2}}{3\pi} \frac{12 + i\pi\tau_D(2E_2(\tau_D) - K_2(\tau_D) - 3L_2(\tau_D))}{\sqrt{K_2(\tau_D) - L_2(\tau_D)}}, \quad (3.2.124)$$

and using the series (3.2.117) of  $\mathbf{a}_D$  in  $q_D$ , one can check that  $\partial_{\mathbf{a}_D} \mathcal{F}_{D,0}^{(0)}$  gives the expansion in (3.2.121). This prepotential is upgraded to a free energy

$$\mathcal{F}_D^{(0)} = \sum_{n=0}^{\infty} \mathcal{F}_{D,0}^{(0)} \hbar^{2n} \quad (3.2.125)$$

via the converse of (3.2.35). In this language, we should use a dual corrected energy  $\xi_D$ , so that instead of (3.2.38) we have

$$\text{vol}(E, \hbar) = \text{vol}_0(\xi_D(E, \hbar)) \quad (3.2.126)$$

and the higher  $\hbar$  free energies satisfy

$$\frac{\partial \mathcal{F}_D^{(0)}}{\partial \text{vol}} = -\mathbf{a}(E(\text{vol}, \hbar), \hbar). \quad (3.2.127)$$

The dual corrected energy must then identified with the modular parameter through (3.2.99),

$$-4\pi\mathbf{a}_D(\tau_D) = \text{vol}_0(\xi_D) = 8\sqrt{\xi_D + 2} \left[ \mathbf{K} \left( \frac{\xi_D - 2}{\xi_D + 2} \right) - \mathbf{E} \left( \frac{\xi_D - 2}{\xi_D + 2} \right) \right] = \text{vol}(E, \hbar). \quad (3.2.128)$$

Like in (3.2.38), we are “hiding” the  $\hbar$  dependence of  $\text{vol}(E, \hbar)$  in  $\xi_D$ . In much the same way, the dual prepotential as a function of  $\xi_D$  is

$$\partial_{\mathbf{a}_D} \mathcal{F}_{D,0}^{(0)}(\tau_D) = 4\pi \mathbf{a}_0(\xi_D) = \frac{2\sqrt{\xi_D}}{\pi} \sqrt{1 + \frac{2}{\xi_D}} \mathbf{E}\left(\frac{4}{2 + \xi_D}\right). \quad (3.2.129)$$

As in (3.2.44), we can get relation between  $\tau_D$  and  $\xi_D$  by doing

$$\tau_D = -\frac{1}{4\pi i} \frac{\partial^2 \mathcal{F}_{D,0}^{(0)}}{\partial \mathbf{a}_D^2} = 4\pi \frac{\mathbf{K}\left(\frac{4}{\xi_D+2}\right)}{\mathbf{K}\left(\frac{\xi_D-2}{\xi_D+2}\right)}, \quad (3.2.130)$$

which unsurprisingly is the reverse of what we found for  $\tau(\xi)$ . In particular, it means that

$$\frac{\xi_D - 2}{\xi_D + 2} = \frac{\vartheta_2^4(q_D)}{\vartheta_3^4(q_D)} \implies \xi_D = 2 \frac{K_2(\tau_D) + 3L_2(\tau_D)}{K_2(\tau_D) - L_2(\tau_D)}. \quad (3.2.131)$$

Beware that  $\tau_D(\xi_D)$  is **not**  $-1/\tau(\xi)$ .  $\xi$  and  $\xi_D$  are resumming completely different  $\hbar$  corrections, respectively those of  $\mathbf{a}(E, \hbar)$  and  $\text{vol}(E, \hbar)$ . However, the modular structures defined by either the prepotential or its dual are equally subject to the constraints of the holomorphic anomaly – whether we use  $\tau_D$  to express the resummed  $\text{vol}(E, \hbar)$  or  $\tau$  for  $\mathbf{a}(E, \hbar)$ . This means that the almost holomorphic free energies giving (3.2.72),

$$\mathcal{F}_n^{(0)}(\mathbf{a}(\tau)) = [F_n^{(0)}]_e(\tau) \quad (3.2.132)$$

also return the dual quasi-modular free energies when taken to the magnetic frame

$$\mathcal{F}_{D,n}^{(0)}(\mathbf{a}_D(\tau_D)) = [F_n^{(0)}]_m(\tau_D). \quad (3.2.133)$$

In short, the modular  $F^{(0)}$  can be used to retrieve the quantum periods as a function of each other,

$$\text{vol}(E, \hbar) = \partial_{\mathbf{a}} \mathcal{F}^{(0)}(\mathbf{a}, \hbar) \Big|_{\mathbf{a}=\mathbf{a}(E, \hbar)} \quad (3.2.134)$$

and

$$4\pi \mathbf{a}(E, \hbar) = \partial_{\mathbf{a}_D} \mathcal{F}_D^{(0)}(\mathbf{a}_D, \hbar) \Big|_{\mathbf{a}_D = -\frac{\text{vol}(E, \hbar)}{4\pi}}. \quad (3.2.135)$$

As a consistency check, consider the first free energy, given in (3.2.40) as a function of the discriminant. Since it does not involve any  $E_2$ , its modular completion should be holomorphic – which is just what the holomorphic anomaly (3.2.86) says for  $n = 1$ . So, with (3.2.92),

$$F_1^{(0)}(\tau, \bar{\tau}) = -\frac{1}{24} \log \left( \frac{K_2^2(\tau)}{L_2^2(\tau)} - 1 \right). \quad (3.2.136)$$

Let us send it to the magnetic frame

$$\mathcal{F}_1^{(0)}(\tau_D) = [F_1^{(0)}]_m = -\frac{1}{24} \log \frac{8L_2(\tau_D) [K_2(\tau_D) + L_2(\tau_D)]}{[K_2(\tau_D) - L_2(\tau_D)]^2}. \quad (3.2.137)$$

With (3.2.131), this becomes

$$\mathcal{F}_{D,1}^{(0)}(\mathbf{a}_D(\tau_D)) = -\frac{1}{24} \log \left( \left( \frac{\xi_D}{2} \right)^2 - 1 \right) = -\frac{1}{24} \log \Delta(\xi_D). \quad (3.2.138)$$

Let us expand (3.2.127) in  $E$  and  $\hbar$  like we did for  $\mathcal{F}^{(0)}$ . If we evaluate  $\mathbf{a}_D$  as the classical volume we simply get

$$\mathcal{F}_{D,1}^{(0)} \left( \mathbf{a}_D = -\frac{\text{vol}_0(E)}{4\pi} \right) = -\frac{1}{24} \log \Delta(E), \quad (3.2.139)$$

the full free energy gives

$$\begin{aligned} \frac{\partial \mathcal{F}_D^{(0)}(\mathbf{a}_D = -\frac{\text{vol}}{4\pi}, \hbar)}{\partial \text{vol}} &= -\mathbf{a}_0(E) + \hbar^2 \left[ \partial_{\text{vol}_0} \mathcal{F}_{D,1}^{(0)} \left( \mathbf{a}_D = -\frac{\text{vol}_0(E)}{4\pi} \right) + \right. \\ &\quad \left. + \text{vol}_1(E) \partial_{\text{vol}_0 \text{ vol}_0} \mathcal{F}_{D,0}^{(0)} \left( \mathbf{a}_D = -\frac{\text{vol}_0(E)}{4\pi} \right) \right] + O(\hbar^4) = \\ &= -\mathbf{a}_0(E) + \hbar^2 \frac{(E-2)\mathbf{K}\left(\frac{4}{E+2}\right) - E\mathbf{E}\left(\frac{4}{E+2}\right)}{6(E-2)\sqrt{E+2}} + O(\hbar^4) = \\ &= -\mathbf{a}_0(E) - \hbar^2 \mathbf{a}_1(E) + O(\hbar^4), \end{aligned} \quad (3.2.140)$$

like it should.

### 3.2.4 Boundary conditions

With all our geometric information neatly packed and built with modular forms, we are ready to attack the holomorphic anomaly equation (3.2.86). It is a recursion relation on  $n$  and a differential equation on  $\tau$ . Both will only give concrete answers if we provide concrete boundary conditions. The most obvious one is the initial value for the recursion. For  $n = 1$ , we simply get the requirement that the first correction to the prepotential should be holomorphic, or in other words, not depend on  $\hat{E}_2$ . We had already come across its form for the Mathieu potential in (3.2.136). As for the conditions on the integration, it gets trickier. Clearly, since  $F_1^{(0)}$  has no anholomorphic dependence, (3.2.86) immediately implies that  $F_2^{(0)}$  is a polynomial of degree one in  $\hat{E}_2$ . Due to the  $\hat{E}_2$  terms introduced in general when acting with the Maass derivative  $D_\tau$ , we can parametrize the  $n$ -th correction as

$$F_n^{(0)}(\tau, \bar{\tau}) = \sum_{r=0}^{2n-3} f_{n,r}(\tau) \hat{E}_2^r(\tau, \bar{\tau}). \quad (3.2.141)$$

All the  $f_{n,r}$  for  $r \geq 1$  are determined by the anomaly once we know the previous free energies. But with regards to  $\partial_{\hat{E}_2}$ ,  $f_{n,0}$  is an integration constant, usually called the *holomorphic ambiguity*. More information is needed to fix it.

### 3.2.4.1 Harmonic behaviour

Let us take a small detour, heading to a (much) simpler model, where all the information on the periods can be computed without fancy technology. Consider then the curve for the harmonic oscillator,

$$\frac{p^2}{2} + \frac{x^2}{2} = E. \quad (3.2.142)$$

We can find the periods in the usual WKB way, using the WKB wavefunction (3.1.3) to compute the quantum corrected phase space volume (3.1.14). The classical cycle, that we will denote as A here, goes between the turning points  $x = \pm\sqrt{2E}$ . Let us call the volume

$$\text{vol}(E, \hbar) = \oint_A P(x) = 2\pi t(E, \hbar) \quad (3.2.143)$$

The A cycle goes around  $x = 0$ , so the integrals for  $P(x)_n$  can be evaluated via minus the residue at  $x = \infty$ ,

$$t(\hbar) = \sum_{n=0}^{\infty} t_n \hbar^{2n}, \quad (3.2.144)$$

$$t_n = \frac{1}{2\pi} (-2\pi i) \text{Res}_{x=\infty} P_n(x).$$

With this we can see that the quantum volume has no  $\hbar$  corrections

$$t_0 = -i \text{Res}_{x=\infty} \left( \sqrt{2E - x^2} \right) = E,$$

$$t_1 = -i \text{Res}_{x=\infty} \left( \frac{4E + 3x^2}{8(2E - x^2)^{5/2}} \right) = 0,$$

$$t_2 = -i \text{Res}_{x=\infty} \left( \frac{-304E^2 - 297x^4 - 1464Ex^2}{128(2E - x^2)^{11/2}} \right) = 0,$$

...

so that as it is well known,  $t(E, \hbar) = E$ . The WKB quantization condition for the harmonic oscillator is

$$\text{vol}(E_n, \hbar) = 2\pi\hbar \left( n + \frac{1}{2} \right) \implies E_n = \hbar \left( n + \frac{1}{2} \right) \quad (3.2.146)$$

which in this very particular example that is the harmonic oscillator, turns out to agree with the exact spectrum of the quantum curve.

On the other hand, there is no proper B cycle for the problem – no tunnelling region, even in the complex plane as is the case for Mathieu. What we can do is consider it to be the limit of a potential where the next turning point has gone to  $\infty$ . Unfortunately (and of course)

$$\int_{\sqrt{2E}}^{\infty} P(x) dx \quad (3.2.147)$$

diverges at infinity, but since this would define the B period, let's take derivatives to regularize it, defining instead the Yuwaka directly as

$$\partial_{ttt}\mathcal{F}^{(0)}(t, \hbar) = -i \oint_B \partial_{tt}P(x) dx, \quad (3.2.148)$$

the  $i$  factor to ensure that it is real-valued for real  $E$ . The first few terms are

$$\begin{aligned} \partial_{ttt}\mathcal{F}_0^{(0)} &= -i \oint_B \frac{-1}{(2t - x^2)^{3/2}} dx = \frac{1}{t}, \\ \partial_{ttt}\mathcal{F}_1^{(0)} &= -i \oint_B \frac{5(12t + 29x^2)}{8(2t - x^2)^{9/2}} dx = -\frac{1}{12t^3}, \\ \partial_{ttt}\mathcal{F}_2^{(0)} &= \frac{7}{240t^5}, \quad \partial_{ttt}\mathcal{F}_3^{(0)} = -\frac{31}{1344t^7}, \quad \partial_{ttt}\mathcal{F}_4^{(0)} = \frac{127}{3840t^9}, \quad \dots \end{aligned} \quad (3.2.149)$$

These can all be expressed as

$$\partial_{ttt}\mathcal{F}_n^{(0)} = (2^{1-2g} - 1) B_{2n} \frac{1}{t^{2n+1}}, \quad n \geq 0, \quad (3.2.150)$$

where  $B_{2n}$  are the Bernoulli numbers. They can be integrated to find

$$\begin{aligned} \mathcal{F}_0^{(0)} &= \frac{1}{2}t^2 \left( \log t - \frac{3}{2} \right), \quad \mathcal{F}_1^{(0)} = -\frac{1}{24} \log t, \\ \mathcal{F}_n^{(0)} &= \frac{(1 - 2^{1-2n}) B_{2n}}{2n(2n-1)(2n-2)} \frac{1}{t^{2n-2}} \quad \text{for } n \geq 2. \end{aligned} \quad (3.2.151)$$

In fact, this behaviour is given from the asymptotic expansion of

$$\partial_t \mathcal{F}^{(0)} = t \log \hbar + \hbar \log \left( \frac{\Gamma\left(\frac{1}{2} + \frac{t}{\hbar}\right)}{\sqrt{2\pi}} \right) \quad (3.2.152)$$

or equivalently

$$\partial_{ttt}\mathcal{F}^{(0)} = \frac{1}{\hbar} \psi^{(1)} \left( \frac{1}{2} + \frac{t}{\hbar} \right) \quad (3.2.153)$$

where  $\psi^{(n)}$  is the polygamma function.

Now let us suppose that we have a potential of the form

$$V(x) = \frac{x^2}{2} + V_p(x), \quad (3.2.154)$$

such that  $x = 0$  is still a quadratic minimum of the potential – so  $\partial_x V_p = \partial_{xx} V_p = 0$ . For very low energies,  $E \rightarrow V_p(0)$ , the system effectively sees only a harmonic oscillator, and its behaviour should approach what we have found just before. The general free energies expanded at  $t(E, \hbar)$ , the quantum period around the classical region, must be

$$\mathcal{F}_n^{(0)} = k_B k_A^n \frac{(1 - 2^{1-2n}) B_{2n}}{2n(2n-1)(2n-2)} \frac{1}{t^{2n-2}} + O(t^0) \quad n \geq 2. \quad (3.2.155)$$

where  $k_A$  and  $k_B$  take into account the normalizations for the A and B periods – which are reflected also in the  $\beta$  coefficient in (3.2.86). The lack of other singular terms is called the *gap condition*, and it was originally applied to the holomorphic anomaly in the context of matrix models [86]. From the point of view of the WKB analysis, this universality of the singular part has already been noticed, for instance see [3].

### 3.2.4.2 First free energy

As previously mentioned, from the holomorphic anomaly (3.2.86), we have that

$$\partial_{\hat{E}_2} F_1^{(0)} = 0, \quad (3.2.156)$$

so it is pure holomorphic ambiguity:  $F_1^{(0)} = f_{1,0}$ . To determine this ambiguity, we require as usual that it is a weight zero modular form, and that its leading behaviour at low energies be given by (3.2.151). Now, in the harmonic oscillator,  $t(E, \hbar)$  was the period giving the volume used for the perturbative quantization condition, obtained from the cycle around the classically allowed region. To match with this harmonic behaviour, expand our curve (3.1.1) at the minimum of the potential, where it has the form

$$H(p, x) = p^2 + 2 + x^2 + \frac{x^4}{12} + \dots \quad (3.2.157)$$

This is a harmonic oscillator (up to scale factors) whose energy minimum lies is at  $E = 2$ . The period defining  $\text{vol}(E, \hbar)$  (3.1.14) vanishes at that value of the energy, as the trajectory collapses to a single point. Since the singularity condition (3.2.151) was written in terms of the *quantum* period  $t(E, \hbar)$ , we should fix the singular behaviour of the free energy as a function of the *quantum* volume  $\text{vol}(E, \hbar)$ . The right variable for this is no one else than  $\mathfrak{a}_D$  (3.2.99), who (up to the factor of  $-4\pi$ ) is  $\text{vol}(E, \hbar)$  when expressing the free energies in the dual magnetic frame. Therefore,

$$\mathcal{F}_{D,1}^{(0)}(\mathfrak{a}_D) = -\frac{1}{24} \log \mathfrak{a}_D + O(\mathfrak{a}_D^0). \quad (3.2.158)$$

This is satisfied precisely by the form we had put forward in (3.2.136),

$$F_1^{(0)} = -\frac{1}{24} \log \Delta, \quad (3.2.159)$$

### 3.2.4.3 Singularity and gap

To fix the ambiguity in the higher corrections,  $f_{n,0}$ , we start again by the requirement that  $F_n^{(0)}$  has weight zero. Since the recursion (3.2.86) introduces a factor of  $Y^2$  at every order, we parametrize the ambiguity by the ansatz

$$f_{n,0} = Y^{2n-2} \times \sum_{i=0}^{\lfloor \frac{3(n-1)}{2} \rfloor} a_{n,i} K_2^{3n-3-2i} L_2^{2i}. \quad (3.2.160)$$

built out of the right number of Yuwakas, and a polynomial on the holomorphic generators ensuring that  $F_n^{(0)}$  has weight zero. The  $a_{n,i}$  are now just constants.

Then we expand  $\mathcal{F}_D^{(0)}$  around  $\mathbf{a}_D$ . As said, near  $\mathbf{a}_D \sim 0$  the potential looks like a harmonic oscillator and the singular behaviour is given by (3.2.151),

$$\mathcal{F}_{D,n}^{(0)}(\mathbf{a}_D) = \left(\frac{1}{2}\right)^{2n-2} \frac{(1-2^{1-2n})B_{2n}}{2n(2n-1)(2n-2)} \frac{1}{\mathbf{a}_D^{2-2n}} + O(\mathbf{a}_D^0), \quad (3.2.161)$$

The factor of  $1/2$  is due to the normalization of the periods (recall  $\beta = -1/2$ ). This sets the singular part, plus the *gap* condition: all further corrections are regular in  $\mathbf{a}_D$ . Enforcing this structure on  $F_n^{(0)}$  is enough to fix completely the ansatz for  $f_{n,0}$ . The first few modular free energies look like this

$$\begin{aligned} F_2^{(0)} &= \frac{Y^2}{4423680} \left[ 10\hat{E}_2 K_2^2 + K_2^3 - 75K_2 L_2^2 \right], \\ F_3^{(0)} &= \frac{Y^4}{10273695989760} \left[ -140\hat{E}_2^3 K_2^3 + \right. \\ &\quad + \hat{E}_2^2 (840K_2^4 + 1260K_2^2 L_2^2) + \\ &\quad + \hat{E}_2 (21K_2^5 - 28287K_2^3 L_2^2 - 9450K_2 L_2^4) + \\ &\quad \left. + 769K_2^6 + 310500K_2^2 L_2^4 + 87012K_2^4 L_2^2 + 43875L_2^6 \right], \\ F_4^{(0)} &= \frac{Y^6}{15149181158660505600} \left[ 2800\hat{E}_2^5 K_2^4 + \right. \\ &\quad + \hat{E}_2^4 (-25200K_2^5 - 33600K_2^3 L_2^2) + \\ &\quad + \hat{E}_2^3 (75320K_2^6 + 671160K_2^4 L_2^2 + 226800K_2^2 L_2^4) + \\ &\quad + \hat{E}_2^2 (-3080K_2^7 - 5677560K_2^5 L_2^2 - 7197120K_2^3 L_2^4 - 756000K_2 L_2^6) + \\ &\quad + \hat{E}_2 (122907K_2^8 + 86435100K_2^2 L_2^6 + 190207143K_2^4 L_2^4 + \\ &\quad \quad + 28382262K_2^6 L_2^2 + 1417500L_2^8) + \\ &\quad + 281309K_2^9 - 69872514K_2^7 L_2^2 - 1513461213K_2^5 L_2^4 - \\ &\quad \left. - 3058152300K_2^3 L_2^6 - 725861250K_2 L_2^8 \right]. \end{aligned} \quad (3.2.162)$$

We can also consider (although it is not necessary to fix the ambiguity in this example) the electric frame

$$[F^{(0)}]_e = \mathcal{F}^{(0)}. \quad (3.2.163)$$

This is the natural frame for using  $\mathbf{a}$  as a variable (3.2.117). In it the free energies also have a simple singular behaviour,

$$\mathcal{F}_n^{(0)}(\mathbf{a}) = (-1)^{n-1} \left(\frac{1}{2}\right)^{2n-2} \frac{2B_{2n}}{2n(2n-1)(2n-2)} \frac{1}{\mathbf{a}^{2n-2}} + O\left(\frac{1}{\mathbf{a}^{2n+2}}\right). \quad (3.2.164)$$

### 3.2.5 Large order of the perturbative sector

In the introduction, we mentioned how the large order of an asymptotic series is intrinsically connected to the non-analytic part of the original function. This is what

is usually called *resurgence*<sup>3</sup>. Let us suppose we have a function with an asymptotic expansion

$$f(\hbar) \simeq \sum_m f_m \hbar^m. \quad (3.2.165)$$

The coefficients can be recovered via residues,

$$f_m = \frac{1}{2\pi i} \oint_{\gamma_0} \frac{f(\hbar)}{\hbar^{m+1}} d\hbar \quad (3.2.166)$$

where  $\gamma_0$  is a loop around  $\hbar = 0$ . We can now deform the contour away from zero. It will have to wrap around possible branch cuts  $\gamma_k$  in the  $\hbar$  plane. This gives integrals that go around both sides of the branch cuts  $\gamma_k$ , capturing the discontinuity,

$$f_m = \frac{1}{2\pi i} \sum_k \int_{\gamma_k} \frac{\text{disc } f(\hbar)}{\hbar^{m+1}} d\hbar. \quad (3.2.167)$$

If we can evaluate this integral as a function of  $m$ , we can obtain the large order of the  $f_m$ .

The key point of resurgence is the following. Suppose we have some non-perturbative definition of the asymptotic series. This can be an integral, as in the introduction, or a differential equation. Then we can use it to find non-analytic exponentially small corrections to the asymptotic series,

$$f(\hbar) \simeq \sum_m f_m \hbar^m + e^{-A/\hbar} \sum_{r=0}^{\infty} f_r^{(1)} \hbar^{r-b} + O(e^{-2A/\hbar}) \quad (3.2.168)$$

No doubt, we might find solutions with different values for the *action*  $A$ . This is important if we want the corrections to be truly exponentially small – a positive  $A$  will make sense for positive real part of  $\hbar$ , but not on the other side of the complex plane. The classical example is the Airy function,

$$\text{Ai}(z) = \frac{1}{2\pi i} \int_{e^{-i\pi/3}\infty}^{e^{i\pi/3}\infty} e^{\frac{t^3}{3} - zt} dt. \quad (3.2.169)$$

To find its asymptotic expansion around  $z = \infty$ , we can either do a saddle point computation of the integral, or see that it satisfies the equation

$$f''(z) - zf(z) = 0. \quad (3.2.170)$$

Then we can pose the ansatz

$$f(z) = \frac{e^A z^{3/2}}{z^{1/4}} \left( c_0 + \frac{c_1}{z^{3/2}} + \frac{c_2}{z^{6/2}} + \dots \right). \quad (3.2.171)$$

The condition at the first order is

$$9A^2 - 4 = 0, \quad (3.2.172)$$

---

<sup>3</sup>Because the non-perturbative sector “resurges” if we know the whole perturbative one

so that there are two different solutions. For positive  $z$ , we find that the asymptotic expansion

$$f_-(z) = \frac{e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}z^{1/4}} \left( 1 - \frac{5}{48z^{3/2}} + \frac{385}{4608z^{6/2}} + \dots \right) \quad (3.2.173)$$

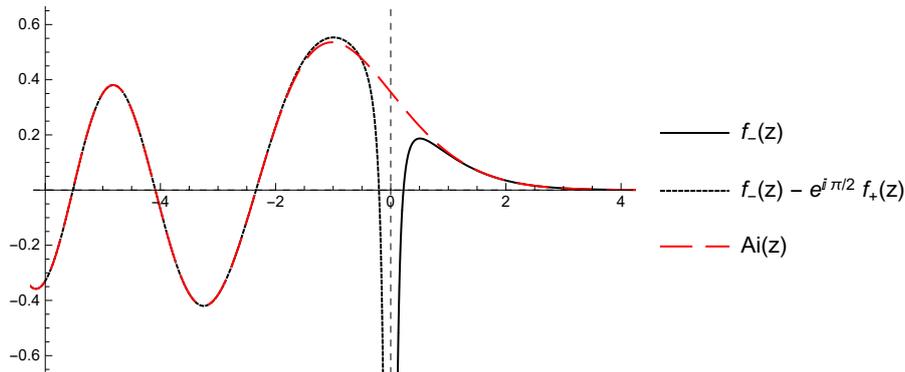
approximates fairly well the Airy function. However, this cannot be the case for negative  $z$ , since first of all, the Airy function in the negative real line is oscillating, and more importantly, real. What we have to do is consider the *other* solution,

$$f_+(z) = \frac{e^{\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}z^{1/4}} \left( 1 + \frac{5}{48z^{3/2}} + \frac{385}{4608z^{6/2}} + \dots \right) \quad (3.2.174)$$

Making a consistent choice for the branch of  $z$ , for instance  $z^{1/4} \mapsto e^{-i\pi/4}(-z)^{1/4}$ , one can see that the Airy function gets approximated very well by

$$f_-(z) - e^{i\pi/2}f_+(z), \quad (3.2.175)$$

which is real and produces the required oscillations. We have plotted them to illus-



**Figure 3.1:** Stokes phenomenon on the Airy function

trate this point in figure 3.1, with just the first subleading correction of  $f_{\pm}$  included. The lesson is that as we move around in the  $z$  complex plane, the different solutions become relevant or disappear from the asymptotic expansion of the function. This is known as the *Stokes phenomenon*.

Let us go back to our  $f(\hbar)$  asymptotic series, and assume that like the Airy function it solves an equation that also determines a family of exponentially small solutions. If there is a discontinuity in the  $\hbar$  plane, it means that on each side a different asymptotic expansion will be relevant – just like different expansions are relevant for  $x > 0$  and  $x < 0$  in the Airy function. Since this difference is captured by the exponentially small solutions, we will have

$$\text{disc } f(\hbar) = S f^{(1)}(\hbar) \quad (3.2.176)$$

where the proportionality  $S$  is called the *Stokes constant*, and the one-instanton correction is

$$f^{(1)} = \hbar^{-b} e^{-A/\hbar} \sum_{r=0}^{\infty} f_r^{(1)} \hbar^r \quad (3.2.177)$$

for some action  $A$ . Suppose for simplicity that there is a single branch cut along the positive real axis. Then,

$$\begin{aligned} f_m &= \frac{S}{2\pi i} \sum_{r=0}^{\infty} f_m^{(1)} \int_0^{\infty} \hbar^{r-b-m-1} e^{-A/\hbar} d\hbar = \\ &= \frac{S}{2\pi i} \sum_{r=0}^{\infty} \frac{\Gamma(m+b-r)}{A^{m+b-r}} f_r^{(1)} = \\ &= \frac{S}{2\pi i} \frac{\Gamma(m+b)}{A^{m+b}} \sum_{r=0}^{\infty} \frac{A^r f_r^{(1)}}{(m-r)_r} \end{aligned} \quad (3.2.178)$$

where  $(x)_r = x(x+1)\dots(x+r-1)$  is the Pochhammer symbol. In general, one would get contributions from each discontinuity. This fully determines the large order of  $f_m$  as a  $1/m$  expansion from the one-instanton term  $\kappa_r^{(1)}$ , up a single undetermined constant  $S$ .

Let us assume then that the free energies  $\mathcal{F}_n^{(0)}$  are indeed the solution to some differential equation, defined non-perturbatively. This will be the subject of the next section: the holomorphic anomaly recursion can in fact be upgraded to a full differential equation that determines exponentially small corrections. For now it will suffice to have faith in that the large order of  $\mathcal{F}_n^{(0)}$  can be parametrized by (3.2.178). Actually, the WKB free energies are actually a series in  $\hbar^2$  instead of  $\hbar$ , so we have a faster rate of divergence<sup>4</sup>,  $(2n)!$ , like the one found in string theory [9]. The large order relation reads

$$\mathcal{F}_n^{(0)} \simeq \sum_{r=0}^{\infty} \mu_r \frac{\Gamma(2n+b-r)}{\mathcal{A}^{2n+b-r}} + O\left[\frac{1}{\tilde{\mathcal{A}}^{2n}}\right] \quad (3.2.179)$$

where  $\mathcal{A}$  is the action which controls the main rate of divergence for  $\mathcal{F}_n^{(0)}$ , and  $\mu_r$  the subleading  $1/n^r$  corrections coming from the gamma function. We have denoted  $\tilde{\mathcal{A}}$  as the next action in absolute value,  $|\tilde{\mathcal{A}}| > |\mathcal{A}|$ .  $\mathcal{A}$  itself can be retrieved via the numeric limit

$$\mathcal{A}^2 = \lim_{n \rightarrow \infty} \frac{4n^2 \mathcal{F}_n^{(0)}}{\mathcal{F}_{n+1}^{(0)}}, \quad (3.2.180)$$

and similar expressions can be found for  $b$ ,  $\mu_0$ ,  $\mu_1$ , etc. The process can be improved with Richardson transforms, a numerical tool defined in the following way. Consider a sequence  $(a_n)$  for which we expect a rate of convergence as

$$a_n = b_0 + \frac{b_1}{n} + \frac{b_2}{n^2} + \dots \quad (3.2.181)$$

<sup>4</sup>Of course, the important thing is the relation between the parameter of the perturbative series,  $\hbar^2$ , and the one appearing in the exponentially small corrections,  $\exp(-A/\hbar)$ . If we had a series in  $\hbar^2$  with instantons like  $\exp(-A/\hbar^2)$ , the divergence would still go as  $n!$ .

The  $j$ -th Richardson transform of  $(a_n)$  is given by

$$\mathcal{R}_j[(a_n)] = \left( a_n + \frac{n}{j}(a_{n+1} - a_n) \right). \quad (3.2.182)$$

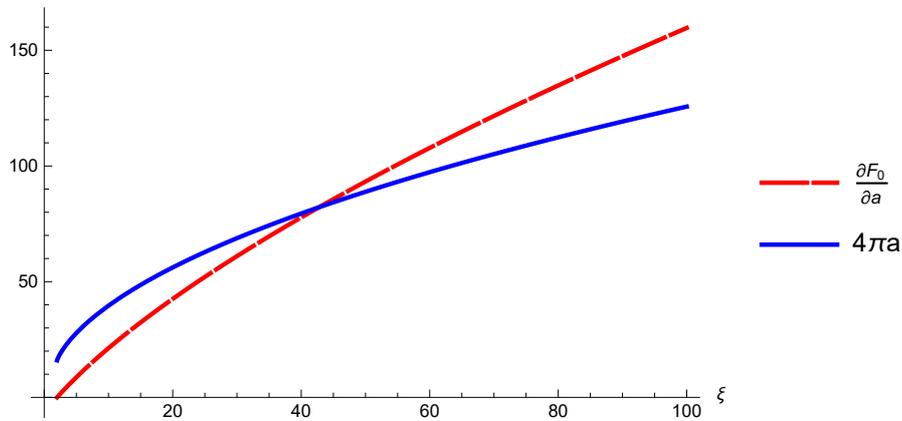
The effect of  $\mathcal{R}_j$  is to eliminate the term converging like  $n^{-j}$ . Then,

$$\mathcal{R}_j[\mathcal{R}_{j-1}[\dots[\mathcal{R}_1[(a_n)]]]] = \left( b_0 + \frac{\#}{n^{j+1}} + \frac{\#}{n^{j+2}} + \dots \right). \quad (3.2.183)$$

In theory, for a complete sequence the iteration would eventually just return the constant sequence  $(b_0)$ . In practice, every Richardson transform reduces a finite sequence by one term, placing an absolute limit on the number of iterations. And normally, much before that, noise takes over, since the  $n(a_{n+1} - a_n)$  will actually amplify any component that is not strictly dominated by the pole we are trying to eliminate. Despite these limitations, it is a very powerful tool to evaluate sequences that are in some measure well behaved, like (3.2.179). Applying this technique to the aforementioned limit, one finds two actions given by the two periods,

$$\begin{aligned} \mathcal{A}_{\text{small } \xi} &= i \partial_{\mathbf{a}} \mathcal{F}_0^{(0)}, \\ \mathcal{A}_{\text{large } \xi} &= 4\pi \mathbf{a}, \end{aligned} \quad (3.2.184)$$

and the value  $b = -2$ . In figure 3.2 we clearly see that there are two regions of



**Figure 3.2:** Absolute value of the large order actions

dominance, at small and large  $\xi$ .

The large order near the singular points can be derived analytically from the harmonic behaviour. Use the electric frame around  $\mathbf{a} \rightarrow 0$ , from (3.2.164), and as we get closer to the singularity, we expect

$$\begin{aligned} \mathcal{A}^2 &= 16\pi^2 \mathbf{a}^2, \\ b &= -2, \\ \mu_0 &\rightarrow \frac{1}{\pi^2}, \quad \mu_1 \rightarrow 0, \quad \mu_2 \rightarrow 0, \quad \dots \end{aligned} \quad (3.2.185)$$

We have used the asymptotic behaviour of the Bernoulli numbers,

$$B_{2n} = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \left( 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \dots \right). \quad (3.2.186)$$

Doing the same limits in the magnetic frame, around  $\mathbf{a}_D \rightarrow 0$ , (3.2.161), one obtains

$$\begin{aligned} \mathcal{A}^2 &= -16\pi^2 \mathbf{a}_D^2, \\ b &= -2, \\ \mu_0 &\rightarrow \frac{1}{2\pi^2}, \quad \mu_1 \rightarrow 0, \quad \mu_2 \rightarrow 0, \quad \dots \end{aligned} \quad (3.2.187)$$

These are precisely the actions that we found numerically. The  $\mu_0$  coefficients also match the numerical results, which are constant, independent of  $\xi$ .

The corrections in (3.2.179) can include further actions, as

$$\mathcal{F}_n^{(0)} \simeq \sum_m \sum_{r=0}^{\infty} \mu_r^{(m)} \frac{\Gamma(2n + b^{(m)} - r)}{[\mathcal{A}^{(m)}]^{2n + b^{(m)} - r}}. \quad (3.2.188)$$

As we will see in the next section, a correction like  $\exp(-\mathcal{A}/\hbar)$  implies a family of smaller ones like  $\exp(-m A/\hbar)$ . The let us focus on

$$\mathcal{A}^{(m)} = m \mathcal{A}_1, \quad b^{(m)} = -2. \quad (3.2.189)$$

The corrections from the singular limit in the electric frame (3.2.164), when  $\xi \rightarrow \infty$ , are

$$\mu_0^{(2)} \rightarrow \frac{1}{4\pi^2}, \quad \mu_1^{(2)} \rightarrow 0, \dots \quad (3.2.190)$$

$$\mu_0^{(3)} \rightarrow \frac{1}{9\pi^2}, \quad \mu_1^{(3)} \rightarrow 0, \dots \quad (3.2.191)$$

$$\dots \quad (3.2.192)$$

$$\mu_0^{(n)} \rightarrow \frac{1}{(2n)^2 \pi^2}. \quad (3.2.193)$$

and for magnetic frame (3.2.161) at  $\xi \rightarrow 2$ ,

$$\mu_0^{(2)} \rightarrow -\frac{1}{8\pi^2}, \quad \mu_1^{(2)} \rightarrow 0, \dots \quad (3.2.194)$$

$$\mu_0^{(3)} \rightarrow \frac{1}{18\pi^2}, \quad \mu_1^{(3)} \rightarrow 0, \dots \quad (3.2.195)$$

$$\dots \quad (3.2.196)$$

$$\mu_0^{(n)} \rightarrow \frac{(-1)^{n+1}}{2(2n)^2 \pi^2}. \quad (3.2.197)$$

### 3.3 NS free energy transseries

#### 3.3.1 Master equation

Since we started analysing the modified Mathieu potential, we have shown what is doubtlessly an interesting tool, but a well known one. The promised purpose of this chapter was to improve it, by going beyond the realm of perturbation. By design, the WKB method and the usual holomorphic anomaly miss any information on the problem that cannot be expressed as an analytic function<sup>5</sup> of  $\hbar$ . Of course, finding exact, closed expressions for, say, the energy levels or the wavefunctions is a dream that one can only afford to pursue in the simplest scenarios (see harmonic oscillators), or wherever we are lucky to find enough symmetries as to reduce the system to something tractable (see, more often than not, harmonic oscillators). The whole reason we use power series is that they are *systematic*, and do not ask from us to have a lucky day in order to tackle a problem. The next logical step is to upgrade power series so that they can at least capture something beyond the perturbative expansion. This is the idea behind *transseries*.

Of course, a priori there are all kinds of extremely pathological functions we could consider as candidates for extending normal power series. Luckily, we are physicists and we can use our physical intuition to make that choice. The typical case of a non-analytic function one often finds around is

$$f(\hbar) = e^{-A/\hbar}, \quad (3.3.198)$$

which cannot be expanded around  $\hbar = 0$ . It appears in the WKB wavefunction itself (3.1.3), through it in tunneling amplitudes, in the path integral formulation of quantum mechanics and quantum field theory, and in many other situations. And not any less than those, in the non-perturbative completion to the WKB periods of difference operators conjectured in the previous chapter (2.1.92). The transseries we will be considering in our construction is then a two-parameter expansion of the form

$$f(\hbar) = \sum_m \sum_n c_n^{(m)} \hbar^n e^{-mA/\hbar}. \quad (3.3.199)$$

Following the ideas of [28] for the standard topological string, we will write a holomorphic anomaly that allows us to compute transseries corrections to the free energy. More than that, we will show that once the perturbative sector is known – what we have been doing until this point – the exponentially small corrections are given with a universal structure, not directly dependent on the details of the perturbative sector. We will provide a systematic way to calculate them, which in a similar spirit to the perturbative holomorphic anomaly consists of transmuting differential equations into algebraic rules. By all means, we will test these results. First, we will use the *resurgent* relation between transseries corrections and large order behaviour, to provide concrete predictions for the asymptotic growth of the

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<sup>5</sup>Or at least an asymptotic series.

WKB free energies and periods. And finally, we will recover the transseries expansion of the previously known quantization condition from the universal structure of our solution.

To begin this program, recall the perturbative free energy,

$$F^{(0)} = \sum_{n=0}^{\infty} \hbar^{2n} F_n^{(0)}. \quad (3.3.200)$$

The recursion (3.2.86) can be retrieved from a general equation for  $F$ , where we no longer use the superindex (0) since we will no longer assume a perturbative expansion. Define the *full quantum corrections* to the prepotential as

$$\tilde{F} = F - F_0^{(0)}. \quad (3.3.201)$$

Then the holomorphic anomaly equation (3.2.75) is a consequence of

$$\partial_S \tilde{F} - \frac{1}{2} \left( D_a \tilde{F} \right)^2 = 0, \quad (3.3.202)$$

that we call *master equation*, as in the naming of [26, 28].

To see this, expand

$$\tilde{F} = \sum_{n=1}^{\infty} F_n^{(0)} \hbar^{2n}, \quad (3.3.203)$$

and it follows that

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} \partial_S F_n^{(0)} \hbar^{2n} - \frac{1}{2} \sum_{n=1}^{\infty} D_a F_n^{(0)} \hbar^{2n} \sum_{m=1}^{\infty} D_a F_m^{(0)} \hbar^{2m} = \\ &= \sum_{n=1}^{\infty} \hbar^{2n} \left[ \partial_S F_n^{(0)} - \frac{1}{2} \sum_{m=1}^{n-1} D_a F_{n-m}^{(0)} D_a F_m^{(0)} \right]. \end{aligned} \quad (3.3.204)$$

### 3.3.2 Expanding the $D_\tau$ algebra for the actions

To get started, we would at least look for something of the form

$$F = F^{(0)} + F^{(1)} e^{-\mathcal{A}/\hbar} + \dots \quad (3.3.205)$$

The term  $F^{(1)}$  is the *one-instanton* correction. As we found in the previous section, the action  $\mathcal{A}$  will be some combination of the periods. However, this poses a problem, since the periods (3.2.96) and (3.2.97) are clearly not modular forms – even worse, they appear in an exponent with a weight different from zero. We need a way to understand how  $D_\tau$  will act upon such objects. Because  $D_\tau$  is a graded derivative, the only way we can make sense of  $\exp(-\mathcal{A}/\hbar)$  is if  $\mathcal{A}$  has zero weight with respect to  $D_\tau$ , *even if* it transforms non-trivially under  $S$ . To do that, we will expand slightly the usual ring of modular forms.

To this end, rewrite the Maass derivative (3.2.69) using  $\hat{E}_2$ ,

$$\begin{aligned} \hat{E}_2 - E_2 &= \frac{3}{\pi \operatorname{Im}\tau} \implies \\ \implies D_\tau &= \frac{1}{2\pi i} \partial_\tau + \frac{k}{12} (\hat{E}_2 - E_2) \end{aligned} \quad (3.3.206)$$

when acting upon an object of weight  $k$ . This introduces  $E_2$  into the algebra, but it does not transform just like a modular form. Define then

$$E_2^D = E_2 + \frac{6}{i\pi\tau}, \quad (3.3.207)$$

so that

$$\begin{aligned} E_2(-1/\tau) &= \tau^2 E_2^D(\tau), \\ E_2^D(-1/\tau) &= \tau^2 E_2(\tau). \end{aligned} \quad (3.3.208)$$

The definition (3.3.207) is valid over the fundamental domain, where

$$|\operatorname{Re} \tau| < \frac{1}{2}, \quad |\tau| > 1. \quad (3.3.209)$$

In order to keep the whole algebra controlled under the modular group, remember that under  $T$  transformations

$$q = e^{i\pi\tau} \mapsto -q \quad (3.3.210)$$

so that

$$E_2(\tau + 1) = E_2(\tau). \quad (3.3.211)$$

Then we extend  $E_2^D$  outside of the fundamental domain by requiring

$$E_2^D(\tau + 1) = E_2^D(\tau). \quad (3.3.212)$$

For instance, with this transformation rules, the combinations

$$E_2 + E_2^D, \quad E_2 E_2^D \quad (3.3.213)$$

are modular forms of weight 2 and 4 respectively. It follows that the weight of  $E_2$  and  $E_2^D$  is 2, in order to be consistent with the grading used in the Maass derivative. This means that

$$\begin{aligned} D_\tau E_2 &= \frac{1}{2\pi i} \partial_\tau E_2 + \frac{2}{12} (\hat{E}_2 - E_2) E_2 = \\ &= \frac{1}{12} (2\hat{E}_2 - E_2) E_2 - \frac{1}{48} (K_2^2 + 3L_2^2). \end{aligned} \quad (3.3.214)$$

To compute  $D_\tau E_2^D$ , notice that  $\tau$  itself has weight  $-2$  when transforming, so

$$D_\tau \left( \frac{1}{i\pi\tau} \right) = -\frac{1}{2} \frac{1}{(i\pi\tau)^2} - \frac{1}{6} \frac{\hat{E}_2 - E_2}{i\pi\tau}, \quad (3.3.215)$$

and then we have an analogous form of the derivation for  $E_2^D$ ,

$$\begin{aligned} D_\tau E_2^D &= D_\tau E_2 + D_\tau \left( \frac{6}{i\pi\tau} \right) = \\ &= \frac{1}{12} \left( 2\hat{E}_2 - E_2^D \right) E_2^D - \frac{1}{48} (K_2^2 + 3L_2^2). \end{aligned} \quad (3.3.216)$$

Finally, as was already mentioned, our objective is to work only with objects which explicitly have weight zero. The periods in (3.2.96) have the “wrong” one to be inside an exponential (seeing forms as a graded space, they are not scalars). Define for this purpose a degree-counting constant of weight one,  $\omega_1$ . For any evaluation it should be taken to  $\omega_1 \mapsto 1$ . Because it has a non-zero weight, its Maass derivative will be non-trivial.

The full differential algebra with of this extended modular ring is

$$\begin{aligned} D_\tau K_2 &= \frac{1}{6} \hat{E}_2 K_2 + \frac{1}{12} (3L_2^2 - K_2^2), \\ D_\tau L_2 &= \frac{1}{6} \hat{E}_2 L_2 + \frac{1}{6} K_2 L_2, \\ D_\tau \hat{E}_2 &= \frac{1}{12} \hat{E}_2 \hat{E}_2 - \frac{1}{48} (K_2^2 + 3L_2^2), \\ D_\tau E_2 &= \frac{1}{6} \hat{E}_2 E_2 - \frac{1}{48} (K_2^2 + 3L_2^2 + 4E_2^2), \\ D_\tau E_2^D &= \frac{1}{6} \hat{E}_2 E_2^D - \frac{1}{48} (K_2^2 + 3L_2^2 + 4(E_2^D)^2), \\ D_\tau \omega_1 &= \frac{1}{12} (\hat{E}_2 - E_2) \omega_1. \end{aligned} \quad (3.3.217)$$

We also need to know how to define the magnetic frame for these objects. The Eisenstein series will be exchanged with their dual. As for  $\omega_1$ , having weight one makes it acquire a  $\tau$  factor, which we can introduce with (3.3.207). The magnetic map is expanded to

$$\begin{aligned} M[K_2] &= \frac{-K_2 - 3L_2}{2}, \quad M[L_2] = \frac{-K_2 + L_2}{2}, \\ M[\hat{E}_2] &= \hat{E}_2, \quad M[E_2] = E_2^D, \quad M[E_2^D] = E_2, \quad M[\omega_1] = \frac{i\pi}{6} \frac{E_2 - E_2^D}{\omega_1}. \end{aligned} \quad (3.3.218)$$

As we wanted, what  $\omega_1$  does is allow us to treat objects with non-zero weight **as if they were of zero weight for the purposes of derivation**, while ensuring that they gain the right  $\tau$  factor when doing an S transform. It can be easily checked that  $M[\cdot]$  commutes with  $D_\tau$ . This is important to maintain the behaviour of  $D_a$  in (3.2.115). We should also write the effect of the  $T$  transform,  $\tau \mapsto \tau + 1$ , which leaves everything invariant except for

$$L_2 \mapsto -L_2, \quad (3.3.219)$$

since  $L_2$  has odd powers of  $q$  in its expansion.

We have already found that the actions are (3.2.185) and (3.2.187). From the periods (3.2.96) and (3.2.97), the zero-weight corresponding objects are

$$\begin{aligned}\mathcal{A}_A(\tau) &= 4\pi\mathfrak{a} \frac{1}{\omega_1} = \frac{8\pi}{3} \frac{E_2 + K_2}{\omega_1 \sqrt{L_2}}, \\ \mathcal{A}_B(\tau) &= i\partial_{\mathfrak{a}}\mathcal{F}_0^{(0)} \omega_1 = \frac{16i}{\sqrt{L_2}} \frac{\omega_1 E_2^D + K_2}{E_2 - E_2^D}.\end{aligned}\tag{3.3.220}$$

Even if it is not obvious at first glance from (3.3.217), their (first) derivatives are just what one would expect,

$$D_a \mathcal{A}_A = -\frac{1}{2} Y D_\tau \mathcal{A}_A = \frac{4\pi}{\omega_1},\tag{3.3.221}$$

and for the B period (compare with (3.2.103)),

$$D_a \mathcal{A}_B = \frac{24i}{E_2 - E_2^D} \omega_1 = 4\pi\tau \omega_1,\tag{3.3.222}$$

both with weight  $-1$ . Here we can also see explicitly how the two are exchanged upon an S-transform.

To see the usefulness of  $\omega_1$  as a degree counting parameter, consider the expansion of the actions in terms of  $\mathfrak{a}$ ,

$$\begin{aligned}\mathcal{A}_A &= 4\pi\mathfrak{a}, \\ -i \mathcal{A}_B &= 8\mathfrak{a} (\log(2\mathfrak{a}) - 1) + \frac{1}{\mathfrak{a}^3} + \frac{15}{32\mathfrak{a}^7} + \frac{15}{32\mathfrak{a}^{11}} + \dots\end{aligned}\tag{3.3.223}$$

Since they are holomorphic to begin with, a frame choice does not change the functions themselves. Still, we can still use the rules (3.3.218) to expand around the dual variable. For instance, to write  $K_2(\tau)$  in terms of  $K_2(\tau_D)$  and  $L_2(\tau_D)$ . The periods themselves, having non zero weight, should get an overall  $\tau$  factor. Precisely  $\omega_1$  keeps track of this.

$$\begin{aligned}\mathcal{A}_A &= \frac{8\pi}{3} \frac{E_2(\tau) + K_2(\tau)}{\sqrt{L_2}} \frac{1}{\omega_1} = \\ &= \frac{8\pi}{3} \frac{E_2^D(\tau_D) - \frac{K_2(\tau_D) + L_2(\tau_D)}{2}}{\sqrt{\frac{L_2(\tau_D) - K_2(\tau_D)}{2}}} \frac{6i}{\pi} \frac{\omega_1}{E_2^D(\tau_D) - E_2(\tau_D)} = \\ &= 8i\sqrt{2} \frac{K_2(\tau_D) + 3L_2(\tau_D) - 2E_2^D(\tau_D)}{[E_2(\tau_D) - E_2^D(\tau_D)] \sqrt{L_2 - K_2}} \omega_1.\end{aligned}\tag{3.3.224}$$

Expanding this (and the equivalent for  $\mathcal{A}_B$ ) with (3.2.117),

$$\begin{aligned}\mathcal{A}_A &= 16 + 4\mathfrak{a}_D \left( \log\left(\frac{\mathfrak{a}_D}{16}\right) - 1 \right) + \frac{3\mathfrak{a}_D^2}{4} + \frac{5\mathfrak{a}_D^3}{32} + \frac{55\mathfrak{a}_D^4}{1024} + \dots \\ -i \mathcal{A}_B &= -4\pi \mathfrak{a}_D,\end{aligned}\tag{3.3.225}$$

which are just the periods of the dual free energy. Notice that the transformation rule for  $\omega_1$  is key to “pass” the log term –or rather,  $\tau$  term– from the  $A$  to the  $B$  period.

Now, something very non trivial happens with the second derivative of the periods. Simply following the algebra (3.3.217),

$$\begin{aligned} D_a \left( D_a \mathcal{A}_A \cdot \left( \hat{E}_2 - E_2 \right) \right) &= D_a \left( 4\pi \frac{\hat{E}_2 - E_2}{\omega_1} \right) = 0, \\ D_a \left( D_a \mathcal{A}_B \cdot \left( \hat{E}_2 - E_2^D \right) \right) &= D_a \left( 24i \omega_1 \frac{\hat{E}_2 - E_2^D}{E_2 - E_2^D} \right) = 0. \end{aligned} \quad (3.3.226)$$

This implies that  $D_{aa}\mathcal{A}$  is just proportional to  $D_a\mathcal{A}$ , a ver remarkable fact we will use later.

### 3.3.3 Instanton recursion

#### 3.3.3.1 One instanton solution

To actually solve (3.3.202), we will write the transseries ansatz in a slightly modified way, as

$$\tilde{F} = \tilde{F}^{(0)} + \hbar^{-b} f^{(1)} e^{-G/\hbar} + \dots \quad (3.3.227)$$

The exponentially  $\hbar$ -suppressing function  $G$  should include the action  $\mathcal{A}$ , but also the  $\hbar$ -corrections of  $F^{(1)}$ , so that  $f^{(1)}$  is just a constant. Introduce it into (3.3.202),

$$\begin{aligned} 0 &= \partial_S \tilde{F}^{(0)} - \frac{1}{2} \left( D_a \tilde{F}^{(0)} \right)^2 + \\ &\quad - \hbar^{-1} \left( \partial_S G - D_a \tilde{F}^{(0)} D_a G \right) \hbar^{-b} f^{(1)} e^{-G/\hbar} + \dots \end{aligned} \quad (3.3.228)$$

and at the first instanton level, the master equation requires

$$\partial_S G - D_a \tilde{F}^{(0)} D_a G = 0. \quad (3.3.229)$$

As we will see later, this equation will play a fundamental role for higher instanton solutions. Consider the ansatz

$$G = \mathcal{A} + \left( (S - \mathcal{S}_A) D_a \mathcal{A} \right) D_a \tilde{F}^{(0)}, \quad (3.3.230)$$

where  $\mathcal{A}$  has weight zero,  $\mathcal{S}_A$  weight two, and both are holomorphic

$$\partial_S \mathcal{A} = \partial_S \mathcal{S}_A = 0. \quad (3.3.231)$$

Of course, we choose this form of the ansatz to make explicit the role of  $\mathcal{A}$  as the action. Since  $\mathcal{A}$  has weight zero, by (3.2.91), we can commute  $\partial_S$  and  $D_a$ ,

$$\partial_S (D_a \mathcal{A}) = D_a (\partial_S \mathcal{A}) = 0. \quad (3.3.232)$$

They also commute on  $\tilde{F}^{(0)}$ , so

$$\begin{aligned}\partial_S G &= D_a \mathcal{A} D_a \tilde{F}^{(0)} + \\ &\quad + \left( (S - \mathcal{S}_A) D_a \mathcal{A} \right) D_{aa} \tilde{F}^{(0)} D_a \tilde{F}^{(0)}, \\ D_a G &= D_a \mathcal{A} + D_a \left( (S - \mathcal{S}_A) D_a \mathcal{A} \right) D_a \tilde{F}^{(0)} + \\ &\quad + \left( (S - \mathcal{S}_A) D_a \mathcal{A} \right) D_{aa} \tilde{F}^{(0)},\end{aligned}\tag{3.3.233}$$

and (3.3.229) is verified if

$$D_a \left( (S - \mathcal{S}_A) D_a \mathcal{A} \right) = 0\tag{3.3.234}$$

which gives the equation for  $\mathcal{S}_A$ , once we know  $\mathcal{A}(\tau)$ . This and the constant  $f^{(1)}$  can be fixed by the large order behaviour (3.2.185) and (3.2.187), which specifies the action and the value of  $\mu_0$ .

Recall that the propagator (3.2.80) is given by

$$S = \hat{E}_2/24.\tag{3.3.235}$$

Then (3.3.234) is verified precisely by taking  $\mathcal{S}_A$  to be a certain holomorphic limit of the propagator<sup>6</sup>, as follows from (3.3.226). In particular,

$$\begin{aligned}S - \mathcal{S}_{A,A} &= \frac{1}{24} \left( \hat{E}_2 - E_2 \right) = \frac{1}{4\pi} \frac{i}{\bar{\tau} - \tau}, \\ S - \mathcal{S}_{A,B} &= \frac{1}{24} \left( \hat{E}_2 - E_2^D \right) = \frac{1}{4\pi\tau} \frac{i\bar{\tau}}{\bar{\tau} - \tau}.\end{aligned}\tag{3.3.236}$$

The one-instanton solutions,  $F^{(1)} e^{-\mathcal{A}/\hbar} = f^{(1)} e^{-G/\hbar}$ , are given by plugging (3.3.221) and (3.3.222) into the ansatz for  $G$ ,

$$G_A(\tau, \bar{\tau}) \omega_1 = 4\pi \mathbf{a}(\tau) + \frac{i}{\bar{\tau} - \tau} D_a \tilde{F}^{(0)}(\tau, \bar{\tau}).\tag{3.3.237}$$

$$G_B(\tau, \bar{\tau}) \omega_1^{-1} = i \partial_a \mathcal{F}_0^{(0)}(\tau) + \frac{i\bar{\tau}}{\bar{\tau} - \tau} D_a \tilde{F}^{(0)}(\tau, \bar{\tau}).\tag{3.3.238}$$

We get the first instanton series with

$$F^{(1)} = \hbar^{-b} f^{(1)} e^{-\frac{G}{\hbar}} = \left( \sum_{n=0}^{\infty} F_n^{(1)} \hbar^{n-b} \right) e^{-\frac{A}{\hbar}}.\tag{3.3.239}$$

The subleading  $\hbar$  corrections to  $F^{(1)}$  are generated precisely by the term  $D_a \tilde{F}^{(0)}$ . In particular, in the corresponding holomorphic limits  $\bar{\tau} \mapsto i\infty$  and  $\bar{\tau} \mapsto 0$ , there are no  $\hbar$  corrections at all for the  $A$  and  $B$  periods. This had better be the case, because it is exactly what is required by the subleading corrections to the large order (3.2.185) and (3.2.187). Let us discuss with more detail the connection between the transseries solution and the large order behaviour in the next section. From now on we will also set the value  $b = -2$ .

<sup>6</sup>This works just as well for other values of  $\beta$ , of course, since (3.3.234) is linear in  $S$ .

### 3.3.3.2 Resurgence of the large order

Now that we know that the free energies and exponentially small corrections to them can be computed from a single, non-perturbative master equation (3.3.202), we are justified to identify the  $\mu_r$  coefficients in (3.2.179) with

$$\mu_n = S \frac{F_n^{(1)}}{2\pi i}. \quad (3.3.240)$$

We will take the Stokes constant  $S$  to just be  $S = 1$ , since it can be reabsorbed in the definition of  $f^{(1)}$ . From the singular behaviour, we know what the large order of  $F_n^{(0)}$  at the singular points must be determined by (3.2.185) and (3.2.187), which fixes

$$f_A^{(1)} = (2\pi i) \frac{1}{\pi^2}, \quad f_B^{(1)} = (2\pi i) \frac{1}{2\pi^2}, \quad b = -2, \quad (3.3.241)$$

since the subleading terms in  $G$  vanish for each action in its respective frame.

The transseries reproduces the large order in the holomorphic limits  $\bar{\tau} \rightarrow \infty$ ,  $\bar{\tau} \rightarrow 0$ ; in which we can also see the trivial asymptotics on the corresponding regions. But it also works in the modular limit where  $\bar{\tau}$  is the conjugate of  $\tau$ , and in general when  $\tau$  and  $\bar{\tau}$  are tested as a pair of arbitrary, independent complex variables, anywhere in moduli space. We now show some concrete examples.

Let us set

$$q = \frac{1}{10}. \quad (3.3.242)$$

This corresponds, via (3.2.92), to

$$\xi/2 = 1.492512187\dots \quad (3.3.243)$$

which from figure 3.2 corresponds to the region of dominance of the action

$$\mathcal{A} = i\omega_1 \partial_a \mathcal{F}_0^{(0)} = \frac{16i\omega_1 (E_2^D + K_2)}{\sqrt{L_2} (E_2 - E_2^D)}. \quad (3.3.244)$$

The corresponding  $G$  function is

$$\begin{aligned} G &= \mathcal{A} + \frac{i\omega_1 (\hat{E}_2 - E_2^D)}{E_2 - E_2^D} D_a \tilde{F}^{(0)} = \\ &= \frac{iK_2 \sqrt{L_2} \omega_1 (\hat{E}_2 - E_2^D)}{6 (E_2^D - E_2) (K_2^2 - L_2^2)} \hbar^2 + \\ &+ \frac{iL_2^{3/2} \omega_1 (\hat{E}_2 - E_2^D)}{51840 (E_2 - E_2^D) (K_2^2 - L_2^2)^3} \left[ K_2^2 (20\hat{E}_2^2 - 80\hat{E}_2 K_2 - K_2^2) + \right. \\ &\quad \left. + 3K_2 L_2^2 (449K_2 - 40\hat{E}_2) + 450L_2^4 \right] \hbar^4 + O(\hbar^6), \end{aligned} \quad (3.3.245)$$

using the free energy formulas from (3.2.162). The one-instanton transseries was related to  $G$  via (3.3.239), which just requires taking the exponential of  $G$ . In the

previous section we also found the value of the constant  $f_B^{(1)} = (2\pi i)/(2\pi^2)$ , so

$$\begin{aligned}
F_0^{(1)} &= \left[ f^{(1)} e^{-G/\hbar} e^{\mathcal{A}/\hbar} \right]_{\hbar^0} = \frac{i}{\pi}, \\
F_1^{(1)} &= \left[ f^{(1)} e^{-G/\hbar} e^{\mathcal{A}/\hbar} \right]_{\hbar} = -\frac{K_2 \sqrt{L_2} \omega_1 (\hat{E}_2 - E_2^D)}{6\pi (E_2 - E_2^D) (K_2^2 - L_2^2)}, \\
F_2^{(1)} &= \left[ f^{(1)} e^{-G/\hbar} e^{\mathcal{A}/\hbar} \right]_{\hbar^2} = -\frac{i K_2^2 L_2 \omega_1^2 (\hat{E}_2 - E_2^D)^2}{72\pi (E_2 - E_2^D)^2 (K_2^2 - L_2^2)^2}, \\
F_3^{(1)} &= \frac{L_2^{3/2} \omega_1 (\hat{E}_2 - E_2^D)}{51840\pi (E_2 - E_2^D)^3 (K_2^2 - L_2^2)^3} \left[ 40 K_2^3 \omega_1^2 (\hat{E}_2 - E_2^D)^2 + \right. \\
&\quad \left. + (E_2 - E_2^D)^2 (K_2^2 (20\hat{E}_2^2 - 80\hat{E}_2 K_2 - K_2^2) + \right. \\
&\quad \left. + 3K_2 L_2^2 (449K_2 - 40\hat{E}_2) + 450L_2^4) \right], \\
&\dots
\end{aligned} \tag{3.3.246}$$

These are related by resurgence to the large order by

$$\mu_m = \frac{F_m^{(1)}}{2\pi i}. \tag{3.3.247}$$

On the other hand, from (3.2.179), we can recover the  $m$ -th correction to the large order with the limit

$$\mu_m = \lim_{n \rightarrow \infty} \left( \frac{2n}{\mathcal{A}} \right)^m \left[ \frac{F_n^{(0)} \mathcal{A}^{2n-2}}{\Gamma(2n-2)} - \sum_{r=0}^{m-1} \frac{\mu_r \mathcal{A}^r}{(2n-2-r)_r} \right], \tag{3.3.248}$$

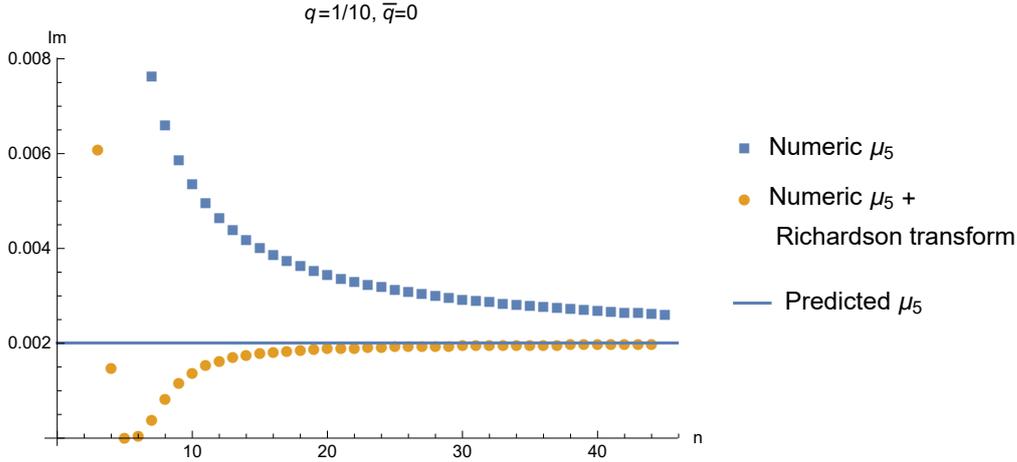
with  $(x)_n = x(x+1)\dots(x+n-1)$  the Pochhammer symbol.

We can test explicitly this relation by setting a concrete value of  $\bar{q}$ . In particular, recall that the electric frame is recovered when  $\bar{q} \rightarrow 0$ , or  $\bar{\tau} \rightarrow i\infty$ . We pick that for now. In terms of  $q$ , recall from (3.2.87) and (3.3.207) that

$$\hat{E}_2(q, \bar{q}) = E_2(q) + \frac{6}{\log(q\bar{q})}, \quad E_2^D(q) = E_2(q) + \frac{6}{\log(q)}. \tag{3.3.249}$$

For  $q = 1/10$ ,  $\bar{q} = 0$ , we get the concrete predictions

$$\begin{aligned}
\frac{F_0^{(1)}}{2\pi i} &= \frac{1}{2\pi^2}, \\
\frac{F_1^{(1)}}{2\pi i} &= i 0.00795571145895174009795449858\dots, \\
\frac{F_2^{(1)}}{2\pi i} &= -0.00062468027457634687835468457\dots, \\
&\dots \\
\frac{F_5^{(1)}}{2\pi i} &= i 0.00200651427778427678850353789\dots
\end{aligned} \tag{3.3.250}$$



**Figure 3.3:** Large order subleading corrections (electric frame)

Using the values of  $F_n^{(0)}(q = 1/10, \bar{q} = 0)$  and the first 4 predicted  $\mu_m$ , we study numerically the limit in (3.3.248) for  $\mu_5$ . In figure 3.3 we show the numerical limit, the first Richardson transform, and the predicted  $\mu_5$  from the transseries. Using the 50 first  $F_n^{(0)}$ , and after 10 Richardson transforms, the limit gives the following significant digits

$$\mu_{5,\text{num}} = i 0.00200651428 \dots \quad (3.3.251)$$

matching 9 digits of the resurgent prediction of  $F_5^{(1)}/(2\pi i)$ , a very strong test of this *and* the preceding large order corrections. The number of significant digits is chosen by taking the number of stable digits compared to 9 and 11 Richardson transforms.

The magnetic frame corresponds to an  $S$  transform, followed by the holomorphic limit  $\bar{q} \rightarrow 0$ . Alternatively, as long as  $\omega_1$  is not involved, as in the perturbative free energies, it is equivalent to just sending  $\bar{q} \rightarrow 1$  (or  $\bar{\tau} \rightarrow 0$ ). According to (3.2.187), all the subleading corrections should be trivial. This also follows from the  $G$  function in (3.3.245). In this  $\bar{q} \rightarrow 1$  limit, we have  $\hat{E}_2 \mapsto E_2^D$ , and all subleading corrections to  $G$  (and therefore to  $F^{(1)}$ ) vanish. The actions for  $q = 1/10$  are

$$\begin{aligned} 4\pi\mathbf{a} &= 21.0208811012 \dots, \\ i \partial_{\mathbf{a}} \mathcal{F}_0^{(0)} &= i 3.00710344354 \dots \end{aligned} \quad (3.3.252)$$

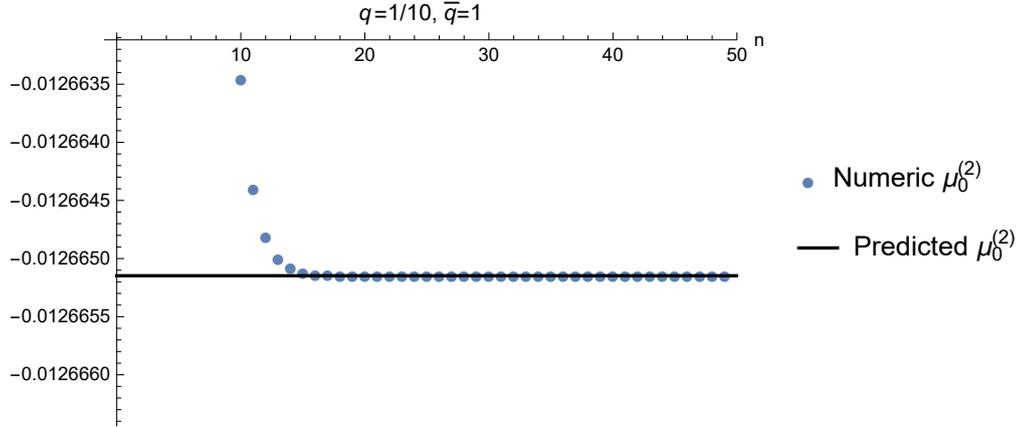
which means the next action in absolute value is still given by the B period,  $2i \partial_{\mathbf{a}} \mathcal{F}_0^{(0)}$ . The large order will look like

$$F_n^{(0)} = \frac{\Gamma(2n-2)}{\mathcal{A}^{2n-2}} \mu_0^{(1)} + \frac{\Gamma(2n-2)}{(2\mathcal{A})^{2n-2}} \mu_0^{(2)} + \dots \quad (3.3.253)$$

with  $\mu_0^{(1)} = 1/(2\pi^2)$  and  $\mu_0^{(2)} = -1/(8\pi^2)$ , according to (3.2.193). It can be recovered numerically with

$$\lim_{n \rightarrow \infty} 2^{2n-2} \left( \frac{F_n^{(0)} \mathcal{A}^{2n-2}}{\Gamma(2n-2)} - \frac{1}{2\pi^2} \right) = -\frac{1}{8\pi^2}. \quad (3.3.254)$$

Notice this is only valid because there are no subleading  $1/n$  corrections! The



**Figure 3.4:** Large order (magnetic frame)

numeric limit gives

$$\mu_{0,\text{num}}^{(2)} = -0.01266514795529222\dots, \quad (3.3.255)$$

with its 16 stable digits agreeing with  $-1/(8\pi^2)$ , confirming that indeed there are no  $1/n$  corrections from one first instanton sector.

We can also take a completely random value of  $\bar{q}$ , such as  $\bar{q} = i/10 + 1/10$ . In this case we find complex corrections (as well as complex free energies), and

$$\begin{aligned} \frac{F_0^{(1)}}{2\pi i} &= \frac{1}{2\pi^2}, \\ \frac{F_1^{(1)}}{2\pi i} &= 0.00076722991800240\dots + i 0.00379562697034436\dots, \\ &\dots, \\ \frac{F_5^{(1)}}{2\pi i} &= 0.000193245115425951\dots + i 0.001019525558246805\dots \end{aligned} \quad (3.3.256)$$

By using (3.3.248), we get after 20 Richardson transformations the numerical limit

$$\mu_{5,\text{num}} = 0.000193245115\dots + i 0.001019525558\dots \quad (3.3.257)$$

its 9 stable digits matching the predicted value.

Now lets move to a different region in moduli space. Take

$$q = \frac{1+i}{20000}, \quad (3.3.258)$$

which corresponds to

$$\xi/2 = 1250.000125\dots - i 1249.999875\dots \quad (3.3.259)$$

The A period dominates here, since

$$\begin{aligned}\mathcal{A} &= 4\pi\mathbf{a} = 690.3235965 \dots - i 285.9413718 \dots, \\ i \partial_{\mathbf{a}}\mathcal{F}_0^{(0)} &= 860.395816 \dots + i 1589.046948 \dots\end{aligned}\quad (3.3.260)$$

The  $G$  function is now

$$\begin{aligned}G &= \mathcal{A} - \frac{\pi \left( \hat{E}_2 - E_2 \right) K_2 \sqrt{L_2}}{36\omega_1 (K_2^2 - L_2^2)} \hbar^2 + \frac{\pi \left( \hat{E}_2 - E_2 \right) L_2^{3/2}}{311040\omega_1 (L_2^2 - K_2^2)^3} \left[ -20\hat{E}_2^2 K_2^2 + \right. \\ &\quad \left. + K_2^4 + 40\hat{E}_2 (2K_2^3 + 3K_2 L_2^2) - \right. \\ &\quad \left. - 1347K_2^2 L_2^2 - 450L_2^4 \right] \hbar^4 + O(\hbar^6).\end{aligned}\quad (3.3.261)$$

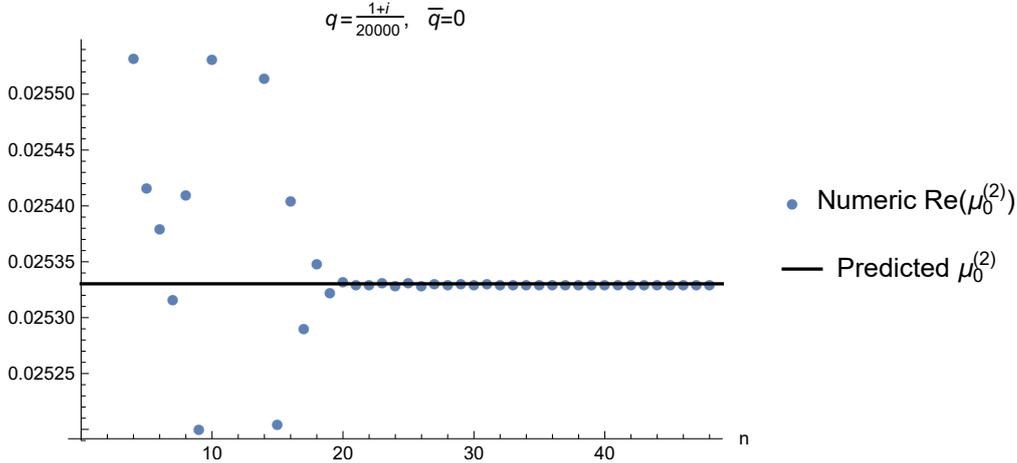
In the electric frame,  $\bar{q} \rightarrow 0$ , this is trivial – as we need for (3.2.185). The propagator is sent to  $\hat{E}_2 \mapsto E_2$ , and the  $(\hat{E}_2 - E_2)$  combinations that appear at every order vanish. As for the actions, we find

$$2|\mathcal{A}| = 2 \cdot |4\pi\mathbf{a}| = 1494.40 \dots < 1807.03 \dots = \left| i \partial_{\mathbf{a}}\mathcal{F}_0^{(0)} \right|, \quad (3.3.262)$$

so the form of the large order is still of the form

$$F_n^{(0)} = \frac{\Gamma(2n-2)}{\mathcal{A}^{2n-2}} \mu_0^{(1)} + \frac{\Gamma(2n-2)}{(2\mathcal{A})^{2n-2}} \mu_0^{(2)} + \dots \quad (3.3.263)$$

From (3.2.193), the constant in front of the one and two instanton corrections are



**Figure 3.5:** Large order (electric frame)

$\mu_0^{(1)} = 1/\pi^2$  and  $\mu_0^{(2)} = 1/(4\pi^2)$ . In figure 3.5 we plot the numerical limit. Its value is

$$\mu_{0,\text{num}}^{(2)} = 0.025330296 \dots \quad (3.3.264)$$

matches its 8 stable digits with  $1/(4\pi^2)$ . This confirms the absence of  $1/n$  corrections to the large order for

$$q = \frac{1+i}{20000}, \quad \bar{q} = 0. \quad (3.3.265)$$

In this region of A period dominance, the  $\bar{q} \rightarrow 1$  limit (that sends the free energies to the magnetic frame) will be nontrivial. We take

$$q = \frac{1+i}{20000}, \quad \bar{q} = 1, \quad (3.3.266)$$

and the  $G$  function in (3.3.261) gives

$$\begin{aligned} \frac{F_0^{(1)}}{2\pi i} &= \frac{1}{\pi^2}, \\ \frac{F_1^{(1)}}{2\pi i} &= -0.00008275698148663271 \dots - i 0.00004252781929691650 \dots, \\ &\dots, \\ \frac{F_5^{(1)}}{2\pi i} &= (1.64426595972547818 \dots - i 3.27963085631266515 \dots) \cdot 10^{-15}. \end{aligned} \quad (3.3.267)$$

Use formula (3.3.248) to compute the numerical value of  $\mu_5$ , and after 10 Richardson transforms,

$$\mu_{5,\text{num}} = (1.64426596 \dots - i 3.27963086 \dots) \cdot 10^{-15}, \quad (3.3.268)$$

with 9 stable digits, again agreeing with the prediction.

For the record, for

$$q = \frac{1+i}{20000}, \quad \bar{q} = 20i, \quad (3.3.269)$$

one finds

$$\frac{F_5^{(1)}}{2\pi i} = (3.11062842267648 \dots - i 3.67479179734533 \dots) \cdot 10^{-15} \quad (3.3.270)$$

and, up to stable digits,

$$\mu_{5,\text{num}} = (3.11062842 \dots - i 3.67479180 \dots) \cdot 10^{-15}, \quad (3.3.271)$$

with 9 matching digits.

Finally, notice we have only looked at the  $|\text{Re } \tau| < 1/2$  region, that is,

$$\text{Re } q > 0, \quad \text{Re } u > 0. \quad (3.3.272)$$

This is because the definition of  $E_2^D$  in (3.3.207) was done in the fundamental domain. Its invariance under the  $T$  transform means that it has a discontinuity along  $|\text{Re } \tau| = 1/2$ , and (3.3.207) is no longer valid. Indeed, we can use the  $T$  transform to access these regions. Recall that

$$L_2 = 16q + 64q^3 + 96q^5 + O(q^7), \quad (3.3.273)$$

so the  $T = \tau \mapsto \tau + 1$  transform, that sends  $q \mapsto -q$ , flips the sign of  $L_2$ . Similarly, it leaves invariant all the other building blocks of the algebra (including by definition, but importantly,  $E_2^D$ ). In fact, what we are doing with this transform is exploring the negative real part of the  $\xi$  plane, since from (3.2.92),

$$\xi \xrightarrow{T} -\xi. \quad (3.3.274)$$

We have from (3.2.103)

$$Y^2 \xrightarrow{T} -Y^2 \quad (3.3.275)$$

and for the free energies (3.2.162)

$$F_n^{(0)} \xrightarrow{T} (-1)^{n-1} F_n^{(0)}. \quad (3.3.276)$$

Effectively, it can be reabsorbed in  $\hbar \mapsto i\hbar$ . This is also reflected in the actions (3.3.220), that have a  $\sqrt{L_2}$  factor so that

$$\mathcal{A}_X \mapsto i \mathcal{A}_X. \quad (3.3.277)$$

Crucially,  $E_2^D$  *must* remain invariant so that the one-instanton transseries for the B period action still reproduces the numerics. Introducing the  $L_2 \mapsto -L_2$  transform in (3.3.246) we get

$$F_n^{(1)} \xrightarrow{T} i^{n-1} F_n^{(1)} \quad (3.3.278)$$

which is indeed equivalent to  $\hbar \mapsto i\hbar$ . For a concrete example, take

$$q_T = \frac{1}{10} = -q, \quad \bar{q}_T = \frac{1+i}{10}, \quad (3.3.279)$$

which is the  $T$  transform of one of the examples we have already seen. In the  $\xi$  plane this is

$$\xi/2 = -1.492512187\dots \quad (3.3.280)$$

controlled by the B period. The action must then be

$$\mathcal{A} = -\partial_a \mathcal{F}_0^{(0)}(q_T) \omega_1 = -3.00710344354202\dots \quad (3.3.281)$$

The prediction for the large order corrections differs also by just a factor of  $i$ ,

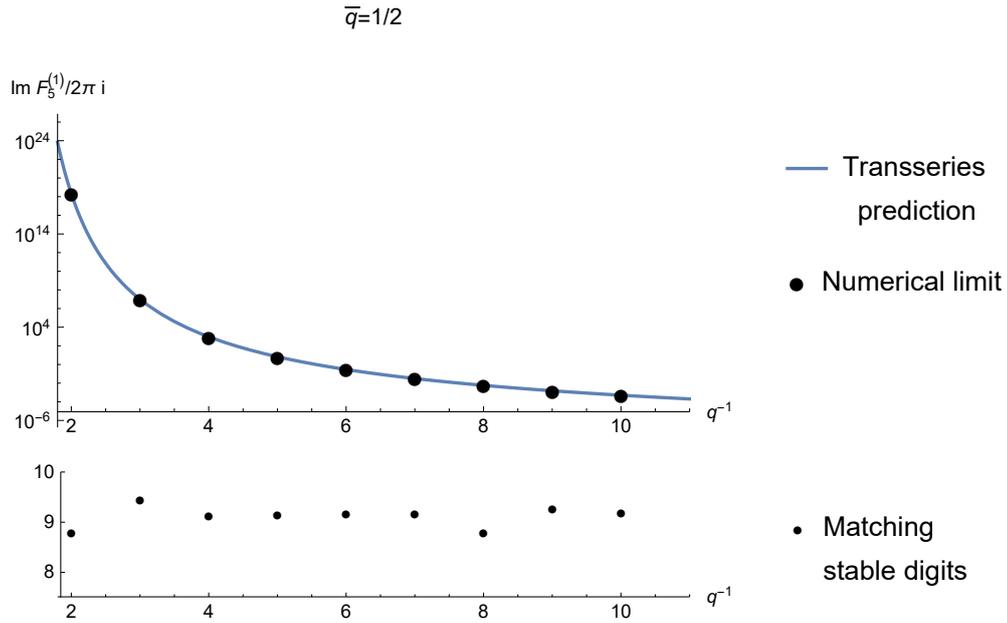
$$\frac{F_5^{(1)}}{2\pi i} = -0.001019525558246805\dots + i 0.000193245115425951\dots \quad (3.3.282)$$

while numerically we find

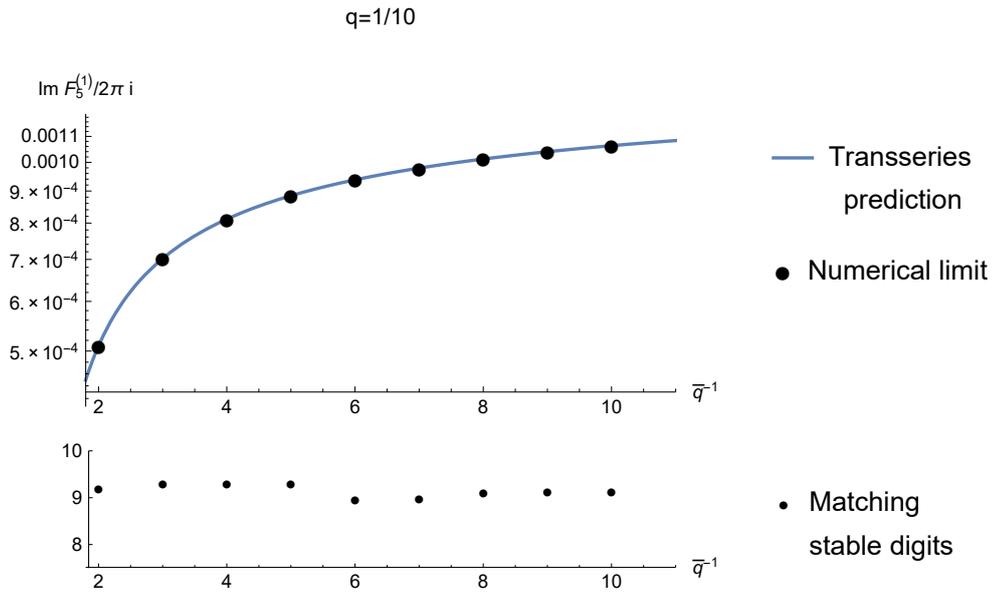
$$\mu_{5,\text{num}} = -0.001019525558\dots + i 0.000193245115\dots \quad (3.3.283)$$

up to stable digits.

In figures 3.6, 3.7, 3.8 we plot transseries predictions for the large order together with the corresponding numerical limit from  $F_n^{(0)}$ . This was done, as in the rest

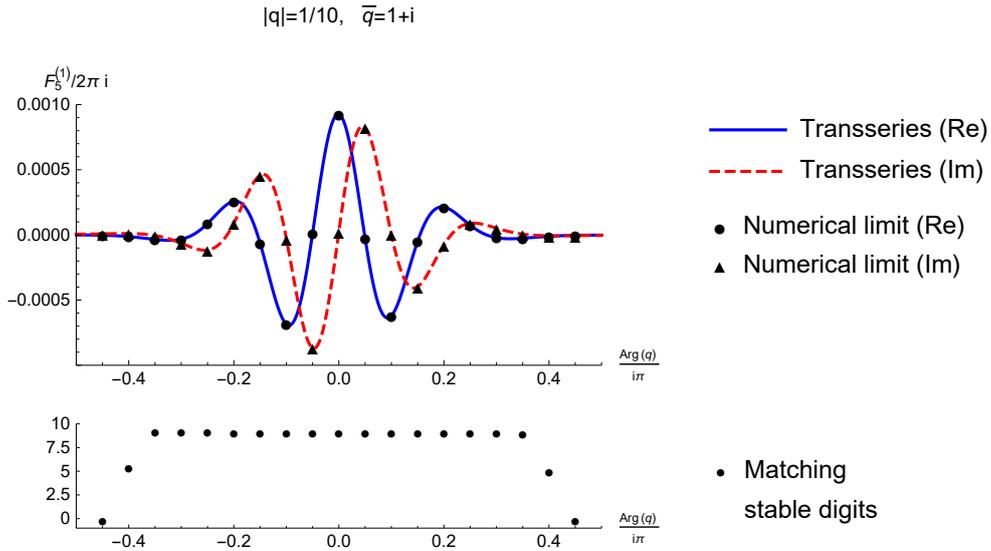


**Figure 3.6:**  $F_n^{(0)}$  large order for several  $q$ , fixed  $\bar{q}$



**Figure 3.7:**  $F_n^{(0)}$  large order for several  $\bar{q}$ , fixed  $q$

of the section, using the  $F_n^{(0)}$  up to  $n = 50$ . All of the plots are at large  $q$ , and therefore controlled by the B period action. Under each plot, we give the number of stable digits (all of which match the predictions). Notably, in figure 3.8 the matching becomes worse as we send  $\arg q \rightarrow \pm i\pi/2$ . This is not surprising since as we saw before in the discussion of the  $T$  transform, in that limit we approach the discontinuity of  $E_2^D$ . Beyond that, when  $\text{Re } q < 0$ , the large order is controlled by



**Figure 3.8:**  $F_n^{(0)}$  large order for several  $\arg q$ , fixed  $\bar{q}$  and  $|q|$

a different action – in particular, the one given after the  $T$  transform in (3.3.277).

We can also use the transseries to study the large order behaviour of the volume. Define

$$V = D_a F. \quad (3.3.284)$$

As per (3.2.115), this will just reduce to the quantum B period in the corresponding frame,

$$[V]_e = \frac{\partial \mathcal{F}}{\partial \mathbf{a}}, \quad [V]_m = i \frac{\partial \mathcal{F}_D}{\partial \mathbf{a}_D} \quad (3.3.285)$$

We take the derivative of (3.3.227) and using (3.3.234),

$$\begin{aligned}
 V &:= V^{(0)} + V^{(1)} e^{-\frac{A}{\hbar}} + \dots = \\
 &= D_a F^{(0)} - \frac{f^{(1)}}{\hbar^{b+1}} D_a G e^{-G/\hbar} + \dots = \\
 &= D_a F^{(0)} - \frac{f^{(1)}}{\hbar^{b+1}} \left( D_a \mathcal{A} + \left( (S - \mathcal{S}_A) D_a \mathcal{A} \right) D_{aa} \tilde{F}^{(0)} \right) e^{-G/\hbar} + \dots
 \end{aligned} \quad (3.3.286)$$

For instance, for the B period action we find

$$V^{(1)} = -\frac{i\hbar}{2\pi} \left( 4\pi\tau + \frac{i\bar{\tau}}{\bar{\tau} - \tau} D_{aa} \tilde{F}^{(0)} \right) \omega_1 \exp \left\{ -\frac{1}{\hbar} \frac{i\bar{\tau}}{\bar{\tau} - \tau} D_a \tilde{F}^{(0)} \omega_1 \right\}. \quad (3.3.287)$$

A similar expression is obtained for  $\mathcal{A}_A = 4\pi\mathbf{a}/\omega_1$ . We can expand this (still with

the B period action), and

$$\begin{aligned}
V^{(1)} &= \hbar \sum_{m=0}^{\infty} V_m^{(1)} \hbar^m, \\
V_0^{(1)} &= \frac{24\omega_1}{\pi (E_2 - E_2^D)}, \\
V_1^{(1)} &= -\frac{4iK_2\sqrt{L_2}\omega_1^2 (E_2^D - \hat{E}_2)}{\pi (E_2 - E_2^D)^2 (K_2^2 - L_2^2)}, \\
V_2^{(2)} &= \frac{L_2 \omega_1 (E_2^D - \hat{E}_2)}{9\pi (E_2 - E_2^D)^3 (K_2^2 - L_2^2)^2} \left[ 3K_2^2\omega_1^2 (\hat{E}_2 - E_2^D) + \right. \\
&\quad \left. + (E_2 - E_2^D)^2 \left( K_2 (\hat{E}_2 - 2K_2) - 3L_2^2 \right) \right], \\
&\dots
\end{aligned} \tag{3.3.288}$$

The most direct prediction we can make for the large order is that the action is still  $\mathcal{A}$ , the same one controlling the free energies, and that  $b = -1$ . We can also test the subleading corrections. Let us take as for the free energies the electric frame at small  $\xi$ ,

$$q = \frac{1}{10}, \quad \bar{q} = 0. \tag{3.3.289}$$

Then,

$$\begin{aligned}
\frac{V_0^{(1)}}{2\pi i} &= -i 0.466601294118718\dots, \\
\frac{V_1^{(1)}}{2\pi i} &= 0.0732748104381767\dots, \\
&\dots \\
\frac{V_5^{(1)}}{2\pi i} &= 0.0162168083090115\dots
\end{aligned} \tag{3.3.290}$$

The large order coefficient  $\mu_{V,m}$ , as in (3.2.179), can be recovered by

$$\mu_{V,m} = \lim_{n \rightarrow \infty} \left( \frac{2n}{\mathcal{A}} \right)^m \left[ \frac{V_n^{(0)} \mathcal{A}^{2n-1}}{\Gamma(2n-1)} - \sum_{r=0}^{m-1} \frac{\mu_{V,r} \mathcal{A}^r}{(2n-1-r)_r} \right], \tag{3.3.291}$$

and we find after 12 Richardson transforms

$$\mu_{V,5 \text{ num}} = 0.016216808299213\dots \tag{3.3.292}$$

with a matching of 9 digits. In the  $\bar{q} \rightarrow 1$  limit, the subleading corrections become once again trivial – although the first one is not a constant, as in the case of the free energies.

At large  $\xi$ , the dominating action is also the A period,  $\mathcal{A} = 4\pi\mathbf{a}$ , and

$$\begin{aligned} V_0^{(1)} &= -\frac{8i}{\omega_1}, \\ V_1^{(1)} &= -\frac{2i\pi \left( \hat{E}_2 - E_2 \right) K_2 \sqrt{L_2}}{9\omega_1^2 (K_2^2 - L_2^2)}, \\ V_2^{(2)} &= -\frac{i \left( \hat{E}_2 - E_2 \right) L_2}{324\omega_1^3 (K_2^2 - L_2^2)^2} \left[ \pi^2 \left( \hat{E}_2 - E_2 \right) K_2^2 + \right. \\ &\quad \left. + 12\omega_1^2 \left( -\hat{E}_2 K_2 + 2K_2^2 + 3L_2^2 \right) \right], \\ &\dots \end{aligned} \tag{3.3.293}$$

Now, the subleading corrections are trivial when  $\bar{q} \rightarrow 0$ , the electric frame where  $\hat{E}_2 \mapsto E_2$ . For

$$q = \bar{q} = \frac{1}{20000}, \tag{3.3.294}$$

we find

$$\frac{V_5^{(1)}}{2\pi i} = 9.16238487545537 \dots \cdot 10^{-15}, \tag{3.3.295}$$

and numerically (after 10 Richardson transforms)

$$\mu_{V,5 \text{ num}} = 9.1623848750296 \dots \cdot 10^{-15}. \tag{3.3.296}$$

### 3.3.3.3 Higher instantons

We have found the one-instanton solution in a relatively closed form in terms of the  $G$  function. We have extracted quite a wealth of information on the large order behaviour of the WKB coefficients from it. But we can do better, and in fact  $G$  will have the central role in what follows. Extend the one-instanton ansatz to

$$\tilde{F} = \sum_{m=0}^{\infty} \tilde{F}^{(m)} e^{-\frac{mG}{\hbar}}. \tag{3.3.297}$$

To plug it into the master equation (3.3.202), we need the propagator derivative,

$$\partial_S \tilde{F} = \sum_{n=0}^{\infty} \left( \partial_S \tilde{F}^{(m)} - \frac{m}{\hbar} \tilde{F}^{(m)} \cdot \partial_S G \right) e^{-\frac{mG}{\hbar}}, \tag{3.3.298}$$

and for the period derivative

$$\begin{aligned} \left( D_a \tilde{F} \right)^2 &= \left( D_a \tilde{F}^{(0)} \right)^2 + 2 D_a \tilde{F}^{(0)} \sum_{m=1}^{\infty} \left( D_a \tilde{F}^{(m)} - \frac{m}{\hbar} \tilde{F}^{(m)} \cdot D_a G \right) e^{-\frac{mG}{\hbar}} + \\ &+ \sum_{m=1}^{\infty} \sum_{r=1}^{m-1} \left( D_a \tilde{F}^{(r)} - \frac{r}{\hbar} \tilde{F}^{(r)} \cdot D_a G \right) \left( D_a \tilde{F}^{(m-r)} - \frac{m-r}{\hbar} \tilde{F}^{(m-r)} \cdot D_a G \right) e^{-\frac{mG}{\hbar}}. \end{aligned} \tag{3.3.299}$$

Then the master equation, order by order in  $\exp\{-G/\hbar\}$ , is expressed as

$$\begin{aligned} m = 0 &\implies \partial_S \tilde{F}^{(0)} - \frac{1}{2} \left( D_a \tilde{F}^{(0)} \right)^2 = 0, \\ m \geq 1 &\implies \partial_S \tilde{F}^{(m)} - D_a \tilde{F}^{(0)} \cdot D_a \tilde{F}^{(m)} = \\ &= \frac{1}{2} \sum_{r=1}^{m-1} \left( D_a \tilde{F}^{(r)} - \frac{r}{\hbar} \tilde{F}^{(r)} \cdot D_a G \right) \left( D_a \tilde{F}^{(m-r)} - \frac{m-r}{\hbar} \tilde{F}^{(m-r)} \cdot D_a G \right). \end{aligned} \quad (3.3.300)$$

The right hand side is clearly recursive on  $m$ , and the left side is the same operator that annihilates  $G$ , which we will call

$$W = \partial_S - D_a \tilde{F}^{(0)} \cdot D_a. \quad (3.3.301)$$

It has the following properties.

- $W(G) = 0$ ,  $W(D_a \tilde{F}^{(0)}) = 0$ ,
- $W$  is linear,
- $W$  is a derivation.

which suggestively makes (3.3.300) have a very similar look to the perturbative holomorphic anomaly (3.2.86). The question now is how to “integrate” with respect to this derivation  $W$ .

Consider the  $m = 2$  instanton. Since the one-instanton term is just the constant  $\tilde{F}^{(1)} = \hbar^2 f^{(1)}$ , equation (3.3.300) gives

$$W(\tilde{F}^{(2)}) = \frac{1}{2} (\hbar f^{(1)})^2 (D_a G)^2. \quad (3.3.302)$$

We mentioned already the combination in (3.3.234). It will have a role in the current construction, so we will name it

$$T = D_a \mathcal{A} (S - \mathcal{S}_A). \quad (3.3.303)$$

Recall from (3.3.234) that

$$D_a T = 0, \quad (3.3.304)$$

and from the definition itself

$$\partial_S T = D_a \mathcal{A}. \quad (3.3.305)$$

We can use it to rewrite the relation between  $D_a \mathcal{A}$  and  $D_{aa} \mathcal{A}$ , using (3.2.91) in terms of  $S$  and  $a$ ,

$$\partial_S D_a = D_a \partial_S - k Y. \quad (3.3.306)$$

Then,

$$D_{aa} \mathcal{A} = D_a \partial_S T = \partial_S D_a T + Y T = Y T, \quad (3.3.307)$$

which is just a compact version of (3.3.331). Now we apply  $W$  on the following weight zero object,

$$\begin{aligned} W(T D_a G) &= D_a \mathcal{A} D_a G + T D_a \partial_S G - D_a \tilde{F}^{(0)} T D_{aa} G = \\ &= D_a \mathcal{A} D_a G + T D_{aa} \tilde{F}^{(0)} D_a G = \\ &= D_a \left( \mathcal{A} + T D_a \tilde{F}^{(0)} \right) D_a G = (D_a G)^2. \end{aligned} \quad (3.3.308)$$

The solution will then be

$$\tilde{F}^{(2)} = \tilde{F}_0^{(2)} + \frac{1}{2} (\hbar f^{(1)})^2 T D_a G. \quad (3.3.309)$$

with  $\tilde{F}_0^{(2)}$  in the kernel of  $W$ .

Since the transseries is related to the large order behaviour, we know that in the corresponding holomorphic limit all subleading corrections must vanish. For the inhomogeneous term we are already sure of this, because

$$S \mapsto \mathcal{S}_A \implies T \mapsto 0. \quad (3.3.310)$$

The same happens with the corrections in  $\exp\{-2G/\hbar\}$ . On the other hand, any non-trivial solution  $\tilde{F}_0^{(2)}$  to  $W(\cdot) = 0$  will have poles at every order in  $\hbar$ . This can be seen because  $G$  (3.3.230) is essentially an arbitrary solution, where there are no *a priori* constraints on  $\mathcal{A}$ , which is the  $\hbar^0$  part. The term  $D_a \tilde{F}^{(0)}$  has poles at every  $\hbar$  when going to its corresponding limit, and they are of arbitrarily high order in  $\alpha^{-1}$  – recall (3.2.161) or (3.2.164). Therefore, the function  $\mathcal{A}$  (which is not itself a power series in  $\hbar$ ) cannot be chosen so that all of them are cancelled.

This means that the solution is

$$\tilde{F}^{(2)}/\hbar^2 = f^{(2)} + \frac{1}{2} (f^{(1)})^2 T D_a G \quad (3.3.311)$$

with  $f^{(2)}$  just a constant.

It is now clear that  $T$  always accompanies the  $D_a$  derivatives so that the full object keeps the correct weight. Define the *instanton derivatives*

$$\begin{aligned} \mathcal{W}(\cdot) &= T^2 W(\cdot), \\ \mathcal{D}(\cdot) &= T D_a(\cdot). \end{aligned} \quad (3.3.312)$$

Just like  $W$  and  $D_a$ , they are both linear derivations.

The refined anomaly instanton sector (3.3.297) is subject to the recursion

$$\mathcal{W} \tilde{F}^{(m)} = \frac{1}{2} \sum_{r=1}^{m-1} \left( \mathcal{D} \tilde{F}^{(r)} - \frac{r}{\hbar} \tilde{F}^{(r)} \cdot \mathcal{D} G \right) \left( \mathcal{D} \tilde{F}^{(m-r)} - \frac{m-r}{\hbar} \tilde{F}^{(m-r)} \cdot \mathcal{D} G \right). \quad (3.3.313)$$

for  $m \geq 1$ , where the  $m$ -th instanton correction to the free energy is given by  $\tilde{F}^{(m)} e^{-\frac{mG}{\hbar}}$ .

Suppose a certain object  $X_0$  has weight zero, so that  $\partial_S$  and  $D_a$  commute. The operators we just defined turn out to have a very nice algebra on such invariant objects. To see that, we take (3.3.230),

$$D_a \tilde{F}^{(0)} = \frac{G - \mathcal{A}}{T}. \quad (3.3.314)$$

Then the action of  $\mathcal{W}$  can be written in terms of  $G$ ,

$$\mathcal{W}X_0 = T^2 \partial_S X_0 - (G - \mathcal{A}) \mathcal{D}X_0. \quad (3.3.315)$$

The commutator follows,

$$\begin{aligned} \mathcal{W}\mathcal{D}X_0 &= T^2 \partial_S (T D_a X_0) - (G - \mathcal{A}) \mathcal{D}^2 X_0 = \\ &= T^2 (D_a \mathcal{A} \cdot D_a X_0 + T D_a \partial_S X_0) - (G - \mathcal{A}) \mathcal{D}^2 X_0 = \\ &= \mathcal{D}\mathcal{A} \cdot \mathcal{D}X_0 + \mathcal{D}(\mathcal{W}X_0 + (G - \mathcal{A}) \mathcal{D}X_0) - (G - \mathcal{A}) \mathcal{D}^2 X_0. \end{aligned} \quad (3.3.316)$$

Most terms cancel out, and in particular, so do all that explicitly involve the action  $\mathcal{A}$ . This would not have been the case if  $X_k$  had weight different from zero, since by (3.3.306),

$$(T^2 \partial_S) \mathcal{D}X_k = \mathcal{D}(T^2 \partial_S) X_k + \mathcal{D}\mathcal{A} \cdot \mathcal{D}X_k - k \mathcal{D}^2 \mathcal{A} \cdot X_k. \quad (3.3.317)$$

In any case, the commutator we are interested in is the one acting over objects of weight zero – the  $\tilde{F}^{(m)}$ .

The differential algebra of the instanton sector is determined by the relations

$$\mathcal{W}\mathcal{D}X_0 = \mathcal{D}\mathcal{W}X_0 + \mathcal{D}G \cdot \mathcal{D}X_0, \quad \mathcal{W}G = 0. \quad (3.3.318)$$

This means that the operator  $\mathcal{W}$  closes over the ring spanned by

$$\langle G, \mathcal{D}G, \mathcal{D}^2 G, \mathcal{D}^3 G, \dots \rangle, \quad (3.3.319)$$

and the recursion (3.3.313) can be built entirely out of these elements.

Let us look then at  $m = 3$ ,

$$\mathcal{W}\tilde{F}^{(3)} = 2\hbar^2 f^{(1)} f^{(2)} (\mathcal{D}G)^2 + \hbar^2 (f^{(1)})^3 (\mathcal{D}G)^3 - \frac{\hbar^3 (f^{(1)})^3}{2} (\mathcal{D}G) (\mathcal{D}^2 G). \quad (3.3.320)$$

The building blocks to solve it can be computed with (3.3.318) and the fact that by construction  $\mathcal{W}G = 0$ ,

$$\begin{aligned} \mathcal{W}\left(\frac{1}{3}\mathcal{D}^2 G\right) &= \mathcal{D}G \cdot \mathcal{D}^2 G, \\ \mathcal{W}\left(\frac{1}{2}(\mathcal{D}G)^2\right) &= (\mathcal{D}G)^3, \\ \mathcal{W}(\mathcal{D}G) &= (\mathcal{D}G)^2. \end{aligned} \quad (3.3.321)$$

Therefore,

$$\begin{aligned} \tilde{F}^{(3)}/\hbar^2 &= f^{(3)} + 2f^{(1)}f^{(2)}\mathcal{D}G + \\ &+ \frac{(f^{(1)})^3}{2}(\mathcal{D}G)^2 - \frac{\hbar(f^{(1)})^3}{6}\mathcal{D}^2G. \end{aligned} \quad (3.3.322)$$

Notice that  $\mathcal{D}(\cdot)$  vanishes if  $S \rightarrow \mathcal{S}_{\mathcal{A}}$ , so that we get the expected trivial asymptotics in that limit. For an example of a higher order instanton,

$$\begin{aligned} \tilde{F}^{(3)}/\hbar^2 &= f^{(5)} - \frac{1}{120}(f^{(1)})^5\hbar^3(\mathcal{D}^4G) + \frac{1}{3}f^{(2)}(f^{(1)})^3\hbar^2(\mathcal{D}^3G) + \\ &+ \frac{1}{8}(f^{(1)})^5\hbar^2(\mathcal{D}^2G)^2 - \frac{1}{2}\left(4(f^{(2)})^2 + 3f^{(1)}f^{(3)}\right)f^{(1)}\hbar(\mathcal{D}^2G) + \\ &+ \frac{25}{24}(f^{(1)})^5(\mathcal{D}G)^4 + \frac{25}{3}f^{(2)}(f^{(1)})^3(\mathcal{D}G)^3 + \\ &+ \frac{5}{2}\left(4(f^{(2)})^2 + 3f^{(1)}f^{(3)}\right)f^{(1)}(\mathcal{D}G)^2 + \\ &+ 2(3f^{(2)}f^{(3)} + 2f^{(1)}f^{(4)})(\mathcal{D}G) + \frac{1}{6}(f^{(1)})^5\hbar^2(\mathcal{D}G)(\mathcal{D}^3G) - \\ &- \frac{5}{4}(f^{(1)})^5\hbar(\mathcal{D}G)^2(\mathcal{D}^2G) - 5f^{(2)}(f^{(1)})^3\hbar(\mathcal{D}G)(\mathcal{D}^2G). \end{aligned} \quad (3.3.323)$$

One can compute higher instanton corrections systematically by using (3.3.318) to relate all the  $\mathcal{D}^n G$ , turning (3.3.313) into an algebraic problem. Like in the perturbative sector, where the  $\hbar^2$  corrections to the perturbative free energy are functions of the A period, here we construct the  $\exp(-\mathcal{A}/\hbar)$  corrections to the full free energy as functions of the resummed *perturbative free energy*, via the  $G$  function.

In fact, by using (3.3.318) and the properties of  $T$ , the master equation (3.3.202) turns into an **instanton master equation** with  $F^{(0)}$  in place of  $F_0^{(0)}$ , and  $\mathcal{W}$  of  $\partial_S$ ,

$$\mathcal{W}(F - F^{(0)}) = \frac{1}{2}\left[\mathcal{D}(F - F^{(0)})\right]^2, \quad (3.3.324)$$

making the parallelism between perturbative and instanton sectors explicit.

### 3.3.3.4 Double recursion

A less efficient approach is to consider the ansatz

$$\tilde{F} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_n^{(m)} \hbar^n e^{-\frac{m\mathcal{A}}{\hbar}}. \quad (3.3.325)$$

Plugging it into (3.3.202), and equating order by order in  $\hbar$  and  $e^{-\mathcal{A}/\hbar}$ ,

$$\begin{aligned} \partial_S C_n^{(m)} &= \frac{1}{2} \sum_{i=0}^m \left[ \sum_{j=0}^n \left( D_a C_j^{(i)} \cdot D_a C_{n-j}^{(m-i)} \right) - \right. \\ &- 2 D_a \mathcal{A} \sum_{j=0}^{n+1} \left( i \cdot C_j^{(i)} \cdot D_a C_{n+1-j}^{(m-i)} \right) + \\ &\left. + (D_a \mathcal{A})^2 \sum_{j=0}^{n+2} \left( i(m-i) \cdot C_j^{(i)} \cdot C_{n+2-j}^{(m-i)} \right) \right]. \end{aligned} \quad (3.3.326)$$

The first line will be the perturbative holomorphic anomaly, the second appears for the instanton corrections, and the third only for  $m \geq 2$ . As the level  $m = 0$  is just the perturbative series,

$$\begin{aligned} C_0^{(0)} &= C_{2j+1}^{(0)} = 0, \\ C_{2j}^{(0)} &= F_j^{(0)}, \end{aligned} \quad (3.3.327)$$

and since  $b = -2$  in (3.2.185),

$$\begin{aligned} C_0^{(i)} &= C_1^{(i)} = 0, \\ C_j^{(i \geq 1)} &= F_{j-2}^{(i)}. \end{aligned} \quad (3.3.328)$$

This constrains the indices on the sums and makes it a explicit recursion,

$$\begin{aligned} \frac{192}{Y^2} \frac{\partial C_n^{(m)}}{\partial \hat{E}_2} &= \sum_{i=0}^m \sum_{j=2}^{n-2} \left( D_\tau C_j^{(i)} \cdot D_\tau C_{n-j}^{(m-i)} \right) - \\ &- 2 D_\tau \mathcal{A} \sum_{i=1}^m \sum_{j=2}^{n-1} \left( i \cdot C_j^{(i)} \cdot D_\tau C_{n+1-j}^{(m-i)} \right) + \\ &+ (D_\tau \mathcal{A})^2 \sum_{i=1}^{m-1} \sum_{j=2}^n \left( i (m-i) \cdot C_j^{(i)} \cdot C_{n+2-j}^{(m-i)} \right) \Big]. \end{aligned} \quad (3.3.329)$$

This is solved just like (3.2.86), except that the boundary condition is simpler, and given by the large order (3.2.185) and (3.2.187).

$$C_j^{(i \geq 1)} \Big|_{\hat{E}_2 \rightarrow E_2^A} = \begin{cases} f^{(i)} & j = 2, \\ 0 & j > 2, \end{cases} \quad (3.3.330)$$

where  $f^{(i)}$  are the constants we defined in the previous section and  $E_2^A$  is either  $E_2$  or  $E_2^D$ , depending on  $\mathcal{A}$ . The boundary condition is satisfied simply by subtracting from the integral of  $\partial_{\hat{E}_2} C_n^{(m)}$  its corresponding holomorphic limit.

The recursion can be solved for an arbitrary action, using (3.3.226),

$$D_{aa} \mathcal{A}_A = - \frac{D_a \left( \hat{E}_2 - E_2 \right)}{\hat{E}_2 - E_2} D_a \mathcal{A}_A = Y \frac{\hat{E}_2 - E_2}{24} D_a \mathcal{A}_A, \quad (3.3.331)$$

This is again just the condition (3.3.234). With the concrete Yuwaka of this problem, (3.2.103),

$$D_\tau D_\tau \mathcal{A} = \frac{E_2^A + 2\hat{E}_2 - 3K_2}{12} D_\tau \mathcal{A}. \quad (3.3.332)$$

One just needs to keep track, on top of the modular generators we have used so far, of  $D_\tau \mathcal{A}$  and of  $E_2^A$ . The derivative of the latter was already in (3.3.217),

$$D_\tau E_2^A = \frac{1}{6} \hat{E}_2 E_2^A - \frac{1}{48} \left( K_2^2 + 3L_2^2 + 4(E_2^A)^2 \right). \quad (3.3.333)$$

The results of this recursion match the ones obtained from (3.3.313). For instance,

$$\begin{aligned} \partial_{\hat{E}_2} F_1^{(1)} &= f^{(1)} \frac{D_a \mathcal{A} K_2 \sqrt{L_2}}{144 (K_2^2 - L_2^2)} \implies \\ &\implies F_1^{(1)} = f^{(1)} \frac{D_a \mathcal{A} K_2 \sqrt{L_2} (\hat{E}_2 - E_2^A)}{144 (K_2^2 - L_2^2)} \end{aligned} \quad (3.3.334)$$

Compare with (3.3.246), after setting

$$\mathcal{A} \mapsto i \partial_a \mathcal{F}_0^{(0)}, \quad f^{(1)} \mapsto (2\pi i) \frac{1}{2\pi^2}, \quad (3.3.335)$$

where in particular

$$D_a \mathcal{A} \mapsto \frac{24i \omega_1}{E_2 - E_2^D}. \quad (3.3.336)$$

Take the result of the direct two-instanton integration (3.3.311), and use the B period  $G$  function (3.3.245),

$$\begin{aligned} F^{(2)} &= \hbar^2 e^{-2\mathcal{A}_B/\hbar} \left[ \frac{12 (f^{(1)})^2 \omega_1^2 (E_2^D - \hat{E}_2)}{(E_2 - E_2^D)^2} + f^{(2)} + \right. \\ &\quad \left. + \frac{i K_2 \sqrt{L_2} \omega_1 (\hat{E}_2 - E_2^D) \left( \frac{12 (f^{(1)})^2 \omega_1^2 (E_2^D - \hat{E}_2)}{(E_2 - E_2^D)^2} + f^{(2)} \right)}{3 (E_2 - E_2^D) (K_2^2 - L_2^2)} \hbar + O(\hbar^2) \right]. \end{aligned} \quad (3.3.337)$$

From the double recursion (3.3.329), we find

$$\begin{aligned} F_0^{(2)} &= \frac{1}{48} (D_a \mathcal{A})^2 (f^{(1)})^2 (\hat{E}_2 - E_2^A) + f^{(2)}, \\ F_1^{(2)} &= \frac{D_a \mathcal{A} K_2 \sqrt{L_2} (\hat{E}_2 - E_2^A) \left( (D_a \mathcal{A})^2 (f^{(1)})^2 (\hat{E}_2 - E_2^A) + 48 f^{(2)} \right)}{3456 (K_2^2 - L_2^2)}, \end{aligned} \quad (3.3.338)$$

who again agree when  $\mathcal{A} = i \partial_a \mathcal{F}_0^{(0)}$ . Notice we can in fact just recover these expressions by using the general form of  $G = \mathcal{A} + (S - \mathcal{S}_A) \cdot D_a \mathcal{A} \cdot D_a \tilde{F}^{(0)}$ . We have tested that the two ways of computing the instanton corrections agree, even at higher instantons like (3.3.322) and beyond. The procedure in this subsection is on the other hand computationally more expensive, since it calculates anew every instanton sector as a perturbative series.

### 3.3.4 Comparison with exact quantization

In conventional quantum mechanics one can find exact quantization conditions for the spectrum by using the exact WKB method [5, 91, 92], instanton calculus [7, 93, 94],

or the uniform WKB approximation [2–4]. These quantization conditions are in fact equations defining implicitly a transseries for the quantum volume  $\text{vol} = 2\pi\hbar\nu$ . This leads, by a formal transseries expansion, to a transseries for any function of the energy level  $\nu = n + 1/2$ , like for example the energy  $E = E(\nu)$ . The exact spectrum is then obtained by applying Borel–Écalle resummation to the resulting transseries.

The transseries obtained from exact quantization conditions are usually based on instanton solutions to the Euclidean EOM. However, there are no real instanton solutions for the  $\cosh(x)$  potential, and one needs complex instantons [95] coming from classical trajectories along the imaginary axis in the complex  $x$  plane [96, 97], where we have a periodic potential. One way to find the appropriate transseries for the modified Mathieu equation is to start with the  $\cos(x)$  potential (i.e., the Mathieu equation). In the  $\cos(x)$  potential the quantization condition was obtained in [7, 93, 94] by using instanton calculus and derived in [4] by using the uniform WKB method. It reads

$$1 + e^{\pm 2\pi i\nu} = f_{SG}(\nu) + 2 \cos \theta \sqrt{f_{SG}(\nu)}. \quad (3.3.339)$$

$\theta$  is the quasimomentum, and  $f_{SG}(\nu)$  can be written as

$$f_{SG}(\nu) = \frac{2\pi}{\Gamma^2(\nu + \frac{1}{2})} \left(\frac{32}{\hbar}\right)^{2\nu} e^{-A_{SG}(\nu, \hbar)}, \quad (3.3.340)$$

where  $A_{SG}(\nu, \hbar)$  is a certain regular function of  $\nu$ ,  $\hbar$ . The  $\pm$  sign in (3.3.339) corresponds to the choice of lateral resummation. To make contact with the modified Mathieu equation, we change  $\hbar \rightarrow -\hbar$  in the function  $A_{SG}(\nu, \hbar)$ , and we eliminate the dependence in  $\theta$  by taking  $\theta = \pi/2$ . We end up with the equation,

$$1 + e^{\pm 2\pi i\nu} = f(\nu), \quad (3.3.341)$$

where

$$f(\nu) = \frac{2\pi}{\Gamma^2(\nu + \frac{1}{2})} \left(\frac{32}{\hbar}\right)^{2\nu} e^{A(\nu, \hbar)}, \quad A(\nu, \hbar) = -A_{SG}(\nu, -\hbar). \quad (3.3.342)$$

The function  $f(\nu)$  turns out to be related to the derivative of the dual quantum prepotential in the following way. The perturbative WKB quantization condition of the modified Mathieu potential can be naturally written in the dual variable,  $\mathbf{a}_D$ , that computes the quantum volume (3.2.128). Its expansion can be obtained from (3.2.92) and (3.2.98).

$$\text{vol} = 2\pi\hbar\nu \implies -\frac{\hbar\nu}{2} = \mathbf{a}_D(\xi) = -\frac{1}{4}(\xi - 2) + \frac{1}{128}(\xi - 2)^2 + \dots \quad (3.3.343)$$

Then one can write the  $f(\nu)$  function as

$$f(\nu, \hbar) = \exp \left\{ \frac{1}{\hbar} \frac{\partial \mathcal{F}_D^{(0)}}{\partial \mathbf{a}_D} \right\}. \quad (3.3.344)$$

The singular behaviour of the free energies, (3.2.151) is precisely the one that generates the asymptotic expansion of  $\Gamma(\nu + 1/2)$  in  $f(\nu)$ .

At the perturbative level,  $\nu$  is just a half integer. So let us keep the notation  $\nu$  for the perturbative part of the energy level, and upgrade it to an exact  $\hat{\nu}$  for the full quantization condition

$$1 + e^{2\pi i \hat{\nu}} = f(\hat{\nu}, \hbar). \quad (3.3.345)$$

This exact  $\hat{\nu}$  will be an exponentially small deviation of the perturbative  $\nu$ ,

$$\hat{\nu} = \nu + \Delta\nu^{(1)} + \Delta\nu^{(2)} + \dots \quad (3.3.346)$$

where every term is suppressed with the corresponding power of  $f$ , and after plugging it into (3.3.345)

$$\begin{aligned} \Delta\nu^{(1)} &= \frac{if(\nu, \hbar)}{2\pi}, \\ \Delta\nu^{(2)} &= \frac{if^2}{4\pi} - \frac{f \partial_\nu f}{4\pi^2}, \\ \Delta\nu^{(3)} &= \frac{if^3}{6\pi} - \frac{3f^2 \partial_\nu f}{8\pi^2} - \frac{if^2 \partial_{\nu\nu} f}{16\pi^3} - \frac{if(\partial_\nu f)^2}{8\pi^3}. \end{aligned} \quad (3.3.347)$$

We want to compare this with the transseries the holomorphic anomaly gives for  $F$ . Notice that we are writing  $\mathbf{a}(\nu, \hbar)$  for the A period as a function of the *quantum* volume, or equivalently the energy level. That is, it should not be confused with  $\mathbf{a}(E, \hbar)$ , as a function of the physical energy. In the former sense it is the derivative of the free energy

$$4\pi \mathbf{a}(\nu; \hbar) = \frac{\partial \mathcal{F}_D^{(0)}(\nu; \hbar)}{\partial \mathbf{a}_D}, \quad (3.3.348)$$

and the non-perturbative correction is

$$f(\nu, \hbar) = e^{\frac{4\pi \mathbf{a}(\nu; \hbar)}{\hbar}}. \quad (3.3.349)$$

To connect the quantization condition with the free energy (which is what we can compute with the holomorphic anomaly), we will use the quantum Matone relation (3.1.25), that as a function of  $\nu$  reads

$$u(\nu, \hbar) - 1 = \frac{\hbar}{32} + \frac{\hbar^3}{8} \partial_\hbar \left[ \frac{\mathcal{F}_D(\nu, \hbar)}{\hbar^2} \right], \quad (3.3.350)$$

where  $E = 2u(\nu, \hbar)$  is the quantum energy as a function of the level  $\nu$ . Of course, simply (3.3.345) does not directly give us the energy  $u$ , but we can still do something about it. We can get the transseries for  $u$  by

$$\hat{u}(\nu, \hbar) = u(\hat{\nu}, \hbar). \quad (3.3.351)$$

By plugging (3.3.347) into  $u$  and Taylor expanding,

$$\begin{aligned}
\hat{u} &= u + u^{(1)} + u^{(2)} + u^{(3)} + \dots = \\
&= u + \frac{if' u'}{2\pi} - \\
&\quad - \frac{ff' u'}{4\pi^2} - \frac{f^2 u''}{8\pi^2} + \frac{if^2 u'}{4\pi} - \\
&\quad - \frac{if^2 f'' u'}{16\pi^3} - \frac{if^2 f' u''}{8\pi^3} - \frac{3f^2 f' u'}{8\pi^2} - \frac{if f'^2 u'}{8\pi^3} - \\
&\quad - \frac{if^3 u'''}{48\pi^3} - \frac{f^3 u''}{8\pi^2} + \frac{if^3 u'}{6\pi} + O(f^4),
\end{aligned} \tag{3.3.352}$$

where we have used the shorthand  $u' = \partial_\nu u(\nu, \hbar)$  and  $f' = \partial_\nu f(\nu, \hbar)$ . From (3.3.344), we can rewrite  $f$  as

$$f = \exp \left\{ -\frac{2}{\hbar^2} \mathcal{F}'_D \right\} \tag{3.3.353}$$

where again we have the shorthand  $\mathcal{F}'_D = \partial_\nu \mathcal{F}_D^{(0)}(\nu, \hbar)$ . The extra  $2/\hbar$  comes from the relation between  $\mathfrak{a}_D$  and  $\nu$ , in (3.3.355). With this we can rewrite the third instanton correction to  $\hat{u}$  as

$$\begin{aligned}
u^{(3)} &= e^{-6/\hbar^2 \mathcal{F}'_D} \left( \frac{i \mathcal{F}_D''' u'}{8\pi^3 \hbar^2} + \frac{i \mathcal{F}_D'' u''}{4\pi^3 \hbar^2} - \frac{3i (\mathcal{F}_D'')^2 u'}{4\pi^3 \hbar^4} + \right. \\
&\quad \left. + \frac{3\mathcal{F}_D'' u'}{4\pi^2 \hbar^2} - \frac{i u'''}{48\pi^3} - \frac{u''}{8\pi^2} + \frac{i u'}{6\pi} \right).
\end{aligned} \tag{3.3.354}$$

This much we get just from the quantization condition.

To recover  $\hat{\nu}$  from  $F$ , the first thing to do is to take the right action and holomorphic limit. First of all, notice that when re-expanding in  $\nu$ , as it corresponds to the quantization condition, we are doing an expansion around

$$\mathfrak{a}_D = -\hbar\nu/2 \sim 0, \tag{3.3.355}$$

since  $\hbar$  is also expanded around zero. The natural variables from the holomorphic anomaly point of view will be  $\mathfrak{a}_D$  itself, and the dual free energy. Also, the  $i \partial_{\mathfrak{a}} \mathcal{F}_0^{(0)}$  action vanishes around this expansion point, so the action must necessarily be given by the other period. Concretely, when re-expanding the action in the  $\nu$  variable, we have

$$i \partial_{\mathfrak{a}} \mathcal{F}_0^{(0)} = -4\pi i \mathfrak{a}_D = 2\pi i \nu \hbar. \tag{3.3.356}$$

When plugged in  $\exp(-\mathcal{A}/\hbar)$ , since the action is the  $1/\hbar$  term, we would have a null action

$$\exp \left( \frac{0}{\hbar} + O(\hbar) \right). \tag{3.3.357}$$

There is another difference with the transseries we used to study the free energy large order. There, the overall sign of  $\mathcal{A}$  was irrelevant – the large order only depends on  $\mathcal{A}^2$ , since the free energy is a series in  $\hbar^2$ . One can see (by expanding

in  $\nu$  with Matone's relation, for instance) that the series of  $E$  with coefficients in  $\nu$  is an alternating series in  $\hbar$ . Since we are using this formal quantization condition to generate the large order, the action must be negative, and

$$\mathcal{A} = -4\pi\mathbf{a}. \quad (3.3.358)$$

Now we need the  $G$  function corresponding to this action. Since  $f(\nu)$  was written in terms of  $\mathcal{F}_D$ , we should write it in the magnetic frame. Notice in particular this requires to do the S transform on  $\omega_1$ . From (3.3.237),

$$G(\tau, \bar{\tau}) = -\frac{4\pi\mathbf{a}}{\omega_1} - \frac{i}{\omega_1(\bar{\tau} - \tau)} D_a \tilde{F}^{(0)}. \quad (3.3.359)$$

This can be rewritten through an S transform, as in (3.3.223) and (3.3.225). Since  $\omega_1$  has weight one, it also transforms, giving

$$G(\tau_D, \bar{\tau}_D) = -\frac{\partial \mathcal{F}_{D,0}^{(0)}}{\partial \mathbf{a}_D} \omega_1 - \omega_1 \left( \frac{-1}{\tau_D} \right) \frac{i D_a \tilde{F}^{(0)}}{1/\tau_D - 1/\bar{\tau}_D}. \quad (3.3.360)$$

Writing it in the standard form

$$G = \mathcal{A} + T D_a \tilde{F}^{(0)}, \quad (3.3.361)$$

from (3.3.230) and (3.3.303), we have

$$\mathcal{A} = -\frac{\partial \mathcal{F}_{D,0}^{(0)}}{\partial \mathbf{a}_D} \omega_1 \quad (3.3.362)$$

and

$$T = \frac{i \omega_1}{1 - \tau_D/\bar{\tau}_D} \quad (3.3.363)$$

The holomorphic limit in the magnetic frame corresponds to  $\bar{\tau}_D \rightarrow i\infty$ . There,

$$[T]_m = i. \quad (3.3.364)$$

The covariant derivative also gets an  $i$  factor in the magnetic frame – see (3.2.115), so

$$[T D_a]_m = -\partial_{\mathbf{a}_D}, \quad (3.3.365)$$

and the  $G$  function

$$[G]_m = -\partial_{\mathbf{a}_D} \mathcal{F}_{0,D}^{(0)} - \partial_{\mathbf{a}_D} \tilde{\mathcal{F}}_D^{(0)} = -\partial_{\mathbf{a}_D} \mathcal{F}_D^{(0)} \quad (3.3.366)$$

Now take the third instanton correction to the free energy (3.3.322). Recall from (3.3.355), (3.3.365) and (3.3.366) that in the magnetic frame

$$[G]_m = \frac{2}{\hbar} \partial_\nu \mathcal{F}_D, \quad [\mathcal{D}]_m = \frac{2}{\hbar} \partial_\nu. \quad (3.3.367)$$

From there,

$$\begin{aligned} \mathcal{F}_D^{(3)}(\nu, \hbar) &= [F^{(3)}]_m = \left[ \hbar^2 \tilde{F}^{(3)} e^{-3G/\hbar} \right]_m = \\ &= e^{-6/\hbar^2} \mathcal{F}'_D \left[ \frac{8 \mathcal{F}_D'' (f^{(1)})^3}{\hbar^2} - \frac{4 \mathcal{F}_D''' (f^{(1)})^3}{3} + 8 \mathcal{F}_D'' f^{(2)} f^{(1)} + (f^{(1)})^3 \hbar^2 \right]. \end{aligned} \quad (3.3.368)$$

Now, from the Matone relation (3.3.350),

$$u^{(3)} = \frac{\hbar^3}{8} \partial_\hbar \left[ \frac{\mathcal{F}_D^{(3)}}{\hbar^2} \right]. \quad (3.3.369)$$

Also from it we get a way to decode derivatives of  $\mathcal{F}_D$  w.r.t.  $\hbar$ ,

$$\partial_\hbar \partial_\nu^n \mathcal{F}_D = \frac{1}{\hbar} \left( 8 \partial_\nu^n u + 2 \partial_\nu^n \mathcal{F}_D \right). \quad (3.3.370)$$

We apply this to the three-instanton free energy. Notice that the exponential term does depend on  $\hbar$ , and it is crucial to obtain the correct result when applying Matone's. We get

$$\begin{aligned} u^{(3)} &= e^{-6/\hbar^2} \mathcal{F}'_D \left[ \frac{8 (f^{(1)})^3 \mathcal{F}_D''' u'}{\hbar^2} + \frac{16 (f^{(1)})^3 \mathcal{F}_D'' u''}{\hbar^2} - \frac{48 (f^{(1)})^3 (\mathcal{F}_D'')^2 u'}{\hbar^4} - \right. \\ &\quad \left. - \frac{48 f^{(2)} f^{(1)} \mathcal{F}_D'' u'}{\hbar^2} - \frac{4 (f^{(1)})^3 u'''}{3} + 8 f^{(2)} f^{(1)} u'' - 6 f^{(3)} u' \right]. \end{aligned} \quad (3.3.371)$$

With the boundary conditions

$$f^{(m)} = \frac{1}{(2m)^2 \pi i} \quad (3.3.372)$$

this reproduces precisely (3.3.354). We have tested all this up to  $m = 5$ , who has 12 terms – involving some rather non-trivial derivatives. It is remarkable that the holomorphic anomaly does not only give us the correct transseries as the quantization condition, but it can also do so with **exactly the same structure**. From the point of view of the quantization (3.3.345), we are resumming the quantum volume in  $\nu$ , and the quantum tunnelling period in  $f(\nu)$ . From the holomorphic anomaly, the quantum volume is already resummed in its natural variable  $\mathbf{a}_D$ , while the tunnelling period is resummed in the  $G$  function that we have introduced for our transseries recursion.

Before we close this chapter, let us say one more time that this structure of the instanton expansion is completely independent of the underlying problem. If we look at (3.3.322), we can see that concrete details appear through the action  $\mathcal{A}$ , the quantum corrections computed by the *perturbative* holomorphic anomaly in  $D_a \tilde{F}^{(0)}$ , and the large order coefficients  $f^{(m)}$ . But they are not needed for the recursion itself. That is, once the perturbative sector is known, **the instantons follow in an universal way**.



## Chapter 4

# Examples and applications

### 4.1 Quantum mechanics

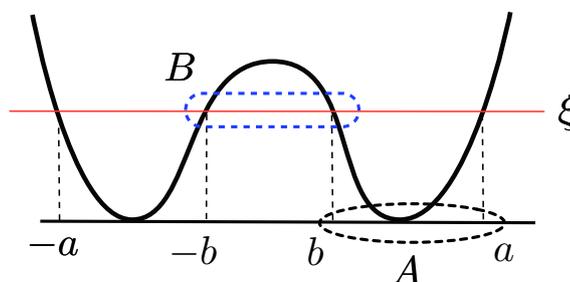
In this chapter we will apply the transseries holomorphic anomaly to other examples. First we will see a couple of examples of pure problems in quantum mechanics – without a gauge theory counterpart as Mathieu has – where the holomorphic anomaly anyway still controls the quantum periods. We can then use our machinery to study the large order of the WKB coefficients. Then, we will make one last return to the operators defined in (2.1.25). For their periods, we know that the perturbative holomorphic anomaly must hold. What is not so obvious is that the transseries obtained from the non-perturbative holomorphic anomaly can have something to say about their large order or resummation.

#### 4.1.1 Double well

The first quantum mechanical example we consider is the double well, describing a particle in the potential

$$V(x) = \frac{x^2}{2} (1 + gx)^2. \quad (4.1.1)$$

This can be made symmetric w.r.t.  $x = 0$  by shifting  $x \rightarrow x - 1/(2g)$ . In this symmetric realization, it has the form shown in figure 4.1. The symmetric double well



**Figure 4.1:** The symmetric double well.

is a textbook example for the importance of non-perturbative effects in quantum mechanics (see for example [98]). In perturbation theory, the model has two degenerate minima, but exponentially small quantum effects, due to instanton tunneling, lift the degeneracy. The exact quantization condition for this model was first conjectured in [7] by using multi-instanton calculus<sup>1</sup>. It can be derived in the context of the all-orders WKB method by using the exact connection formula of Voros–Silverstone (a related derivation appears in [92]). Another derivation, based on the uniform WKB method, was presented by Álvarez in [2]. The resulting condition reads,

$$1 + \exp\left(\pm \frac{i}{\hbar} \oint_A P(x) dx\right) = \epsilon i \exp\left(\frac{i}{2\hbar} \oint_B P(x) dx\right). \quad (4.1.2)$$

Here,  $\epsilon = \pm 1$  refers to the parity of the state, while the  $\pm$  sign in the l.h.s. refers to the choice of lateral Borel resummation. This quantization condition reduces to the previous one when the “tunneling” integral over the  $B$  cycle is neglected. The tunneling cycle gives a non-perturbative, exponentially small correction to the perturbative result. The function appearing in the r.h.s.,

$$f(\nu) = \exp\left(\frac{i}{2\hbar} \oint_B P(x) dx\right), \quad (4.1.3)$$

has the structure

$$f(\nu) = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2} + \nu\right)} \left(\frac{2}{g^2}\right)^\nu e^{-A(\nu)/2}, \quad (4.1.4)$$

where

$$\hbar\nu = \frac{1}{2\pi} \oint_A P(x) dx \quad (4.1.5)$$

is the quantum volume and  $A(\nu)$  is the function appearing in the quantization condition of [7, 93]. This structural result can be obtained in the framework of [92]. It was also derived by Álvarez in [2] by using the uniform WKB method<sup>2</sup>. The function  $A(\nu)$  has an expansion

$$A(\nu, \hbar = 1) = \frac{1}{3g^2} + \sum_{k \geq 1} c^{(k)}(\nu) g^{2k}, \quad (4.1.6)$$

where the coefficients  $c^{(k)}(\nu)$  are polynomials in  $\nu$  with rational coefficients. They can be computed by performing the WKB integrals, as in [93, 94], or by using asymptotic matching in the uniform WKB method of [2] (this method has the advantage of producing the function  $A(\nu)$  directly as a function of the quantum period, while the standard WKB technique, as applied in [93, 94], gives  $A(\nu)$  as a

<sup>1</sup>We should however note that a structurally identical, exact quantization condition for the pure quartic oscillator was already written down explicitly by Voros in [5].

<sup>2</sup>In order to make contact with [2], we note that: i) the function  $f(\nu)$ , as defined in [2], includes the factor of  $i$  appearing in the r.h.s. of (4.1.2), ii) our variable  $\nu$  has to be identified with his  $\nu/\sqrt{2}$ , and iii) his  $g$  has to be set to  $\sqrt{2}g^2$ .

function of  $E$ ). Still, in both cases the calculation is involved and time-consuming. One obtains, at the very first orders (see also [4])

$$\begin{aligned} c^{(1)}(\nu) &= \frac{19}{12} + 17\nu^2, \\ c^{(2)}(\nu) &= \frac{153\nu}{4} + 125\nu^3, \\ c^{(3)}(\nu) &= \frac{22709}{576} + \frac{23405\nu^2}{24} + \frac{17815\nu^4}{12}. \end{aligned} \quad (4.1.7)$$

A more comprehensive list can be found in [99].

#### 4.1.1.1 From holomorphic anomaly

Let us see how this information is contained in the holomorphic anomaly. To avoid unnecessary clutter, we set  $g = 1$ . The turning points of the potential (4.1.1) are

$$x_{\text{tp}} = \frac{1}{2} \left( -1 \pm \sqrt{1 \pm \sqrt{32E}} \right), \quad (4.1.8)$$

given as usual by  $p(x) = 0$ . Also like in Mathieu, the classical WKB period integrals can be written in terms of elliptic integrals. Over the classically allowed region, called A in figure 4.1,

$$t_0(E) = \frac{1}{2\pi} \oint_A p(x) dx = \frac{\sqrt{1 + \sqrt{32E}}}{12\pi} \left[ \mathbf{E}(k^2(E)) - (\sqrt{32E} - 1) \mathbf{K}(k^2(E)) \right] \quad (4.1.9)$$

where  $t_0$  is the classical part of the quantum A period  $t(E, \hbar)$  defining the quantum volume,

$$\frac{\text{vol}}{2\pi} = t = t_0 + t_1 \hbar^2 + O(\hbar^4). \quad (4.1.10)$$

We have also written

$$k^2(E) = 2 - \frac{2}{1 + \sqrt{32E}}. \quad (4.1.11)$$

Over the tunnelling region we have the B period,

$$\begin{aligned} \partial_t \mathcal{F}_0^{(0)} \Big|_{t=t_0} &= -i \oint_B p(x) dx = \\ &= \frac{\sqrt{1 + \sqrt{32E}}}{3} \left[ \mathbf{E}(1 - k^2(E)) - \sqrt{32E} \mathbf{K}(1 - k^2(E)) \right]. \end{aligned} \quad (4.1.12)$$

Again, we write it like this to make clear that  $\mathcal{F}_0^{(0)}(t)$  is itself a function of the *quantum* A period  $t$ , and so are all the corrections generated by the holomorphic anomaly. To get the B quantum period as a function of the physical energy  $E$  we should reexpand

$$\begin{aligned} \partial_t \mathcal{F}^{(0)}(t) &= \partial_t \mathcal{F}^{(0)}(t_0 + t_1 \hbar^2 + \dots) = \\ &= \partial_t \mathcal{F}_0^{(0)} \Big|_{t_0} + \left( \partial_t \mathcal{F}_1^{(0)} \Big|_{t_0} + t_1 \partial_{tt} \mathcal{F}_0^{(0)} \Big|_{t_0} \right) \hbar^2 + \dots \end{aligned} \quad (4.1.13)$$

which is now directly connected to  $E$  via the  $t_n$ . Like in Mathieu, we can make good use of a corrected energy  $\xi$  such that

$$t(E, \hbar) = \frac{\sqrt{1 + \sqrt{32\xi}}}{12\pi} \left[ \mathbf{E}(k^2(\xi)) - \left( \sqrt{32\xi} - 1 \right) \mathbf{K}(k^2(\xi)) \right]. \quad (4.1.14)$$

From now on we will just let  $k^2$  mean  $k^2(\xi)$ . We can obtain this  $\xi$  from (4.1.11), giving

$$\xi = \frac{L_2^2}{32K_2^2}. \quad (4.1.15)$$

To put the elliptic integrals in terms of modular ingredients too, recall the identity we used to find the periods of Mathieu,

$$\mathbf{K}(k^2) = \frac{\pi}{2} \vartheta_3^2(q) = \frac{\pi}{\sqrt{8}} \sqrt{K_2 + L_2}. \quad (4.1.16)$$

The quotient with the dual integral gives the elliptic parameter  $\tau$ ,

$$-\log q = \pi \frac{\mathbf{K}(k^2)}{\mathbf{K}(1 - k^2)}, \quad (4.1.17)$$

Actually, we can use (4.1.17) and the definition of  $E_2^D$  in (3.3.207) to rewrite it directly with our  $E_2^D$  generator,

$$\mathbf{K}(1 - k^2) = \frac{3}{\sqrt{2}} \frac{\sqrt{K_2 + L_2}}{E_2 - E_2^D}. \quad (4.1.18)$$

We also had

$$\mathbf{E}(k^2) = \frac{\pi}{3\sqrt{2}} \frac{E_2 + K_2}{\sqrt{K_2 + L_2}}. \quad (4.1.19)$$

And finally, we use Legendre's relation (3.2.43) to get

$$\mathbf{E}(1 - k^2) = \frac{1}{\sqrt{2}} \frac{3L_2 + K_2 - 2E_2^D}{(E_2 - E_2^D) \sqrt{K_2 + L_2}}. \quad (4.1.20)$$

With all these equalities, it is straightforward to write the periods as

$$t = \frac{2E_2K_2 + 3L_2^2 - K_2^2}{72\sqrt{2} K_2^{3/2}} \quad (4.1.21)$$

and

$$\partial_t \mathcal{F}_0^{(0)} = \frac{2E_2^D K_2 + 3L_2^2 - K_2^2}{3\sqrt{2} (E_2^D - E_2) K_2^{3/2}}. \quad (4.1.22)$$

Now, the Maass derivative on zero weight objects is just a standard derivative  $\partial_\tau$ . We can use it to compute

$$\partial_{tt} \mathcal{F}_0^{(0)} = \frac{\partial_\tau \left( \partial_t \mathcal{F}_0^{(0)} \right)}{\partial_\tau (t)} = \frac{D_\tau \left( \partial_t \mathcal{F}_0^{(0)} \cdot \omega_1 \right) / \omega_1}{D_\tau (t/\omega_1) \cdot \omega_1} = \frac{24}{E_2^D - E_2} = 4\pi i \tau, \quad (4.1.23)$$

and again to find the Yuwaka

$$Y = \partial_{ttt}\mathcal{F}_0^{(0)} = \frac{128\sqrt{2}K_2^{5/2}}{K_2^2L_2^2 - L_2^4}. \quad (4.1.24)$$

This means that in (3.2.44) we have  $\beta = 1/2$ .

From the boundary conditions derived in (3.2.151), we can set the first free energy to be the logarithm of the discriminant,

$$F_1^{(0)} = -\frac{1}{24} \log \Delta(\xi) = -\frac{1}{24} \log \left( \frac{(K_2^2 - L_2^2) L_2^4}{16K_2^6} \right), \quad (4.1.25)$$

where the discriminant of  $H(p, x) - E = \frac{p^2}{2} + V(x) - E$  for (4.1.1) is

$$\Delta(E) = E^2(1 - 32E). \quad (4.1.26)$$

Not we can proceed to integrate the  $F_n^{(0)}$ . We parametrize the ambiguity at every order, as in the case of Mathieu, by an overall term with the singularity of the Yuwaka times a polynomial in the holomorphic generators.

$$\tilde{f}_{n,0} = \left( \frac{K_2}{(K_2^2 - L_2^2) L_2^2} \right)^{2(n-1)} \times \sum_{i=0}^{3(n-1)} a_i K_2^{6(n-1)-2i} L_2^{2i}. \quad (4.1.27)$$

We fix the  $a_i$  coefficients with the gap condition that follows from (3.2.151), and (with the normalization of the periods for this potential) set

$$[F_n^{(0)}]_{\bar{\tau} \rightarrow i\infty} = 2 \frac{S_n}{t^{2n-2}} + O(t^0) \quad (4.1.28)$$

with the singularity coefficient

$$S_n = \frac{(1 - 2^{1-2n})(2n-3)!}{(2n)!} B_{2n}. \quad (4.1.29)$$

However, it turns out that this is not enough for this problem. What happens is that the double well curve has two singular points, as we see in the discriminant (4.1.26). We can therefore get (and need) more conditions from the other singularity. Physically, when we perform an S transformation, we exchange the periods – effectively turning the problem into a quartic potential with a single minimum, where the maximum of the double well's barrier used to be. In that problem, the classical period is the double well B period, which should vanish at low energies. Since in the definition of the A and B periods there was a difference in a factor of  $2\pi i$ , namely

$$t_D = \partial_t \mathcal{F}_0^{(0)} = 2\pi i s, \quad (4.1.30)$$

and  $s$  has the normalization of an A period. The corresponding condition for the dual problem is

$$[F_n^{(0)}]_{\bar{\tau}_D \rightarrow i\infty} = \mathcal{F}_{D,n}^{(0)} = \frac{S_n}{s^{2n-2}} + O(s^0) \quad (4.1.31)$$

With this we can compute the following  $F_n$ , such as

$$\begin{aligned} F_2 &= -\frac{2(2K_2^2 - 3K_4)^2 K_2^3}{27(K_2^2 - K_4)^2 K_4^2} \hat{E}_2 - \frac{2(158K_2^5 - 330K_4K_2^3 + 135K_4^2K_2) K_2^3}{135(K_2^2 - K_4)^2 K_4^2}, \\ F_3 &= \frac{32(2K_2^2 - 3K_4)^3 K_2^7}{2187(K_2^2 - K_4)^4 K_4^4} \hat{E}_2^3 + \frac{16(2K_2^2 - 3K_4)^2 (22K_2^4 - 39K_4K_2^2 + 27K_4^2) K_2^6}{729(K_2^2 - K_4)^4 K_4^4} \hat{E}_2^2 + \\ &+ \frac{8(2K_2^2 - 3K_4)(7604K_2^6 - 23088K_4K_2^4 + 23625K_4^2K_2^2 - 6345K_4^3) K_2^7}{3645(K_2^2 - K_4)^4 K_4^4} \hat{E}_2 + f_3(\tau), \end{aligned} \quad (4.1.32)$$

where the holomorphic ambiguity at order  $n = 3$  reads

$$\begin{aligned} f_3(\tau) &= \frac{4K_2^6}{76545(K_2^2 - K_4)^4 K_4^4} (7384904K_2^{10} - 31999716K_4K_2^8 + 53857062K_4^2K_2^6 - \\ &- 43355655K_4^3K_2^4 + 16924950K_4^4K_2^2 - 1927233K_4^5). \end{aligned} \quad (4.1.33)$$

By expanding around  $t = 0$ , we obtain

$$\begin{aligned} F_2(t) &= -\frac{7}{2880t^2} + \frac{131}{192} + \frac{22709t}{576} + \frac{217663t^2}{128} + \frac{61936297t^3}{960} + \frac{581912191t^4}{256} + O(t^5), \\ F_3(t) &= \frac{31}{80640t^4} + \frac{10483}{256} + \frac{20182631t}{3840} + \frac{628163783t^2}{1536} + \frac{178000040363t^3}{7168} + O(t^4). \end{aligned} \quad (4.1.34)$$

The explicit expressions for  $F_4$  and the higher  $n$  free energies become too long to be copied here, but we can write the first terms of their expansion around  $t = 0$ . We have, for example,

$$F_4(t) = -\frac{127}{645120t^6} + \frac{193438987}{24576} + \frac{553607616973t}{344064} + \frac{12106775869037t^2}{65536} + O(t^3). \quad (4.1.35)$$

The dependence on  $g$  is restored again by

$$t \rightarrow g^2t, \quad F_n \rightarrow (g^2)^{2n-2} F_n. \quad (4.1.36)$$

The  $A(\nu)$  function appearing in (4.1.6) generates the exponentially suppressed terms in the quantization condition, and is in fact given by (the regular part of) the quantum B period. Then, we should recover it from the regular part of the derivative of the free energies w.r.t.  $t$ . The expansions of the  $F_n$  obtained above agree with previous results in quantum mechanics, as listed in for example [2] (see also [4]). For instance, after setting  $t = \hbar\nu$ ,

$$\begin{aligned} \partial_t \mathcal{F}^{(0)} \Big|_{t=\hbar\nu} &= \hbar^0 \left( \frac{1}{3} + 2\nu\hbar \log\left(\frac{\nu\hbar}{2}\right) - 2\nu\hbar + 17\nu^2\hbar^2 + 125\nu^3\hbar^3 + O(\hbar^4) \right) + \\ &+ \hbar^2 \left( -\frac{1}{12\nu\hbar} + \frac{19}{12} + \frac{153\nu\hbar}{4} + \frac{23405\nu^2\hbar^2}{24} + \frac{50715\nu^3\hbar^3}{2} + O(\hbar^4) \right) + \\ &+ \hbar^4 \left( \frac{7}{1440\nu^3\hbar^3} + \frac{22709}{576} + \frac{217663\nu\hbar}{64} + \frac{61936297\nu^2\hbar^2}{320} + O(\hbar^3) \right) + O(\hbar^6) = \\ &= (\text{singular}) + \frac{1}{3} + \left( 17\nu^2 + \frac{19}{12} \right) \hbar^2 + \left( 125\nu^3 + \frac{153\nu}{4} \right) \hbar^3 + O(\hbar^4) \end{aligned} \quad (4.1.37)$$

which are the coefficients<sup>3</sup> in (4.1.7). Interestingly, the above ansatz for the holomorphic ambiguity generates a non-trivial constant term  $\mu_n$  in the regular part of the free energies  $F_n^r(\nu)$ , which has the form

$$F_n^r(\nu) = \mu_n + O(\nu). \quad (4.1.38)$$

The presence of this constant term is at first glance surprising, since the WKB method only determines the derivative of  $F_n(\nu)$  w.r.t.  $\nu$ , so the constant  $\mu_n$  is not fixed by quantum special geometry. What is its meaning?

In the same spirit of the Matone relation, the A and B periods are not completely independent. This fact is usually expressed with the *Perturbative-Non Perturbative* relation. This PNP relation for the double well, found by Álvarez in [2], can be written as

$$\frac{dE}{d\nu} = -3g^2 \left( 2\nu + g^2 \frac{\partial A}{\partial g^2} \right). \quad (4.1.39)$$

after setting  $\hbar = 1$ . After integration, we find that

$$E(\nu, \hbar = 1) = -3g^2\nu^2 - 3g^4 \frac{\partial \mathcal{F}^r(\nu)}{\partial g^2} + \text{constant}. \quad (4.1.40)$$

up to a  $\nu$  independent term. The quantum energy at  $\nu = 0$  is given by

$$E(0, \hbar = 1) = -\frac{g^2}{4} - \frac{131g^6}{32} - \frac{31449g^{10}}{64} - \frac{580316961g^{14}}{4096} - \dots \quad (4.1.41)$$

This can be easily calculated for example with the Bender-Wu algorithm of [100]. The constant term  $\mu_n$  in  $F_n$  should be such that (4.1.40) is satisfied for the  $\nu$ -independent part, i.e. such that

$$E(0, \hbar = 1) + \frac{g^2}{4} = -3g^4 \sum_{n \geq 2} (2n - 2) \mu_n (g^2)^{2n-3}. \quad (4.1.42)$$

This are precisely the values of  $\mu_n$  obtained with the holomorphic anomaly equations. This means one can use the full quantum free energy to write the PNP relation already integrated as with the Matone relation,

$$E(\nu) = -\frac{g^2 \hbar^2}{2} - 3g^4 \frac{\partial \mathcal{F}}{\partial g^2}, \quad (4.1.43)$$

where we reintroduce both  $\hbar$  and  $g$  with (4.1.36) before the derivative. With  $t = \hbar\nu$ , one finds

$$\begin{aligned} \mathcal{F}^{(0)} = & (g\text{-independent}) + \frac{\hbar}{g^2} \frac{\nu}{3} + \hbar^2 \frac{12\nu^2 \log\left(\frac{g^2\nu}{2}\right) - \log(8g^2\nu)}{12} + \\ & + \hbar^3 g^2 \left( \frac{17\nu^3}{3} + \frac{19\nu}{12} \right) + \hbar^4 g^4 \left( \frac{125\nu^4}{4} + \frac{153\nu^2}{8} + \frac{131}{192} \right) + O(\hbar^5). \end{aligned} \quad (4.1.44)$$

<sup>3</sup>The parameters  $g^2$  and  $\hbar$  are not independent, and by a simple rescaling, they can be used interchangeably.

and the first orders for the energy levels are

$$E(\nu) = \hbar\nu + \hbar^2 g^2 \left( -3\nu^2 - \frac{1}{4} \right) + \hbar^3 g^4 \left( -17\nu^3 - \frac{19\nu}{4} \right) + O(\hbar^4), \quad (4.1.45)$$

They are precisely what is found from a direct computation with the Bender-Wu recursion [100]. This means that the holomorphic anomaly provides **more information than just the WKB periods**, since it also computes a critical ingredient to relate them via the PNP relation.

Finally, we note that results for the quantum free energies of the (unstable) quartic oscillator with potential

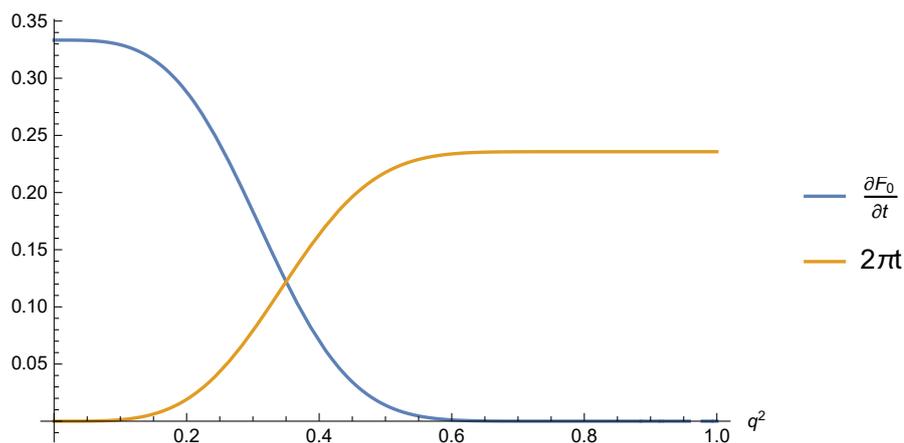
$$V(x) = \frac{x^2}{2} - gx^4 \quad (4.1.46)$$

can be obtained from the results for the double well, after performing the  $S$  transformation. The quantum free energies of the quartic oscillator are precisely the dual free energies, while  $2\pi i s = t_D$  plays the role of the quantum volume. One obtains, for example,

$$\begin{aligned} \mathcal{F}_{D,2}^{(0)}(s) &= -\frac{7}{5760s^2} + \frac{513}{512} + \frac{305141s}{9216} + \frac{3105983s^2}{4096} + \frac{912774217s^3}{61440} + \dots, \\ \mathcal{F}_{D,3}^{(0)}(s) &= \frac{31}{161280s^4} + \frac{485523}{8192} + \frac{1056412343s}{245760} + \frac{34978331399s^2}{196608} + \dots \end{aligned} \quad (4.1.47)$$

This and other results agree with the calculations using uniform WKB in [3, 99]. They also agree with the calculation of the  $A$  function in [101], after expressing it in terms of the quantum period as in [102].

#### 4.1.1.2 Large order



**Figure 4.2:** Double well actions

Like with modified Mathieu, it is definitely interesting to reproduce known results in a more efficient manner, but it is better to provide something new. Now we can

study the large order of the free energies (or their derivatives, the corrections to the quantum WKB volume) by using the general solution (3.3.239), where  $G$  is (3.3.230). Numerically, we find the actions

$$\begin{aligned}\mathcal{A}_A \cdot \omega_1 &= 2\pi i t, \\ \mathcal{A}_B / \omega_1 &= t_D = \partial_t \mathcal{F}_0^{(0)}.\end{aligned}\tag{4.1.48}$$

In figure 4.2 we plot their absolute value as a function of  $q^2$ , and we clearly see their regions of dominance. The normalization of the transseries was determined by the singular behaviour, so indeed it remains

$$f_A^{(1)} = (2\pi i) \frac{1}{\pi^2}, \quad f_B^{(1)} = (2\pi i) \frac{1}{2\pi^2}.\tag{4.1.49}$$

Finally, we need the  $T$  function. With (3.2.80) we get

$$T = D_t \mathcal{A} (S - \mathcal{S}_A) = D_t \mathcal{A} \frac{1}{2} \frac{\hat{E}_2 - E_2^A}{-12} = \begin{cases} \frac{2\pi i}{24} \frac{\hat{E}_2 - E_2}{\omega_1} & \text{if } \mathcal{A} = \mathcal{A}_A, \\ \frac{\hat{E}_2 - E_2^D}{E_2 - E_2^D} \omega_1 & \text{if } \mathcal{A} = \mathcal{A}_B. \end{cases}\tag{4.1.50}$$

Then, from (3.3.239), for the  $\mathcal{A}_B$  action

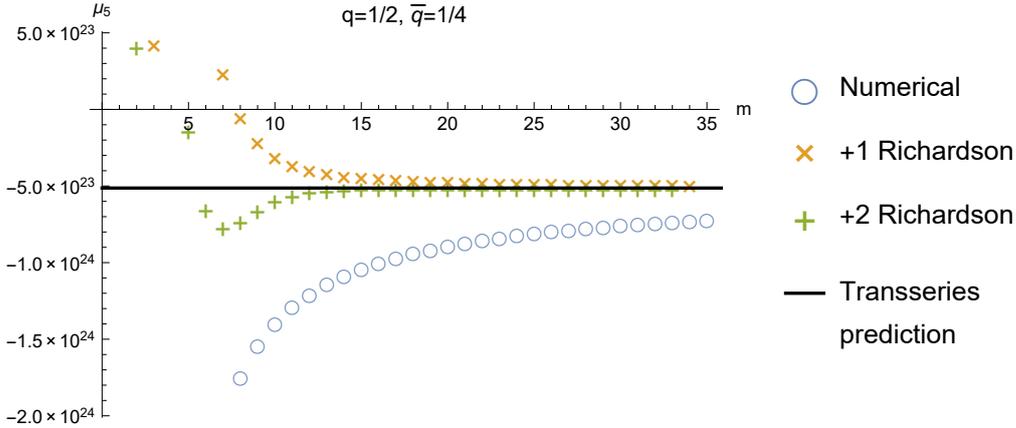
$$\begin{aligned}F_B^{(1)} &= \frac{i}{\pi} \left( 1 + \frac{4\sqrt{2}K_2^{3/2}\omega_1 (\hat{E}_2 - E_2^D) (2K_2^2 - 3L_2^2)}{3L_2^2 (E_2^D - E_2) (L_2^2 - K_2^2)} \hbar + \right. \\ &\quad \left. + \frac{16K_2^3\omega_1^2 (\hat{E}_2 - E_2^D)^2 (2K_2^2 - 3L_2^2)^2}{9L_2^4 (E_2 - E_2^D)^2 (K_2^2 - L_2^2)^2} \hbar^2 + O(\hbar^3) \right)\end{aligned}\tag{4.1.51}$$

and

$$\begin{aligned}F_A^{(1)} &= \frac{2i}{\pi} \left( 1 + \frac{i\sqrt{2}\pi (\hat{E}_2 - E_2) K_2^{3/2} (2K_2^2 - 3L_2^2)}{9L_2^2\omega_1 (K_2^2 - L_2^2)} \hbar - \right. \\ &\quad \left. - \frac{\pi^2 (\hat{E}_2 - E_2)^2 K_2^3 (2K_2^2 - 3L_2^2)^2}{81L_2^4\omega_1^2 (K_2^2 - L_2^2)^2} \hbar^2 + O(\hbar^2) \right).\end{aligned}\tag{4.1.52}$$

The complexity of the expressions grows fast. For instance, the next order is

$$\begin{aligned}
F_{A,3}^{(1)} = & \frac{i\sqrt{2}(\hat{E}_2 - E_2)K_2^{9/2}}{L_2^6\omega_1^3(L_2^2 - K_2^2)^3} \left( \frac{352}{243}\pi\hat{E}_2K_2^6\omega_1^2 + \frac{32}{243}\pi\hat{E}_2^2K_2^5\omega_1^2 + \right. \\
& + \frac{8\pi^3E_2^2K_2^6}{2187} + \frac{8\pi^3\hat{E}_2^2K_2^6}{2187} - \frac{16\pi^3E_2\hat{E}_2K_2^6}{2187} + \frac{15208\pi K_2^7\omega_1^2}{1215} - \\
& - \frac{128}{27}\pi\hat{E}_2K_2^4L_2^2\omega_1^2 - \frac{32}{81}\pi\hat{E}_2^2K_2^3L_2^2\omega_1^2 + \frac{152}{27}\pi\hat{E}_2K_2^2L_2^4\omega_1^2 + \\
& + \frac{8}{27}\pi\hat{E}_2^2K_2L_2^4\omega_1^2 - \frac{4}{243}\pi^3E_2^2K_2^4L_2^2 - \frac{4}{243}\pi^3\hat{E}_2^2K_2^4L_2^2 + \\
& + \frac{8}{243}\pi^3E_2\hat{E}_2K_2^4L_2^2 + \frac{2}{81}\pi^3E_2^2K_2^2L_2^4 + \frac{2}{81}\pi^3\hat{E}_2^2K_2^2L_2^4 - \\
& - \frac{4}{81}\pi^3E_2\hat{E}_2K_2^2L_2^4 - \frac{8}{3}\pi\hat{E}_2L_2^6\omega_1^2 - \frac{1}{81}\pi^3E_2^2L_2^6 - \\
& - \frac{1}{81}\pi^3\hat{E}_2^2L_2^6 + \frac{2}{81}\pi^3E_2\hat{E}_2L_2^6 - \frac{15392}{405}\pi K_2^5L_2^2\omega_1^2 + \\
& \left. + \frac{350}{9}\pi K_2^3L_2^4\omega_1^2 - \frac{94}{9}\pi K_2L_2^6\omega_1^2 \right)
\end{aligned} \tag{4.1.53}$$



**Figure 4.3:** Double well subleading large order corrections

Now we can do some tests. Take

$$q = 1/2, \quad \bar{q} = 1/4. \tag{4.1.54}$$

From figure 4.2 we know that we are in the region where the B period dominates. Using (3.3.248) we can compute the numerical value of  $\mu_n$ . Define as  $\mu_m^{(n,r)}$  the  $m$ -th correction to the large order computed by taking the perturbative free energies up to order  $n$ , and using  $r$  Richardson transforms. We take as significant those digits that remain stable when compared to the previous and next Richardson transforms. Then,

$$\mu_5^{(40,11)} = -5.13138 \dots \cdot 10^{23} \tag{4.1.55}$$

The prediction from the transseries is

$$\left. \frac{F_5^{(1)}}{2\pi i} \right|_{\substack{q=1/2 \\ \bar{q}=1/4}} = -5.13138091 \dots \cdot 10^{23}, \quad (4.1.56)$$

all the 6 significant digits agreeing. We show the convergence to this values in figure 4.3.

The holomorphic limit corresponds to  $\bar{q} \rightarrow 0$ , or  $\hat{E}_2 \rightarrow E_2$ . In this limit, the  $T$  function we found before for the B period goes to

$$[T]_{\bar{q} \rightarrow 0} = 1 \quad (4.1.57)$$

and consequently the large order of  $\mathcal{F}^{(0)}$  is generated simply by

$$\exp\left(\partial_t \mathcal{F}_0^{(0)} + \partial_t \tilde{\mathcal{F}}^{(0)}\right). \quad (4.1.58)$$

In this case, for  $q = 1/3$ ,

$$\mu_5^{(40,12)} = -4.68766 \dots \cdot 10^{12}. \quad (4.1.59)$$

Compare to

$$\left. \frac{\mathcal{F}_5^{(1)}}{2\pi i} \right|_{q=1/3} = -4.68765525 \dots \cdot 10^{12} \quad (4.1.60)$$

to find another 6 matching digits.

The dual holomorphic limit is obtained by performing an S transform and sending  $\bar{\tau}_D \rightarrow i\infty$ , or  $\bar{q}_D \rightarrow 0$ . When doing this

$$[T]_{\bar{q}_D \rightarrow 0} = 0, \quad (4.1.61)$$

since  $\hat{E}_2 \mapsto E_2^D$ . Then we expect the large order to be trivial,

$$\mu_0 = \frac{1}{2\pi^2}, \quad \mu_{m \geq 1} = 0, \quad (4.1.62)$$

which is precisely what we find numerically.

We go to the other region of dominance when at small  $q$ , where the energy is small, and the A period  $2\pi i t$  dominates. In this case the large order is trivial when  $\bar{q} \rightarrow 0$ , since from (4.1.50),

$$[T]_{\hat{E}_2 \mapsto E_2} = 0. \quad (4.1.63)$$

This is again what happens numerically. Notice that the boundary conditions that we use to fix  $\mathcal{S}_A$  in the ansatz (3.3.230) follow from the harmonic singularity (3.2.151), as in (3.2.185), and say that indeed the asymptotics must become trivial as we send  $t \mapsto 0$ . However, what the result from the holomorphic anomaly tells us is much stronger: not only is the large order trivial in the harmonic limit, but over the *whole* region of dominance of the corresponding action.

For a more interesting behaviour, take  $q = 1/30$  and  $\bar{q} = i$ . Then we should find a large order controlled by  $\mathcal{A}_A$ , and with subleading corrections given by

$$\frac{F_0^{(1)}}{2\pi i} = \frac{1}{\pi^2}, \dots, \frac{F_5^{(1)}}{2\pi i} \Bigg|_{\substack{q=1/30 \\ \bar{q}=i}} = (5.431150468\dots - i 12.574197519\dots) \cdot 10^8. \quad (4.1.64)$$

Indeed, numerically,

$$\mu_5^{(40,9)} = (5.43115\dots - i 12.57419\dots) \cdot 10^8. \quad (4.1.65)$$

### 4.1.1.3 Exact WKB quantization

As we already wrote in the introduction, the exact quantization condition for the double well is given in [2] in terms of period integrals,

$$1 - \exp\left(\frac{i}{\hbar} \oint_A Q dx\right) = \epsilon i \exp\left(\frac{i}{2\hbar} \oint_B Q dx\right), \quad (4.1.66)$$

The right hand side is exponentially suppressed in  $\hbar$ , since the period is evaluated around the classically forbidden zone. The first order in  $e^{-1/\hbar}$  is just the perturbative WKB condition,

$$1 + e^{2\pi i t/\hbar} = 0 \implies t = \hbar \left(n + \frac{1}{2}\right). \quad (4.1.67)$$

The full condition can be seen as an equation on a non-perturbative period  $\hat{t}$ ,

$$1 + e^{2\pi i \hat{t}/\hbar} = \epsilon i e^{-\frac{t_D(\hat{t}, \hbar)}{2\hbar}}, \quad (4.1.68)$$

where  $t_D(t, \hbar)$  is the perturbative quantum B period  $t_D(\xi, \hbar)$  re-expressed in terms of the quantum A period  $t(\xi, \hbar)$ . Indeed,  $t_D(t, \hbar)$  is precisely what the perturbative holomorphic anomaly gives us,

$$t_D = \partial_t \mathcal{F}^{(0)}. \quad (4.1.69)$$

We can rename the right hand side of (4.1.68) as

$$f(t, \hbar) = \epsilon i \exp\left[-\frac{1}{2\hbar} t_D(t, \hbar)\right] \quad (4.1.70)$$

From now on we will set  $\epsilon = +1$  for simplicity. Then we set up an ansatz for the subleading corrections,

$$\hat{t}(t, \hbar) = t + t^{(1)}(t, \hbar) + t^{(2)}(t, \hbar) + \dots \quad (4.1.71)$$

and plug it into (4.1.68).

$$\begin{aligned} t^{(1)} &= \frac{i\hbar f}{2\pi}, \\ t^{(2)} &= -\frac{\hbar^2 f f'}{4\pi^2} + \frac{i\hbar f^2}{4\pi}, \\ t^{(3)} &= -\frac{i\hbar^3 f^2 f''}{16\pi^3} - \frac{i\hbar^3 f f'^2}{8\pi^3} - \frac{3\hbar^2 f^2 f'}{8\pi^2} + \frac{i\hbar f^3}{6\pi}, \\ &\dots \end{aligned} \quad (4.1.72)$$

where it is implied that  $f' = \partial_t f(t, \hbar)$ .

As a function of  $n$ , the exact energy levels<sup>4</sup>  $\widehat{E}$  are given at first order in  $e^{-1/\hbar}$  by the perturbative energy  $\xi$ ,

$$\widehat{E}(t, \hbar) = E(t, \hbar) + O(e^{-1/\hbar}). \quad (4.1.73)$$

We can obtain it by inverting the quantum perturbative period  $t$ ,

$$t(E, \hbar) \implies E(t, \hbar). \quad (4.1.74)$$

The full energy is then given by using the exact  $\hat{t}(t, \hbar)$  instead of just the perturbative result  $n$ ,

$$\widehat{E}(t, \hbar) = E(\hat{t}, \hbar). \quad (4.1.75)$$

Expanding  $\hat{t}$  in its powers of  $f$ , and explicitly writing the  $f$  function,

$$\begin{aligned} \widehat{E}(t, \hbar) = & E + \left( -\frac{\hbar E'}{2\pi} \right) e^{-t_D/2\hbar} + \\ & + \left( -\frac{\hbar t'_D E'}{8\pi^2} + \frac{\hbar^2 E''}{8\pi^2} - \frac{i\hbar E'}{4\pi} \right) e^{-t_D/\hbar} + \\ & + \left( \frac{\hbar^2 t'_D E''}{16\pi^3} + \frac{\hbar^2 t''_D E'}{32\pi^3} - \frac{3\hbar t'_D{}^2 E'}{64\pi^3} \right. \\ & \left. - \frac{3i\hbar t'_D E'}{16\pi^2} - \frac{\hbar^3 E'''}{48\pi^3} + \frac{i\hbar^2 E''}{8\pi^2} + \frac{\hbar E'}{6\pi} \right) e^{-3t_D/2\hbar} + \\ & + O(e^{-2t_D/\hbar}), \end{aligned} \quad (4.1.76)$$

again with  $t'_D = \partial_t t_D(t, \hbar)$  and  $E' = \partial_t E(t, \hbar)$ .

To reproduce these results from the transseries for the free energy, we only need the fact that

$$\frac{\partial E}{\partial t} = \alpha \partial_\lambda \frac{\partial \mathcal{F}^{(0)}}{\partial t}, \quad (4.1.77)$$

which is just the PNP relation for some constant  $\alpha$ . From (4.1.68) it is clear that the action we need in the transseries is

$$\mathcal{A} = \frac{1}{2} \partial_t \mathcal{F}_0^{(0)}. \quad (4.1.78)$$

Once the action is chosen, we turn our attention to the  $T$  function in (3.3.303). Suppose, more generally, we choose an action  $\mathcal{A} = \alpha \partial_t \mathcal{F}_0^{(0)}$ . Since the form of the holomorphic anomaly is the same up to a rescaling of  $S$  (3.2.80)

$$T = (S - \mathcal{S}_A) D_t \mathcal{A} = \frac{\widehat{E}_2 - E_2^D}{-12/\beta} \alpha D_t \left( \partial_t \mathcal{F}_0^{(0)} \right). \quad (4.1.79)$$

In the electric frame (that is,  $\widehat{E}_2 \mapsto E_2$ )

$$[T]_e = \alpha \frac{-6/(i\pi\tau)}{-12/\beta} \frac{\partial t_D}{\partial t} = \alpha \frac{\beta}{2i\pi\tau} \frac{2\pi i\tau}{\beta} = \alpha. \quad (4.1.80)$$

<sup>4</sup>Do not mistake with the modular form  $\widehat{E}_2$ .

The  $G$  function of (3.3.230), in the electric frame, will then go to

$$[G]_e = \alpha \partial_t \mathcal{F}_0^{(0)}(t) + \alpha \partial_t \tilde{\mathcal{F}}^{(0)}(t, \hbar) = \alpha \partial_t \mathcal{F}^{(0)}. \quad (4.1.81)$$

Then for the action (4.1.78) of the quantization condition for the double well

$$[T]_e = \frac{1}{2}, \quad [G]_e = \frac{1}{2} t_D(t, \hbar) \implies \alpha [\partial_\lambda G]_e = \frac{1}{2} \partial_t E(t, \hbar). \quad (4.1.82)$$

Notice that from (3.3.312), this also means that

$$[\mathcal{D}]_e = \frac{1}{2} \partial_t. \quad (4.1.83)$$

Take the  $m = 2$  instanton, (3.3.311). Following the PNP relation, the corresponding correction to the energy should be given by

$$\partial_\lambda F^{(2)} = \left( \frac{\hbar^2 (f^{(1)})^2}{2} \mathcal{D} \partial_\lambda G - \hbar (f^{(1)})^2 \mathcal{D} G \partial_\lambda G - 2\hbar f^{(2)} \partial_\lambda G \right) e^{-2G/\hbar}. \quad (4.1.84)$$

and

$$E^{(2)} = \alpha [\partial_\lambda F^{(2)}]_e = \left( \frac{\hbar^2 (f^{(1)})^2}{8} E'' - \frac{\hbar (f^{(1)})^2}{8} t'_D E' - \hbar f^{(2)} E' \right) e^{-t_D/\hbar}. \quad (4.1.85)$$

Using boundary conditions similar to (3.2.193),

$$f^{(m)} = \frac{i^{m-1}}{m^2 \pi}, \quad (4.1.86)$$

we get

$$E^{(2)} = \left( \frac{\hbar^2}{8\pi^2} E'' - \frac{\hbar}{8\pi^2} t'_D E' - \frac{i\hbar}{4\pi} E' \right) e^{-t_D/\hbar}, \quad (4.1.87)$$

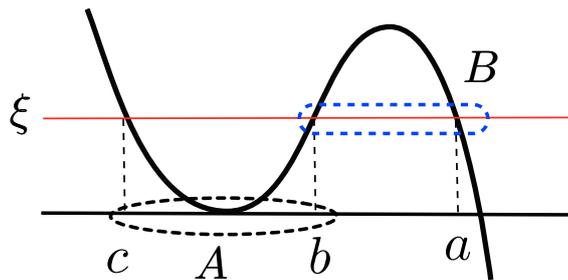
precisely what appears in (4.1.76). For compactness we only show  $m = 2$ , but we have tested the matching of several higher instanton corrections.

### 4.1.2 Cubic well

Another interesting example with a genus one curve is the cubic oscillator, that puts a one-dimensional quantum particle in the cubic potential

$$V(x) = \frac{x^2}{2} - gx^3. \quad (4.1.88)$$

This is also a much studied example in quantum mechanics (see for example [103] for a textbook presentation). The potential does not support bound states, but by imposing Gamow–Siegert boundary conditions, one finds an infinite tower of resonances which can be computed numerically by using complex dilatation techniques [104, 105]. The resonant energies can be also obtained by Borel resummation



**Figure 4.4:** The cubic oscillator.

of the stationary perturbative series in the coupling  $g$  [10,105–107]. In the all-orders WKB approach, one can still derive an exact quantization condition involving two different periods. To write down this condition, we denote the turning points of the potential by

$$c < b < a, \quad (4.1.89)$$

as shown in figure 4.4. The A cycle goes around the points  $c$  and  $b$ , while the B cycle goes around the points  $b$  and  $a$ . The exact quantization condition can be obtained in a simple way by using the Voros–Silverstone connection formulae [5,6,91] (this is essentially the method followed in [92]). Alternatively, it can be derived from the uniform WKB method [3,107]. Since the resulting series is not Borel summable, there are two different quantization conditions, depending on the choice of lateral resummation. One choice gives the perturbative quantization condition,

$$\oint_A P(x)dx = 2\pi\hbar \left( n + \frac{1}{2} \right) = 2\pi\hbar \nu, \quad n \in \mathbb{Z}_{\geq 0}, \quad (4.1.90)$$

while the other choice gives

$$1 + \exp\left(\frac{i}{\hbar} \oint_A P(x)dx\right) + \exp\left(\frac{i}{\hbar} \oint_B P(x)dx\right) = 0. \quad (4.1.91)$$

The Voros multiplier associated to the tunneling period is

$$f(\nu) = \exp\left(\frac{i}{\hbar} \oint_B P(x)dx\right). \quad (4.1.92)$$

where again  $\hbar\nu = t(E, \hbar)$ , the quantum A period. It has the following structure

$$f(\nu) = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2} + \nu\right)} \left(\frac{8}{g^2}\right)^\nu e^{-A(\nu)}. \quad (4.1.93)$$

As in the double well,  $A(\nu)$  is given by the power series in  $g$ ,

$$A(\nu) = \frac{2}{15g^2} + \sum_{k=1}^{\infty} c^{(k)}(\nu)g^{2k}, \quad (4.1.94)$$

and, the concrete coefficients can be obtained by using the uniform WKB method [3, 107], or the techniques of [92]. At the very first orders, one finds

$$\begin{aligned} c^{(1)}(\nu) &= \frac{141\nu^2}{8} + \frac{77}{32}, \\ c^{(2)}(\nu) &= \frac{7717\nu^3}{32} + \frac{13937\nu}{128}, \\ c^{(3)}(\nu) &= \frac{2663129\nu^4}{512} + \frac{5153379\nu^2}{1024} + \frac{43147783}{122880}. \end{aligned} \quad (4.1.95)$$

By using the same method of (3.1.32), we can get a Picard-Fuchs equation for the period integrals  $\Pi$  on the spectral curve  $H(p, x) = E$ ,

$$\left[ \frac{15}{2} - \Delta(E) \frac{\partial^2}{\partial E^2} \right] \Pi = 0, \quad (4.1.96)$$

where the discriminant is

$$\Delta(E) = E(1 - 54E). \quad (4.1.97)$$

The singularity at  $E = 1/54$  corresponds to the top of the potential wall in figure 4.4, where the tunnelling region disappears and there are not even resonant states. In fact, with a change of variables

$$E = \frac{1}{54} \left( \sin \frac{3\phi_E}{2} \right)^2, \quad \phi_E \in \left[ 0, \frac{\pi}{3} \right]. \quad (4.1.98)$$

we can write the turning points explicitly as

$$\begin{aligned} c &= \frac{1 - \sqrt{3} \sin(\phi_E) - \cos(\phi_E)}{6}, \\ b &= \frac{1 + \sqrt{3} \sin(\phi_E) - \cos(\phi_E)}{6}, \\ a &= \frac{1 + 2 \cos(\phi_E)}{6}. \end{aligned} \quad (4.1.99)$$

With this we can write the periods

$$\begin{aligned} t_0(E) &= \frac{1}{2\pi} \oint_A p(x) dx = \frac{1}{\pi} \int_c^b p(x) dx = \\ &= \frac{2\sqrt{2}}{\pi} \frac{(b-c)^2 \sqrt{a-c}}{15k^4} \left( (1-k^2)(k^2-2) \mathbf{K}(k^2) + 2(1-k^2+k^4) \mathbf{E}(k^2) \right), \\ \partial_t \mathcal{F}_0^{(0)} \Big|_{t=t_0} &= -2i \int_b^a p(x) dx = \frac{4\sqrt{2}(a-b)^2(b-c)}{\sqrt{a-c}} \times \\ &\times \frac{2(1-k^2+k^4) \mathbf{E}(1-k^2) - k^2(1+k^2) \mathbf{K}(1-k^2)}{15k^2(k^2-1)^2} \end{aligned} \quad (4.1.100)$$

where the elliptic modulus is

$$k^2 = \frac{b-c}{a-c}. \quad (4.1.101)$$

Again, naming  $\xi$  the corrected energy such that  $t(E, \hbar) = t_0(\xi)$ , the prepotential follows from expanding  $t_0(\xi)$ ,

$$\begin{aligned} t &= \xi + \frac{15\xi^2}{4} + \frac{1155\xi^3}{16} + \frac{255255\xi^4}{128} + \dots \\ \mathcal{F}_0^{(0)}(t) &= \frac{t^2}{2} \left( \log\left(\frac{t}{8}\right) - \frac{3}{2} \right) + \frac{2t}{15} + \frac{47t^3}{8} + \frac{7717t^4}{128} + \frac{2663129t^5}{2560} + \dots \end{aligned} \quad (4.1.102)$$

To make contact with modular forms, we compute its second derivative, that in terms of the elliptic integrals is

$$\frac{\partial \mathcal{F}_0^{(0)}}{\partial t} = -2\pi \frac{\mathbf{K}(1-k^2)}{\mathbf{K}(k^2)} = 2\pi i \tau. \quad (4.1.103)$$

From (3.2.46), this sets  $\beta = 1$  for our parametrization of the cubic oscillator, and we get the map between  $t$  and the elliptic parameter

$$q^2 = \frac{t}{8} + \frac{141t^2}{32} + \frac{5379t^3}{32} + O(t^4). \quad (4.1.104)$$

The first free energy will be no surprise by now,

$$F_1 = -\frac{1}{24} \log \Delta(\xi) = -\frac{1}{24} \log(t) + \frac{77t}{32} + \frac{13937t^2}{256} + \frac{1717793t^3}{1024} + O(t^4). \quad (4.1.105)$$

The discriminant can be written in terms of modular generators with

$$\Delta(\xi) = \frac{(K_2^2 - K_4)^2 K_4}{(K_2^2 + 3K_4)^3}, \quad (4.1.106)$$

and the Yuwaka for this geometry is

$$Y = \partial_{ttt} \mathcal{F}_0^{(0)} = \frac{16\sqrt{2} (K_2^2 + 3K_4)^{9/4}}{(K_2^2 - K_4)^2 K_4}. \quad (4.1.107)$$

At higher genera, the following ansatz for the ambiguity

$$f_{n,0} = \frac{(K_2^2 + 3K_4)^{\frac{3(n+1)}{2}}}{(K_2^2 - K_4)^{4(n-1)} K_4^{2(n-1)}} \sum_{i=0}^{\lfloor \frac{9n-15}{2} \rfloor} a_i K_2^{9n-15-2i} K_4^i \quad (4.1.108)$$

is enough to solve the constraints coming from the gap condition (3.2.155),

$$\lim_{\bar{\tau} \rightarrow \infty} F_n^{(0)} = \mathcal{F}_n^{(0)} = \frac{(1 - 2^{1-2n}) B_{2n}}{2n(2n-1)(2n-2)} \frac{1}{t^{2-2n}} + O(t)^0, \quad (4.1.109)$$

where since  $\beta = 1$  as in the harmonic oscillator, no normalization coefficients are needed. Like in the double well, there are two singular points, one at  $E = 0$  and one at  $E = 1/54$ . We need to impose another gap condition on

$$\lim_{\bar{\tau}_D \rightarrow \infty} F_n^{(0)} = \mathcal{F}_{D,n}^{(0)} = \frac{(1 - 2^{1-2n}) B_{2n}}{2n(2n-1)(2n-2)} \frac{1}{t_D^{2-2n}} + O(t)^0. \quad (4.1.110)$$

where  $t_D$  is simply the quantum B period,

$$t_D = \frac{1}{2\pi} \partial_t \mathcal{F}^{(0)}(E, \hbar). \quad (4.1.111)$$

As usual, we define a corrected dual energy  $\xi_D$  such that

$$\begin{aligned} t_D &= \frac{1}{2\pi} \partial_t \mathcal{F}_0^{(0)} \left( E = \frac{1}{54} - \xi_D \right) = \\ &= \xi_D + \frac{15\xi_D^2}{4} + \frac{1155\xi_D^3}{16} + \frac{255255\xi_D^4}{128} + \dots \end{aligned} \quad (4.1.112)$$

Notice this is the same expansion as  $t(\xi)$ . Therefore, in terms of the dual elliptic parameter,

$$q_D^2 = \frac{t_D}{8} + \frac{141t_D^2}{32} + \frac{5379t_D^3}{32} + O(t_D^4). \quad (4.1.113)$$

There is good reason for this self-duality. From the physical point of view, if we exchange the two periods in figure 4.4, we are left with exactly the same problem: a cubic oscillator facing the opposite way, but with identical shape and therefore identical spectrum.

All this is enough to find, for instance,

$$F_2 = -\frac{K_2^2 (K_2^2 - 9K_4)^2 (K_2^2 + 3K_4)^{5/2}}{108 (K_2^2 - K_4)^4 K_4^2} E_2 - \frac{79K_2 (K_2^2 - 9K_4) (K_2^2 + 3K_4)^{9/2}}{1080 (K_2^2 - K_4)^4 K_4^2} \quad (4.1.114)$$

and so on.

Again, the free energies can be used with a PNP relation to get the perturbative energy levels,

$$E(\nu) = -\frac{3g^2 \hbar^2}{4} - \frac{15g^4}{2} \frac{\partial \mathcal{F}}{\partial g^2}, \quad (4.1.115)$$

after identifying  $\hbar\nu = t(E, \hbar)$ , and reintroducing the  $g$  dependence with

$$\begin{aligned} t &\rightarrow g^2 t, \\ t_D &\rightarrow g^2 t_D, \\ F_g &\rightarrow (g^2)^{2g-2} F_g. \end{aligned} \quad (4.1.116)$$

One finds

$$E(\nu) = \hbar\nu + \hbar^2 g^2 \left( -\frac{15\nu^2}{4} - \frac{7}{16} \right) + \hbar^3 g^4 \left( -\frac{705\nu^3}{16} - \frac{1155\nu}{64} \right) + O(\hbar^4), \quad (4.1.117)$$

just as with the Bender-Wu algorithm [100].

Finally, one can also look at the transseries generated by the quantization condition (4.1.91), written as

$$1 + e^{2\pi i \hat{\nu}} = -f(\nu). \quad (4.1.118)$$

The non-perturbative part of course means that the action should be

$$\mathcal{A} = t_D = \partial_t \mathcal{F}_0^{(0)}. \quad (4.1.119)$$

From that we also have

$$[T]_e = 1, \quad [G]_e = t_D, \quad \alpha [\partial_\lambda G]_e = \partial_t \xi, \quad [\mathcal{D}]_e = \partial_t. \quad (4.1.120)$$

Together with the boundary conditions

$$f^{(m)} = \frac{i^{2m-1}}{2m^2\pi}, \quad (4.1.121)$$

the transseries of  $\partial_\lambda F$  matches in exactly the same way as in the double well the transseries for  $E$ .

### 4.1.3 Remarks on the PNP relation

Strictly speaking, the PNP relation found in [2] was observed to be valid at the perturbative level. Indeed, in [108] a derivation order by order in  $\hbar$  for arbitrary polynomial potentials is given. Still, there is nothing in the argument that intrinsically requires the perturbative expansion. Partially for the sake of the exercise and partially for rigor, let us sketch a derivation that does not assume a power series in  $\hbar$ . Let the A and B periods be

$$t(E, \hbar) := \frac{1}{2\pi} \oint_A Q \, dx - \frac{\hbar}{2}, \quad t_D(E, \hbar) := \frac{\partial \mathcal{F}}{\partial t} := -i \oint_B Q \, dx. \quad (4.1.122)$$

We will consider a potential that is a regular function with asymptotics

$$V(x) = \lambda x^{2d} + o(x^{2d}), \quad (4.1.123)$$

where  $o(x^{2d})$  stands for terms smaller than  $x^{2d}$  when  $x \mapsto \infty$ . The whole idea is to not care explicitly about  $\hbar$ , so we will forget about it for the moment. By inverting (4.1.122), we can get

$$E = E(t, \lambda), \quad Q = Q(x; t, \lambda), \quad \text{and} \quad t_D = t_D(t, \lambda). \quad (4.1.124)$$

The PNP relation is the statement, from the point of view of geometry, that the two periods of the curve are not independent. But of course, this has been known for a long time: it is nothing else than the Riemann bilinear identity for meromorphic differentials. Let us set

$$\omega_1 = \left( \frac{\partial Q}{\partial E} \right)_\lambda \, dx, \quad (4.1.125)$$

where the derivative is w.r.t.  $E$  keeping  $\lambda$  constant, and

$$\omega_2 = \left( \frac{\partial Q}{\partial \lambda} \right)_t \, dx. \quad (4.1.126)$$

Now we state the bilinear identity. For a genus one curve – with two periods, A and B – let  $\omega_1$  be a meromorphic differential,  $\Omega_1$  such that  $d\Omega_1 = \omega_1$ , and  $\omega_2$  a residueless meromorphic differential. Then

$$\oint_A \omega_1 \oint_B \omega_2 - \oint_A \omega_2 \oint_B \omega_1 = 2\pi i \sum_{\text{poles}} \text{Res}(\Omega_1 \omega_2). \quad (4.1.127)$$

Since  $V(x)$  is a regular function, the solution of the Riccati equation  $Q(x)$  will also be regular, with the possible exception of the point at infinity, making  $\omega_i$  suitable to apply the bilinear identity. The derivatives w.r.t. the parameters commute with the integrals, and we can explicitly extract them from  $\omega_i$  to leave period integrals of the differential  $Q dx$ .

$$\begin{aligned}
& \frac{\partial}{\partial E} \left( \oint_A Q dx \right)_\lambda \frac{\partial}{\partial \lambda} \left( \oint_B Q dx \right)_t - \\
& \frac{\partial}{\partial E} \left( \oint_B Q dx \right)_\lambda \frac{\partial}{\partial \lambda} \left( \oint_A Q dx \right)_t = \\
& = \left( \frac{\partial(2\pi t + \pi \hbar)}{\partial E} \right)_\lambda \left( \frac{\partial(i t_D)}{\partial \lambda} \right)_t - \left( \frac{\partial(i t_D)}{\partial E} \right)_\lambda \left( \frac{\partial(2\pi t + \pi \hbar)}{\partial \lambda} \right)_t = \\
& = 2\pi i \left( \frac{\partial t}{\partial E} \right)_\lambda \left( \frac{\partial t_D}{\partial \lambda} \right)_t.
\end{aligned} \tag{4.1.128}$$

As we said, the only pole  $Q$  and its derivatives may have is at infinity. To compute it, notice that the asymptotics of the right-hand side on the Riccati equation (3.1.4) must match that of the left. If it was the case that

$$\partial_x Q \sim x^{2d} + o(x^{2d}), \tag{4.1.129}$$

then we would have

$$Q^2 \sim x^{4d+2} + o(x^{4d+2}), \tag{4.1.130}$$

which would overgrow  $V(x)$ . Therefore, the asymptotics of  $Q$  must be

$$Q = i\epsilon\sqrt{2\lambda}x^d + o(x^d), \tag{4.1.131}$$

with  $\epsilon = \pm 1$ . That the derivative is subleading at infinity is precisely why we can forget about  $\hbar$ . Taking derivatives of the Riccati equation,

$$\begin{aligned}
& -\frac{i\hbar}{2} \partial_x \partial_E Q + Q \partial_E Q = 1, \\
& -\frac{i\hbar}{2} \partial_x \partial_\lambda Q + Q \partial_\lambda Q = -x^{2d} + o(x^{2d}).
\end{aligned} \tag{4.1.132}$$

By applying the same reasoning, and since we know the asymptotics of  $Q$ , we get

$$\begin{aligned}
\partial_E Q &= \frac{1}{i\epsilon\sqrt{2\lambda}} x^{-d} + o(x^{-d}), \\
\partial_\lambda Q &= \frac{i\epsilon}{\sqrt{2\lambda}} x^d + o(x^d).
\end{aligned} \tag{4.1.133}$$

We can see that  $\partial_E Q$  has residue zero at infinity, so it takes the role of  $\omega_2$  in (4.1.127). We need the residue of

$$(\partial_E Q dx) d^{-1}(\partial_\lambda Q dx) = \left( \frac{x}{\lambda r} + o(x) \right) dx. \tag{4.1.134}$$

where  $r \in \mathbb{Q}$  is fixed by the the antiderivative  $d^{-1}$ . It indeed has a pole, going like  $x$ , at infinity. And it is, as expected, independent of  $E$  or  $\hbar$ . Using the bilinear identity on (4.1.128),

$$2\pi i \left( \frac{\partial t}{\partial E} \right)_\lambda \left( \frac{\partial t_D}{\partial \lambda} \right)_t = -2\pi i \frac{1}{\lambda r}. \quad (4.1.135)$$

Writing the energy as a function of  $t$ , we arrive at the by now well known PNP relation

$$\frac{\partial E}{\partial t} = -r \lambda \frac{\partial t_D}{\partial \lambda}, \quad (4.1.136)$$

or its integrated version

$$E(t, \hbar, \lambda) = c(\hbar, \lambda) - r \lambda \partial_\lambda \mathcal{F}(t, \hbar, \lambda). \quad (4.1.137)$$

Critically, in this construction  $t$  is the quantum A period, so this gives us the map between the energy  $E$  and the quantum A period if we know  $\mathcal{F}$ , or equivalently the quantum B period, *as a function of the quantum A period  $t$* . Which is nice, because it is exactly what the holomorphic anomaly has been providing us in a natural way.

## 4.2 Quantum mirror curves

In this section, we will study the spectral problem for two different toric CY manifolds, local  $\mathbb{P}^2$  and local  $\mathbb{F}_0$ , in the all-orders WKB expansion, through the refined holomorphic anomaly equations. We will also calculate the corresponding transseries. For these spectral problems, there are no transseries results in quantum mechanics to compare with, so the holomorphic anomaly gives the only concrete approach to understand their resurgent structure.

### 4.2.1 Local $\mathbb{P}^2$

In the case of the local  $\mathbb{P}^2$  geometry, the corresponding quantum-mechanical operator is

$$\mathcal{O}_{\mathbb{P}^2} = e^x + e^y + e^{-x-y}. \quad (4.2.138)$$

It was conjectured in [23] and then proved in [47, 52] that this operator has a discrete spectrum and its inverse  $\rho = \mathcal{O}^{-1}$  is trace class. Although these operators are not of Schrödinger type, and they lead to difference equations instead of differential equations, one can still use the all-orders WKB approximation [109], as it has been done in [18, 36, 37, 62]. The associated Riemann surface is just the mirror curve of local  $\mathbb{P}^2$ , which is a genus one curve with the form

$$e^x + e^y + e^{-x-y} + \kappa = 0. \quad (4.2.139)$$

The calculation of the classical volume reduces to the calculation of classical periods on this curve. To get the WKB corrections, we will need the holomorphic anomaly.

The general, two-parameter refined holomorphic anomaly as in [27] is given by

$$\frac{\partial F^{[g_1, g_2]}}{\partial S^{zz}} = \frac{1}{2} \left[ D_z \partial_z F^{[g_1, g_2 - 1]} + \sum_{h_1, h_2} \partial_z F^{[h_1, h_2]} \partial_z F^{[g_1 - h_1, g_2 - h_2]} \right] \quad (4.2.140)$$

with  $(g_1, g_2) > (h_1, h_2) > (0, 0)$ . The NS free energies<sup>5</sup> related to the WKB periods are given by the  $g_1 = 0$  sector, and the standard energies (the ones in (2.1.81)) by the  $g_2 = 0$ . The classical periods are governed by the Picard-Fuchs equation

$$[\theta^3 + 3z(3\theta - 1)(3\theta - 2)\theta] \Pi = 0, \quad \theta := z\partial_z. \quad (4.2.141)$$

where the moduli  $z$  is related to the eigenvalue  $\kappa$  by

$$z = \frac{1}{\kappa^3}. \quad (4.2.142)$$

The standard, classical periods in the large radius frame (which is appropriate for the point  $z = 0$ ) are given by

$$\begin{aligned} -t &= \log(z) + \tilde{\omega}_1(z), \\ \frac{\partial \mathcal{F}_0^{(0)}}{\partial t} &= \frac{1}{6} (\log^2(z) + 2\tilde{\omega}_1(z) \log(z) + \tilde{\omega}_2(z)), \end{aligned} \quad (4.2.143)$$

where

$$\begin{aligned} \tilde{\omega}_1(z) &= \sum_{j \geq 1} 3 \frac{(3j-1)!}{(j!)^3} (-z)^j, \\ \tilde{\omega}_2(z) &= \sum_{j \geq 1} \frac{18}{j!} \frac{\Gamma(3j)}{\Gamma(1+j)^2} \{\psi(3j) - \psi(j+1)\} (-z)^j. \end{aligned} \quad (4.2.144)$$

After integration, we find the prepotential

$$\mathcal{F}_0^{(0)}(t) = \frac{t^3}{18} + 3e^{-t} - \frac{45}{8}e^{-2t} + \dots \quad (4.2.145)$$

For consistency with the naming we have used during the analysis of the holomorphic anomaly, we will use  $\mathcal{F}$  for the *large radius frame holomorphic limit*<sup>6</sup> of  $F$ . Then, a simple calculation shows that (see for example [23, 27] for more details)

$$\text{vol}_0(E) = 3 \frac{d\widehat{\mathcal{F}}_0^{(0)}}{dt} - \frac{\pi^2}{2}, \quad (4.2.146)$$

where the  $\widehat{\mathcal{F}}_0^{(0)}$  symbol means, as in (2.1.92), that we changed  $e^{-t} \rightarrow -e^{-t}$  in the exponentially small corrections appearing in the expansion (4.2.145). The relation between  $t$  and  $E$  is given by

$$t = 3E - \tilde{\omega}_1(-e^{-3E}). \quad (4.2.147)$$

<sup>5</sup>Do not mistake the notation  $F^{[g_1, g_2]}$  from [27] referring to the perturbative free energies with our  $F^{(m)}$  referring to the instanton sectors.

<sup>6</sup>And do not confuse this with chapter 2 on the TS/ST correspondence, where we used it for the *conifold frame* holomorphic limit, keeping the plain  $F$  for the large radius frame since there was no risk of mistaking it for the anholomorphic version.

Explicitly, one finds [37]

$$\text{vol}_0(E) = \frac{9E^2 - \pi^2}{2} + 9 \sum_{j \geq 1} \frac{(3j-1)!}{j!^3} \{\psi(3j) - \psi(j+1) - E\} e^{-3jE}. \quad (4.2.148)$$

To solve the holomorphic anomaly, we will need the discriminant of the curve and the Yuwaka, that are given by

$$\Delta = 1 + 27z, \quad C_{zzz} = -\frac{1}{3} \frac{1}{z^3 \Delta} = (\partial_z t)^3 \partial_{ttt} \mathcal{F}_0^{(0)}, \quad (4.2.149)$$

The first free energies are

$$\begin{aligned} F^{[0,1]} &= \frac{1}{2} \log \left| \frac{\partial t}{\partial z} \right| + \frac{1}{12} \log(z^7 \Delta), \\ F^{[1,0]} &= -\frac{1}{24} \log(z^{-1} \Delta). \end{aligned} \quad (4.2.150)$$

We can recover the propagator by

$$\partial_z F^{[0,1]} = -\frac{1}{2} C_{zzz} S^{zz} \quad (4.2.151)$$

which can be used to start the recursion, together with

$$\partial_z F^{[1,0]} = -\frac{1}{8} C_{zzz} z^2. \quad (4.2.152)$$

The covariant derivative is given by

$$D_z = \partial_z - \Gamma_z, \quad \Gamma_z = C_{zzz} \left( \frac{1}{2} z^2 (7 + 216z) - S^{zz} \right), \quad (4.2.153)$$

and together with

$$\partial_z S^{zz} = C_{zzz} \left[ (S^{zz})^2 - S^{zz} z^2 (7 + 216z) + \frac{z^4}{4} \right] \quad (4.2.154)$$

generates all the remaining  $F^{[g_1, g_2]}$  as polynomials in  $S^{zz}, z$ . The holomorphic ambiguity after every integration is fixed by requiring To fix the ambiguity, we need to expand around the singular point at the conifold, where we know the universal behaviour. From the Picard-Fuchs we can obtain  $t_c$ , the period vanishing at the conifold, as

$$\begin{aligned} t_c &= \Delta + \frac{11\Delta^2}{18} + \frac{109\Delta^3}{243} + \frac{9389\Delta^4}{26244} + O(\Delta^5), \\ &= \frac{3\sqrt{3}}{2\pi} \left( \partial_t \mathcal{F}^{(0)} - \frac{\pi^2}{6} \right). \end{aligned} \quad (4.2.155)$$

Then holomorphic ambiguity is fixed in the standard and NS limit by

$$\begin{aligned}\mathcal{F}^{[g,0]} &= \frac{3^{g-1}}{(2g)(2g-1)(2g-2)} \frac{(1-2^{1-2g})B_{2g}}{t_c^{2g-2}} + O(t_c^0), \\ \mathcal{F}^{[0,g]} &= \frac{3^{g-1}}{(2g)(2g-2)} \frac{B_{2g}}{t_c^{2g-2}} + O(t_c^0).\end{aligned}\tag{4.2.156}$$

Some concrete examples<sup>7</sup>:

$$\begin{aligned}\mathcal{F}^{[2,0]} &= Y^2 \left( \frac{Sz^4}{128} - \frac{729z^8}{80} + \frac{27z^7}{80} - \frac{z^6}{256} \right), \\ \mathcal{F}^{[1,1]} &= Y^2 \left( \frac{S^2z^2}{16} + S \left( \frac{27z^5}{8} - \frac{z^4}{32} \right) - \frac{729z^8}{80} - \frac{27z^7}{20} \right), \\ \mathcal{F}^{[0,2]} &= Y^2 \left( \frac{5S^3}{24} - \frac{3S^2z^2}{16} + \frac{Sz^4}{16} - \frac{243z^8}{80} - \frac{27z^7}{160} - \frac{z^6}{96} \right).\end{aligned}\tag{4.2.157}$$

$$\begin{aligned}\mathcal{F}^{[3,0]} &= -Y^4 \left( \frac{S^3z^6}{3072} + S^2 \left( \frac{27z^9}{512} - \frac{z^8}{2048} \right) + S \left( \frac{6561z^{12}}{640} - \frac{1053z^{11}}{5120} + \frac{z^{10}}{4096} \right) - \right. \\ &\quad \left. - \frac{1594323z^{16}}{1120} + \frac{5137263z^{15}}{4480} - \frac{1644381z^{14}}{35840} + \frac{2223z^{13}}{17920} - \frac{z^{12}}{24576} \right), \\ \mathcal{F}^{[2,1]} &= -Y^4 \left( \frac{S^4z^4}{128} + S^3 \left( \frac{27z^7}{32} - \frac{7z^6}{512} \right) + S^2 \left( \frac{5103z^{10}}{80} - \frac{4023z^9}{1280} + \frac{9z^8}{1024} \right) + \right. \\ &\quad \left. + S \left( \frac{177147z^{13}}{80} - \frac{98415z^{12}}{256} + \frac{9747z^{11}}{2560} - \frac{5z^{10}}{2048} \right) + \right. \\ &\quad \left. + \frac{4782969z^{16}}{4480} - \frac{334611z^{15}}{140} + \frac{149445z^{14}}{896} - \frac{18063z^{13}}{17920} + \frac{z^{12}}{4096} \right), \\ \mathcal{F}^{[1,2]} &= -Y^4 \left( \frac{5S^5z^2}{64} + S^4 \left( \frac{135z^5}{32} - \frac{21z^4}{128} \right) + S^3 \left( \frac{1215z^8}{16} - \frac{1143z^7}{64} + \frac{35z^6}{256} \right) + \right. \\ &\quad \left. + S^2 \left( -\frac{172773z^{10}}{320} + \frac{8073z^9}{320} - \frac{29z^8}{512} \right) + S \left( -\frac{59049z^{13}}{40} + \frac{665577z^{12}}{1280} - \frac{14931z^{11}}{1280} + \frac{3z^{10}}{256} \right) - \right. \\ &\quad \left. - \frac{177147z^{16}}{2240} + \frac{4809213z^{15}}{4480} - \frac{2324781z^{14}}{17920} + \frac{31347z^{13}}{17920} - \frac{z^{12}}{1024} \right), \\ \mathcal{F}^{[0,3]} &= -Y^4 \left( \frac{5S^6}{16} - \frac{25S^5z^2}{32} + S^4 \left( \frac{13z^4}{16} - \frac{189z^5}{16} \right) + S^3 \left( -\frac{729z^8}{16} + \frac{747z^7}{32} - \frac{173z^6}{384} \right) + \right. \\ &\quad \left. + S^2 \left( \frac{44469z^{10}}{320} - \frac{5049z^9}{320} + \frac{9z^8}{64} \right) + S \left( \frac{19683z^{13}}{160} - \frac{60507z^{12}}{640} + \frac{2943z^{11}}{640} - \frac{3z^{10}}{128} \right) + \right. \\ &\quad \left. + \frac{177147z^{16}}{2240} - \frac{177147z^{15}}{2240} + \frac{42039z^{14}}{2240} - \frac{1107z^{13}}{2240} + \frac{5z^{12}}{3072} \right),\end{aligned}\tag{4.2.158}$$

(4.2.159)

More relevantly for the NS limit and the WKB periods,

$$\begin{aligned}\mathcal{F}^{[4,0]} &= Y^6 \left( \frac{S^5z^8}{32768} + S^4 \left( \frac{27z^{11}}{4096} - \frac{5z^{10}}{65536} \right) + S^3 \left( \frac{21627z^{14}}{20480} - \frac{2493z^{13}}{81920} + \frac{5z^{12}}{65536} \right) + \right. \\ &\quad \left. S^2 \left( \frac{531441z^{17}}{5120} - \frac{649539z^{16}}{81920} + \frac{10557z^{15}}{163840} - \frac{5z^{14}}{131072} \right) + \right. \\ &\quad \left. + S \left( \frac{3065883129z^{20}}{179200} - \frac{241057701z^{19}}{102400} + \frac{229450563z^{18}}{5734400} - \frac{132597z^{17}}{2293760} + \frac{5z^{16}}{524288} \right) - \right. \\ &\quad \left. - \frac{45328197213z^{24}}{44800} + \frac{5179955427z^{23}}{1792} - \frac{8562754539z^{22}}{17920} + \right. \\ &\quad \left. + \frac{73229007177z^{21}}{5734400} - \frac{157356351z^{20}}{2867200} + \frac{464283z^{19}}{22937600} - \frac{z^{18}}{1048576} \right),\end{aligned}\tag{4.2.160}$$

<sup>7</sup>In which we write  $S := S^{zz}$ ,  $Y := C_{zzz}$  to simplify the already rather long expressions.

$$\begin{aligned}
\mathcal{F}^{[5,0]} = & -Y^8 \left( \frac{S^7 z^{10}}{262144} + S^6 \left( \frac{135z^{13}}{131072} - \frac{7z^{12}}{524288} \right) + S^5 \left( \frac{59049z^{16}}{327680} - \frac{7533z^{15}}{1310720} + \frac{21z^{14}}{1048576} \right) \right) + \\
& + S^4 \left( \frac{1712421z^{19}}{81920} - \frac{1966113z^{18}}{1310720} + \frac{19413z^{17}}{1310720} - \frac{35z^{16}}{2097152} \right) + \\
& + S^3 \left( \frac{6793410303z^{22}}{2867200} - \frac{132092613z^{21}}{409600} + \frac{314459253z^{20}}{45875200} - \frac{407727z^{19}}{18350080} + \frac{35z^{18}}{4194304} \right) + \\
& + S^2 \left( \frac{297151515063z^{25}}{1433600} - \frac{631643669109z^{24}}{11468800} + \frac{11370888783z^{23}}{4587520} - \right. \\
& \quad \left. - \frac{48833523z^{22}}{2293760} + \frac{1449657z^{21}}{73400320} - \frac{21z^{20}}{8388608} \right) + \\
& + S \left( \frac{3634391607309z^{28}}{89600} - \frac{47976531931269z^{27}}{2867200} + \frac{53208775838259z^{26}}{45875200} - \right. \\
& \quad \left. - \frac{39544190199z^{25}}{2293760} + \frac{2053439181z^{24}}{45875200} - \frac{7639731z^{23}}{734003200} + \frac{7z^{22}}{16777216} \right) - \\
& - \frac{312649496884467z^{32}}{197120} + \frac{20877473766212001z^{31}}{1971200} - \frac{135423883001736657z^{30}}{31539200} + \\
& \quad + \frac{89960188055512167z^{29}}{252313600} - \frac{7761923380259343z^{28}}{1009254400} + \frac{1644364094733z^{27}}{40370176} - \\
& \quad - \frac{58965464619z^{26}}{1614807040} + \frac{7841529z^{25}}{4037017600} - \frac{z^{24}}{33554432} \Big). \tag{4.2.161}
\end{aligned}$$

#### 4.2.1.1 Large order WKB asymptotics

Let us call the set of perturbative NS free energies simply as  $\mathcal{F}_n^{(0)} := \mathcal{F}^{[n,0]}$ . We will find out these are non-Borel summable for real, positive  $z$ . To avoid the problems at negative  $z$ , where we will work, we make a choice for the branch cut by taking  $\log(-z) := \log(z) - i\pi$ . We also perform a change of frame,

$$\begin{aligned}
t & \mapsto t - i\pi, \\
\frac{\partial \mathcal{F}_0^{(0)}}{\partial t} & \mapsto \frac{\partial \mathcal{F}_0^{(0)}}{\partial t} - \frac{i\pi}{3}t - \frac{\pi^2}{6}. \tag{4.2.162}
\end{aligned}$$

This change of frame plus the branch cut choice can be implemented easily in the  $z$ -expansions by sending

$$\log(z) \mapsto \log(-z), \tag{4.2.163}$$

and for instance we will now have

$$t = -\log(-z) + 6z - 45z^2 + \dots, \quad \frac{\partial \mathcal{F}_0^{(0)}}{\partial t} = \frac{\log^2(-z)}{6} - (2\log(-z) + 3)z + \dots \tag{4.2.164}$$

There is a natural reason to choose this frame, since we want to evaluate  $z$  between the large radius point,  $z = 0$ , and the conifold  $z = -1/27$ . Also, the action controlling the large order behaviour is simply given (up to factors) by the B-period of the new frame. As usual, this large order behaviour is parametrized by

$$F_n^{(0)} = \left[ \frac{\Gamma(2n-b)}{\mathcal{A}^{2n-b}} \mu_0 + \frac{\Gamma(2n-b-1)}{\mathcal{A}^{2n-b-1}} \mu_1 + O\left(\frac{1}{n^2}\right) \right] + O\left(\frac{1}{(\mathcal{A})^{2n-b}}\right). \tag{4.2.165}$$

The resurgent dictionary is then

$$\mu_k = \frac{S_1}{2\pi i} F_k^{(1)}. \tag{4.2.166}$$

We can extract

$$\mathcal{A}^2 = \lim_{n \rightarrow \infty} \left[ 4n^2 \frac{F_n^{(0)}}{F_{n+1}^{(0)}} \right], \quad b = \lim_{n \rightarrow \infty} \left[ \frac{1}{2} + n - \frac{\mathcal{A}^2 F_{n+1}^{(0)}}{n F_n^{(0)}} \right]. \quad (4.2.167)$$

By using

$$B_{2n} = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} (1 + 4^{-n} + \dots) \quad (4.2.168)$$

and (4.2.156) we get in the conifold

$$\mathcal{A}_c = \frac{2\pi i}{\sqrt{3}} t_c = 3i \left( \partial_t \mathcal{F}_0^{(0)} - \frac{\pi^2}{6} \right), \quad b = 2. \quad (4.2.169)$$

This is an holomorphic quantity and as long as we stay near its region of dominance, we will have  $\mathcal{A} = \mathcal{A}_c$ . The prediction from the transseries ansatz for  $F_0^{(1)}$  is that it is also purely holomorphic. To get the subleading coefficients, we take the usual

$$\mu_0 = \lim_{n \rightarrow \infty} \frac{\mathcal{A}^{2n-2} F_n^{(0)}}{\Gamma(2n-2)} \quad (4.2.170)$$

which in the conifold limit (4.2.156) gives

$$\mu_{0,c} = \frac{1}{2\pi^2}. \quad (4.2.171)$$

Now, since this should have no propagator dependence, it will (just like the action) hold without modification away from the conifold,

$$F_0^{(1)} = \frac{2\pi i}{S_1} \frac{1}{2\pi^2}. \quad (4.2.172)$$

The first less trivial correction will come from  $F_1^{(1)}$ . The transseries recursion of the holomorphic anomaly tells us that

$$F_1^{(1)} = f_1^{(1)}(z) - \partial_z \mathcal{A} \partial_z F_1^{(0)} F_0^{(1)} S^{zz}. \quad (4.2.173)$$

On the other hand, (4.2.156) means that for  $\mu_1$

$$\mu_{1,c} = 0, \quad (4.2.174)$$

which fixes the holomorphic ambiguity as

$$\frac{S_1}{2\pi i} F_1^{(1)} = \frac{1}{16\pi^2} z^2 (\partial_z \mathcal{A}) C_{zzz} (S^{zz} - S_{c,hol}^{zz}) \quad (4.2.175)$$

where [110]

$$S_{c,hol}^{zz} = \frac{z^2}{2} \left( -1 - 54z + 2 \frac{\pi P_{2/3}(1+54z) + 2\sqrt{3}Q_{2/3}(1+54z)}{\pi P_{-1/3}(1+54z) + 2\sqrt{3}Q_{-1/3}(1+54z)} \right). \quad (4.2.176)$$

The next correction satisfies

$$\frac{\partial F_2^{(1)}}{\partial S^{zz}} = -\partial_z \mathcal{A} \partial_z F_1^{(0)} F_1^{(1)} + \partial_z F_1^{(0)} \partial_z F_0^{(1)}, \quad (4.2.177)$$

and the corresponding conifold condition is

$$\mu_{2,c} = 0, \quad (4.2.178)$$

giving

$$\frac{S_1}{2\pi i} F_2^{(1)} = \frac{1}{256\pi^2} \left[ z^2 (\partial_z \mathcal{A}) C_{zzz} (S^{zz} - S_{c,hol}^{zz}) \right]^2. \quad (4.2.179)$$

To go beyond that, it is better to directly use the exponential form of the instanton corrections to the free energy. Namely,

$$F^{(1)} = f^{(1)} \hbar^2 \exp \left[ \frac{\mathcal{A} + D_t \mathcal{A} \cdot (S^{tt} - S_{hol}^{tt}) \cdot D_t (F^{(0)} - F_0^{(0)})}{\hbar} \right] \quad (4.2.180)$$

Of course, we want to set

$$f^{(1)} = \frac{2\pi i}{S_1} \frac{1}{2\pi^2}. \quad (4.2.181)$$

Notice that the expression is coordinate independent, and we can change the  $t$  derivatives in the exponent into

$$\partial_z \mathcal{A} \cdot (S^{zz} - S_{hol}^{zz}) \cdot \partial_z (F^{(0)} - F_0^{(0)}) \quad (4.2.182)$$

just being careful to also use the  $S^{zz}$  propagator as in (4.2.151). Also, both  $\mathcal{A}$  and  $F^{(0)}$  are zero weight objects so that we need not worry about covariant derivatives. Using the free energies from (4.2.157) and the relation (4.2.154),

$$\begin{aligned} F^{(1)} / \hbar^2 f^{(1)} &= 1 + \frac{1}{8} (\partial_z \mathcal{A}) Y z^2 (S - S_c) \hbar + \frac{1}{128} (\partial_z \mathcal{A})^2 Y^2 z^4 (S - S_c)^2 \hbar^2 + \\ &+ \frac{(\partial_z \mathcal{A}) Y^3 z^4 (S - S_c)}{15360} \left( 5 (\partial_z \mathcal{A})^2 z^2 (S - S_c)^2 - \right. \\ &\quad \left. - 6 (20S^2 + 20S(108z - 1)z^2 + (209952z^2 - 4212z + 5)z^4) \right) \hbar^3 + \\ &+ \frac{(\partial_z \mathcal{A})^2 Y^4 z^6 (S - S_c)^2}{491520} \left( 5 (\partial_z \mathcal{A})^2 z^2 S_c^2 - \right. \\ &\quad - 10S z^2 ((\partial_z \mathcal{A})^2 S_c + 5184z - 48) + 5S^2 ((\partial_z \mathcal{A})^2 z^2 - 96) - \\ &\quad \left. - 24 (209952z^2 - 4212z + 5) z^4 \right) \hbar^4 + O(\hbar^5). \end{aligned} \quad (4.2.183)$$

From (4.2.152), in this notation,

$$\partial_z F_1^{(0)} = -\frac{z^2}{8} Y, \quad (4.2.184)$$

Figure 4.5: Large order tests,  $z = -2^{-\xi}$

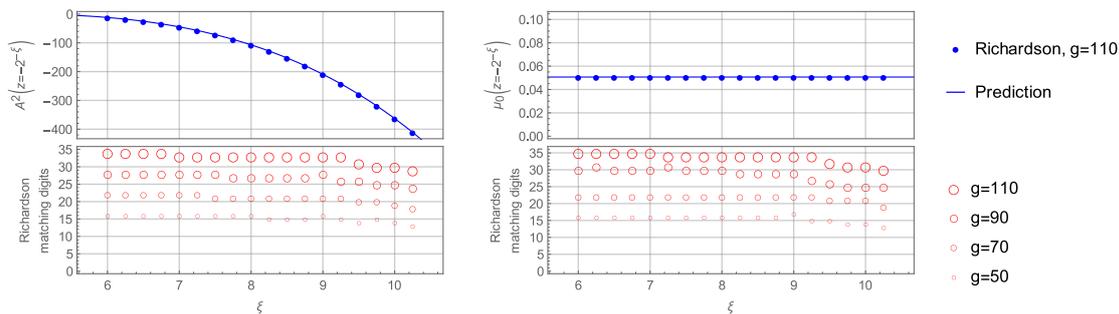


Figure 4.6: Large order tests,  $z = -2^{-\xi}$

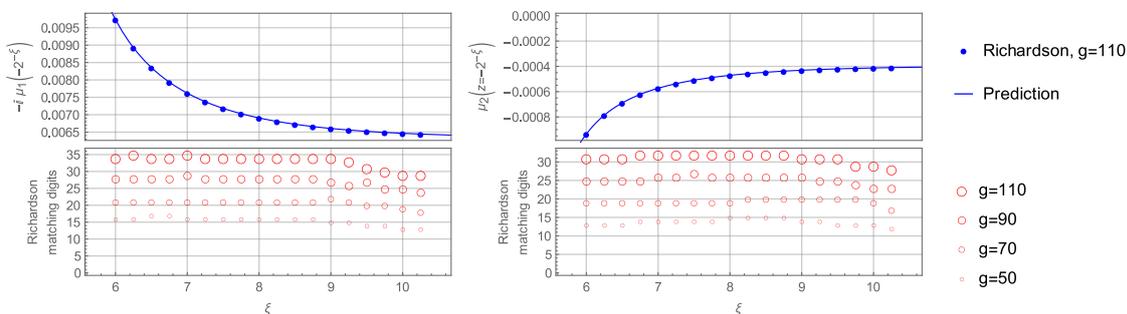
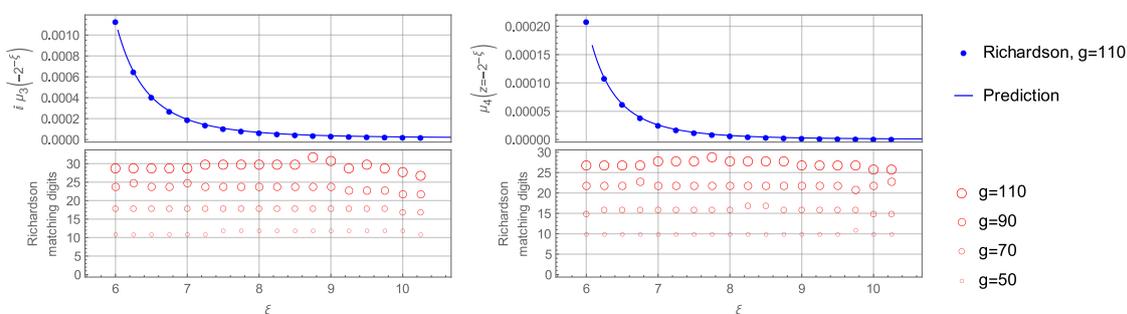


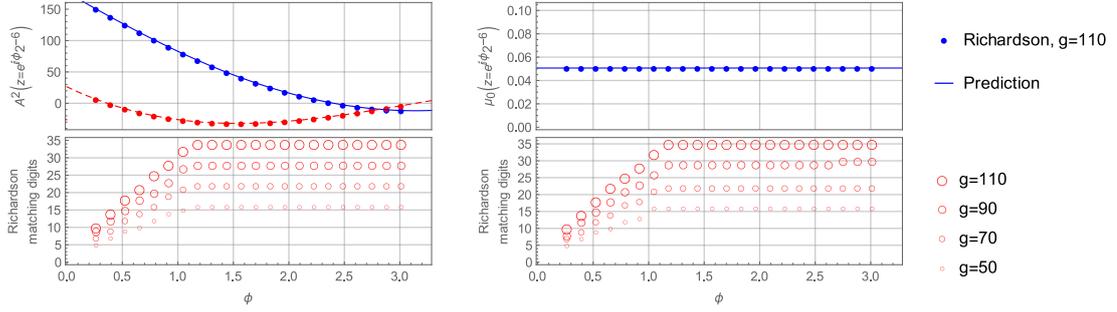
Figure 4.7: Large order tests,  $z = -2^{-\xi}$



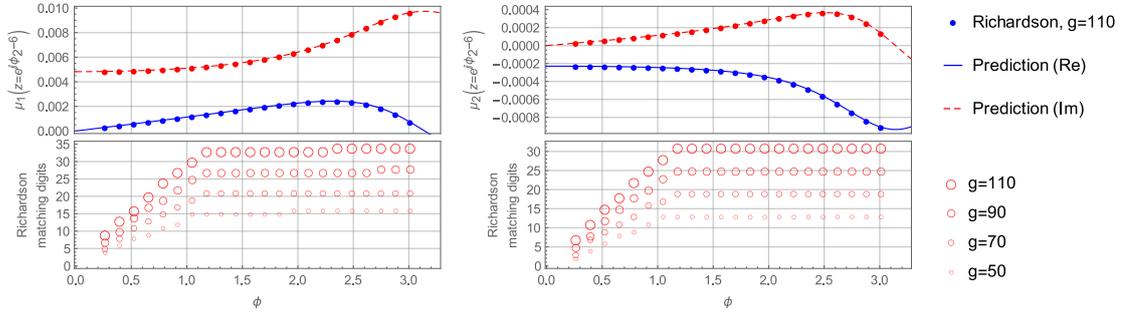
so that the first two terms indeed reproduce what we found with the recursion.

With the dictionary (4.2.166) we have some very concrete predictions for the large order. So we test them. In figures 4.5-4.13 we plot the predictions as lines compared to the (Richardson transformed) numerical asymptotics of  $F_n^{(0)}$ . The Richardson transforms are taken up to the highest number of stable digits, all of which match the prediction. We also accompany each plot with the number of such matching digits, since the very high precisions we obtain are not clearly seen on the plots. We do this for several maximum values of  $n \leq g$ , to confirm that including more terms improves the convergence. The actual values plotted correspond to the highest  $g$ . Figures 4.5, 4.6, 4.7 study in the large radius holomorphic limit the

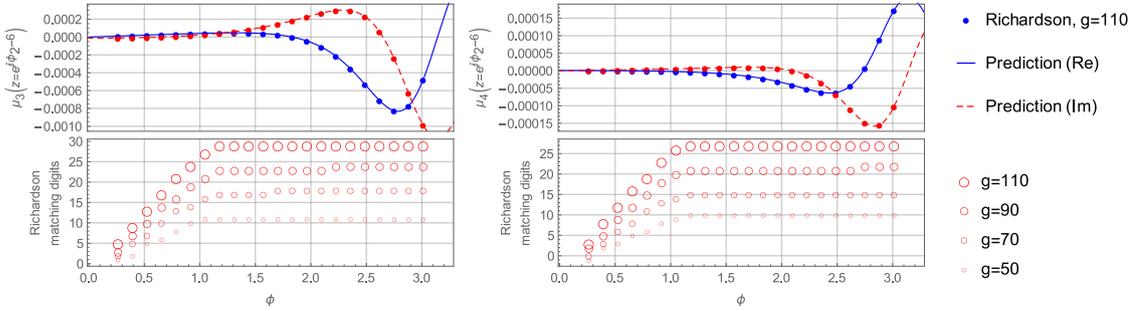
**Figure 4.8:** Large order tests,  $z = 2^{-6} e^{i\phi}$



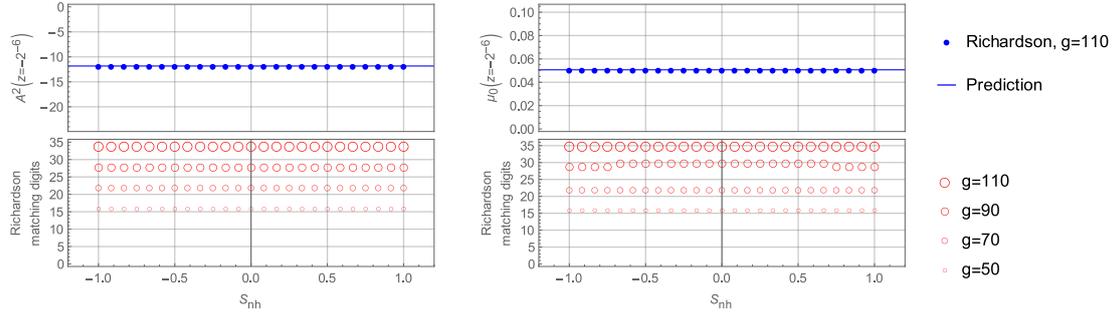
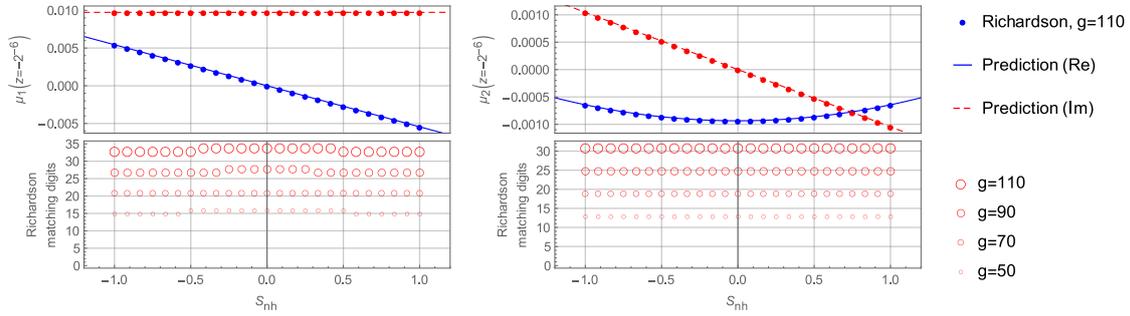
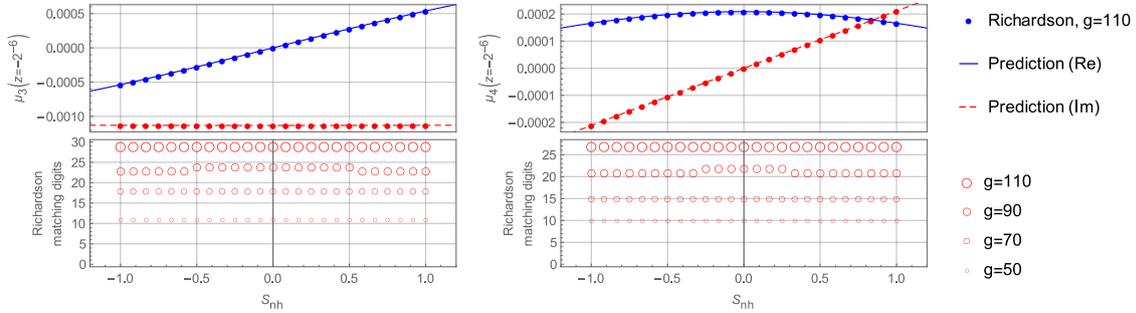
**Figure 4.9:** Large order tests,  $z = 2^{-6} e^{i\phi}$



**Figure 4.10:** Large order tests,  $z = 2^{-6} e^{i\phi}$



action  $A^2$ , and the first five  $\mu_k$  corrections to the large order for real, negative  $z$ , from halfway to the conifold ( $z = -1/27$ ) into large radius. Figures 4.8, 4.9, 4.10 show the same comparisons, now fixing the modulus of  $z$  and varying its argument. We can see a clear breakdown of the matching around  $\phi = \pi/3$ , a symptom of the Stokes phenomenon (the large order for real, positive  $z$  is oscillating, which requires a combination of conjugate actions). Finally in 4.11, 4.12, 4.13, for fixed  $z$ , we introduce a non holomorphic propagator value given by  $S^{zz} = S_{hol}^{zz} (1 + iS_{nh})$ . Notice that both the action and the first correction remain constant throughout, as expected.

**Figure 4.11:** Large order tests,  $S = (1 + i S_{nh})S_{hol}$ **Figure 4.12:** Large order tests,  $S = (1 + i S_{nh})S_{hol}$ **Figure 4.13:** Large order tests,  $S = (1 + i S_{nh})S_{hol}$ 

#### 4.2.1.2 Exact free energy and Borel sum

The main lesson of the chapter on the TS/ST correspondence for quantum mirror curves was that the operators associated to toric CY can be given a well defined spectral determinant via a non-perturbative completion. For genus one, the spectral determinant was shown to be equivalent to a simpler quantization condition using only BPS invariants in the NS limit [111] (the equivalence between both formulations in many cases was derived in [112]). Let us denote

$$f(t, \hbar) = 3 \frac{\partial \widehat{\mathcal{F}}_{inst}^{(0)}}{\partial t} \quad (4.2.185)$$

as the series in  $Q$  of  $\widehat{\mathcal{F}}^{(0)}$ . Then, the non-perturbative volume is simply given by

$$\text{vol}_{np}(E, \hbar) = \hbar f\left(\frac{2\pi t}{\hbar}, \frac{4\pi^2}{\hbar}\right). \quad (4.2.186)$$

The total, exact quantum volume is defined as

$$\text{vol}_{exact}(E, \hbar) = \text{vol}_p(E, \hbar) + \text{vol}_{np}(E, \hbar). \quad (4.2.187)$$

There is strong evidence that the above expression defines a function of  $E$  and  $\hbar$ , for real  $\hbar$  and  $E$  sufficiently large, which gives the actual spectrum of the operator (4.2.138) through the exact quantization condition

$$\text{vol}_{exact}(E, \hbar) = 2\pi\hbar\nu. \quad (4.2.188)$$

In some cases, this exact volume function can be derived from a first principles, resummed WKB calculation [113]. It is clear (see e.g. [114]) that the above procedure also defines an exact function  $\mathcal{F}_{exact}(t, \hbar)$  by

$$\mathcal{F}_{exact}(t, \hbar) = \frac{t^3}{18} - \hbar^2 \frac{t}{24} + \mathcal{F}_{inst}^{(0)}(t, \hbar) + \frac{\hbar}{2\pi} \mathcal{F}_{inst}^{(0)}\left(\frac{2\pi t}{\hbar}, \frac{4\pi^2}{\hbar}\right). \quad (4.2.189)$$

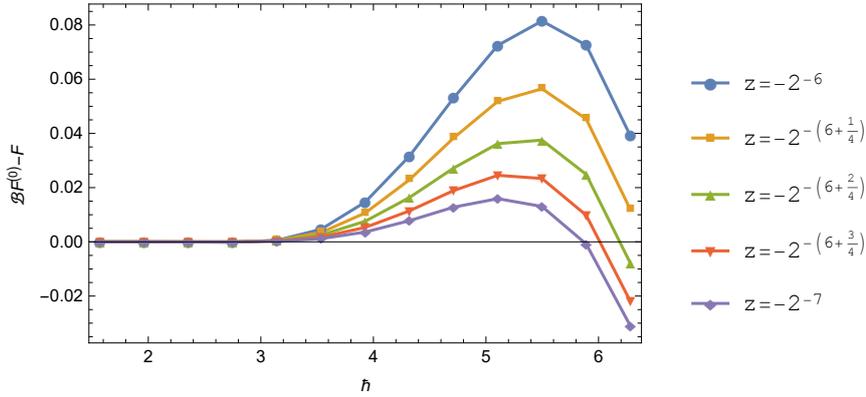
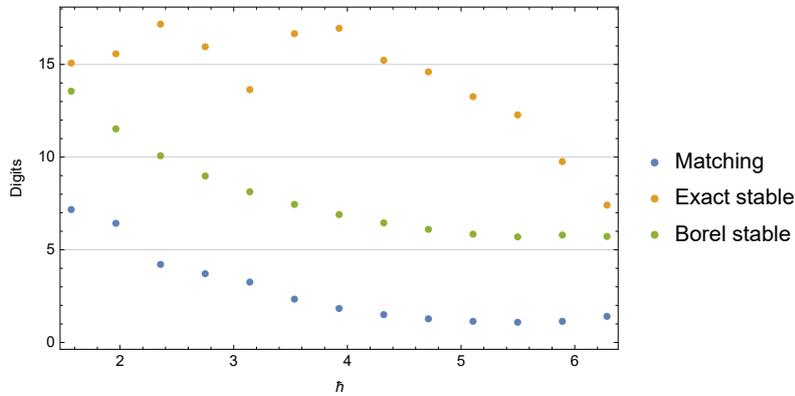
The asymptotic expansion of this function for small  $\hbar$  and fixed  $t$  is precisely the quantum NS free energy of local  $\mathbb{P}^2$ , in the large radius frame:

$$\mathcal{F}_{exact}(t, \hbar) \sim \sum_{n \geq 0} \mathcal{F}_n^{(0)}(t) \hbar^{2n}. \quad (4.2.190)$$

It is then natural to ask whether the asymptotic series in the r.h.s. is Borel summable, and in case it is, whether its Borel resummation, as introduced in (1.26), agrees with the exact function in the l.h.s. Borel summability in the region of negative  $z$  follows from the large order analysis above, since  $\mathcal{A}^2$  is negative. We have found numerically that the Borel resummation of the  $\mathcal{F}_n^{(0)}$ , which we denote by  $\mathcal{BF}^{(0)}$ , differs from the exact result (4.2.189), as shown in figure 4.14. For example, we obtain

$$\begin{aligned} \mathcal{BF}^{(0)}(z = -2^{-6}, \hbar = \pi) &= 2.0571102\dots, \\ \mathcal{F}_{exact}(z = -2^{-6}, \hbar = \pi) &= 2.0565565\dots \end{aligned} \quad (4.2.191)$$

For other values, in figure 4.15, we plot for a particular value of  $z$  the number of stable digits (to the right of the decimal point) for both the Borel sum and the exact formula. The worse performer is clearly the Borel sum, but still it is enough to see that we have a difference of around 5 significant digits for all values of  $\hbar$ . This is in contrast to what happens with the perturbative series in  $\hbar$  calculating the energy levels of the spectral problem of (4.2.138). This series is Borel summable and can be Borel-resummed to the exact values of the energies [114, 115]. At the same time, the mismatch we find is not surprising, and it seems to be the default behavior for “stringy” series which diverge doubly-factorially, as it has been realized recently in related examples [31, 116].

**Figure 4.14:** Borel mismatch**Figure 4.15:**  $\mathcal{B}\mathcal{F}^{(0)}$  and  $\mathcal{F}_{exact}$  digits for  $z = -2^{-6}$ 

In [31], on the other hand, it was also realized that the difference between both was controlled by the one instanton correction,  $\mathcal{F}^{(1)}$ . We can compute and resum it from the general formula (3.3.239). Of course, the action we found at large order cannot be the one controlling this correction, since it is purely imaginary (see figure 4.5, for instance). Following what was done for the standard case, we consider the actions in a different branch of the  $z$  plane,

$$\mathcal{A}_+ = \mathcal{A}_c \Big|_{\log(-z) \mapsto \log(-z) + 2\pi i}, \quad (4.2.192)$$

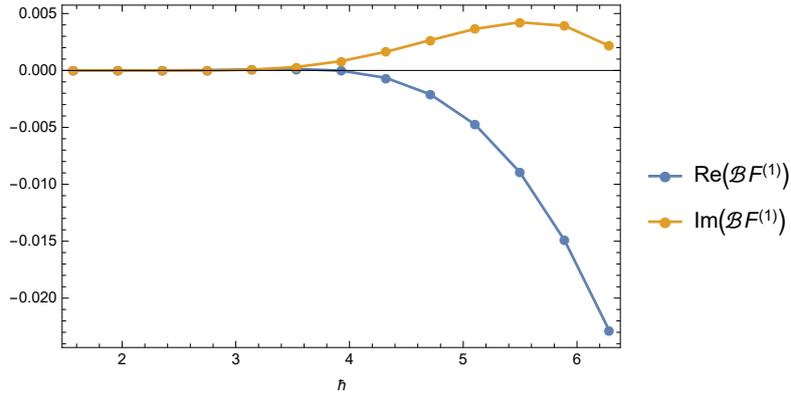
$$\mathcal{A}_- = -\mathcal{A}_c \Big|_{\log(-z) \mapsto \log(-z) - 2\pi i}, \quad (4.2.193)$$

which both have the same (positive) real part. Then we expect

$$\mathcal{F}_{exact} - \mathcal{B}\mathcal{F}^{(0)} = \sigma_+ e^{-\frac{\mathcal{A}_+}{\hbar}} \mathcal{B}\mathcal{F}^{(1,+)} + \sigma_- e^{-\frac{\mathcal{A}_-}{\hbar}} \mathcal{B}\mathcal{F}^{(1,-)} + O\left(e^{-\frac{2\mathcal{A}}{\hbar}}\right), \quad (4.2.194)$$

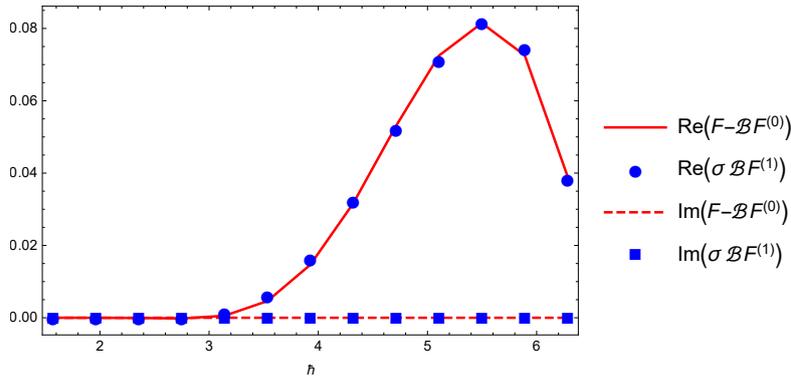
where  $F^{(1,\pm)}$  denotes the first instanton series evaluated with the corresponding action. Figure 4.16 shows the behaviour of the  $A_+$  correction, including terms in  $F^{(1)}$  up to genus 30. The two  $\sigma_{\pm}$  are Stokes constants, and lacking a better ansatz, we can fix them through a least squares fit. For real  $z$ ,  $A_+$  and  $A_-$  are complex

**Figure 4.16:**  $\mathcal{BF}^{(1,+)}$  for  $z = -2^{-6}$



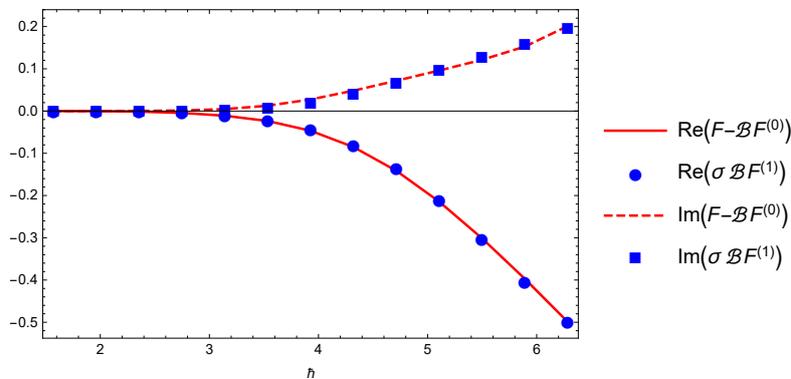
conjugates and the result is, like the difference, real – see figure 4.17, where we plot the computed differences and the fitted  $\mathcal{BF}^{(1)}$ . A complex  $z$  example, in figure 4.18,

**Figure 4.17:** One instanton correction for  $z = -2^{-6}$



shows how the interaction between  $A_+$  and  $A_-$  (which are no longer conjugates) still reproduces the difference. Similar plots are obtained for other values of  $z$ . In figure

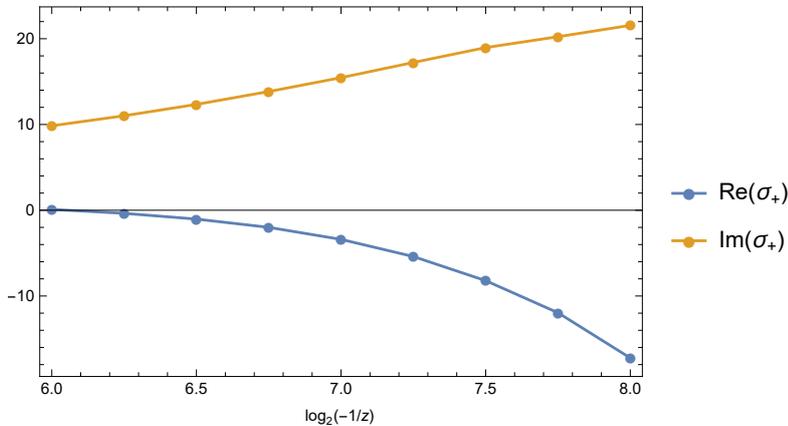
**Figure 4.18:** One instanton correction for  $z = e^{\frac{i\pi}{6}} 2^{-6}$



4.19 we plot the values of  $\sigma_+$  obtained through the fit, for a range of  $z$ -values. For real modulus,  $\sigma_-$  is just the conjugate. Unfortunately, the holomorphic anomaly

tells us nothing about the  $z$  behaviour of the Stokes constants, so the best we can do is observe a non trivial functional dependence.

**Figure 4.19:** Stokes constants as a function of  $z$



We can do another set of tests by inverting the condition

$$\text{vol}_{\text{exact}}(z; \hbar) = 2\pi\hbar \left( n + \frac{1}{2} \right) \quad (4.2.195)$$

that computes the spectrum of  $z$  to define

$$z(n; \hbar) \quad (4.2.196)$$

for  $n$  fixed. Let us take  $n = 5$ , and consider again the exact free energy and the Borel sum (at order 120), for various values of  $\hbar$ . Again we estimate the error of the exact  $F$  and the Borel sum by comparing with the truncation at the previous order (in  $z$  for the exact  $\mathcal{F}_{\text{exact}}$ , and in  $\hbar$  for the Borel sum). In figure 4.20 one can already appreciate that there is a very significant<sup>8</sup> difference between the two, except for the region around  $\hbar = 3\pi/2$ , where the Borel sum seems to have some serious trouble converging.

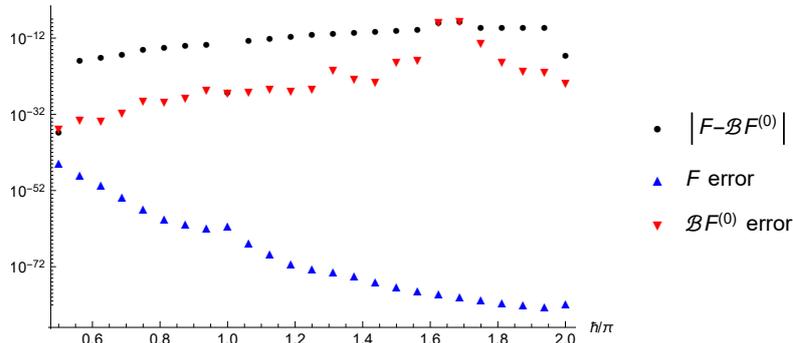
The difference between exact and resummation should still be given by  $\sigma \mathcal{BF}^{(1)}$ , with the action (4.2.193). We find, for instance in figure 4.21, that they agree nicely<sup>9</sup> across the values of  $z$ . It is important to remark that we really need the Borel resummation of the one instanton  $\mathcal{BF}^{(1)}$  (computed up to order 20 in this examples). If we just used the first term in  $\mathcal{F}^{(1)}$  as an approximation, the fit breaks completely for  $\hbar > \pi$ . If we take all the coefficients but do not even do the Borel sum, it just quickly diverges like the asymptotic series that it is.

Another thing to notice is that at  $\hbar = \pi/2$ ,  $\hbar = \pi$ , and  $\hbar = 2\pi$  the difference vanishes (at least, it becomes smaller than the error of the Borel sum, as we can see in figure 4.20), so in these cases the resummation seems to be exact. This behaviour is also reproduced by the one instanton correction. Although  $n$  is to be

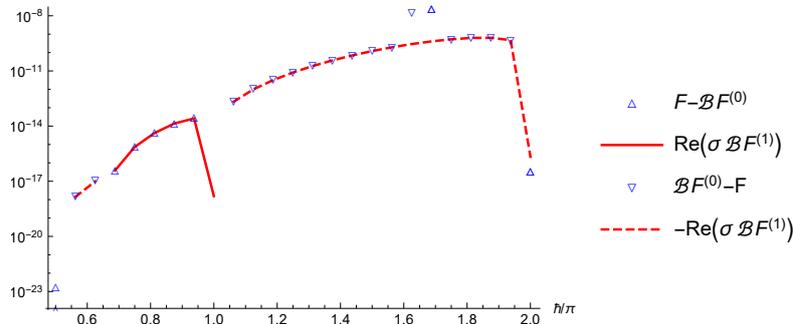
<sup>8</sup>Not only in the qualitative but in the purely numerical sense.

<sup>9</sup>Except, of course, for the two points around  $\hbar = 3\pi/2$  where the Borel sum was converging badly.

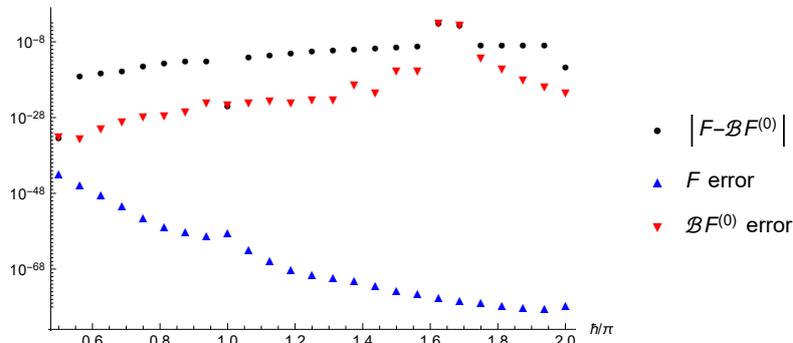
**Figure 4.20:** Exact free energy / Borel sum difference,  $n = 5$



**Figure 4.21:** One instanton correction,  $n = 5$



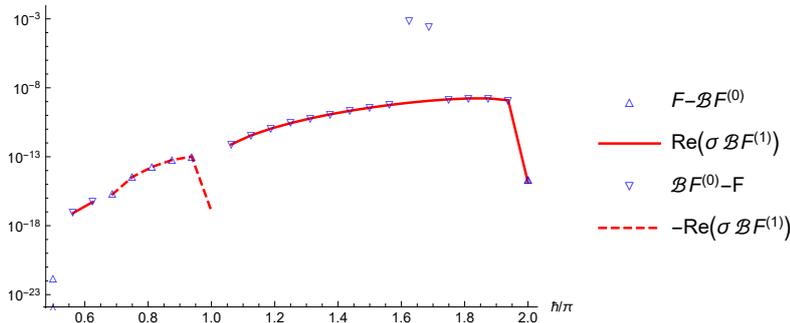
**Figure 4.22:** Exact free energy / Borel sum difference,  $n = 9/2$



identified with the energy level when evaluated as a positive integer, we can just as well evaluate it at other values, in the same way that the WKB coefficients can be evaluated at arbitrary  $E$ . Nothing particularly different happens then. For instance, with  $n = 9/2$ , we get figures 4.22 and 4.23. Notice in 4.23 that the sign of the one instanton correction is inverted with respect to the integer  $n$  case. For other values of  $n$ , we have found that these are the signs for integer and half integer values of  $n$ , which suggests that the Stokes constant has at least a dependence like

$$\sigma \propto (-1)^{(2n)} \tag{4.2.197}$$

although that is not enough to completely determine it.

**Figure 4.23:** One instanton correction,  $n = 9/2$ 

## 4.2.2 Local $\mathbb{P}^1 \times \mathbb{P}^1$

Let us now focus on another genus one toric CY, namely local  $\mathbb{P}^1 \times \mathbb{P}^1$ , also known as local  $\mathbb{F}_0$ . Its mirror curve is

$$e^x + me^{-x} + e^y + e^{-y} + \kappa = 0. \quad (4.2.198)$$

and for simplicity we will just take the parameter to be  $m = 1$ . It thus corresponds to the operator

$$(e^x + e^p + e^{-x} + e^{-p}) \psi = \hat{\kappa} \psi, \quad [x, p] = i\hbar. \quad (4.2.199)$$

It has been proved rigorously [47, 52] that the operator is trace class and positive when  $m > 0$ . Its all-orders WKB corrections have been studied extensively in [18, 36, 37, 62]. In particular, in [115] a variation of the Bender-Wu algorithm for difference operators was provided, that computes the  $\hbar$  expansion of the energies as a function of  $\nu$ , the energy level. The main objective of this section is to study the large order behaviour of this coefficients. As of now, the non-perturbative holomorphic anomaly is the only way of accessing such information.

Concretely, consider the perturbative part of the  $n$ -th eigenvalue,

$$\hat{\kappa}(n, \hbar) = \kappa(n, \hbar) + O(e^{-1/\hbar}). \quad (4.2.200)$$

where

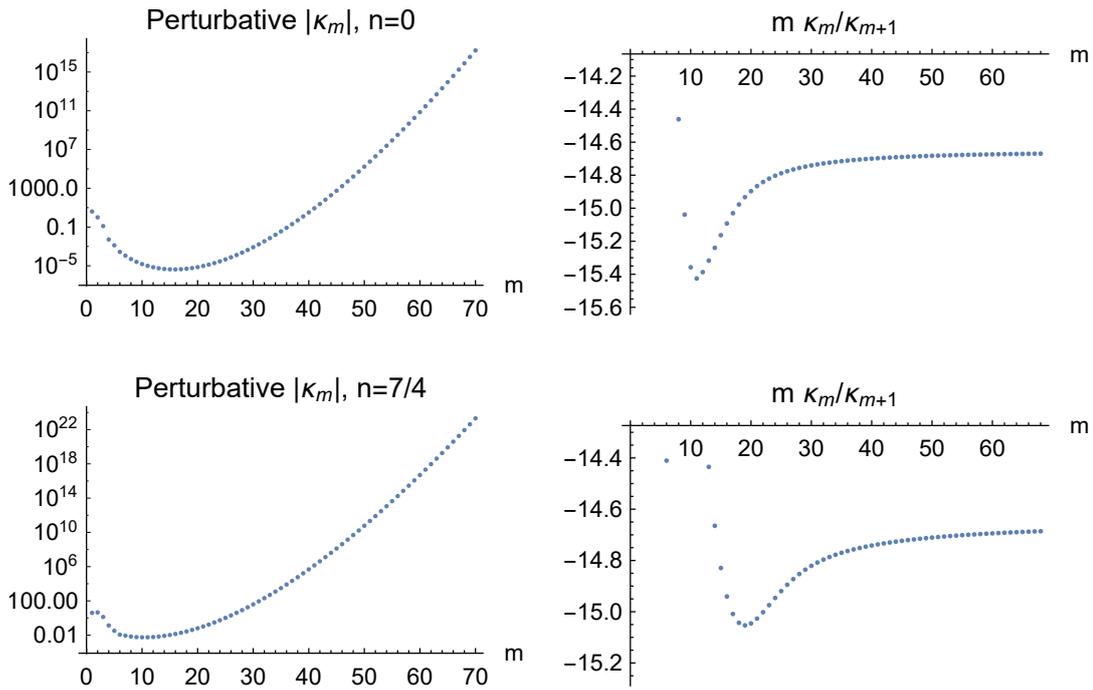
$$\kappa(n, \hbar) = \sum_{m=0}^{\infty} \kappa_m(n) \hbar^m, \quad (4.2.201)$$

The difference operator Bender-Wu algorithm gives the  $\kappa_m(n)$  coefficients for fixed integer  $n$ . Knowing that each such coefficient is a polynomial in  $n$ , we parametrize them with the ansatz

$$\kappa_m(n) = \sum_{i=0}^m \kappa_{m,i}(n-i+1)_i \quad (4.2.202)$$

where  $(x)_i = x(x+1)\dots(x+i-1)$  is the Pochhammer symbol. Like this we can efficiently get  $\kappa_m(n)$  by computing the first  $m$  series,  $\kappa_m(0), \dots, \kappa_m(m)$ . We find the coefficients

$$\{\kappa_m(n)\} = \left\{ 4, 2n+1, \frac{n^2}{4} + \frac{n}{4} + \frac{1}{8}, \frac{n^3}{96} + \frac{n^2}{64} + \frac{n}{64} + \frac{1}{192}, \dots \right\}. \quad (4.2.203)$$



**Figure 4.24:** Local  $\mathbb{P}^1 \times \mathbb{P}^1$  energy coefficients

The large order of these coefficients is

$$\kappa_m = \frac{\Gamma(m+b)}{A^{m+b}} \mu_0 + \frac{\Gamma(m+b-1)}{A^{m+b-1}} \mu_1 + \frac{\Gamma(m+b-2)}{A^{m+b-2}} \mu_2 + \dots, \quad (4.2.204)$$

and from resurgence we expect them to be related to the first instanton correction,

$$\hat{\kappa} = \kappa + \hbar^{-b} e^{-A/\hbar} \left( \kappa_0^{(1)} + \kappa_1^{(1)} \hbar + \dots \right) + O(e^{-2A/\hbar}) \quad (4.2.205)$$

by

$$\mu_m \propto \kappa_m^{(1)}. \quad (4.2.206)$$

Notice that this is *not the same* large order as we find for WKB coefficients. The latter is an expansion in  $\hbar^2$ , while these energies are an expansion in  $\hbar$ . It is for this reason that the large order depends on the action<sup>10</sup>  $A$  and not  $A^2$ . The action  $A$  can be easily retrieved with the limit

$$\lim_{m \rightarrow \infty} \frac{\kappa_m}{\kappa_{m+1}} = A. \quad (4.2.207)$$

In figure 4.24 we see that, numerically, the quotient of coefficients does converge to a certain value. Remarkably, it is independent of level  $n$ .

<sup>10</sup>Incidentally, this is also why we have written  $A$  and not  $\mathcal{A}$ , since we will want to stress the difference between the action of the energies,  $A(\nu)$  and that of WKB,  $\mathcal{A}(t)$ . In any case,  $A$  is *still* a holomorphic quantity – actually, a number.

Let us look at it from the topological string point of view. First resum the energy levels given by (4.2.203) as a function of  $\hbar\nu$ , with

$$\nu = n + \frac{1}{2}. \quad (4.2.208)$$

One finds

$$\begin{aligned} \kappa &= \left( 4 + 2(\hbar\nu) + \frac{(\hbar\nu)^2}{4} + \frac{(\hbar\nu)^3}{96} + \dots \right) + \\ &+ \left( \frac{1}{16} + \frac{(\hbar\nu)}{128} + \frac{3(\hbar\nu)^2}{1024} + \dots \right) \hbar^2 + \\ &+ \left( \frac{13}{24576} - \frac{151(\hbar\nu)}{393216} + \dots \right) \hbar^4 + \dots \end{aligned} \quad (4.2.209)$$

The link with the spectral theory comes through the modulus  $z$ ,

$$\kappa|_{\hbar^0} = z^{-1/2}. \quad (4.2.210)$$

When  $\hbar\nu \mapsto 0$ ,  $z$  goes to the conifold point. Define a conifold coordinate  $\rho$ ,

$$z + \rho = \frac{1}{16}, \quad (4.2.211)$$

and we can invert

$$(\hbar\nu) = 16\rho + 160\rho^2 + \frac{5696\rho^3}{3} + 24384\rho^4 + O(\rho^5). \quad (4.2.212)$$

Since  $\hbar\nu$  is essentially the quantum volume, it must be some solution of the Picard-Fuchs system. First, the large radius periods of  $\mathbb{P}^1 \times \mathbb{P}^1$  are given by

$$\begin{aligned} \partial_z t &= -\frac{2}{\pi z \sqrt{1-16z}} \mathbf{K} \left( \frac{16z}{16z-1} \right), \\ \partial_z \frac{\partial \mathcal{F}_0^{LR}}{\partial t} &= -\frac{2}{z} \mathbf{K}(1-16z), \end{aligned} \quad (4.2.213)$$

where the integration is fixed by the leading order

$$\begin{aligned} t &= -\log z - 4z - 18z^2 + \dots, \\ \frac{\partial \mathcal{F}_0^{LR}}{\partial t} &= \frac{1}{2} (\log z)^2 + 4(1 + \log z)z + \dots \end{aligned} \quad (4.2.214)$$

The A period can actually be integrated exactly to the hypergeometric function

$$t = -\log z - 4z {}_4F_3 \left( 1, 1, \frac{3}{2}, \frac{3}{2}; 2, 2, 2; 16z \right). \quad (4.2.215)$$

Now, define  $t_c$  as follows and expand around  $\rho = 0$ ,

$$t_c := \frac{1}{\pi} \left( \frac{\partial \mathcal{F}_0^{LR}}{\partial t} - \frac{\pi^2}{3} \right) = 16\rho + 160\rho^2 + \frac{5696\rho^3}{3} + \dots = \hbar\nu. \quad (4.2.216)$$

This is a period that vanishes around the conifold, i.e., the conifold A period, which is identified with the energy level through  $\hbar\nu$ .

The other period gives the derivative of the conifold frame prepotential. Its leading behaviour, in terms of  $t_c$ , should be

$$\frac{\partial \mathcal{F}_0^C}{\partial t_c} = t_c \log t_c + \dots \quad (4.2.217)$$

With this condition, we can relate it to the large radius period by

$$\frac{\partial \mathcal{F}_0^C}{\partial t_c} = -\pi t + 8C. \quad (4.2.218)$$

where  $C$  is Catalan's constant. Its value is determined by evaluating (4.2.215) at the conifold, since  $\partial_{t_c} \mathcal{F}_0^C$  vanishes there. Expanding (4.2.213) in  $\rho$

$$\begin{aligned} \frac{\partial \mathcal{F}_0^C}{\partial t_c} &= 16 \left( \log \rho - 1 \right) \rho + \left( 160 \log \rho - 16 \right) \rho^2 + O(\rho^3) = \\ &= t_c \left( \log \frac{t_c}{16} - 1 \right) - \frac{t_c^2}{16} + \frac{5t_c^3}{1152} - \frac{7t_c^4}{12288} + O(t_c^5). \end{aligned} \quad (4.2.219)$$

We now go back to the transseries. Consider the eigenvalue  $\hat{\kappa}$  as a function of  $n$ , or equivalently,  $\nu$ . The non-perturbative corrections to  $\kappa$  can be encoded in non-perturbative corrections to  $\nu$ ,

$$\hat{\nu} = \nu + \Delta\nu^{(1)} + \dots \quad (4.2.220)$$

so that

$$\hat{\kappa}(\nu) = \kappa(\hat{\nu}) = \kappa(\nu) + \frac{\partial \kappa}{\partial \nu} \Delta\nu^{(1)} + \dots \quad (4.2.221)$$

The problem is reduced to computing  $\Delta\nu^{(1)}$ . The good news is that  $\nu$  is related to the free energy, and we can get transseries corrections to the latter via the holomorphic anomaly.

Begin by inverting (4.2.209),

$$\begin{aligned} \hbar\nu = t_c(\kappa, \hbar) &= \frac{\kappa - 4}{2} - \frac{(\kappa - 4)^2}{32} + \frac{5(\kappa - 4)^3}{1536} + \dots + \\ &+ \left( -\frac{1}{32} + \frac{\kappa - 4}{512} - \frac{5(\kappa - 4)^2}{8192} + \frac{45(\kappa - 4)^3}{262144} \right) \hbar^2 + \\ &+ \left( -\frac{13}{49152} + \frac{275(\kappa - 4)}{1572864} + \dots \right) \hbar^4 + \dots \end{aligned} \quad (4.2.222)$$

which basically gives a quantum mirror map. Similarly, there is a map involving the large radius A period,  $t(\kappa, \hbar)$ . The perturbative refined holomorphic anomaly gives a quantum free energy such that

$$\pi t_c(\kappa, \hbar) = \frac{\partial \mathcal{F}^{LR}}{\partial t} \Big|_{t=t(\kappa, \hbar)} - \frac{\pi^2}{3}, \quad (4.2.223)$$

where

$$\mathcal{F}^{LR}(t) = \mathcal{F}_0^{LR}(t) + \hbar^2 \mathcal{F}_1^{LR}(t) + \dots \quad (4.2.224)$$

At the beginning we wrote the energies,  $\kappa$ , as a function of the energy level. Remember this is directly related to the conifold A period,  $\hbar\nu = t_c$ . For simplifying the notation, write

$$t(\kappa(t_c, \hbar), \hbar) := t(t_c, \hbar). \quad (4.2.225)$$

The relation between  $\nu$  and the quantum free energy in terms of these variables is

$$\nu = \frac{1}{\pi\hbar} \frac{\partial \mathcal{F}^{LR}}{\partial t(t_c, \hbar)} - \frac{\pi}{3\hbar}. \quad (4.2.226)$$

Here is where we upgrade the solution of the HA to a transseries,

$$\hat{F} = F + F^{(1)} + \dots \quad (4.2.227)$$

where  $F$  is the perturbative free energy, and

$$F^{(1)} \propto e^{-A/\hbar}. \quad (4.2.228)$$

The transseries of  $\hat{F}$  naturally extends to  $\hat{\nu}$  by (4.2.226),

$$\Delta\nu^{(1)} = \frac{1}{\pi\hbar} \frac{\partial \mathcal{F}^{(1),LR}}{\partial t(t_c, \hbar)} \quad (4.2.229)$$

The general solution of the one instanton sector of the refined holomorphic anomaly was

$$F^{(1)} = f^{(1)} \hbar^2 \exp \left\{ -\frac{\mathcal{A} + (S - \mathcal{S}_{\mathcal{A}}) D_t \mathcal{A} D_t \tilde{F}}{\hbar} \right\}. \quad (4.2.230)$$

where the action  $\mathcal{A}$  is a period, and  $f^{(1)}$  an overall constant. To recover the energies, we must reexpand  $t_c$  as  $\hbar\nu$ , and so an  $\hbar$  expansion will send, at leading order,  $t_c \mapsto 0$ . Since we want a non-vanishing action, take the non-trivial one

$$\mathcal{A} = -2\pi t = -16C + 2 \frac{\partial \mathcal{F}_0^C}{\partial t_c}. \quad (4.2.231)$$

The normalization is chosen to reproduce, as we will see, the asymptotics of the energies in figure 4.24.

To compute  $\mathcal{F}^{(1),LR}$  we must take  $F^{(1)}$  to the large radius frame. The action is the conifold B-period, so as we found for the QM problems, the quantum corrections in the exponential go precisely to the quantum corrections of the free energy derivatives,

$$\begin{aligned} \mathcal{F}^{(1),LR} &= f^{(1)} \hbar^2 \exp \left\{ -\frac{-16C + 2 \frac{\partial \mathcal{F}_0^C}{\partial t_c} + 2 \frac{\partial \mathcal{F}_1^C}{\partial t_c} + \dots}{\hbar} \right\} = \\ &= f^{(1)} \hbar^2 \exp \left( 2\pi t(t_c, \hbar)/\hbar \right). \end{aligned} \quad (4.2.232)$$

This results in

$$\begin{aligned}\Delta\nu^{(1)} &= \frac{1}{\pi\hbar} f^{(1)} \hbar^2 \frac{2\pi}{\hbar} \exp\left(2\pi t(t_c, \hbar)/\hbar\right) = \\ &= 2f^{(1)} e^{16C/\hbar} \left[ \exp\left(-\frac{1}{\hbar} \frac{\partial \mathcal{F}^C}{\partial t_c}\right) \right]^2.\end{aligned}\quad (4.2.233)$$

The first correction to the free energy is given by

$$\mathcal{F}_1^C = -\frac{1}{24} \log\left(\frac{1-16z}{z^2}\right) = -\frac{1}{24} \log\left(256 t_c\right) - \frac{11t_c}{192} + \frac{49t_c^2}{9216} + \dots \quad (4.2.234)$$

Higher corrections can be obtained by using the perturbative refined holomorphic anomaly. For this geometry we have

$$\frac{\partial F_n}{\partial \hat{E}_2} = \frac{Y^2}{3} \sum_{r=1}^{n-1} D_\tau F_r \cdot D_\tau F_{n-r}.\quad (4.2.235)$$

It can be solved recursively in terms of modular forms

$$b = \vartheta_2^4(q), \quad c = \vartheta_3^4(q), \quad d = \vartheta_4^4(q), \quad \hat{E}_2(q, \bar{q}), \quad (4.2.236)$$

where

$$\log(q) = -\frac{\partial^2 \mathcal{F}_0^{LR}}{\partial t^2} = \log z + 8z + 52z^2 + \frac{1472z^3}{3} + 5402z^4 + O(z^5). \quad (4.2.237)$$

and  $Y$  is the Yukawa

$$Y = \frac{2}{d\sqrt{c}} = 2 \partial_{ttt} \mathcal{F}_0^{LR}.\quad (4.2.238)$$

This provides the map between the modular variables and the geometry. In particular, we can recover the modulus as

$$z = \frac{1}{16} \frac{b}{c}.\quad (4.2.239)$$

We can relate these to the conifold by the S-transform

$$b(q) = -d_c(q_c), \quad c(q) = -c_c(q_c), \quad d(q) = -b_c(q_c), \quad (4.2.240)$$

which are now given in terms of the variable

$$\log(q_c) = \frac{\partial^2 \mathcal{F}_0^C}{\partial t_c^2} = \log\left(\frac{t_c}{16}\right) - \frac{t_c}{8} + \frac{5t_c^2}{384} - \frac{7t_c^3}{3072} + O(t_c^4), \quad (4.2.241)$$

so that

$$\rho = \frac{1}{16} - z = \frac{1}{16} - \frac{1}{16} \frac{d_c}{c_c} = \frac{t_c}{16} - \frac{5t_c^2}{128} + \dots \quad (4.2.242)$$

as we knew from the expansion in  $\rho$  in (4.2.216). Finally, there is the orbifold frame. The natural variable is

$$x := z^{-1/2}.\quad (4.2.243)$$

The A period is given by

$$t_o = \frac{i}{4\pi} \left( 2\pi i t - 2 \frac{\partial \mathcal{F}_0^{LR}}{\partial t} \right) = \frac{x}{4} + \frac{x^3}{768} + \frac{9x^5}{327680} + \frac{25x^7}{29360128} + O(x^9), \quad (4.2.244)$$

and the B period by

$$\frac{\partial \mathcal{F}_0^O}{\partial t_o} = -\frac{1}{2} \frac{\partial \mathcal{F}_0^{LR}}{\partial t} = \left( \log \frac{t_o^2}{16} - 2 \right) t_o + \frac{t_o^3}{9} + \frac{283t_o^5}{14400} + \frac{961t_o^7}{158760} + O(t_o^9). \quad (4.2.245)$$

This is connected to the modular forms by

$$\log(q_o) = \frac{\partial^2 \mathcal{F}_0^O}{\partial t_o^2} = \log \left( \frac{t_o^2}{16} \right) + \frac{t_o^2}{3} + O(t_o^4). \quad (4.2.246)$$

The corresponding modular transform is

$$b = c_o, \quad d = -d_o, \quad c = b_o. \quad (4.2.247)$$

With all this we write the starting point for the recursion, the (holomorphic) first free energy

$$F_1 = -\frac{1}{24} \log \left( \frac{256 c d}{b^2} \right) \implies D_\tau F_1 = \frac{c+d}{48}. \quad (4.2.248)$$

Let us work out in detail the second correction,  $F_2$ . We can integrate the recursion to get

$$F_2 = \frac{1}{1728} \frac{(c+d)^2 \hat{E}_2 + \tilde{f}_2}{c d^2}. \quad (4.2.249)$$

The holomorphic ambiguity  $\tilde{f}_2$  must be of weight 6, so we parametrize it by

$$\tilde{f}_2 = \sum_{i=0}^3 \alpha_{2,i} b^i d^{3-i}. \quad (4.2.250)$$

We need to fix four coefficients. One is determined by requiring the large radius expansion to have no constant term, since the constant map contribution will be irrelevant to us. The rest must come from gap conditions. With the normalization of (4.2.235), the conifold energies are recovered by

$$\mathcal{F}_n^C = \left( \frac{1}{2} \right)^{2g-2} \left[ F_n \right]_{\hat{E}_2 \mapsto E_2(q_c)}. \quad (4.2.251)$$

The gap condition on the conifold is, as usual

$$\mathcal{F}_n^C = \frac{(1 - 2^{1-2n}) B_{2n}}{(2n)(2n-1)(2n-2)} \frac{1}{t_c^{2n-2}} + O(t_c^0). \quad (4.2.252)$$

For  $n = 2$  this gives another two conditions, at orders  $t_c^{-2}$  and  $t_c^{-1}$ . The last one can be fixed with the orbifold singularity,

$$\begin{aligned}\mathcal{F}_n^O &= \frac{(-1)^{n-1}}{2} [F_n]_{\hat{E}_2 \rightarrow E_2(q_o)} = \\ &= \frac{(1 - 2^{1-2n}) B_{2n}}{(2n)(2n-1)(2n-2)} \frac{1}{t_o^{2n-2}} + O(t_o^0)\end{aligned}\quad (4.2.253)$$

Since  $q_o$  has an expansion in powers of  $t_o^2$ , this gives one last condition, completely determining the  $\alpha_{2,i}$ . One gets

$$\begin{aligned}F_2 &= \frac{(c+d)^2}{1728cd^2} \hat{E}_2 - \frac{37b^3 + 51b^2d + 18bd^2 + 20d^3}{8640cd^2}, \\ \mathcal{F}_2^C &= -\frac{7}{5760t_c^2} - \frac{101}{221184} - \frac{889t_c}{2949120} + \frac{181981t_c^2}{707788800} + O(t_c^3), \\ \mathcal{F}_2^O &= -\frac{7}{5760t_o^2} + \frac{13}{4320} + \frac{9631t_o^2}{2764800} + \frac{1489t_o^4}{311040} + O(t_o^6).\end{aligned}\quad (4.2.254)$$

The general ambiguity is

$$\tilde{f}_n = \sum_{i=0}^{3n-3} \alpha_{n,i} b^i d^{3n-3-i}, \quad (4.2.255)$$

and by repeating the integration,

$$\begin{aligned}\mathcal{F}_3^C &= \frac{31}{161280t_c^4} + \frac{549181}{29727129600} - \frac{112573t_c}{3170893824} + O(t_c^2), \\ \mathcal{F}_4^C &= -\frac{127}{1290240t_c^6} + \frac{446478743}{255700877967360} - \frac{1910609149t_c}{324699527577600} + O(t_c^2), \\ \mathcal{F}_5^C &= \frac{511}{4866048t_c^8} + \frac{290229163247}{771486077524377600} - \frac{83074158847t_c}{47023913296723968} + O(t_c^2), \\ &\dots\end{aligned}\quad (4.2.256)$$

We now use this and (4.2.219) to re-expand the exponent of (4.2.233) in terms of  $\nu$ . It can be done by substituting  $t_c = \nu/\hbar$ ,

$$\begin{aligned}-\frac{1}{\hbar} \frac{\partial \mathcal{F}^C}{\partial t_c} &= -\frac{1}{\hbar} \frac{\partial \mathcal{F}_0^C}{\partial t_c} - \hbar \frac{\partial \mathcal{F}_1^C}{\partial t_c} - \hbar^3 \frac{\partial \mathcal{F}_2^C}{\partial t_c} + O(\hbar^5) = \\ &= \left[ \nu - \nu \log\left(\frac{\nu}{16}\right) + \frac{1}{24\nu} - \frac{7}{2880\nu^3} + O\left(\frac{1}{\nu^5}\right) \right] - \\ &\quad - \nu \log \hbar + \frac{12\nu^2 + 11}{192} \hbar - \frac{20\nu^3 + 49\nu}{4608} \hbar^2 + \\ &\quad + \frac{1680\nu^4 + 9240\nu^2 + 889}{2949120} \hbar^3 + O(\hbar^4).\end{aligned}\quad (4.2.257)$$

Every term in the  $\hbar$  expansion is a finite polynomial in  $\nu$ . However, the singular part, that does not depend on  $\hbar$ , receives contributions from every free energy. This

behaviour is universal, and as in quantum mechanics it can in fact be resummed to

$$\log f(\nu) := \log \left[ \frac{\sqrt{2\pi} 16^\nu}{\Gamma\left(\frac{1}{2} + \nu\right)} \right] = \nu - \nu \log\left(\frac{\nu}{16}\right) + \frac{1}{24\nu} - \frac{7}{2880\nu^3} + O\left(\frac{1}{\nu^5}\right). \quad (4.2.258)$$

Putting everything together,

$$\begin{aligned} \Delta\nu^{(1)} = & 2f^{(1)} \left[ 1 + \frac{12\nu^2 + 11}{96} \hbar + \frac{144\nu^4 - 160\nu^3 + 264\nu^2 - 392\nu + 121}{18432} \hbar^2 + \right. \\ & + \frac{8640\nu^6 - 28800\nu^5 + 54000\nu^4 - 96960\nu^3 + 188100\nu^2 - 64680\nu + 22657}{26542080} \hbar^3 + \\ & \left. + O(\hbar^3) \right] \hbar^{-2\nu} f(\nu)^2 e^{16C/\hbar}. \end{aligned} \quad (4.2.259)$$

For  $\kappa^{(1)}$  we also need  $\partial_\nu \kappa$ , which easily can be obtained from (4.2.203),

$$\frac{\partial \kappa}{\partial \nu} = 2\hbar + \frac{\nu}{2} \hbar^2 + \frac{4\nu^2 + 1}{128} \hbar^3 + O(\hbar^4). \quad (4.2.260)$$

The final ingredient is the normalization  $f^{(1)}$ . This is fixed by looking at the large order behaviour of the free energies. Similarly to the local  $\mathbb{P}^2$  case, for the large radius A period action we need

$$f^{(1)} = 2\pi i \cdot \frac{1}{\pi^2}. \quad (4.2.261)$$

Notice that the alternating factor is just 1 for integer energy levels. It can, in fact, be reabsorbed with a sign change on  $t_c$ , being generated from the  $t_c \log t_c$  term of the conifold prepotential derivative.

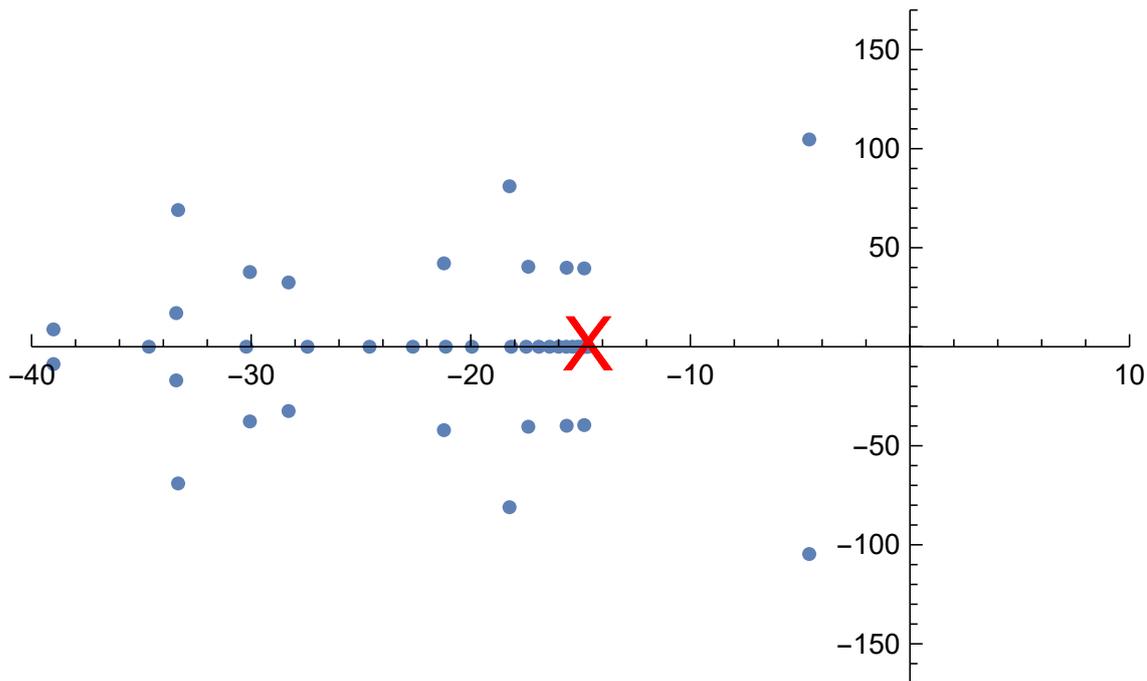
From (4.2.221), we get the one instanton correction to the *energy* transseries.

$$\begin{aligned} \kappa^{(1)} = & \frac{\partial \kappa}{\partial \nu} \Delta\nu^{(1)} = 2\pi i f(\nu)^2 \left[ \frac{2}{\pi^2} + \frac{12\nu^2 + 24\nu + 11}{48\pi^2} \hbar + \right. \\ & + \frac{144\nu^4 + 416\nu^3 + 552\nu^2 + 136\nu + 193}{9216\pi^2} \hbar^2 + \\ & + \frac{8640\nu^6 + 23040\nu^5 + 48240\nu^4 + 32640\nu^3 + 107460\nu^2 + 56640\nu + 34537}{13271040\pi^2} \hbar^3 + \\ & \left. + O(\hbar^3) \right] \frac{e^{16C}}{\hbar^{2\nu-1}}. \end{aligned} \quad (4.2.262)$$

We now want to retrieve the large order from this transseries. In (3.2.178) we got the expression for the large order of an  $\hbar$  power series, assuming the discontinuity was along the real axis,

$$\kappa_m = \frac{S}{2\pi i} \sum_{r=0}^{\infty} \frac{\Gamma(m+b-r)}{A^{m+b-r}} \kappa_r^{(1)}. \quad (4.2.263)$$

In figure 4.25, we plot the poles on Borel plane for the  $\kappa_m$ , and the value of the action



**Figure 4.25:** Local  $\mathbb{P}^1 \times \mathbb{P}^1$  Borel-Padé energy plane poles

as a cross. This is the complex plane in which we integrate the Borel transform  $\hat{\kappa}(\hbar)$  along the real axis. Recall that the Borel transform was defined in (1.24), and with it we obtain the Borel resummation. Typically  $\hat{\kappa}(\hbar)$  cannot be obtained explicitly, so one uses a *Padé approximant*. This is a rational function of  $\hbar$  with degree  $d$  in numerator and denominator, whose Taylor series agrees with the one of  $\hat{\kappa}(\hbar)$  up to order  $2d$ . Because of this, the branch cuts of the function  $\hat{\kappa}(\hbar)$  manifest as accumulations of poles of the rational approximant.

In the  $d \rightarrow \infty$  limit, one such branch cut forms along the negative axis. Its head is located precisely at the value of the action  $A = -16C$ , as follows from the argument in (3.2.178). We expect something similar to (4.2.263), but since  $A < 0$  and the branch cut is on the negative real axis, the integral in (3.2.167) will wrap around the negative axis too. In particular the  $\hbar^{-b}$  term as appears in (3.2.178) will give a complex large order for non-integer  $b$ , while we know all the  $\kappa_m$  coefficients are real. After all, they come from the (real) eigenvalues of a self-adjoint operator. However, we still need to set the Stokes constant, which should precisely cancel this. We need

$$S = (-1)^b \tag{4.2.264}$$

so that the large order is finally

$$\kappa_m = \sum_{r=0}^{\infty} (-1)^{m-r} \frac{\Gamma(m+b-r)}{(-A)^{m+b-r}} \frac{\kappa_r^{(1)}}{2\pi i}. \tag{4.2.265}$$

The subleading corrections to this large order are then

$$\mu_r = \frac{\kappa_r^{(1)}}{2\pi i}. \quad (4.2.266)$$

From the one-instanton series for the energy  $\kappa$ , (4.2.262), we can identify

$$\begin{aligned} A &= -16C, \\ b &= 2\nu - 1, \\ \mu_0 &= f(\nu)^2 \frac{2}{\pi^2}, \\ \mu_1 &= f(\nu)^2 \frac{12\nu^2 + 24\nu + 11}{48\pi^2}, \\ &\dots \end{aligned} \quad (4.2.267)$$

Notice we can then write the Stokes constant as

$$S = (-1)^{2n}, \quad (4.2.268)$$

which is just  $S = 1$  for the actual  $\kappa$  eigenvalues.

Numerically, we can recover the large order behaviour from (4.2.265) by taking limits, such as

$$\begin{aligned} A &= \lim_{m \rightarrow \infty} \frac{m \kappa_m}{\kappa_{m+1}}, \\ b &= \lim_{m \rightarrow \infty} m \left( 1 - \frac{m \kappa_m}{A \kappa_{m+1}} \right), \\ \mu_0 &= \lim_{m \rightarrow \infty} \frac{(-A)^b A^m \kappa_m}{\Gamma(m+b)}, \\ &\dots \\ \mu_i &= \lim_{m \rightarrow \infty} \left( \frac{m}{A} \right)^i \left( \frac{(-A)^b A^m \kappa_m}{\Gamma(m+b)} - \sum_{r=0}^{i-1} \frac{\mu_r A^r}{(m+b-r)_r} \right), \end{aligned} \quad (4.2.269)$$

with  $(x)_n = x(x+1)\dots(x+n-1)$  the Pochhammer symbol.

For instance, take  $n = 3$ . We get the concrete predictions

$$\begin{aligned} b &= 6, \\ \mu_0 &= \frac{268435456}{9\pi}, \\ \mu_1 &= \frac{2030043136}{27\pi}, \\ &\dots \\ \mu_8 &= -\frac{178302865566172531}{190468454400\pi} = -297978.817665105740\dots \end{aligned} \quad (4.2.270)$$

We evaluate the series for  $\kappa$  up to order 120 in  $\hbar$ , and use it to compute the numerical  $\mu_8$  following (4.2.269). With 30 Richardson transforms, we get

$$\mu_{8,\text{num}} = -297978.817665106864\dots \quad (4.2.271)$$

which has 15 stable, agreeing digits. Notice this result is also highly sensitive to the correctness of the previous  $\mu_i$ . For the ground state,  $n = 0$ , the predictions are

$$\begin{aligned}
 b &= 0, \\
 \mu_0 &= \frac{64}{\pi}, \\
 &\dots, \\
 \mu_5 &= \frac{65131771}{3344302080\pi} = 0.00619922666015825876905655974809\dots \quad (4.2.272) \\
 &\dots, \\
 \mu_8 &= -\frac{13253217104875}{3550589334061056\pi} = -0.00118814924264887833570888\dots
 \end{aligned}$$

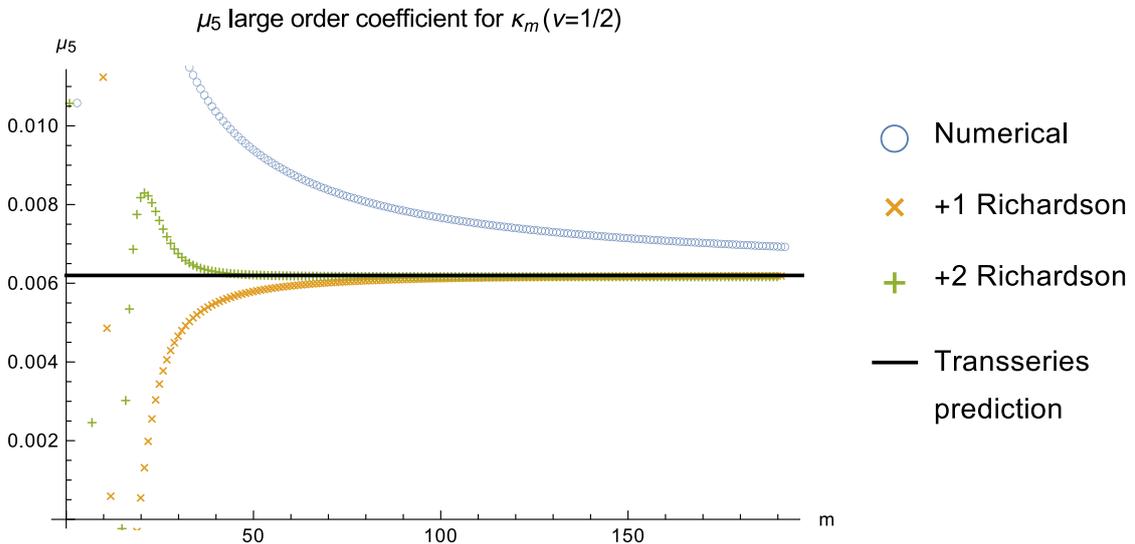
With the numerical limit, order 120 and 47 Richardson transforms, we get

$$\mu_{5,\text{num}} = 0.0061992266601582587690566\dots \quad (4.2.273)$$

and with 40 Richardson transforms

$$\mu_{8,\text{num}} = -0.00118814924264887833573125\dots \quad (4.2.274)$$

They agree on 23 and 20 digits respectively. To see how Richardson transforms improve the convergence of formula (4.2.269), we plot in figure 4.26 the numerical value of  $\mu_5$  according to the formula together with the first two Richardson transforms and the prediction.



**Figure 4.26:** Convergence of a subleading large order correction for the ground state

Finally, we can also consider non-integer values of  $n$  by using (4.2.203), although this severely limits the  $\hbar$  order, since to get the  $m$  term we need the series for the

first  $m$  eigenvalues. Set  $n = 4 + 1/7$ , and we should have

$$\begin{aligned}
 b &= 8 + 1/7, \\
 \mu_0 &= \frac{549755813888\sqrt[7]{2}}{\pi \Gamma(36/7)^2}, \\
 &\dots \\
 \mu_4 &= \frac{14313686339037279813632\sqrt[7]{2}}{7004233215\pi \Gamma(36/7)^2} = \\
 &= (8.0722791\dots) \cdot 10^8.
 \end{aligned} \tag{4.2.275}$$

Numerically, up to  $m = 70$  and with 10 Richardson transforms,

$$\mu_{4,\text{num}} = (8.0721551\dots) 10^8, \tag{4.2.276}$$

with 5 stable, matching digits.

Summarizing, we have reproduced the large order behaviour of the energy levels of the local  $\mathbb{P}^1 \times \mathbb{P}^1$  quantum mirror curve by using the general one instanton solution (3.3.227, 3.3.230) to the refined holomorphic anomaly. It is particularly remarkable in that, by opposition to the Schrödinger equation examples, there is no other known way of computing the transseries for this problem.

## Concluding remarks

In this thesis we have studied the non-perturbative  $\hbar$  structure for two types of spectral problems built out of Heisenberg operators  $[\mathbf{x}, \mathbf{p}] = i\hbar$ , namely quantum mirror curves and one dimensional Schrödinger equations.

Our main result is the development of a transseries generalization of the refined holomorphic anomaly recursion [27] controlling the WKB periods of quantum mirror curves [18], together with an efficient method to solve it order by order in  $e^{-1/\hbar}$ . We have shown that these exponentially suppressed corrections are universal as a function of the  $\hbar$  perturbative sector. To do this, we have proposed an extension of the modular ring over which the holomorphic anomaly is solved, that has allowed us to consistently treat objects in the transseries that are not necessarily modular invariant.

On quantum mirror curves, a conjecture [23] relating the spectral determinant of certain difference operators and the free energies of topological string theory was presented in its extended form [21]. A concrete example was studied here for a genus two spectral curve with a so called mass parameter,  $Y^{3,0}$ . With it we built a set of two operators –corresponding to the genus of curve– together with a one parameter family –given by the mass parameter– of perturbations for each. We have presented extensive evidence both in analytical and numerical form that the spectral traces of such operators are encoded in the topological string theory living in the mirror Calabi-Yau to the spectral curve. The matching is remarkable in that we are able to retrieve exact quantities from a theory –topological strings– that is in principle only defined perturbatively in  $\hbar$ , by choosing an appropriate non-perturbative completion, based on the ideas of [42] for the ABJM theory.

Also for quantum mirror curves, we have used the refined holomorphic anomaly to compute transseries  $\hbar$  corrections to the NS free energies [34], which are related to their WKB coefficients. With the example of the local  $\mathbb{P}^2$  geometry, we have shown that the large order behaviour of the NS free energy genus expansion is controlled by resurgence. That is to say, their rate of divergence and subleading corrections thereof are encoded in the first instanton, or exponentially suppressed, correction to the free energies. We have also computed a non-perturbative completion of this NS free energy  $\hbar$  expansion by using a conjectural definition [111] of the exact quantization condition for the quantum mirror curve operator, given in terms of the derivative of the NS free energies. As in the case for standard strings [116] this definition does not match the usual regularization of asymptotic series given

by Borel resummation. The mismatch we have found reaches at points more than ten significant digits, and it has an interesting functional dependence on both the modulus of this geometry and  $\hbar$ . Following the ideas in [31], we expected the difference to be controlled by exponentially small corrections in  $\hbar$ . We have tested whether it was given in particular by the first instanton correction computed by our transseries refined holomorphic anomaly. Up to an undetermined Stokes constant, we have seen that this is indeed the case. Finally, in a different example, the local  $\mathbb{P}^1 \times \mathbb{P}^1$  geometry, we have used the transseries to reproduce the large order behaviour of the energy levels of the associated quantum mirror curve. We have found agreement with very high numerical accuracy. These perturbative energy levels can be computed purely from the spectral theory side [115]. But the transseries calculations from the holomorphic anomaly are particularly relevant insofar as there is no other known method to systematically obtain this kind of non-perturbative corrections to the genus expansion of topological string theories.

On Schrödinger problems, we have started by analysing the modified Mathieu or  $\cosh x$  potential, directly linked to topological string theories and SW theory [34, 79]. Because of that relation, the WKB periods of the Schrödinger problem must be controlled by the holomorphic anomaly. With help from our transseries framework, we reproduced their large order behaviour from the one instanton correction. The most interesting result, however, was finding that the holomorphic anomaly applies to other quantum mechanical problems without an obvious relation to topological strings, such as the double and cubic well oscillators.

First, we were able to compute their perturbative energy levels by means of the holomorphic anomaly and the so called PNP relation. Of course we reproduced, with the by now usual tool of resurgence, the large order of the perturbative WKB coefficients from the one instanton correction. More interestingly, for all these problems –including modified Mathieu– exact quantization conditions [2, 7, 92] can be found. They can be used to generate a transseries expansion for the energy levels that includes exponentially suppressed  $\hbar$  corrections. Remarkably, the solutions of the transseries holomorphic anomaly developed in this thesis reproduce these transseries for the energy levels. This was a very non-trivial success for our framework. Tests using resurgence and the large order make use only of the first instanton sector, that while far from obvious, is by no means the whole story. The comparison with the quantization transseries has allowed us to verify analytically that the expressions for the higher instanton sectors also contain information about the quantum mechanical problem. This is an interesting result, because finding the exact quantization condition is a problem dependent task that is not always precisely easy. But if the holomorphic anomaly does apply in general to the WKB periods of quantum mechanical problems, then from our construction it follows that the structure of the transseries associated to the quantization condition should be universal. After all, one of the main consequences of the higher instanton recursion we have built is that the refined holomorphic anomaly transseries can be solved explicitly, in the general case, as a function of just the perturbative sector.

On that note, the most important work that could hopefully be done in the future would be proving that the relations between geometry and spectral theory found in this thesis and previous works do not apply only to our examples and a few selected others [21, 48, 49, 56, 117], but are rather a general feature. In particular, it should be proven that the conjecture of [23] applies to general quantum mirror curves, or at least constrain for which subset of them it is true. Also, hopefully the holomorphic anomaly for WKB periods in Schrödinger problems can be derived only from within the context of the all-orders WKB method. A hint of what might be needed in that direction can be found in [54], where the structure of the NS free energies is recovered by having some assumptions on the modular transformation laws for the quantum periods.

Something surely key to understand the connection between quantum mechanics and the anomaly is what the anholomorphic free energies mean physically. In the case of the double well, two different holomorphic limits give two different problems – the double well itself, and the quartic oscillator. The full free energies interpolate between them, but it does not seem obvious what they interpolate through.

It might be the case that the anholomorphic version of the free energies should simply be taken as a convenient way of writing their quasi-modular transformation rules. This quasi-modular structure arises from the exchange and combination of the WKB periods on the curve, which in itself defines a modular group. Since these are the periods defining the free energies, in a sense what their anholomorphic dependence tells us is which part of the latter transforms vectorially under the group, as the periods do. And of course which part – the holomorphic one – is left invariant as a scalar. In the double well and quartic oscillator case, both problems are equivalent under the exchange of periods, that would physically correspond to swapping the classically allowed and forbidden regions. Because of that, the two holomorphic limits are the same up to the period swap.

But the anholomorphic free energies as used in the original holomorphic anomaly of [26] do actually have a meaning in the underlying supersymmetric gauge theory. In that sense we would expect the work in [118] to have a lot to say, where somehow in reverse, a quantum Hilbert space is built out of the holomorphic anomaly for topological string theories. Their question is to what physical problem, if any, does this Hilbert space correspond. Instead, in this thesis we know exactly what the quantum problem is, while lacking a meaning for the general anholomorphic free energies. At this stage, we can only state our hope for a similarly rich configuration space to make sense in quantum mechanics.

Although those would be the ideal next steps, there are a couple of simpler alleys of research left open here. One would be, as in [30, 31], to find the closed form of the Stokes constant that accounts for the difference between the NS exact free energy and the resummation of its asymptotic expansion in the local  $\mathbb{P}^2$  geometry (or any other, for that matter). The one instanton correction fully determined, a disagreement should still be found at an even smaller scale. We would expect it to

be given by the two instanton correction found in this thesis.

There is another immediate question that this work leaves open, and that is how to apply the holomorphic anomaly to quantum mechanics when the underlying spectral curve has genus two or higher. Luckily, the titanic work of writing the holomorphic anomaly in such case has already been taken care of in [70], but it remains to be understood how would it exactly connect with Schrödinger problems. In particular, in our development there has been a one to one correspondence between the modular parameter and the energy level. The higher genus scenario will bring us several moduli, while there will be still a single energy that should get quantized. Our expectation is that something similar to the generalized spectral determinant of chapter 2 and [21] is at play here, where several problems are manifestations of the same modular structure and the energies of some are perturbations of the potential for the others.

All these will surely be fundamental pieces of the puzzle to properly understand what is the full picture that encompasses the WKB method for quantum mechanics and the theory of modular forms.

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