Absolute Spectroscopic Factors

The empirical determination of absolute spectroscopic factors is crucial for an assessment of the validity of the shell model and its limitations. Recent data appear to produce reliable and consistent information on this issue.

Key Words: absolute spectroscopic factor, electron scattering, shell model, stripping and pickup

1. INTRODUCTION

Absolute spectroscopic factors and occupation numbers of quantum states of nucleons are of fundamental interest for our understanding of nuclei. They describe the degree to which reality overlaps with a simple and intuitively appealing description of the nucleus, the shell model. Nuclear static and dynamic properties at low excitation energies are surprisingly well described by the shell model based on the mean-field approximation. This seems to imply that the nucleus can essentially be treated as an assembly of independent nucleons, subject mainly to the constraints of a mean field.

Historically, single-nucleon transfer reactions such as \((d, {^3}\text{He})\) or \((p, d)\) have been used to determine spectroscopic factors. Much of our knowledge on the single-particle structure of nuclei is based on these quantities. The spectroscopic factor \(s_\alpha\) is defined as the probability to reach a final single-particle state \(\alpha\) when a nucleon is removed from (added to) the target nucleus. The occupation
number $n_\alpha$ equals the number of nucleons in the quantum state $\alpha$ in the target nucleus.

The determination of absolute spectroscopic factors $s_\alpha$ and occupation numbers $n_\alpha$ appears very difficult. Transfer reactions are sensitive to the wave function $\Psi_{\alpha=n_{l,j}}(r)$ of the transferred nucleon at large radii $r$ where $|\Psi_{\alpha}(r)|^2$ is typically a few percent of its maximum value. To obtain integral information, the shape of the radial wave function $R(r)$ must be known. For a 10% determination of $s$ the rms-radius of $R(r)$ has to be known to typically 1%; this is the case for a very few states only. In addition, a truly quantitative treatment of the reaction mechanism for composite projectiles and/or reaction products is difficult to achieve. While the one-step pickup (stripping) process can be described quite accurately, the role of two-step processes (of importance for the weaker transitions) is harder to assess.

In order to nevertheless determine absolute quantities, most results are based on a tacit assumption: if for a given orbit with total angular momentum $j$ all particle (hole) strength is observed in pickup (stripping) reactions, one can use a sum rule\(^1\) to normalize the summed strength to $2j + 1$. Higher-order spin-dependent sum rules have been used to provide additional constraints.\(^2\)

The picture that has emerged from this approach: occupation numbers of shell model orbits are 90–100% of the sum rule value $(2j + 1)$. This would imply that the effects of nucleon–nucleon correlations, in particular the ones due to the short range nucleon–nucleon repulsion that depopulate the normally occupied states, are small.

This picture is contradicted by results for the few cases where absolute values can be derived since the $R(r)$ is known from elastic magnetic electron scattering. For these cases, an important lack of spectroscopic strength observed in transfer reactions is found.\(^3\)

Additional experimental evidence comes from two sources: results from high resolution $(e,e'p)$ knockout reactions, and a sum-rule analysis of both transfer data and charge-density differences of isotone pairs:

- The $(e,e'p)$ reaction can measure absolute spectroscopic factors with complications due to sensitivity to $R(r)$ and reaction
mechanism that are much reduced relative to transfer reactions. For the prototype case that we discuss in this Comment, the 3s orbital in $^{208}$Pb, the spectroscopic factor (normalized to a maximum value of unity) was found to be $0.5 \pm 0.05$. This value is remarkably low compared to the hadron-probe results.

- The sum-rule analysis $^5$–$^7$ combines the information on relative spectroscopic factors (from transfer or knockout data) with the absolute difference of spectroscopic factors measurable via elastic electron scattering. For the 3s orbit, an occupation of $0.85 \pm 0.09$ has been determined. This value is quite high.

Recent theoretical calculations of spectroscopic factors indicate rather low values. We here will refer to results from two different approaches:

- Mahaux and collaborators $^8$–$^9$ exploit the analytical properties of the optical potential (or integral quantities thereof) to extrapolate from the positive energies accessible by proton–nucleus scattering to negative energies (bound states). From the potential they derive spectroscopic factors and occupation numbers of single-particle states. For the 3s state in $^{208}$Pb they find $n_{3s} = 0.77$ and $s_{3s} = 0.66$, respectively.

- Pandharipande et al. $^{10}$ use a variational calculation of nuclear matter, which allows us to get a realistic estimate for the effect of short-range correlations. The longer-range correlations near the Fermi-edge are calculated in correlated basis perturbation theory, and finite-nucleus effects are added via an RPA estimate. The occupation number for the 3s state in $^{208}$Pb is again found to be rather low, $=0.63 \pm 0.1$, where the error reflects an estimate of the model dependence.

While theory gives values for occupation numbers 0.63–0.77, the most reliable experiments seem to lie significantly higher and lower, respectively. We thus continue to be faced with apparent inconsistencies. In this Comment, we try to sketch a unified picture, pointing out where some of the apparent differences are due to confusion of terms, and to which degree new data remove the discrepancy.
A missing mass resolution of 100 keV is nowadays available with a $10^{-4}$ resolution spectrometer pair; this allows us to study the quasi-free proton knockout reaction $A(e, e'p)A-1$ with full separation of energetically nearby states in heavy nuclei.\textsuperscript{11} Precision measurements of cross sections for the $(e, e'p)$ reaction induced by electrons of an energy of $\approx 500$ MeV allow us to reconstruct the spectral function $S(E_m, P_m)$, defined as the joint probability of having a proton with binding energy $E_m$ and momentum $P_m$ in the target nucleus $A$. With a choice of the momentum and energy transferred by the virtual photon appropriate to satisfy the quasi-free scattering condition (typically $2 \text{ fm}^{-1}$ and $100$ MeV), the reaction can be treated with confidence in the Distorted Wave Impulse Approximation (DWIA) framework. An experiment on $^{208}\text{Pb}$, the nucleus of main interest for the present Comment, has recently been performed by Quint \textit{et al.}\textsuperscript{4}

For a heavy nucleus such as lead it appears mandatory to use a full phase shift treatment for the distortion induced both by the Coulomb interaction acting on the electron, and by the strong interaction plus Coulomb repulsion acting on the knocked-out proton. The full treatment of the Coulomb distortion of the electron\textsuperscript{12} remedies a problem that was previously encountered in the form of a shift in $P_m$ between fit and data\textsuperscript{4} at small $P_m$ values.

Since the sampling function $\Sigma(P_m, r)$ of the virtual photon field overlaps with the bound state wave function $\Psi_\alpha$, where it has a large amplitude, the spectroscopic strength $s$ has a minor dependence on the Ansatz for $\Psi_\alpha$. In fact by treating as a free parameter the radius of a Woods Saxon potential well that binds the proton, the radial extent of $\Psi_\alpha$ can be extracted from $S(E_\alpha, P_m)$ for every quantum state $\alpha = (n, l, j)$.

The $P_m$-dependence of the spectral function for $3s_{1/2}$ knockout from $^{208}\text{Pb}$, for the transition to the $^{207}\text{Tl}$ ground state ($J^\pi = 1/2^+$), is illustrated in Fig. 1. Clearly the shape of $\rho(P_m)$ is well reproduced by the full phase shift calculation. Distortions smear out the triple-maximum structure, a signature of the radial quantum number $n = 3$, but the feature of predominance of $S(P_m)$ for $l = 0$ quantum states at low-$P_m$ prevails. This feature is valuable in detecting $l = 0$ spectroscopic strength at large excitation energy.
Using a full partial-wave calculation and a $\Psi(r)$ of Hartree-Fock type the match with the data is achieved if the normalization of $\Psi^2$, i.e., $s_{3s}$, is treated as a free parameter when fitting the data in the momentum range $-50$ MeV/c $\leq p_m \leq 130$ MeV/c. Data with $p_m$ larger than 130 MeV/c were excluded in the fit in order to avoid dependence on complex partial wave cancellations and on subtle details (e.g., non-locality effects) of the bound state wave function in the nuclear interior; these properties of $\Psi(r)$ at small values of $r$ preferentially influence the momentum distribution at large values of $p_m$.

The resulting value for the $3s$ spectroscopic strength is $s_{3s} = 0.63 \pm 0.06$. It is this value we will use below when comparing the absolute spectroscopic factors coming from different sources.
3. THE CERES METHOD

Charge density differences $\Delta \rho(r)$ between isotone pairs obtained from elastic electron scattering data provide additional information on the change of proton occupation numbers. For special cases, such as $^{205}\text{Tl}-^{206}\text{Pb}$, information on the occupation of the $3s$ shell can be extracted due to the particular radial shape of the $3s_{1/2}$ wave function. The additional contributions due to core polarization and other shells can be separated. In this case, the absolute charge density difference deduced from $(e, e)$ data can provide the normalization of relative spectroscopic factors from other sources.

The approach developed by the Tübingen group\textsuperscript{5-7} is based on a Combined Evaluation of Relative spectroscopic factors and Electron Scattering (CERES). The charge density difference between $^{206}\text{Pb}$ and $^{205}\text{Tl}$ is assumed to be given by

$$
\Delta \rho(r) = \sum_\alpha \Delta n_\alpha \cdot \rho_\alpha(r) + \Delta \rho_{cp}(r)
$$

(1)

where $n_\alpha$ and $\rho_\alpha$ are the occupation numbers and single-particle radial density of state $\alpha$, $\Delta n_\alpha$ the change $n^{(206)} - n^{(205)}$, and $\Delta \rho_{cp}$ is the change of the charge distribution of the core due to the polarization by the added nucleon. From electron scattering, $z = n^{(206)} - n^{(205)}$ of the $3s$-shell has been extracted\textsuperscript{13} (for a detailed discussion see Section 4).

In the above expression the off-diagonal terms that appear when expanding $\rho$ as a sum over single-particle densities have been neglected.\textsuperscript{14} We will assume that this introduces an error that is not significant compared to the final errors we will quote, as in the expression for $\Delta \rho$ much of the off-diagonal elements will drop out. We also note that part of these off-diagonal terms contribute to the core polarization term, the influence of which is suppressed by the analysis of the $(e, e)$ data in $q$-space (see Section 4).

Introducing the spectroscopic factor for $3s$ protons in $^{208}\text{Pb}$, $s^{(208)}$, with $n^{(208)} = \sum s^{(208)}$ one obtains

$$
n^{(208)} = z \left/ \left( \frac{\sum s^{(206)}}{\sum s^{(208)}} - \frac{\sum s^{(205)}}{\sum s^{(208)}} \right) \right.
$$

(2)
In this equation, only relative spectroscopic factors occur, thus making the right-hand side appropriate for an application to transfer data. The sums, however, in practice must be truncated as the analyses of transfer data are typically limited to excitation energies of 5 MeV in the (A-1) system. A significant fraction of the strength may lie at excitation energy \( \epsilon^* \geq 5 \text{ MeV} \) due to short-range N-N correlations and coupling to giant-resonance excitations. For an approximate solution of this difficulty one assumes equal percentage depletions of all shells, a plausible assumption borne out by the theory of short-range correlations of the Jastrow type,\textsuperscript{10,15} and by the weak A-dependence of giant resonances. With this assumption one obtains

\[
n^{(208)} = \frac{z}{\left( \frac{\sum' s^{(208)}}{\sum' s^{(205)}} - \frac{\sum' s^{(205)}}{\sum' s^{(208)}} \right)}
\]

where the primed sums are taken over a range of excitation energies of typically less than 1 \( \hbar \omega \) (i.e., 8 MeV for \( ^{208}\text{Pb} \)).

The above equation now on the right-hand side only contains quantities that are experimentally measurable in transfer reactions, i.e., relative spectroscopic strength in a limited energy interval. To calculate an absolute occupation number, these quantities are normalized with the \( z \)-value.

This CERES approach has been used\textsuperscript{5,6,16} to analyze the spectroscopic factors measured by \( (d, \text{ } ^3\text{He}) \) on the Hg, Tl, Pb isotopes. This analysis has led to the rather large absolute occupation number referred to in the Introduction. We will show below that, when using the best-fit value for \( z \), and when eliminating the contribution of those levels that, according to \( (e, e'p) \), do not correspond to single-particle states,\textsuperscript{17} values compatible with \( (e, e'p) \) and theory are found.

CERES is a powerful method that needs only fairly well founded assumptions to go from relative spectroscopic factors to an absolute occupation number.

4. 3s-OCCUPATION DIFFERENCE FROM \((e, e)\)

In the CERES analysis of spectroscopic factors in the lead-region, the absolute normalization depends entirely on the difference \( z \) of
the 3s-occupation numbers of $^{206}\text{Pb}$ and $^{205}\text{Tl}$. As the absolute value of this number has turned out to be very important, we reassess its value below.

In the work of Cavedon et al.$^{13}$ the elastic electron scattering cross sections for $^{206}\text{Pb}$ and $^{205}\text{Tl}$ were measured. The experiment showed that, at momentum transfers $q_0 = 2$ fm$^{-1}$, the ratio of $^{205}\text{Tl}$ to $^{206}\text{Pb}$ cross sections exhibits a pronounced peak, a clear signal of a change in the occupation of the 3s-shell; the radial wave function $R(r)$ of a 3s-proton having a $j_0(q_0r)$-type radial dependence with 3 maxima and 2 nodes, the form factor (Fourier transform of $R(r)$) exhibits a $\delta(q_0r)$-type $q$-dependence that can easily be identified and quantitatively interpreted.

From the amplitude of the peak in $\sigma_{\text{Tl}^{205}}/\sigma_{\text{Pb}^{206}}$ one can therefore deduce the change in the 3s-occupation. The change of the occupation of the other open shells or the change due to core polarization lead to a relatively small change of $X = \sigma_{\text{Tl}^{205}}/\sigma_{\text{Pb}^{206}}$ at $q_0 = 2$ fm$^{-1}$, with a smooth dependence on $q$.

Cavedon et al. have shown that a Hartree–Fock calculation with finite-range density dependent effective force for $z = 1$ significantly overpredicts the cross section ratio near 2 fm$^{-1}$, whereas a calculation for $z = 0.7$ explained the data. No explicit fit of $z$, or a detailed study of its determination from $(e, e)$, was carried out.

Since absolute values of spectroscopic factors crucially depend on $z$, we have reconsidered a number of points affecting its determination. For an accurate extraction of $z$, improvements can be made on the following items:

- Study of the model dependence of both the shape of the radial wave function $R(r)$ used, and the size of the core polarization.
- An actual fit of $z$ to the cross section ratios.

Concerning the first point, several authors$^{18-20}$ have studied the effect of employing different $R(r)$ and core polarization terms. These studies all have been performed using different Skyrme forces to calculate the $^{206}\text{Pb}$ and $^{205}\text{Tl}$ charge densities and the $^{205}\text{Tl}/^{206}\text{Pb}$ cross section ratio $X$. These authors find a considerable variation of $X$, depending on the particular Skyrme force used. In extreme cases, the data can be explained with $z = 1$, thus suggesting that the $(e, e)$ data on $X$ do not allow a reliable (model independent) determination of $z$.  

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Two caveats need to be mentioned, however:

(i) These calculations all use zero-range effective forces. For a zero-range force, the response of the nucleus to an added nucleon is very localized; at places where the density of the added nucleon(s) is large, the polarization of the core will lead to a large and opposite change in the core density, with similar radial shape. In the more realistic calculations that employ finite-range forces, the response of the nucleus to the polarizing effect of the added nucleon is much more spread out in coordinate space.

As the 3s radial wave function has narrow peaks of large amplitude (the largest and narrowest of any occupied shell) a localized response of the nucleus leads to a core polarization that largely compensates the local density change introduced by the 3s proton. As a result, to explain the experimental $X$ which represents the combined effect of $R(r)$ and of the core polarization, one needs a large $z$.

This observation is illustrated in Fig. 2 which shows the $^{206}\text{Pb} - ^{205}\text{Tl}$ cross section ratios, for the case of a finite-range force (Campi-Sprung) and a zero-range force (Skyrme 3). While in the case of a finite-range force the shape of the core polarization differs from the one of the single particle wave function (for the 3s piece the node occurs at larger $r$, leading to a small change of $F_{\text{Tl}}/F_{\text{Pb}}$ at $q = 2 \text{ fm}^{-1}$), the core polarization for a zero-range force follows closely the shape of the single-particle contribution; it compensates almost half of the 3s contribution in both the density difference and the form factor ratio near $2 \text{ fm}^{-1}$ where the 3s dominates.

Hartree–Fock calculations with zero-range effective interactions therefore are not suitable to reliably calculate the subtle effect of core polarization.

(ii) Many of the Skyrme forces employed have been fitted to the total charge density of nuclei, $^{208}\text{Pb}$ in particular. The parameters of these forces have been optimized via HF calculations that do not allow for a significant depopulation of shells due to configuration mixing or short range N–N correlations. For the case of $^{208}\text{Pb}$ in particular, the parameters of some of the forces have been trimmed to fit the reduced central density and the small oscillations of $\rho(r)$ resulting in part from configuration mixing.

These HF calculations manage to mimic open-shell nuclei in
FIGURE 2 (a) Single-particle density difference between $^{206}$Pb and $^{205}$Tl, together with the core polarization, both calculated using the zero-range $M^*$ force. (b) Corresponding effects on the $^{206}$Pb/$^{205}$Tl cross section ratio. (c) Same quantity for the finite-range Campi–Sprung effective force.
terms of a closed-shell representation, by an appropriate choice of the free parameters of the effective force. It must be expected that they also allow us to reproduce density differences within the framework used to fit the force, i.e., \( z = 1.0 \); anything else actually would indicate an inconsistency.

Realistic finite-range forces derived to a large degree from the N–N force do not share this difficulty. While they may give a worse overall fit to total densities—at least as long as they ignore configuration mixing and short range N–N correlations—they are more reliable tools for a determination of \( z \).

For these two reasons, we feel justified to reject the results obtained with zero-range forces for a determination of \( z \).

In order to extract from the data the best value for \( z \), one can perform a more detailed analysis than had been done by Cavedon et al. The data in the region \( q = 2 \text{ fm}^{-1} \) can be fit with a variable \( z \). The accurate knowledge of the charge density of \(^{206}\text{Pb}\) can be exploited by calculating the cross section ratios from the experimental density of \(^{206}\text{Pb}\) and the \(^{206}\text{Pb}–^{205}\text{Tl}\) density difference obtained from the density dependent, finite-range HF calculation. In the work of Cavedon et al. the ratio of cross sections was calculated from Hartree–Fock densities for both \(^{206}\text{Pb}\) and \(^{205}\text{Tl}\); for a determination of \( z \) this is not optimal, 21 as the result would depend on the accuracy of the DDHF form factor of \(^{206}\text{Pb}\) near \( q = 2 \text{ fm}^{-1} \).

The result of this analysis is \( z = 0.64 \pm 0.06 \), where the error bar contains the errors of the data, and an estimate of the model dependence of the Hartree–Fock input (obtained from the difference between the Gogny and Campi–Sprung calculations). The value of \( z \) was extracted from the data in the region \( 1.6 \leq q \leq 2.4 \text{ fm}^{-1} \) only, thus emphasizing the contribution from the 3s-shell. The contribution of core polarization at \( 1.6 \leq q \leq 2.4 \text{ fm}^{-1} \) amounts to \( \approx 5\% \) of the 3s contribution, the one of the other shells \((2d, 1g)\) being populated to \( \approx 3\% \). 22 No attempt was made to fit \( X \) in the other regions of \( q \), nor to fit the experimental charge density difference, both of which are much more subject to contamination from core polarization and other shells. The resulting fit is shown in Fig. 3.

In the comparison between absolute spectroscopic factors discussed below we will employ \( z = 0.64 \pm 0.06 \).
FIGURE 3 The experimental $^{206}\text{Pb}/^{205}\text{Tl}$ cross section ratios (Refs. 13 and 29) compared to the best fit with $z_w = 0.64$ obtained using finite-range DDHF wave functions and core polarization.

5. SINGLE PARTICLE TRANSITION FORM FACTORS

Form factors for single-particle transitions of high multipolarity give additional information on the occupation of shell model orbitals.\textsuperscript{23,24} These observables do, unfortunately, not concern the main example we discuss here, the $3s$ state in $^{208}\text{Pb}$; the information nevertheless corroborates the occupation numbers we will find below for lead.

The observables of interest here, concerning the amplitude of the form factor of single-particle transitions from the ground state to a particle–hole state, in general are not easy to interpret due to the effects of configuration mixing. Electromagnetic form factors for particle–hole transitions are sensitive to $2p2h \rightarrow 1p1h$ transitions that (destructively) interfere with the $0p0h \rightarrow 1p1h$ transition, provided the $2p2h$ admixture in the ground state has the same $p$, $h$ quantum numbers as the $1p1h$ state of interest. Among the various observables studied, the magnetic elastic form factors of high multipolarity are the least sensitive to these com-

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plications due to configuration mixing, and not hampered by the fragmentation that is expected for inelastic transitions according to RPA calculations.

Magnetic elastic form factors of the highest multipolarity, $\Lambda = 2j$, for nuclei where the valence shell has the highest $j = l + \frac{1}{2}$ of all normally occupied states, measures two quantities of interest$^{25}$: the radial wave function $R(r)$ of the valence shell, and the one-body density matrix element $\Phi_{MA}$. The former can be used as input for the analysis of transfer reaction data in terms of absolute spectroscopic factors, the latter is related via a sum rule to spectroscopic factors.$^{24}$

The studies of a number of nuclei in the $f/g$ shell have shown two systematic trends$^3$:

1. The $MA$ moments reach 84% of the single-particle value on average. These high values for non-magic nuclei can be explained by the fact that here we deal with states ($l = 3, 4, j = l + \frac{1}{2}$) with a surface-peaked wave function, with most of the density located in a region where the nuclear density is low. Accordingly, the effects of short-range correlations are somewhat smaller than for the $3s$ orbit discussed in the previous sections.

2. The occupation of valence nucleon states, found from the pickup + stripping sum rule using the spectroscopic factors determined for those cases where $R(r)$ is known from $(e, e)$, is much less than 100%; the average value for the cases studied is 66%. To the extent that the exact finite-range DWBA calculations correctly describe the reaction, this shows that in the shell model space usually studied a relatively low fraction of the strength is found.

6. ABSOLUTE NORMALIZATION FROM SUM RULES?

The shortcomings of attempts to determine absolute spectroscopic factors and occupation numbers from transfer reactions have been known for a long time. Besides the complications in the description of the reaction mechanism (optical potentials, coupled channel effects), the calculated cross sections are extremely sensitive to the shape of the bound-state wave function of the transferred nucleon.
Typically, the cross section (hence the extracted spectroscopic factor) changes 10% for a 1% change of the rms-radius of \( R(r) \). As these radii are unknown within several percent, data for pickup or stripping yield relative spectroscopic factors only.

This situation could be greatly improved if a complete set of data for both pickup and stripping on the same nucleus were available. When measuring the relative spectroscopic factors of both particles \((s_p)\) and holes \((s_h)\) of given \( j^n \), the sum rule \( \Sigma s_p + \Sigma s_h = 2j + 1 \) can be used to provide an absolute normalization.\(^1\)

The standard recipe is to assume that \( \Sigma s_p = n_p \) and \( \Sigma s_h = n_h \), where the sum extends over all observed transitions. For the given example of \(^{208}\)Pb (with no hole strength observed) this would imply that the 3s shell is full. This argument, however, ignores that transfer experiments in general are restricted to the first \( =5\) MeV of excitation energy. At larger \( \epsilon^* \), the level densities get too high and levels too broad to allow for an identification of strength. In this section we want to address the question of whether a normalization via a sum rule that does account for this fact is incompatible with the occupation numbers arrived at from other experimental data or recent theoretical calculations.

A discussion of this question is most interesting for the nucleus \(^{208}\)Pb and the 3s shell, where we have detailed information from various sources. For this nucleus, however, no 3s hole strength has yet been detected via stripping reactions (the results of a recent \((^3\text{He, d})\) experiment\(^7\) are not yet available). We therefore will tacitly assume zero 3s hole strength \((\epsilon^* \leq 5\) MeV\) for the discussion of whether transfer data are compatible with low occupation numbers.

In order to estimate the effect of this strength at \( \epsilon^* \geq 5\) MeV we start from the distribution of strength expected from theory, as given by Pandharipande et al.\(^10\) (see Fig. 4). To do so, we will use integrals over the calculated continuum strength, even though in the experimentally accessible region the strength is concentrated in discrete states. The true particle strength \( n_p \) is given by the sum of \( \Sigma s_p \) summed up to \( \Sigma' = 5\) MeV and \( \Sigma'' \) summed over all larger \( \epsilon^* \geq 5\) MeV. The 3s particle strength amounts to 0.70 when summed over the first 5 MeV. The hole strength amounts to 0.085, of which 0.01 comes from the strength moved to \( 0 \leq \epsilon^* \leq 5\) MeV by short range correlations; the rest corresponds to strength moved to \( \epsilon^* > 0\)
FIGURE 4 Occupation probabilities (right-hand scale) and hole strengths (left-hand scale) for nuclear matter (dashed) and $^{208}$Pb (solid). The long-dash curve includes short-range correlations only (Ref. 10).

by finite-nucleus effects (treated via RPA correlations), 75% of which is recovered when summing over 5 MeV. The total strength $\Sigma' s_p + \Sigma' s_h$ thus amounts to 0.78, not 1.0 as implied by an appli-
cation of the sum rule that does not distinguish between $\Sigma$ and $\Sigma'$. From the existing experimental information on strength in $^{208}\text{Pb}$, and the correction factor 0.78 derived above, one then would arrive at an absolute occupation number of 0.78 for the 3s particle state. If some 3s hole strength would be detected via stripping reactions, this number would decrease somewhat. In either case, the transfer reaction results are compatible with the occupation probabilities we arrive at below from other sources.

The dependence on a sum rule normalization can be avoided if the radial wave functions are known from $(e, e'p)$, and if modern, more reliable finite-range DWBA codes are used to analyze transfer reaction data. Such a reanalysis, done recently for a series of nuclei by Kramer et al.\textsuperscript{26} yields absolute spectroscopic strengths that are far below the IPSM limit; typically the spectroscopic factors of valence nucleon states are, for non-magic nuclei, around 0.6–0.7.

7. CONCLUSIONS

We now want to employ the updated information from $(e, e)$, $(e, e'p)$ and transfer reactions discussed above, and extract the best information on absolute spectroscopic factors and occupation numbers.

For $(e, e'p)$ we start from the new result that is obtained by fitting the 3s momentum distribution with a full DWBA code that allows for both electron- and proton-distortion. The result for the 3s strength in the $^{208}\text{Pb}$ ground state: $0.63 \pm 0.06$.

As discussed in the section on $(e, e'p)$, the 3s strength in $(e, e'p)$ corresponds to the discontinuity of the single particle occupation probability at the Fermi edge. The integral up to 20 MeV of Ref. 10 amounts to $0.50 \pm 0.05$ (where the quoted error bar corresponds to the part of the finite-nucleus correction subject to double-counting). The result of Mahaux et al. is $0.66 \pm 0.1$. These results agree with the experimental number of $0.63 \pm 0.06$.

For the CERES interpretation we start from the z-value of $0.64 \pm 0.06$ obtained from the fit of the $^{206}\text{Pb}$ to $^{205}\text{Tl}$ elastic cross section ratios. We use the relative spectroscopic factors from $(d, ^3\text{He})$, but omit the level at 3.89 MeV in $^{207}\text{Tl}$ which does not
appear in the \((e, e'p)\) spectra and hence cannot be interpreted as single-nucleon pickup. Via the CERES approach the occupation of the 3s shell in \(^{208}\text{Pb}\) of \(0.78 \pm 0.12\) is obtained. Using the \textit{relative} spectroscopic factors from \((e, e'p)\) yields \(0.75 \pm 0.08\).

The occupation number of the 3s state in \(^{208}\text{Pb}\) is 0.63 and 0.77 for the calculations of Pandharipande \textit{et al.} and Mahaux \textit{et al.}, respectively, with the same uncertainties as above. These numbers agree within errors with the occupation probability of \(0.78 \pm 0.12\) and \(0.75 \pm 0.08\) derived above via the CERES analysis.

The direct comparison between \((e, e'p)\) and CERES results is not possible without some theoretical input. According to the calculation of Pandharipande \textit{et al.} the spectroscopic factor and occupation number differ by 0.14. Augmenting the \((e, e'p)\) strength by 0.14 with an estimated error of 0.05 yields \(0.77 \pm 0.08\), in good agreement with the \(0.75 \pm 0.08\) from CERES.

The discussion in the previous sections has shown that the information from other sources—single-particle form factors, transfer reaction sum rules— are compatible with occupation numbers in the range 0.7–0.8 provided proper distinction is made between strength integrated over the full versus the experimentally accessible energy range.

Thus a rather consistent picture emerges. For the 3s state in lead—the prototype example for a state near the Fermi edge in a magic nucleus—the absolute occupation probability is quite low, of order \(0.76 \pm 0.07\).

It is clear that the above discussed analyses of data employ concepts of independent-particle models that do not account for several of the complications that arise in a more rigorous theoretical description\(^{14}\); in particular, the validity of Eq. (1) should be studied in more detail with correlated basis models if one wants to push the accuracy beyond the one presently achieved.

In many respects the shell model appears to be working much better than one would naively expect for occupation numbers of valence shells of \(~75\%\). The success of the shell model can probably be assigned to the fact that, in many observables, the “background” due to excitation of states far above the Fermi surface does not play an important role. The partial occupation of states near the Fermi surface can be accounted for by the phenomeno-
logical renormalizations inherent to many practical applications of the shell model.

The realization that spectroscopic strength and occupation numbers in nuclei may be quite low is not new; theoretical studies in the past, notably by G. E. Brown,\textsuperscript{27,28} have pointed this out. We now seem to finally have enough reliable and consistent experimental data in hand to verify this insight.

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