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Strings, Branes, and Gravity Duals of Gauge Theories

Kenneth John Lovis

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A Thesis presented for the degree of
Doctor of Philosophy



Centre for Particle Theory
Department of Mathematical Sciences
University of Durham
United Kingdom

August 2002



18 DEC 2002

FOR MY PARENTS

“Physical laws should have mathematical beauty.”

P. A. M. Dirac.

Strings, Branes, and Gravity Duals of Gauge Theories

Kenneth John Lovis

Submitted for the degree of Doctor of Philosophy

August 2002

Abstract

We study the correspondence between certain supersymmetric gauge theories and their dual supergravity descriptions. Using low-energy brane probes of the supergravity geometries we find moduli spaces of vacua, as expected from considering the dual gauge theories. The metrics on these spaces can be put into a form consistent with field theory expectations. This provides a non-trivial check on the supergravity solutions, in addition to strong-coupling predictions for the gauge theories.

In the case of $\mathcal{N} = 2$ supersymmetric gauge theory, proposed supergravity duals have previously been shown, using brane probe techniques, to display the ‘enhancement mechanism’. In particular, the supergravity geometries correctly reproduce the perturbative behaviour of the gauge theory. We calculate exact non-perturbative results at low-energies using the method of Seiberg & Witten. These correctly reproduce the perturbative results in the supergravity limit, but also make predictions for when the supergravity approximation is not valid.

Finally, we study the Penrose limit of a geometry that is dual to a known $\mathcal{N} = 1$ superconformal gauge theory. The resulting spacetime is a new plane-wave solution with constant three-form fluxes. We quantize type IIB superstrings on this background using the Green-Schwarz formalism. We find the spectrum of string excitations and discover that it is particularly simple, due to the specific form of the plane-wave background. Using the gauge theory/gravity duality, we make predictions (beyond the supergravity approximation) for gauge theory quantities in the corresponding limit.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, Department of Mathematical Sciences, University of Durham, United Kingdom. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Chapter 1 contains necessary background material and no claim of originality is made there. Chapters 2, 3 and 4 also contain some background material which is clearly indicated as such in the text. Appendices A, B and C.3 contain reference material and do not contain original work.

Chapter 2 contains original work, done in collaboration with my supervisor Clifford V. Johnson and David C. Page, some of which has been published in [1, 2]. Chapter 3 contains original, unpublished work done in collaboration with Clifford V. Johnson. Chapter 4 and appendices C.1 and C.2 contain original work done in collaboration with Dominic Brecher, Clifford V. Johnson and Robert C. Myers which has been presented in [3].

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I would also like to thank Dominic Brecher, Robert Myers and David Page for the collaborations upon which much of the research presented here is based. I wish them all success in the future.

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Chapter 1

Introduction and Review of Background Material

1.1 Introduction

Through the standard model, gauge theories currently form the basis of our understanding of particle physics¹. Most calculations are based on perturbative techniques where the gauge coupling is required to be small. For instance, since QCD is asymptotically free at high energies, perturbation theory can be used in that regime to make very accurate predictions. However at low energies the coupling is strong and we are hampered by the need to use non-perturbative methods.

Although the standard model has been incredibly successful at describing the interactions we observe in particle colliders, it is somewhat incomplete. The choice of gauge group, particle content and couplings is not predicted (although there are restrictions) and must be chosen using experimental data. One might hope that a truly fundamental theory would explain, for instance, why there are three generations of quarks and leptons. There are other limitations, in particular the standard model does not include gravity. In order to understand the universe at the very high energies and large spacetime curvatures that we believe occurred in the early universe, we will need a quantum theory of gravity that reduces to general

¹See, for example, [4–6].



relativity at low energies.

String theory is believed to be a consistent theory of quantum gravity. It also naturally gives rise to gauge theories. For these reasons, and many others, string theory is being studied intensively to try to understand whether it can go beyond the standard model and correctly describe our universe.

The central theme of this thesis is how string theory can be used to understand strongly-coupled gauge theories, and vice versa. To be more specific, we will mainly be concerned with supersymmetric field theories because they naturally appear in string theory. Furthermore, some of their non-perturbative behaviour can be often deduced exactly.

The AdS-CFT correspondence [7–9] (and its generalizations) has been central to much of the recent progress in non-perturbative field theory and string theory. It gives a precise way to make non-perturbative field theory calculations using string theory and will form the basis of much of the material presented here.

In this introductory chapter we first discuss the field content and properties of various four-dimensional supersymmetric field theories. We will then introduce string theories and describe how gauge theories appear from the dynamics of objects called ‘D-branes’. Finally, we briefly review the AdS-CFT correspondence and its derivation using D-branes.

In chapter 2 we will study various proposed gravity duals of gauge theories using the low-energy dynamics of a probe D-brane in the supergravity geometry. We first describe the general probe technique, before studying a specific supergravity solution which has been proposed to be dual to a deformation of $\mathcal{N} = 4$ super-Yang-Mills theory that preserves $\mathcal{N} = 1$ supersymmetry. We find that the result of the probe calculation can be interpreted in terms of the gauge theory and make a strong-coupling prediction for the metric on moduli space. We then apply the probe technique to other supergravity duals of supersymmetric theories and find that the results match field theory expectations. We conclude the chapter with a summary of our results and mention some possible avenues for future investigation.

In chapter 3 we start by reviewing the ‘enhancement mechanism’ and supergravity solutions that attempt to encapsulate some of the physics of $\mathcal{N} = 2$ gauge theory. We

then calculate exact, non-perturbative results for the gauge theory at low energies and compare them to those derived using perturbative methods and the proposed supergravity duals. We summarize by comparing our results to those in the existing literature and identify some possibilities for how the work presented here could be extended and applied.

In chapter 4 we study string quantization on non-trivial ‘plane-wave’ backgrounds and relate the results to a special limit of the corresponding dual gauge theories. We start by reviewing the maximally-supersymmetric plane-wave solution of type IIB supergravity and how it can be obtained as a limit of $\text{AdS}_5 \times S^5$. We also review the quantization of strings on this background and how this has provided new predictions for $\mathcal{N} = 4$ gauge theory beyond the supergravity approximation. We then apply similar methods to the supergravity dual of an $\mathcal{N} = 1$ superconformal field theory studied in chapter 2. We find a new plane-wave solution of type IIB supergravity by taking a limit of the supergravity dual. The spectrum of string excitations on this background is calculated and found to be of a special form. We then attempt to match some of the string excitations to operators in the corresponding limit of the gauge theory. Finally, we present the plane-wave limit of a supergravity solution dual to a renormalization group flow and make some conclusions on our results. The appendices contain some reference material and an original observation based on the work in chapter 4.

1.2 Four-dimensional globally supersymmetric field theories

In this section we will review some basic facts about four-dimensional field theories that have supersymmetry². These theories will be the main focus of this thesis and we will need to understand some of the restrictions that supersymmetry puts on a quantum field theory in terms of its field content and its dynamics.

Supersymmetry is an extension of the Poincaré spacetime symmetry algebra to

²Some books, reviews and lecture notes are [10–18].

include anticommuting spinorial generators. Part of its importance lies in the fact that, under some reasonable assumptions, it is the only possible extension of the known spacetime symmetries of particle physics [17, 19, 20]. Supersymmetric field theories, as we shall see, appear naturally in superstring theories. Furthermore, the restrictions put on quantum field theories by supersymmetry can allow one to make powerful statements about their strong-coupling behaviour and have been key to much of the recent progress made in understanding non-perturbative field theory and string theory.

The four-dimensional supersymmetry algebra (with central charges) includes the following anti-commutation relations:

$$\{Q_\alpha^A, \bar{Q}_{\dot{\beta}B}\} = 2\sigma_{\alpha\dot{\beta}}^\mu P_\mu \delta_B^A, \quad \{Q_\alpha^A, Q_\beta^B\} = 2\epsilon_{\alpha\beta} Z^{AB}, \quad (1.1)$$

The Q_α^A are spinor supercharges with $A = 1, \dots, \mathcal{N}$ and their adjoints are $(Q_\alpha^A)^\dagger = \bar{Q}_{\dot{\alpha}}^A$. $P_\mu = i\partial_\mu$ is the generator of spacetime translations and the σ^μ are 2×2 matrices defined in terms of the the Pauli matrices: $\sigma^\mu = (1, \vec{\sigma})$. The Z^{AB} are Lorentz scalars which commute with all the generators of the algebra — they are called central charges. Since they are antisymmetric in A and B , they vanish for $\mathcal{N} = 1$.

It is also possible for the $\mathcal{N} = 1$ supersymmetry algebra to be invariant under a global $U(1)_R$ rotation of the supercharges called an R-symmetry. If this is generated by R , one can take

$$[Q_\alpha, R] = Q_\alpha, \quad [\bar{Q}_{\dot{\alpha}}, R] = -\bar{Q}_{\dot{\alpha}}. \quad (1.2)$$

This particular global symmetry is special because any other global symmetry must commute with the supersymmetry generators. For extended supersymmetry (*i.e.* $\mathcal{N} > 1$) the R-symmetry group can be larger. However, one should note that the R-symmetry is not part of the supersymmetry algebra. One can have supersymmetric theories with no R-symmetry or theories with an R-symmetry that is broken by a quantum anomaly.

1.2.1 Field contents of supersymmetric field theories

A direct consequence of requiring that a theory have supersymmetry is that particles of different spin (or helicity) fall into particle supermultiplets. The supersymmetry

generators relate a bosonic field to a fermionic superpartner, and vice versa. In particular the number of bosonic degrees of freedom match the number of fermionic ones. Furthermore, if a state is charged under a global symmetry that commutes with supersymmetry then its superpartner will have the same charge. In the case of R-symmetries, states in the same supermultiplet have different R-charges.

In the following we shall state the supermultiplets that will arise in our study of gauge theories³. These all have particles with spins or helicities less than or equal to one.

Massless multiplets

First we shall study some massless representations. The basic idea in this case is to choose a frame where the four-momentum is $P_\mu = (E, 0, 0, E)$. Then (1.1) becomes

$$\{Q_\alpha^A, \bar{Q}_{\dot{\beta}B}\} = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} \delta_B^A, \quad (1.3)$$

so that since the operators act on positive-definite Hilbert space, we have that $Q_2^A = 0$. Furthermore, one can show that the central charges vanish. This leaves Q_1^A which lowers the helicity of a state by $\frac{1}{2}$, and \bar{Q}_1^A which raises helicity by $\frac{1}{2}$. Therefore, we can use the supercharges as creation and annihilation operators. Starting on a state of lowest helicity one can find all $2^{\mathcal{N}}$ states in the multiplet by acting with the \bar{Q}_i .

So in the case of $\mathcal{N} = 1$ supersymmetry, starting with a helicity-zero state one obtains a state with helicity equal to $\frac{1}{2}$. In order for the theory to be CPT-invariant we require that the spectrum should be symmetric in helicities. Therefore, we add its CPT conjugate to obtain a multiplet with states of helicity $(\frac{1}{2}, 0, 0, -\frac{1}{2})$. This is the massless chiral multiplet, which is listed along with other massless multiplets⁴

³For more details see, for instance, [10].

⁴One might have expected that the $\mathcal{N} = 2$ multiplet one obtains by acting on a state of (lowest) helicity $-\frac{1}{2}$ would form a multiplet with helicities $(-\frac{1}{2}, 0, 0, \frac{1}{2})$. In fact, because of the way supersymmetry acts on quantum fields (rather than single particle states) the hypermultiplet must have eight degrees of freedom rather than four [17] (see table 1.1). One might also wonder why there is no $\mathcal{N} = 3$ multiplet listed. One finds that upon requiring CPT-invariance, the $\mathcal{N} = 3$

in table 1.1.

Helicity	$\mathcal{N} = 1$		$\mathcal{N} = 2$		$\mathcal{N} = 4$
	Chiral	Gauge	Hyper-	Gauge	Gauge
1	0	1	0	1	1
$\frac{1}{2}$	1	1	2	2	4
0	2	0	4	2	6
$-\frac{1}{2}$	1	1	2	2	4
-1	0	1	0	1	1

Table 1.1: The content of some massless supersymmetric multiplets. The entries indicate the number of states with a given helicity in the multiplet.

Since states with helicity one correspond to gauge fields, the ‘gauge’ multiplets transform in the adjoint representation of the gauge group (since gauge fields do). In a supersymmetric gauge theory, the chiral multiplet and the hypermultiplet can act as ‘matter’ multiplets and can transform in an arbitrary representation of the gauge group.

From the table it is clear how the $\mathcal{N} = 2, 4$ multiplets decompose under an $\mathcal{N} = 1$ subalgebra. The hypermultiplet decomposes into two chiral multiplets in complex-conjugate representations of the gauge group (see *e.g.* [12, 13, 17]) whereas the $\mathcal{N} = 2$ gauge multiplet is the sum of an $\mathcal{N} = 1$ gauge multiplet plus a chiral multiplet in the adjoint representation of the gauge group. Similarly, the $\mathcal{N} = 4$ gauge multiplet decomposes into an $\mathcal{N} = 2$ gauge multiplet plus a hypermultiplet in the adjoint representation of the gauge group.

Massive multiplets

Having seen the massless multiplets, we can now consider the massive multiplets. One finds that generically there are $2^{2\mathcal{N}}$ creation operators, that come in spin doublets. Acting on $(2j + 1)$ states that make up a representation with spin j , one

multiplet is exactly the same as the one for $\mathcal{N} = 4$.

obtains a multiplet with $2^{2\mathcal{N}}(2j+1)$ states. (One can work out the spins of the states using the usual angular momentum addition rules.)

However, for $\mathcal{N} > 1$ there is the possibility of *short* multiplets. Let us consider the case of $\mathcal{N} = 2$. Here, we can have a non-vanishing central charge Z . Using the supersymmetry algebra, it is possible to show that for a massive multiplet with mass M one must have⁵

$$M \geq |Z|. \quad (1.4)$$

For $M > |Z|$ there are $2^{2\mathcal{N}} = 16$ operators as discussed above. However, in the case of multiplets with $M = Z$ (so-called BPS-saturated states), there are only $2^{\mathcal{N}} = 4$ operators so that the multiplet only has $4(2j+1)$ states. These multiplets are of importance in supersymmetric theories because their masses are fixed by their central charges. This is a consequence of the spacetime supersymmetry and is not modified by quantum corrections (although the central charges may receive quantum corrections themselves). In order for the equality not to hold, the states would have to no longer be in a short representation and it is not expected that quantum corrections could generate the extra degrees of freedom needed to fill out a long multiplet. Therefore, this can often allow one to extend results deduced at weak coupling to the strong-coupling regime⁶ [23]. Some examples of massive multiplets are given in table 1.2.

The massive gauge multiplets in table 1.2 can be understood in terms of the Higgs mechanism [4–6] (assuming that supersymmetry remains unbroken). For instance, a gauge field in a massless $\mathcal{N} = 1$ multiplet can ‘eat’ a scalar in a massless chiral multiplet and become a massive spin 1 field in an $\mathcal{N} = 1$ massive gauge multiplet. Exactly the same thing can happen when an $\mathcal{N} = 2$ massless vector plus an $\mathcal{N} = 2$ massless hypermultiplet combine to form an $\mathcal{N} = 2$ massive gauge multiplet. However, an $\mathcal{N} = 2$ massless vector multiplet itself contains a scalar field, and so there is the possibility that the vector can eat this scalar. In this case, by counting

⁵This is often referred to as the BPS bound because of its relationship to BPS monopoles (see chapter 3).

⁶However, there are subtleties that allow the strong-coupling BPS spectrum of a theory to differ from the BPS spectrum at weak coupling [21, 22].

Spin irrep. (No. of states)	$\mathcal{N} = 1$		$\mathcal{N} = 2$			$\mathcal{N} = 4$
	Chiral	Gauge	Gauge	BPS Hyper-	BPS Gauge	BPS Gauge
1	0 (0)	1 (3)	1 (3)	0 (0)	1 (3)	1 (3)
$\frac{1}{2}$	1 (2)	2 (4)	4 (8)	2 (4)	2 (4)	4 (8)
0	2 (2)	1 (1)	5 (5)	4 (8)	1 (1)	5 (5)

Table 1.2: The content of some massive supersymmetric multiplets. The first entry indicates the number of a particular spin irrep. in a multiplet. The second (in brackets) indicates the corresponding number of states.

states one can see that the Higgs mechanism must result in an $\mathcal{N} = 2$ BPS multiplet. Similarly, in the case of the $\mathcal{N} = 4$ massless gauge multiplet the Higgs mechanism must result in the BPS massive gauge multiplet⁷.

1.2.2 $\mathcal{N} = 1$ supersymmetric field theories

In order to construct Lagrangians that are supersymmetric, it is very useful to introduce the notion of superspace. This is an extension of the normal Minkowski space that we are familiar with, that allows one to write down superfields (which contain all the component fields that make up a supermultiplet) on which the action of supersymmetry is realized linearly. The basic idea is to construct composite superfields out of ones containing the elementary fields one wishes to consider, and then write down actions that are automatically supersymmetric. Here we shall only consider $\mathcal{N} = 1$ superspace (although one can also consider $\mathcal{N} = 2$ superspace) and shall follow [11, 13].

Superspace is defined by introducing spin- $\frac{1}{2}$ Grassmann coordinates θ_α and $\bar{\theta}^{\dot{\alpha}} = (\theta_\alpha)^\dagger$, where $\alpha, \dot{\alpha} = 1, 2$. They satisfy

$$[x^\mu, \theta_\alpha] = \{\theta_\alpha, \theta_\beta\} = \{\theta_\alpha, \bar{\theta}^{\dot{\beta}}\} = \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = 0, \quad (1.5)$$

and the spinor indices can be raised and lowered using the ϵ -tensor (for notation and conventions see, for example, [10]). As usual, Grassmann differentiation and integration

⁷The generic $\mathcal{N} = 4$ massive multiplet will contain massive spin-2 particles.

are defined to be identical:

$$\begin{aligned}\frac{\partial}{\partial\theta^\alpha}\theta^\beta &\equiv \int d\theta^\alpha\theta^\beta \equiv \delta_\alpha^\beta, \\ \frac{\partial}{\partial\theta^\alpha}(1) &\equiv \int d\theta^\alpha(1) \equiv 0, \\ \frac{\partial}{\partial\theta^\alpha}\bar{\theta}^{\dot{\beta}} &\equiv \int d\theta^\alpha\bar{\theta}^{\dot{\beta}} \equiv 0.\end{aligned}\tag{1.6}$$

Given this, we can now take the supersymmetry transformations to be

$$\begin{aligned}x^\mu &\longrightarrow x'^\mu = x^\mu + i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}, \\ \theta &\longrightarrow \theta' = \theta + \xi, \\ \bar{\theta} &\longrightarrow \bar{\theta}' = \bar{\theta} + \bar{\xi},\end{aligned}\tag{1.7}$$

which are generated by $\xi^\alpha Q_\alpha + \bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}$ given by

$$Q_\alpha = \partial_\alpha - i\sigma_{\alpha\dot{\alpha}}^\mu\bar{\theta}^{\dot{\alpha}}\partial_\mu, \quad \bar{Q}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} + i\theta^\alpha\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu.\tag{1.8}$$

These then satisfy the correct relation, $\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2i\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu$. Although these generators commute with the momentum operator, $P_\mu = i\partial_\mu$, they do not commute with the other superspace derivatives, ∂_α and $\partial_{\dot{\alpha}}$. Therefore, it is convenient to introduce the following super-covariant derivatives:

$$D_\alpha = \partial_\alpha + i\sigma_{\alpha\dot{\alpha}}^\mu\bar{\theta}^{\dot{\alpha}}\partial_\mu, \quad \bar{D}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} - i\theta^\alpha\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu,\tag{1.9}$$

which do commute with Q and \bar{Q} .

A superfield is defined to be an analytic function on superspace. Because the θ -coordinates are Grassmann, the Taylor expansion of a superfield in terms of the θ is quite simple:

$$\begin{aligned}S(x, \theta, \bar{\theta}) &= \phi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \bar{\theta}\bar{\sigma}^\mu\theta A_\mu(x) + \theta\theta f(x) + \bar{\theta}\bar{\theta}g^*(x) \\ &\quad + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\rho(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x).\end{aligned}\tag{1.10}$$

A superfield can be bosonic or fermionic. For instance, if it is bosonic then ϕ , A_μ , f , g and D will be bosonic fields while the other components are fermionic fields. The supersymmetry generators act on S by

$$\delta_\xi S = (\xi Q + \bar{\xi}\bar{Q})S.\tag{1.11}$$

Now, from the form of the supersymmetry generators one can see that the variation of the highest component in the superfield, D , is a total derivative of the other components. Therefore, the space-time integral of this component is invariant under supersymmetry (if we ignore surface terms), and can be used as a possible action for a supersymmetric theory

$$\mathcal{L} = \frac{1}{2} D = \int d^2\theta d^2\bar{\theta} S. \quad (1.12)$$

In general, a superfield (1.10) is in a reducible representation of $\mathcal{N} = 1$ supersymmetry. In order to describe the multiplets above, it is very useful to impose conditions on superfields so that they describe chiral and vector multiplets.

Chiral superfields

A chiral superfield Φ is defined to be a superfield that satisfies

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \quad (1.13)$$

(An anti-chiral superfield, Φ^\dagger , is defined by $D_\alpha\Phi^\dagger = 0$.) Since \bar{D} anti-commutes with the supersymmetry generators, this condition is invariant under supersymmetry transformations. If we let $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$, and notice that

$$\bar{D}_{\dot{\alpha}}y^\mu = 0, \quad \bar{D}_{\dot{\alpha}}\theta^\beta = 0, \quad (1.14)$$

then any function of y and θ is a chiral superfield (and vice versa). In these variables, a chiral superfield $\Phi(y, \theta)$ can be expanded as

$$\Phi(y, \theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y), \quad (1.15)$$

which gives

$$\Phi(x, \theta, \bar{\theta}) = A + i\theta\sigma^\mu\bar{\theta}\partial_\mu A - \frac{1}{4}\theta^2\bar{\theta}^2\Box A + \sqrt{2}\theta\psi - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi\sigma^\mu\bar{\theta} + \theta^2 F. \quad (1.16)$$

Here, A is a complex scalar field and ψ is a Weyl spinor field that make up a chiral multiplet, while F is an auxiliary field.

Given this, we can now write down supersymmetric Lagrangians for chiral superfields. We will be interested in writing Lagrangians for both renormalizable field

theories and low-energy *effective* Lagrangians. This means that we shall restrict to Lagrangians with terms that have no more than two derivatives on bosonic fields and no more than one derivative on fermion fields.

Let us first consider constructing a kinetic term for some chiral superfields Φ_i . The D-term of $\Phi_i^\dagger \Phi_i$ (where we sum over i) is, after integrating by parts,

$$\mathcal{L} = \Phi_i^\dagger \Phi_i \Big|_{\theta^2 \bar{\theta}^2} = \partial_\mu A_i^\dagger \partial^\mu A_i + F_i^\dagger F_i - i \bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_i, \quad (1.17)$$

which is precisely the Lagrangian for free massless scalars and free massless fermions (once the auxiliary fields have been integrated out). In fact, more generally one can take

$$\mathcal{L}_K = \int d^4\theta K(\Phi_i^\dagger, \Phi_j), \quad (1.18)$$

where K is a real function called the Kähler potential. The bosonic part of the resulting Lagrangian is

$$\mathcal{L}_{K,\text{bosonic}} = \frac{\partial^2 K(A_i, \bar{A}_j)}{\partial \bar{A}_i \partial A_j} (F_i F_j + \partial_\mu \bar{A}_i \partial^\mu A_j), \quad (1.19)$$

so that the kinetic term for the scalar fields is governed by a Kähler metric. (Note that K is a function of the Φ_i and Φ_j^\dagger , but not their spacetime derivatives in order to avoid getting higher-derivative terms in the Lagrangian.)

In the case of a chiral superfield, one can also construct a supersymmetric Lagrangian from the auxiliary field F :

$$\mathcal{L} = F = \int d^2\theta \Phi. \quad (1.20)$$

In fact, since a holomorphic function of a chiral superfield is itself a chiral superfield, one can write down another term in the Lagrangian

$$\begin{aligned} \mathcal{L}_W &= \int d^2\theta W(\Phi_i) + \int d^2\bar{\theta} \bar{W}(\Phi_i^\dagger) \\ &= \sum_i F_i \frac{\partial W(A_j)}{\partial A_i} - \frac{1}{2} \sum_{i,j} \psi_i \psi_j \frac{\partial^2 W}{\partial A_i \partial A_j} + \text{complex conj.}, \end{aligned} \quad (1.21)$$

where W is called the superpotential. This will provide mass and interaction terms for both the scalars and the fermions.

Gauge superfields

In order to describe gauge fields using superspace it is convenient to define a vector superfield as a superfield V that satisfies $V = V^\dagger$. Therefore, in the component expansion of (1.10) one has $\chi = \psi$, $g = f$, $\rho = \lambda$ with ϕ , A_μ and D all real. Starting with a $U(1)$ gauge group, Abelian gauge transformations (that preserve the reality condition) are given by $V \rightarrow V + \Lambda + \Lambda^\dagger$, where Λ is a chiral superfield. Using these, one can go to Wess-Zumino gauge where

$$V(x, \theta, \bar{\theta}) = \bar{\theta}\bar{\sigma}^\mu\theta A_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x). \quad (1.22)$$

From this, one can construct a (fermionic) chiral superfield, using the super-covariant derivatives:

$$\begin{aligned} W_\alpha &= -\frac{1}{4}\bar{D}^2 D_\alpha V \\ &= -i\lambda_\alpha + \theta_\alpha D - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F_{\mu\nu} + \theta^2(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha. \end{aligned} \quad (1.23)$$

W_α is called the field-strength superfield because it contains $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Since W_α is a chiral superfield we can immediately write down the following supersymmetric Lagrangian (as we did for the superpotential):

$$\begin{aligned} \mathcal{L}_V &= \frac{1}{4g^2} \left(\int d^2\theta W^\alpha W_\alpha + \int d^2\bar{\theta} \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \right) \\ &= -\frac{1}{4g^2} F^{\mu\nu} F_{\mu\nu} - \frac{i}{g^2} \lambda^\sigma \partial_\mu \bar{\lambda} + \frac{1}{2g^2} D^2, \end{aligned} \quad (1.24)$$

which is precisely (once the auxiliary field has been integrated out) the correct action for a supersymmetric (free) $U(1)$ gauge theory.

The above can be generalized to the non-abelian case so that $V = V^a T^a$ is in the adjoint representation of the gauge group. Now one can also include a $F\tilde{F}$ (or ‘ θ -angle’) term:

$$\begin{aligned} \mathcal{L} &= \frac{1}{8\pi} \text{Im} \left(\tau \text{Tr} \int d^2\theta W^\alpha W_\alpha \right) \\ &= -\frac{1}{4g^2} F_{\mu\nu}^a F^{a\mu\nu} + \frac{\theta}{32\pi^2} F_{\mu\nu}^a \tilde{F}^{a\mu\nu} - \frac{i}{g^2} \lambda^a \sigma^\mu D_\mu \bar{\lambda}^a + \frac{1}{2g^2} D^a D^a, \end{aligned} \quad (1.25)$$

where D_μ is the appropriate gauge-covariant derivative and

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}, \quad (1.26)$$

is the complexified coupling.

We are now in a position to write down the full Lagrangian for a $\mathcal{N} = 1$ supersymmetric gauge theory with ‘matter’ chiral superfields:

$$\begin{aligned} \mathcal{L} = & \frac{1}{8\pi} \text{Im} \left(\tau(\Phi) \text{Tr} \int d^2\theta W^\alpha W_\alpha \right) + \int d^2\theta d^2\bar{\theta} K(e^{2V} \Phi_i, \Phi_j^\dagger) \\ & + \int d^2\theta W(\Phi_i) + \int d^2\bar{\theta} \bar{W}(\Phi_i^\dagger), \end{aligned} \quad (1.27)$$

where we have allowed τ to be a holomorphic function of the Φ_i . (Note that in the scalar kinetic term $V = V^a T^a$ now acts on the Φ_i which can be in an arbitrary representation of the gauge group.)

Non-renormalization theorems and exact β -functions

The fact that the superpotential W is a holomorphic function of the chiral superfields has some important consequences. Using holomorphicity and symmetry arguments with some other assumptions, Seiberg [24] found that the form of quantum corrections to the superpotential in the Wilsonian effective action was highly restricted. In particular, the UV superpotential is not renormalized perturbatively. Additional non-perturbative terms can appear, but their form is restricted. Sometimes this can imply that the superpotential receives no corrections at all. (A detailed review of the arguments and assumptions used can be found in [14, 25].)

Similar arguments can be applied to the gauge field term — it is only renormalized through the coupling $\tau(\Phi)$. In perturbation theory the one-loop correction to τ is exact, but it can receive non-perturbative corrections (as we shall see in chapter 3).

The non-renormalization theorems can be used [26] to find the exact form of the β -functions for the gauge coupling, g , and the couplings in the superpotential. When the β -functions are zero, the couplings do not run and the theory is scale-invariant (and under some weak conditions, conformally invariant). The theory is said to be at a fixed point of the renormalization group flow. (If the theory is at fixed point in the space of couplings then it will remain there during a renormalization group flow.)

A case of particular interest is when the space of couplings has a line of fixed points. This implies that there is a coupling corresponding to an exactly-marginal operator — changing the value of the coupling moves the theory along the line without breaking scale invariance. Furthermore, the existence of an exactly-marginal operator implies the theory has non-trivial interactions.

Let us consider a gauge theory with some chiral superfields Φ_i that transform in representations R_i of the gauge group. For an operator $\mathcal{O}_\lambda = \prod_i (\Phi_i)^{n_i}$ that appears in the superpotential with coupling λ , the exact β -functions are of the form [26–29]:

$$\begin{aligned}\beta_g &\propto 3T(\text{adj}) - \sum_i T(R_i)(1 - 2\gamma_i) , \\ \beta_\lambda &\propto 3 - \sum_i n_i(1 - \gamma_i) ,\end{aligned}\tag{1.28}$$

where the γ_i are the anomalous dimensions of the Φ_i , *i.e.* for a chiral superfield the scaling dimension is quantum corrected from its canonical value of 1 to $(1 - \gamma_i)$. The index of a representation R_i is defined by $\text{Tr}(T^a T^b) = T(R_i)\delta^{ab}$. (In the case of the adjoint representation of $SU(N)$, $T(\text{adj}) = N$.)

We can now review some of the arguments of Leigh & Strassler [29] who studied the existence of non-perturbative fixed points and marginal operators using (1.28). As an example, let us consider the case of three chiral superfields Φ_1 , Φ_2 and Φ_3 each in the adjoint representation, with a superpotential [18, 29]

$$W = \text{Tr} [a\Phi_1\Phi_2\Phi_3 + b\Phi_1\Phi_3\Phi_2 + c(\Phi_1^3 + \Phi_2^3 + \Phi_3^3)] .\tag{1.29}$$

Using cyclic symmetry one can see that the anomalous dimensions of the chiral superfields are all equal, $\gamma_i = \gamma$ say. In this case the β -function relations (1.28) reduce to

$$\beta_a \propto \beta_b \propto \beta_c \propto \beta_g \propto \gamma,\tag{1.30}$$

so that in the space of couplings (a, b, c, g) one expects a three-dimensional submanifold of couplings, given by $\gamma(a, b, c, d) = 0$, that give rise to exactly marginal operators. Furthermore, because the subspace passes through the origin, where the theory is at weak coupling, we can be sure that this manifold exists. Although γ is not known away from the origin, we will see that the case when $c = 0$ and the other couplings are related in a specific way, gives rise to the $\mathcal{N} = 4$ supersymmetric

theory. We shall also use this method to study an $\mathcal{N} = 1$ superconformal fixed point that arises from a deformation of the $\mathcal{N} = 4$ theory.

In chapter 2 we will obtain results for the Kähler potential of certain theories using the AdS-CFT correspondence. These results are of particular interest because the above non-renormalization arguments do not hold for the Kähler potential (since it is not holomorphic).

1.2.3 $\mathcal{N} = 2$ supersymmetric field theories

To study theories with $\mathcal{N} = 2$ supersymmetry one can extend $\mathcal{N} = 1$ superspace to $\mathcal{N} = 2$ superspace (see [12, 17] and references therein). In the case of pure $\mathcal{N} = 2$ supersymmetric Yang-Mills theory (*i.e.* no matter hypermultiplets) one has the following low-energy Lagrangian in terms of $\mathcal{N} = 1$ superfields:

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left[\frac{1}{2} \int d^2\theta \frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi^a \partial \Phi^b} W^{\alpha a} W_\alpha^b + \int d^4\theta (\Phi^\dagger e^{2V})^a \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi^a} \right], \quad (1.31)$$

where the $\mathcal{N} = 2$ gauge superfield is composed of an $\mathcal{N} = 1$ gauge superfield in the adjoint representation of the gauge group (with field strength superfield W^α) and a chiral multiplet (also in the adjoint representation), Φ . Note that now both the gauge coupling and the Kähler potential for the chiral multiplet are determined in terms of a single holomorphic function called the prepotential. This restriction, the fact that the theory contains BPS states, and other known properties, allowed Seiberg & Witten to write down the exact low-energy effective action for the case with $SU(2)$ gauge group. (This work was then extended to the case of other gauge groups and theories with hypermultiplets (see [12, 13, 16, 30] and references therein).)

If we take $\mathcal{F} = \frac{1}{2}\tau\Phi^2$, with τ a constant, so that the above Lagrangian is renormalizable, then one finds

$$\begin{aligned} \mathcal{L} &= \frac{1}{4\pi} \text{Im} \text{Tr} \left[\tau \left(\frac{1}{2} \int d^2\theta W^\alpha W_\alpha + \int d^2\theta d^2\bar{\theta} \Phi^\dagger e^{2V} \Phi \right) \right] \\ &= \text{Tr} \left(-\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{g^2} (D_\mu A)^\dagger D^\mu A - \frac{1}{2g^2} [A^\dagger, A]^2 \right) \\ &\quad + \text{fermions}. \end{aligned} \quad (1.32)$$

where in the second line we have integrated out the auxiliary fields. From this one can see that the scalar potential has minima given by $[A^\dagger, A] = 0$, *i.e.* when A and A^\dagger

commute. These give (after taking into account the action of the unbroken gauge symmetry) a continuous set of inequivalent vacua, or moduli space. In fact, this moduli space is not lifted by quantum corrections, although the metric on it (given by the scalar kinetic terms) does receive corrections through the prepotential.

1.2.4 $\mathcal{N} = 4$ super-Yang-Mills gauge theory

If one now requires $\mathcal{N} = 4$ supersymmetry (with fields of helicity or spin ≤ 1) then the theory is uniquely determined by the choice of gauge group up to the (constant) complex gauge coupling τ . In terms of $\mathcal{N} = 1$ superfields, the $\mathcal{N} = 4$ theory has one gauge superfield and three chiral superfields, Φ_i ($i = 1, 2, 3$), all in the adjoint representation of the gauge group. In $\mathcal{N} = 1$ superspace the Lagrangian is:

$$\begin{aligned} \mathcal{L} = & \frac{1}{4\pi} \text{ImTr} \left[\tau \left(\frac{1}{2} \int d^2\theta W^\alpha W_\alpha + \int d^2\theta d^2\bar{\theta} \Phi_i^\dagger e^{2V} \Phi_i \right) \right] \\ & + \text{Tr} \left[ig \int d^2\theta \Phi_1[\Phi_2, \Phi_3] - ig \int d^2\bar{\theta} \bar{\Phi}_1[\bar{\Phi}_2, \bar{\Phi}_3] \right] \end{aligned} \quad (1.33)$$

where now every coupling is determined in terms of τ . The theory has an $SU(4) \simeq SO(6)$ R-symmetry, although in this representation only an $SU(3) \times U(1)$ subgroup is explicit. In terms of component fields the Lagrangian is

$$\begin{aligned} \mathcal{L}_{\text{bosonic}} = & \text{Tr} \left(-\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{1}{2g^2} D_\mu X^I D^\mu X^I \right. \\ & \left. + \frac{1}{4g^2} \sum_{I,J} [X^I, X^J]^2 \right) + \text{fermions}. \end{aligned} \quad (1.34)$$

Here the X^I are in the **6** representation of the $SU(4)$ R-symmetry group, while the four Weyl fermions are in the fundamental representation. This form of the $\mathcal{N} = 4$ Lagrangian can be derived by dimensional reduction of ten-dimensional super-Yang-Mills theory [31].

Remarkably, there are no perturbative ultraviolet divergences and it is believed that the $\mathcal{N} = 4$ Yang-Mills theory is UV finite, even non-perturbatively [26–28]. This implies that the β -function vanishes and so the theory is scale-invariant, and is in fact invariant under the conformal group $SO(2, 4) \sim SU(2, 2)$. The supersymmetry, R-symmetry and conformal symmetry actually generate a larger supergroup called $SU(2, 2|4)$.

The form of the scalar potential implies that the theory has a moduli space given by $[X^I, X^J] = 0$ for $I, J = 1, \dots, 6$. By a gauge transformation, one can take the X^I to be diagonal. For $\langle X^I \rangle = 0$ the theory remains conformally invariant, whereas for $\langle X^I \rangle \neq 0$ the conformal symmetry is broken (although the $\mathcal{N} = 4$ supersymmetry remains unbroken). From the scalar kinetic term one can see that the metric on moduli space is required to be flat (although there are orbifold singularities due to unbroken, discrete gauge transformations [32]).

Another important property of his theory is that it is conjectured to display exact Montonen-Olive, or S-duality [23, 33, 34]. The S-duality acts on τ as $\tau \rightarrow -1/\tau$, which for $\theta = 0$ takes $g \rightarrow 4\pi/g$, so that the theory at strong coupling is dual to the theory at weak coupling. The theory is also invariant under a shift of the theta angle $\theta \rightarrow \theta + 2\pi$, which implies $\tau \rightarrow \tau + 1$. Together these generate the full $SL(2, \mathbb{Z})$ duality group which acts as:

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}. \quad (1.35)$$

The same sort of S-duality symmetry, but in more complicated realizations, plays a role in $\mathcal{N} = 2$ gauge theory, as shown by Seiberg & Witten [21].

1.2.5 The Leigh-Strassler fixed point

Having briefly considered the $\mathcal{N} = 4$ supersymmetric theory we shall now consider a specific mass deformation of it that breaks $\mathcal{N} = 4$ supersymmetry to $\mathcal{N} = 1$. It will be useful to again use the superfield formalism to describe the deformation in an explicitly $\mathcal{N} = 1$ supersymmetric way. In particular, consider the superpotential

$$W = k \text{Tr}(\Phi_3[\Phi_1, \Phi_2]) + \frac{1}{2} m \text{Tr}(\Phi_3^2) \quad (1.36)$$

where we have chosen to give a mass to the chiral superfield Φ_3 , while keeping Φ_1 and Φ_2 massless, and in the UV we have $k \sim g$. Using the exact β -functions one finds that in this case [11, 29, 35, 36]:

$$\begin{aligned} \beta_g &\propto \beta_k \propto \gamma_1 + \gamma_2 + \gamma_3, \\ \beta_m &\propto 1 - 2\gamma_3, \end{aligned} \quad (1.37)$$

where we again consider the γ_i to be functions of the couplings g , k and m . The theory has a global $SU(2)$ ‘flavour’ symmetry which acts on Φ_1 and Φ_2 , so that $\gamma_1 = \gamma_2$. (It also has an unbroken $U(1)_R$ R-symmetry.) Note that since $\beta_g \propto \beta_k$, we have two constraints on three couplings and so expect a line of fixed points in the coupling space. The β -functions are zero if $\gamma_1 = \gamma_2 = -1/4$ and $\gamma_3 = +1/2$. We should now note that the line of exactly marginal couplings given by these relations does not pass through the origin and we cannot be sure that it exists. However, the conjecture made by Leigh & Strassler is that in the infra-red the deformed theory approaches a conformal fixed point on this line of marginal couplings. Because one has an exactly marginal coupling for this theory, one can then deform this theory by adding the corresponding operator to the superpotential without breaking conformal invariance. Furthermore, the fact that the theory has an exactly marginal operator implies that the infra-red theory is an interacting conformal theory. In fact at low energies, the fields in Φ_3 will be integrated out and the resulting superpotential will be of the form

$$W = h\text{Tr}([\Phi_1, \Phi_2]^2) \quad (1.38)$$

where the marginal coupling h is related to g and m . This theory will again have a moduli space, this time given by $[\phi_1, \phi_2] = 0$ (where the ϕ_i are the lowest components of the $\Phi_i, i = 1, 2$).

Since the infra-red theory (we shall call it the Leigh-Strassler theory or Leigh-Strassler fixed point) is $\mathcal{N} = 1$ supersymmetric, the conformal symmetry is extended to an $\mathcal{N} = 1$ superconformal symmetry (the notation for the supergroup is $SU(2, 2|1)$). This implies that operators in the theory must be in superconformal multiplets⁸. Each superconformal multiplet contains an operator of lowest dimension called a superconformal primary operator. A special class of operators are called chiral operators. They are analogous to BPS states — they are annihilated by some of the supercharges and lie in short representations of the superconformal algebra. For the superconformal primaries of these multiplets (superconformal chiral primaries) the superconformal algebra relates the scaling dimension of the operator

⁸For brief reviews of representations of superconformal symmetry see [11, 37].

to its R-charge by $\Delta = \frac{3}{2}R$. This allows an alternative way of deducing the scaling dimensions of the Φ_i at a fixed point. For the R-symmetry to be a symmetry of the theory it must leave the superpotential term $\int d^2\theta W$ invariant. Since $d^2\theta$ has R-charge -2 (see e.g. [14]), W must have charge $+2$. Using (1.36) and the $SU(2)$ symmetry implies that $R(\Phi_3) = 1$ and $R(\Phi_{1,2}) = \frac{1}{2}$. Therefore the scaling dimensions are $\Delta_3 = \frac{3}{2}$ and $\Delta_{1,2} = \frac{3}{4}$, as derived above.

1.3 String theory and D-branes

Now that we have briefly reviewed supersymmetric gauge theories in four dimensions, we shall describe how they appear in some superstring theories as describing the dynamics of solitonic objects called D-branes⁹.

The basic idea of string theory is that instead of regarding the fundamental excitations of the theory as being particles, as in quantum field theory, one considers the quantization of one-dimensional objects, or strings. At low energies one probes length scales above the characteristic length of the strings (the string length, l_s) and the strings appear to be point-like. Since the strings actually have spatial extent they can vibrate, and the different oscillation modes of the string give rise to a whole spectrum of particles (or more precisely quantum fields) in spacetime.

1.3.1 Bosonic strings

Firstly, before considering superstrings, let us remind ourselves of the case of the bosonic string. In D -dimensional Minkowski space the bosonic string action is

$$S = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma (-\gamma)^{1/2} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \quad (1.39)$$

where we have worldsheet coordinates $\sigma^a = (\tau, \sigma)$ and worldsheet metric γ_{ab} . The X^μ are embedding coordinates in Minkowski space. In this formula the tension of the string is $T = \frac{1}{2\pi\alpha'}$ which is related to the string length by $l_s^2 = \alpha'$. If one

⁹Standard reference texts on string theory are [31, 38]. Some reviews and lecture notes on strings and branes are [39–42], while a text on string theory with a modern D-brane viewpoint is [43] (based on [44]).

solves for the worldsheet metric using the equations of motion and puts the answer back into (1.39), then the action is simply proportional to the area of the string worldsheet. (This can be compared to the case of a point particle where the action is proportional to the proper length of its worldline.)

So far we have not specified any boundary conditions to supplement the action (1.39). There are two basic possibilities — one can either have closed strings (which have periodic boundary conditions) or open strings. For the moment, let us concentrate on the case of closed strings.

Upon quantization of (1.39), one finds that in the critical dimension $D = 26$ the string spectrum contains at the massless level a spin-2 field, a two-form gauge field $B_{\mu\nu}$, and a scalar (called the dilaton). Since the spectrum includes a massless spin-2 field the theory must include gravity. However, the bosonic string is rather ‘sick’ in that the spectrum also includes a tachyon field, which implies that the vacuum is unstable. Furthermore, in terms of matching to the real world, the spacetime spectrum does not include fermions.

Despite these problems, it is interesting to consider quantizing the bosonic string in a non-trivial background where vacuum expectation values have been given to the massless fields. The correct action to consider is

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma g^{1/2} [(g^{ab}G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}(X)) \partial_a X^\mu \partial_b X^\nu + \alpha' R\Phi(X)] , \quad (1.40)$$

where $(G_{\mu\nu}, B_{\mu\nu}, \Phi)$ are the background values for the metric, two-form field and dilaton respectively¹⁰. In order for this action to define a consistent string theory it should be Weyl-invariant. Requiring this at the quantum level implies that the following β -functions should vanish:

$$\begin{aligned} 0 &= \beta_{\mu\nu}^G = \alpha' \left(R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H_{\mu\lambda\omega} H_\nu^{\lambda\omega} \right) + \mathcal{O}(\alpha'^2) \\ 0 &= \beta_{\mu\nu}^B = \alpha' \left(-\frac{1}{2} \nabla^\omega H_{\omega\mu\nu} + \nabla^\omega \Phi H_{\omega\mu\nu} \right) + \mathcal{O}(\alpha'^2) \\ 0 &= \beta^\Phi = \frac{D-26}{6} + \alpha' \left(-\frac{1}{2} \nabla^2 \Phi + \nabla_\lambda \Phi \nabla^\lambda \Phi + \frac{1}{2} R - \frac{1}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right) + \mathcal{O}(\alpha'^2). \end{aligned} \quad (1.41)$$

¹⁰Here we have taken the worldsheet metric, g , to have Euclidean signature. R is the corresponding Ricci scalar.

Here, we have made an expansion in α' , which is equivalent to an expansion in spacetime derivatives. At zeroth order in α' we have the requirement that $D = 26$. At order α' the above equations are just spacetime equations of motion, which can be derived from a generalized Einstein-Hilbert action, with corrections that are higher order in α' . This is one of the most striking results in string theory — that the consistency of the worldsheet string model implies sensible spacetime field equations.

1.3.2 Superstrings and supergravity

Given the problems with bosonic string theory (tachyon field, no fermions, etc.), it is reasonable to see if there are other theories that are better behaved. There are, and they are called superstring theories. They can be obtained by including world-sheet fermions (in the Neveu-Schwarz-Ramond (NSR) formalism) and then imposing a projection condition. This results in a tachyon-free spectrum which also includes spacetime fermions. In fact the spectrum is spacetime supersymmetric¹¹. Furthermore, the spacetime is now required to be ten-dimensional.

There are five distinct (perturbative) string theories called type I, type IIA, type IIB, $E_8 \times E_8$ heterotic and $SO(32)$ heterotic. The type I theory contains both open and closed strings whereas the others only contain closed strings. It is believed that the superstring theories are consistent theories of quantum gravity, at least in perturbation theory.

At low energies superstring theories are described by supergravity (possibly coupled to ten-dimensional Yang-Mills theory). In this thesis we shall be mainly concerned with the type IIB theory, which is described at low-energies by Type IIB supergravity. It has $\mathcal{N} = 2$ supersymmetry in ten-dimensions (giving 32 supercharges) with the supersymmetry generated by two Majorana-Weyl spinors of the same chirality. The type IIB supergravity field equations can be derived from the

¹¹One can also use the Green-Schwarz formalism where spacetime supersymmetry is manifest. We shall use this in chapter 4.

following action (ignoring all fermion fields):

$$\begin{aligned}
S_{\text{IIB}} = & \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \left(R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} |H_3|^2 \right) \\
& - \frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2} |\tilde{F}_5|^2 \right) \\
& - \frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3 ,
\end{aligned} \tag{1.42}$$

where

$$\begin{aligned}
\tilde{F}_3 &= F_3 - C_0 \wedge H_3 , \\
\tilde{F}_5 &= F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3 .
\end{aligned} \tag{1.43}$$

Here, we are in the ‘string frame’ so that the Einstein-Hilbert term involves a factor of the dilaton, Φ . In ‘Einstein frame’ this term is of the standard form, and perturbative string calculations imply Newton’s constant is given by

$$16\pi G_N = 2\kappa_{10}^2 g_s^2 = (2\pi)^7 \alpha'^4 g_s^2 , \tag{1.44}$$

where the string coupling is given in terms of the asymptotic value of the dilaton, $g_s = e^{\Phi_0}$. Together with the metric and the dilaton, the two-form gauge field B_2 is in the NS-NS sector of the theory. Its field strength is $H_3 = dB_2$. The other fields in the action are R-R $(p+1)$ -form gauge fields $C_{(p+1)}$, which have field strengths $F_{(p+2)} = dC_{(p+1)}$. There is a condition on C_4 that is not implied by the equations of motion but must put in by hand — it has self-dual¹² field strength, $F_5 = \star F_5$.

1.3.3 p -branes

Having seen that type IIB supergravity contains gauge fields of various ranks, it is reasonable to ask whether one can describe objects that are charged under those fields. If we consider the familiar case of a point particle, then it couples to a gauge field A_μ via

$$S = \mu \int_{\mathcal{M}_1} \mathcal{P}[A] , \tag{1.45}$$

¹²Here, for a p -form ω , $\star\omega$ is the corresponding Hodge-dual $(10-p)$ -form.

where \mathcal{M}_1 is the worldline of the particle and here $\mathcal{P}[A]$ is the pullback of A_μ onto \mathcal{M}_1 . Note that equation (1.40) shows that the fundamental string couples to the field $B_{\mu\nu}$ in a natural generalization of (1.45), showing that it is a source for this field. Similarly, for the R-R $C_{(p+1)}$ fields one considers an object with p spatial directions called a p -brane. It has a $(p+1)$ -dimensional worldvolume, $\mathcal{M}_{(p+1)}$, and the coupling is given by

$$S = \mu_p \int_{\mathcal{M}_{p+1}} \mathcal{P}[C_{(p+1)}] . \quad (1.46)$$

It is reasonable to ask whether one can find corresponding p -brane solutions in supergravity. One can, and (in the case of the extremal p -brane solution) the metric and dilaton are given by [37, 45–47]

$$\begin{aligned} ds^2 &= H(r)^{-1/2} \left(-dt^2 + \sum_{i=1}^p dx^i dx^i \right) + H(r)^{1/2} \sum_{a=1}^{9-p} dr^a dr^a , \\ e^\Phi &= g_s H(r)^{\frac{3-p}{4}} , \end{aligned} \quad (1.47)$$

where

$$H(r) = 1 + \frac{L^{7-p}}{r^{7-p}} , \quad L^{7-p} = 2^{5-p} \pi^{\frac{5-p}{2}} \Gamma\left(\frac{7-p}{2}\right) g_s N l_s^{7-p} . \quad (1.48)$$

Here, N is the charge of the p -brane located at $r = 0$, given by

$$N = \int_{S^{8-p}} \star F_{(p+2)} . \quad (1.49)$$

The ‘extremal’ p -brane solution is invariant under half of the 32 supersymmetries and is therefore a BPS solution. The supersymmetry algebra then fixes the tension, T_p , of the brane in terms of its charge, which can be directly computed using standard methods from the fields given in (1.47) to be

$$T_p = \frac{N}{g_s} \mu_p = \frac{N}{g_s (2\pi)^p \alpha'^{\frac{p+1}{2}}} . \quad (1.50)$$

1.3.4 D-branes

String theory is relatively well understood in the limit where g_s is small, where one considers a genus expansion in string worldsheets. It is then interesting to consider what happens to the above supergravity solutions for $g_s \rightarrow 0$. One can see that the metric approaches that of flat space, except on the $(p+1)$ -dimensional hyperplane

at $r = 0$ where it appears to be singular. In this limit the p -brane reduces to a localized defect in flat space. Closed strings propagate as normal except near the brane. It is then interesting to consider whether the brane can be described directly in terms of how strings propagate near the brane.

It is possible to do this by introducing open strings. One defines a ‘D p -brane’ to be a $(p + 1)$ -dimensional hyperplane (with p spatial dimensions) where open strings can end. In the directions along the brane the strings obey standard open string Neumann boundary conditions, whereas in the directions transverse to the brane the string endpoints are constrained to be on the D-brane and obey Dirichlet boundary conditions. In this way one can consistently introduce open strings into a closed string background that describe the dynamics of the D-brane.

D-branes appear in the type I theory and both type II theories. In the case of the type IIB theory, there are supersymmetric D p -branes¹³ for $p = -1, 1, 3, 5, 7$ that break half the supersymmetry and are therefore BPS states¹⁴.

In a key calculation, Polchinski [48] showed using string worldsheet techniques that D-branes act as sources for closed string fields. In particular they have a tension (they act as gravitational sources) and they carry charges for R-R fields. A single D-brane has tension τ_p and charge μ_p given by

$$\tau_p = \frac{\mu_p}{g_s} = \frac{1}{g_s (2\pi)^p \alpha'^{\frac{p+1}{2}}}. \quad (1.51)$$

so that comparing to the expression (1.50) we can see that the supergravity solution corresponds to N D-branes. Furthermore, Polchinski pointed out that the charges obey the correct Dirac quantization condition and that the quantization is such that a D-brane carries the minimal unit of charge.

D-branes are dynamical objects in that they interact with the background closed string fields. In particular, one can write down the Dirac-Born-Infeld-Wess-Zumino

¹³The D(-1)-brane is a type of instanton.

¹⁴One can also consider D-branes with other dimensionalities. These are non-BPS branes and are unstable.

action for a D-brane in a non-trivial background [49, 50]:

$$\begin{aligned}
 S = & -\tau_p \int_{\mathcal{M}_{(p+1)}} d^{p+1}\xi e^{-\Phi} \sqrt{-\det(G_{ab} + \mathcal{F}_{ab})} \\
 & + \mu_3 \int_{\mathcal{M}_{(p+1)}} \mathcal{P} \left[\sum_n C_{(n+1)} \right] \wedge e^{\mathcal{F}}.
 \end{aligned} \tag{1.52}$$

Here $\mathcal{F} = \mathcal{P}[B] + 2\pi\alpha'F$, where F is the field strength of a $U(1)$ gauge field that propagates on the brane. (In the second term the sum over forms of differing rank is to be understood in sense that one integrates using the forms of rank $(p+1)$. In the type IIA theory n is even, while for type IIB n is odd.) The form of the second term (the Wess-Zumino term) can be understood from various points of view including anomaly arguments and T-duality. Furthermore, this action can be generalized to a supersymmetric form [51, 52], and when added to the supergravity action generates the correct source term for the p -brane solutions.

Upon quantization of the open strings that end on a single D-brane one finds the Dirac-Born-Infeld action which reduces at low energies to a supersymmetric $U(1)$ Yang-Mills theory that can be thought of ‘living’ on the brane. In particular there are $(9-p)$ scalar fields that describe the embedding of the brane in spacetime.

Concentrating on the case of D3-branes in type IIB supergravity, the low-energy dynamics of the brane is governed by $\mathcal{N} = 4$ $U(1)$ super-Yang-Mills theory, that we saw earlier. (Since the D-brane leaves 16 unbroken supercharges, one can see that the field theory must be $\mathcal{N} = 4$ supersymmetric.) This is consistent because the $\mathcal{N} = 4$ supersymmetry requires that there be six scalar fields X^I to fill out the gauge multiplet (see table 1.1) — these correspond to the six transverse directions of the brane. In particular, the vevs $\langle X^I \rangle$ of the fields X^I give the position of the brane in the transverse space.

We can also consider the case of N parallel D3-branes [53]. If they are all well-separated then the low-energy degrees of freedom arising from the open strings are N copies of the $\mathcal{N} = 4$ $U(1)$ gauge theory described above, one for each brane. Now consider the case when all the branes are coincident, *i.e.* they are all located at the same point in the transverse space. Then there are additional massless degrees of freedom arising from open strings that start on one brane but end on another. One finds that the field theory that governs the dynamics of the branes is now a non-

abelian gauge theory — $\mathcal{N} = 4$ super-Yang-Mills with gauge group $U(N)$. In this case the fields are in the adjoint representation and one can think of the scalar fields X^I as being $N \times N$ matrices. As described in section 1.2.4 the moduli space is given by mutually commuting matrices X^I which can then be put into diagonal form¹⁵. The diagonal entries then give the positions of the branes. Because the branes are separated the open strings that run between different branes become massive (with mass proportional to their length). But from the point of view of the gauge theory, this is just the Higgs mechanism where the gauge group is broken and some of the gauge bosons become massive. One can also observe that the metric on moduli space is flat (because the branes are in flat space) as it should be from requiring $\mathcal{N} = 4$ supersymmetry.

1.3.5 Dualities and M-theory

D-branes have been crucially important in understanding how the five perturbative string theories are related non-perturbatively by dualities¹⁶. We will not need to review many of the conjectured dualities for our purposes, but we shall mention a few that relate to theories that appear in this thesis.

Type IIB string theory is conjectured to have an $SL(2, \mathbb{Z})$ duality group that acts on $\tau = 2(C_0 + ig_s^{-1})$ as

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}. \quad (1.53)$$

The S-duality transformation is given by $\tau \rightarrow -1/\tau$, which for $C_0 = 0$ implies $g_s \rightarrow 4/g_s$. Therefore, this is a strong-weak duality transformation in the string coupling constant. On the other fields of type IIB supergravity¹⁷, the metric and C_4 are invariant under S-duality while the B_2 and C_2 fields are interchanged. In terms of branes this implies that the D3-brane is invariant while the fundamental string and the D1-brane (or D-string), which are sources for B_2 and C_2 as previously

¹⁵Here we are ignoring any remaining (discrete) gauge transformations.

¹⁶For a review see [54].

¹⁷Type IIB supergravity is actually invariant under $SL(2, \mathbb{R})$, which is broken when one considers branes (because they carry quantized charges).

stated, are interchanged. The D5-brane, which couples to C_6 (where $dC_6 = \star dC_2$) is interchanged with the NS5-brane which couples to B_6 (where $dB_6 = \star dB_2$).

If this type of $SL(2, \mathbb{Z})$ duality seems familiar, it is because we have seen it before in section 1.2.4 where it was a duality of the $\mathcal{N} = 4$ gauge theory. One can now see that the gauge theory duality can be realized by applying the string duality to a set of N parallel D3-branes in flat space. Monopoles and W-bosons are interchanged by S-duality because, from the string theory viewpoint, strings and D1-branes are interchanged. Applying string theory dualities to various configurations of D-branes is often the easiest way to visualize conjectured field theory dualities¹⁸.

Another remarkable string theory duality is that type IIA string theory is equivalent to an eleven-dimensional theory, called M-theory, compactified on a circle. A fully satisfactory formulation of M-theory is not yet known, although it is known that at low-energies M-theory should reduce to eleven-dimensional supergravity. Since this supergravity only has a single gauge field A_3 , it has p -brane solutions for $p = 2, 5$ called M2-branes and M5-branes. The worldvolume theory of N coincident M2-branes is a three-dimensional superconformal field theory, which can be realized as the infra-red fixed point of three-dimensional pure $SU(N)$ $\mathcal{N} = 8$ supersymmetric gauge theory. The worldvolume theory of N coincident M5-branes is a superconformal theory with $(2, 0)$ supersymmetry in six dimensions, a strong coupling completion of five-dimensional $SU(N)$ gauge theory with sixteen supercharges.

1.4 The AdS-CFT correspondence

In the previous sections we have seen that string theory contains D-branes and that they can be described in quite different ways. On one hand they can be seen to be surfaces where open strings can end and give rise to gauge theories. On the other hand we have a description in terms of p -brane solutions of supergravity. Given this, it is possible to relate the two descriptions directly and argue that, in a certain

¹⁸A good review of brane constructions of supersymmetric field theories is [55].

limit, they are in fact equivalent¹⁹.

In order to do this, let us return to the configuration of N coincident D3-branes in Minkowski space. In this case there are closed strings propagating in the ten-dimensional spacetime. There are also open strings with their end-points restricted to lie on the D3-branes.

It is interesting to take the low-energy limit of this system where we only need to consider the dynamics of the massless modes. In order to do this it is sometimes easier to fix energies and let $\alpha' \rightarrow 0$ (*i.e.* send the string length to zero) keeping dimensionless quantities, such as N and g_s , fixed. The closed string massless excitations give rise to supergravity on flat space. The massless excitations of the open strings give $\mathcal{N} = 4 U(N)$ gauge theory at low energies²⁰. Furthermore, in this limit the two sectors decouple leaving $\mathcal{N} = 4$ super-Yang-Mills theory on the branes and free gravity in the bulk.

Now let us consider this same limit, but this time from the point of view of the supergravity p -brane. In the case of $p = 3$ the extremal p -brane metric becomes

$$ds^2 = H(r)^{-1/2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + H(r)^{1/2} \sum_{a=1}^6 dr^a dr^a, \quad (1.54)$$

where

$$H(r) = 1 + \frac{L^4}{r^4}, \quad L^4 = 4\pi g_s N \alpha'^2. \quad (1.55)$$

and the dilaton is now constant $e^\Phi = g_s$. Since G_{tt} is not constant, the energy of an object as measured by an observer at some position r , E_r say, is different to the energy measured by an observer at infinity, E . The two are related by

$$E = E_r H^{-1/4}(r), \quad (1.56)$$

so that as an object approaches $r = 0$ it has lower and lower energy as observed from infinity, due to the redshift factor. So now we can consider the low-energy limit in this system. There are two types of low-energy excitation. One is gravity

¹⁹Reviews of the derivation of the AdS-CFT correspondence can be found in [11, 37].

²⁰The diagonal $U(1)$ factor in $U(N) = SU(N) \times U(1)$ corresponds to the overall position of the branes and decouples from the rest of the gauge theory dynamics. We shall ignore it in the following.

propagating far from the brane on flat space. The other is *any* excitation near to $r = 0$, including not only supergravity modes but also higher string excitations. Furthermore, these two types of excitations decouple from each other as $\alpha' \rightarrow 0$ (see [37] and references therein). Therefore, we have at low energies a decoupled system consisting of gravity in ten-dimensional Minkowski space and superstring theory in the near-horizon limit of (1.54). One might imagine that this limit does not make sense because the p -brane solution is singular at $r = 0$. However, the near-horizon geometry of the D3-brane solution is in fact regular and for $r \ll L$ the metric is given by

$$ds^2 = \frac{r^2}{L^2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{L^2}{r^2} dr^2 + L^2 d\Omega_5^2, \quad (1.57)$$

which is the metric of the direct product of five-dimensional anti-de Sitter space with a five-sphere, or $\text{AdS}_5 \times S^5$.

Therefore we now have two different descriptions of the $\alpha' \rightarrow 0$ limit in terms of decoupled systems. On one side we have $\mathcal{N} = 4$ $SU(N)$ Yang-Mills theory plus ten-dimensional gravity in flat space, while we have just seen that from considering the extremal p -brane solution one has superstring theory in the near-horizon region plus gravity in ten-dimensional flat space. This led Maldacena [7] to propose that $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills theory is dual (or equivalent) to type IIB string theory propagating on $\text{AdS}_5 \times S^5$. The parameters of the two theories are related by

$$g_{\text{YM}}^2 = 2\pi g_s, \quad L^4 = 2(g_{\text{YM}}^2 N) \alpha'^2. \quad (1.58)$$

When we take the $\alpha' \rightarrow 0$ limit on the near-horizon geometry we want to keep all the string excitations in the low-energy theory, due to the redshift factor. This implies that $E_r \sim 1/l_s$, *i.e.* $\sqrt{\alpha'} E_r$ should remain fixed. However, from (1.56) one can see that $E \sim E_r r / \sqrt{\alpha'}$, so in order to keep E (the energy as measured from infinity) fixed, we need to keep r/α' fixed. Therefore, we can take the near-horizon limit of (1.54) by defining a new coordinate $u = r/\alpha'$

$$ds^2 = \alpha' \left[\frac{u^2}{\sqrt{4\pi g_s N}} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \sqrt{4\pi g_s N} \frac{du^2}{u^2} + \sqrt{4\pi g_s N} d\Omega_5^2 \right]. \quad (1.59)$$

This makes sense in gauge theory terms because a brane located at r corresponds to giving a vev to the scalar fields. Because gauge theory quantities should remain constant in the decoupling limit, the mass of the W-boson should remain constant. But this mass is just the mass of a string stretched from the brane to $r = 0$, which is proportional to r/α' .

The strongest form of the Maldacena conjecture is that the equivalence between $\mathcal{N} = 4$ $SU(N)$ Yang-Mills and string theory on $AdS_5 \times S^5$ should hold for all values of N and g_s . However, at present string quantization on the $AdS_5 \times S^5$ background is not well understood. Therefore one usually takes certain limits to make comparisons between the two theories.

The 't Hooft limit [56] is given by taking $N \rightarrow \infty$ while keeping $\lambda = g_{YM}^2 N$ (the 't Hooft coupling) fixed. This limit is well defined in perturbative Yang-Mills theory and 't Hooft showed that the series of Feynman diagrams can be arranged in terms of topology, in analogy to the expansion in worldsheet genus one has in perturbative string theories. In terms of string theory on $AdS_5 \times S^5$ one has $g_s N$ held fixed as $N \rightarrow \infty$ so $g_s \rightarrow 0$. Therefore, the 't Hooft limit corresponds to a classical string theory limit where string loop corrections are suppressed.

Once we have taken the 't Hooft limit, the only available parameter is the 't Hooft coupling $\lambda = g_{YM}^2 N$. This is the effective coupling constant (rather than g_{YM}^2) in Yang-Mills perturbation theory. For perturbation theory to be a good approximation, we should have $\lambda \ll 1$. In contrast, for classical supergravity to be a good approximation to string theory we require $L \gg l_s$, which implies $\lambda \gg 1$. In this way we can see that perturbative field theory and supergravity are good descriptions in completely different regimes. Therefore the correspondence is a duality in the sense that it relates the strong-coupling limit of one theory to the weak-coupling limit of another.

1.4.1 Matching symmetries

Having argued that two theories that appear so different are in fact equivalent, there are various checks that one can make. Perhaps the most obvious one is to make sure that the theories have the same symmetries.

Firstly, as we mentioned before, the $\mathcal{N} = 4$ Yang-Mills theory is in fact invariant under the conformal group $SO(2, 4)$. This symmetry arises on the gravity side because it is the isometry group of AdS_5 . Similarly, the $SU(4) \simeq SO(6)$ R-symmetry acts as the obvious symmetry of the S^5 .

From the gauge theory viewpoint the obvious $\mathcal{N} = 4$ supersymmetries of the gauge theory are supplemented by extra fermionic generators of the superconformal group $SU(2, 2|4)$ to give 32 conserved supercharges in all. It turns out that $\text{AdS}_5 \times S^5$ is a maximally supersymmetric solution of type IIB supergravity and also has symmetry group $SU(2, 2|4)$. Therefore, the symmetries of the two theories match.

1.4.2 The field-operator correspondence

Having seen that the symmetries of the two theories match up, it is reasonable to ask whether the various representations of the symmetry group on either side of the correspondence match as well.

It was shown by Gubser, Klebanov & Polyakov [8] and Witten [9] that one could match operators in the gauge theory to fields in the gravitational description. To see this, let us consider Euclidean AdS_5 given by $z_0 > 0$ with metric

$$ds^2 = \frac{1}{z_0^2} (dz_0^2 + d\vec{x}^2) . \quad (1.60)$$

This metric diverges at $z_0 = 0$ which is the boundary of the anti-de Sitter space. It turns out that one should think of the field theory as ‘living’ on this boundary [9]. The radial coordinate of AdS space, *e.g.* u defined previously, corresponds to an energy scale. (This can be seen from the action of the conformal group.) Therefore, the UV limit of the gauge theory corresponds to large u (near to the boundary) while $u \rightarrow 0$ is the IR limit²¹.

Consider a scalar field $\varphi(z, y)$ where z^μ are coordinates on AdS_5 and y^i are coordinates on S^5 . Then one can decompose φ in terms of spherical harmonics on S^5 :

$$\varphi(z, y) = \sum_{k_1, k_2, k_3} \varphi_{k_1 k_2 k_3}(z) Y_{k_1 k_2 k_3}(y) . \quad (1.61)$$

²¹For some papers on the ‘UV-IR’ correspondence see [57–59].

The scalar fields $\varphi_{k_1 k_2 k_3}$ are then massive scalar fields on AdS_5 with $m^2 = m^2(k_i)$. By considering the asymptotic behaviour of solutions of the scalar field equation, one can match a scalar field on AdS_5 with mass m to an operator in the gauge theory with scaling dimension Δ by

$$\Delta = 2 \pm \sqrt{4 + m^2} . \quad (1.62)$$

One finds when decomposing the type IIB supergravity fields according to (1.61) that negative values of m^2 are obtained. At first sight this signals an instability in the theory. However, for stability in AdS space one requires that $m^2 \geq -4$ (this is called the Breitenlohner-Freedman bound [60, 61]). Therefore the dimension Δ is always real as it should be for a unitary theory.

Using this one can match the spectrum of type IIB supergravity on $\text{AdS}_5 \times S^5$ to operators which are in chiral (or short) representations. These operators have dimensions that are given in terms of their $SO(6)$ representation and so we can be sure of their dimensions even at strong coupling. Their presence in the supergravity description at large λ is therefore a strong test of the correspondence. Furthermore, higher string states will have dimensions of order $\lambda^{1/4}$ and so for large λ the corresponding operators are not expected to be in short representations of the superconformal algebra.

For a scalar field with mass m (corresponding to an operator with conformal dimension Δ) there are generically two independent asymptotic solutions for $z_0 \rightarrow 0$. One is normalizable while the other is non-normalizable. A non-zero normalizable mode corresponds to giving a vacuum expectation value to the corresponding operator, while the non-normalizable mode corresponds to an insertion of source for an operator in the theory.

These ideas can be generalized to asymptotically AdS supergravity solutions [35, 62–83] by considering, for instance, supergravity solutions with non-zero values for scalar fields at large u (*i.e.* in the UV). The evolution of the supergravity fields with u then corresponds to a renormalization group flow, since u corresponds to an energy scale.

In addition to matching fields to operators, Gubser *et al.* and Witten also proposed a method to calculate gauge theory correlation functions from the classical

supergravity. The basic idea is that the generating functional for correlation functions in the gauge theory is given in terms of the full string theory partition function:

$$\langle e^{\int d^4x \phi_0(\vec{x}) \mathcal{O}(\vec{x})} \rangle_{\text{CFT}} = \mathcal{Z}_{\text{string}} \left[\phi(z_0, \vec{x}) \Big| \phi(0, \vec{x}) = \phi_0(\vec{x}) \right]. \quad (1.63)$$

On the left-hand side ϕ_0 acts as source for a particular operator \mathcal{O} . On the right-hand side ϕ_0 is a boundary value for the corresponding scalar field. Of course the full partition function is not known, so in the classical supergravity approximation one uses $\mathcal{Z} \simeq \exp(-I_{\text{sugra}})$. (In fact the prescription given in (1.63) is not well-defined in that one needs to use a cutoff.) One evaluates this on the solution of the supergravity field equations, subject to the boundary conditions. Reviews of the calculation of correlation functions using the above method can be found in [11, 37].

Chapter 2

Probing Holographic Renormalization-Group Flows

2.1 Probing AdS backgrounds with D3-branes

The AdS-CFT correspondence has given us remarkable new calculational methods with which to study strongly-coupled gauge theory. However, as mentioned in the introduction, in its current form it is limited in various ways. One of the most difficult problems is that currently we are unable to quantize strings in the $\text{AdS}_5 \times S^5$ background. Therefore, many of the tests and predictions that have been made are based on the supergravity approximation (large λ). Indeed, this is the regime that we will use throughout this chapter.

Another important problem is to extend the correspondence to more realistic theories — non-conformal theories with less or no supersymmetry. A step in this direction has been to study backgrounds that correspond to deformations of the $\mathcal{N} = 4$ superconformal gauge theory. One reason for this is that properties of the new supergravity duals can easily be compared to those of the original $\text{AdS}_5 \times S^5$. Furthermore, these deformations are often to $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetric field theories where some strong-coupling predictions can be made using field theoretic techniques.

Many of these supergravity duals have been found, although sometimes it is rather unclear how to test them or how to make predictions of the field theory at

strong coupling. The aim of the research presented in this chapter is to try to understand how some field theory physics can be easily extracted from the supergravity solution.

The method employed here is very straightforward. The idea is that since $\text{AdS}_5 \times S^5$ is the near-horizon geometry of N D3-branes (where N is large), and that $SU(N)$ gauge theory governs the dynamics of those branes, useful information about the gauge theory can be extracted by considering the dynamics of a “probe” D3-brane in the ten-dimensional bulk theory. By saying that the brane is a probe, we mean that we can neglect its back-reaction on the spacetime. The results obtained from this method have a very direct interpretation in the dual field theory.

We will consider a variety of geometries that are asymptotically $\text{AdS} \times M$ (for some Euclidean space M), and can be thought of as being “sourced” by branes. The probing method has been very successful in correctly understanding a variety of proposed supergravity duals. In particular the enhançon mechanism [84–90] has resolved various spacetime singularities by relating them to the low-energy physics of $\mathcal{N} = 2$ supersymmetric gauge theories. We shall study this further in chapter 3.

2.1.1 The generalized Born-Infeld action

The low-energy effective action for a single D3-brane is given by a generalized form of the Born-Infeld action:

$$S = -\tau_3 \int_{\mathcal{M}_4} d^4\xi \left[-\det(G_{ab} + e^{-\Phi/2} \mathcal{F}_{ab}) \right]^{1/2} + \mu_3 \int_{\mathcal{M}_4} \left(C_{(4)} + C_{(2)} \wedge \mathcal{F} + \frac{1}{2} C_{(0)} \mathcal{F} \wedge \mathcal{F} \right). \quad (2.1)$$

Here $\mathcal{F}_{ab} = B_{ab} + 2\pi\alpha' F_{ab}$, and \mathcal{M}_4 is the world-volume of the D3-brane, with coordinates ξ^0, \dots, ξ^3 . Also, G_{ab} and B_{ab} are the pull-backs of the ten dimensional metric (in Einstein frame) and the NS-NS two-form potential, respectively. They are defined as *e.g.*:

$$G_{ab} = G_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b}. \quad (2.2)$$

The other fields that appear in the action are the dilaton Φ , and the R-R p -form gauge fields $C_{(p)}$. As usual, the parameters μ_3 and τ_3 are the basic [48] R-R charge

and tension of the D3-brane:

$$\mu_3 = \tau_3 g_s = \frac{1}{(2\pi)^3 (\alpha')^2} . \quad (2.3)$$

In this chapter we will be studying solutions of type IIB supergravity with the following general form

$$ds_{10}^2 = \Omega^2 (e^{2A(r)} \eta_{ab} dx^a dx^b + dr^2) + ds_5^2 . \quad (2.4)$$

This is a warped product of AdS₅ and a five-dimensional compact space¹ that is parametrized by five coordinates ψ_i , $i = 1, \dots, 5$. This means that the “warp factor” Ω^2 and the metric components G_{ij} can be functions of r and ψ_i .

The strategy we shall employ is to insert the form of the supergravity solution into the Born-Infeld action and write it in terms of the transverse scalars y^m and the field strength F_{ab} . These both depend on the worldvolume coordinates ξ^a . We can then write the action as a derivative expansion, only keeping the potential and kinetic terms. Physically, this corresponds to only considering a slowly-moving D3-brane probe.

First of all we partition the spacetime coordinates, x^μ , as $x^a = \{x^0, x^1, x^2, x^3\}$, and $y^m = \{r, \psi_i\}$. We choose the following gauge²

$$x^a = \xi^a , \quad y^m = y^m(x^a) . \quad (2.5)$$

This is a natural choice because the x^a are coordinates on the Minkowski space that the dual $SU(N)$ gauge theory is supposed to live on.

It turns out that the p -form fields in the supergravity solutions have the following general form:

$$\begin{aligned} C_{abcd}^{(4)} &= w(r, \psi_i) \varepsilon_{abcd} \\ C_{abci}^{(4)} &= C_{abij}^{(4)} = 0 \\ C_{ab}^{(2)} &= C_{ai}^{(2)} = 0 \\ B_{ab}^{(2)} &= B_{ai}^{(2)} = 0 \end{aligned} \quad (2.6)$$

¹This will often be a deformed five-sphere.

²This is more general than “static gauge” (where $y^m = y^m(x^0)$) but is explicitly Lorentz invariant and allows the possibility of studying interaction terms which are higher order in the derivative expansion.

Inserting the form of the background given in (2.4) and (2.6) into (2.1), and expanding to quadratic order in y^m and F_{ab} , one finds the following result:

$$\mathcal{L} = -\frac{\tau_3}{2}\Omega^2 e^{2A} G_{mn} \partial_a y^m \partial^a y^n - V(y^m) - \frac{1}{8\pi g_s e^\Phi} F_{ab} F^{ab} + \frac{C_{(0)}}{8\pi} F_{ab} \tilde{F}^{ab} \quad (2.7)$$

$$= -\frac{\tau_3}{2}\Omega^2 e^{2A} G_{mn} \partial_a y^m \partial^a y^n - V(y^m) - \frac{1}{4g_{\text{eff}}^2} F_{ab} F^{ab} + \frac{\theta_{\text{eff}}}{32\pi^2} F_{ab} \tilde{F}^{ab} . \quad (2.8)$$

One can see directly that the dilaton gives the effective coupling, g_{eff} , for the $U(1)$ gauge field on the probe. Similarly the axion, $C_{(0)}$, gives the effective theta angle. Specifically,

$$g_{\text{eff}}^2 = 2\pi g_s e^\Phi , \quad \theta_{\text{eff}} = 4\pi C_{(0)} . \quad (2.9)$$

The scalar fields, y^m , have a kinetic term and also a potential given by:

$$V(y^m) = \tau_3(\Omega^4 e^{4A} - w g_s) . \quad (2.10)$$

The Lagrangian takes this simple form because of the restrictions (2.4), (2.6) we have placed on the form of the supergravity solution. If we had allowed terms like $C_{abij}^{(4)} \neq 0$, then this would have resulted in extra terms quadratic in $\partial_a y^m$.

2.1.2 A D3-brane in $\text{AdS}_5 \times S^5$

Before we look at more complicated supergravity geometries, we should consider the standard $\text{AdS}_5 \times S^5$ case (1.57). The dilaton satisfies $\Phi = 0$, so that equation (2.9) implies that³

$$g_{\text{eff}}^2 = g_{\text{YM}}^2 = 2\pi g_s . \quad (2.11)$$

Therefore the effective (running) coupling is constant, as it should be for a conformal theory. The potential in the probe Lagrangian vanishes identically and so there is a six-dimensional moduli space (or Coulomb branch). This is the whole space transverse to the brane, parametrized by $\{r, \psi_i\}$. The kinetic term for these scalars gives the metric on moduli space (the metric “seen” by the probe):

$$ds^2 = \frac{1}{8\pi^2 g_{\text{YM}}^2} [dv^2 + v^2 d\Omega_5^2] , \quad \text{with} \quad v = \frac{L}{\alpha'} e^{r/L} , \quad (2.12)$$

³In fact we can set $\Phi = \text{constant}$, but this can always be absorbed into a redefinition of g_s .

where we have used the relations (2.11) and (2.3) and defined an energy scale v . Here, $d\Omega_5^2$ is the standard metric on S^5 , and the metric in (2.12) is the flat metric on \mathbb{R}^6 . All of these properties match those expected from the discussion in section 1.2.4.

One should note how the extended nature of the brane probe was required for the potential $V(y^m)$ to cancel. The second term in equation (2.10) arises because the D3-brane couples to the R-R four-form potential. Without this cancellation, the probe would not have a six-dimensional moduli space.

2.2 The holographic Leigh-Strassler RG flow

2.2.1 Vacua of $\mathcal{N} = 8$, $D = 5$ gauged supergravity

The field content of $\mathcal{N} = 8$, $D = 5$ gauged supergravity is a single graviton multiplet (see table 2.1). The 42 scalars parametrize the coset space $E_{6(6)}/USp(8)$ and the

Spin	Field	Number of Fields
2	$g_{\mu\nu}$	1
3/2	ψ_μ	8
1	A_μ	27
1/2	λ	48
0	φ	42

Table 2.1: The field content of $\mathcal{N} = 8$, $D = 5$ gauged supergravity.

scalar potential $V(\varphi)$ of the theory is invariant under the $SO(6)$ R-symmetry and also under $SL(2, \mathbb{R})$ (which is inherited from the $SL(2, \mathbb{R})$ symmetry of type IIB supergravity). Therefore V is a function of 24 independent variables. A critical point φ_c of $V(\varphi)$ gives a solution of the gauged supergravity satisfying [91]

$$R_{\mu\nu} = V_c g_{\mu\nu}, \quad (2.13)$$

so that V_c is the cosmological constant at the critical point. For the cases we are interested in $V_c < 0$ and the supergravity vacua are anti-de Sitter spaces.

In [64] $V(\varphi)$ was studied with the intention of finding new critical points φ_c . Finding local extrema of the full supergravity potential would be an extremely hard task and therefore the authors of [64] studied a sector of the scalar fields invariant under a particular $SU(2) \subset SU(4)_R$. This reduced the problem to studying the potential as a function of four independent variables.⁴ The results found in [64] are reproduced in table 2.2.

	Unbroken gauge symmetry	Unbroken supersymmetry	Perturbatively stable
(i)	$SO(6)$	$\mathcal{N} = 8$	Yes
(ii)	$SO(5)$	$\mathcal{N} = 0$	No
(iii)	$SU(3)$	$\mathcal{N} = 0$	No
(iv)	$SU(2) \times U(1) \times U(1)$	$\mathcal{N} = 0$	No
(v)	$SU(2) \times U(1)$	$\mathcal{N} = 2$	Yes

Table 2.2: Some known vacua of $D = 5$ gauged supergravity.

Case (i) in table 2.2 is simply the normal, maximally supersymmetric, AdS_5 solution (with all the scalar fields except for the dilaton and axion set to zero). Cases (ii) and (iii) were originally found in [91, 92] and were studied in terms of the AdS-CFT correspondence in [62, 63]. However, cases (ii–iv) are unstable (see [63, 93] and references within) since not all of the supergravity fields satisfy the Breitenlohner-Freedman bound. This means that the dual conformal field theory would have an operator with complex conformal dimension, violating unitarity. It seems as though these fixed points are a pathology of the supergravity description of large N gauge theory.

The final entry in table 2.2 is $\mathcal{N} = 2$ supersymmetric and therefore corresponds to an $\mathcal{N} = 1$ superconformal field theory. A natural question to ask is what can be said about the field theory dual to this new fixed point. This question was addressed in [35, 36]. Since the unbroken *gauge* symmetry in the five-dimensional

⁴Another reason for making this choice was that $\mathcal{N} = 2$ field theories have $SU(2)$ R-symmetry. Instead of finding an $\mathcal{N} = 2$ superconformal field theory Khavaev *et al.* found an $\mathcal{N} = 1$ superconformal field theory with an $SU(2)$ flavour symmetry.

theory is $SU(2) \times U(1)$ the corresponding field theory has a $SU(2) \times U(1)$ global symmetry. This should include an $U(1)$ R-symmetry because the field theory is $\mathcal{N} = 1$ supersymmetric.

These global symmetries led the authors of [35, 36] to conjecture that the new critical point of $D = 5$ gauged supergravity is dual to the Leigh-Strassler conformal fixed point described in section 1.2.5. The superconformal and global symmetries match exactly. They were able, however, to perform a quantitative check using a property of the new fixed point. As mentioned above, $V_c = V(\varphi_c)$ is the cosmological constant of the AdS vacuum at the critical point. In the AdS-CFT correspondence the cosmological constant of the bulk theory is related to the central charge of the corresponding field theory [8, 9, 94, 95]. In fact for two AdS spaces with cosmological constants Λ_A and Λ_B the following formula holds:

$$\frac{c_A}{c_B} = \left(\frac{\Lambda_A}{\Lambda_B} \right)^{-3/2}, \quad (2.14)$$

where $c_{A,B}$ are the central charges of the respective dual field theories. Applying this formula to the $\mathcal{N} = 2$ and $\mathcal{N} = 8$ supersymmetric critical points, Khavaev *et al.* found that

$$\frac{c_{\mathcal{N}=2}}{c_{\mathcal{N}=8}} = \frac{27}{32}. \quad (2.15)$$

This was reproduced exactly from the dual Leigh-Strassler field theory in [35, 36]. However, we should realize that this is only a consistency check that the $\mathcal{N} = 2$ critical point is dual to the Leigh-Strassler theory. The result in (2.15) can also occur for other supergravity solutions. For example, $AdS_5 \times T^{1,1}$ (which is dual to a theory with $SU(N) \times SU(N)$ gauge group [96]) gives exactly the same result.

2.2.2 RG flows from D=5 gauged supergravity

Given that the Leigh-Strassler field theory is naturally obtained after a renormalization group flow from $\mathcal{N} = 4$ super-Yang-Mills theory, it is sensible to ask whether it is possible to construct this flow in terms of five-dimensional gauged supergravity. Indeed, the supergravity solution which is the holographic RG flow to the Leigh-Strassler point was constructed in [35]. We shall summarize the results here.

Freedman *et al.* take the metric to have the following form

$$ds_{1,4}^2 = e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu + dr^2, \quad (2.16)$$

where $\eta_{\mu\nu}$ is the Minkowski metric.⁵ In this parametrization, AdS space has $A(r) = r/l$, with l the radius of curvature. They also allow the four scalars parametrizing the $SU(2)$ -invariant sector of the full scalar coset to be functions of r only, in order for the solution to have 4-dimensional Poincaré invariance. To preserve $\mathcal{N} = 1$ supersymmetry and retain a $U(1)$ R-symmetry along the flow, two of these scalars are set to zero. The other two, $\varphi_{1,3}$, satisfy⁶

$$\frac{d\varphi_j}{dr} = \frac{1}{L} \frac{\partial W}{\partial \varphi_j}, \quad j = 1, 3 \quad (2.17)$$

and the function $A(r)$ in the metric satisfies

$$\frac{dA}{dr} = -\frac{2}{3L} W, \quad (2.18)$$

where W is a “superpotential”,

$$W = \frac{1}{4\rho^2} [\cosh(2\varphi_1)(\rho^6 - 2) - (3\rho^6 + 2)]. \quad (2.19)$$

Here we have defined $\rho \equiv e^\alpha$ and $\alpha \equiv \frac{1}{\sqrt{6}}\varphi_3$. One property of W that can be noted immediately is that it is even in φ_1 . In fact the ten-dimensional solution found by Pilch & Warner [82] only depends on $|\varphi_1|$, so without loss of generality, we shall take $\varphi_1 \geq 0$.

The critical points occur at the following values of the scalars:

$$\mathcal{N} = 8 \text{ critical point: } \quad \varphi_1 = 0, \quad \varphi_3 = 0 \quad (2.20)$$

$$\mathcal{N} = 2 \text{ critical point: } \quad \varphi_1 = \frac{1}{2} \log(3), \quad \varphi_3 = \frac{1}{\sqrt{6}} \log(2) \quad (2.21)$$

Near $\varphi_j = 0$ one can expand W as a series:

$$W = -\frac{3}{2} - \frac{\varphi_1^2}{2} - \varphi_3^2 + \sqrt{\frac{8}{3}} \varphi_1^2 \varphi_3 + \dots \quad (2.22)$$

At a critical point of W , the mass matrix for the φ_j is related to the Hessian of W . The conformal dimensions of the corresponding field theory operators can then be found using equation (1.62):

⁵Note that we use the “mostly-plus” signature for the metric.

⁶Here, L is the radius of curvature of the $\mathcal{N} = 8$ AdS₅ solution.

Scalar	Mass ²	Δ	Operator
φ_1	-3	3	$\mathcal{O}_3 = \text{Tr}(\lambda_3 \lambda_3 + \phi_1[\phi_2, \phi_3]) + \text{h.c.}$
φ_3	-4	2	$\mathcal{O}_2 = -\sum_{i=1}^4 \text{Tr}(X_i^2) + 2\sum_{i=5}^6 \text{Tr}(X_i^2)$

This implies that φ_1 is dual to a fermion bilinear operator whereas φ_3 is dual to a scalar bilinear operator. This is what one would expect — in order to reach the Leigh-Strassler theory one has to switch on mass terms for λ_3 and $\phi_3 = X_5 + iX_6$. However, adding a positive multiple of the operator \mathcal{O}_2 to the Lagrangian would result in negative mass terms for X_1, X_2, X_3 and X_4 . What happens from a field theory point of view is that an appropriate amount of the Konishi operator $\mathcal{O}_K = \text{Tr}(\sum_i X_i^2)$ is added to \mathcal{O}_2 to ensure that the deformation preserves positivity and $\mathcal{N} = 1$ supersymmetry [77,83]⁷. Therefore the correct form for the scalar mass term

$$\frac{1}{3}(\mathcal{O}_2 + \mathcal{O}_K) = X_5^2 + X_6^2, \quad (2.23)$$

is added to the Lagrangian.

A similar analysis has been made at the $\mathcal{N} = 2$ fixed point [35]. Again, the eigenvalues of the mass matrix for φ_1 and φ_3 were calculated. The conformal dimensions for the corresponding operators were then found — they are $\Delta = 3 - \sqrt{7}$ and $\Delta = 3 + \sqrt{7}$. Since φ_1 and φ_3 are no longer mass eigenstates the operators are now dual to linear combinations of φ_1 and φ_3 . Furthermore, only one operator is now relevant ($\Delta < 4$) while the other is irrelevant ($\Delta > 4$).

It is also interesting to consider the asymptotic behaviour of $\varphi_{1,3}$ near the $\mathcal{N} = 8$ critical point for large r . Using (2.17) and (2.22) gives the following solutions [35]:

$$\begin{aligned} \varphi_1 &\simeq a_0 e^{-r/L} \\ \varphi_3 &\simeq \sqrt{\frac{8}{3}} a_0^2 \frac{r}{L} e^{-2r/L} + a_1 e^{-2r/L} \end{aligned} \quad (2.24)$$

This is consistent with the discussion in section 1.4.2. For $a_0 \neq 0$, $\varphi_{1,3}$ are non-normalizable modes and therefore correspond to adding mass terms to the field

⁷This is very interesting since the Konishi operator is not dual to a field in gauged (or even type IIB) supergravity, but is dual to a string state and has large conformal dimension in the supergravity (large λ) limit.

theory Lagrangian. However, depending on the choice of a_1 , a particular scalar VEV can be switched on as well. We shall discuss this a bit later in section 2.2.4. First of all we shall consider the holographic RG flow to the Leigh-Strassler conformal field theory, *i.e.* switching on a mass for Φ_3 without setting any scalar VEVs to be non-zero.

2.2.3 The pure mass deformation

We now wish to study the holographic RG flow from the $\mathcal{N} = 8$ fixed point to the $\mathcal{N} = 2$ fixed point described above. In the language of the renormalization group, this will involve switching on relevant operators in the UV (*i.e.* perturbing the field theory Lagrangian) and then evolving down in energy to the IR. In order to approach a fixed point in the IR, the operator governing the RG flow will then need to be irrelevant there.

From now on we will find it easier to write in terms of slightly different variables in order to match the literature. They are

$$\chi(r) \equiv \varphi_1(r) , \quad (2.25)$$

$$\rho(r) \equiv e^{\alpha(r)} , \quad \text{where } \alpha(r) \equiv \frac{1}{\sqrt{6}}\varphi_3(r) . \quad (2.26)$$

The flow equations in (2.17) and (2.18) are then:

$$\begin{aligned} \frac{d\rho}{dr} &= \frac{1}{6L}\rho^2\frac{\partial W}{\partial\rho} = \frac{1}{6L\rho}(\rho^6(\cosh(2\chi) - 3) + \cosh(2\chi) + 1) \\ \frac{d\chi}{dr} &= \frac{1}{L}\frac{\partial W}{\partial\chi} = \frac{1}{2L\rho^2}((\rho^6 - 2)\sinh(2\chi)) \\ \frac{dA}{dr} &= -\frac{2}{3L}W = -\frac{1}{6L\rho^2}(\rho^6(\cosh(2\chi) - 3) - 2(\cosh(2\chi) + 1)) , \end{aligned} \quad (2.27)$$

for which no explicit analytic solution is known. However, they can be studied numerically.

Rewriting (2.24) in terms of α and χ , one finds that we will require for $r \rightarrow \infty$:

$$\chi(r) \rightarrow a_0 e^{-r/L} + \dots , \quad \alpha \rightarrow \frac{2a_0^2}{3} \frac{r}{L} e^{-2r/L} + \frac{a_1}{\sqrt{6}} e^{-2r/L} + \dots \quad (2.28)$$

In order to study the required flow numerically, one might make various choices for the initial conditions, (a_0, a_1) , and then find the corresponding solution using

(2.27). However, one doesn't know the correct initial conditions required to reach the $\mathcal{N} = 2$ fixed point as $r \rightarrow \infty$. In fact physically inequivalent solutions of (2.27) are parametrized by

$$\hat{a} = \frac{a_1}{a_0^2} + \sqrt{\frac{8}{3}} \log a_0. \quad (2.29)$$

Even with this knowledge, finding the correct numerical value for \hat{a} is quite hard. Instead, it is much easier to start at the infra-red fixed point (as $r \rightarrow -\infty$) and integrate up to the ultra-violet. Since we know that the flow should approach the IR fixed point with an irrelevant operator, the functions $(\alpha(r), \chi(r))$ should approach the fixed point from the direction corresponding to the $\Delta = 3 + \sqrt{7}$. This implies as $r \rightarrow -\infty$:

$$\begin{aligned} \chi(r) \rightarrow \frac{1}{2} \log 3 - b_0 e^{\lambda r/L} + \dots, \quad \alpha(r) \rightarrow \frac{1}{6} \log 2 - \frac{\sqrt{7}-1}{6} b_0 e^{\lambda r/L} + \dots, \\ \text{where } \lambda = \frac{2^{5/3}}{3} (\sqrt{7} - 1). \end{aligned} \quad (2.30)$$

Using this, one can then easily find the numerical solution and in fact find that $\hat{a}_c \simeq -1.4694$ corresponds to the flow to the fixed point.

2.2.4 The mass deformation with non-zero vevs

The critical value [35] $\hat{a}_c \simeq -1.4694$ represents the particular flow which starts out at the $\mathcal{N} = 8$ critical point and ends precisely on the $\mathcal{N} = 2$ critical point. In [77] it was proposed that the solutions with $\hat{a} > \hat{a}_c$ describe the gauge theory at different points on the Coulomb branch of moduli space. This makes sense because from (2.24) one can see that it is possible to vary a_1 which controls a normalizable contribution to φ_3 . In fact the choice of \hat{a} controls the vev to be switched on with the mass deformation, with $\hat{a} = \hat{a}_c$ specifying zero vev, *i.e.* the Leigh-Strassler conformal fixed point (specifying a vev would break conformal invariance at the very least).

It is important to note that the flows with $\hat{a} \neq \hat{a}_c$ terminate at some finite value of r , r_0 say. In particular they are not defined for $r < r_0$. This is because at least one of the supergravity scalar fields diverges as $r \rightarrow r_0^+$. We shall see later that in terms of the probe calculation, we will need to consider the geometry at $r = r_0$.

The solutions with $\hat{a} < \hat{a}_c$ have the property that the gauged supergravity scalar potential is not bounded above. The criterion suggested in [77] implies that they

are unphysical. In fact these flows correspond to giving a vev to $\text{Tr}(X_5^2 + X_6^2)$, *i.e.* giving a vev to massive scalars. This is neither a sensible vacuum state nor supersymmetric.

On the other hand, if one chooses $\hat{a} > \hat{a}_c$ then the supergravity potential is bounded above and the solutions are physical. These correspond to giving a vev to $\text{Tr}(X_1^2 + X_2^2 + X_3^2 + X_4^2)$, which is a perfectly valid choice of vacuum state on the full moduli space of the $SU(N)$ Leigh-Strassler theory. Some different flows, including the flow to the fixed point are presented in figure 2.1.

It will be very interesting to see later in section 2.4 how the probe calculation can determine between physical and unphysical RG flows.

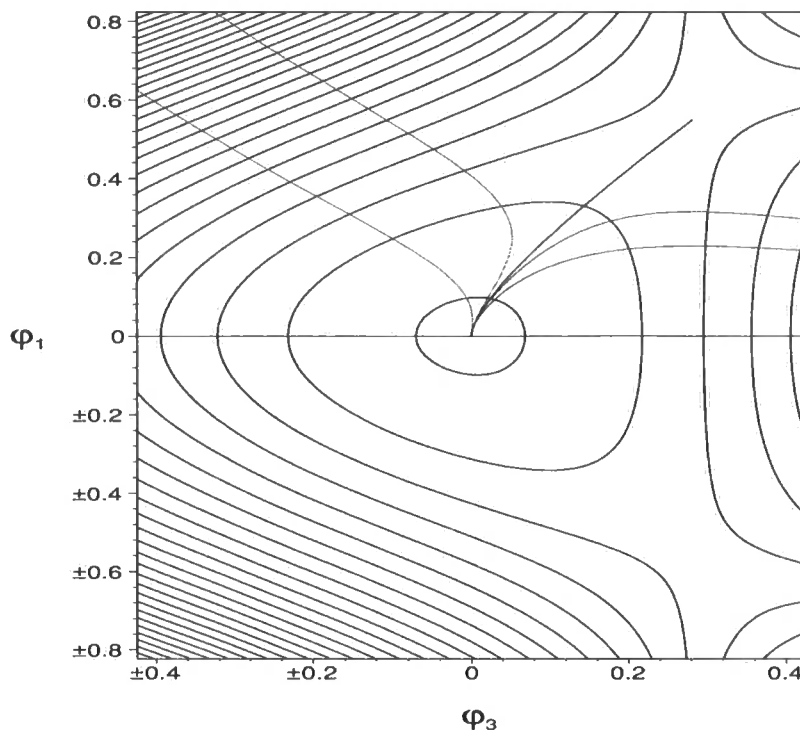


Figure 2.1: A contour plot of $W(\varphi_1, \varphi_3)$ with some physical and unphysical RG flows superposed. The flows along the φ_3 -axis (starting at the origin and tending to $\varphi = \pm\infty$) are physical, and in fact are a special case of the $\mathcal{N} = 4$ Coulomb branch. The flow to the saddle point is the Leigh-Strassler flow. Those to the right are physical; those to left are unphysical (see text).

2.2.5 The ten dimensional solution

The full lift to ten dimensions of the five-dimensional solution found in [35] was carried out by Pilch & Warner [82], after parts of the solution had been found previously [64, 97, 98]. The Einstein metric is

$$ds_{10}^2 = \Omega^2 ds_{1,4}^2 + ds_5^2, \quad (2.31)$$

where $ds_{1,4}^2$ is as in equation (2.16) and

$$ds_5^2 = L^2 \frac{\Omega^2}{\rho^2 \cosh^2 \chi} \left[d\theta^2 + \frac{\rho^6 \cos^2 \theta}{\bar{X}_1} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \frac{\sin^2 \theta}{\bar{X}_1} d\phi^2 + \frac{\rho^6 \sinh^2 \chi}{\bar{X}_1^2} (\cos^2 \theta \sigma_3 - \sin^2 \theta d\phi)^2 \right], \quad (2.32)$$

with

$$\begin{aligned} \Omega^2 &= \frac{\bar{X}_1^{1/2} \cosh \chi}{\rho} \\ \bar{X}_1 &= \cos^2 \theta + \rho^6 \sin^2 \theta \end{aligned} \quad (2.33)$$

The σ_i are the standard $SU(2)$ left-invariant forms, normalized such that $d\Omega_3 = \sigma_i \sigma_i$ is the metric on the unit 3-sphere (see appendix A).

As mentioned in section 2.1.1, we will not require the full form of the two-form gauge fields $B_{(2)}$ and $C_{(2)}$. We will need, however, the explicit form for the R-R four form potential $C_{(4)}$, to which the D3-brane naturally couples. The partial derivatives of this field, which appear in the field strength, are presented in [82], and checks are made there on the mixed second derivatives in order to ensure consistency. However, it is possible to integrate the equations to yield a closed form for $C_{abcd}^{(4)}$:

$$C_{abcd}^{(4)} = -\frac{e^{4A}}{2g_s \rho^2} [\rho^6 \sin^2 \theta (\cosh(2\chi) - 3) - \cos^2 \theta (1 + \cosh(2\chi))] \varepsilon_{abcd}. \quad (2.34)$$

Finally, the dilaton, Φ , and the axion, C_0 , are constant throughout the ten-dimensional solution.

Symmetries of the Pilch-Warner solution

It is useful at this point to check that the symmetries of the Pilch-Warner solution match those of the field theory it is supposed to be dual to.

The solution should preserve 4 real supercharges in order to correspond to $\mathcal{N} = 1$ supersymmetry in the field theory. Showing the existence of Killing spinors should reduce to requiring that $\chi(r)$ and $\rho(r)$ satisfy the flow equations (2.27). Although (to my knowledge) this calculation has not been done, the five-dimensional solution is supersymmetric by construction (indeed this is how equations (2.27) were found). However, it has been checked that the ten-dimensional solution corresponding to the Leigh-Strassler *fixed point* [98] has $\mathcal{N} = 2$ supersymmetry — this is sensible as the field theory is $\mathcal{N} = 1$ superconformal.

The metric has an $SU(2) \times U(1) \times U(1)$ global symmetry. The $SU(2)$ acts on the σ_i in the natural way, while one of the $U(1)$ s rotates σ_1 and σ_2 into each other. (It shifts the coordinate β , as described in appendix A.) The other $U(1)$ is generated by ∂_ϕ . However, only a linear combination of the two $U(1)$ s is a symmetry of the two-form gauge fields. In particular,

$$C_{(2)} = e^{-i\phi}(a_1 d\theta - a_2 \sigma_3 - a_3 d\phi) \wedge (\sigma_1 + i\sigma_2). \quad (2.35)$$

has a $U(1)$ symmetry under

$$\phi \rightarrow \phi + \psi, \quad \sigma_1 + i\sigma_2 \rightarrow e^{i\psi}(\sigma_1 + i\sigma_2). \quad (2.36)$$

Therefore, the full solution only has an $SU(2) \times U(1)$ global symmetry, as it should.

2.3 Probing the Leigh-Strassler RG flow

2.3.1 The probe moduli space

As mentioned in section 2.1, it is quite natural to probe asymptotically $\text{AdS}_5 \times M$ supergravity duals with a D3-brane. Indeed, the Maldacena conjecture was motivated by considering the near-horizon geometry of a stack of N coincident D3-branes. The dynamics of these branes is governed by an $SU(N)$ gauge theory. In flat space, separating a single D-brane from the stack breaks the gauge group to $SU(N-1) \times U(1)$ by the Higgs mechanism. Therefore in the case of $\text{AdS}_5 \times S^5$ it is reasonable to suppose that putting a probe brane in the geometry again corresponds

to breaking the gauge group to $SU(N-1) \times U(1)$. The probe interacts with the other branes via the background geometry produced by them.

More precisely, the action presented in (2.8) only contains the transverse scalars and the $U(1)$ gauge field that “live on” the probe. It does not explicitly contain the degrees of freedom associated to the rest of the original $SU(N)$ theory and so we should interpret it as a low-energy action in which the massive W-bosons have been integrated out. That is, it should not be valid at energies above the mass of the W-bosons, which is set by the size of the vev given to the probe. (This expectation value exactly corresponds to the position of the brane in its six-dimensional transverse space.) In fact we shall argue that the Lagrangian in (2.8) can be interpreted quite sensibly as a low energy action.

Given these ideas, we can probe the ten-dimensional supergravity solution corresponding to the Leigh-Strassler flow described in section 2.2.5. In this background, (2.8) becomes

$$\mathcal{L} \equiv T - V = -\frac{\tau_3}{2} \Omega^2 e^{2A} G_{mn} \partial_a y^m \partial^a y^n - \tau_3 \sin^2 \theta e^{4A} \rho^4 (\cosh(2\chi) - 1), \quad (2.37)$$

where we have only kept the terms involving the scalar fields. The first difference from the case of a D3-brane in $\text{AdS}_5 \times S^5$ is that now we have a non-trivial potential given by

$$V = \tau_3 \sin^2 \theta e^{4A} \rho^4 (\cosh(2\chi) - 1). \quad (2.38)$$

The potential is always non-negative and vanishes for $\theta = 0$. This condition defines a moduli space of inequivalent vacua⁸. The metric on moduli space is given by the kinetic term evaluated on this space:

$$ds^2 = \frac{\tau_3 e^{2A}}{2} \left\{ \frac{\cosh^2 \chi}{\rho^2} dr^2 + L^2 \rho^2 (\cosh^2 \chi \sigma_3^2 + \sigma_1^2 + \sigma_2^2) \right\}. \quad (2.39)$$

As $r \rightarrow \infty$ one finds that $\chi \rightarrow 0$, $\rho \rightarrow 1$ and so this tends to a flat metric on \mathbb{R}^4 . For $r \rightarrow -\infty$ the supergravity scalars χ and ρ again tend to constant values, so near the origin the metric is not flat and in fact has a conical singularity.

⁸There is a possibility that the r -dependent function in (2.38) could vanish for some r — it does, but only at $r = -\infty$, where the kinetic term vanishes as well. This vacuum is actually included in the space given by $\theta = 0$.

We will see in section 2.3.2 that by a suitable change of coordinates it is possible to make some interesting gauge theory predictions using this result. However, before doing that, let us consider some of the more qualitative features of the probe Lagrangian (2.37).

To understand the probe result we should consider the energy scales in the problem. First, there is the mass given to Φ_3 , let us call it m_3 . Secondly, there is the mass of the W-bosons that are integrated out in the process of obtaining the Lagrangian for the probe theory. This is proportional to the size of the vev that breaks $SU(N) \rightarrow SU(N-1) \times U(1)$, *i.e.* $|\langle X^i \rangle|$, where X^i is the vev of the brane. Finally, there is the energy for which the effective action is valid at or below, Λ , say. As argued above, we should have $\Lambda \lesssim |\langle X^i \rangle|$.

Although the ten dimensional geometry of the flow solution becomes arbitrarily close to that of pure $\text{AdS}_5 \times S^5$ for large r , “the UV”, the physics of the D3-brane probing the flow geometry does not approach that seen by a D3-brane in pure $\text{AdS}_5 \times S^5$. In fact, for large r the potential behaves like

$$V \sim e^{2r/L} \sin^2 \theta . \quad (2.40)$$

In fact we can interpret the behaviour of V in equation (2.40) as a mass term for ϕ_3 with $|\phi_3| \sim e^{r/L} \sin \theta$.

This is slightly counterintuitive — one might expect that for large positive r , that a probe could not tell the difference between normal $\text{AdS}_5 \times S^5$ and the Pilch-Warner geometry⁹. Indeed, if placing the brane at large r meant that (2.37) was valid at high energies, then perhaps one would expect to see behaviour similar to the $\mathcal{N} = 4$ theory. But as emphasized above, the action is really a low-energy action and considering the brane at large r is setting $|\langle X^i \rangle|$ to be large.

However, one should note that for large r that the kinetic term in (2.37) does indeed tend to a flat metric on \mathbb{R}^6 — the result found for pure $\text{AdS}_5 \times S^5$ in section 2.1.2. How we can interpret this from a field theory point of view? Taking r to be large means that we are setting $|\langle X^i \rangle|$ to be large — therefore the energy scale at which the gauge group is broken to $SU(N-1) \times U(1)$ is larger than the mass

⁹See [57–59, 99, 100] for some discussion of the issue of probes in AdS and holography.

m_3 . In terms of obtaining a low-energy action by integrating out massive fields this means that the massive W-bosons are integrated out first at an energy where the mass term can be neglected. This leaves a Lagrangian of the form

$$\mathcal{L} \simeq \frac{1}{2} \sum_{i=1}^3 \partial_a \phi_i \partial^a \bar{\phi}_i - \frac{1}{2} m_3^2 \phi_3 \bar{\phi}_3 - \frac{1}{4g_{\text{YM}}^2} F_{ab} F^{ab} + \frac{\theta_{\text{YM}}}{32\pi^2} F_{ab} \tilde{F}^{ab}. \quad (2.41)$$

which is indeed of the form that we see.

What happens if we now take r (roughly $|\langle X^i \rangle|$) to be small? This implies that in obtaining the low-energy action, the mass term for Φ_3 will be integrated out before the massive gauge bosons. *I.e.* the gauge theory will flow to the Leigh-Strassler fixed point first, and then be broken to $SU(N-1) \times U(1)$. So in this case the metric on moduli space should be that of the Leigh-Strassler theory. However, there is now a slight problem in interpreting the potential, V . Since the action is valid at energies at or below $\Lambda \lesssim |\langle X^i \rangle| \lesssim m_3$, ϕ_3 should have been integrated out. However, the brane is still able to fluctuate in the “ ϕ_3 ” directions $\theta \neq 0$. Therefore the interpretation of these directions is unclear¹⁰. Having mentioned this, from now on we shall concentrate on the Lagrangian for the massless fields $\Phi_{1,2}$, A_a only.

The effective coupling and inherited duality

So far, we have not mentioned the $U(1)$ gauge theory on the probe. From the analysis in section 2.1, one can immediately see that the effective coupling, g_{eff} , and the effective theta-angle, θ_{eff} , are both constant in this background. It is quite remarkable that this is the case — one would expect that adding a mass term to the $\mathcal{N} = 4$ theory will cause the coupling to run. This would have implied that g_{eff} would be a non-trivial function of (X^i) at low energies.

This interesting property of the Leigh-Strassler theory has been explained by a simple argument in [101]. Furthermore, the authors argue that the Leigh-Strassler theory should inherit a form of duality from the $\mathcal{N} = 4$ theory. This manifests itself in the supergravity dual as the usual action of $SL(2, \mathbb{Z})$ on the dilaton and axion.

¹⁰Some progress has been made in interpreting the potential term by Evans, Johnson & Petrini (unpublished work).

2.3.2 A Kähler metric for moduli space

Having considered the probe Lagrangian qualitatively, it is sensible to ask whether the result for the metric on moduli space can be put into a form where the connection to field theory is more apparent. As described in section 1.2.2, the metric on moduli space in an $\mathcal{N} = 1$ supersymmetric theory should be a Kähler metric. In this section we shall show that it *is* possible to put the metric into Kähler form, and furthermore that this constitutes a non-trivial test of the ten-dimensional geometry.

The moduli space is parametrized by the vevs of the massless scalars ϕ_1 and ϕ_2 , which we shall write as z_1 and z_2 , respectively. The z_i transform in the fundamental of $SU(2)$, while their complex conjugates transform in the anti-fundamental representation. The $SU(2)$ flavour symmetry implies that the Kähler potential is a function of u^2 only, where

$$u^2 = z_1 \bar{z}_1 + z_2 \bar{z}_2 . \quad (2.42)$$

Dividing the coordinates (and indices) into holomorphic and anti-holomorphic (those with and those without a bar), if the metric is Kähler, then it is given by

$$ds^2 = g_{\mu\bar{\nu}} dz^\mu dz^{\bar{\nu}} = g_{1\bar{1}} dz_1 d\bar{z}_1 + g_{1\bar{2}} dz_1 d\bar{z}_2 + g_{2\bar{1}} dz_2 d\bar{z}_1 + g_{2\bar{2}} dz_2 d\bar{z}_2 , \quad (2.43)$$

where

$$\begin{aligned} g_{\mu\bar{\nu}} &= \partial_\mu \partial_{\bar{\nu}} K(u^2) \\ &= \partial_\mu (\partial_{\bar{\nu}}(u^2) K') \\ &= \partial_\mu (\partial_{\bar{\nu}}(u^2)) K' + \partial_\mu (u^2) \partial_{\bar{\nu}}(u^2) K'' , \end{aligned} \quad (2.44)$$

where the primes denote differentiation with respect to u^2 , and we have used the fact that K only depends on u^2 . Notice that since

$$\partial_i(u^2) = \bar{z}_i \quad \text{and} \quad \bar{\partial}_i(u^2) = z_i , \quad (2.45)$$

we have,

$$\begin{aligned} g_{1\bar{1}} &= \partial_1 \bar{\partial}_1 K = K' + z_1 \bar{z}_1 K'' , \\ g_{1\bar{2}} &= \bar{z}_1 z_2 K'' , \end{aligned} \quad (2.46)$$

and so on. Therefore the metric can be written as

$$ds^2 = (dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2)K' + (\bar{z}_1 dz_1 + \bar{z}_2 dz_2)(z_1 d\bar{z}_1 + z_2 d\bar{z}_2)K'' . \quad (2.47)$$

Now notice that (see appendix A)

$$\begin{aligned} du &= \frac{1}{2u}(\bar{z}_1 dz_1 + \bar{z}_2 dz_2 + z_1 d\bar{z}_1 + z_2 d\bar{z}_2) \quad \text{and} \\ u\sigma_3 &= \frac{1}{2u}(-i\bar{z}_1 dz_1 - i\bar{z}_2 dz_2 + iz_1 d\bar{z}_1 + iz_2 d\bar{z}_2) , \end{aligned} \quad (2.48)$$

which is convenient, since we can write

$$du + iu\sigma_3 = \frac{1}{u}(\bar{z}_1 dz_1 + \bar{z}_2 dz_2) \quad \text{and} \quad du - iu\sigma_3 = \frac{1}{u}(z_1 d\bar{z}_1 + z_2 d\bar{z}_2) . \quad (2.49)$$

This implies that

$$ds^2 = (K' + u^2 K'')du^2 + u^2(K'(\sigma_1^2 + \sigma_2^2) + (K' + u^2 K'')\sigma_3^2) . \quad (2.50)$$

2.3.3 Comparison with probe result

The result derived using the brane probe should be written out at this stage, to give:

$$ds^2 = \frac{\tau_3}{2} \left\{ \frac{\cosh^2 \chi}{\rho^2} e^{2A} dr^2 + L^2 \rho^2 e^{2A} (\cosh^2 \chi \sigma_3^2 + \sigma_1^2 + \sigma_2^2) \right\} . \quad (2.51)$$

The explicit $SU(2)$ invariance in this equation is that of the flavour symmetry, so in order to put the metric into Kähler form we need a change of radial coordinate relating r and u .

Comparing equations (2.50) and (2.51) we obtain three equations:

$$(K' + u^2 K'')du^2 = \frac{\tau_3 \cosh^2 \chi}{2 \rho^2} e^{2A} dr^2 , \quad (2.52)$$

$$u^2(K' + u^2 K'') = \frac{\tau_3}{2} L^2 \rho^2 e^{2A} \cosh^2 \chi , \quad (2.53)$$

$$u^2 K' = \frac{\tau_3}{2} L^2 \rho^2 e^{2A} . \quad (2.54)$$

Using the first two equations we find

$$dr^2 = \frac{L^2 \rho^4}{u^2} du^2 . \quad (2.55)$$

A solution is:

$$u = \frac{L}{\alpha'} e^{f(r)/L}, \quad \text{with} \quad \frac{df}{dr} = \frac{1}{\rho^2}. \quad (2.56)$$

The latter is always positive and so defines a sensible radial coordinate u .

We can now define K by the differential equation (2.54):

$$K' = \frac{dK}{d(u^2)} = \frac{\tau_3 L^2 \rho^2 e^{2A}}{2 u^2}, \quad (2.57)$$

and we have to check that such a K obeys equation (2.53), which can be written as:

$$u^2 \frac{d}{d(u^2)} (u^2 K') = \frac{\tau_3}{2} L^2 \rho^2 e^{2A} \cosh^2 \chi. \quad (2.58)$$

From the definition of u in equation (2.56), we have that:

$$\frac{d}{d(u^2)} = \frac{L \rho^2}{2 u^2} \frac{d}{dr}, \quad (2.59)$$

and so we need to show

$$\frac{L \rho^2}{2} \frac{d}{dr} (u^2 K') = \frac{\tau_3}{2} L^2 \rho^2 e^{2A} \cosh^2 \chi. \quad (2.60)$$

From our definition of K in equation (2.57) this amounts to requiring us to show that:

$$\frac{d}{dr} (\rho^2 e^{2A}) = \frac{2}{L} e^{2A} \cosh^2 \chi, \quad (2.61)$$

which seems quite unlikely. Amazingly, performing the derivative on the left hand side and substituting the flow equations for $\rho(r)$ and $A(r)$ listed in (2.27) gives precisely the result on the right. This demonstrates the existence of Kähler potential. In fact, using the equation (2.59) we can write an alternative form for the definition of K , to accompany (2.57), which is:

$$\frac{dK}{dr} = \tau_3 L e^{2A(r)}. \quad (2.62)$$

After some thought, one can write down an exact solution to the equation (2.62) for the Kähler potential, for all r . Up to additive constants, it is:

$$K = \frac{\tau_3 L^2 e^{2A}}{4} \left(\rho^2 + \frac{1}{\rho^4} \right). \quad (2.63)$$

2.3.4 A few asymptotic results

Large u

For large u (*i.e.*, in the limit of large vevs), $\rho \sim 1$ so that, from equation (2.56) we have $u \sim \frac{L}{\alpha'} \exp(r/L)$, and to leading order:

$$K \sim \frac{\tau_3}{2} L^2 e^{2r/L} = \frac{1}{8\pi^2 g_{YM}^2} u^2, \quad (2.64)$$

which implies the expected flat metric:

$$ds^2 = \frac{1}{8\pi^2 g_{YM}^2} (dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2). \quad (2.65)$$

We can also look at next-to-leading order corrections to the Kähler potential. Recalling the asymptotic solutions for α and χ in equations (2.28) and also the flow equations (2.27) one can show:

$$A(r) \simeq \frac{r}{L} - \frac{a_0^2}{6} e^{-2r/L} + O(e^{-4r/L}), \quad (2.66)$$

so that

$$K \simeq \tau_3 L^2 \left(\frac{1}{2} e^{2r/L} - \frac{a_0^2}{3} \frac{r}{L} \right), \quad (2.67)$$

where we have now discarded terms of order $\exp(-2r/L)$ as well as constant terms. Similarly, the corresponding expression for u^2 is

$$u^2 \simeq \frac{L^2}{\alpha'^2} \left(e^{2r/L} + \frac{4a_0^2}{3} \frac{r}{L} \right). \quad (2.68)$$

Returning to the Kähler potential, we find:

$$K \simeq \frac{1}{8\pi^2 g_{YM}^2} \left[u^2 - \frac{a_0^2 L^2}{\alpha'^2} \ln \left(\frac{\alpha'^2 u^2}{L^2} \right) \right]. \quad (2.69)$$

This expression looks similar to that which one might obtain from a one-loop calculation in field theory. To compare with such a result we need to know how a_0^2 corresponds to the mass for Φ_3 . To deduce this we can look at the probe result at large u more closely.

To leading order, we have

$$\begin{aligned} |z_1|^2 + |z_2|^2 &= \frac{L^2}{\alpha'^2} e^{2r/L} \cos^2 \theta, \quad \text{and} \\ |z_3|^2 &= \frac{L^2}{\alpha'^2} e^{2r/L} \sin^2 \theta, \end{aligned} \quad (2.70)$$

and so

$$\mathcal{L} = \frac{-1}{8\pi^2 g_{YM}^2} \left(\partial_a z_1 \partial^a \bar{z}_1 + \partial_a z_2 \partial^a \bar{z}_2 + \partial_a z_2 \partial^a \bar{z}_2 + \frac{4a_0^2}{L^2} |z_3|^2 \right), \quad (2.71)$$

where the asymptotic solution for α and for χ have again been used. The mass of Φ_3 is therefore

$$m_3 = \frac{2a_0}{L}. \quad (2.72)$$

Inserting this into the Kähler potential, we obtain

$$K \simeq \frac{1}{8\pi^2 g_{YM}^2} u^2 - \frac{Nm_3^2}{16\pi^2} \ln \left(\frac{\alpha'^2 u^2}{L^2} \right). \quad (2.73)$$

It would be very interesting to match this result to one obtained using conventional field-theoretic techniques. Indeed, using the result of [102] (see also [103, 104]), one finds that the one-loop correction to the Kähler potential is of the following form¹¹:

$$\Delta K = -\frac{Nm_3^2}{k^2} \ln \left(\frac{u^2}{c^2} \right) \quad (2.74)$$

where c and k are real constants. Although I have not matched the coefficients of (2.73) and (2.74), it is encouraging that the result based on a perturbative approach appears to give a result that is consistent with that derived from the probe result.

Small u

For small u , $\rho \rightarrow 2^{1/6}$ and we have

$$u \sim \frac{L}{\alpha'} \exp \left(\frac{r}{2^{1/3} L} \right). \quad (2.75)$$

This gives us the Kähler potential:

$$K \sim \frac{\tau_3}{2} L^2 \frac{3}{2^{5/3}} \left(\frac{u^2 \alpha'^2}{L^2} \right)^{4/3} = \frac{1}{8\pi^2 g_{YM}^2} \frac{3}{2^{5/3}} \left(\frac{\alpha'^2}{L^2} \right)^{1/3} (u^2)^{4/3}, \quad (2.76)$$

¹¹One should note that corrections such as $u^2 \ln u^2$ are not expected because the Kähler potential should return to its $\mathcal{N} = 4$ form for $m_3 = 0$. However, this requirement does not preclude correction terms such as $m_3 u$ or $m_3 u \ln u^2$. It is interesting to note that if one expands the perturbative prepotential for the $\mathcal{N} = 2$ mass-deformed theory considered in [79, 105, 106], then one obtains a similar correction to that in (2.74).

and so the metric in the IR is:

$$ds^2 \sim \frac{1}{8\pi^2 g_{YM}^2} 2^{1/3} \left(\frac{u^2 \alpha'^2}{L^2} \right)^{1/3} \left(\frac{4}{3} du^2 + u^2 \left(\sigma_1^2 + \sigma_2^2 + \frac{4}{3} \sigma_3^2 \right) \right). \quad (2.77)$$

Therefore, near the origin of moduli space the metric has a conical singularity. This may seem rather strange, but we will see that this is quite reasonable based on a simple (classical) scaling argument. (Furthermore, at the origin of moduli space the unbroken gauge group is the full $SU(N)$, so that the probe approximation is not really valid.)

2.3.5 Scaling dimensions

Now that we have found coordinates on the probe moduli space that appear to be suitable for field-theoretic considerations, we can perform a consistency check on them. As described in section 1.2.5, the scaling dimensions of the fields ϕ_i are well-defined at the conformal fixed points of the RG flow, and are related to the R-charges. We shall find they are easily reproduced from the supergravity description.

First, we will find the correct coordinates to cover the space $\theta = \pi/2$, which corresponds to $z_1 = z_2 = 0$. The probe metric is

$$ds^2 \Big|_{\theta=\frac{\pi}{2}} = \frac{\tau_3}{2} \rho^4 \cosh^2 \chi e^{2A} \left(dr^2 + \frac{L^2}{\rho^8} d\phi^2 \right). \quad (2.78)$$

Since $0 \leq \phi < 2\pi$, we should take $z_3 = w(r)e^{i\phi}$. With this choice of sign z_3 has charge 1 under the R-symmetry (2.36), whereas z_1 and z_2 both have charge 1/2. These charges match those stated in section 1.2.5. If we then put the metric into a natural form, $ds^2 = g dz_3 d\bar{z}_3$ for some function g , we find that $dw/dr = \rho^4 w/L$.

To find the scaling dimensions we notice that the supergravity solution and probe action have the following scaling symmetry:

$$x \rightarrow \frac{1}{\alpha} x, \quad e^A \rightarrow \alpha e^A. \quad (2.79)$$

At either end of the RG flow the gauge theory is approaching a conformal fixed point and we can find the scaling dimensions of the z_i . For instance, at large r , both $u, w \simeq \frac{L}{\alpha'} e^A$, so that

$$x \rightarrow \frac{1}{\alpha} x, \quad z_i \rightarrow \alpha z_i, \quad (2.80)$$

for $i = 1, 2, 3$. Therefore the z_i each have $\Delta = 1$ in the UV limit¹². In the IR limit, $r \rightarrow -\infty$, $u \sim (e^A)^{3/4}$ and $w \sim (e^A)^{3/2}$ so that

$$x \rightarrow \frac{1}{\alpha}x, \quad z_{1,2} \rightarrow \alpha^{3/4}z_{1,2}, \quad z_3 \rightarrow \alpha^{3/2}z_3, \quad (2.81)$$

and the correct scaling dimensions are recovered.

Having found a very simple form for the Kähler potential (2.77) for small u , we should perhaps ask ourselves whether it can be derived by a different method. It can, by a rather heuristic scaling argument.

From the $SU(2)$ flavour symmetry of the theory, we know that K is a function of u^2 only. The scaling dimension of u at the UV end of the flow is 1; at the IR end of the flow it is $3/4$. Now, in the action the scalars ϕ_i have the following term

$$S|_{\phi} = \int d^4x \partial_{\phi} \partial_{\bar{\phi}} K \partial_{\alpha} \phi \partial^{\alpha} \bar{\phi} \quad (2.82)$$

For S to be invariant under the scaling symmetry, classically $K(u^2)$ must have scaling dimension 2. Therefore for large u , $K \sim u^2$. Similarly for small u (at the IR end of the flow solution), u has scaling dimension $3/4$ and so $K \sim (u^2)^{4/3}$. This matches the result found in section 2.3.4. It is therefore possible to recover the form of the Kähler potential at either end of the flow from a classical scaling argument.

2.4 Probing the Pilch-Warner geometry with non-zero vevs

Having studied the behaviour of the probe brane in the Pilch-Warner geometry (corresponding to a pure mass deformation) and found a Kähler metric on moduli space, we now turn our attention to the other flows discussed in section 2.2.4.

2.4.1 Physical RG flows

First, consider the flows for which $\hat{a} > \hat{a}_c$, that are physical. These correspond to giving a mass to $\phi_3 = X_5 + iX_6$ (and its superpartner) and switching on a vev for

¹²As mentioned before, for the brane probe the large r limit is a limit of large vevs. However, one can read off UV properties because the gauge group is broken at high energies.

$\text{Tr}(X_1^2 + X_2^2 + X_3^2 + X_4^2)$. For these flows, the geometry terminates at some finite value of r , r_0 say. As $r \rightarrow r_0$, one finds the following:

$$\chi(r) \rightarrow 0, \quad \rho(r) \rightarrow \infty, \quad A(r) \rightarrow -\infty, \quad (2.83)$$

such that the following holds

$$e^{4A} \rho^4 (\cosh(2\chi) - 1) \rightarrow 0, \quad e^{2A} \rho^2 \rightarrow C \neq 0, \quad (2.84)$$

where C is a constant. We will see that the behaviour of these function gives a sensible result for the probe as $r \rightarrow r_0$. Indeed, $e^{4A} \rho^4 (\cosh(2\chi) - 1)$ appears in the expression for V in equation (2.38). Since the potential vanishes for $r = r_0$ (where the supergravity geometry terminates), we have a second locus that did not appear in the pure mass deformation. Therefore the probe now has a moduli space made up of two loci:

$$\text{Locus I:} \quad \theta = 0,$$

$$\text{Locus II:} \quad r = r_0. \quad (2.85)$$

The form of the metric on Locus I is exactly the same as in section 2.3.2:

$$ds^2|_{\text{I}} = \frac{\tau_3 e^{2A}}{2} \left\{ \frac{\cosh^2 \chi}{\rho^2} dr^2 + L^2 \rho^2 (\cosh^2 \chi \sigma_3^2 + \sigma_1^2 + \sigma_2^2) \right\}. \quad (2.86)$$

The choice of coordinates to make this Kähler is also the same, and equation (2.65) holds for large u .

As $r \rightarrow r_0$, one finds the following

$$ds^2|_{\text{I} \cap \text{II}} = \frac{\tau_3 CL^2}{2} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) = \frac{\tau_3 CL^2}{2} d\Omega_3^2. \quad (2.87)$$

So locus I is no longer a full \mathbb{R}^4 — it is an \mathbb{R}^4 with a ball of radius

$$\sqrt{\frac{\tau_3 CL^2}{2}} \quad (2.88)$$

removed.

On locus II, the kinetic term in the Lagrangian implies the following metric on moduli space

$$ds^2|_{\text{II}} = \frac{\tau_3 CL^2}{2} (\sin^2 \theta d\theta^2 + \cos^2 \theta (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)). \quad (2.89)$$

It is important to note here that although the ten-dimensional metric contains $\rho(r)$, which diverges as $r \rightarrow r_0$, the metric on moduli space is well-behaved. Furthermore, this locus is again four-dimensional. Indeed, because $0 \leq \theta \leq \pi/2$, one can see that locus II fills out the ball not included in locus I! In particular, we can change coordinates to $u = u_0 \cos \theta$, with $u_0 = \frac{L}{\alpha'} e^{f(r_0)/L}$, so that the definitions of u match at the intersection of the loci. This gives

$$ds^2 \Big|_{\text{II}} = \frac{\gamma}{8\pi^2 g_{\text{YM}}^2} (dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2) , \quad (2.90)$$

where

$$\gamma = \lim_{r \rightarrow r_0} \rho^2 e^{2A} e^{2f/L} . \quad (2.91)$$

The metric can obviously be put into Kähler form because it is simply a flat metric on \mathbb{R}^4 . The Kähler potential in this case is given by

$$K = \frac{\tau_3 C L^2}{4} (2 \cos^2 \theta - 1) , \quad (2.92)$$

where we have included a constant term so that the two definitions of K also match on the intersection of the loci.

It is interesting to notice that although the overall constant factors in both (2.90) and (2.65) are not physical (because both can be absorbed into a redefinition of u), their ratio, γ , is. It is easy to show using the flow equations that the function $\rho^2 e^{2A} e^{-2f/L}$ has non-negative r -derivative. This implies that $0 \leq \gamma < 1$. In fact it appears (from brief study of numerical solutions) that γ is monotonic as a function $(\hat{a} - \hat{a}_c)$.

From the above analysis it is quite clear that the N D3-branes that “source” the ten-dimensional geometry are now located at the intersection of the two loci, *i.e.* on the S^3 defined by $r = r_0$, $\theta = 0$. This is quite sensible because, as described above, a non-zero vev has been given to the operator $\text{Tr}(X_1^2 + X_2^2 + X_3^2 + X_4^2)$.

In the above analysis we have found a coordinate u such that $u^2 = z_1 \bar{z}_1 + z_2 \bar{z}_2$ and the metric is Kähler over both loci. The Kähler potential is continuous across the intersection, as is the metric — this implies that dK/du and d^2K/du^2 are continuous there as well. However, it seems that the higher derivatives of K , $d^{(n)}K/du^{(n)}$, $n \geq 3$, are not continuous there. In fact, on locus II these derivatives

are zero (since the potential is quadratic in u) whereas they remain non-zero on locus I as $r \rightarrow r_0$. These discontinuities are familiar in spacetime physics where there is a shell of matter¹³, but their direct interpretation in terms of low-energy lagrangians and renormalization theory remains to be clarified.

2.4.2 Unphysical RG flows

In the case of the unphysical flows with $\hat{a} < \hat{a}_c$, one again finds that the flow terminates at some finite $r = r_0$. In these cases one has

$$\chi(r) \rightarrow \infty, \quad \rho(r) \rightarrow -\infty, \quad A(r) \rightarrow -\infty. \quad (2.93)$$

Numerically, one can find that, as for the pure mass deformation, the moduli space is given by $\theta = 0$ only. However, for these flows the metric on moduli space diverges, which seems to be indicative of unphysical behaviour. Therefore, the probe calculation can correctly discriminate between physical and unphysical RG flows.

2.5 A more general $\mathcal{N} = 1$ flow in $D = 4$

2.5.1 The ten-dimensional solution and probe result

We shall now study a slightly more general solution, presented in [108], that allows a new scalar field, $\beta = \log \nu$, to vary. This field corresponds to the operator $\text{Tr}(X_1^2 + X_2^2 - X_3^2 - X_4^2)$, and so in a way similar to that described before, the holographic RG flow will now have different vevs for $\text{Tr}(X_1^2 + X_2^2)$ and $\text{Tr}(X_3^2 + X_4^2)$. This breaks the $SU(2) \times U(1)$ global symmetry to $U(1)^2$.

The new metric will again be of the form given in (2.31) and (2.16), while the warp factor is given by [108]:

$$\Omega^2 = \cosh \chi \left((\nu^2 \cos^2 \phi + \nu^{-2} \sin^2 \phi) \frac{\cos^2 \theta}{\rho^2} + \rho^4 \sin^2 \theta \right)^{\frac{1}{2}}, \quad (2.94)$$

¹³These have been studied in the enhançon scenario [107].

and the deformed sphere metric is given as follows [108]:

$$\begin{aligned}
ds_5^2 = & \frac{L^2}{\Omega^2} \left[\rho^{-4} (\cos^2 \theta + \rho^6 \sin^2 \theta (\nu^{-2} \cos^2 \phi + \nu^2 \sin^2 \phi)) d\theta^2 \right. \\
& + \rho^2 \cos^2 \theta (\nu^2 \cos^2 \phi + \nu^{-2} \sin^2 \phi) d\phi^2 \\
& - 2\rho^2 (\nu^2 - \nu^{-2}) \sin \theta \cos \theta \sin \phi \cos \phi d\theta d\phi \\
& + \rho^2 \cos^2 \theta (\nu^{-2} \cos^2 \phi d\varphi_1^2 + \nu^2 \sin^2 \phi d\varphi_2^2) + \rho^{-4} \sin^2 \theta d\varphi_3^2 \left. \right] \\
& + \frac{L^2}{\Omega^6} \sinh^2 \chi \cosh^2 \chi (\cos^2 \theta (\cos^2 \phi d\varphi_1 - \sin^2 \phi d\varphi_2) - \sin^2 \theta d\varphi_3)^2 \quad (2.95)
\end{aligned}$$

This metric has a $U(1)^3$ symmetry generated by the Killing vectors $\partial/\partial\varphi_1$, $\partial/\partial\varphi_2$ and $\partial/\partial\varphi_3$. However, the two-form fields of the solution are only invariant under shifts of the φ_i satisfying $\delta\varphi_1 + \delta\varphi_2 + \delta\varphi_3 = 0$, so the actual symmetry is $U(1)^2$.

The superpotential for this flow is given by [108]:

$$W = \frac{1}{4}\rho^4(\cosh 2\chi - 3) - \frac{1}{4\rho^2}(\nu^2 + \nu^{-2})(\cosh 2\chi + 1), \quad (2.96)$$

which generalizes the superpotential in equation (2.19). The equations of motion for the supergravity fields are:

$$\begin{aligned}
\frac{d\rho}{dr} &= \frac{1}{6L}\rho^2 \frac{\partial W}{\partial \rho} = \frac{1}{12L\rho} (2\rho^6(\cosh 2\chi - 3) + (\nu^2 + \nu^{-2})(\cosh 2\chi + 1)), \\
\frac{d\nu}{dr} &= \frac{1}{2L}\nu^2 \frac{\partial W}{\partial \nu} = -\frac{1}{4L\rho^2}(\cosh 2\chi + 1)\nu(\nu^2 - \nu^{-2}), \\
\frac{d\chi}{dr} &= \frac{1}{L} \frac{\partial W}{\partial \chi} = \frac{\sinh 2\chi}{2L\rho^2} (\rho^6 - (\nu^2 + \nu^{-2})), \\
\frac{dA}{dr} &= -\frac{2}{3L}W = -\frac{1}{6L\rho^2} (\rho^6(\cosh 2\chi - 3) - (\nu^2 + \nu^{-2})(\cosh 2\chi + 1)). \quad (2.97)
\end{aligned}$$

The authors of [108] probed the metric with a D3-brane, and found that the probe potential V was the same as in equation (2.38). In the following, we shall consider a RG flow from the $\mathcal{N} = 8$ critical point such that for large r each of α , β and χ are small and positive. Then for some r_0 one finds that as $r \rightarrow r_0$:

$$\chi(r) \rightarrow 0, \quad \rho(r) \rightarrow \infty, \quad A(r) \rightarrow -\infty, \quad (2.98)$$

as for the flows considered in section 2.4.1, and

$$\nu \rightarrow \nu_0 \neq 0, \quad e^{4A}\rho^4(\cosh(2\chi) - 1) \rightarrow 0, \quad e^{2A}\rho^2 \rightarrow C \neq 0, \quad (2.99)$$

where now both ν_0 and C are constants. As before we have two loci, given by (2.85). The metric on locus I is now

$$\begin{aligned} ds^2 \Big|_{\text{I}} = & \frac{1}{2} \tau_3 e^{2A} \left[\zeta (\rho^{-2} \cosh^2 \chi dr^2 + L^2 \rho^2 d\phi^2) \right. \\ & + L^2 \rho^2 (\nu^{-2} \cos^2 \phi d\varphi_1^2 + \nu^2 \sin^2 \phi d\varphi_2^2) \\ & \left. + L^2 \rho^2 \sinh^2 \chi \zeta^{-1} (\cos^2 \phi d\varphi_1 - \sin^2 \phi d\varphi_2)^2 \right] , \end{aligned} \quad (2.100)$$

where $\zeta \equiv (\nu^2 \cos^2 \phi + \nu^{-2} \sin^2 \phi)$. The metric for locus II is:

$$\begin{aligned} ds^2 \Big|_{\text{II}} = & \frac{\tau_3 C L^2}{2} (\sin^2 \theta (\nu_0^{-2} \cos^2 \phi + \nu_0^2 \sin^2 \phi) d\theta^2 \\ & + \cos^2 \theta (\nu_0^2 \cos^2 \phi + \nu_0^{-2} \sin^2 \phi) d\phi^2 \\ & - 2(\nu_0^2 - \nu_0^{-2}) \sin \theta \cos \theta \sin \phi \cos \phi d\theta d\phi \\ & + \cos^2 \theta (\nu_0^{-2} \cos^2 \phi d\varphi_1^2 + \nu_0^2 \sin^2 \phi d\varphi_2^2) . \end{aligned} \quad (2.101)$$

2.5.2 Finding a Kähler potential

As in section 2.3.2, we would now like to find a set of coordinates such that the metric on the probe moduli space is a Kähler metric. In order to do this we will use the $U(1)^2$ global symmetries of the problem to impose conditions on the Kähler potential. Since φ_3 does not appear in either (2.100) or (2.101), the $U(1)^2$ symmetry is generated by constant shifts in φ_1 and φ_2 . In the supergravity solution presented in section 2.5.1, one has $0 \leq \varphi_{1,2} < 2\pi$. Therefore, it is quite natural to make the following ansatz:

$$z_1 = u(r, \phi) e^{i\varphi_1} , \quad z_2 = v(r, \phi) e^{-i\varphi_2} , \quad (2.102)$$

with

$$K = K(z_1 \bar{z}_1, z_2 \bar{z}_2) = K(u^2, v^2) . \quad (2.103)$$

This results in the following form of the Kähler metric:

$$\begin{aligned} ds^2 = & \frac{1}{4u^2} (du^2 + u^2 d\varphi_1^2) (u\partial_u)^2 K + \frac{1}{4v^2} (dv^2 + v^2 d\varphi_2^2) (v\partial_v)^2 K \\ & + \frac{1}{2uv} (dudv - uv d\varphi_2 d\varphi_1) (u\partial_u)(v\partial_v) K . \end{aligned} \quad (2.104)$$

We can compare this to the metric on locus I, (2.100), to obtain the following:

$$\begin{aligned}\frac{1}{4}(u\partial_u)^2 K &= \frac{\tau_3}{2}e^{2A}L^2\rho^2(\nu^{-2}\cos^2\phi + \sinh^2\chi\zeta^{-1}\cos^4\phi), \\ \frac{1}{4}(v\partial_v)^2 K &= \frac{\tau_3}{2}e^{2A}L^2\rho^2(\nu^2\sin^2\phi + \sinh^2\chi\zeta^{-1}\sin^4\phi), \\ \frac{1}{4}(u\partial_u)(v\partial_v)K &= \frac{\tau_3}{2}e^{2A}L^2\rho^2\sinh^2\chi\zeta^{-1}\cos^2\phi\sin^2\phi.\end{aligned}\quad (2.105)$$

Given the form of the lift ansatz that was used in [108] to find the ten-dimensional supergravity solution,¹⁴ it is quite natural to make a further ansatz:

$$u = F(r)\cos\phi, \quad v = G(r)\sin\phi. \quad (2.106)$$

It is then straightforward to show that $F(r)$ and $G(r)$ must satisfy

$$\frac{dF}{dr} = \frac{\nu^2}{L\rho^2}F, \quad \frac{dG}{dr} = \frac{1}{L\rho^2\nu^2}F. \quad (2.107)$$

In fact using equations (2.104 – 2.107), it is possible to reproduce the probe metric (2.100). However, we still need to check that equations (2.105 – 2.107) are themselves consistent. The first step is to note that

$$\begin{aligned}\partial_r &= \frac{\nu^2}{L\rho^2}(u\partial_u) + \frac{1}{L\rho^2\nu^2}(v\partial_v), \\ \partial_\phi &= -\tan\phi(u\partial_u) + \cot\phi(v\partial_v),\end{aligned}\quad (2.108)$$

so that, for instance,

$$\begin{aligned}\partial_r(v\partial_v K) &= 2\tau_3 L e^{2A} \sin^2\phi \cosh^2\chi, \\ \partial_\phi(v\partial_v K) &= 2\tau_3 L^2 e^{2A} \rho^2 \nu^2 \cos\phi \sin\phi.\end{aligned}\quad (2.109)$$

This implies that

$$v\partial_v K = \tau_3 L^2 e^{2A} \rho^2 \nu^2 \sin^2\phi + b, \quad (2.110)$$

¹⁴The lift ansatz involves deforming the metric for a 5-sphere in \mathbb{R}^6 . The authors of [108] choose the sphere to be parametrized by

$$\begin{aligned}u_1 &\equiv \cos\theta \cos\phi e^{i\varphi_1}, \\ u_2 &\equiv \cos\theta \sin\phi e^{-i\varphi_2}, \\ u_3 &\equiv \sin\theta e^{-i\varphi_3},\end{aligned}$$

so on locus I, $u_1 = \cos\phi e^{i\varphi_1}$ and $u_2 = \sin\phi e^{-i\varphi_2}$.

where b is a constant. Repeating this process for $u\partial_u K$ and then K itself, one finds that

$$K = \frac{\tau_3}{2} L^2 e^{2A} \left(\rho^2 (\nu^2 - \nu^{-2}) \sin^2 \phi + \frac{1}{2} (\rho^2 \nu^{-2} + \rho^{-4}) \right) + a \log(u) + b \log(v) + d, \quad (2.111)$$

where a , b and d are constants. As before, the equations of motion (2.97) were needed in order to find a solution. As in section 2.3.3, the required formulae are relatively neat:

$$\begin{aligned} \frac{d(e^{2A} \rho^2 \nu^{-2})}{dr} &= \frac{d(e^{2A} \rho^2 \nu^2)}{dr} = \frac{2}{L} e^{2A} \cosh^2 \chi, \\ \frac{d(e^{2A} \rho^{-4})}{dr} &= \frac{2e^{2A}}{L} (2 - \cosh^2 \chi). \end{aligned} \quad (2.112)$$

This implies a solution for A as a function of ρ and ν when $\nu \neq 1$:

$$e^{2A} = \frac{k}{\rho^2 (\nu^2 - \nu^{-2})}, \quad (2.113)$$

where k is a constant. Using this expression and setting $a = b = d = 0$ gives a relatively simple form for the Kähler potential:

$$K = \frac{\tau_3}{2} L^2 k \sin^2 \phi + \frac{\tau_3 L^2 e^{2A}}{4} \left(\frac{\rho^2}{\nu^2} + \frac{1}{\rho^4} \right). \quad (2.114)$$

It is interesting to note that this Kähler potential also satisfies equation (2.62) (because we have set $a = b = 0$).

Extension to locus II

Having found a set of coordinates so that the metric on locus I of the moduli space is Kähler, we can now do the same for locus II. As in section 2.4.1, the coordinates and the Kähler potential should match across the intersection of the two loci.

If we again use the ansatz in (2.102) then the general form of the metric in (2.104) is still valid. Comparing (2.104) with the probe metric on locus II (2.101), results in the following expressions:

$$\begin{aligned} \frac{1}{4} (u\partial_u)^2 K &= \frac{\tau_3 C L^2}{2\nu_0^2} \cos^2 \theta \cos^2 \phi, \\ \frac{1}{4} (v\partial_v)^2 K &= \frac{\tau_3 C L^2 \nu_0^2}{2} \cos^2 \theta \sin^2 \phi, \\ (u\partial_u)(v\partial_v) K &= 0. \end{aligned} \quad (2.115)$$

Given the form of u and v on locus I, and our experience of the simpler example in section 2.4.1, it is quite reasonable to propose the following:

$$\begin{aligned} u &= u_0 \cos \theta \cos \phi, \\ v &= v_0 \cos \theta \sin \phi, \end{aligned} \tag{2.116}$$

where u_0 and v_0 are chosen to match the definitions used on locus I. Given this ansatz, one can reproduce (2.104) using (2.115). Furthermore, it is straightforward to find a Kahler potential:

$$\begin{aligned} K|_{\text{II}} &= \frac{\tau_3 CL^2}{2} \left(\frac{1}{\nu_0^2} \cos^2 \theta \cos^2 \phi + \nu_0^2 \cos^2 \theta \sin^2 \phi - \frac{1}{2\nu_0^2} \right), \\ &= \frac{\tau_3 CL^2}{2} \left(\frac{1}{\nu_0^2 u_0^2} u^2 + \frac{\nu_0^2}{v_0^2} v^2 - \frac{1}{2\nu_0^2} \right), \end{aligned} \tag{2.117}$$

where again the constant term has been chosen so that the Kähler potential is continuous across the intersection of the two loci¹⁵.

2.6 The Coulomb branch of $\mathcal{N} = 4$ gauge theory

In this section we shall consider a rather simpler situation from the point of view of gauge theory. Instead of deforming the $\mathcal{N} = 4$ theory by adding a mass term to the its Lagrangian, the supergravity duals we will look at correspond to just giving non-zero vevs to the scalar fields¹⁶, X^i . As discussed before, this separates the branes in their six-dimensional transverse space. Firstly, we shall briefly present the general type IIB supergravity solution that corresponds to the $\mathcal{N} = 4$ theory away from the superconformal point on its moduli space. We shall then discuss a subclass of solutions found using five-dimensional gauged supergravity.

¹⁵It is now natural to ask whether $v_0 = u_0 \nu_0^2$, so that the metric in (2.117) is flat in (z_1, z_2) -coordinates. It appears that this is not the case.

¹⁶More precisely, vevs are given to gauge-invariant operators constructed out of the X^i .

2.6.1 General supergravity duals

The general solution to type IIB supergravity that corresponds to $\mathcal{N} = 4$ gauge theory in some vacuum on the Coulomb branch is given by [109]:

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}} \eta_{ab} dx^a dx^b + H^{\frac{1}{2}} (dy_1^2 + dy_2^2 + \dots + dy_6^2) , \\ C_{(4)} &= H^{-1} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 , \end{aligned} \quad (2.118)$$

for any harmonic function $H(y_i)$. The harmonic function is given by¹⁷

$$H(y_i) = L^4 \int d^6 y' \frac{\sigma(y_i)}{|y - y'|^4} , \quad (2.119)$$

This background preserves 16 supersymmetries, which matches that of the $\mathcal{N} = 4$ gauge theory in a non-conformal phase. A D3-brane probe in this background has a flat metric on moduli space [44]:

$$ds^2 = \frac{\tau_3}{2} (dy_1^2 + dy_2^2 + \dots + dy_6^2) = \frac{\tau_3}{2} (dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 + dz_3 d\bar{z}_3) , \quad (2.120)$$

with a very simple Kähler potential

$$K = \frac{\tau_3}{2} (z_1 \bar{z}_1 + z_2 \bar{z}_2 + z_3 \bar{z}_3) . \quad (2.121)$$

Furthermore, because the dilaton is constant, the effective gauge coupling on the brane is constant on moduli space.

2.6.2 The Coulomb branch from gauged supergravity

A simple case

Before we consider the general case of Coulomb branch flows lifted from gauged supergravity, let us consider the same solution that we studied in section 2.5 but this time setting $\chi = 0$. This is a consistent solution to the flow equations (2.97) and corresponds to not having a mass deformation, but to a state on the Coulomb branch of the theory¹⁸.

¹⁷It should be noted that we are working in the near-horizon limit, otherwise the harmonic function would have a 1 added which would give asymptotically flat space for large $|y|$.

¹⁸The solution is similar to one studied in [110].

In particular, for $\chi = 0$ the probe potential is identically zero, and so its moduli space is six-dimensional, as expected. The metric on moduli space is given by:

$$\begin{aligned} ds^2 = & \frac{\tau_3 e^{2A}}{2} \left[\Omega^4 dr^2 + L^2 \left((\rho^{-4} \cos^2 \theta + \rho^2 \sin^2 \theta (\nu^{-2} \cos^2 \phi + \nu^2 \sin^2 \phi)) d\theta^2 \right. \right. \\ & + \rho^2 \cos^2 \theta (\nu^2 \cos^2 \phi + \nu^{-2} \sin^2 \phi) d\phi^2 \\ & - 2\rho^6 (\nu^2 - \nu^{-2}) \sin \theta \cos \theta \sin \phi \cos \phi d\theta d\phi \\ & \left. \left. \rho^2 \cos^2 \theta (\nu^{-2} \cos^2 \phi d\varphi_1^2 + \nu^2 \sin^2 \phi d\varphi_2^2) + \rho^{-4} \sin^2 \theta d\varphi_3^2 \right) \right], \quad (2.122) \end{aligned}$$

where now $\Omega^4 = \rho^{-2} (\nu^2 \cos^2 \phi + \nu^{-2} \sin^2 \phi) \cos^2 \theta + \rho^4 \sin^2 \theta$. In this case, we have

$$\frac{d}{dr} (e^{2A} \rho^2 \nu^2) = \frac{d}{dr} (e^{2A} \rho^2 \nu^{-2}) = \frac{d}{dr} (e^{2A} \rho^{-4}) = \frac{2e^{2A}}{L}, \quad (2.123)$$

and using these one can show that

$$ds^2 = \frac{1}{8\pi^2 g_{\text{YM}}^2} (dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 + dz_3 d\bar{z}_3) \quad (2.124)$$

where

$$\begin{aligned} z_1 &= \frac{L e^A \rho}{\alpha' \nu} \cos \theta \cos \phi e^{i\varphi_1}, \\ z_2 &= \frac{L}{\alpha'} e^A \rho \nu \cos \theta \sin \phi e^{-i\varphi_2}, \\ z_3 &= \frac{L e^A}{\alpha' \rho^2} \cos \theta e^{-i\varphi_3}. \end{aligned} \quad (2.125)$$

So in the case where $\rho \rightarrow \infty$, $\nu \rightarrow \nu_0$ as $r \rightarrow r_0$, the subspace defined by $r = r_0$ is spanned by z_1, z_2 . This is the same as in the case studied in section 2.5 where a mass deformation was also made (although the definitions of z_1 and z_2 were, of course, slightly different). It is now very clear that the geometry studied in section 2.5 approaches the Coulomb branch geometry considered here, as $r \rightarrow r_0$. However, it is important to recognize that in the case where a mass has been given to Φ_3 , the probe only ever has a four-dimensional moduli space — at no point does it have a (locally) six-dimensional space of vacua.

More general solutions from gauged supergravity

The previous example of a Coulomb branch supergravity solution was found by lifting a solution from gauged supergravity. In fact the complete ansatz for lifting

solutions from five-dimensional gauged supergravity to ten dimensions has been found in the case when only scalars in the $20'$ representation are non-zero [74].

The ten-dimensional solution is given by:

$$ds^2 = \Omega^2(e^{2A}\eta_{ab}dx^a dx^b + dr^2) + ds_5^2, \quad (2.126)$$

$$= H_0^{-1/2}\eta_{ab}dx^a dx^b + H_0^{1/2}(dy_1^2 + dy_2^2 + \dots + dy_6^2), \quad (2.127)$$

where the harmonic function H_0 is given by

$$H_0^{-1} = \frac{1}{R^4} f^{1/2} \sum_{i=1}^6 \frac{y_i^2}{(F - b_i)^2}, \quad f = \prod_{i=1}^6 (F - b_i). \quad (2.128)$$

The b_i are real constants and F is determined from the y_i by:

$$\sum_{i=1}^6 \frac{y_i^2}{F - b_i} = 1. \quad (2.129)$$

The deformed S^5 in (2.126) is parametrized by $\hat{x}_i = (F - b_i)^{-1/2}y_i$ and the function F is related to r by

$$\frac{dF}{dr} = 2Le^{2A}. \quad (2.130)$$

In terms of the y_i , the Kähler potential is given by

$$K = \frac{\tau_3}{2} \sum_{i=1}^6 y_i^2 = \frac{\tau_3}{2} \sum_{i=1}^6 (F - b_i) \hat{x}_i^2 = \frac{\tau_3}{2} \left(F - \sum_{i=1}^6 b_i \hat{x}_i^2 \right), \quad (2.131)$$

so that

$$\frac{\partial K}{\partial r} = \frac{dF}{dr} \frac{\partial K}{\partial F} = \tau_3 L e^{2A}, \quad (2.132)$$

which is the same as equation (2.62).¹⁹

2.7 An analogous Leigh-Strassler flow in $D = 3$

In this section we shall study a family of supergravity solutions that are analogous to those studied in section 2.3. They have been constructed [112] using

¹⁹These results have been generalized to also include the analogous cases for M2- and M5-branes, using the results of [111]. It was found that [2]:

$$\frac{\partial K}{\partial r} = \tau_p L e^{(p-1)A}, \quad (2.133)$$

where $p = 2, 3$ or 5 .

four-dimensional $SO(8)$ gauged supergravity and then lifting to solutions of eleven-dimensional supergravity. In this way they are holographic RG flows of the $(2+1)$ -dimensional field theory on a stack of N coincident M2-branes. This theory is an $\mathcal{N} = 8$ superconformal theory with $8N$ scalar fields (corresponding to the transverse positions of the branes) and $8N$ fermionic superpartners [32]. Much of the analysis in this section is exactly analogous to that in section 2.3.

In this case the probe action is given by:

$$S = -\tau_{M2} \int_{M_3} d^3\xi \sqrt{-\det(G_{ab})} + \tau_{M2} \int_{M_3} A_{(3)}. \quad (2.134)$$

The eleven-dimensional metric [112] is written in terms of cartesian coordinates parametrizing an S^7 , $\sum_{I=1}^8 X^I X^I = 1$:

$$\begin{aligned} ds_{11}^2 &= \Delta^{-1}(e^{2A(r)}\eta_{ab}dx^a dx^b + dr^2) + ds_7^2, \\ ds_7^2 &= \Delta^{1/2}L^2 \left[dX^I Q_{IJ}^{-1} dX^J + \frac{\sinh^2 \chi}{\xi^2} (X^I J_{IJ} dX^J)^2 \right], \\ \Delta &= (\xi \cosh \chi)^{-4/3}, \\ \xi^2 &= X^I Q_{IJ} X^J, \\ Q &= \text{diag}(\rho^{-2}, \rho^{-2}, \rho^{-2}, \rho^{-2}, \rho^{-2}, \rho^{-2}, \rho^6, \rho^6), \end{aligned} \quad (2.135)$$

where J is an antisymmetric matrix satisfying $J_{12} = J_{34} = J_{56} = J_{78} = 1$. The functions $(\chi(r), \rho(r), A(r))$ are given by the following supersymmetric flow equations:

$$\begin{aligned} \frac{d\rho}{dr} &= \frac{1}{8L\rho} ((\cosh(2\chi) + 1) + (\cosh(2\chi) - 3)\rho^8), \\ \frac{d\chi}{dr} &= \frac{1}{2L\rho^2} (\rho^8 - 3) \sinh(2\chi), \\ \frac{dA}{dr} &= -\frac{2}{L}W = -\frac{1}{4L\rho^2} (\rho^8(\cosh(2\chi) - 3) - 3(\cosh(2\chi) + 1)). \end{aligned} \quad (2.136)$$

We also require the form of the three-form potential $A_{(3)}$, [112] :

$$A_{(3)} = 2\widetilde{W}e^{3A}dx^0 \wedge dx^1 \wedge dx^2 + \dots, \quad (2.137)$$

$$\widetilde{W} = \frac{1}{4\rho^2} \left[(\cosh(2\chi) + 1) \sum_{I=1}^6 X^I X^I - \rho^8(\cosh(2\chi) - 3) \sum_{I=7}^8 X^I X^I \right] \quad (2.138)$$

Here the other terms in the expression for $A_{(3)}$ do not lie in the x^0, x^1, x^2 directions, and will be not needed for our purposes.

Inserting the background (2.135), (2.138) into the action (2.134) and expanding the square-root gives [112]:

$$\mathcal{L} = -\frac{\tau_{\text{M2}}}{2} e^A \Delta^{-1/2} G_{mn} \partial_a y^m \partial^a y^n - V(y^m), \quad (2.139)$$

$$\begin{aligned} V &= \tau_{\text{M2}} e^{3A} (\Delta^{-3/2} - 2\widetilde{W}) \\ &= 2e^{3A} \rho^6 \sinh^2 \chi \sum_{I=7}^8 X^I X^I, \end{aligned} \quad (2.140)$$

2.7.1 The flow to the conformal fixed point

As in section 2.3, we shall start by considering the flow to the conformal fixed point. Firstly, in the UV, we have as $r \rightarrow \infty$,

$$\rho \rightarrow 1, \quad \chi \rightarrow 0, \quad \frac{dA}{dr} \rightarrow \frac{2}{L}. \quad (2.141)$$

As $r \rightarrow -\infty$, the theory approaches the IR fixed point,

$$\rho \rightarrow 3^{1/8}, \quad \cosh(2\chi) \rightarrow 2, \quad \frac{dA}{dr} \rightarrow \frac{3^{3/4}}{L}. \quad (2.142)$$

It is possible to check that the moduli space is given by $X^7 = X^8 = 0$. Then the metric on moduli space is:

$$ds^2 = \frac{\tau_{\text{M2}}}{2} e^A \left(\frac{\cosh^2 \chi}{\rho^2} dr^2 + L^2 \rho^2 dX^I dX^I + L^2 \rho^2 \sinh^2 \chi (X^I J_{IJ} dX^J)^2 \right), \quad (2.143)$$

where now $\sum_{i=1}^6 X^i X^i = 1$.

2.7.2 A Kähler metric

Having found the metric on moduli space from a probe computation, we should now try to rewrite it in coordinates that are more suitable for field theory interpretation. As before, we first find the correct form for the Kähler metric. This time the moduli space is parametrized by three complex coordinates z_1, z_2 and z_3 which transform as a triplet under the $SU(3)$ flavour symmetry. This implies that the Kähler potential K is a function of $u^2 = z_1 \bar{z}_1 + z_2 \bar{z}_2 + z_3 \bar{z}_3$ only. Following the same steps as in section 2.3.2, we find that:

$$ds^2 = K'(dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 + dz_3 d\bar{z}_3) + K'' |z_1 d\bar{z}_1 + z_2 d\bar{z}_2 + z_3 d\bar{z}_3|^2. \quad (2.144)$$

Since the z_i are rotated into each other under the $SU(3)$ symmetry, we can write

$$z_1 = u(X^1 + iX^2), \quad z_2 = u(X^3 + iX^4), \quad z_3 = u(X^5 + iX^6), \quad (2.145)$$

where $\sum_{I=1}^6 X^I X^I = 1$. In these new coordinates:

$$ds^2 = (K' + u^2 K'') du^2 + u^2 K' \sum_{I=1}^6 dX^I dX^I + u^4 K'' (X^I J_{IJ} dX^J)^2, \quad (2.146)$$

where J_{IJ} is defined as before. Matching this metric to that in equation (2.143), we find the following equations:

$$(K' + u^2 K'') du^2 = \frac{\tau_{M2} e^A \cosh^2 \chi}{2 \rho^2} dr^2, \quad (2.147)$$

$$u^2 K' = \frac{\tau_{M2}}{2} L^2 e^A \rho^2, \quad (2.148)$$

$$u^4 K'' = \frac{\tau_{M2}}{2} L^2 e^A \rho^2 \sinh^2 \chi. \quad (2.149)$$

Therefore we can define u and K by:

$$\frac{du}{dr} = \frac{u}{L \rho^2}, \quad (2.150)$$

$$\frac{dK}{d(u^2)} = \frac{\tau_{M2} L^2 e^A \rho^2}{2 u^2}. \quad (2.151)$$

This leaves us to check that equation (2.149) holds. For this to be true, one requires:

$$\frac{d}{dr}(e^A \rho^2) = \frac{2}{L} e^A \cosh^2 \chi, \quad (2.152)$$

which can be proved using the flow equations (2.136). Therefore, we have shown that the metric (2.143) can be put into Kähler form. In fact, as before, it is possible to find an exact expression for K in terms of the functions ρ and A :

$$K = \frac{\tau_{M2}}{4} L^2 e^A \left(\rho^2 + \frac{1}{\rho^6} \right). \quad (2.153)$$

2.7.3 The flow with non-zero vevs

Having considered the holographic flow to the non-trivial fixed point, we can now study a class of flows where one switches on a vev, in analogy to section 2.4.

It is possible to study solutions of (2.136) numerically, such that as $r \rightarrow \infty$, $\chi \rightarrow 0^+$ and $\rho \rightarrow 1^+$. One finds that the flow geometry terminates at some finite $r = r_0$. As $r \rightarrow r_0$,

$$\chi \rightarrow 0, \quad \rho \rightarrow \infty, \quad A \rightarrow -\infty. \quad (2.154)$$

Furthermore,

$$e^{3A} \rho^6 \sinh^2 \chi \rightarrow 0, \quad e^A \rho^2 \rightarrow C \neq 0, \quad (2.155)$$

where C is a constant. Therefore, we again have two loci:

$$\begin{aligned} \text{Locus I:} \quad & X^7 = X^8 = 0, \\ \text{Locus II:} \quad & r = r_0. \end{aligned} \quad (2.156)$$

We can put the metric on locus I into Kähler form by the change of coordinates described above for the flow to the conformal fixed point. As $r \rightarrow r_0$, one finds that

$$ds^2|_{\text{I} \cap \text{II}} = \frac{\tau_{\text{M2}} CL^2}{2} \left(\sum_{I=1}^6 dX^I dX^I \right) = \frac{\tau_{\text{M2}} CL^2}{2} d\Omega_5^2. \quad (2.157)$$

where $d\Omega_5^2$ is the metric on a unit 5-sphere. Now we have the situation that locus I is an \mathbb{R}^6 with an S^5 removed. The metric on locus II is given by

$$ds^2|_{\text{II}} = \frac{\tau_{\text{M2}} CL^2}{2} \left(\sum_{I=1}^6 dX^I dX^I \right). \quad (2.158)$$

where now we have the constraint that $\sum_{I=1}^6 X^I X^I \leq 1$, so that locus II is a six-dimensional ball that fills out the rest of the \mathbb{R}^6 .

On locus II we should now choose coordinates so that

$$z_1 = u_0(X^1 + iX^2), \quad z_2 = u_0(X^3 + iX^4), \quad z_3 = u_0(X^5 + iX^6), \quad (2.159)$$

so that the metric on locus II is (up to constant factors) $\sum_i dz_i d\bar{z}_i$.

2.8 Summary and future possibilities

In this chapter we have used the simple technique of considering a brane probe in a background corresponding to a RG flow to elucidate new predictions for the gauge theory. In the cases we considered the probe calculation was able to reproduce the expected moduli space from the ten-dimensional supergravity solution. This included (in the case of non-zero vevs) the probe exhibiting sensible physics²⁰ at the singularity $r = r_0$.

²⁰It would be interesting to consider further how this can be interpreted in terms of holography. In particular, for the example studied in section 2.4.1, the radial coordinate on moduli space (which is an energy scale) is a non-trivial function of both r and θ .

We were also able to make predictions for the Kähler potential for various supersymmetric gauge theories. For these theories holomorphy arguments and global symmetries can be used to make quite powerful predictions about the form of the superpotential. This makes predictions for the Kähler potential rather difficult to check, but very interesting for that reason.

It is remarkable that in each case, it was possible to find relatively simple expressions for the Kähler potential in terms of the supergravity fields. (However, in the case of the Coulomb branch flows, the supergravity form of the Kähler potential was more complicated than the very simple form in terms of gauge theory variables.) Furthermore, we were able to find a Kähler potential in each case that satisfied (2.133). It would be interesting to understand the relevance of this equation in terms of gauge theory and gauged supergravity.

Having found new coordinates, it was quite easy to rederive the scaling dimensions of the complex scalar fields ϕ_i at an IR fixed point. Normally in the AdS-CFT correspondence, finding the scaling dimensions of gauge invariant operators requires finding the masses of linearized fluctuations about the background which can be rather cumbersome.

It is possible to extend this type of analysis to other RG flow supergravity solutions. In [2] a solution (found in [79]) corresponding to a mass deformation of the $\mathcal{N} = 4$ theory preserving $\mathcal{N} = 2$ supersymmetry was considered. This solution had previously been studied in [105, 106]. Again, we were able to find a Kähler potential in terms of the supergravity fields. Another example in [2] was that of the same four-dimensional (gauged supergravity) flow studied in 2.7, but lifted to a *different* supergravity solution in eleven dimensions. (Instead of using a deformed S^5 metric the lift ansatz uses a deformed $T^{1,1}$.)

An entirely natural question to ask is whether the results presented here for Kähler metrics on probe moduli spaces can be extended to the *whole* transverse space in which the probe moves. Unfortunately, for the initial example studied in section 2.3, we have been unable to put the metric on the six-dimensional transverse space into Kähler form. (At some point in the calculation, one finds that consistency conditions on the partial derivatives of K are not satisfied by applying the flow

equations (2.27).) This might be a signal that supergravity is unable to reproduce all the expected features of the gauge theory. On the other hand, there may be a mistake in our calculations or even in the supergravity solution itself. Another feature of the probe result presented here that would be interesting to fully understand is the form of the potential in the probe Lagrangian (2.38).²¹

Another calculation that could be made is finding the exact form of the one-loop correction to the Kähler potential (2.74). The exact numerical coefficient in that correction is possibly an important test of the results presented here.

A natural extension to the work presented in this chapter would be to include world-volume fermions in the probe action (after all, it is supposed to be describing a supersymmetric theory). This would probably not only give results on the gauge theory superpotential but also provide an independent check on the supergravity solution (especially the two-form fields). Work on probing similar backgrounds using this method has been done by Graña and Polchinski [113, 114].

²¹I understand some progress has been made on this by Evans, Johnson and Petrini (unpublished work).

Chapter 3

$\mathcal{N} = 2$ Supersymmetric $SU(N)$

Gauge Theory and the Enhancement

Having considered supergravity duals of gauge theories with $\mathcal{N} = 1$ supersymmetry, let us now turn to studying supergravity backgrounds that are connected to the physics of $\mathcal{N} = 2$ gauge theories. In order to do this it is reasonable to consider supergravity solutions that arise from brane configurations which at low energies are governed by $\mathcal{N} = 2$ gauge theories. In particular, in this section we shall concentrate on the case of pure $\mathcal{N} = 2$ supersymmetric Yang-Mills theory (*i.e.* no matter hypermultiplets). Finding a supergravity dual for this theory (and other theories with 8 conserved supercharges) was part of the motivation for the paper by Johnson, Peet and Polchinski [84]. Although a precise supergravity dual was not found, a certain type of naked spacetime singularity was resolved by studying the behaviour of a D-brane probe in the geometry. It was found that the singularity is unphysical and that the geometry is well-behaved once the D-branes that source the geometry are correctly taken into account — this was dubbed the ‘enhancement mechanism’.

Furthermore, it was suggested [84] that the solution of low-energy $\mathcal{N} = 2$ gauge theories found by Seiberg & Witten [21, 115] (extended to the $SU(N)$ case in [116, 117]) can be related to the enhancement.

In this chapter we shall first review the enhancement argument in the case related to four-dimensional field theory. Because this particular geometry is rather patho-

logical, we then review geometries found in [118, 119] that correspond to wrapped branes and are rather better behaved. We shall see that these supergravity solutions are able to reproduce the perturbative behaviour of the field theory.

We then calculate some exact non-perturbative results using the $SU(N)$ Seiberg-Witten curve and relate these to the enhancement viewpoint. In particular we point out that the relationship between Seiberg-Witten theory and the enhancement proposed in [84] is incomplete. The results found by the Seiberg-Witten method reproduce those found from supergravity at large N , but are also valid for any N .

3.1 The D7/D3 enhancement

Consider N D7-branes wrapped on a K3 manifold in type IIB string theory [84, 120]. Since K3 is a compact four-dimensional manifold¹, a wrapped D7-brane looks like a type of 3-brane carrying a $(3 + 1)$ -dimensional worldvolume theory, at least when the volume of the K3 is small. The branes have a two-dimensional transverse space which we shall parametrize with a complex scalar w . Since the K3 breaks half the supersymmetries and adding D7-branes breaks another half, it turns out that the field theory on the branes is $\mathcal{N} = 2$ $U(N)$ super-Yang-Mills with no hypermultiplets² [84, 120]. Because the K3 manifold has a non-trivial curvature class, the wrapped D7 has an induced D3-brane charge of -1 [122]. Let us therefore denote a wrapped D7 by $D3^*$. It is also possible to add normal D3-branes without breaking supersymmetry further, although we shall consider the case of only having $D3^*$ s.

	x^0	x^1	x^2	x^3	w	K3
$D7 = D3^*$	-	-	-	-	•	•
$D3$	-	-	-	-	•	•

Table 3.1: The D7/D3-brane enhancement configuration.

In the case of parallel D_p -branes in flat space the scalars that appear in the

¹For a discussion of K3 surfaces see, for instance, [121].

²Usually we will ignore the diagonal $U(1)$ in $U(N)$ and consider $SU(N)$ only.

world-volume theory on the branes are interpreted as transverse positions of the branes. We can do the same here because the $\mathcal{N} = 2$ field theory has a complex scalar field, Φ , which corresponds to w . In fact at the classical level, the field theory has a moduli space of vacua parametrized by the vev of Φ . One might expect that for generic values of $\langle \Phi \rangle$ the gauge group would be broken to $U(1)^{N-1}$, but by setting $\langle \Phi \rangle = 0$ the $SU(N)$ gauge symmetry would remain unbroken. However, this is not the case when quantum corrections to the low-energy field theory are considered — it is not possible to set $\langle \Phi \rangle = 0$ and so the low-energy theory never has unbroken $SU(N)$ gauge symmetry.

It is possible to see this effect from the supergravity solution for the D7/D3 system outlined above. Initially, we can find a supergravity solution which corresponds to attempting to put all the branes at the origin of the w -plane. The geometry is given by:

$$\begin{aligned}
 ds^2 &= \frac{1}{\sqrt{Z_3 Z_7}} \eta_{\mu\nu} dx^\mu dx^\nu + (\alpha')^2 \sqrt{Z_3 Z_7} dw d\bar{w} + V_0^{1/2} \sqrt{\frac{Z_3}{Z_7}} ds_{K3}^2, \\
 e^\Phi &= \frac{g}{Z_7}, \\
 C_{(4)} &= \frac{1}{g Z_3} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \\
 C_{(8)} &= \frac{V_0}{g Z_7} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge \epsilon_{K3},
 \end{aligned} \tag{3.1}$$

where V_0 is a (dimensionful) constant that determines the volume of the K3 and

$$\begin{aligned}
 Z_3 &= \frac{gN}{2\pi} \frac{(2\pi)^4 (\alpha')^2}{V_0} \ln \left(\frac{W}{\rho_3} \right), \\
 Z_7 &= \frac{gN}{2\pi} \ln \left(\frac{\rho_7}{W} \right),
 \end{aligned} \tag{3.2}$$

and $W = |w|$. This geometry is only valid for $\rho_3 < W < \rho_7$. At $W = \rho_7$, we have $Z_7 = 0$ and therefore the dilaton diverges. For a sensible supergravity dual we should have that the dilaton is small because it is the effective string coupling. For $W = \rho_3$, $Z_3 = 0$ and this is in fact a naked “repulson” singularity.

If this solution is sourced by a stack N coincident D3*-branes then it should be possible to bring in branes from infinity to construct the geometry³. This can be

³This process is not very sensible in this case because the geometry is not defined for $W > \rho_7$.

checked by studying the behaviour of a probe D3*. (Already we can see that because the geometry is not valid for $W < \rho_3$, it will not be possible to put the branes at $W = 0$.)

The probe computation [84] reveals that although a probe D3* has a two-dimensional moduli space, the metric on that moduli space is given by:

$$\begin{aligned} ds^2 &= \frac{1}{2g}(\mu_7 V_0 Z_3(W) - \mu_3 Z_7(W)) dw d\bar{w} \\ &= \frac{\mu_7 Z_7(W)}{2g}(V_{K3}(W) - V_*) dw d\bar{w} . \end{aligned} \quad (3.3)$$

Therefore, the probe cannot move below a certain value of W given by

$$V_{K3}(W) = V_* . \quad (3.4)$$

Here, $V_* = (2\pi)^4(\alpha')^2$ is a volume defined by the string scale and V_{K3} is the volume of the K3 at radius W :

$$V_{K3}(W) = V_0 \frac{Z_3}{Z_7} . \quad (3.5)$$

From the point of view of the probe D3*, it sees the following metric on moduli space:

$$ds^2 = \frac{N}{(2\pi)^4} \log \left(\frac{W}{\sqrt{\rho_3 \rho_7}} \right) dw d\bar{w} . \quad (3.6)$$

The interpretation of all this is that the branes are not situated at the origin of the W -plane, but form a ring, the enhancement, given by:

$$W = W_e \equiv \sqrt{\rho_3 \rho_7} > \rho_3 . \quad (3.7)$$

Therefore, the geometry for $W < W_e$ is unphysical. It can be consistently excised and replaced by a flat, sourceless geometry. (This still leaves the difficulty of the singularity at $W = \rho_7$. In the next section we shall study a set of supergravity solutions that do not suffer from this problem.)

Now consider the more general situation of trying to write down the geometry when one attempts to place the branes in a circle of radius W_0 . In the region outside the branes the solution would be as before, whereas inside the metric would

In the case of wrapped D6 or D5 branes the geometry does not have such a singularity and moving branes in from large W is quite reasonable.

be constant. If $W_0 < W_e$ then one again has an enhancement at $W = W_e$ and the same excision should be carried out. However, if one tries to place the branes at $W_0 > W_e$ then the enhancement does not appear because $V_{K3}(W)$ never reaches V_* . Therefore the actual radius of the ring of branes is $\max(W_0, W_e)$. To summarize, if one tries to place the branes on a circle of radius W_0 then the correct geometry is given by

$$\begin{aligned} Z_3 &= \frac{gN V_*}{2\pi V_0} \ln\left(\frac{\rho}{\rho_3}\right), \\ Z_7 &= \frac{gN}{2\pi} \ln\left(\frac{\rho_7}{\rho}\right), \end{aligned} \quad (3.8)$$

where

$$\rho(W) = \begin{cases} W & \text{if } W > \max(W_0, W_e), \\ \max(W_0, W_e) & \text{otherwise.} \end{cases} \quad (3.9)$$

The metric on moduli space seen by the probe D3* is then

$$ds^2 = \frac{N}{(2\pi)^4} \log\left(\frac{\rho(W)}{\sqrt{\rho_3\rho_7}}\right) dw d\bar{w}. \quad (3.10)$$

3.2 Supergravity duals from wrapped D5-branes

Although the D7/D3 configuration described above is relatively simple, it suffers from various problems. Because D7-branes only have a two-dimensional transverse space they induce a deficit angle on the spacetime geometry. For $N = 24$, the deficit angle is such that the transverse space is a 2-sphere. However for large N , this becomes a serious problem. Furthermore, the solution is only sensible for $\rho_3 < W < \rho_7$. Near $W = \rho_7$ the dilaton diverges and one would need to include non-perturbative corrections to remove the singularity [84].

Therefore, we shall briefly reproduce some of results obtained by Gauntlett *et al.* [118] and Bigazzi *et al.* [119]. These papers present solutions⁴ of type IIB supergravity that correspond to large numbers of D5-branes wrapped on a two-sphere in a Calabi-Yau two-fold, *i.e.* a K3 manifold. (This configuration can be related to the one studied previously by T-dualities.) The low-energy field theory of these branes is $\mathcal{N} = 2$ supersymmetric $SU(N)$ Yang-Mills theory with no hypermultiplets.

⁴These solutions were also studied in [123].

Therefore these solutions should be an alternative set of supergravity duals to the problematic D7/D3 solutions described in section 3.1.

3.2.1 A ten-dimensional solution

We shall use the solution and conventions of Gauntlett *et al.* [118]. Their ten-dimensional metric is:

$$ds^2 = e^\Phi \left(\eta_{ab} dx^a dx^b + z(d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2) + \frac{e^{2x}}{N} dz^2 + Nd\theta^2 + \frac{Ne^{-x}}{\Omega} \cos^2 \theta (d\phi_1 + \cos \tilde{\theta} d\tilde{\phi})^2 + \frac{Ne^x}{\Omega} \sin^2 \theta d\phi_2^2 \right), \quad (3.11)$$

where the dilaton is given by⁵

$$e^{2\Phi} = e^{-2\Phi_0} e^{2z/N} e^{-x} \Omega, \quad (3.12)$$

and

$$\Omega = e^x \cos^2 \theta + e^{-x} \sin^2 \theta, \quad (3.13)$$

$$e^{-2x} = 1 - \frac{N(1 + ke^{-2z/N})}{2z}. \quad (3.14)$$

Here k is a constant and the D5-branes wrap the $\tilde{\theta}, \tilde{\phi}$ directions. The qualitative behaviour of the solution is different for varying values of k and is determined through the behaviour of e^{-2x} :

$$\begin{aligned} e^{-2x} &\rightarrow \infty & \text{as } z &\rightarrow 0 & \text{for } k < -1, \\ e^{-2x} &\rightarrow 0 & \text{as } z &\rightarrow 0 & \text{for } k = -1, \\ e^{-2x} &\rightarrow 0 & \text{as } z &\rightarrow z_0 > 0 & \text{for } k > -1. \end{aligned} \quad (3.15)$$

Gauntlett *et al.* argue that the solutions with $k \geq -1$ are related to gravity descriptions of a part of the Coulomb branch of $\mathcal{N} = 2$ Yang-Mills theory, whilst the $k < -1$ solutions are unphysical.

⁵This formula has $e^{-\Phi_0}$, rather than e^{Φ_0} , because the solution has been obtained from a background corresponding to NS5-branes by a S-duality transformation that takes $\Phi \rightarrow -\Phi$.

3.2.2 The probe computation

The above solution has been probed using a wrapped D5-brane [118, 119] to match the geometry to gauge theory expectations. We briefly present the results here, which are somewhat similar to the results in chapter 2. One finds that the moduli space is given by:

$$\begin{aligned} \text{Locus I: } & \theta = \frac{\pi}{2} \quad \text{for all } k, \\ \text{Locus II: } & z = z_0 \quad \text{for } k \geq -1. \end{aligned} \quad (3.16)$$

The metric on moduli space can be put into the following form⁶ on locus I:

$$ds^2|_{\text{I}} = \frac{N}{4\pi^2} \log\left(\frac{|w|}{\Lambda}\right) dw d\bar{w}, \quad (3.17)$$

where w is defined by

$$w = \Lambda e^{z/N} e^{i\phi_2}, \quad \Lambda = \frac{\sqrt{N}}{2\pi e^{\Phi_0}}. \quad (3.18)$$

and so in the physical case $k \geq -1$ satisfies $w \geq w_0 = \Lambda e^{z_0/N}$. For $k \geq -1$ one also has locus II which fills out the disc missing from locus I:

$$ds^2|_{\text{II}} = \frac{N}{4\pi^2} \log\left(\frac{w_0}{\Lambda}\right) dw d\bar{w}, \quad (3.19)$$

where now

$$w = w_0 \sin \theta e^{i\phi_2}, \quad w_0 = \Lambda e^{z_0/N}. \quad (3.20)$$

In fact the brane action can be put into a form consistent with $\mathcal{N} = 2$ supersymmetry:

$$S = \frac{1}{8\pi} \int d^4x \left(-\text{Im} \tau(w) \partial_a w \partial^a \bar{w} + \frac{1}{2} \text{Re} \left[\tau(w) (iF^2 + F\tilde{F}) \right] \right). \quad (3.21)$$

where

$$\tau(w) = \begin{cases} \frac{2Ni}{\pi} \log\left(\frac{w}{\Lambda}\right) & \text{if } |w| > w_0 \\ \frac{2Ni}{\pi} \log\left(\frac{w_0}{\Lambda}\right) & \text{if } |w| \leq w_0 \end{cases}. \quad (3.22)$$

⁶Here we have set $\alpha' = 1$.

3.3 $\mathcal{N} = 2$ $SU(N)$ gauge theory

3.3.1 Matching to the probe results

In this section we will review how the results from the probe computations can be matched to perturbative results of $\mathcal{N} = 2$ $SU(N)$ gauge theory [84, 105, 118, 119].

The field content of pure four-dimensional $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory is a massless vector A_μ , two Weyl fermions (λ, ψ) , and a complex scalar field Φ that all transform in the adjoint representation of $SU(N)$. The D-term gives rise to a classical scalar potential:

$$V = \frac{1}{g^2} \text{Tr}[\Phi, \Phi^\dagger]^2, \quad (3.23)$$

which has the following supersymmetric vacua:

$$\langle \Phi \rangle = \text{diag}(a_1, \dots, a_N), \quad (3.24)$$

subject to

$$\sum_{i=1}^N a_i = 0. \quad (3.25)$$

For generic values of the a_i the $SU(N)$ gauge group is broken to $U(1)^{N-1}$, with the W-bosons having masses $m_{ij} = \sqrt{2}|a_i - a_j|$. If some of the a_i coincide then the gauge symmetry is enhanced.

In the generic case of $U(1)^{N-1}$, the low-energy Lagrangian is of the following form [21, 105, 118]:

$$S = \frac{1}{8\pi} \int d^4x \left(-\text{Im} \tau_{ij} \partial a_i \partial \bar{a}_j + \frac{1}{2} \text{Re} \left[\tau_{ij} (iF^i F^j + F^i \tilde{F}^j) \right] \right). \quad (3.26)$$

where the couplings are derived from an holomorphic prepotential

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}. \quad (3.27)$$

The one-loop prepotential is given by

$$\mathcal{F} = \frac{i}{8\pi} \sum_{i \neq j} (a_i - a_j)^2 \log \frac{(a_i - a_j)^2}{\mu^2}, \quad (3.28)$$

and in fact this is exact in perturbation theory. There are non-perturbative corrections from instantons, but these vanish in the large N limit provided $|a_i - a_j| > O(1/N)$ [124].

In the probe approximation we should take the following

$$\langle \Phi \rangle = \text{diag}(w, a_1 - w/N, \dots, a_N - w/N), \quad (3.29)$$

so that the coupling on the probe is given by

$$\tau(w) = \frac{\partial^2 \mathcal{F}}{\partial w^2} = \frac{i}{2\pi} \sum_i \log \frac{(w - a_i)^2}{\mu^2}. \quad (3.30)$$

At large N this can be approximated by an integral:

$$\tau(w) = \frac{i}{2\pi} \int d^2 a \rho(a) \log \frac{(w - a)^2}{\mu^2}, \quad (3.31)$$

where the density $\rho(a)$ is normalized so that $\int d^2 a \rho(a) = N$.

For the case discussed above, where the branes are distributed on a circle of radius w_0 , we can take

$$\rho(a) = \frac{N}{2\pi w_0} \delta(|a| - w_0), \quad (3.32)$$

giving

$$\tau(w) = \begin{cases} \frac{Ni}{\pi} \log \left(\frac{w}{\mu} \right) & \text{if } |w| > w_0 \\ \frac{Ni}{\pi} \log \left(\frac{w_0}{\mu} \right) & \text{if } |w| \leq w_0 \end{cases} \quad (3.33)$$

This matches the form found by the probe computations outlined in sections 3.1 and 3.2 (with some differences in normalizations).

3.3.2 The Seiberg-Witten solution

Quantum mechanically, the moduli space is parametrized by vevs of the following gauge invariant operators:

$$u_k \equiv \text{Tr}(\Phi^k). \quad (3.34)$$

These can be expressed in terms new parameters ϕ_i :

$$u_k = \sum_{i=1}^N \phi_i^k \quad \text{where} \quad \sum_{i=1}^N \phi_i = 0. \quad (3.35)$$

The a_i then become non-trivial functions of the ϕ_i , with $a_i \simeq \phi_i$ only at weak coupling, *i.e.* for large ϕ_i . The interpretation of this in terms of the enhançon is that ϕ_i are the classical positions of the branes in the transverse space, while the $a_i(\phi_j)$ are the physical (quantum corrected) transverse positions of the branes. In

particular, the a_i are the quantities that appear in the BPS formula for the W-boson masses⁷. Generally, the masses of BPS states are given by

$$M_{\text{BPS}} = \sqrt{2}|Z| , \quad (3.36)$$

where the central charge Z is

$$Z = \sum_{i=1}^N (q_i a_i + h_i a_{D_i}) , \quad (3.37)$$

and the integers q_i (electric charges) and h_i (magnetic charges) satisfy

$$\sum_{i=1}^N q_i = 0 , \quad \sum_{i=1}^N h_i = 0 . \quad (3.38)$$

The dual variables a_{D_i} are derived from the prepotential \mathcal{F} by [21]:

$$a_{D_i} = \frac{\partial \mathcal{F}}{\partial a_i} , \quad (3.39)$$

and the complex coupling matrix is

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} = \frac{\partial a_{D_j}}{\partial a_i} . \quad (3.40)$$

The question is then how to calculate the a_i , a_{D_i} from the parameters ϕ_i . This was solved for the $SU(2)$ case by Seiberg & Witten [21, 115]. Some reviews of this subject are [12, 13, 16, 30].

Following Seiberg and Witten's method⁸, Argyres and Faraggi [116] and Klemm, Lerche, Yankielowicz and Theisen [117] proposed a solution of the $SU(N)$ case based on the following hyperelliptic curve⁹:

$$y^2 = p(x)^2 - \Lambda^{2N} , \quad (3.41)$$

where the $p(x)$ is a order N polynomial in x with the ϕ as its roots:

$$p(x) = \prod_{i=1}^N (x - \phi_i) = x^N - \sum_{i=2}^N u_i x^{N-i} . \quad (3.42)$$

⁷Strictly, the equivalence of brane positions and W-boson masses is a result derived in Minkowski space, and will not be true in more general backgrounds.

⁸A good textbook on complex function theory is [125].

⁹For the rest of this chapter we will take Λ to be real and positive.

Here, we have introduced Λ which is the dynamically generated scale of the theory.

The a_i and the a_{Di} can then be found using the curve by performing certain period integrals:

$$a_i(\phi_j) = \oint_{\alpha_i} \lambda_{\text{SW}} , \quad a_{Di}(\phi_j) = \oint_{\beta_i} \lambda_{\text{SW}} , \quad (3.43)$$

where the α_i are ‘electric’ contours and the β_i are ‘magnetic’ contours. The Seiberg-Witten differential is given by:

$$\begin{aligned} \lambda_{\text{SW}} &= \frac{1}{2\pi i} \frac{x \, dp}{y} \\ &= \frac{1}{2\pi i} \frac{x p'(x) \, dx}{y} . \end{aligned} \quad (3.44)$$

Actually performing the contour integrals is rather impractical for most cases. However, it has been found that the period integrals satisfy certain ‘Picard-Fuchs’ equations which are partial differential equations in the u_i defined in (3.42):

$$\mathcal{L}_n a(\phi_j) = 0 , \quad (3.45)$$

where the differential operators \mathcal{L}_n are given by:

$$\begin{aligned} \mathcal{L}_0 &= N \partial_2 \partial_{N-1} - \sum_{j=2}^N (N-j) u_j \partial_{j+1} \partial_N , \\ \mathcal{L}_n &= -n \partial_{N+1-n} + N \partial_2 \partial_{N-1-n} - \sum_{j=2}^N (N-j) u_j \partial_{N+1-n} \partial_j \\ &\quad (\text{for } n = 1, \dots, N-3) , \\ \mathcal{L}_{N-2} &= 1 + \sum_{j=2}^N j(j-2) u_j \partial_j + \sum_{i,j=2}^N i j u_i u_j \partial_i \partial_j - N^2 \Lambda^{2N} \partial_N^2 , \end{aligned} \quad (3.46)$$

where

$$\partial_i = \frac{\partial}{\partial u_i} . \quad (3.47)$$

3.3.3 A circular distribution of branes

In order to determine the correct strong coupling results (*i.e.* for small ϕ_i) one should be able to compare to known results at weak coupling (large ϕ_i). To do this it is useful to consider various one (complex) dimensional subspaces of the moduli space.

One way of doing this is to fix the classical shape of the distribution of branes and then scale and rotate this distribution with a parameter ϕ :

$$\begin{aligned}\phi_i &= \gamma_i \phi, \quad i = 1, \dots, N, \\ p(x) &= \prod_{i=1}^N (x - \gamma_i \phi).\end{aligned}\tag{3.48}$$

Although the Picard-Fuchs equations simplify with this restriction, it is still unclear how to extract useful results from them. Given this, let us consider the following choice for the γ_i :

$$\begin{aligned}\phi_i &= \omega^{i-1} \phi \quad \text{where} \quad \omega = e^{2\pi i/N} \quad \text{and} \quad i = 1, \dots, N. \\ p(x) &= \prod_{i=1}^N (x - \omega^{i-1} \phi) \\ &= x^N - u, \quad \text{so} \quad u = \phi^N.\end{aligned}\tag{3.49}$$

Here, we are trying to put the branes in a circle, with the first brane at position ϕ (see figure (3.2)). In this case the final Picard-Fuchs equation simplifies to¹⁰:

$$\left(N^2(u^2 - \Lambda^{2N}) \frac{d^2}{du^2} + N(N-2)u \frac{d}{du} + 1 \right) a(u) = 0.\tag{3.53}$$

Now consider the following change of variables

$$\alpha = \frac{u^2}{\Lambda^{2N}}.\tag{3.54}$$

After some algebra one finds that equation (3.53) becomes

$$\left(\alpha(1-\alpha) \frac{d^2}{d\alpha^2} + \left(\frac{1}{2} - \left(1 - \frac{1}{N} \right) \alpha \right) \frac{d}{d\alpha} - \frac{1}{4N^2} \right) a(\alpha) = 0.\tag{3.55}$$

This is in fact a hypergeometric differential equation with $a = b = -1/2N$ and $c = 1/2$, and has the following general solution [126]:

$$a(u) = A(u^2 - \Lambda^{2N})^{\mu/2} P_\nu^\mu \left(\frac{u}{\Lambda^N} \right) + B(u^2 - \Lambda^{2N})^{\mu/2} Q_\nu^\mu \left(\frac{u}{\Lambda^N} \right),\tag{3.56}$$

¹⁰It is possible to check equation (3.53) independently using Maple. One finds:

$$\left(N^2(u^2 - \Lambda^{2N}) \frac{d^2}{du^2} + N(N-2)u \frac{d}{du} + 1 \right) \lambda_{\text{sw}}(u) = \frac{\partial g}{\partial x},\tag{3.51}$$

where

$$g = \frac{x^{N+1}(x^{2N} - (N+2)x^N u + (N+1)u^2 + (N-1)\Lambda^{2N})}{y^3}.\tag{3.52}$$

with

$$\mu = \frac{1}{2} + \frac{1}{N}, \quad \nu = -\frac{1}{2}, \quad (3.57)$$

and where P_ν^μ and Q_ν^μ are the associated Legendre functions [126]. This can be checked by putting into equation (3.53) the following:

$$a(u) = (u^2 - \Lambda^{2N})^{\mu/2} f(u), \quad (3.58)$$

which results in the Legendre differential equation [126]:

$$\left((1-v^2) \frac{d^2}{dv^2} - 2v \frac{d}{dv} + \left[\nu(\nu+1) - \frac{\mu^2}{1-v^2} \right] \right) f(v) = 0, \quad (3.59)$$

where we have changed variable to $v = u/\Lambda^N$. Therefore it is possible to find the general form of the period integrals a_i, a_{Di} from the Picard-Fuchs equations.

3.3.4 Direct calculation of period integrals

Interestingly, in the case of the circular distribution of branes it is possible to evaluate the period integrals directly, without resorting to the Picard-Fuchs equations. Let us first concentrate on the calculation of the $a_i(\phi)$:

$$a_i(\phi) = \oint_{\alpha_i} \lambda_{\text{SW}}, \quad (3.60)$$

where the α_i are ‘electric’ contours and are shown in figure (3.1). They encircle the branch cuts in the x -plane. The branch cuts run between singularities of the integrand which are where $y^2 = p^2 - \Lambda^{2N}$ vanishes. These are given by

$$\begin{aligned} x_{i,+} &= \omega^{i-1} (\phi^N + \Lambda^N)^{1/N} & \text{where } i = 1, \dots, N. \\ x_{i,-} &= \omega^{i-1} (\phi^N - \Lambda^N)^{1/N}, \end{aligned} \quad (3.61)$$

and $\omega = e^{2\pi i/N}$. Before we evaluate the above integrals one can make the following observation. Using the form of λ_{SW} given in (3.44):

$$\begin{aligned} a_i(\phi) &= \oint_{\alpha_i} \frac{1}{2\pi i} \frac{x p' dx}{\sqrt{p^2 - \Lambda^{2N}}} \\ &= \oint_{\alpha_1} \frac{1}{2\pi i} \frac{N(\omega^{i-1} x)^N d(\omega^{i-1} x)}{\sqrt{((\omega^{i-1} x)^N - \phi^N)^2 - \Lambda^{2N}}} \\ &= \omega^{i-1} a_1(\phi). \end{aligned} \quad (3.62)$$

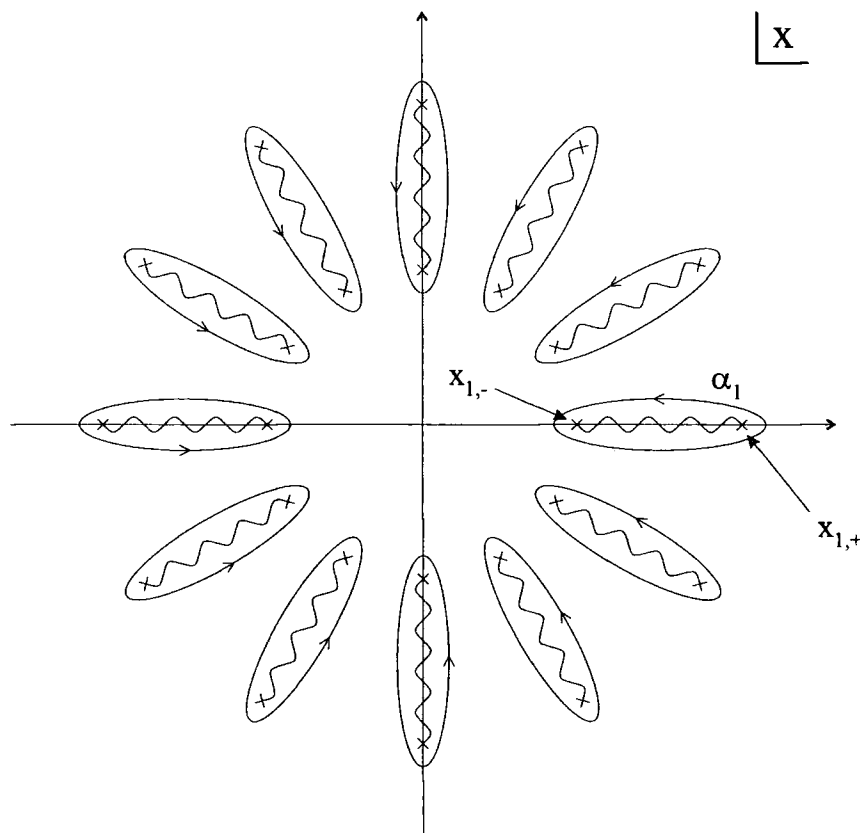


Figure 3.1: The electric contours α_i in the x -plane. Here we have taken ϕ to be real and positive with $\phi > \Lambda$.

Therefore the branes are still arranged on a circle if one takes the a_i to be the branes' transverse positions rather than the ϕ_i (see figure 3.2). Therefore, we only need to calculate $a_1(\phi)$.

Furthermore, it is interesting to consider the case $\Lambda = 0$. This is the classical limit where the dynamically generated scale, Λ , can be ignored. The integrals a_i can then be evaluated for any choice of ϕ_i (and not just the circular case we are studying here) by using Cauchy's theorem. One finds that $a_i = \phi_i$, so the period integral reproduces the correct result in the classical regime of $\Lambda = 0$.

Returning to the case of the ϕ lying on a circle, with $\Lambda \neq 0$, we can now consider:

$$\begin{aligned} a_1(\phi) &= \oint_{\alpha_1} \frac{1}{2\pi i} \frac{x p' dx}{\sqrt{p^2 - \Lambda^{2N}}} \\ &= \oint_{\alpha_1} \frac{1}{2\pi i} \frac{x z' dx}{\sqrt{z^2 - 1}}, \end{aligned} \quad (3.63)$$

where we have introduced z such that $p = \Lambda^N z$. It is clear that we can rewrite this

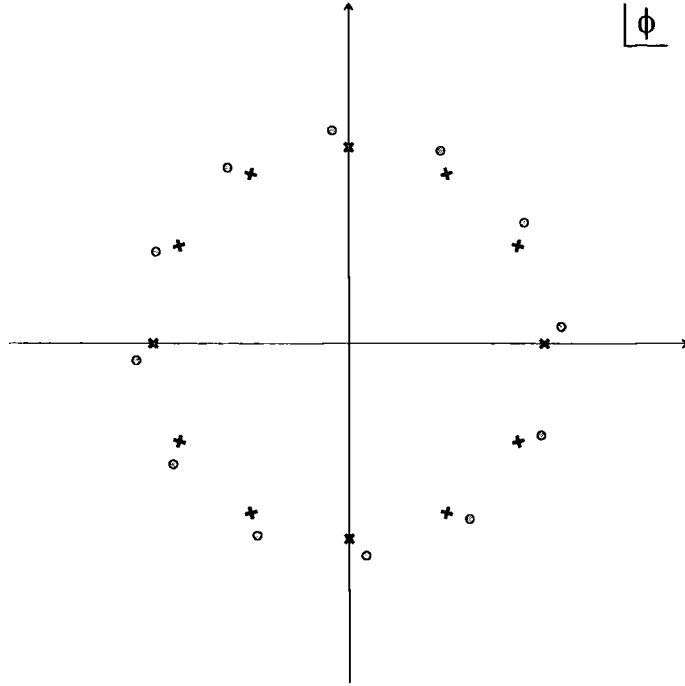


Figure 3.2: The points ϕ_i , $i = 1, \dots, N$, which correspond to the classical positions of the branes are denoted by crosses. We have chosen ϕ to be real. The quantum-corrected positions are denoted by circles.

integral as twice the integral along the bottom of the branch cut. Choosing a sign for the square root,

$$\begin{aligned}
 a_1(\phi) &= \frac{1}{\pi} \int_{x_{1,-}}^{x_{1,+}} \frac{x z' dx}{(1-z^2)^{1/2}} \\
 &= \frac{1}{\pi} \int_{x_{1,-}}^{x_{1,+}} \frac{(z\Lambda^N + \phi^N)^{1/N}}{(1-z^2)^{1/2}} z' dx \\
 &= \frac{1}{\pi} \int_{-1}^1 \frac{(z\Lambda^N + \phi^N)^{1/N}}{(1-z^2)^{1/2}} dz \\
 &= \frac{1}{\pi} \int_{-1}^1 \frac{(z\Lambda^N + \phi^N)^{1/N}}{(1-z)^{1/2}(1+z)^{1/2}} dz .
 \end{aligned} \tag{3.64}$$

Changing variable to w , where $z = 2w - 1$, gives:

$$\begin{aligned}
 a_1(\phi) &= \frac{1}{\pi} \int_0^1 \frac{((2w-1)\Lambda^N + \phi^N)^{1/N}}{(2-2w)^{1/2}(2w)^{1/2}} 2 dw \\
 &= \frac{(\phi^N - \Lambda^N)^{1/N}}{\pi} \int_0^1 w^{-1/2}(1-w)^{-1/2} \left(1 + \frac{2\Lambda^N w}{\phi^N - \Lambda^N}\right)^{1/N} dw.
 \end{aligned} \tag{3.65}$$

Now recall the integral representation of the hypergeometric function [126]:

$$F(a, b; c; \zeta) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dx x^{b-1} (1-x)^{c-b-1} (1-\zeta x)^{-a}, \quad (3.66)$$

where $\text{Re}(c) > \text{Re}(b) > 0$. This function is defined on the ζ -plane with a branch cut on the real axis between 1 and ∞ . So in particular

$$\begin{aligned} a_1(\phi) &= (\phi^N - \Lambda^N)^{1/N} F\left(-\frac{1}{N}, \frac{1}{2}; 1; \frac{-2\Lambda^N}{\phi^N - \Lambda^N}\right) \\ &= (\phi^N + \Lambda^N)^{1/N} F\left(-\frac{1}{N}, \frac{1}{2}; 1; \frac{2\Lambda^N}{\phi^N + \Lambda^N}\right), \end{aligned} \quad (3.67)$$

where the second line follows from the following identity [126]:

$$F(a, b; c; \zeta) \equiv (1-\zeta)^{-a} F\left(a, c-b; c; \frac{\zeta}{\zeta-1}\right). \quad (3.68)$$

In calculating the integral in terms of a hypergeometric function we now seem to have a problem in matching this result with that obtained using the Picard-Fuchs equations which was in terms of Legendre functions. Thankfully, there are equations relating one form to the other [126]:

$$F(a, b; 2b; \zeta) = \frac{2^{2b}}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2} + b)}{\Gamma(2b - a)} \zeta^{-b} (1-\zeta)^{(b-a)/2} e^{i\pi(b-a)} P_{b-1}^{b-a}\left(\frac{2}{\zeta} - 1\right). \quad (3.69)$$

In order to check that we have calculated the actual position of the brane we should check that the position of the brane is correctly reproduced in the weak-coupling region, *i.e.* for large ϕ . There we expect the quantum corrections to be small and $a_1 \simeq \phi$. Taking $\phi \gg \Lambda$ one finds

$$a_1(\phi) \simeq \phi \left(1 - \frac{(N-2)(2N-1)\Lambda^{2N}}{N^2(N-1)\phi^{2N}}\right). \quad (3.70)$$

The branch cut involved in the definition of F is now a branch cut for a_1 lying on the real axis of the u -plane between $-\Lambda^N$ and Λ^N .

One might worry that because the roots $x_{i,-}$ degenerate at $\phi = \Lambda$ we might not be able to match up weak-coupling results (large ϕ) to those at strong-coupling (*e.g.* ϕ small). One could envisage problems if we had to choose a different set of contours α_i for particular values of ϕ . One solution is to consider the case when $u = \phi^N$ is imaginary. In this case one can go from $|\phi| \gg \Lambda$ to $\phi = 0$ without the singularities in the x -plane degenerating and therefore check that it is possible to

continue using the same contour down to $\phi = 0$. Furthermore, the result in equation (3.67) does not depend on ϕ being real. A plot of $|a_1(\phi)|$ for this case is presented in figure 3.3 for $N = 3, 10, 100$. It is clear that for large N , equation (3.67) reproduces the form of the supergravity results. If one tries to place the branes in a circle of radius $|\phi|$ then they will actually lie in a circle of radius $\max(|\phi|, \Lambda)$, where now Λ can be interpreted as the enhançon radius.

A natural definition of the enhançon radius is given by $|a_1(0)|$, *i.e.* the actual radius of the branes when the classical radius is zero. Using equation (3.67) one finds a relatively simple expression [126]:

$$\Lambda \frac{\sqrt{\pi}}{\Gamma\left(1 + \frac{1}{2N}\right) \Gamma\left(\frac{1}{2} + \frac{1}{2N}\right)} \quad (3.71)$$

which in the limit $N \rightarrow \infty$ simply gives Λ .

Direct calculation of a_{D_i} by integration

Having found the $a_i(\phi)$ by carrying out the relevant period integrals directly, we can now attempt the same for a combination of the dual variables a_{D_i} . Many of the steps are similar in nature to those done before and again result in an answer involving an hypergeometric function.

For this we shall use the contour β_1 shown in figure 3.4. It has intersection numbers -1 with α_1 and $+1$ with α_2 . Therefore, the corresponding quantity is $(a_{D_2} - a_{D_1})$, which gives the mass of a monopole between the $U(1)_1$ and $U(1)_2$ factors. (Recall that the sum of the magnetic charges vanishes, $\sum h_i = 0$.) Similar expressions can be found by using different contours.

$$\begin{aligned} (a_{D_2} - a_{D_1})(\phi) &= \oint_{\beta_1} \frac{1}{2\pi i} \frac{x p' dx}{\sqrt{p^2 - \Lambda^{2N}}} \\ &= \frac{1}{\pi i} \int_{x_{1,-}}^{\omega x_{1,-}} \frac{x p' dx}{\sqrt{p^2 - \Lambda^{2N}}} \\ &= \frac{2}{\pi i} \int_{(\phi^N - \Lambda^N)^{1/2}}^{-(\phi^N - \Lambda^N)^{1/2}} \frac{q^{1+2/N} dq}{\sqrt{(q^2 - \phi^N)^2 - \Lambda^{2N}}} \\ &= \frac{2(\omega - 1)}{\pi i} \int_0^{(\phi^N - \Lambda^N)^{1/2}} \frac{q^{1+2/N} dq}{\sqrt{(q^2 - \phi^N)^2 - \Lambda^{2N}}} \end{aligned} \quad (3.72)$$

where we have defined $q = x^{N/2}$ in order to allow us to change the integration range.

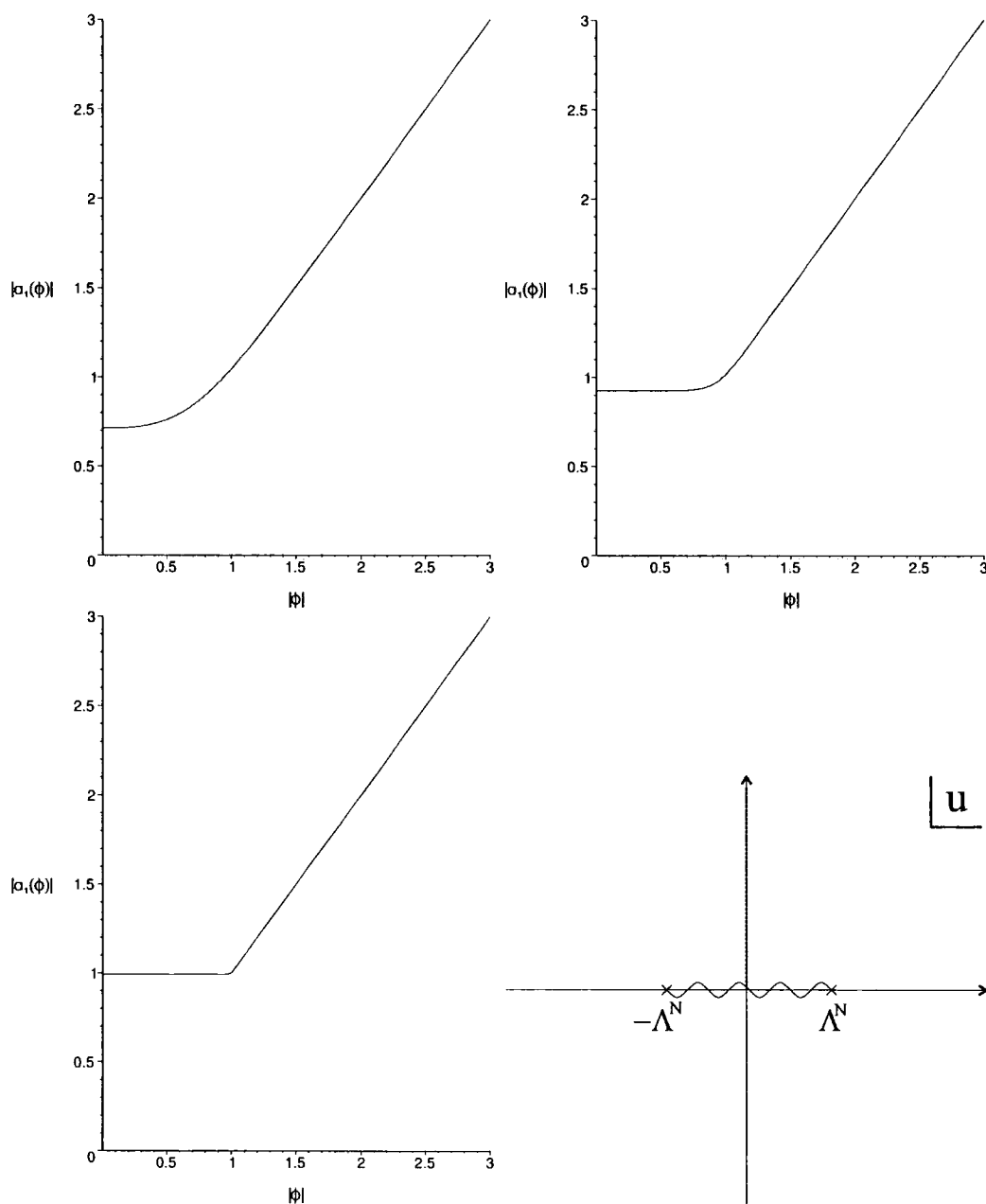
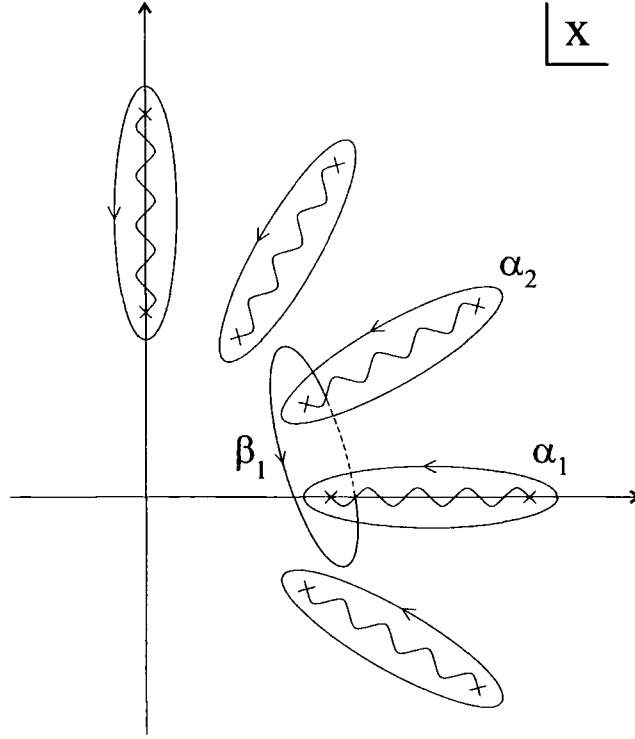


Figure 3.3: Plots of the physical radius of the branes against the ‘classical’ radius. Here we have chosen $u = \phi^N$ to be imaginary and $\Lambda = 1$. The different plots have $N = 3, 10, 100$, respectively. The u -plane has a branch cut from $-\Lambda^N$ to Λ^N .


 Figure 3.4: The contour β_1 which intersects α_1 and α_2 .

If we now use p as our integration variable, recalling that $p = q^2 - \phi^N$, then we find:

$$(a_{D2} - a_{D1})(\phi) = \frac{(\omega - 1)}{\pi i} \int_{-\phi^N}^{-\Lambda^N} \frac{(p + \phi^N)^{1/N}}{\sqrt{p^2 - \Lambda^{2N}}} dp. \quad (3.73)$$

As before, we rescale variables and define $z = p/\Lambda^N$ and $v = \phi^N/\Lambda^N$:

$$\begin{aligned} (a_{D2} - a_{D1})(\phi) &= \frac{(\omega - 1)\Lambda}{\pi i} \int_{-v}^{-1} \frac{(z + v)^{1/N}}{\sqrt{z^2 - 1}} dz. \\ &= \frac{(\omega - 1)\Lambda}{\pi i} \int_1^v \frac{(v - z)^{1/N}}{\sqrt{z^2 - 1}} dz. \end{aligned} \quad (3.74)$$

We can now put the integral into a form suitable for expression in terms of a hypergeometric function by defining t by $z = (v - 1)t + 1$:

$$\begin{aligned} (a_{D2} - a_{D1})(\phi) &= \frac{(\omega - 1)\Lambda}{\sqrt{2\pi i}} (v - 1)^{1/N+1/2} \int_0^1 \frac{(1 - t)^{1/N}}{t^{1/2} \left(1 + \frac{(v-1)t}{2}\right)^{1/2}} dt, \\ &= \frac{(\omega - 1)\Lambda}{\sqrt{2\pi i}} (v - 1)^{1/N+1/2} \frac{\Gamma(1 + \frac{1}{N})}{\Gamma(\frac{3}{2} + \frac{1}{N})} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2} + \frac{1}{N}; \frac{1 - v}{2}\right) \\ &= \frac{(\omega - 1)}{\sqrt{\pi i}} (\phi^N - \Lambda^N)^{1/N} \left(\frac{\phi^N - \Lambda^N}{\phi^N + \Lambda^N}\right)^{1/2} \frac{\Gamma(1 + \frac{1}{N})}{\Gamma(\frac{3}{2} + \frac{1}{N})} \\ &\quad \times F\left(\frac{1}{2}, 1 + \frac{1}{N}; \frac{3}{2} + \frac{1}{N}; \frac{\phi^N - \Lambda^N}{\phi^N + \Lambda^N}\right), \end{aligned} \quad (3.75)$$

where to obtain the final result we have again used equation (3.68). (This result is presumably related to the general solution found in section 3.3.3 by formulae such as (15.4.16) and (15.4.17) in [126].) It is clear that $(a_{D2} - a_{D1})(\Lambda)$ vanishes, as it should, because all the magnetic contours degenerate to zero at $\phi = \Lambda$.

3.4 Summary and future possibilities

The original interpretation of the enhançon in terms of Seiberg-Witten theory given in [84] is that the transverse positions of the branes can be related to the branch points of the $SU(N)$ Seiberg-Witten curve, $x_{\pm,i}(\phi_j)$. Indeed, these quantities give a good heuristic picture of the enhançon. For the $SU(N+1)$ case where we consider a probe:

$$\langle \Phi \rangle = \text{diag}(w, -w/N, \dots, -w/N), \quad \text{and } w \gg \Lambda, \quad (3.76)$$

one finds that $2N$ of the branch points lie approximately on a circle of radius Λ , while the remaining two lie near $x = w$. Similarly, if one considers the case where $w < \Lambda$ then all the branch points lie on the circle. Therefore the behaviour of the branch points reproduces the behaviour of the probe.

However, as an example consider the case of $\phi = \Lambda$. This is the case briefly mentioned above where the magnetic contours degenerate. Half of the branch points are at the origin, $x_{-,i} = 0$, whereas the other half lie on a circle of radius $2^{1/N}\Lambda$. What precisely does this mean for the branes? It is rather unclear how this should be interpreted.

In this chapter, we have proposed a different set of quantities, $a_i(\phi_j)$, to be identified with the transverse positions of the branes. One reason for this is that the $a_i(\phi_j)$ appear in physical quantities, such as W-boson masses, whereas the $x_{\pm,i}$ do not. Another, simpler, reason is that there are N transverse positions whereas there are $2N$ branch points. Given that brane positions are usually identified with vevs of scalar fields, the a_i seem to be the most appropriate quantities to calculate.

Furthermore this is supported by the matching between various supergravity duals and one-loop results, such as that in [118, 119] (which we reviewed in section 3.3.1) which followed probe analyses [105, 106] of another solution [79] corresponding

to a different $\mathcal{N} = 2$ gauge theory.

However, this is not to say that the branch points are not a useful guide. They are also much easier to calculate. Furthermore, from Witten's construction [127] of the Seiberg-Witten curve by uplifting configurations of D4- and NS5-branes to an M5-brane wrapped on a holomorphic curve, one can see that the branch points give an indication of the positions of the (uplifted) D4-branes. These then correspond to the vevs of the scalar fields.

Eleven-dimensional supergravity solutions have been found [128] that correspond to the above construction of M5-branes wrapped on the Seiberg-Witten curve. It would be interesting to see if they could provide further insight into the enhancement mechanism.

In addition, Witten's construction of Seiberg-Witten curves using M-theory can be related [129] to the appearance of the Seiberg-Witten solution in F-theory [130, 131]. Given that the results derived here are exact (they do not rely on large N), it may be possible to apply similar methods to F-theory configurations.

An important question not addressed here is that of calculating the matrix τ_{ij} which contains the effective couplings and theta-angles and also gives the metric on moduli space¹¹. Some progress has been made in this respect by the recent paper by Alberghi *et al.* [132].

Further extensions to this work could include studying the theory near Argyres-Douglas points [133] or relating it to previous work on large N Seiberg-Witten theory by Douglas & Shenker [124]. It would also be interesting to see how changes in the BPS spectrum [21, 22] can be seen in the enhancement picture.

Finally, one should note that the supergravity solutions we have studied are limited by the fact that they do not include all the relevant degrees of freedom in their derivation. For instance, in the case of D6-branes wrapped on K3 manifold (giving a three-dimensional field theory) there are also degrees of freedom arising from D4-branes wrapping the K3. These degrees of freedom become light as the

¹¹It is interesting to note that in general this matrix will not be diagonal. Therefore the metric on moduli space and the specific form of the low-energy theories on the separated branes will be quite complicated.

volume of the K3 shrinks to V_* , and should be included as fields in the supergravity Lagrangian. Work towards including these non-perturbative effects has been done by Wijnholt & Zhukov [134] and also by Mohaupt and Zagermann [135].

Chapter 4

Penrose Limits of Supersymmetric Gauge Theories

4.1 Introduction

Recently, there has been a flurry of interest in a particular limit of the AdS-CFT correspondence. This arose from the realization that there exists a plane-wave solution to type IIB supergravity that preserves all 32 supersymmetries [136] which can be related to the $\text{AdS}_5 \times S^5$ solution by the concept of the “Penrose limit” [137–139]. Remarkably, type IIB string theory can be quantized on this background [139–141]. (Quantizing strings on a background with non-trivial R-R flux has proven to be a rather hard problem [142–144].) The Penrose limit was given a gauge-theoretic interpretation by Berenstein, Maldacena and Nastase (BMN) [139] using the AdS-CFT correspondence. This work is particularly important in the context of gauge theory/gravity dualities because it represents one of the few examples where we are not restricted to using supergravity techniques, but can compare gauge theory to stringy calculations.

In this chapter we shall first review the above material, before applying it to the study of the Pilch-Warner solution that is dual to the Leigh-Strassler fixed point studied in chapter 2. We find its Penrose limit which is a new plane-wave solution with constant three-form fluxes. On this background the string action takes a simple form and can be quantized using the same method as in the maximally

supersymmetric case, although the details are different. We find the string spectrum and then study the corresponding limit of the Leigh-Strassler theory. Finally, we find the Penrose limit of the supergravity solution corresponding to the RG flow from the $\mathcal{N} = 4$ $SU(N)$ gauge theory to the Leigh-Strassler fixed point¹.

4.2 Penrose limits and plane-waves

4.2.1 The Penrose-Güven limit

Consider the following solution [136] of type IIB supergravity [147]:

$$\begin{aligned} ds^2 &= 2dudv + H(u, x)du^2 + ds^2(\mathbb{E}^8), \\ F_5 &= (1 + \star)du \wedge \omega_4, \end{aligned} \tag{4.1}$$

where for each u , $\omega_4(u, x)$ is a closed and co-closed² four-form on \mathbb{E}^8 (it can depend on u in an arbitrary way). The equations of motion relate H and ω_4 as

$$\nabla^2 H = -\frac{2}{3}\omega_4^2 \equiv -\frac{2}{3}\omega_{ijkl}\omega^{ijkl}, \tag{4.2}$$

where $x^i, i = 1, \dots, 8$, are the coordinates, and ∇^2 the Laplacian, on \mathbb{E}^8 . This solution is an example of what is called a “pp-wave”³. (Later, we will study pp-wave solutions of type IIB supergravity with non-zero three-form fields as well.) If we now take H to be quadratic in the x^i , $H = A_{ij}(u)x^i x^j$ say, then the solution is called a “plane-wave”. If we further take A_{ij} to be a constant, symmetric matrix then the metric is that of a Cahen-Wallach space [149]. These solutions are of interest for various reasons. One of these is that plane-wave solutions of type IIB supergravity preserve at least 16 supersymmetries. In fact, Blau *et al.* [136] found that for $\omega_4 = \mu\epsilon(\mathbb{E}^4)$ a constant multiple of the volume form on one of the transverse \mathbb{E}^4 s,

¹This chapter is based on work done in collaboration with Dominic Brecher, Clifford Johnson and Robert Myers and presented in [3]. Whilst that paper was in final preparation, two other papers appeared [145, 146] that address some of the same issues.

²*I.e.* $d_8\omega_4 = d_8(\star_8\omega_4) = 0$.

³The “pp-” stands for “plane-fronted wave with parallel rays” [148].

there is a unique choice⁴ of $A_{ij} = -\mu^2 \delta_{ij}$ for which the solution is maximally supersymmetric, *i.e.* it preserves all 32 supercharges⁵. Therefore, the known solutions of type IIB supergravity that preserve 32 supercharges are $\text{AdS}_5 \times S^5$, the maximally-supersymmetric plane-wave and Minkowski space (which is the $\mu = 0$ case of the pp-wave).

The maximally-supersymmetric plane-wave was related to the $\text{AdS}_5 \times S^5$ solution by Blau *et al.* [137] and Berenstein *et al.* [139] by the notion of the ‘‘Penrose limit’’. In [152], Penrose showed that any spacetime has a limit in which it becomes a plane-wave⁶. This can be seen by first considering a null geodesic, γ , in a spacetime⁷. Then it is possible [152, 153] (under some reasonable conditions [137, 138]) to introduce local coordinates (U, V, Y^i) such that the metric takes the following form:

$$ds^2 = 2dV \left(dU + \alpha dV + \sum_i dY^i \right) + \sum_{i,j} C_{ij} dY^i dY^j, \quad (4.3)$$

where α , β_i and C_{ij} are functions of all the coordinates and the matrix C_{ij} is positive-definite. The coordinate U is the affine parameter along a congruence of null geodesics labelled by V and Y^i , such that γ has $V = Y^i = 0$. The next step is to introduce a real constant $\Omega > 0$ and rescale the coordinates in the following way:

$$U = u, \quad V = \Omega^2 v, \quad Y^i = \Omega y^i, \quad (4.4)$$

so that the metric is now Ω -dependent. The fact that we can put the metric into the form (4.3) ensures that the Penrose limit

$$d\bar{s}^2 = \lim_{\Omega \rightarrow 0} \frac{ds^2(\Omega)}{\Omega^2}, \quad (4.5)$$

is well-defined. The resulting metric is of the form

$$d\bar{s}^2 = 2dudv + \sum_{i,j} C_{ij}(u) dy^i dy^j \quad (4.6)$$

which only depends on the coordinate u and not on v or y^i . Here, the metric is in what is called Rosen form. One can also find Brinkman coordinates⁸ so that the

⁴There are differences between our conventions and those of [136]. (See [3].)

⁵A similar solution in eleven-dimensional supergravity was already known [150, 151].

⁶Note that this is a stronger claim than saying it has a limit in which it is a pp-wave.

⁷For a detailed description of the Penrose-Güven limit and its mathematical properties, see [138].

⁸For the explicit change of variables see, for instance, [138].

metric takes the form

$$d\bar{s}^2 = 2dudv + \sum_{i,j} A_{ij}(u)x^i x^j du^2 + \sum_i dx^i dx^i, \quad (4.7)$$

i.e. precisely the form of a plane-wave (4.2) described above.

So far we have not mentioned what happens when one introduces other spacetime fields, such as the ones that appear in supergravity theories. Güven [153] extended this analysis to supergravity theories by also rescaling the gauge fields in a way analogous to (4.4).

Importantly, the Penrose limit of a supergravity solution is again a solution of the supergravity equations of motion. This follows from the fact [153] that the supergravity actions transform homogeneously under the rescaling (4.4).

Another important point is that the Penrose limit depends on the choice of null geodesic. Therefore, later on when we look at taking Penrose limits in the context of the AdS-CFT correspondence, the choice of null geodesic will be important. However, the Penrose limit has the following *covariance property*: if two null geodesics in a spacetime are related by an isometry, then their associated Penrose limits are isometric.

One interpretation of the Penrose limit is that it gives a ‘first order approximation’ to the spacetime along the null geodesic (*cf.* the tangent space to a point). Another, physical, interpretation [138, 152] is that one can consider a series of observers in the spacetime whose worldlines approach the null geodesic γ more and more closely. As these observers travel faster and faster, approaching the speed of light, they recalibrate their clocks so that in the limit the clocks measure the affine parameter along the geodesic. Then the argument given above implies that in the Penrose limit the observers see the spacetime as being of plane-wave form.

The Penrose limit also has some interesting mathematical properties [138]. These are often hereditary — any Penrose limit of a spacetime inherits certain properties from that spacetime. For instance, Geroch [154] showed that the number of linearly independent Killing vectors can never decrease in the Penrose limit. This can be generalized [137, 138] in the case of supergravity theories to the statement that the number of supersymmetries preserved by a supergravity solution does not decrease

in the Penrose limit. However, it is important to note that the Penrose limit of a supergravity configuration may have more Killing vectors and/or supersymmetries than the original solution. In fact, the Penrose limit of a supergravity solution preserves at least one half of the maximum possible supersymmetry [138]. A plane-wave solution that preserves more than the generic 16 supersymmetries is sometimes said to have supernumerary supersymmetries⁹. We shall see an example of such a plane-wave solution in section 4.5.

4.2.2 The Penrose limits of $\text{AdS}_5 \times S^5$

Having discussed the Penrose limit and some of its properties, let us now return to the case of immediate interest — using it to relate the maximally-supersymmetric pp-wave of type IIB supergravity to the $\text{AdS}_5 \times S^5$ solution.

The Penrose limits of $\text{AdS}_5 \times S^5$ were classified in [138] by Blau *et al.*. Using the covariance property they found that there are two cases. The first (non-generic) case is when the null geodesic γ is tangent to the AdS-space, *i.e.* when the tangent of γ vanishes in the S^5 directions. For this type of geodesic the resulting Penrose limit is Minkowski space (which is the $\mu = 0$ case of the maximally supersymmetric plane-wave). However, the tangent of a generic null geodesic in $\text{AdS}_5 \times S^5$ has a non-vanishing component in the S^5 directions. In this case it was shown [138] that the Penrose limit results in the maximally-supersymmetric plane-wave (with $\mu \neq 0$). This makes sense because since the amount of unbroken supersymmetry does not decrease in the Penrose limit, any Penrose limit of $\text{AdS}_5 \times S^5$ must still preserve 32 supercharges.

To illustrate this, let us carry out a Penrose limit of $\text{AdS}_5 \times S^5$ which does not result in flat space. This is most easily done if one chooses to write the AdS-space

⁹Some plane-waves in ten and eleven dimensions that have supernumerary supersymmetries are discussed in [155–158].



in global coordinates¹⁰:

$$ds^2 = L^2 \left(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + \cos^2 \theta d\psi^2 + d\theta^2 + \sin^2 \theta d\widehat{\Omega}_3^2 \right), \quad (4.8)$$

where $d\Omega_3^2, d\widehat{\Omega}_3^2$ are metrics on unit S^3 s. The Lagrangian for geodesics on this space is then simply (in an obvious notation)

$$\mathcal{L} = L^2 \left(-\cosh^2 \rho \dot{t}^2 + \dot{\rho}^2 + \sinh^2 \rho \dot{\Omega}_3^2 + \cos^2 \theta \dot{\psi}^2 + \dot{\theta}^2 + \sin^2 \theta \dot{\widehat{\Omega}}_3^2 \right), \quad (4.9)$$

where now dots indicate derivatives with respect to the affine parameter along the geodesic. From the Euler-Lagrange equations one finds that it is consistent to set $\rho = \theta = 0$ and that for a null geodesic with affine parameter λ

$$t(\lambda) = \psi(\lambda) = E\lambda, \quad (4.10)$$

so that the energy and angular momentum of the massless particle travelling on the geodesic are equal. (Note that we have left the coordinates on both S^3 s arbitrary.) We can now find coordinates near the geodesic that are suitable for taking the Penrose limit. To do this we require coordinates (u, v) such that (for $\rho = \theta = 0$) we have

$$g_{uu} = g(\partial_u, \partial_u) = 0, \quad g_{uv} = g(\partial_u, \partial_v) = 1. \quad (4.11)$$

These are satisfied by taking¹¹

$$\begin{aligned} \partial_u &= t\partial_t + \dot{\psi}\partial_\psi = E(\partial_t + \partial_\psi) \\ \partial_v &= \frac{1}{2EL^2}(-\partial_t + \partial_\psi), \end{aligned} \quad (4.12)$$

which are solved by

$$t = Eu - \frac{1}{2EL^2}v, \quad \psi = Eu + \frac{1}{2EL^2}v. \quad (4.13)$$

We have identified u with the parameter λ along the geodesic. To take the Penrose limit we will take the limit $L \rightarrow \infty$ (which can be thought of as ‘flattening’ the

¹⁰For a discussion of different coordinate systems on AdS-space see [37].

¹¹We have made a specific choice here in the definition of v , but a more general definition leads to the same results here and in section 4.3.

space). Since in our choice of geodesic we have set $\rho = \theta = 0$, the corresponding coordinates are rescaled as follows (*cf.* equation (4.4)):

$$\rho = \frac{r}{L}, \quad \theta = \frac{y}{L}. \quad (4.14)$$

Inserting these expressions back into the metric (4.8), one finds that the Penrose limit given by $L \rightarrow \infty$ is well-defined:

$$ds^2 = 2dudv - (r^2 + y^2)E^2du^2 + dr^2 + r^2d\Omega_3^2 + dy^2 + y^2d\widehat{\Omega}_3^2, \quad (4.15)$$

which is of course the metric of the maximally-supersymmetric plane-wave. The corresponding calculation of this Penrose limit, but using Poincaré coordinates¹² on the AdS-space is presented in appendix C.3.

4.3 Type IIB string theory on plane-waves

Another reason for studying plane-wave solutions of type IIB supergravity is that the Green-Schwarz action [159, 160] for superstrings drastically simplifies in these backgrounds. In fact it has been known for some time that the model for strings propagating on plane-wave backgrounds with only non-trivial NS-NS fields is exactly solvable [161–167].

Metsaev [140] studied the Green-Schwarz action in the maximally-supersymmetric plane-wave background and found that the resulting light-cone gauge action was quadratic in both bosonic and fermionic fields and could therefore be quantized exactly. This quantization was carried out in detail by Metsaev & Tseytlin [141] and the spectrum was also found by Berenstein *et al.* [139].

Let us now review the method and results of [141]. First of all, let us recall the explicit form of the maximally-supersymmetric pp-wave:

$$\begin{aligned} ds^2 &= 2dudv - E^2 \sum_{i=1}^8 x^i x^i du^2 + \sum_{i=1}^8 dx^i dx^i, \\ F_5 &= E(1 + \star) du \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4. \end{aligned} \quad (4.16)$$

¹²It is useful to also take the Penrose limit in these coordinates because the Pilch-Warner flow solution that we study in section 4.8 is presented in Poincaré-type coordinates.

We shall initially consider the bosonic part of the world-sheet action:

$$S_B = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \left\{ \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} + \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} \right\} \quad (4.17)$$

where $\mu, \nu = 0, \dots, 9$ and in the case of the background (4.16) it simplifies to:

$$S_B = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \sqrt{-g} g^{\alpha\beta} (2\partial_\alpha U \partial_\beta V - E^2 X^i X^i \partial_\alpha U \partial_\beta U + \partial_\alpha X^i \partial_\beta X^i). \quad (4.18)$$

Here $\epsilon^{01} = 1$, and we shall use the familiar world-sheet gauge choice $g_{\alpha\beta} = \eta_{\alpha\beta}$. We have also used world-sheet coordinates σ^α , where $\alpha, \beta = 0, 1$, and $\sigma^0 = \tau, \sigma^1 = \sigma$.

Variation of V gives rise to the equation of motion for U , namely $\square U = 0$. So we can work in the standard light-cone gauge with $U = \alpha' p^+ \tau + \text{constant}$. In that case, the worldsheet scalars obey the following equations:

$$\square X^i - M^2 X^i = 0,$$

where $i = 1, \dots, 8$ and we have set $M = E\alpha' p^+$. The two independent components of the standard constraint from world-sheet reparametrizations, $T_{\alpha\beta} = 0$, are

$$\partial_\sigma V = -\frac{1}{\alpha' p^+} \partial_\tau X^i \partial_\sigma X^i, \quad (4.20)$$

$$\partial_\tau V = -\frac{1}{2\alpha' p^+} (\partial_\tau X^i \partial_\tau X^i + \partial_\sigma X^i \partial_\sigma X^i - M^2 X^i X^i), \quad (4.21)$$

which allow for the elimination of V in the usual way. Integrating the former over σ gives

$$\int_0^{2\pi} d\sigma \partial_\tau X^i \partial_\sigma X^i = 0. \quad (4.22)$$

In the light-cone gauge, the action becomes

$$S_B = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \left\{ -2\alpha' p^+ \partial_\tau V + M^2 X^i X^i + \eta^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^i \right\}, \quad (4.23)$$

from which it is easy to derive the Hamiltonian:

$$H_B = \frac{1}{4\pi\alpha'} \int_0^{2\pi} d\sigma \left\{ (2\pi\alpha')^2 \Pi^i \Pi^i + M^2 X^i X^i + \partial_\sigma X^i \partial_\sigma X^i \right\}, \quad (4.24)$$

where the conjugate variable to X^i is

$$\Pi^i = \frac{1}{2\pi\alpha'} \partial_\tau X^i. \quad (4.25)$$

Note that (4.24) only contains the transverse degrees of freedom (X^i, Π^i) and not U or V . The only non-zero (equal- τ) Poisson bracket is

$$\left[X^i(\sigma), \dot{X}^j(\sigma') \right]_{\text{P.B.}} = 2\pi\alpha' \delta^{ij} \delta(\sigma - \sigma'). \quad (4.26)$$

To solve for the eigenmodes of the system, subject to the usual periodic boundary conditions $X^i(\tau, \sigma + 2\pi) = X^i(\tau, \sigma)$, we Fourier expand

$$X^i(\tau, \sigma) = \sum_n C_n^i e^{i(\omega_n \tau + n\sigma)}, \quad (4.27)$$

for some unknown coefficients C_n^i . The normal modes are given by

$$\omega_n^2 = n^2 + M^2. \quad (4.28)$$

A key feature of this spectrum is that even the zero modes ($n = 0$) have an oscillator frequency $\omega_0 = M$ set by the plane-wave background, corresponding to the mass of the world-sheet bosons associated with those directions. The mode expansion for these coordinates is then [141]

$$\begin{aligned} X^i(\tau, \sigma) = & \cos M\tau x_0^i + \frac{\alpha'}{M} \sin M\tau p_0^i \\ & + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{\omega_n} (\alpha_n^i e^{in\sigma} + \tilde{\alpha}_n^i e^{-in\sigma}) e^{-i\omega_n \tau}, \end{aligned} \quad (4.29)$$

where, to ensure reality, we have

$$\omega_n = \text{sgn}(n) \sqrt{n^2 + M^2} \quad (n \neq 0), \quad (4.30)$$

and

$$(\alpha_n^i)^\dagger = \alpha_{-n}^i, \quad (\tilde{\alpha}_n^i)^\dagger = \tilde{\alpha}_{-n}^i. \quad (4.31)$$

The non-vanishing Poisson brackets are found to be

$$[x_0^i, p_0^j]_{\text{P.B.}} = \delta^{ij}, \quad [\alpha_m^i, \alpha_n^j]_{\text{P.B.}} = [\tilde{\alpha}_m^i, \tilde{\alpha}_n^j]_{\text{P.B.}} = -i\omega_m \delta_{m+n,0} \delta^{ij}. \quad (4.32)$$

The constraint (4.22) becomes $N = \tilde{N}$, where

$$N = \sum_{n \neq 0} \frac{n}{\omega_n} \alpha_{-n}^i \alpha_n^i, \quad \tilde{N} = \sum_{n \neq 0} \frac{n}{\omega_n} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i, \quad (4.33)$$

and the Hamiltonian (4.24) is

$$H_B = \frac{1}{2\alpha'} (\alpha'^2 p_0^i p_0^i + M^2 x_0^i x_0^i) + \frac{1}{2} \sum_{n \neq 0} \left(\alpha_{-n}^i \alpha_n^i + \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i \right). \quad (4.34)$$

To quantize the system, we replace the Poisson brackets with commutators in the usual way. We further take, for $n > 0$,

$$a_n^i \equiv \frac{\alpha_n^i}{\sqrt{\omega_n}}, \quad \bar{a}_n^i \equiv \frac{\alpha_{-n}^i}{\sqrt{\omega_n}}, \quad (4.35)$$

and similarly for the the independent set of operators with a tilde, and combine the zero modes as

$$a_0^i = \frac{1}{\sqrt{2M\alpha'}} (\alpha' p_0^i - iM x_0^i), \quad \bar{a}_0^i = \frac{1}{\sqrt{2M\alpha'}} (\alpha' p_0^i + iM x_0^i). \quad (4.36)$$

The new creation and annihilation operators obey the standard harmonic oscillator commutation relations

$$[a_m^i, \bar{a}_n^j] = \delta_{mn} \delta^{ij}, \quad n, m \geq 0, \quad (4.37)$$

and similarly for the tilded set of operators. In this basis, the Hamiltonian (4.34) becomes

$$H = \Delta E + MN_0 + \sum_{n>0} \omega_n N_n, \quad (4.38)$$

where ΔE is the zero point energy. The occupation numbers are given by

$$\begin{aligned} N_n &= \sum_{i=1}^8 (\bar{a}_n^i a_n^i + \tilde{\bar{a}}_n^i \tilde{a}_n^i), \quad n > 0, \\ N_0 &= \sum_{i=1}^8 \bar{a}_0^i a_0^i. \end{aligned} \quad (4.39)$$

We now turn our attention to the fermionic sector which requires us to include contributions from the background R-R fields in the world-sheet action. The problem as to how to include R-R fields in the world-sheet analysis of the superstring is a difficult one. Techniques utilizing coset superspaces have been used in an attempt to construct actions for superstrings in AdS backgrounds (*e.g.* [142]), although the resulting action is difficult to quantize explicitly. However, in the case of pp-waves, the superstring action simplifies considerably in the light-cone gauge. For

the maximally-supersymmetric plane-wave it can be easily quantized, since it turns out that the five-form field strength only gives rise to mass terms for the fermions.¹³ More heuristically, since this background admits a null Killing vector, it can be argued [141] that the fermionic action is a direct covariantization of the flat action, at least in the standard light-cone gauge.

In our conventions, the light-cone gauge is implemented *via*

$$\Gamma^- \theta = \Gamma_+ \theta = 0, \quad (4.40)$$

in which case the fermionic action is simply [141]

$$S_F = \frac{i}{\pi} \int d\sigma d\tau (\eta^{\alpha\beta} \delta_{IJ} - \epsilon^{\alpha\beta} \rho_{IJ}) \partial_\alpha X^a \partial_\beta X^b \bar{\theta}^I \Gamma_a (\mathcal{D}_b \theta)^J, \quad (4.41)$$

where $I, J = 1, 2$ denote the two 16-component Majorana-Weyl spinors¹⁴. In terms of the Pauli matrices, τ_i , the two-dimensional gamma matrices are $\rho^0 = i\tau_2$ and $\rho^1 = \tau_1$, so that $\rho = \rho^0 \rho^1 = \tau_3$. With $G_3 = H_3 + iF_3$, and viewed as acting on a column matrix, the supercovariant derivative¹⁵ in the string frame then takes the form [141]

$$\mathcal{D}_a = D_a + \frac{1}{8} H_a{}^{bc} \Gamma_{bc} \rho + \frac{1}{48} F_{bcd} \Gamma^{bcd} \Gamma_a \rho_1 + \frac{1}{480} F_{b_1 \dots b_5} \Gamma^{b_1 \dots b_5} \Gamma_a \rho_0. \quad (4.42)$$

In the light-cone gauge, the action simplifies considerably, and we have [141, 156]

$$S_F = -\frac{i}{\pi} \alpha' p^+ \int d\sigma d\tau \left\{ \bar{\theta} \Gamma_- (\partial_\tau \theta + \rho \partial_\sigma \theta) + \frac{1}{8} \alpha' p^+ \bar{\theta} \Gamma_- \not{H} \rho \theta + \frac{1}{8} \alpha' p^+ \bar{\theta} \Gamma_- \not{F}_3 \rho_1 \theta + \frac{1}{240} \alpha' p^+ \bar{\theta} \Gamma_- \not{F}_5 \rho_0 \theta \right\}, \quad (4.43)$$

¹³Other techniques can be used to derive the relevant action [156]: since the eleven-dimensional supermembrane action is known to $\mathcal{O}(\theta^2)$ [168], dimensional reduction will give rise [169] to the superstring action to the same order in the fermions; and this is all that is required in the case at hand.

¹⁴It is important to realize that the supercovariant derivative appearing in this action is written in the string frame, as opposed to the Einstein frame. Since the gravitino is shifted by a multiple of the dilatino when going from one frame to the other [170], what we mean by the supercovariant derivative is also changed.

¹⁵Here, $\Gamma_{a_1 \dots a_k}$ are antisymmetrized products of gamma matrices, *e.g.* $\Gamma_{123} = \frac{1}{6} \Gamma_1 \Gamma_2 \Gamma_3 \pm 5 \text{ terms} = \Gamma_1 \Gamma_2 \Gamma_3$.

where

$$\mathcal{H} = H_{uij}\Gamma_{ij}, \quad F_3 = F_{uij}\Gamma_{ij}, \quad F_5 = F_{uijkl}\Gamma_{ijkl}. \quad (4.44)$$

We should note that the NS-NS three-form gives rise to a chiral interaction, whereas the R-R three-form field strength gives further mass terms [141].

In the case of the maximally-supersymmetric plane-wave we have $F_{u1234} = F_{u5678} = E$, and so rewriting in terms of θ^1 and θ^2 , gives

$$S_F = -\frac{i}{\pi}\alpha'p^+ \int d\sigma d\tau \left\{ \theta^1\Gamma_-\partial_+\theta^1 + \theta^2\Gamma_-\partial_-\theta^2 + 2M\theta^1\Gamma_-\Gamma_{1234}\theta^2 \right\}, \quad (4.45)$$

where

$$\partial_{\pm} = \partial_{\tau} \pm \partial_{\sigma}. \quad (4.46)$$

The equations of motion for θ^1 and θ^2 are then

$$\begin{aligned} \partial_+\theta^1 + M\Gamma_{1234}\theta^2 &= 0, \\ \partial_-\theta^2 - M\Gamma_{1234}\theta^1 &= 0. \end{aligned} \quad (4.47)$$

which imply

$$\square\theta^I - M^2\theta^I = 0. \quad (4.48)$$

The next step is again to Fourier expand

$$\theta^I(\tau, \sigma) = \sum_n \theta_n^I(\tau) e^{in\sigma}, \quad (4.49)$$

giving

$$\begin{aligned} \dot{\theta}_n^1 + in\theta_n^1 + M\Gamma_{1234}\theta_n^2 &= 0, \\ \dot{\theta}_n^2 - in\theta_n^2 - M\Gamma_{1234}\theta_n^1 &= 0. \end{aligned} \quad (4.50)$$

We also have

$$\ddot{\theta}_n^I + \omega_n^2\theta_n^I = 0, \quad (4.51)$$

where ω_n is defined as before (and $\omega_0 = M$). Therefore, the fermionic sector of the Green-Schwarz superstring in the maximally-supersymmetric plane-wave background has the same set of frequencies as the bosonic sector [139–141]. In fact, it

was found by Berenstein *et al.* [139] and Metsaev & Tseytlin [141] that this matching extends to the form of the complete light-cone Hamiltonian:

$$H = 4 + MN_0 + \sum_{n>0} \omega_n N_n \quad (4.52)$$

(and also the constraint), so that now the number operators N_0 and the N_n contain contributions from both bosonic and fermionic oscillators. The expression for the light-cone Hamiltonian contains the zero-mode energy, which arises from the bosonic zero-modes ($8 \times \frac{1}{2} = 4$) only. The fermionic zero modes do not contribute, whereas for $n > 0$ the contribution vanishes because there are equal numbers of bosonic and fermionic oscillators.

The vacuum state is the direct product of a zero-mode vacuum and the vacuum for string ($n > 0$) modes. If we concentrate on the zero-mode part of (4.52), there are eight bosonic zero-mode creation operators that raise the value of H by M , as is familiar from the superstring in flat space. However, to obtain this form of the Hamiltonian we have assumed that four of the fermionic oscillators raise the value of H of a state by M , whereas the other four lower the value of H by M . Therefore, by applying the fermionic zero-modes to the vacuum state one obtains states with light-cone energy from 0 to 8 (in units of M).

However, as pointed out in [141], the choice of fermionic vacuum is not unique. An equivalent choice is to take the Hamiltonian to be

$$H = MN_0 + \sum_{n>0} \omega_n N_n \quad (4.53)$$

where now the 8 fermionic zero-mode creation operators all raise the value of H by M . This gives exactly the same light-cone energy spectrum¹⁶.

Metsaev & Tseytlin were able to reproduce the zero-mode spectrum from an analysis of the fluctuations of the type IIB supergravity fields around the plane wave background (4.16). Therefore, one can identify the zero-modes of the string with supergravity modes, and the modes with $n > 0$ with string excitations, as one would expect.

¹⁶The original choice of Hamiltonian is useful because it has a natural flat space limit [141].

4.4 The BMN limit of $\mathcal{N} = 4$ gauge theory

Having identified the maximally supersymmetric plane-wave as a Penrose limit of $\text{AdS}_5 \times S^5$, and having also found the string spectrum on the plane-wave, it is natural to ask whether it is possible to relate string theory on the plane-wave background to $\mathcal{N} = 4$ gauge theory *via* the AdS-CFT correspondence. This matching was made by Berenstein, Maldacena & Nastase (BMN) [139], by identifying the corresponding ‘Penrose limit’ in the field theory. They were able to match certain gauge theory operators to string states on the plane-wave. Furthermore, they were able to check a prediction of the string spectrum to first order in the ’t Hooft coupling using a gauge theory computation. In this section we will briefly review some of their arguments.

The quantities we will initially consider are the light-cone energy, which in the light-cone gauge of the previous section is given by

$$H = i\partial_\tau = i\alpha' p^+ \partial_u, \quad (4.54)$$

and the light-cone momentum $p_- = i\partial_v$. Using the definitions of u and v used in deriving the Penrose limit (4.12) we can write

$$\begin{aligned} H &= i\alpha' p^+ \partial_u = \alpha' p^+ E (i\partial_t + i\partial_\psi) = M(\Delta - J), \\ p_- &= i\partial_v = -\frac{1}{2EL^2}(\Delta + J). \end{aligned} \quad (4.55)$$

Here, $M = \alpha' p^+ E$, and we have used the fact that ∂_t is the dilatation operator of the conformal group and that ∂_ψ is a generator of a $U(1)$ subgroup of the full $SO(6)$ R-symmetry group of $\mathcal{N} = 4$ Yang-Mills theory.

The Penrose limit is given by $L \rightarrow \infty$, so that since $L^4 = g_{\text{YM}}^2 N \alpha'^2$, in gauge theory variables we have $N \rightarrow \infty$ with $g_{\text{YM}}^2 = 2\pi g_s$ kept fixed. Furthermore, for H and p_- to remain finite and non-zero in this limit, we must have $\Delta \sim J \sim N^{1/2}$ with $\Delta - J$ fixed.

Let us now try to match some of the lowest lying states of the string spectrum on the pp-wave background to the corresponding Yang-Mills operators. All of these operators will have large J and Δ with $(\Delta - J)$ given by (4.55). Because the string spectrum on the pp-wave background has a unique ground state with $H = 0$ this implies that there should be a unique operator with large J and $(\Delta - J) = 0$. Indeed,

by choosing a particular S^5 coordinate ψ when taking the Penrose limit, we have chosen a particular $U(1)$ subgroup of $SO(6)$ which has generator J . We can take the complex scalar field that has R-charge +1 under $U(1)_J$ to be $Z = \phi_5 + i\phi_6$, with the remaining ϕ_i , $i = 1, 2, 3, 4$, uncharged. It is then clear that that we can make the following matching:

$$\frac{1}{\sqrt{JN^{J/2}}} \text{Tr}[Z^J] \longleftrightarrow |0, p^+\rangle \quad (4.56)$$

(from now on we will drop normalization factors). This is the unique operator that has R-charge J and $\Delta = J$ (since it is a chiral operator its conformal dimension is given by the free-field value).

We can now consider the states which have $H = M$. There are eight bosonic oscillators which raise the value of H by M and eight fermionic counterparts. This implies there should be eight bosonic operators with $(\Delta - J) = 1$ and large J . These arise from inserting the following operators into the “string of Z s”:

$$\begin{aligned} \phi_i, & \quad i = 1, 2, 3, 4, \\ D_a Z = \partial_a Z + [A_i, Z], & \quad a = 1, 2, 3, 4. \end{aligned} \quad (4.57)$$

There should also be eight fermionic operators with $(\Delta - J) = 1$. These can be constructed by inserting any of the eight components of the sixteen-component gaugino that have $J = \frac{1}{2}$ (since they increase the conformal dimension by $\frac{3}{2}$).

Having matched some of the lowest supergravity modes to their corresponding large J operators in the gauge theory, we can now consider some of the string states. The contribution of a single $n > 0$ oscillator to $(\Delta - J)$ is

$$\begin{aligned} (\Delta - J)_n &= \frac{1}{M} \sqrt{M^2 + n^2} = \sqrt{1 + \frac{n^2}{(E\alpha'p^+)^2}} \\ &= \sqrt{1 + \frac{n^2 g_{\text{YM}}^2 N}{J^2}} = 1 + \frac{n^2 g_{\text{YM}}^2 N}{2J^2} + \dots \end{aligned} \quad (4.58)$$

where we have used the fact that¹⁷

$$p^+ = p_- \sim -\frac{J}{EL^2}. \quad (4.59)$$

In fact, because of the level-matching constraint, one has to act with at least two stringy oscillators. In order to match these predictions, BMN proposed that acting

¹⁷Note that $p^+ = p_-$, but that $p^- \neq p_+$.

with an oscillator mode with $n > 0$ corresponded to inserting an operator into the string of Z 's with a position-dependent phase. For instance acting with \bar{a}_n^8 would correspond to:

$$\frac{1}{\sqrt{J}} \sum_{l=1}^J \frac{1}{\sqrt{J} N^{J/2+1/2}} \text{Tr} [Z^l \phi_4 Z^{J-l}] e^{\frac{2\pi i n l}{J}} \longleftrightarrow \bar{a}_n^8 |0, p^+\rangle. \quad (4.60)$$

This vanishes by the cyclicity of the trace — this makes sense because the constraint is not satisfied. An example of where the constraint is satisfied is:

$$\sum_{l=1}^J \frac{1}{\sqrt{J} N^{J/2+1}} \text{Tr} [\phi_3 Z^l \phi_4 Z^{J-l}] e^{\frac{2\pi i n l}{J}} \longleftrightarrow \bar{a}_n^7 \tilde{a}_n^8 |0, p^+\rangle, \quad (4.61)$$

where now there are two insertions. Remarkably, BMN were able to reproduce the correct first-order correction to $(\Delta - J)$, which they predicted using (4.58), by a gauge theory calculation. This was checked to the two-loop level by Gross *et al.* [171]. However, as pointed out by Constable *et al.* [172], this analysis does not quite match our expectations. The string theory on the plane-wave background is an *interacting* string theory and so we would expect the masses of string states to be renormalized accordingly. But BMN were able to reproduce the *free* string spectrum from field theory. In fact, BMN only considered planar diagrams — only these contribute in the usual 't Hooft limit. However, the limit considered here is different and in fact non-planar diagrams do contribute. In the BMN limit Yang-Mills perturbation theory can be organized as a double expansion in an effective loop-counting parameter $\lambda' = g_{\text{YM}}^2 N / J^2$ and an effective genus-counting parameter $g_2^2 = J^4 / N^2$.

In a paper by Santambrogio & Zanon [173], it was found that one can reproduce the complete square-root that appears in (4.58) from the gauge theory using superspace techniques (in the planar limit). The calculation relies on the constraints imposed on two-point functions by the superconformal invariance of the theory. It would be very interesting to see how this result, and the other results mentioned above, will improve our understanding of the relationship between strings and large N gauge theories.

4.5 A Penrose limit of the Pilch-Warner geometry

Having seen how the Penrose limit can be interpreted in terms of $\mathcal{N} = 4$ supersymmetric gauge theory, it is natural to wonder whether one can do the same for other supergravity duals. Some papers that have studied problems of this type (especially Penrose limits of $\text{AdS}_5 \times T^{1,1}$) are [174–176]. In this section we will take a Penrose limit of the supergravity dual of the Leigh-Strassler fixed point theory that we studied in chapter 2. In section 4.6 we will compute the free string spectrum on the resulting plane-wave and then compare to the gauge theory in section 4.7.

For completeness, and to fix our conventions, we present the fixed point solution in appendix B. As discussed there, we want to work in coordinates for which the $U(1)_R$ symmetry is simplest so we shift the S^3 Euler angle $\beta \rightarrow \beta + 2\phi$, to give a solution with a global $U(1)_R = U(1)_\phi$ symmetry. Performing this coordinate transformation on the solution (B.9), (B.10), (B.12) and writing the AdS space in global coordinates gives

$$ds^2 = \hat{L}^2 \Omega^2 (-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2) + ds_5^2, \quad (4.62)$$

$$ds_5^2 = \frac{2}{3} \hat{L}^2 \Omega^2 \left[d\theta^2 + \frac{4 \cos^2 \theta}{(3 - \cos 2\theta)} (\sigma_1^2 + \sigma_2^2) + \frac{4 \sin^2(2\theta)}{(3 - \cos 2\theta)^2} (\sigma_3 + d\phi)^2 + \frac{8}{3} \left(\frac{2 \sin^2 \theta - \cos^2 \theta}{3 - \cos(2\theta)} \right)^2 \left(d\phi - \frac{2 \cos^2 \theta}{2 \sin^2 \theta - \cos^2 \theta} \sigma_3 \right)^2 \right], \quad (4.63)$$

$$F_5 = -\frac{2^{5/3}}{3} \hat{L}^4 \cosh \rho \sinh^3 \rho (1 + \star) d\tau \wedge d\rho \wedge \epsilon(S^3), \quad (4.64)$$

$$G_3 = -iL^2 \cos \theta \left[d\theta \wedge d\phi - \frac{8 \cos^2 \theta}{(3 - \cos(2\theta))^2} d\theta \wedge (\sigma_3 + d\phi) - \frac{2i \sin(2\theta)}{(3 - \cos(2\theta))} \sigma_3 \wedge d\phi \right] \wedge (\sigma_1 + i\sigma_2). \quad (4.65)$$

where

$$\Omega^2 = \frac{2^{1/3}}{\sqrt{3}} \sqrt{3 - \cos(2\theta)}, \quad (4.66)$$

and the AdS radius, \hat{L} , is given in terms of L , the AdS radius of the UV spacetime, by

$$\hat{L} = \frac{3}{2^{5/3}} L. \quad (4.67)$$

Note that the three-form field strength G_3 could include an arbitrary constant phase, which we have set to -1 here. The global isometry group of the metric is $SU(2) \times$

$U(1)_\beta \times U(1)_\phi$, where $U(1)_\beta$ denotes the shift in the Euler angle β , rotating σ_1 into σ_2 . However, from the three-form one sees that the $U(1)_R$ R-symmetry of the solution as a whole is $U(1)_R = U(1)_\phi$, as required.

Since we are working in global coordinates, we can consider the simple null geodesics for which $\rho = 0$. An examination of the θ geodesic equation shows that one can also consistently set either $\theta = 0$ or $\theta = \pi/2$. We will consider taking the Penrose limit along a null geodesic in the probe moduli space, with $\theta = 0$. We do also consider the other class of geodesics, with $\theta = \pi/2$, but since we have not been able to see the relevance of the latter to the gauge theory, we have consigned the analysis of this case to appendix C.1. This may be because $\theta = \pi/2$ is the massive direction from the point of view of a brane probe and it is precisely this direction which is to be “integrated out” in the $\mathcal{N} = 1$ gauge theory at the IR fixed point

We thus take $\theta = 0$, in which case the effective Lagrangian is

$$\mathcal{L} = \hat{L}^2 \Omega_0^2 \left[-\dot{\tau}^2 + \frac{1}{3} (\dot{\alpha}^2 + \sin^2 \alpha \dot{\gamma}^2) + \frac{4}{9} (\dot{\phi} + \dot{\beta} + \cos \alpha \dot{\gamma})^2 \right], \quad (4.68)$$

where $\Omega_0^2 = 2^{1/3} \sqrt{2/3}$ and a dot denotes differentiation with respect to the affine parameter. We can thus also consider geodesics for which $\alpha = 0$, giving¹⁸

$$\mathcal{L} = \hat{L}^2 \Omega_0^2 \left(-\dot{\tau}^2 + \frac{4}{9} \dot{\psi}^2 \right), \quad (4.69)$$

where we have defined

$$\psi = \phi + \beta + \gamma, \quad (4.70)$$

to be the direction in which our geodesics have an angular momentum. The natural light-cone coordinates are then

$$u = \frac{1}{2E} \left(\tau + \frac{2}{3} \psi \right), \quad v = -E \hat{L}^2 \Omega_0^2 \left(\tau - \frac{2}{3} \psi \right), \quad (4.71)$$

where E is the conserved energy associated with the Killing vector $\partial/\partial\tau$. If h is the conserved angular momentum associated with the Killing vector $\partial/\partial\psi$, we have $E = (2/3)h$. We implement the fact that we are considering $\rho = \theta = \alpha = 0$ geodesics

¹⁸It would be interesting to study more general geodesics, since other geodesics could give a Hamiltonian with an alternate linear combination of the charges J and J_3 (to be defined below).

by taking

$$\rho = \frac{r}{\hat{L}}, \quad \theta = \frac{y}{\hat{L}}, \quad \alpha = \frac{w}{\hat{L}}, \quad (4.72)$$

and considering the $\hat{L} \rightarrow \infty$ limit.

Dropping terms of $\mathcal{O}(1/\hat{L}^2)$, defining two new angular coordinates as

$$\hat{\phi} = \phi - \frac{1}{3}\psi, \quad \hat{\gamma} = \gamma - \frac{2}{3}\psi, \quad (4.73)$$

and rescaling r , y and w , the metric becomes

$$\begin{aligned} ds^2 = & 2dudv - E^2 (r^2 + w^2 + 4y^2) du^2 + dr^2 + r^2 d\Omega_3^2 \\ & + dy^2 + y^2 d\hat{\phi}^2 + dw^2 + w^2 d\hat{\gamma}^2. \end{aligned} \quad (4.74)$$

With the same definitions of $\hat{\phi}$ and $\hat{\gamma}$, and rescalings of coordinates, taking the Penrose limit of the form fields gives

$$\begin{aligned} F_5 &= -E(1 + \star) du \wedge \epsilon(\mathbb{E}^4), \\ G_3 &= -\sqrt{3}E e^{i\beta} du \wedge (dy - iy d\hat{\phi}) \wedge (dw - iwd\hat{\gamma}) \\ &= \sqrt{3}E du \wedge dz^1 \wedge dz^2, \end{aligned} \quad (4.75)$$

where $\epsilon(\mathbb{E}^4) = r^3 dr \wedge \epsilon(S^3)$ and we have defined complex coordinates on the remaining \mathbb{E}^4 . It turns out that this new plane-wave solution preserves 20 supersymmetries [3]. Because it is of relatively simple form, we will again be able to carry out string quantization on this background. Furthermore, because the gauge theory dual to the original supergravity solution is known, we can attempt to match the string spectrum to the corresponding limit of the gauge theory.

4.6 String propagation on the plane-wave

4.6.1 World-sheet analysis: bosonic sector

The fields which will contribute to our discussion of the world-sheet bosons are the NS-NS fields, *i.e.* the metric and the antisymmetric tensor field B_2 . A convenient choice of gauge for the B -field is

$$B_2 = -\sqrt{3}E(x^1 du \wedge dx^3 - x^2 du \wedge dx^4). \quad (4.76)$$

The relevant part of the world-sheet action is:

$$S_B = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \left\{ \sqrt{-g} g^{\alpha\beta} (2\partial_\alpha U \partial_\beta V + A_{ij} X^i X^j \partial_\alpha U \partial_\beta U + \partial_\alpha X^i \partial_\beta X^i) - 2\sqrt{3} E \epsilon^{\alpha\beta} (X^1 \partial_\alpha U \partial_\beta X^3 - X^2 \partial_\alpha U \partial_\beta X^4) \right\}, \quad (4.77)$$

where A_{ij} may be read off from (4.74).

As before, variation of V gives $\square U = 0$. So we can again take light-cone gauge with $U = \alpha' p^+ \tau + \text{const}$, so that the worldsheet scalars obey the following equations:

$$\begin{aligned} \square X^1 - 4M^2 X^1 + \sqrt{3} M \partial_\sigma X^3 &= 0, \\ \square X^2 - 4M^2 X^2 - \sqrt{3} M \partial_\sigma X^4 &= 0, \\ \square X^3 - M^2 X^3 - \sqrt{3} M \partial_\sigma X^1 &= 0, \\ \square X^4 - M^2 X^4 + \sqrt{3} M \partial_\sigma X^2 &= 0, \\ \square X^p - M^2 X^p &= 0, \end{aligned} \quad (4.78)$$

where $p, q = 5, 6, 7, 8$ will label the directions which are unaffected by the B -field. The structure of these equations is similar to those from other plane-wave systems (see for example [141, 167, 177], which follow on from [161–163, 166]), but the crucial difference arises from the asymmetry between the 1–3 plane and the 2–4 plane (visible in (4.74)), which will produce a mass splitting in the spectrum. The two independent components of the standard constraint from world-sheet reparametrizations, $T_{\alpha\beta} = 0$, are

$$\partial_\sigma V = -\frac{1}{\alpha' p^+} \partial_\tau X^i \partial_\sigma X^i, \quad (4.79)$$

$$\partial_\tau V = -\frac{1}{2\alpha' p^+} (\partial_\tau X^i \partial_\tau X^i + \partial_\sigma X^i \partial_\sigma X^i + (\alpha' p^+)^2 A_{ij} X^i X^j), \quad (4.80)$$

which we can use to eliminate V . The constraint is again found by integrating the former over σ :

$$\int_0^{2\pi} d\sigma \partial_\tau X^i \partial_\sigma X^i = 0. \quad (4.81)$$

In the light-cone gauge, the action becomes

$$S_B = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \left\{ -2\alpha' p^+ \partial_\tau V - (\alpha' p^+)^2 A_{ij} X^i X^j + \eta^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^i - 2\sqrt{3} M (X^1 \partial_\sigma X^3 - X^2 \partial_\sigma X^4) \right\}, \quad (4.82)$$

from which it is easy to derive the Hamiltonian:

$$H_B = \frac{1}{4\pi\alpha'} \int_0^{2\pi} d\sigma \left\{ (2\pi\alpha')^2 \Pi^i \Pi^i - (\alpha' p^+)^2 A_{ij} X^i X^j + \partial_\sigma X^i \partial_\sigma X^i - 2\sqrt{3}M (X^1 \partial_\sigma X^3 - X^2 \partial_\sigma X^4) \right\}. \quad (4.83)$$

Again, the conjugate variable to X^i is

$$\Pi^i = \frac{1}{2\pi\alpha'} \partial_\tau X^i. \quad (4.84)$$

The usual periodic boundary conditions apply in this case so we can Fourier expand,

$$X^i(\tau, \sigma) = \sum_n C_n^i e^{i(\omega_n \tau + n\sigma)}, \quad (4.85)$$

from which we get the following system of equations:

$$\begin{aligned} [-\omega_n^2 + (n^2 + 4M^2)] C_n^1 - in\sqrt{3}M C_n^3 &= 0, \\ [-\omega_n^2 + (n^2 + 4M^2)] C_n^2 + in\sqrt{3}M C_n^4 &= 0, \\ [-\omega_n^2 + (n^2 + M^2)] C_n^3 + in\sqrt{3}M C_n^1 &= 0, \\ [-\omega_n^2 + (n^2 + M^2)] C_n^4 - in\sqrt{3}M C_n^2 &= 0, \\ [-\omega_n^2 + (n^2 + M^2)] C_n^p &= 0, \end{aligned} \quad (4.86)$$

for some unknown coefficients C_n^i . For the X^p , the analysis is the same as in section 4.3, and so we can turn our attention to the $i = 1, 2, 3, 4$ directions. The form of the equations is slightly more complicated because although the 1 and 3 directions are coupled, the contributions from A_{ij} are different. The frequencies of these modes are:

$$\omega_n^2 = \frac{1}{2} \left(2n^2 + 5M^2 \pm \sqrt{12n^2 M^2 + 9M^4} \right) \equiv (\omega_n^\pm)^2. \quad (4.87)$$

It is interesting to note that this expression is positive for all n , and so we can expect to obtain a string spectrum that is qualitatively familiar. In appendix C.2 we will study slightly more general backgrounds where this is not the case (the equation equivalent to (4.87) gives imaginary frequencies), and some possible interpretations.

The natural frequencies of the zero modes are $\omega_0^- = \omega_0 = M$ and $\omega_0^+ = 2M$, as

expected. Explicitly, the mode expansions for these coordinates are

$$X^1(\tau, \sigma) = \cos 2M\tau x_0^1 + \frac{\alpha'}{2M} \sin 2M\tau p_0^1 + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left[\frac{1}{\omega_n^+} \left(\beta_n^1 e^{in\sigma} + \tilde{\beta}_n^1 e^{-in\sigma} \right) e^{-i\omega_n^+ \tau} + \frac{1}{\omega_n^-} \left(\gamma_n^1 e^{in\sigma} + \tilde{\gamma}_n^1 e^{-in\sigma} \right) e^{-i\omega_n^- \tau} \right], \quad (4.88)$$

$$X^3(\tau, \sigma) = \cos M\tau x_0^3 + \frac{\alpha'}{M} \sin M\tau p_0^3 + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left[\frac{c_n^+}{\omega_n^+} \left(\beta_n^1 e^{in\sigma} - \tilde{\beta}_n^1 e^{-in\sigma} \right) e^{-i\omega_n^+ \tau} + \frac{c_n^-}{\omega_n^-} \left(\gamma_n^1 e^{in\sigma} - \tilde{\gamma}_n^1 e^{-in\sigma} \right) e^{-i\omega_n^- \tau} \right], \quad (4.89)$$

where ω_n^\pm is given by the positive (negative) root of (4.87) for positive (negative) n and

$$c_n^\pm = \frac{i}{2\sqrt{3nM}} \left(-3M^2 \pm \sqrt{12n^2M^2 + 9M^4} \right), \quad (4.90)$$

which obeys $c_n^+ c_n^- = 1$. It is important to realize that X^1 and X^3 still have four families of oscillators, but that now all four appear in both expansions. Similar expressions hold for X^2 and X^4 , with $\{\beta^1, \gamma^1, \tilde{\beta}^1, \tilde{\gamma}^1\}$ replaced by $\{\beta^2, \gamma^2, \tilde{\beta}^2, \tilde{\gamma}^2\}$ and c_n^\pm replaced by $-c_n^\pm$. With $A = 1, 2$, reality of the coordinates implies

$$(\beta_n^A)^\dagger = \beta_{-n}^A, \quad (\gamma_n^A)^\dagger = \gamma_{-n}^A, \quad (\tilde{\beta}_n^A)^\dagger = \tilde{\beta}_{-n}^A, \quad (\tilde{\gamma}_n^A)^\dagger = \tilde{\gamma}_{-n}^A. \quad (4.91)$$

The remaining non-vanishing Poisson brackets are then

$$\begin{aligned} [x_0^A, p_0^B]_{PB} &= [x_0^{A+2}, p_0^{B+2}]_{PB} = \delta^{AB}, \\ [\beta_m^A, \beta_n^B]_{PB} &= [\tilde{\beta}_m^A, \tilde{\beta}_n^B]_{PB} = -i\omega_m^+ \frac{c_m^-}{c_m^- - c_m^+} \delta_{m+n,0} \delta^{AB}, \\ [\gamma_m^A, \gamma_n^B]_{PB} &= [\tilde{\gamma}_m^A, \tilde{\gamma}_n^B]_{PB} = -i\omega_m^- \frac{c_m^+}{c_m^+ - c_m^-} \delta_{m+n,0} \delta^{AB}, \end{aligned} \quad (4.92)$$

The constraint (4.81) becomes $N = \tilde{N}$, where

$$\begin{aligned} N &= \sum_{n \neq 0} n \left[\frac{1}{\omega_n} \alpha_{-n}^p \alpha_n^p + \frac{1}{\omega_n^+} (1 - c_n^{+2}) \beta_{-n}^A \beta_n^A + \frac{1}{\omega_n^-} (1 - c_n^{-2}) \gamma_{-n}^A \gamma_n^A \right], \\ \tilde{N} &= \sum_{n \neq 0} n \left[\frac{1}{\omega_n} \tilde{\alpha}_{-n}^p \tilde{\alpha}_n^p + \frac{1}{\omega_n^+} (1 - c_n^{+2}) \tilde{\beta}_{-n}^A \tilde{\beta}_n^A + \frac{1}{\omega_n^-} (1 - c_n^{-2}) \tilde{\gamma}_{-n}^A \tilde{\gamma}_n^A \right], \end{aligned} \quad (4.93)$$

and the Hamiltonian (4.83) is

$$\begin{aligned}
H_B = & \frac{1}{2\alpha'} \left(\alpha'^2 p_0^i p_0^i + 4M^2 \sum_{i=1,2} x_0^i x_0^i + M^2 \sum_{i=3}^8 x_0^i x_0^i \right) \\
& + \frac{1}{2} \sum_{n \neq 0} \left(\alpha_{-n}^p \alpha_n^p + \bar{\alpha}_{-n}^p \bar{\alpha}_n^p + (1 - c_n^{+2}) \left(\beta_{-n}^A \beta_n^A + \tilde{\beta}_{-n}^A \tilde{\beta}_n^A \right) \right. \\
& \left. + (1 - c_n^{-2}) \left(\gamma_{-n}^A \gamma_n^A + \tilde{\gamma}_{-n}^A \tilde{\gamma}_n^A \right) \right). \quad (4.94)
\end{aligned}$$

As before we replace the Poisson brackets with commutators and take for $n > 0$,

$$\begin{aligned}
\alpha_n^p &= \sqrt{\omega_n} a_n^p, & \alpha_{-n}^p &= \sqrt{\omega_n} \bar{a}_n^p, \\
\beta_n^A &= \sqrt{\omega_n^+ \frac{c_n^-}{c_n^- - c_n^+}} b_n^A, & \beta_{-n}^A &= \sqrt{\omega_n^+ \frac{c_n^-}{c_n^- - c_n^+}} \bar{b}_n^A, \\
\gamma_n^A &= \sqrt{\omega_n^- \frac{c_n^+}{c_n^+ - c_n^-}} c_n^A, & \gamma_{-n}^A &= \sqrt{\omega_n^- \frac{c_n^+}{c_n^+ - c_n^-}} \bar{c}_n^A. \quad (4.95)
\end{aligned}$$

Analogous formulae hold for the independent set of operators with a tilde, and we combine the zero modes as

$$\begin{aligned}
a_0^i &= \frac{1}{\sqrt{4M\alpha'}} (\alpha' p_0^i - 2iMx_0^i), \\
\bar{a}_0^i &= \frac{1}{\sqrt{4M\alpha'}} (\alpha' p_0^i + 2iMx_0^i), & (i = 1, 2), \quad (4.96) \\
a_0^i &= \frac{1}{\sqrt{2M\alpha'}} (\alpha' p_0^i - iMx_0^i), \\
\bar{a}_0^i &= \frac{1}{\sqrt{2M\alpha'}} (\alpha' p_0^i + iMx_0^i), & (i = 3, \dots, 8). \quad (4.97)
\end{aligned}$$

The new creation and annihilation operators obey the standard harmonic oscillator commutation relations

$$[a_0^i, \bar{a}_0^j] = \delta^{ij}, \quad [a_m^p, \bar{a}_n^q] = \delta_{mn} \delta^{pq}, \quad [b_n^A, \bar{b}_m^B] = [c_n^A, \bar{c}_m^B] = \delta_{nm} \delta^{AB}, \quad (4.98)$$

and similarly for the tilded set of operators. In this basis, the Hamiltonian (4.94) becomes

$$H = \Delta E + 2M \sum_{i=1,2} N_0^{(i)} + M \sum_{i=3}^8 N_0^{(i)} + \sum_{n>0} (\omega_n N_n^{(a)} + \omega_n^+ N_n^{(b)} + \omega_n^- N_n^{(c)}), \quad (4.99)$$

where ω_n and ω_n^\pm are given by (4.28) and (4.87) respectively and ΔE is the zero point energy. The occupation numbers are given by

$$N_n^{(a)} = \bar{a}_n^p a_n^p + \bar{\tilde{a}}_n^p \tilde{a}_n^p, \quad (4.100)$$

and similarly for $N_n^{(b)}$ and $N_n^{(c)}$, and we have defined

$$N_0^{(i)} = \bar{a}_0^i a_0^i. \quad (4.101)$$

The spectrum of the bosonic string is thus of the same form as in the maximally supersymmetric case reviewed in section 4.3, the only difference being the more complicated frequencies ω_n^\pm . We will see shortly that precisely the same frequencies appear in the normal modes of some of the fermions.

4.6.2 World-sheet analysis: fermionic sector

In order to carry out the quantization of the fermionic fields, our starting point is again the action in light-cone gauge (4.41). In the case of the Penrose limit of the Leigh-Strassler point we have $H_{u13} = -H_{u24} = F_{u14} = F_{u23} = \sqrt{3}E$ and $F_{u1234} = F_{u5678} = -E$, so that rewriting in terms of θ^1 and θ^2 , gives

$$\begin{aligned} S_F = & -\frac{i}{\pi} \alpha' p^+ \int d\sigma d\tau \left\{ \theta^1 \Gamma_- \partial_+ \theta^1 + \theta^2 \Gamma_- \partial_- \theta^2 + \frac{\sqrt{3}}{2} M \theta^1 \Gamma_- (\Gamma_{14} + \Gamma_{23}) \theta^2 \right. \\ & + \frac{\sqrt{3}}{4} M \theta^1 \Gamma_- (\Gamma_{13} - \Gamma_{24}) \theta^1 - \frac{\sqrt{3}}{4} M \theta^2 \Gamma_- (\Gamma_{13} - \Gamma_{24}) \theta^2 \\ & \left. - 2M \theta^1 \Gamma_- \Gamma_{1234} \theta^2 \right\}, \end{aligned} \quad (4.102)$$

where

$$\partial_\pm = \partial_\tau \pm \partial_\sigma. \quad (4.103)$$

The equations of motion for θ^1 and θ^2 are then

$$\begin{aligned} \partial_+ \theta^1 - M \Gamma_{1234} \theta^2 + \frac{\sqrt{3}}{4} M (\Gamma_{14} + \Gamma_{23}) \theta^2 + \frac{\sqrt{3}}{4} M (\Gamma_{13} - \Gamma_{24}) \theta^1 &= 0, \\ \partial_- \theta^2 + M \Gamma_{1234} \theta^1 + \frac{\sqrt{3}}{4} M (\Gamma_{14} + \Gamma_{23}) \theta^1 - \frac{\sqrt{3}}{4} M (\Gamma_{13} - \Gamma_{24}) \theta^2 &= 0. \end{aligned} \quad (4.104)$$

The next step is again to Fourier expand

$$\theta^I(\tau, \sigma) = \sum_n \theta_n^I(\tau) e^{in\sigma}, \quad (4.105)$$

giving

$$\begin{aligned} \dot{\theta}_n^1 + M \left(\frac{\sqrt{3}}{4} (\Gamma_{14} + \Gamma_{23}) - \Gamma_{1234} \right) \theta_n^2 + \left(\frac{\sqrt{3}}{4} M (\Gamma_{13} - \Gamma_{24}) + in \right) \theta_n^1 &= 0, \\ \dot{\theta}_n^2 + M \left(\frac{\sqrt{3}}{4} (\Gamma_{14} + \Gamma_{23}) + \Gamma_{1234} \right) \theta_n^1 - \left(\frac{\sqrt{3}}{4} M (\Gamma_{13} - \Gamma_{24}) + in \right) \theta_n^2 &= 0. \end{aligned} \quad (4.106)$$

Differentiating with respect to τ and using (4.106) again to eliminate the first derivatives, results in

$$\ddot{\varepsilon}_n + A_n \varepsilon_n = 0, \quad (4.107)$$

where

$$A_n = \left(n^2 + \frac{7M^2}{4} \right) I - \frac{3M^2}{4} \Gamma_{1234} - \frac{i\sqrt{3}Mn}{2} (\Gamma_{13} - \Gamma_{24}) - \frac{i3M^2}{4} (\Gamma_{12} + \Gamma_{34}), \quad (4.108)$$

and we have re-combined θ^1 and θ^2 into a single complex spinor $\varepsilon = \theta^1 + i\theta^2$. In order to solve (4.107) we need to find the eigenspinors of the matrix A_n . To do this, we consider constant eigenspinors of $i\Gamma_{12}$ and $i\Gamma_{34}$, and denote them as $\varepsilon^{\pm\pm}$ where

$$\begin{aligned} i\Gamma_{12} \varepsilon^{\pm(\cdot)} &= \pm \varepsilon^{\pm(\cdot)}, \\ i\Gamma_{34} \varepsilon^{(\cdot)\pm} &= \pm \varepsilon^{(\cdot)\pm}. \end{aligned} \quad (4.109)$$

so that

$$\begin{aligned} A_n \varepsilon^{++} &= (M^2 + n^2) \varepsilon^{++} - \sqrt{3} M n \varepsilon^{--}, \\ A_n \varepsilon^{+-} &= (M^2 + n^2) \varepsilon^{+-}, \\ A_n \varepsilon^{-+} &= (M^2 + n^2) \varepsilon^{-+}, \\ A_n \varepsilon^{--} &= (4M^2 + n^2) \varepsilon^{--} - \sqrt{3} M n \varepsilon^{++}. \end{aligned} \quad (4.110)$$

Therefore, at each level n , there are four fermionic oscillators with frequency given by

$$\omega_n^2 = n^2 + M^2, \quad (4.111)$$

and there are four fermionic oscillators with frequencies

$$(\omega_n^\pm)^2 = \frac{1}{2} \left(2n^2 + 5M^2 \pm \sqrt{12n^2M^2 + 9M^4} \right). \quad (4.112)$$

Remarkably, these exactly match the frequencies (4.28) and (4.87) found for the bosonic oscillators above. This is presumably required by the supernumerary supersymmetries.

Having seen that the bosonic and fermionic contributions match up in the case of the maximally-supersymmetric plane-wave, it is reasonable to expect that the same thing happens in this case. Therefore the Hamiltonian takes the following form:

$$H = \Delta E + 2M \sum_{i=1,2} N_0^{(i)} + M \sum_{i=3}^8 N_0^{(i)} + \sum_{n>0} (\omega_n N_n^1 + \omega_n^+ N_n^2 + \omega_n^- N_n^3), \quad (4.113)$$

where, in analogy with [141], the zero point energy is

$$\Delta E = (6 \times 1/2 + 2 \times 2 \times 1/2)M = 5M, \quad (4.114)$$

the fermion zero modes appear in $N_0^{(i)}$ and the operators $N_n^{1,2,3}$ now also include the relevant contributions from the fermions.

As before, one can take a different definition of the fermionic vacuum to remove the zero-point energy. In this case, one then has six fermionic zero-mode creation operators that raise the value of H by M , and two that raise it by $2M$.

4.7 The BMN limit of the Leigh-Strassler fixed point

Now that we have found the string spectrum on the background (4.74), we can attempt to match it to the Leigh-Strassler theory, following the work of BMN.

Let us first consider the light-cone Hamiltonian

$$H = i\alpha' p^+ \partial_u = \alpha' p^+ \left(\frac{\partial \tau}{\partial u} i\partial_\tau + \frac{\partial \phi}{\partial u} i\partial_\phi + \frac{\partial \beta}{\partial u} i\partial_\beta + \frac{\partial \gamma}{\partial u} i\partial_\gamma \right), \quad (4.115)$$

where

$$\begin{aligned} \tau &= Eu - \frac{1}{2\Omega_0^2} \frac{v}{E\hat{L}^2}, & \psi &= \frac{3}{2} \left(Eu + \frac{1}{2\Omega_0^2} \frac{v}{E\hat{L}^2} \right), \\ \phi &= \hat{\phi} + \frac{1}{3}\psi, & \gamma &= \hat{\gamma} + \frac{2}{3}\psi. \end{aligned} \quad (4.116)$$

Since $U(1)_\beta$ is not a symmetry of the gauge theory superpotential, there is no conserved charge associated with the operator $i\partial_\beta$, and so it would not make sense to have this term present in the Hamiltonian. However, we have

$$\beta = \psi - (\phi + \gamma) = -(\hat{\phi} + \hat{\gamma}), \quad (4.117)$$

so that $\partial\beta/\partial u = 0$ as required. The scaling dimension, Δ , the R-charge, J , and the ‘flavour’ charge, J_3 , associated with the $U(1)_\gamma$ diagonal subgroup of the global ‘flavour’ $SU(2)$, are given by

$$\Delta = i\partial_\tau, \quad J = -i\partial_\phi, \quad J_3 = i\partial_\gamma, \quad (4.118)$$

	Δ	J	J_3	$\frac{H}{M} = \Delta - \frac{1}{2}J + J_3$
ϕ_1	3/4	1/2	1/2	1
ϕ_2	3/4	1/2	-1/2	0
ϕ_3	3/2	1	0	1
$\bar{\phi}_1$	3/4	-1/2	-1/2	1/2
$\bar{\phi}_2$	3/4	-1/2	1/2	3/2
$\bar{\phi}_3$	3/2	-1	0	2

Table 4.1: The conformal dimensions, charges and light-cone energies of the complex scalar fields appearing as the lowest-order components in the expansions of the three chiral and three anti-chiral superfields.

so that

$$H = M \left(\Delta - \frac{1}{2}J + J_3 \right). \quad (4.119)$$

Likewise, the light-cone momentum is given by

$$p^+ = i\partial_v = -\frac{1}{2\Omega_0^2} \frac{1}{E\hat{L}^2} \left(\Delta + \frac{J}{2} - J_3 \right). \quad (4.120)$$

Since both of these quantities should remain fixed after taking the Penrose limit, in analogy with [139], we are interested in operators with large R- and flavour-charges:

$$|J|, |J_3| \sim \hat{L}^2 \sim N^{1/2}, \quad (4.121)$$

as we take the $N \rightarrow \infty$ limit, keeping g_{YM}^2 fixed and small. In this limit of *infinite* 't Hooft coupling, we must further demand that $\Delta - (J/2) + J_3$ is kept fixed, so that the light-cone Hamiltonian remains finite.

The values of Δ, J, J_3 and H for the complex scalar fields appearing as the lowest-order components in the expansion of the three chiral and three anti-chiral superfields are listed in table 4.1. Remembering that it is Φ_3 which is massive, and can be integrated out as $\Phi_3 \sim [\Phi_1, \Phi_2]$, the values of H which we find make sense: the energy of ϕ_3 is equal to the sum of the energies of ϕ_1 and ϕ_2 .

The first prediction from the spectrum found in section 4.6 is that there should be a *unique* light-cone ground state with large Δ, J and J_3 . It is simply that state for which all the occupation numbers in (4.113) vanish. This corresponds in the

gauge theory to the operator $\text{Tr}(\phi_2^{2J})$. It has $H = 0$ since it is chiral — its conformal dimension is simply the naïve value $\Delta = 3J/2$. The second prediction (and this is where we depart from the previous results concerning both the $\mathcal{N} = 4$ [139] and $\mathcal{N} = 1$ [174–176] theories) is that there should be precisely *six* bosonic operators with $H = M$ and *two* bosonic operators with $H = 2M$, corresponding to the zero modes of the worldsheet scalars. Four are straightforward to write down; they are simply derivatives of the ground state operator

$$\text{Tr}(D_a \phi_2^{2J}), \quad a = 1, 2, 3, 4, \quad (4.122)$$

coming from inserting covariant derivatives along any of the four spacetime directions. This follows straightforwardly from the descendant of the action of the conformal group.

Another operator with $H = M$ is $\text{Tr}(\phi_1 \phi_2^{2J})$. This is again chiral and so its conformal dimension is the sum of those of its constituents. This leaves a sixth bosonic operator to be found, which we can look for in analogy with the analysis of Itzhaki *et al.* [174] in the $T^{1,1}$ case. We propose

$$\text{Tr}(\bar{\phi}_1 \phi_2^{2J}), \quad (4.123)$$

as our sixth operator with $H = M$. Since it is not chiral, its conformal dimension is not necessarily the naïve one found using the values in table 4.1.

To understand why this operator should have the required conformal dimension, consider

$$\text{Tr}(\bar{\phi}_1 \phi_2). \quad (4.124)$$

Perusal of the tables in [35] shows that this operator is in the same $\mathcal{N} = 2$ supermultiplet as the conserved $SU(2)$ current and therefore its dimension is the same as its free-field value, *i.e.* $\Delta = 2$, which gives $H = M$. Unfortunately, there does not seem to be a field theory method to derive the conformal dimension of $\text{Tr}(\bar{\phi}_1 \phi_2^{2J})$. For the equivalent operator in the $T^{1,1}$ case, Itzhaki *et al.* [174] were able to find the relevant conformal dimension using a standard AdS/CFT formula relating the conformal dimension to the Laplacian on $T^{1,1}$. In the case of the Pilch-Warner geometry considered here, we do not have such a formula and so the proposed conformal dimension of $\text{Tr}(\bar{\phi}_1 \phi_2^{2J})$ is somewhat more conjectural than one might have liked.

	Δ	J	J_3	$\frac{H}{M} = \Delta - \frac{1}{2}J + J_3$
χ_1	5/4	-1/2	1/2	2
χ_2	5/4	-1/2	-1/2	1
ψ	3/2	1	0	1
$\bar{\chi}_1$	5/4	1/2	-1/2	1/2
$\bar{\chi}_2$	5/4	1/2	1/2	3/2
$\bar{\psi}$	3/2	-1	0	2

Table 4.2: The conformal dimensions, charges and light-cone energies of the gauginos, ψ , and the fermionic components, χ_1 and χ_2 , of the chiral superfields Φ_1 and Φ_2 , and their anti-chiral counterparts. We do not consider the components of Φ_3 here, since it should not enter our discussion at all.

We should also consider the two operators with $H = 2M$, which contribute at the same ‘level’ as the six operators with $H = M$. However, there are various candidates, and none of them seem to have protected conformal dimensions. Therefore, we have not been able to identify them and have to leave their existence in the appropriate BMN limit as a conjecture.

Given this limited success with the bosonic operators at the lowest lying levels, let us now turn our attention to the fermionic ones. From table 4.2 one can immediately see that the following operators have $H = M$:

$$\text{Tr}(\chi_2 \phi_2^{2J}), \quad \text{Tr}(\psi \phi_2^{2J}). \quad (4.125)$$

These give four fermionic operators since both χ and ψ are two-component Weyl fermions. The first two are the supersymmetry variation of the ground state operator. The second two involve the gaugino, ψ . The remaining two operators to be found are the fermionic counterparts of $\text{Tr}(\bar{\phi}_1 \phi_2^{2J})$:

$$\text{Tr}(\bar{\chi}_1 \phi_2^{2J}). \quad (4.126)$$

Again, we do not consider fermionic operators with $H = 2M$, but expect that there are precisely two of them as above.

Of course, our ultimate aim should be to reproduce the form of the string spectrum (4.113) from the gauge theory, along the lines of BMN [139]. Rewriting this

in gauge theory variables, we would want to derive that, for the 5, 6, 7, 8 directions,

$$(\Delta - \hat{J})_n = \sqrt{1 + \left(\frac{27}{32}\right) \frac{n^2 g_{YM}^2 N}{j^2}}, \quad (4.127)$$

where $\hat{J} = J/2 - J_3$, and we have used the fact that

$$\hat{L}^4 = \left(\frac{3}{2^{5/3}}\right)^4 g_{YM}^2 N \alpha'^2 \quad \text{and} \quad p^+ \sim -\frac{1}{\Omega_0^2 E \hat{L}^2}. \quad (4.128)$$

For the 1, 2, 3, 4 directions, we should have the very interesting result

$$(\Delta - \hat{J})_n = \sqrt{\frac{5}{2} + \left(\frac{27}{32}\right) \frac{n^2 g_{YM}^2 N}{j^2} \pm \frac{3}{2} \sqrt{1 + \frac{4}{3} \left(\frac{27}{32}\right) \frac{n^2 g_{YM}^2 N}{j^2}}}. \quad (4.129)$$

Having considered the IR fixed point geometry and its Penrose limits in some detail, we now move away from the fixed point, turning to the flow geometry of [82].

4.8 Taking a Penrose limit along the RG flow

If we are to take the Penrose limit of the flow geometry away from the fixed points, we are forced to use ‘Poincare’ coordinates on the ‘AdS’ space. Using the the same coordinates on the squashed five-sphere as in section 4.5 above, the flow geometry is described by the metric [82]

$$\begin{aligned} ds^2 &= \frac{X^{1/2} \cosh \chi}{\rho} (e^{2A} ds^2(\mathbb{M}^4) + dr^2) + ds_5^2, \\ ds_5^2 &= L^2 \frac{X^{1/2} \text{sech} \chi}{\rho^3} \left[d\theta^2 + \frac{\rho^6 \cos^2 \theta}{X} (\sigma_1^2 + \sigma_2^2) + \frac{\rho^{12} \sin^2(2\theta)}{4X^2} \left(\sigma_3 + \frac{(2 + \rho^6)}{2\rho^6} d\phi \right)^2 \right. \\ &\quad \left. + \frac{\rho^6 \cosh^2 \chi}{4X^2} (2 \sin^2 \theta - \cos^2 \theta)^2 \left(d\phi - \frac{2 \cos^2 \theta}{(2 \sin^2 \theta - \cos^2 \theta)} \sigma_3 \right)^2 \right], \quad (4.130) \end{aligned}$$

where

$$X = \cos^2 \theta + \rho^6 \sin^2 \theta. \quad (4.131)$$

We are still free to choose geodesics for which $\theta = 0$ and $\alpha = 0$. Considering also a constant point on \mathbb{E}^3 , the effective Lagrangian is

$$\mathcal{L} = \frac{\cosh \chi}{\rho} \left(-e^{2A} \dot{t}^2 + \dot{r}^2 + \frac{L_0^2}{4} \rho^4 \dot{\psi}^2 \right), \quad (4.132)$$

where ψ is defined in (4.70). As above, there are two conserved quantities, E and h , associated with the Killing vectors $\partial/\partial t$ and $\partial/\partial\psi$ respectively. The t and ψ equations then give

$$\dot{t} = EL \frac{\rho}{\cosh \chi} e^{-2A}, \quad \dot{\psi} = \frac{h}{\rho^3 \cosh \chi}, \quad (4.133)$$

and the null condition is

$$\dot{r} = EL \frac{\rho}{\cosh \chi} \sqrt{e^{-2A} - \frac{h^2}{E^2} \frac{1}{4\rho^4}}, \quad (4.134)$$

where we have chosen the arbitrary sign in the above to be positive. Of course, we cannot integrate to find $r(\lambda)$, but we do not need to.

Following [137, 138], we introduce coordinates $\{u, v, x\}$ such that $g_{uu} = 0 = g_{ux}$ and $g_{uv} = 1$. In other words, just as in (C.25), we have

$$\begin{aligned} \partial_u &= \dot{r} \partial_r + \dot{t} \partial_t + \dot{\psi} \partial_\psi, \\ \partial_v &= -\frac{1}{EL} \partial_t, \\ \partial_x &= \frac{1}{L} \partial_\psi + \frac{1}{4} \frac{h}{E} \partial_t, \end{aligned} \quad (4.135)$$

which gives

$$\begin{aligned} dr &= EL \frac{\rho}{\cosh \chi} \left(e^{-2A} - \frac{h^2}{E^2} \frac{1}{4\rho^4} \right)^{1/2} du, \\ dt &= EL \frac{\rho}{\cosh \chi} e^{-2A} du - \frac{dv}{EL} + \frac{1}{4} \frac{h}{E} dx, \\ d\psi &= \frac{h}{\rho^3 \cosh \chi} du + \frac{dx}{L}. \end{aligned} \quad (4.136)$$

Substituting for these in the metric (4.130), taking

$$\theta = \frac{y}{L}, \quad \alpha = \frac{w}{L}, \quad (4.137)$$

and dropping all terms of $\mathcal{O}(1/L)$, we find

$$\begin{aligned} ds^2 &= 2dudv + \frac{1}{4} \rho^3 \cosh \chi e^{2A} \left(e^{-2A} - \frac{h^2}{E^2} \frac{1}{4\rho^4} \right) dx^2 + \frac{\cosh \chi}{\rho} e^{2A} ds^2(\mathbb{E}^3) \\ &\quad + \frac{\operatorname{sech} \chi}{\rho^3} \left(dy^2 + y^2 d\hat{\phi}^2 \right) + \frac{1}{4} \operatorname{sech} \chi \rho^3 (dw^2 + w^2 d\hat{\gamma}^2) \\ &\quad - \frac{h^2}{4} \cosh \chi \left(\rho^3 y^2 + \frac{1}{4} \frac{w^2}{\rho^3} \right) du^2, \end{aligned} \quad (4.138)$$

where

$$d\hat{\phi} = d\phi - \rho^6 \sinh^2 \chi \frac{d\psi}{2}, \quad d\hat{\gamma} = d\gamma - \cosh^2 \chi \frac{d\psi}{2}. \quad (4.139)$$

Note that in the IR, these reduce to the angular variables in (4.73) as required.

To write this in terms of Brinkman coordinates, define

$$E(u) = \frac{1}{2} \rho^{3/2} \cosh^{1/2} \chi e^A \sqrt{e^{-2A} - \frac{h^2}{E^2} \frac{1}{4\rho^4}} = \frac{1}{2EL} \rho^{1/2} \cosh^{3/2} \chi e^{A\dot{r}}, \quad (4.140)$$

$$F(u) = e^A \sqrt{\frac{\cosh \chi}{\rho}}, \quad G(u) = \sqrt{\frac{\operatorname{sech} \chi}{\rho^3}}, \quad H(u) = \frac{1}{2} \sqrt{\rho^3 \operatorname{sech} \chi}, \quad (4.141)$$

and consider the metric

$$ds^2 = 2dudv + E(u)^2 dx^2 + F(u)^2 dx^i dx^i + G(u)^2 dz^1 d\bar{z}^1 + H(u)^2 dz^2 d\bar{z}^2 - \frac{h^2}{4} \cosh \chi \left(\rho^3 |z^1|^2 + \frac{|z^2|^2}{4} \right) du^2, \quad (4.142)$$

where $i = 1, 2, 3$ and z^1, z^2 are complex coordinates on the obvious \mathbb{E}^2 s. Then, with a dot denoting $\partial/\partial u$, the relevant Brinkman coordinates are

$$\begin{aligned} \hat{u} &= u, & \hat{x} &= Ex, & \hat{x}^i &= Fx^i, & \hat{z}^1 &= Gz^1, & \hat{z}^2 &= Hz^2, \\ \hat{v} &= v - \frac{1}{2} \left(E\dot{E}x^2 + F\dot{F}x^i x^i + G\dot{G}|z^1|^2 + H\dot{H}|z^2|^2 \right), \end{aligned} \quad (4.143)$$

in terms of which the metric (4.142) becomes, dropping the hats,

$$ds^2 = 2dudv + ds^2(\mathbb{E}^8) - \left[-\frac{\ddot{E}}{E} x^2 - \frac{\ddot{F}}{F} x^i x^i + \left(\frac{h^2}{4} \rho^3 - \frac{\ddot{G}}{G} \right) |z^1|^2 + \left(\frac{h^2}{16} \frac{1}{\rho^3} - \frac{\ddot{H}}{H} \right) |z^2|^2 \right] du^2. \quad (4.144)$$

We will not consider the form fields explicitly, but it is easy to see that an application of the Penrose limit will give the same fields (4.75) as for the IR solution of section 4.5, but with a u -dependent amplitude.

At any rate, the resulting metric is certainly in the form of a one-half supersymmetric pp-wave, but with a complicated u -dependent profile. It seems unlikely that string theory on this background is tractable. Moreover, it is somewhat difficult to see what statements about the dual gauge theory can be made. The immediate observation in this regard is, of course, that there is no concept of operators with a definite conformal dimension at a general point along the flow. However,

in the maximally supersymmetric case, dual to the $\mathcal{N} = 4$ Yang-Mills theory, we know [178–180] that evolution in light-cone time u corresponds to changes of scale in the gauge theory (the original holographic radial direction is a monotonic function of u). It is thus tempting to argue that string theory on the above pp-wave is dual to an “RG flow” between the Penrose limit of the $\mathcal{N} = 4$ Yang-Mills theory ($u = \infty$) and the Penrose limit of the $\mathcal{N} = 1$ fixed point theory ($u = -\infty$). Evolution in light-cone time would then induce a flow between the relevant sectors of the two gauge theories.

However, the interpretation must be more subtle as is apparent from considering the Penrose limit of $\text{AdS}_5 \times S^5$ in Poincaré coordinates. For example in (C.23), one finds that the usual null trajectories start at $r = 0$, travel out to some maximum $r = L \ln(E/h)$ and then fall back to $r = 0$. Hence these geodesics sample a finite range of energies extending from the far IR to some maximum, which depends solely on the choice of the initial conditions for the geodesic¹⁹. Therefore the correct interpretation of the Penrose limit of a non-conformal theory is rather unclear.

4.9 Conclusions

One of the most interesting unresolved problems arising from the work presented in this chapter is the complete matching of the string spectrum (4.113) to the corresponding set of operators in the field theory. Initially, one would want to match the supergravity modes first (including those single-oscillator modes with $H = 2M$), but eventually the aim would be to match the stringy modes as well. At first, the double square-root formula of (4.129) appears very difficult to reproduce from perturbative Yang-Mills theory. However, an approach similar to that taken by Santambrogio & Zanon [173] may be quite promising. They were able to reproduce the square-root formula of BMN using superspace techniques. (In fact they found the square-root arose simply as a solution of a quadratic equation.) One of the key elements of their analysis was that they were able to use properties of the operators implied by $\mathcal{N} = 1$

¹⁹We should also add that similar geodesics in nonconformal backgrounds were considered in [181, 182], and a discussion of RG flows in this context also appeared in [179].

superconformal invariance. In the case we have studied, the Leigh-Strassler theory, we also have $\mathcal{N} = 1$ superconformal symmetry and so one might imagine that with some assumptions one could use a similar argument. (Perhaps it is possible to derive the double square-root formula in the same way that we found it in the string analysis — as the solution of two coupled quadratic equations.)

Other interesting questions concern the appearance of supernumerary supersymmetries. Does the existence of supernumerary supersymmetries always imply that the frequencies of the bosonic and fermionic modes of the superstring match? How can the appearance of 20 supersymmetries be seen from the gauge theory?²⁰

Finally, it would be attractive to extend the work of Gubser, Klebanov & Polyakov [185] (and also Russo [186]) who studied highly excited string states on $\text{AdS}_5 \times S^5$ using semi-classical soliton solutions and reproduced the square-root formula (4.58) of BMN. Applying this technique to other supergravity duals would hopefully give further insights into a variety of gauge theories.

²⁰It has been suggested [3] that this could be related to “Inonu-Wigner contractions” [183,184].

Appendix A

The $SU(2)$ -invariant one-forms

In the following we shall use the conventions of Eguchi *et al.* [187], (for instance see p. 377).

The group manifold of $SU(2)$ is S^3 – this can easily be seen by the following parametrization of $SU(2)$:

$$g = \begin{pmatrix} v^1 & -\bar{v}^2 \\ v^2 & \bar{v}^1 \end{pmatrix}, \quad (\text{A.1})$$

where $v^1\bar{v}^1 + v^2\bar{v}^2 = 1$. Let us define three one-forms, σ_i , $i = 1, 2, 3$, by the following:

$$g^{-1}dg = i\sigma_i\tau_i \quad (\text{A.2})$$

where the τ_i are the Pauli matrices, that satisfy $\text{Tr}(\tau_i\tau_j) = 2\delta_{ij}$. Therefore,

$$\sigma_i = -\frac{i}{2}\text{Tr}(\tau_i g^{-1}dg) \quad (\text{A.3})$$

so that

$$\begin{aligned} \sigma_1 &= \frac{i}{2}(v^2dv^1 - v^1dv^2 - \bar{v}^2d\bar{v}^1 + \bar{v}^1d\bar{v}^2), \\ \sigma_2 &= \frac{1}{2}(v^2dv^1 - v^1dv^2 + \bar{v}^2d\bar{v}^1 - \bar{v}^1d\bar{v}^2), \\ \sigma_3 &= -\frac{i}{2}(\bar{v}^1dv^1 + \bar{v}^2dv^2 - v^1d\bar{v}^1 - v^2d\bar{v}^2). \end{aligned} \quad (\text{A.4})$$

Then if we choose the following parametrization of v^1 and v^2 (where $0 \leq \alpha < \pi$, $0 \leq \beta < 4\pi$, $0 \leq \gamma < 2\pi$),

$$\begin{aligned} v^1 &= \cos\left(\frac{\alpha}{2}\right) e^{\frac{i}{2}(\beta+\gamma)}, \\ v^2 &= \sin\left(\frac{\alpha}{2}\right) e^{\frac{i}{2}(\beta-\gamma)}, \end{aligned} \quad (\text{A.5})$$

one finds the following:

$$\begin{aligned}\sigma_1 &= \frac{1}{2}(\sin \beta d\alpha - \sin \alpha \cos \beta d\gamma) , \\ \sigma_2 &= -\frac{1}{2}(\cos \beta d\alpha + \sin \alpha \sin \beta d\gamma) , \\ \sigma_3 &= \frac{1}{2}(d\beta + \cos \alpha d\gamma) .\end{aligned}\tag{A.6}$$

In particular

$$\sigma_1 + i\sigma_2 = -\frac{ie^{i\beta}}{2}(d\alpha - i \sin \alpha d\gamma) ,\tag{A.7}$$

so that $\sigma_1^2 + \sigma_2^2 = \frac{1}{4}d\Omega_2^2$, where $d\Omega_2^2$ is the metric on the unit two-sphere. Furthermore, one has

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = dv^1 d\bar{v}^1 + dv^2 d\bar{v}^2 = d\Omega_3^2 .\tag{A.8}$$

The corresponding volume forms on S^2 and S^3 are:

$$\omega_2 = 4\sigma_1 \wedge \sigma_2 = \sin \alpha d\gamma \wedge d\alpha ,\tag{A.9}$$

$$\omega_3 = \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = \frac{1}{8} \sin \alpha d\alpha \wedge d\beta \wedge d\gamma .\tag{A.10}$$

Another useful fact is

$$d\sigma_i = \epsilon_{ijk} \sigma_j \wedge \sigma_k .\tag{A.11}$$

Given, this we can parametrize \mathbb{R}^4 in terms of the σ_i and a radial coordinate u . Let

$$z^1 = uv^1 , \quad z^2 = uv^2 ,\tag{A.12}$$

so that

$$\begin{pmatrix} du \\ u\sigma_1 \\ u\sigma_2 \\ u\sigma_3 \end{pmatrix} = \frac{1}{2u} \begin{pmatrix} \bar{z}^1 & \bar{z}^2 & z^1 & z^2 \\ iz^2 & -iz^1 & -i\bar{z}^2 & i\bar{z}^1 \\ z^2 & -z^1 & \bar{z}^2 & -\bar{z}^1 \\ -i\bar{z}^1 & -i\bar{z}^2 & iz^1 & iz^2 \end{pmatrix} \begin{pmatrix} dz^1 \\ dz^2 \\ d\bar{z}^1 \\ d\bar{z}^2 \end{pmatrix} ,\tag{A.13}$$

and so

$$ds^2 \Big|_{\mathbb{R}^4} = dz^1 d\bar{z}^1 + dz^2 d\bar{z}^2 = du^2 + u^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) .\tag{A.14}$$

Appendix B

The Pilch-Warner geometry

The ten-dimensional Pilch-Warner geometry has the metric [82, 98]

$$ds^2 = \Omega^2 (e^{2A} ds^2(\mathbb{M}^4) + dr^2) + ds_5^2, \quad (\text{B.1})$$

where, in terms of Cartesian coordinates $x^I, I = 1, \dots, 6$, on \mathbb{E}^6 such that $x^I x^I = 1$, the five-dimensional “internal” metric is

$$ds_5^2 = L_0^2 \frac{\text{sech} \chi}{\xi^3} \left[\xi^2 dx^I Q_{IJ}^{-1} dx^J + \sinh^2 \chi (x^I J_{IJ} dx^J)^2 \right], \quad (\text{B.2})$$

L_0 being the radius of the AdS space at the UV fixed point. The complex structure $J_{IJ} = -J_{JI}$ has non-zero components $J_{14} = J_{23} = J_{65} = 1$ and

$$\Omega^2 = \xi \cosh \chi, \quad \xi^2 = x^I Q_{IJ} x^J, \quad Q = \text{diag} (\rho^{-2}, \rho^{-2}, \rho^{-2}, \rho^{-2}, \rho^4, \rho^4). \quad (\text{B.3})$$

The supergravity scalars $\chi(r)$ and $\rho(r)$ obey [35], together with the metric function $A(r)$, the equations in (2.27). As explained in [98], one uses the complex coordinates

$$u^1 = x^1 + ix^4, \quad u^2 = x^2 + ix^3, \quad u^3 = x^5 - ix^6, \quad (\text{B.4})$$

parametrized as

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = e^{-i\phi/2} \cos \theta \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \quad u^3 = e^{-i\phi} \sin \theta, \quad (\text{B.5})$$

where (v^1, v^2) are as in appendix A.

Our choice of these one-forms differs from that of [82, 98] by a factor of 1/2, which gives rise to various discrepancies between the form-fields given here and those in [82, 98].

There is thus a natural $U(1)$ action $\beta \rightarrow \beta + \text{const.}$, under which the $SU(2)$ doublet in (B.5) picks up an overall phase. This global $U(1)_\beta$ rotates σ_1 into σ_2 and leaves σ_3 invariant. The local version can be used to choose different parametrization of the five-sphere directions. Thus, to go from the coordinates used in [98] to those used in [82], one shifts $\beta \rightarrow \beta + \phi$, which removes the overall phase from the $SU(2)$ doublet in (B.5), and induces the shift $\sigma_3 \rightarrow \sigma_3 + d\phi/2$. There is a further $U(1)$ action $\gamma \rightarrow \gamma + \text{const.}$ which will be of interest to us. This $U(1)_\gamma$ is the diagonal subgroup of the obvious global $SU(2)$.

Putting this all together, we have (cf. [98])

$$ds^2 = \frac{X^{1/2} \cosh \chi}{\rho} (e^{2A} ds^2(\mathbb{M}^4) + dr^2) + ds_5^2, \quad (\text{B.6})$$

$$ds_5^2 = L^2 \frac{X^{1/2} \text{sech} \chi}{\rho^3} \left[d\theta^2 + \frac{\rho^6 \cos^2 \theta}{X} (\sigma_1^2 + \sigma_2^2) + \frac{\rho^{12} \sin^2(2\theta)}{4X^2} \left(\sigma_3 + \frac{(2 - \rho^6)}{2\rho^6} d\phi \right)^2 + \frac{\rho^6 \cosh^2 \chi}{16X^2} (3 - \cos(2\theta))^2 \left(d\phi - \frac{4 \cos^2 \theta}{(3 - \cos(2\theta))} \sigma_3 \right)^2 \right], \quad (\text{B.7})$$

where

$$X(r, \theta) = \cos^2 \theta + \rho^6 \sin^2 \theta. \quad (\text{B.8})$$

Note that the global isometry group of the metric is $SU(2) \times U(1)_\beta \times U(1)_\phi$, although only a combination of the two $U(1)$ s is preserved by the form-fields. In the IR ($r \rightarrow -\infty$), we have $\chi \rightarrow 2/\sqrt{3}$, $\rho \rightarrow 2^{1/6}$ and $A(r) \rightarrow r/\hat{L}$, where $\hat{L} = 3L/2^{5/3}$. The metric becomes [98]

$$ds^2(\text{IR}) = \frac{2^{1/3}}{\sqrt{3}} (3 - \cos 2\theta)^{1/2} (e^{2r/\hat{L}} ds^2(\mathbb{M}^4) + dr^2) + ds_5^2(\text{IR}),$$

$$ds_5^2(\text{IR}) = \frac{\sqrt{3}L^2}{4} (3 - \cos 2\theta)^{1/2} \left[d\theta^2 + \frac{4 \cos^2 \theta}{(3 - \cos 2\theta)} (\sigma_1^2 + \sigma_2^2) + \frac{4 \sin^2 2\theta}{(3 - \cos 2\theta)^2} \sigma_3^2 + \frac{2}{3} \left(d\phi - \frac{4 \cos^2 \theta}{(3 - \cos 2\theta)} \sigma_3 \right)^2 \right]. \quad (\text{B.9})$$

Concentrating for the time being on this fixed point geometry, our self-dual five-form is

$$F_5(\text{IR}) = -\frac{2}{3} \frac{2^{2/3}}{\hat{L}} e^{4r/\hat{L}} (1 + \star) \epsilon(\mathbb{E}^4) \wedge dr, \quad (\text{B.10})$$

which differs by a factor of 2 to that in [98]. To determine the correct ansatz for the three-form, one considers the linear $G_3 = du^1 \wedge du^2 \wedge du^3$, which depends on ϕ only

through the overall phase $e^{-2i\phi}$, and which includes an overall factor of $(\sigma_1 + i\sigma_2)$ (this is $(\sigma_1 - i\sigma_2)$ in [82, 98] due to different conventions for the left-invariant one-forms). The two-form potential is thus

$$A_2(\text{IR}) = A e^{-2i\phi} \frac{L^2 \cos \theta}{2} \left(d\theta - \frac{2i \sin(2\theta)}{(3 - \cos(2\theta))} \sigma_3 \right) \wedge (\sigma_1 + i\sigma_2), \quad (\text{B.11})$$

where there is an overall arbitrary constant phase, A which is set to $-i$ in [98]. The field strength $G_3 = dA_2$ is

$$G_3(\text{IR}) = iA e^{-2i\phi} L^2 \cos \theta \left(d\theta \wedge d\phi - \frac{8 \cos^2 \theta}{(3 - \cos(2\theta))^2} d\theta \wedge \sigma_3 - \frac{2i \sin(2\theta)}{(3 - \cos(2\theta))} \sigma_3 \wedge d\phi \right) \wedge (\sigma_1 + i\sigma_2). \quad (\text{B.12})$$

It should be obvious that the global $U(1)$ symmetry group of the solution as a whole is the combination $U(1)_R = U(1)_\phi + 2U(1)_\beta$. Of course, by shifting β as discussed above, one is free to choose the R-symmetry to be any combination of the two $U(1)$ s. For example, in the text we are interested in coordinates for which $U(1)_R = U(1)_\phi$, so we perform the coordinate transformation $\beta \rightarrow \beta + 2\phi$ on the above solution. This removes the overall ϕ -dependent phase in the two-form potential precisely as required.

Appendix C

Penrose Limits and PP-Waves

C.1 The $\theta = \pi/2$ geodesics

Here we will consider the Penrose limit of the IR fixed point solution along a null geodesic with $\theta = \pi/2$, corresponding to the massive direction orthogonal to the moduli space. We start with the solution (4.62–4.65), and take

$$\rho = \frac{r}{\hat{L}}, \quad \theta = \frac{\pi}{2} + \frac{y}{\hat{L}}, \quad (\text{C.1})$$

with $\hat{L} \rightarrow \infty$. Defining the light-cone coordinates,

$$u = \frac{1}{2E} \left(\tau + \frac{2}{3}\phi \right), \quad v = -\frac{2^{4/3}}{\sqrt{3}} E \hat{L}^2 \left(\tau - \frac{2}{3}\phi \right). \quad (\text{C.2})$$

the solution becomes

$$\begin{aligned} ds^2 &= 2dudv + \frac{E^2}{4} (1 - 4(r^2 + |v|^2)) du^2 + ds^2(\mathbb{E}^8) + iE du (v^1 d\bar{v}^1 + v^2 d\bar{v}^2), \\ F_5 &= -\frac{E}{2} du \wedge \epsilon(\mathbb{E}^4), \\ G_3 &= -\sqrt{3}E du \wedge dv^1 \wedge dv^2. \end{aligned} \quad (\text{C.3})$$

where we have rescaled r and y and defined the coordinates

$$v^1 = e^{-iEu/2} z^1, \quad v^2 = e^{-iEu/2} z^2, \quad (\text{C.4})$$

z^1, z^2 being complex coordinates on one of the two \mathbb{E}^4 s. The constant in g_{uu} is unimportant as far as the field equations and supersymmetry transformations are

concerned, and has been discussed in [155]. The metric has a $dudx^i$ cross-term, but it is clear that this can be traded with explicit u -dependence in the three-form.

In the original coordinates, that is, the cross term is of the form $\sigma_3 du$ and can be removed by shifting the Euler angle β , to give

$$\begin{aligned} ds_2^2 &= 2dudv - E^2(r^2 + |z|^2)du^2 + ds^2(\mathbb{E}^8), \\ F_3 &= -\frac{E}{2}du \wedge \epsilon(\mathbb{E}^4), \\ G_3 &= -\sqrt{3}Ee^{-iEu}du \wedge dz^1 \wedge dz^2. \end{aligned} \tag{C.5}$$

Substituting for $\lambda = -E/2$ and $\mu = -\sqrt{3}E e^{-iEu}$ in the field equation (C.8), one can verify that the solution is valid. The u -dependence in G_3 drops out of the field equations since it is just an overall phase. We note that the above pp-wave will give rise to worldsheet scalars of the same mass, unlike the case we have considered in the text.

We have not been able to understand the significance of this particular Penrose limit with respect to the dual gauge theory. In the original coordinates with a cross term in the metric, the light-cone Hamiltonian one finds is

$$H \sim \Delta - \frac{3}{2}J, \tag{C.6}$$

so that all six scalar fields have $H = 0$ — there is certainly no unique ground state. Moreover, this fact does not seem to be mirrored in the string theory spectrum on this background, which *does* seem to show a unique ground state. Furthermore, the frequencies of the bosonic and fermionic modes do not seem to match, in which case it seems unlikely that a simple Hamiltonian can be written down at all. On the other hand, after shifting β to remove the cross-term, one finds an $i\partial_\beta$ term in the Hamiltonian and, as discussed in section 4.7, there is no conserved charge in the gauge theory associated with this differential operator.

Taking the Penrose limit along geodesics with angular momentum in the massive directions is perhaps an odd thing to try to do anyway, since at the IR fixed point, one can simply integrate out these directions. As far as a D-brane probe would be concerned, motion in these directions is energetically disfavoured and simply not to be described in the dual picture by the effective low energy $\mathcal{N} = 1$ field theory.

C.2 Possible instabilities for large B -fields

Let us consider the general solution

$$\begin{aligned} ds^2 &= 2dudv + A_{ij}x^i x^j du^2 + ds^2(\mathbb{E}^8), \\ F_5 &= \mu (1 + \star) du \wedge \epsilon(\mathbb{E}^4), \\ G_3 &= \zeta du \wedge dz^1 \wedge dz^2, \end{aligned} \tag{C.7}$$

where $z^1 = x^1 + ix^2$, $z^2 = x^3 + ix^4$ are complex coordinates on one of the transverse \mathbb{E}^4 s, μ and ζ are real and complex constants respectively, and the equations of motion demands that

$$\text{tr}A = -8\mu^2 - 2|\zeta|^2. \tag{C.8}$$

Furthermore, let A be diagonal so that $A_{ij} = -\delta_{ij}E_i^2$, so that (C.8) becomes:

$$\sum E_i^2 = 8\mu^2 + 2|\zeta|^2. \tag{C.9}$$

Following the analysis of section 4.6.1, the equations of motion for X^1 and X^3 are then

$$\nabla^2 X^1 - M_1^2 X^1 + b\partial_\sigma X^3 = 0, \tag{C.10}$$

$$\nabla^2 X^3 - M_3^2 X^3 - b\partial_\sigma X^1 = 0, \tag{C.11}$$

where $b = p^+ \alpha' \zeta$ and $M_i = p^+ \alpha' E_i$. (Similar equations hold for the X^2 and X^4 directions.) Fourier expanding as in (4.85) above, one finds that the frequencies of the normal modes are:

$$\omega_n^2 = n^2 + \frac{M_1^2 + M_3^2}{2} \pm \sqrt{\frac{1}{4}(M_1^2 - M_3^2)^2 + b^2 n^2} \tag{C.12}$$

Now note that *all* of the above ω_n 's are real and non-zero if and only if

$$b^2 < n^2 + M_1^2 + M_3^2 + \frac{M_1^2 M_3^2}{n^2}. \tag{C.13}$$

In particular, by minimizing the right-hand side with respect to n , one is guaranteed real ω_n for

$$b^2 < (M_1 + M_3)^2. \tag{C.14}$$

However, for larger values of b^2 , it is possible that some of the frequencies are imaginary, resulting in exponentially growing string modes.

Generalizing the analysis of section 4.6.2, one finds no such instability in the fermionic spectrum. As well as the standard four fermions with frequency $\omega_n^2 = n^2 + M^2$ at each level, the remaining fermionic oscillators have

$$\omega_n^2 = n^2 + M^2 + \frac{b^2}{2} \pm \frac{1}{2} \sqrt{b^4 + 4b^2 n^2} \quad (\text{C.15})$$

where $M = p^+ \alpha' \mu$. It is straightforward to show that this expression always yields real frequencies. Note that the metric coefficients E_i^2 do not appear directly in the fermionic spectrum. Further, for the general background, the bosonic and fermionic spectra no longer match.

The interpretation of the unstable modes is somewhat unclear, although their existence is quite interesting. One might think of them as some sort of (classical) instability in the string theory in these backgrounds. The appearance of these modes is particularly curious because the supergravity background still appears to be at least one-half supersymmetric, *i.e.*, the 16 standard Killing spinors one has in the case of plane-wave solutions will yield vanishing supersymmetry variations, irrespective of the value of the three-form field. One might suspect that these Killing spinors are ill-behaved in some way, *e.g.*, exponentially diverging in u , if the three-form is too large. However, this is not the case since the equation for standard Killing spinors is independent of G_3 .

One might imagine that solutions with the B -field too large, in the above sense, are excluded by the supergravity field equations C.9. However, it is easy to see that this is not the case, as the inequality (C.14) only refers to three of the nine parameters appearing in the former equation. Hence these unstable modes apparently appear in valid supergravity backgrounds. Further it seems that given the null form of these gravity wave solutions, they will be solutions of the string equations of motion to all orders in α' [162, 163, 188]. In particular, it seems the general discussion of [162] should apply even with the appearance of R-R fields in the background.

Let us make several further observations. First, these solutions are in no sense asymptotically flat in any directions, rather the field strengths and R_{uiuj} are constant throughout the spacetime. Hence one may wonder whether or not the generic

backgrounds are relevant in string theory. Certainly we have found that certain plane-wave solutions with constant three-form fluxes appear as Penrose limits of asymptotically AdS backgrounds, and so play a role in string theory. It could be that these “unstable” supergravity solutions are simply pathological backgrounds as far as the string theory is concerned and are not useful spacetimes to consider from this point of view. A second observation is that the unstable modes in the bosonic spectrum only occur at finite, non-zero n . Roughly, one may think that oscillator modes need to be excited so that the string is spatially extended and can “see” the B -field. Hence the instability is inherently stringy in origin. This feature is somewhat reminiscent of the instabilities discussed in [189]. Finally, we note that the instability only appears for a finite set of modes, *i.e.*, for a finite range of n^2 . It is straightforward to derive the exact range, however, let us make some qualitative statements. Generically if we take $M_1^2 \simeq M_3^2 \simeq M^2$ then the instability sets in for $b^2 \gtrsim M^2$. In this case the unstable modes appear in a certain range, $n_-^2 < n^2 < n_+^2$, where $n_\pm^2 = \mathcal{O}(M^2)$ and $n_+^2 - n_-^2 = \mathcal{O}(M^2)$. However, recall the definitions above, $M_i^2 = (p^+ \alpha' E_i)^2$. Now in studying supergravity backgrounds, we would ask that typical curvatures are small which in this case corresponds to $(l_s E_i)^2 \ll 1$. If this inequality applies and $(l_s p^+)^2 \lesssim 1$, then the unstable range will lie entirely within the range 0 and 1. That is, there will not actually be any integer values of n for which the frequencies (C.12) become imaginary. Hence the appearance of an actual unstable mode requires that either the background is highly curved on the string scale and/or the p^+ component of the momentum is very large (which corresponds to a highly excited string state). This once again emphasizes the stringy nature of this potential instability.

C.3 Penrose limit of $\text{AdS}_5 \times S^5$ in Poincaré coordinates

Although the Penrose limit of $\text{AdS}_5 \times S^5$ in Poincaré coordinates has already been discussed in [138], it is worth reviewing the analysis here: firstly, we use different coordinates on the five-sphere, which leads initially to a “mixed” Rosen-Brinkman

form of the maximally supersymmetric pp-wave; and secondly, it will be useful to compare this simple case with the more complicated geometry in section 4.8. The metric on $\text{AdS}_5 \times S^5$, with the AdS factor in global coordinates, is

$$ds^2 = L^2 \left[-\cosh^2 \rho \, d\tau^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_3^2 + \cos^2 \theta \, d\phi^2 + d\theta^2 + \sin^2 \theta \, d\hat{\Omega}_3^2 \right], \quad (\text{C.16})$$

where $d\Omega_3^2$ and $d\hat{\Omega}_3^2$ denote metrics on unit three-spheres. In these coordinates, a simple class of null geodesics is that for which $\rho = 0 = \theta$. Taking the Penrose limit along such a geodesic which has angular momentum in the ϕ direction gives rise [137–139] to the maximally supersymmetric pp-wave of type IIB supergravity [136]. Poincaré coordinates on AdS_5 are defined by

$$\begin{aligned} y &= \frac{1}{L} (\cosh \rho \cos \tau - \sinh \rho \Omega_4), \\ t &= \frac{1}{y} \cosh \rho \sin \tau, \\ x^i &= \frac{1}{y} \sinh \rho \Omega_i, \end{aligned} \quad (\text{C.17})$$

where $x^i, i = 1, 2, 3$, are the coordinates on \mathbb{E}^3 and where $\Omega_i \Omega_i + \Omega_4 \Omega_4 = 1$ gives an embedding of S^3 in \mathbb{E}^4 . Defining a new radial coordinate

$$r = L \ln(Ly), \quad (\text{C.18})$$

the metric on $\text{AdS}_5 \times S^5$ is thus

$$ds^2 = e^{2r/L} [-dt^2 + ds^2(\mathbb{E}^3)] + dr^2 + L^2 \left[\cos^2 \theta \, d\phi^2 + d\theta^2 + \sin^2 \theta \, d\hat{\Omega}_3^2 \right]. \quad (\text{C.19})$$

For geodesics at a constant point in \mathbb{E}^3 , the effective Lagrangian is

$$\mathcal{L} = -e^{2r/L} \dot{t}^2 + \dot{r}^2 + L^2 \left[\cos^2 \theta \, \dot{\phi}^2 + \dot{\theta}^2 + \sin^2 \theta \, \dot{\hat{\Omega}}_3^2 \right], \quad (\text{C.20})$$

where, if λ is the affine parameter, a dot denotes $d/d\lambda$. One is still free to consider the class of geodesics for which $\theta = 0$. Then the t and ϕ equations give

$$\dot{t} = EL e^{-2r/L}, \quad \dot{\phi} = h, \quad (\text{C.21})$$

where E and h are the conserved energy and angular momentum associated with the Killing vectors $\partial/\partial t$ and $\partial/\partial \phi$ respectively. The null condition $\mathcal{L} = 0$ then gives

$$\dot{r} = \pm EL \sqrt{e^{-2r/L} - \frac{h^2}{E^2}}. \quad (\text{C.22})$$

If we choose the $-$ sign in the above, then the resulting geodesic matches onto the $\rho = 0$ geodesics in global coordinates.¹ Integrating (C.22) and (C.21) gives

$$r(\lambda) = L \ln \left(\frac{E}{h} \cos \lambda h \right), \quad t(\lambda) = L \frac{h}{E} \tan \lambda h. \quad (\text{C.23})$$

Transforming back to the y coordinate defined in (C.17) above, gives

$$y(\lambda) = \frac{1}{L} \frac{E}{h} \cos \lambda h, \quad (\text{C.24})$$

which, with $h = E$, matches onto the $\rho = 0$ null geodesics in global coordinates, as promised (and these latter do indeed have $h = E$).

Following [138], we introduce coordinates $\{u, v, x\}$ such that u is the affine parameter along the null geodesics. Demanding that $g_{uu} = 0 = g_{ux}$ and $g_{uv} = 1$, a possible choice is

$$\begin{aligned} \partial_u &= \dot{r} \partial_r + \dot{t} \partial_t + \dot{\phi} \partial_\phi, \\ &= -Lh \tan uh \partial_r + \frac{h^2 L}{E} \sec^2 uh \partial_t + h \partial_\phi, \\ \partial_v &= -\frac{1}{EL} \partial_t, \\ \partial_x &= \frac{1}{L} \partial_\phi + \frac{h}{E} \partial_t, \end{aligned} \quad (\text{C.25})$$

which can be integrated to give

$$\begin{aligned} r(u) &= L \ln \left(\frac{E}{h} \cos uh \right), \\ t(u, v, x) &= \frac{hL}{E} \tan uh - \frac{v}{EL} + \frac{h}{E} x, \\ \phi(u, x) &= \frac{x}{L} + hu. \end{aligned} \quad (\text{C.26})$$

We now write the original metric (C.19) in terms of $\{u, v, x\}$ and implement the fact that $\theta = 0$ by defining $\theta = y/L$ and taking the limit $L \rightarrow \infty$. Dropping terms of $\mathcal{O}(1/L)$ gives the metric

$$ds^2 = 2dudv - h^2 y^2 du^2 + \sin^2 uh dx^2 + \frac{E^2}{h^2} \cos^2 uh ds^2(\mathbb{E}^3) + ds^2(\mathbb{E}^4), \quad (\text{C.27})$$

¹The choice of sign either gives ingoing or outgoing geodesics, which does not change the effect of the Penrose limit on the spacetime.

where y is the radial coordinate on \mathbb{E}^4 . The coordinate singularities in this metric appear because of degeneracies in the choice of vectors in (C.25). For example, $\partial_u = hL\partial_x$ at $\sin uh = 0$. Further, we note in passing that working in global coordinates, and the analogue thereof on the sphere, gives rise to the pp-wave in Brinkman coordinates. Use of Poincaré coordinates, however, and the parametrization of the five-sphere used in [138], gives rise to the pp-wave in Rosen coordinates. The “mixed” coordinates used here has given rise to the above pp-wave in “mixed” Brinkman-Rosen coordinates. At any rate, introducing

$$\begin{aligned} x^- &= u, & z &= \sin uh x, & z^i &= \frac{E}{h} \cos uh x^i, \\ x^+ &= v + \frac{1}{4} \left(\frac{E}{h} x^i x^i - x^2 \right) \sin(2uh), \end{aligned} \quad (\text{C.28})$$

gives

$$ds^2 = 2dx^- dx^+ - h^2 |x|^2 dx^{-2} + ds^2(\mathbb{E}^8), \quad (\text{C.29})$$

where now $|x|$ denotes the radial coordinate on \mathbb{E}^8 .

As to the R-R five-form field strength which, in Poincaré coordinates, has the form

$$F_5 = \frac{C}{L} (1 + \star) \epsilon(\text{AdS}_5) = \frac{C}{L} e^{4r/L} (1 + \star) dt \wedge \epsilon(\mathbb{E}^3) \wedge dr, \quad (\text{C.30})$$

for some constant C and where $\epsilon(\mathcal{M})$ denotes the volume form on \mathcal{M} , we find

$$F_5 = C dx^- \wedge \epsilon(\mathbb{E}^4). \quad (\text{C.31})$$

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