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Off-shell Amplitudes in the $\mathcal{N} = 4$ Super Yang-Mills theory

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Contents

Introduction

The $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory can be considered as the most symmetric gauge theory. It has the maximal possible number of supersymmetry charges for a gauge theory without gravity and has only two parameters, i.e. the number of colors N of the gauge group $SU(N)$ and the gauge coupling constant g. It is a conformal field theory even at the quantum level [1] since its β-function vanishes to all orders of perturbation theory. This is why one refers to $\mathcal{N} = 4$ SYM as a finite quantum field theory.
The AdS/CFT correspondence [2] claims that it is dual to string theory on an $AdS_5 \times S_5$ space-time background and relates the local gauge invariant operators of $\mathcal{N} = 4$ SYM theory to string states. This has motivated a great work in studying the correlation functions of gauge invariant operators in $\mathcal{N} = 4$ SYM.
In the last years a remarkable progress has been achieved also in the study of scattering amplitudes which in $\mathcal{N} = 4$ SYM theory have a very simple structure.

At one loop an amplitude of a non-supersymmetric theory can be decomposed in a linear combination of bubble, triangle and box scalar integrals[0.1] , i.e. integrals with two, three and four internal propagators respectively, plus a rational function of the kinematical invariants. The integrals are called scalar since no loop-momentum factors appear in the numerator of the integrand.
In an amplitude of a supersymmetric gauge theory the rational function is vanishing. Moreover, in $\mathcal{N} = 4$ SYM the high degree of supersymmetry implies that in the on shell amplitudes only boxes have-non vanishing coefficients. These can be determined only by the knowledge of the branch cut singularities of the amplitude. This is done by the so called (generalized) unitarity method [46]-[53] which extracts all information from on shell physical states without using Feynman diagrams which involve unphysical off shell states.

In the large N (or planar) limit, the $\mathcal{N} = 4$ SYM on shell scattering amplitudes exhibit remarkable properties such as a duality between gluon amplitudes and the expectation value of Wilson loops for closed polygons bounded by light-like edges both at strong [3] and at weak coupling [4], [76]-[78].
Moreover, additional symmetries emerge for the on shell planar amplitudes. In fact, the computation of the planar four gluon amplitude [98]-[101] by generalized unitarity methods has allowed to discover a new symmetry not manifest at the Lagrangian level. This is the dual conformal symmetry [75]-[78], which is called dual since it acts on the momentum variables. It is not related, at least not in an obvious way, to the conventional conformal symmetry of $\mathcal{N} = 4$ SYM, but it is connected to the conformal symmetry of the dual light-like Wilson loop. Later it was discovered that this symmetry extends to a dual superconformal symmetry [86]-[87], which is an exact symmetry of all planar tree level amplitudes of $\mathcal{N} = 4$ SYM.
Both the algebra of the conventional and the dual superconformal symmetry have finite dimension. But it has been shown [88] that the commutation of the generators of these two algebras gives rise to the infinite dimensional algebra of a Yangian symmetry, under which tree level amplitudes are invariant. One could expect to have an infinite-dimensional symmetry algebra as a manifestation of the integrability of the theory. In $\mathcal{N} = 4$ SYM integrability has been observed in the study of the spectrum of the scaling anomalous dimensions of gauge invariant composite local operators. This spectrum is governed by the Hamiltonian of a quantum spin chain which is integrable since it has an infinite number of conserved charges [5].
These extra symmetries give rise to the prospect to find an exact result for all the on shell-amplitudes of the theory. In fact, a recursive formula for the all loop integrand of planar scattering amplitudes in $\mathcal{N} = 4$ SYM with manifest Yangian symmetry is given in [6].

But at the loop level the fate of these symmetries is not clear. Even if $\mathcal{N} = 4$ SYM is finite in the ultraviolet, its scattering amplitudes have infrared (IR) divergences since they involve massless particles. The generalized unitarity method, which has been employed to obtain the amplitudes of the $\mathcal{N} = 4$ SYM,

[0.1] In the following we will refer to bubble, triangle and box scalar integrals also simply as bubbles, triangles and box.

assumes from the beginning of the computation that the particles are on-shell, i.e $p^2 = 0$, and uses dimensional regularization to regularize these IR divergences. Dimensional regularization breaks dual conformal symmetry since to be unbroken this symmetry requires that the space-time dimension is kept equal to four. In fact, the on shell planar four gluon amplitude is expressed in terms of dimensionally regularized Feynman integrals. On the other hand, if one allows the external particles to be off-shell, i.e. $p^2 \neq 0$, and keeps the space-time dimension equal to four then the integrals appearing in the computation up to four loop are finite in the IR and are exactly covariant under the dual conformal symmetry. Thus, a regulator for IR divergences which preserves dual conformal symmetry is given by the off-shell regularization, i.e. by letting the external particles to have $p^2 \neq 0$. The use of this regulator in the computation of amplitudes implies that one loses manifest gauge invariance and can no more employ unitarity techniques which are intrinsically on shell but instead has to employ the conventional Feynman diagrams.
Up to now the only off shell four point amplitude which has been computed is that with four gluons in the background field gauge [17]. As happens in the on shell dimensional regularized amplitude, in this off shell version of the amplitude appears only the box scalar integral which is dual conformal covariant and so dual conformal symmetry is present even in the off shell regime in this gauge.
Hence, it is important to know if in a different (supersymmetric) gauge this symmetry is still present or is lost, or in other words if the dual conformal symmetry in the off shell regime depends or not on the choice of the gauge.
We have computed the off shell four scalar amplitude $\mathcal{A}^{off\, shell}_{1\, loop}\Big(\phi\phi^\dagger\phi\phi^\dagger\Big)$ and the off shell four gluon amplitude $\mathcal{A}^{off\, shell}_{1\, loop}\Big(A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}\Big)$ in a $\mathcal{N}=1$ supersymmetric gauge at one loop. We have found that in these amplitudes there are integrals which are not dual conformal covariant. In fact, the decomposition in scalar integrals of $\mathcal{A}^{off\, shell}_{1\, loop}\Big(\phi\phi^\dagger\phi\phi^\dagger\Big)$ contains triangles, while that of $\mathcal{A}^{off\, shell}_{1\, loop}\Big(A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}\Big)$ contains triangles as well as bubbles. Both triangles and bubbles are not dual conformal covariant. Therefore, the presence of the dual conformal symmetry for the off shell amplitudes depends on the choice of the gauge.
Moreover, triangles are finite in the ultraviolet (UV), while bubbles are UV divergent integrals. In spite of the presence of these UV divergent integrals in its decomposition, the gluon amplitude is UV finite since the sum of all the divergent terms arising from the bubbles vanishes (see sections 3.6.2, 3.6.5).

There is another issue related to the off shell regime. It may happen that the on shell limit $p^2 \to 0$ of an off-shell amplitude differs from the on-shell dimensional regularized version of the amplitude where the on shell condition is imposed from the beginning, i.e.

$$\lim_{p^2\to 0} \mathcal{A}^{off\, shell} \not\equiv \mathcal{A}^{on\, shell}_{dim.\, reg.}. \tag{0.0.1}$$

This is due to terms which are absent if one assumes the on shell condition from the beginning, but give a non-vanishing result if one computes the off shell amplitude and then considers its on shell limit (see section (3.4)).
However, we have found that for both $\mathcal{A}^{off\, shell}_{1\, loop}\Big(\phi\phi^\dagger\phi\phi^\dagger\Big)$ and $\mathcal{A}^{off\, shell}_{1\, loop}\Big(A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}\Big)$, the on-shell limit of the off-shell amplitudes coincides with the on-shell dimensional regularized version of the amplitudes, i.e.

$$\begin{aligned}
&\lim_{p^2\to 0} \mathcal{A}^{off\, shell}_{1\, loop}\Big(\phi\phi^\dagger\phi\phi^\dagger\Big) \equiv \mathcal{A}^{on\, shell}_{dim.\, reg.}\Big(\phi\phi^\dagger\phi\phi^\dagger\Big),\\
&\lim_{p^2\to 0} \mathcal{A}^{off\, shell}_{1\, loop}\Big(A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}\Big) \equiv \mathcal{A}^{on\, shell}_{dim.\, reg.}\Big(A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}\Big).
\end{aligned} \tag{0.0.2}$$

Another theory we have studied in this thesis is the so called β deformation of $\mathcal{N}=4$ SYM. It is a theory obtained by modifying the superpotential of $\mathcal{N}=4$ SYM in such a way to break SUSY down to $\mathcal{N}=1$ but maintaining the property of conformal invariance and finiteness. The superpotential of the β deformation depends on two complex parameters, i.e. h and β which gives the name of the theory.
In [96] it has been shown that in the planar limit and with β real, all the amplitudes of the β deformation coincide with the ones of $\mathcal{N}=4$ up to phase factors.
We have studied some $n-$point correlation functions with $n \geq 4$ (or equivalently off shell amplitudes) in the case of complex β. More precisely, we have considered the correlation functions with four and six vector superfields $< V^{a_1}V^{a_2}V^{a_3}V^{a_4} >$ and $< V^{a_1}V^{a_2}V^{a_3}V^{a_4}V^{a_5}V^{a_6} >$ respectively. We have also considered the

'mixed' chiral-vector correlation functions with a chiral, an antichiral and two or three vector superfields $< \Phi^{a_1} \Phi^{\dagger\, a_2} V^{a_3} V^{a_4} >$ and $< \Phi^{a_1} \Phi^{\dagger\, a_2} V^{a_3} V^{a_4} V^{a_5} >$ respectively (the flavor indices are omitted). We have found that these Green's functions are not equal to their value in $\mathcal{N} = 4$ SYM, but receive non planar corrections.
The thesis is organized as follows.
In the first chapter we review super Feynman diagrams, i.e. the perturbation theory in superspace which represents the technique we have used to derive our results.
In the second chapter we discuss the decomposition of one loop amplitudes with particular attention to the case of supersymmetric gauge theories. We describe the Passarino-Veltman method to reduce one loop integrals in a basis of scalar integrals. We also discuss the decomposition of amplitudes as far as color factors are concerned.
The third chapter is devoted to the presentation of our results.

Chapter 1

Superspace perturbation theory

This chapter deals with superspace, superfields and Super Feynman rules.
Superspace is an extension of ordinary space-time by the introduction of fermionic coordinates. It allows to keep supersymmetry manifest and to calculate the quantum behavior of supersymmetric theories more easily since a single superdiagram correspond to many component fields diagrams.
In section 1.1 we briefly review supersymmetry transformations, chiral and vector superfields and the form of the lagrangian with manifest $\mathcal{N}=1$ supersymmetry .
Section 1.2 presents the algebra of supercovariant derivatives $\mathcal{D}$, also named D-algebra, which is useful in the calculations of superdiagrams.
Sections 1.3 and 1.4 respectively treat the propagators and the interactions vertices as originally constructed by Salam and Strathdee in [7] and for $\mathcal{N}=1$ Super Yang-Mills in [8].
In section 1.5, we review a new version of super Feynman-rules formulated in ref. [19] (also named improved super Feynman rules).
In section 1.6 we present some examples of computation of superdiagrams by applying both the old and the new versions of super Feynman rules. Also, we give a brief discussion about the regularization of supersymmetric theories.
In this chapter we have followed closely [14] and [17].

1.1 Supersymmetry and Superspace

According to the Coleman-Mandula theorem [9], the most general symmetry group with bosonic generators of a quantum field theory having a mass gap is the direct product $P \times G$, where P is the Poincaré group and G is an internal symmetry group.
The direct product implies at the level of algebra that the generators of the Poincaré group, i.e. space-time translations P^μ and Lorentz transformations $M^{\mu\nu}$, commute with the generators T^a of G

$$[T^a, P^\mu] = [T^a, M^{\mu\nu}] = 0. \tag{1.1.1}$$

Hence, for a generic quantum field theory the Poincaré algebra cannot be extended in a non trivial way .
If one allows the presence of fermionic generators, one can enlarge the Poincaré algebra. In fact supersymmetry (for a review see [10] - [18]) is obtained adding to the Poincaré generators the fermionic generators Q^I_α and $\bar{Q}_{\dot{\alpha}I}$, $(\alpha, \dot{\alpha} = 1, 2)$, which transform as spinors under the Lorentz group (for the conventions see Appendix A).
The supersymmetry (or in brief SUSY) algebra is

$$\begin{aligned}
&\{Q^I_\alpha, \bar{Q}_{\dot{\alpha}J}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^I_J, \\
&\{Q^I_\alpha, Q^J_\beta\} = 0,\ \{\bar{Q}_{\dot{\alpha}I}, \bar{Q}_{\dot{\beta}J}\} = 0 \\
&[Q^I_\alpha, P_\mu] = 0,\ [\bar{Q}_{\dot{\alpha}I}, P_\mu] = 0 \\
&[M_{\mu\nu}, Q^I_\alpha] = -(\sigma_{\mu\nu})_\alpha^{\ \beta} Q^I_\beta \\
&[M_{\mu\nu}, \bar{Q}^{\dot{\alpha}I}] = -(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} \bar{Q}^{\dot{\beta}I}.
\end{aligned} \tag{1.1.2}$$

where the indices I, J run from 1 to the total number of supersymmetries $\mathcal{N}$. This chapter deals only with $\mathcal{N} = 1$ SUSY, so in the following the indices I, J will be dropped. Moreover in the algebra (1.1.2) we have neglected possible central charges.
The Poincaré group acts naturally on the space-time coordinates x^μ. Thus, having added fermionic generators, it is straightforward to enlarge the space-time by introducing two anticommuting fermionic coordinates θ_α and $\bar\theta_{\dot\alpha}$. This extended space is called superspace. An arbitrary function $\mathcal{F}(x,\theta,\bar\theta)$ on superspace (called also superfield) can always be expanded as a polynomial in θ and $\bar\theta$:

$$\mathcal{F}(x,\theta,\bar\theta) = f_0(x)+\theta^\alpha f_{1\alpha}(x)+\bar\theta_{\dot\alpha}\bar f^{2\dot\alpha}(x)+\theta\theta f_3(x)+\bar\theta\bar\theta f_4(x)+\theta\sigma^\mu\bar\theta f_{5\mu}(x)+\theta\theta\bar\theta_{\dot\alpha}\bar f_6^{\dot\alpha}(x)+\bar\theta\bar\theta\theta^\alpha f_{7\alpha}(x)+\theta\theta\bar\theta\bar\theta\, D(x), \tag{1.1.3}$$

since the product of any three or more components of θ (or $\bar\theta$) vanishes

$$\theta_\alpha\theta_\beta\theta_\gamma = \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta}\bar\theta_{\dot\gamma} = 0. \tag{1.1.4}$$

A representation of Q_α and $\bar Q_{\dot\alpha}$ as differential operators in superspace can be found in such a way that an infinitesimal susy transformation is:

$$\delta_{\bar\epsilon,\epsilon}\mathcal{F} \equiv \mathcal{F}(x+\delta x, \theta+\epsilon, \bar\theta+\bar\epsilon) - \mathcal{F}(x,\theta,\bar\theta) = i(\epsilon Q + \bar\epsilon\bar Q)\mathcal{F}. \tag{1.1.5}$$

Here δx and the representation of Q_α and $\bar Q_{\dot\alpha}$ can be determined using the SUSY algebra (1.1.2) and the Baker-Campbell-Hausdorff formula $e^A e^B = e^{A+B+[A,B]/2}$ (valid if the commutator $[A,[A,B]]$ vanishes [1.1]) for the product of two susy transformations $U(x_2,\epsilon_2,\bar\epsilon_2)\cdot U(x_1,\epsilon_1,\bar\epsilon_1)$, where $U(x_i,\epsilon_i,\bar\epsilon_i) = e^{i(x_\mu P^\mu + \epsilon_i Q + \bar\epsilon_i \bar Q)}$
Thus one obtains

$$\begin{aligned}
\delta x^\mu &= -i\theta\sigma^\mu\bar\epsilon + i\epsilon\sigma^\mu\bar\theta,\\
Q_\alpha &= -i(\partial_\alpha + i\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu),\\
\bar Q_{\dot\alpha} &= i(\bar\partial_{\dot\alpha} + i\theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu),
\end{aligned} \tag{1.1.6}$$

where $\partial_\alpha \equiv \partial/\partial\theta^\alpha$ and $\bar\partial_{\dot\alpha} \equiv \partial/\partial\bar\theta^{\dot\alpha}$ (see Appendix A). The product of any three or more components of ∂ (or $\bar\partial$) vanishes and we will write $\partial\partial$ and $\bar\partial\bar\partial$ for $\partial^\alpha\partial_\alpha$ and $\bar\partial_{\dot\alpha}\bar\partial^{\dot\alpha}$ respectively. Dimensional analysis requires that θ and ϵ have the mass dimension $[\theta] = [\epsilon] = -1/2$, while $[\partial_\alpha] = 1/2$.
The SUSY transformation of a superfield induces for the component field $D(x)$ in (1.1.3) the transformation $\delta D(x) = \partial_\mu K^\mu(x)$ where $K^\mu(x)$ is a vector function on space-time. Since δD is a four divergence and assuming that the surface terms can be discarded, any $D-$term, i.e the coefficient of $\theta\theta\bar\theta\bar\theta$ in a superfield, produces in the lagrangian density an action invariant under susy.
A $D-$term is also denoted as $D(x) = [\mathcal{F}]_D = [\mathcal{F}]_{\theta\theta\bar\theta\bar\theta} = \int d^4\theta \mathcal{F}(x,\theta,\bar\theta)$.

1.1.1 Chiral and Vector superfields

The superfield (1.1.3) is reducible in the sense that one can impose on it constraints which are preserved by SUSY transformations.
We shall consider two kinds of constraints: chirality and reality. To this end let us introduce the super-covariant derivatives $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_{\dot\alpha}$

$$\begin{aligned}
\mathcal{D}_\alpha &\equiv \partial_\alpha - i\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu,\\
\bar{\mathcal{D}}_{\dot\alpha} &\equiv -\bar\partial_{\dot\alpha} + i\theta^\beta\sigma_{\beta\dot\alpha}\partial_\mu.
\end{aligned} \tag{1.1.7}$$

They anticommute with the SUSY generators Q and $\bar Q$, i.e. $\{\mathcal{D}_\alpha, Q_\beta\} = \{\mathcal{D}_\alpha, \bar Q_{\dot\beta}\} = \{\bar{\mathcal{D}}_{\dot\alpha}, Q_\beta\} = \{\bar{\mathcal{D}}_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0$ which implies $[\mathcal{D}_\alpha, \epsilon Q] = 0$, etc... if one contracts Q_α with the fermionic parameter ϵ^α
These relations imply that $\mathcal{D}_\alpha\mathcal{F} = 0$ and $\bar{\mathcal{D}}_{\dot\alpha}\mathcal{F} = 0$ are SUSY invariant constraints. In fact under a SUSY transformation one has that $\mathcal{F} \to \mathcal{F}' = \mathcal{F} + \delta_\epsilon\mathcal{F}$, hence if $\mathcal{D}_\alpha\mathcal{F} = 0$ then

[1.1] Since in this case, schematically, A and B are of the form $P + Q + \bar Q$, from $[P,Q] = 0$, $[P,P] = 0$ and $\{Q,\bar Q\} = P$ we have $[A,B] = P$ and indeed $[A,[A,B]] = 0$

$$\mathcal{D}_\alpha \mathcal{F}' = \mathcal{D}_\alpha \mathcal{F} + \mathcal{D}_\alpha(\delta_\epsilon \mathcal{F}) = \mathcal{D}_\alpha \mathcal{F} + \mathcal{D}_\alpha(i\epsilon Q\mathcal{F}) = 0 + i\epsilon Q(\mathcal{D}_\alpha \mathcal{F}) = 0. \tag{1.1.8}$$

Moreover the product of any three or more components of $\mathcal{D}_\alpha$ ($\bar{\mathcal{D}}_{\dot\alpha}$) is equal to zero

$$\mathcal{D}_\alpha \mathcal{D}_\beta \mathcal{D}_\gamma = 0, \quad \bar{\mathcal{D}}_{\dot\alpha} \bar{\mathcal{D}}_{\dot\beta} \bar{\mathcal{D}}_{\dot\gamma} = 0. \tag{1.1.9}$$

We will write $\mathcal{D}\mathcal{D}$ ($\bar{\mathcal{D}}\bar{\mathcal{D}}$) for $\mathcal{D}^\alpha \mathcal{D}_\alpha$ ($\bar{\mathcal{D}}_{\dot\alpha} \bar{\mathcal{D}}^{\dot\alpha}$) respectively. A superfield $\Phi(x,\theta,\bar\theta)$ which satisfies the constraint

$$\bar{\mathcal{D}}_{\dot\alpha} \Phi(x,\theta,\bar\theta) = 0, \tag{1.1.10}$$

is called chiral (left-handed) superfield, while a superfield $\Phi^\dagger$ that satisfies $D_\alpha \Phi^\dagger = 0$ is called antichiral (right-handed) superfield. To find the general expression of a chiral superfield, it is useful to define new coordinates $y^\mu = x^\mu - i\theta\sigma^\mu\bar\theta$. It results that $\bar{\mathcal{D}}_{\dot\alpha} y^\mu = 0$ and $\bar{\mathcal{D}}_{\dot\alpha}\theta^\alpha = 0$. Thus a left-handed chiral superfield is a function of y and θ only: $\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)$ (the $\sqrt{2}$ is a convention). Expanding this expression in θ and $\bar\theta$ gives

$$\Phi(y,\theta) = \phi(x) - i\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^\mu\partial_\mu\phi(x) + \sqrt{2}\theta\psi(x) + \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi\sigma^\mu\bar\theta + \theta\theta F(x). \tag{1.1.11}$$

The SUSY transformation induces on the component field $F(x)$ in (1.1.11) the transformation $\delta F(x) = \partial_\mu H^\mu$ where H^μ is a vector function. This implies that $\int d^4x F(x)$ is invariant under susy. It is called F-term and is also denoted by $F(x) = [\mathcal{F}]_F = [\mathcal{F}]_{\theta\theta} = \int d^2\theta\Phi$.
Similar expressions are valid for a right-handed superfield, which is a function of $\bar y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$ and $\bar\theta$, i.e. $\Phi^\dagger(\bar y,\bar\theta) = \phi^*(\bar y) + \sqrt{2}\bar\theta\bar\psi(\bar y) + \bar\theta\bar\theta F^*(\bar y)$. Expanding this in θ and $\bar\theta$ gives

$$\Phi^\dagger(\bar y,\bar\theta) = \phi^*(x) + i\theta\sigma^\mu\bar\theta\partial_\mu\phi^*(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^\mu\partial_\mu\phi^*(x) + \sqrt{2}\bar\theta\bar\psi(x) - \frac{i}{\sqrt{2}}\bar\theta\bar\theta\theta\sigma^\mu\partial_\mu\bar\psi + \bar\theta\bar\theta F^*(x). \tag{1.1.12}$$

Chiral superfields do not contain spin-1 bosons. To describe gauge fields another kind of superfield (called the vector superfield) is introduced. It is characterized by the reality condition $V = V^\dagger$. This leads to the following decomposition for V

$$\begin{aligned} V(x,\theta,\bar\theta) &= C(x) + \sqrt{2}\theta\chi(x) + \sqrt{2}\bar\theta\bar\chi(x) + \theta\theta S(x) + \bar\theta\bar\theta S^*(x) + \theta\sigma^\mu\bar\theta A_\mu(x) \\ &+ \theta\theta\,\bar\theta\left(\bar\lambda(x) - \frac{i}{\sqrt{2}}\bar\sigma^\mu\partial_\mu\chi(x)\right) + \bar\theta\bar\theta\,\theta\left(\lambda(x) - \frac{i}{\sqrt{2}}\sigma^\mu\partial_\mu\bar\chi(x)\right) + \frac{1}{2}\theta\theta\,\bar\theta\bar\theta\left(D(x) - \frac{1}{2}\partial^\mu\partial_\mu C(x)\right) \end{aligned} \tag{1.1.13}$$

One can also find a supersymmetric generalization of a guage transformation. First we note that if Λ is a left-handed chiral superfield, $i\Lambda - i\Lambda^\dagger$ is a vector superfield. Second, replacing in (1.1.11) Φ with $i\Lambda$, the transformation[1.2]

$$V \to V' = V + i\Lambda - i\Lambda^\dagger \tag{1.1.14}$$

implies that A_μ transforms like an abelian gauge field

$$\begin{aligned} &A_\mu \to A'_\mu = A_\mu - 2\partial_\mu \mathrm{Im}(\phi) \\ &C \to C' = C + 2\mathrm{Re}(\phi) \\ &\chi \to \chi' = \chi + \psi \\ &S \to S' = S + F \\ &\lambda \to \lambda' = \lambda \\ &D \to D' = D. \end{aligned} \tag{1.1.15}$$

So λ and D are invariant. The parametrization of the coefficients of $\theta\theta\bar\theta$, $\bar\theta\bar\theta\theta$ and $\theta\theta\bar\theta\bar\theta$ in the expansion (1.1.13) is chosen so as to have these simple transformations.

[1.2] In the following discussion, we will also use the notation V^Λ to indicate a gauge transformed superfield instead of V'.

One can choose a particular gauge, called the Wess-Zumino gauge, where $C^{'}$, $\chi^{'}$, $S^{'}$ in $V^{'}$ vanish by imposing that $2\,\mathrm{Re}(\phi) = -2C$, $\psi = -\chi$, $F = -S$. Thus the vector supermultiplet reduces to D, A_μ and λ

$$V_{WZ}(x,\theta,\bar\theta) = \theta\sigma^\mu\bar\theta A_\mu(x) + \theta\theta\,\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\,\theta\lambda(x) + \frac{1}{2}\theta\theta\bar\theta\bar\theta D(x) \tag{1.1.16}$$

This gauge-fixing leaves the abelian $U(1)$ gauge freedom, since no condition is imposed on $\mathrm{Im}(\phi)$. The Wess-Zumino gauge breaks manifestly supersymmetry because the conditions $C^{'} = \chi^{'} = S^{'} = 0$ cannot be maintained under a supersymmetry transformation. In fact an infinitesimal susy transformation $\delta_\epsilon V_{WZ} = i\epsilon Q\big(V_{WZ}\big)$ generates, among many others, a term $i\epsilon Q\big(\bar\theta\bar\theta\,\theta\lambda(x)\big) = \bar\theta\bar\theta\epsilon\lambda$. Such a term corresponds to a $S(x)$ component field that is absent in (1.1.16).
However, simultaneously with a supersymmetry transformation $V \to V^{'} = V + \delta_{\epsilon\bar\epsilon}V$, we can perform a compensating gauge transformation generated by a chiral superfield Λ in order to leave the Wess-Zumino gauge invariant: $V^{'} \to V^{''}_{WZ} = V^{'} - i\Lambda^\dagger + i\Lambda$ with $V^{''}_{WZ}$ being in the form (1.1.16). Let us note that eq.(1.1.16) implies that in this gauge $V^n = 0$ if $n \geq 3$

1.1.2 The $\mathcal{N} = 1$ supersymmetric Lagrangian

Now we turn to the case of a non-abelian symmetry. We shall consider a gauge group G with the generators T^a of the group in the representation $\mathcal{R}$ of G. The generators T^a satisfy

$$[T^a, T^b] = if^{abc}T_c, \;\; \mathrm{Tr}(T^aT^b) = \tau_{\mathcal{R}}\delta^{ab}, \tag{1.1.17}$$

where f^{abc} are the real antisymmetric structure constants of G and $\tau_{\mathcal{R}}$ is a normalization constant. We define Λ and V as matrices having elements $\Lambda_{ij} \equiv \Lambda^a T^a_{ij}$, $V_{ij} \equiv V^a T^a_{ij}$, where Λ^a is a chiral superfield and V^a is a vector superfield, both in the adjoint representation of G. [1.3]

Assume that there is a set of chiral superfields transforming in the representation $\mathcal{R}$ of G, i.e.

$$\Phi^\Lambda_i = [e^{-ig\Lambda^a\,T^a}]_{i\,j}\Phi_j, \tag{1.1.18}$$

or infinitesimally $\delta\Phi^\Lambda_i = -ig\Lambda^a(T^a)_{i\,j}\Phi_j$. Since Λ^a is a superfield, Φ^Λ is a superfield as well.
To construct a Lagrangian invariant under supersymmetry and under gauge transformations, one can observe that V can be exponentiated since from (1.1.13) one sees that it has zero mass dimension.Then the term

$$Tr[\Phi^\dagger e^{gV}\Phi]_{\theta\theta\bar\theta\bar\theta} = [\Phi^\dagger_i (e^{gV})_{ij}\Phi_j]_{\theta\theta\bar\theta\bar\theta} \tag{1.1.19}$$

is invariant under supersymmetry because is a $D-$term. Also it is invariant under a gauge transformation (1.1.18) provided that e^{gV} transforms according to

$$e^{gV} \to e^{gV^\Lambda} = e^{-i\Lambda^\dagger}\,e^{gV}\,e^{i\Lambda}. \tag{1.1.20}$$

To obtain an infinitesimal gauge transformation up to terms linear in $\Lambda\,(\Lambda^\dagger)$, one has to apply the Baker-Campbell-Haussdorff formula to eq.(1.1.20)

$$e^{gV^\Lambda} = e^{gV + i(\Lambda-\Lambda^\dagger) + \frac{i}{2}g[V,\Lambda+\Lambda^\dagger] + \frac{i}{12}g^2[V,[V,\Lambda-\Lambda^\dagger]] + \dots}. \tag{1.1.21}$$

Hence, logarithm of eq.(1.1.21) gives

$$\delta V^\Lambda \equiv V^\Lambda - V = \widehat{H}(V)\Lambda + \widehat{H}^\dagger(V)\Lambda^\dagger + O(\Lambda^2) \tag{1.1.22}$$

where the linear operators $\widehat{H}(V)$, $\widehat{H}^\dagger(V)$ on Λ, $\Lambda^\dagger$ are defined as

$$\begin{aligned}\widehat{H}(V)\Lambda &\equiv i\Lambda + \frac{i}{2}g[V,\Lambda] + \frac{i}{12}g^2[V,[V,\Lambda]] + \dots \\ \widehat{H}^\dagger(V)\Lambda^\dagger &\equiv -i\Lambda^\dagger + \frac{i}{2}g[V,\Lambda^\dagger] - \frac{i}{12}g^2[V,[V,\Lambda^\dagger]] + \dots .\end{aligned} \tag{1.1.23}$$

[1.3] Although V and Λ are matrices the expansion (1.1.13) and the discussion on the Wess-Zumino gauge remain the same.

If the group G is abelian, then eq.(1.1.22) is equal to eq.(1.1.14).
In literature, one can also find this formal expression for an infinitesimal gauge transformation[1.4]

$$\delta V^{\Lambda} = i\mathcal{L}_{\frac{gV}{2}}[(\Lambda^{\dagger}+\Lambda) + \coth \mathcal{L}_{\frac{gV}{2}}(\Lambda^{\dagger}+\Lambda)] \tag{1.1.24}$$

where the Lie derivative $\mathcal{L}_X Y$ is defined as

$$\mathcal{L}_X Y = [X, Y]. \tag{1.1.25}$$

Applying this definitions and expanding in power series the function coth in eq.(1.1.24) , one can prove that eq.(1.1.24) coincides with eq.(1.1.22).

Eq.(1.1.19) provides the kinetic terms for the fields in the chiral supermultiplet and their interaction with the fields in the vector supermultiplet.
The kinetic terms for the fields in the vector supermultiplet and the interaction terms between them are constructed from the supersymmetric field strengths

$$W_{\alpha} = \bar{\mathcal{D}}\bar{\mathcal{D}}e^{-gV}\mathcal{D}_{\alpha}e^{gV}, \quad \bar{W}_{\dot{\alpha}} = \mathcal{D}\mathcal{D}e^{gV}\bar{\mathcal{D}}_{\dot{\alpha}}e^{-gV} \tag{1.1.26}$$

Eq.(1.1.9) implies that these superfields are chiral $\bar{\mathcal{D}}_{\dot{\alpha}}W_{\alpha} = \mathcal{D}_{\alpha}\bar{W}_{\dot{\alpha}} = 0$. Furthermore, it can be shown that under a guage transformation (1.1.20) they transform as

$$W_{\alpha} \to W_{\alpha}^{\Lambda} = e^{-i\Lambda}W_{\alpha}e^{i\Lambda}, \quad \bar{W}_{\dot{\alpha}} \to \bar{W}_{\dot{\alpha}}^{\Lambda} = e^{-i\Lambda^{\dagger}}\bar{W}_{\dot{\alpha}}e^{i\Lambda^{\dagger}} \tag{1.1.27}$$

Therefore a term of the form $\mathrm{Tr}[W^{\alpha}W_{\alpha}]_{\theta\theta}$ (and similarly $Tr[\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}]_{\bar{\theta}\bar{\theta}}$) is gauge-invariant and is also susy invariant because it is a F-term.
Finally, the mass and interaction terms for the component fields of the chiral superfields are obtained from the so called superpotential $\mathcal{W}(\Phi_i)$:

$$\mathcal{W}(\Phi_i) = h_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3!}\lambda_{ijk}\Phi_i\Phi_j\Phi_k, \tag{1.1.28}$$

In order to have a renormalizable Lagrangian, $\mathcal{W}(\Phi_i)$ can contain at most cubic terms , i.e. coupling constants with mass dimension equal or bigger than zero. In (1.1.28) m_{ij} and λ_{ijk} are symmetric in their indices and the factors 1/2 and 1/3! are only a conventional choice.
Note that $\mathcal{W}(\Phi_i)$ is a chiral superfield since it is a product of chiral superfields . Thus the F-term $\mathcal{W}(\Phi_i)_{\theta\theta} = \int d^2\theta\mathcal{W}(\Phi_i)$ is susy invariant and is also invariant under the group G if each term in (1.1.28) is gauge invariant. (For instance, h_i can be non zero only for fields Φ_i invariant under G).

From the discussion above it follows that the most general Lagrangian with $\mathcal{N}=1$ susy is[1.5]

$$\begin{aligned}\mathcal{L}_{\mathcal{N}=1} &= \frac{1}{128g^2\tau_{\mathcal{R}}}\mathrm{Tr}\left[W^{\alpha}W_{\alpha} + \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\right]_F + \left[\Phi_i^{\dagger}(e^{gV})_{ij}\Phi_j\right]_D + \left[\mathcal{W}(\Phi_i) + \mathcal{W}^{\dagger}(\Phi_i^{\dagger})\right]_F \\ &= \frac{1}{128g^2\tau_{\mathcal{R}}}\mathrm{Tr}\left[\int d^2\theta W^{\alpha}W_{\alpha} + \int d^2\bar{\theta}\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\right] + \int d^4\theta\Phi_i^{\dagger}(e^{gV})_{ij}\Phi_j + \int d^2\theta\mathcal{W}(\Phi_i) + \int d^2\bar{\theta}\mathcal{W}^{\dagger}(\Phi_i^{\dagger})\end{aligned} \tag{1.1.29}$$

1.2 D-algebra

The computation of correlation functions and Feynman diagrams in superspace is greatly simplified if one makes use of the relations involving supercovariant derivatives (see also the Appendix B).

[1.4]For a derivation of eq.(1.1.24) see [15].
[1.5]the factor 1/128 is chosen to have a simple form for the propagator of the vector superfield.

Integration relations

First, there are some identities regarding the integration by parts with supercovariant derivatives. Assuming that surface terms $\int d^4x\partial_\mu(\ldots)$ vanish, under a space-time integral $\int d^4x$ the superderivatives $\mathcal{D}_\alpha, \mathcal{D}^\alpha, \bar{\mathcal{D}}_{\dot\alpha}$ and $\bar{\mathcal{D}}^{\dot\alpha}$ can be substituted with $\partial_\alpha, -\partial^\alpha, -\bar\partial_{\dot\alpha}$ and $\bar\partial^{\dot\alpha}$ respectively. Moreover, the product of any three or more components of ∂ (or $\bar\partial$) vanishes and integration with respect to the Grassmann variables is equivalent to differentiation (see the Appendix A). Thus, if $\mathcal{F}(x,\theta,\bar\theta)$ is a generic superfield, then

$$\int d^2\theta\mathcal{F} = \frac{1}{4}\partial\partial\mathcal{F} \tag{1.2.1a}$$

$$\int d^2\bar\theta\mathcal{F} = \frac{1}{4}\bar\partial\bar\partial\mathcal{F} \tag{1.2.1b}$$

$$\int d^2\theta\partial_\alpha\mathcal{F} = \frac{1}{4}\partial\partial\partial_\alpha\mathcal{F} = 0 \tag{1.2.1c}$$

$$\int d^2\bar\theta\bar\partial_{\dot\alpha}\mathcal{F} = \frac{1}{4}\bar\partial\bar\partial\bar\partial_{\dot\alpha}\mathcal{F} = 0. \tag{1.2.1d}$$

From eqs.(1.2.1), it follows also that the following integrals vanish

$$\int d^4xd^2\theta\,\mathcal{D}_\alpha\mathcal{F} = \int d^4xd^2\theta\,\partial_\alpha\mathcal{F} = 0 \tag{1.2.2a}$$

$$\int d^4xd^2\bar\theta\,\bar{\mathcal{D}}_{\dot\alpha}\mathcal{F} = -\int d^4xd^2\bar\theta\,\bar\partial_{\dot\alpha}\mathcal{F} = 0 \tag{1.2.2b}$$

$$\int d^4xd^2\theta\,\mathcal{D}\mathcal{D}\mathcal{F} = -\int d^4xd^2\theta\,\partial\partial\mathcal{F} = 0 \tag{1.2.2c}$$

$$\int d^4xd^2\bar\theta\,\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{F} = -\int d^4xd^2\bar\theta\,\bar\partial\bar\partial\mathcal{F} = 0 \tag{1.2.2d}$$

In addition, there are two important identities which allow to extend the partial superspace integrations $\int d^4xd^2\theta$, $(\int d^4xd^2\bar\theta)$ to the full one $\int d^4xd^2\theta d^2\bar\theta$ if in the integrand there is an operator $\bar{\mathcal{D}}\bar{\mathcal{D}}(\mathcal{D}\mathcal{D})$

$$-\frac{1}{4}\int d^4xd^2\theta\,\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{F} = \frac{1}{4}\int d^4xd^2\theta\,\bar\partial\bar\partial\mathcal{F} = \int d^4xd^2\theta d^2\bar\theta\,\mathcal{F} \tag{1.2.2e}$$

$$-\frac{1}{4}\int d^4xd^2\bar\theta\,\mathcal{D}\mathcal{D}\mathcal{F} = \frac{1}{4}\int d^4xd^2\bar\theta\,\partial\partial\mathcal{F} = \int d^4xd^2\bar\theta d^2\theta\,\mathcal{F} \tag{1.2.2f}$$

It is also possible to reduce an integral over $d^4xd^2\theta(d^2\bar\theta)$ as an integral over d^4x

$$\int d^4xd^2\theta\mathcal{F} = \frac{1}{4}\int d^4x\partial\partial\mathcal{F} = -\int d^4x\frac{\mathcal{D}\mathcal{D}}{4}\mathcal{F} \tag{1.2.2g}$$

$$\int d^4xd^2\bar\theta\mathcal{F} = \frac{1}{4}\int d^4x\bar\partial\bar\partial\mathcal{F} = -\int d^4x\frac{\bar{\mathcal{D}}\bar{\mathcal{D}}}{4}\mathcal{F} \tag{1.2.2h}$$

$$\int d^4xd^2\theta d^2\bar\theta\mathcal{F} = \frac{1}{16}\int d^4x\mathcal{D}\mathcal{D}\,\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{F} \tag{1.2.2i}$$

$$\int d^4xd^2\theta d^2\bar\theta\mathcal{F} = \frac{1}{16}\int d^4x\bar{\mathcal{D}}\bar{\mathcal{D}}\,\mathcal{D}\mathcal{D}\mathcal{F} \tag{1.2.2j}$$

Leibnitz rules and integration by parts

From the definition (1.1.7) of the supercovariant derivatives, from the Leibnitz rule for the space-time derivative ∂_μ and for the Grassmann derivatives (A.5.4a)-(A.5.4b), a generalization of the Leibnitz rule for $\mathcal{D}_\alpha$ can be derived

$$\mathcal{D}_\alpha(\mathcal{B}_1\mathcal{B}_2) = (\mathcal{D}_\alpha\mathcal{B}_1)\mathcal{B}_2 + \mathcal{B}_1\mathcal{D}_\alpha(\mathcal{B}_2), \tag{1.2.3a}$$

$$\mathcal{D}_\alpha(\mathcal{F}_1\mathcal{B}_2) = (\mathcal{D}_\alpha\mathcal{F}_1)\mathcal{B}_2 - \mathcal{F}_1\mathcal{D}_\alpha(\mathcal{B}_2), \tag{1.2.3b}$$

$$\mathcal{D}_\alpha(\mathcal{B}_1\mathcal{F}_2) = (\mathcal{D}_\alpha\mathcal{B}_1)\mathcal{F}_2 + \mathcal{B}_1\mathcal{D}_\alpha(\mathcal{F}_2), \tag{1.2.3c}$$

$$\mathcal{D}_\alpha(\mathcal{F}_1\mathcal{F}_2) = (\mathcal{D}_\alpha\mathcal{F}_1)\mathcal{F}_2 - \mathcal{F}_1\mathcal{D}_\alpha(\mathcal{F}_2), \tag{1.2.3d}$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are bosonic superfields and $\mathcal{F}_1$ and $\mathcal{F}_2$ are fermionic ones. The same identities (1.2.3a)-(1.2.3d) hold if one substitutes $\mathcal{D}_\alpha$ with $\bar{\mathcal{D}}_{\dot\alpha}$.

Furthermore, under an integral $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_{\dot\alpha}$ can be integrated by parts. In fact from the Leibnitz rules (1.2.3a)-(1.2.3d) after discarding the surface terms (see eqs.(1.2.2a)-(1.2.2b) with $\mathcal{F} = \mathcal{B}_1\mathcal{B}_2\mathcal{B}_3$), one obtains

$$\int d^4xd^4\theta\, \mathcal{B}_1(\mathcal{D}_\alpha\mathcal{B}_2)\mathcal{B}_3 = -\int d^4xd^4\theta\, (\mathcal{D}_\alpha\mathcal{B}_1)\mathcal{B}_2\mathcal{B}_3 - \int d^4xd^4\theta\, \mathcal{B}_1\mathcal{B}_2(\mathcal{D}_\alpha\mathcal{B}_3) \tag{1.2.4a}$$

$$\int d^4xd^4\theta\, \mathcal{B}_1(\bar{\mathcal{D}}_{\dot\alpha}\mathcal{B}_2)\mathcal{B}_3 = -\int d^4xd^4\theta\, (\bar{\mathcal{D}}_{\dot\alpha}\mathcal{B}_1)\mathcal{B}_2\mathcal{B}_3 - \int d^4xd^4\theta\, \mathcal{B}_1\mathcal{B}_2(\bar{\mathcal{D}}_{\dot\alpha}\mathcal{B}_3), \tag{1.2.4b}$$

where $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_3$ are all bosonic superfields.
Applying repeatedly eqs.(1.2.4a)-(1.2.4b) and considering that $\mathcal{D}_\alpha\mathcal{B}$ and $\bar{\mathcal{D}}_{\dot\alpha}\mathcal{B}$ are anticommuting fermionic superfields if $\mathcal{B}$ is a bosonic one, it follows that (see the Appendix B for other similar formulas)

$$\int d^4xd^4\theta\, \mathcal{B}_1(\mathcal{D}\mathcal{D}\mathcal{B}_2)\mathcal{B}_3 = \int d^4xd^4\theta\, (\mathcal{D}\mathcal{D}\mathcal{B}_1)\mathcal{B}_2\mathcal{B}_3 \,+ 2\int d^4xd^4\theta\, (\mathcal{D}^\alpha\mathcal{B}_1)\mathcal{B}_2(\mathcal{D}_\alpha\mathcal{B}_3) + \int d^4xd^4\theta\, \mathcal{B}_1\mathcal{B}_2(\mathcal{D}\mathcal{D}\mathcal{B}_3) \tag{1.2.5a}$$

$$\int d^4xd^4\theta\, \mathcal{B}_1(\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{B}_2)\mathcal{B}_3 = \int d^4xd^4\theta\, (\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{B}_1)\mathcal{B}_2\mathcal{B}_3 \,+ 2\int d^4xd^4\theta(\bar{\mathcal{D}}_{\dot\alpha}\mathcal{B}_1)\mathcal{B}_2(\bar{\mathcal{D}}^{\dot\alpha}\mathcal{B}_3) \,+ \int d^4xd^4\theta\, \mathcal{B}_1\mathcal{B}_2(\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{B}_3). \tag{1.2.5b}$$

D-algebra in momentum space

To treat correlation functions in momentum space, it is useful to define the Fourier transform of the superfield $\mathcal{F}(x,\theta,\bar\theta)$:

$$\mathcal{F}(x,\theta,\bar\theta) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}\mathcal{F}(p,\theta,\bar\theta) \tag{1.2.6}$$

Thus, a derivative ∂_μ acting on $\mathcal{F}(x,\theta,\bar\theta)$ in momentum space becomes $-ip_\mu\mathcal{F}(p,\theta,\bar\theta)$. Hence, to pass from coordinate to momentum space it is necessary to substitute ∂_μ with $-ip_\mu$ (and viceversa p_μ with $i\partial_\mu$ from momentum to coordinate space).
We adopt the convention that the momentum p_μ appearing in the relation

$$\partial_{1\mu} \equiv -ip_\mu \tag{1.2.7}$$

with $\partial_{1\mu} = \partial/\partial x_1$, is the ingoing momentum corresponding to the superspace point $(x_1, \theta_1, \bar\theta_1)$.

$$\mathbf{2} \xrightarrow{\quad p^\mu \quad} \mathbf{1} \qquad \begin{array}{l} \partial_1^\mu \equiv -ip^\mu \\ \partial_2^\mu \equiv ip^\mu \end{array}$$

In momentum space the substitution (1.2.7) in eqs.(1.1.7) gives [1.6]

$$\mathcal{D}^p_\alpha = \partial_\alpha - \sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}p_\mu \tag{1.2.8a}$$

$$\mathcal{D}^{p,\alpha} = -\partial^\alpha + \bar\theta_{\dot\beta}\bar\sigma^{\mu\,\dot\beta\alpha}p_\mu \tag{1.2.8b}$$

$$\bar{\mathcal{D}}^p_{\dot\alpha} = -\bar\partial_{\dot\alpha} + \theta^\beta\sigma^\mu_{\beta\dot\alpha}p_\mu \tag{1.2.8c}$$

$$\bar{\mathcal{D}}^{p,\dot\alpha} = \bar\partial^{\dot\alpha} - \bar\sigma^{\mu\,\dot\alpha\beta}\theta_\beta p_\mu \tag{1.2.8d}$$

[1.6] The superscript p on $\mathcal{D}^p$ and $\bar{\mathcal{D}}^p$ indicates the ingoing momentum p corresponding to the superspace point $z \equiv (x,\theta,\bar\theta)$. If the theta variables $\theta_1, \bar\theta_1$ have a further label, such as 1 in this case, then we will also use the notation $\mathcal{D}^p_{1\alpha}, \mathcal{D}^{p,\alpha}_1$, etc..., to indicate $\mathcal{D}^p_{1\alpha} = \partial_{1\alpha} - \sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}_1 p_\mu$, $\quad \mathcal{D}^{p,\alpha}_1 = -\partial^\alpha_1 + \bar\theta_{1\,\dot\beta}\bar\sigma^{\mu\,\dot\beta\alpha}p_\mu$, etc... respectively.
In addition, $\mathcal{D}^\alpha = \epsilon^{\alpha\beta}\mathcal{D}_\beta$ and $\bar{\mathcal{D}}^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar{\mathcal{D}}_{\dot\beta}$.

From (1.2.8a)-(1.2.8d) one can verify the anticommutation relations

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = 0 \tag{1.2.9a}$$

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} p_\mu \tag{1.2.9b}$$

and the identities

$$\mathcal{D}\sigma^\mu\bar{\mathcal{D}} + \bar{\mathcal{D}}\bar\sigma^\mu\mathcal{D} = 4p^\mu \tag{1.2.10a}$$

$$[\mathcal{D}_\alpha, \bar{\mathcal{D}}\bar{\mathcal{D}}] = 4\sigma^\mu_{\alpha\dot\beta}\bar{\mathcal{D}}^{\dot\beta}\, p_\mu \tag{1.2.10b}$$

$$[\mathcal{D}^\alpha, \bar{\mathcal{D}}\bar{\mathcal{D}}] = -4\bar{\mathcal{D}}_{\dot\beta}\bar\sigma^{\mu\dot\beta\alpha}\, p_\mu \tag{1.2.10c}$$

$$[\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{D}\mathcal{D}] = -4\mathcal{D}^\beta\sigma^\mu_{\beta\dot\alpha}\, p_\mu \tag{1.2.10d}$$

$$[\bar{\mathcal{D}}^{\dot\alpha}, \mathcal{D}\mathcal{D}] = 4\bar\sigma^{\mu\dot\alpha\beta}\mathcal{D}_\beta\, p_\mu, \tag{1.2.10e}$$

$$[\mathcal{D}\mathcal{D}, \bar{\mathcal{D}}\bar{\mathcal{D}}] = 8\mathcal{D}\sigma^\mu\bar{\mathcal{D}}\, p_\mu - 16p^2 \tag{1.2.10f}$$

$$[\bar{\mathcal{D}}\bar{\mathcal{D}}, \mathcal{D}\mathcal{D}] = 8\bar{\mathcal{D}}\bar\sigma^\mu\mathcal{D}\, p_\mu - 16p^2 \tag{1.2.10g}$$

$$\mathcal{D}^\alpha\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}_\alpha = \bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}\mathcal{D}\bar{\mathcal{D}}^{\dot\alpha} = -8p^2 + \frac{1}{2}\{\mathcal{D}\mathcal{D}, \bar{\mathcal{D}}\bar{\mathcal{D}}\} \tag{1.2.10h}$$

$$\mathcal{D}\mathcal{D}\,\bar{\mathcal{D}}\bar{\mathcal{D}}\,\mathcal{D}\mathcal{D} = 16p^2\mathcal{D}\mathcal{D} \tag{1.2.10i}$$

$$\bar{\mathcal{D}}\bar{\mathcal{D}}\,\mathcal{D}\mathcal{D}\,\bar{\mathcal{D}}\bar{\mathcal{D}} = 16p^2\bar{\mathcal{D}}\bar{\mathcal{D}} \tag{1.2.10j}$$

(See the Appendix B for a derivation)
One can use an analogue of the Leibnitz rules (1.2.3a)-(1.2.3d) and of the integrations by parts (1.2.4a)-(1.2.5b) also in momentum space (see the Appendix B) .
For example, eqs.(1.2.3a), (1.2.4a) and (1.2.5a) become

$$\mathcal{D}^p_\alpha(\mathcal{B}_1\mathcal{B}_2) = (\mathcal{D}^q_\alpha\mathcal{B}_1)\mathcal{B}_2 + \mathcal{B}_1\mathcal{D}^{p-q}_\alpha(\mathcal{B}_2) \tag{1.2.11a}$$

$$\int d^4\theta\, \mathcal{B}_1(\mathcal{D}^{p_2}_\alpha\mathcal{B}_2)\mathcal{B}_3 = -\int d^4\theta\, (\mathcal{D}^{p_1}_\alpha\mathcal{B}_1)\mathcal{B}_2\mathcal{B}_3 - \int d^4\theta\, \mathcal{B}_1\mathcal{B}_2(\mathcal{D}^{p_3}_\alpha\mathcal{B}_3) \tag{1.2.11b}$$

$$\int d^4\theta\, \mathcal{B}_1(\mathcal{D}^{p_2}\mathcal{D}^{p_2}\mathcal{B}_2)\mathcal{B}_3 = \int d^4\theta\, (\mathcal{D}^{p_1}\mathcal{D}^{p_1}\mathcal{B}_1)\mathcal{B}_2\mathcal{B}_3\, + 2\int d^4\theta\, (\mathcal{D}^{p_1,\alpha}\mathcal{B}_1)\mathcal{B}_2(\mathcal{D}^{p_3}_\alpha\mathcal{B}_3) + \int d^4\theta\, \mathcal{B}_1\mathcal{B}_2(\mathcal{D}^{p_3}\mathcal{D}^{p_3}\mathcal{B}_3) \tag{1.2.11c}$$

where $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ are the Fourier transforms of bosonic superfields. In eq. (1.2.11a) $\mathcal{B}_1$ and $\mathcal{B}_2$ depend on the momenta q and $p-q$ respectively. In eqs.(1.2.11b) and (1.2.11c) $p_1+p_2+p_3=0$ and $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ depend on the momenta p_1,p_2 and p_3 respectively . Also, the momentum in the supercovariant derivative can always be taken as the momentum of the superfield on which it acts.

The fermionic delta function δ

The fermionic delta function can be defined as (see Appendix A) :

$$\delta^{(2)}(\theta) = \theta^2, \quad \delta^{(2)}(\bar\theta) = \bar\theta^2 \tag{1.2.12a}$$

$$\delta^{(4)}(\theta) = \delta^{(2)}(\theta)\,\delta^{(2)}(\bar\theta) = \theta^2\bar\theta^2 \tag{1.2.12b}$$

$$\int d^4\theta\, \delta^{(4)}(\theta) = \int d^4\theta\, \theta\theta\,\bar\theta\bar\theta = 1, \tag{1.2.12c}$$

where $d^4\theta = d^2\theta d^2\bar\theta$. Using the symbol θ_{12} for $\theta_{12} \equiv \theta_1 - \theta_2$ and δ_{12} for $\delta_{12} \equiv \delta^{(4)}(\theta_{12}) = \theta_{12}^2\bar\theta_{12}^2$, one can write

$$\int d^4\theta_1\, \delta_{12} = \int d^4\theta_2\, \delta_{12} = 1. \tag{1.2.12d}$$

From the definitions of $\mathcal{D}$, $\bar{\mathcal{D}}$ and δ_{12} it follows that

$$\mathcal{D}^p_{1\alpha}\,\delta_{12} = -\mathcal{D}^{-p}_{2\alpha}\,\delta_{12} \tag{1.2.13a}$$

$$\bar{\mathcal{D}}^p_{1\dot\alpha}\,\delta_{12} = -\bar{\mathcal{D}}^{-p}_{2\dot\alpha}\,\delta_{12} \tag{1.2.13b}$$

$$\mathcal{D}^p_1\mathcal{D}^p_1\,\delta_{12} = \mathcal{D}^{-p}_2\mathcal{D}^{-p}_2\,\delta_{12} \tag{1.2.13c}$$

$$\bar{\mathcal{D}}^p_1\bar{\mathcal{D}}^p_1\,\delta_{12} = \bar{\mathcal{D}}^{-p}_2\bar{\mathcal{D}}^{-p}_2\,\delta_{12} \tag{1.2.13d}$$

(See the Appendix B for a derivation.)

Finally, there are the relations

$$0 = \delta_{12}\delta_{12} = \delta_{12}\mathcal{D}_\alpha\delta_{12} = \delta_{12}\bar{\mathcal{D}}_{\dot\alpha}\delta_{12} = \delta_{12}\mathcal{D}\mathcal{D}\delta_{12} = \delta_{12}\bar{\mathcal{D}}\bar{\mathcal{D}}\delta_{12} = \delta_{12}\mathcal{D}_\alpha\bar{\mathcal{D}}\bar{\mathcal{D}}\delta_{12} = \delta_{12}\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}\mathcal{D}\delta_{12}\ \text{etc.} \tag{1.2.14a}$$

while

$$\delta_{12}\mathcal{D}\mathcal{D}\,\bar{\mathcal{D}}\bar{\mathcal{D}}\delta_{12} = 16\delta_{12} \tag{1.2.14b}$$

$$\delta_{12}\bar{\mathcal{D}}\bar{\mathcal{D}}\,\mathcal{D}\mathcal{D}\delta_{12} = 16\delta_{12} \tag{1.2.14c}$$

To put it differently only the product of two $\mathcal{D}'$s and two $\bar{\mathcal{D}}'$s between two δ_{12}s gives non-vanishing result.

1.3 Propagators

1.3.1 The propagator of chiral superfield

The main subject of the next sections will be the computation of correlation functions (in momentum space) at one loop. The terms correlation functions, correlators, n-points functions and Green's functions will be used as synonyms to indicate the same object that is the functional integral given by is

$$< T\phi_1 \dots \phi_n >= N \int [d\phi]\, \phi_1 \dots \phi_n\, e^{iS} \tag{1.3.1}$$

where $\phi_i \equiv \phi(x_i)$ is a generic field belonging to any representation of the Lorentz group and of a gauge group G. S is the classical action for the field ϕ and N is a normalization constant. The symbol T indicates time ordering and will be omitted below. In the correlator one could also introduce any composite operator $O[\phi]$ build from elementary fields ϕ. Usually, the classical action can be divided in a free part which is quadratic in the fields and in an interaction part:

$$S = S_{free} + gS_{int}, \tag{1.3.2}$$

where g is a coupling constant. If $g << 1$, one can treat S_{int} as a perturbation and expand $e^{iS_{int}}$ in Taylor series in (1.3.1)

$$< \phi_1 \dots \phi_n >= \sum_{k=0}^{+\infty} \frac{(ig)^k}{k!} \quad N \int [d\phi]\, \phi_1 \dots \phi_n\, (S_{int})^k\, e^{iS_{free}} = \sum_{k=0}^{+\infty} \frac{(ig)^k}{k!} < \phi_1 \dots \phi_n\, (S_{int})^k >_{free} . \tag{1.3.3}$$

This allows to express the correlator in the interacting theory in terms of the correlators in the free theory. The Wick theorem gives an algorithm to compute a free correlator, that is a correlator with the action of the free theory (also named correlator at tree level), such as $< \phi \dots \phi\, S^k_{int} >_{free},$. In fact a free correlator is given by the sum of products of two-point free correlators, also called free propagators or contractions, for example $< \phi\phi >_{free}$.

Since the free propagators are the building blocks of the correlators in perturbation theory, this section will be devoted to the computation of the propagator of the chiral superfield $< \Phi\Phi^\dagger >_{free}$, while the next one to the propagator of the vector superfield $< VV >_{free}$. A peculiarity of a supersymmetric theory is that both the propagators of Φ and V involve the fermionic delta function $\delta^{(4)}(\theta_{12})$, while the propagators of Φ involves also the supercovariant derivative $\mathcal{D}$.

Functional derivatives in superspace

In a generic quantum field theory, a correlation function can be obtained from a generating functional by functional differentiation. To do so in superspace, one needs to extend the definition of functional derivatives also to superfields.

In the discussion the following notation will be used: the points 1 and 2 in superspace will be represented by supercoordinates $z_1 = (x_1, \theta_1, \bar{\theta}_1)$ and $z_2 = (x_2, \theta_2, \bar{\theta}_2)$, while the integration measures are $d^8z \equiv d^4x\, d^2\theta d^2\bar{\theta}$, $d^6z \equiv d^4x\, d^2\theta$ and $d^6\bar{z} \equiv d^4x d^2\bar{\theta}$.

A functional derivative for a vector superfield $V(z)$ can be defined with these properties :

$$\frac{\delta V(z_2)}{\delta V(z_1)} = \delta^{(8)}(z_1 - z_2) \equiv \delta^{(4)}(x_1 - x_2)\delta^{(4)}(\theta_{12}) \tag{1.3.4}$$

and

$$\int d^8z_2 \frac{\delta V(z_2)}{\delta V(z_1)} = 1 \tag{1.3.5}$$

The functional product $j \cdot V$ of two vector superfields V and j is defined as

$$j \cdot V \equiv \int d^8z j(z)V(z). \tag{1.3.6}$$

We shall use the symbols j or $J(J^\dagger)$ to denote the sources for the superfields in what follows. A source of a vector superfield is a vector superfield and a source of a chiral (antichiral) superfield is a chiral (antichiral) superfield.

Integration over all the superspace of a chiral superfield as integrand gives zero. Indeed eqs.(1.2.2e)-(1.2.2f) and the chirality condition $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0\,(\mathcal{D}_\alpha\Phi^\dagger = 0)$ imply that

$$\int d^8z\Phi(z) = -\frac{1}{4}\int d^6z\bar{\mathcal{D}}\bar{\mathcal{D}}\Phi = 0 \tag{1.3.7}$$

$$\int d^8z\Phi^\dagger(z) = -\frac{1}{4}\int d^6\bar{z}\mathcal{D}\mathcal{D}\Phi^\dagger = 0. \tag{1.3.8}$$

So if one has a chiral (antichiral) integrand, one needs to restrict the integration only over $d^6z(d^6\bar{z})$. Thus for chiral and antichiral superfields the functional product is defined as

$$J \cdot \Phi \equiv \int d^6z J(z)\Phi(z) \tag{1.3.9}$$

$$J^\dagger \cdot \Phi^\dagger \equiv \int d^6\bar{z} J^\dagger(z)\Phi^\dagger(z), \tag{1.3.10}$$

with

$$\int d^6z_2 \frac{\delta\Phi(z_2)}{\delta\Phi(z_1)} = 1 \tag{1.3.11}$$

$$\int d^6\bar{z}_2 \frac{\delta\Phi^\dagger(z_2)}{\delta\Phi^\dagger(z_1)} = 1 \tag{1.3.12}$$

From eq.(1.2.2e), one has that $\int d^6z_2\left(-\frac{1}{4}\right)\bar{\mathcal{D}}_2\bar{\mathcal{D}}_2\delta^8(z_2 - z_1) = \int d^8z_2\delta^8(z_2 - z_1) = 1$ and similarly for the case with $\int d^6\bar{z}$. Hence, the equations (1.3.11) and (1.3.12) are solved by:

$$\frac{\delta\Phi(z_2)}{\delta\Phi(z_1)} = -\frac{1}{4}\bar{\mathcal{D}}_2\bar{\mathcal{D}}_2\delta^{(8)}(z_1 - z_2) \equiv -\frac{1}{4}\bar{\mathcal{D}}_1\bar{\mathcal{D}}_1\delta^{(8)}(z_1 - z_2) \tag{1.3.13a}$$

$$\frac{\delta\Phi^\dagger(z_2)}{\delta\Phi^\dagger(z_1)} = -\frac{1}{4}\mathcal{D}_2\mathcal{D}_2\delta^{(8)}(z_1 - z_2) \equiv -\frac{1}{4}\bar{\mathcal{D}}_1\bar{\mathcal{D}}_1\delta^{(8)}(z_1 - z_2) \tag{1.3.13b}$$

where we have used eq.(B.1.5c).

The presence of supercovariant derivatives in the propagator of chiral superfield is due to the presence of $\mathcal{D}\mathcal{D}$ and $\bar{\mathcal{D}}\bar{\mathcal{D}}$ in the previous functional derivatives.

To find a common formalism for chiral and vector superfields that allows to integrate the former over all the superspace, one makes use of

$$\bar{\mathcal{D}}\bar{\mathcal{D}}\,\mathcal{D}\mathcal{D}\Phi = -16\partial^2\Phi \tag{1.3.14}$$

$$\mathcal{D}\mathcal{D}\bar{\mathcal{D}}\bar{\mathcal{D}}\Phi^\dagger = -16\partial^2\Phi^\dagger \tag{1.3.15}$$

which are a consequence of (B.1.2i)-(B.1.2j). Let us introduce a set of projection operators $\Pi_i = (\Pi_+, \Pi_0, \Pi_-)$ defined by [1.7]

$$\Pi_+ \equiv -\frac{\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}\mathcal{D}}{16\partial^2} \tag{1.3.16a}$$

$$\Pi_- \equiv -\frac{\mathcal{D}\mathcal{D}\bar{\mathcal{D}}\bar{\mathcal{D}}}{16\partial^2} \tag{1.3.16b}$$

$$\Pi_0 \equiv \frac{\mathcal{D}^\alpha\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}_\alpha}{8\partial^2} \equiv \frac{\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}\mathcal{D}\bar{\mathcal{D}}^{\dot\alpha}}{8\partial^2} \tag{1.3.16c}$$

The form of Π_+ and Π_- is motivated by eqs.(1.3.14)-(1.3.15) which imply, if Φ is a chiral superfield, that

$$\Pi_+\Phi = \Phi, \qquad \Pi_-\Phi^\dagger = \Phi^\dagger, \tag{1.3.17}$$

.

Moreover, if $\mathcal{F}$ is a generic superfield, $\Pi_+\mathcal{F}$ is a chiral superfield since $\bar{\mathcal{D}}_{\dot\alpha}\Pi_+\mathcal{F} = 0$ and similarly $\Pi_-\mathcal{F}$ is an antichiral one.
From the $D-$algebra eqs.(B.1.2), it follows that the operators Π_i satisfy

$$\sum \Pi_i = 1 \tag{1.3.18a}$$

$$\Pi_i\Pi_j = \delta_{ij}\Pi_j. \tag{1.3.18b}$$

This justifies the name of projectors. Using the chirality of both the superfield Φ and its source J, eq.(1.3.9) can be written as

$$J\cdot\Phi = \int d^6z\Phi\Pi_+J = -\int d^6z\Phi\frac{\bar{\mathcal{D}}\bar{\mathcal{D}}}{4}\frac{\mathcal{D}\mathcal{D}}{4\partial^2}J = -\int d^6z\frac{\bar{\mathcal{D}}\bar{\mathcal{D}}}{4}(\Phi\frac{\mathcal{D}\mathcal{D}}{4\partial^2}J). \tag{1.3.19}$$

Then, using again eq.(1.2.2e), one has

$$J\cdot\Phi = \int d^8z\Phi\frac{\mathcal{D}\mathcal{D}}{4\partial^2}J = \int d^8z(\frac{\mathcal{D}\mathcal{D}}{4\partial^2}\Phi)J \tag{1.3.20}$$

where the last equality is a consequence of the fact in eq.(1.3.19) one could apply the operator Π_+ to Φ instead of J[1.8]. Similarly for the antichiral superfields, one obtains

$$J^\dagger\cdot\Phi^\dagger = \int d^8z\Phi^\dagger\frac{\bar{\mathcal{D}}\bar{\mathcal{D}}}{4\partial^2}J^\dagger = \int d^8z(\frac{\bar{\mathcal{D}}\bar{\mathcal{D}}}{4\partial^2}\Phi^\dagger)J^\dagger \tag{1.3.21}$$

Generating Superfunctionals

An n-point correlation function of superfields V,Φ and $\Phi^\dagger$ can be obtained by functional derivation from the following superfunctional

$$Z[j, J, J^\dagger] = N\int dVd\Phi d\Phi^\dagger \exp i(S + j\cdot V + J\cdot\Phi + J^\dagger\cdot\Phi^\dagger), \tag{1.3.22}$$

[1.7] the non local operator $1/\partial^2$ is defined in such a way that $(1/\partial^2)g = f \Rightarrow g = \partial^2 f$.
Partial superspace integrations such as eq.(1.2.4a) in general do not hold if the supercovariant derivatives are multiplied by the non local operator $1/\partial^2$.

[1.8] here one cannot naively employ partial integration because of the presence of the non local operator $1/\partial^2$: see footnote (1.7).

where j,J and $J^\dagger$ are the sources of V,Φ and $\Phi^\dagger$, N is a normalization constant and S is the action. In fact from the definition (1.3.1), it follows that

$$\begin{aligned} & i^n < T\, V(z_1)\ldots V(z_k)\Phi(z_{k+1})\ldots\Phi(z_m)\Phi^\dagger(z_{m+1})\ldots\Phi^\dagger(z_n)>= \\ & = \left.\frac{\delta^n Z[j,J,J^\dagger]}{\delta j(z_1)\ldots\delta j(z_k)\delta J(z_{k+1})\ldots\delta J(z_m)\delta J^\dagger(z_{m+1})\ldots\delta J^\dagger(z_n)}\right|_{j=J=J^\dagger=0} \end{aligned} \tag{1.3.23}$$

The **connected** correlation functions $< TV(z_1)\ldots V(z_k)\Phi(z_{k+1})\ldots\Phi(z_m)\Phi^\dagger(z_{m+1})\ldots\Phi^\dagger(z_n) >_C$ are given by functional derivation of the superfunctional W

$$W[j,J,J^\dagger] \equiv -i\ln Z[j,J,J^\dagger], \tag{1.3.24}$$

and then by evaluating it at $j = J = J^\dagger = 0$

$$\begin{aligned} & i^n < TV(z_1)\ldots V(z_k)\Phi(z_{k+1})\ldots\Phi(z_m)\Phi^\dagger(z_{m+1})\ldots\Phi^\dagger(z_n)>_C= \\ & = i\left.\frac{\delta^n W[j,J,J^\dagger]}{\delta j(z_1)\ldots\delta j(z_k)\delta J(z_{k+1})\ldots\delta J(z_m)\delta J^\dagger(z_{m+1})\ldots\delta J^\dagger(z_n)}\right|_{j=J=J^\dagger=0} \end{aligned} \tag{1.3.25}$$

The third important superfunctional Γ is the generator of the one particle irreducible correlation functions (1PI) and is the quantum analog of the classical action. 1PI Green's functions derive their name from the fact that they are associated to Feynman graphs that cannot be separated into two disconnected parts by cutting only one internal line.
Γ is obtained by a Legendre transformation. First, one defines the 'classical' superfields $\tilde\Phi$,$\tilde\Phi^\dagger$ and $\tilde V$ which are solutions of the equations of motion

$$\tilde\Phi \equiv \frac{\delta W}{\delta J},\ \tilde\Phi^\dagger \equiv \frac{\delta W}{\delta J^\dagger},\ \tilde V \equiv \frac{\delta W}{\delta j}. \tag{1.3.26}$$

Then one inverts these relations to obtain the expressions for J, $J^\dagger$ and j in terms of $\tilde\Phi$, $\tilde\Phi^\dagger$ and $\tilde V$. After substituting them in W, Γ is obtained from

$$\Gamma[\tilde\Phi,\tilde\Phi^\dagger,\tilde V] \equiv W[j(\tilde\Phi,\tilde\Phi^\dagger,\tilde V),J(\tilde\Phi,\tilde\Phi^\dagger,\tilde V),J^\dagger(\tilde\Phi,\tilde\Phi^\dagger,\tilde V)] - j\cdot\tilde V - J\cdot\tilde\Phi - J^\dagger\cdot\tilde\Phi^\dagger. \tag{1.3.27}$$

The 1PI Green's functions are given by

$$\begin{aligned} & < T\, V(z_1)\ldots V(z_k)\Phi(z_{k+1})\ldots\Phi(z_m)\Phi^\dagger(z_{m+1})\ldots\Phi^\dagger(z_n)>_{1PI}= \\ & = i\left.\frac{\delta^n\Gamma[\tilde\Phi,\tilde\Phi^\dagger,\tilde V]}{\delta\tilde V(z_1)\ldots\delta\tilde V(z_k)\delta\tilde\Phi(z_{k+1})\ldots\delta\tilde\Phi(z_m)\delta\tilde\Phi^\dagger(z_{m+1})\ldots\delta\tilde\Phi^\dagger(z_n)}\right|_{\tilde\Phi=\tilde\Phi^\dagger=\tilde V=0}. \end{aligned} \tag{1.3.28}$$

Since our goal is to find the free propagators, we need only to compute the free part of the generating superfunctionals $Z_0[j,J,J^\dagger]$ and $W_0[j,J,J^\dagger]$

$$Z_0[j,J,J^\dagger] = N_0\int dV d\Phi d\Phi^\dagger \exp i(S_{free} + j\cdot V + J\cdot\Phi + J^\dagger\cdot\Phi^\dagger) \tag{1.3.29}$$

$$W_0[j,J,J^\dagger] = -i\ln Z_0[j,J,J^\dagger] \tag{1.3.30}$$

where S_{free} is the free part of the action which is quadratic in the superfields . From eq.(1.3.25) the free propagator for the chiral superfield will be

$$< \Phi(z_1)\Phi^\dagger(z_2) >_{C\,(free)} \equiv -i\left.\frac{\delta^2 W_0}{\delta J(z_1)\delta J^\dagger(z_2)}\right|_{j=J=J^\dagger=0}. \tag{1.3.31}$$

Free two-points correlation functions

The free part of the action for a **massless** chiral superfield is simply $S_{free} = \int d^8z\Phi(z)\Phi^\dagger(z)$ (cf. eqs.(1.1.28)-(1.1.29) with all the couplings equal to zero ($h_i = m_{ij} = \lambda_{ijk} = g = 0$) and neglecting the terms which contain the vector superfields).
This leads to

$$\begin{aligned} Z_0[J, J^\dagger] &= N_0 \int d\Phi d\Phi^\dagger \exp i\{\Phi^\dagger \cdot \Phi + J \cdot \Phi + J^\dagger \cdot \Phi^\dagger\} \\ &= N_0 \int d\Phi d\Phi^\dagger \exp i\Big\{ \int d^8z \Big[\Phi^\dagger(z)\Phi(z) + \Phi(z)\frac{\mathcal{D}\mathcal{D}}{4\partial^2}J(z) + \Phi^\dagger(z)\frac{\bar{\mathcal{D}}\bar{\mathcal{D}}}{4\partial^2}J^\dagger(z)\Big]\Big\} \\ &= N_0 \int d\Phi d\Phi^\dagger \exp i\Big\{ \int d^8z \Big[\frac{1}{2}\begin{pmatrix}\Phi(z) & \Phi^\dagger(z)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Phi(z)\\ \Phi^\dagger(z)\end{pmatrix} + \begin{pmatrix}\Phi(z) & \Phi^\dagger(z)\end{pmatrix}\begin{pmatrix}\frac{\mathcal{D}\mathcal{D}}{4\partial^2}J(z)\\ \frac{\bar{\mathcal{D}}\bar{\mathcal{D}}}{4\partial^2}J^\dagger(z)\end{pmatrix}\Big]\Big\} \end{aligned} \tag{1.3.32}$$

In the second step has been made use of eqs.(1.3.20)-(1.3.21) which allow to express all the integrals as full superspace integrals $\int d^8z$. In the last one, a matrix notation has been employed. To solve the gaussian integral in eq.(1.3.32), one can use the identity

$$\int dx_1 \dots dx_n \exp i\Big\{\frac{1}{2}\vec{x}^T A\vec{x} + \vec{x}^T \cdot \vec{y}\Big\} = c\exp\Big\{-\frac{i}{2}\vec{y}^T A^{-1}\vec{y}\Big\} \tag{1.3.33}$$

where $\vec{x}$ and $\vec{y}$ are $n-$dimensional vectors, A is a symmetric $n \times n$ matrix with nonzero determinant and c is a constant independent on y_i . In our case

$$A = A^{-1} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{1.3.34}$$

thus eq.(1.3.32) becomes

$$\begin{aligned} Z_0[J, J^\dagger] &= Z_0[0,0]\exp\left\{ -\frac{i}{2}\int d^8z \begin{pmatrix}\frac{\mathcal{D}\mathcal{D}}{4\partial^2}J(z) & \frac{\bar{\mathcal{D}}\bar{\mathcal{D}}}{4\partial^2}J^\dagger(z)\end{pmatrix} A^{-1}\begin{pmatrix}\frac{\mathcal{D}\mathcal{D}}{4\partial^2}J(z)\\ \\ \frac{\bar{\mathcal{D}}\bar{\mathcal{D}}}{4\partial^2}J^\dagger(z)\end{pmatrix}\right\}. \\ &= Z_0[0,0]\exp\left\{ -\frac{i}{2}\int d^8z\frac{1}{8}\left(\frac{\bar{\mathcal{D}}\bar{\mathcal{D}}}{\partial^2}J^\dagger(z)\right)\frac{\mathcal{D}\mathcal{D}}{\partial^2}J(z)\right\} \end{aligned} \tag{1.3.35}$$

where J and $\bar{\mathcal{D}}\bar{\mathcal{D}}(\partial^2)^{-1}J^\dagger$ are chiral superfields, while $J^\dagger$ and $\mathcal{D}\mathcal{D}(\partial^2)^{-1}J$ are antichiral ones. In eq.(1.3.35) one can shift the nonlocal operator $\bar{\mathcal{D}}\bar{\mathcal{D}}/\partial^2$ from one factor to the other one, as in eq.(1.3.21) (see also footnote (1.7)). Thus one can reconstruct the projector Π_+ and from eq.(1.3.17) it follows that

$$Z_0[J, J^\dagger] = Z_0[0,0]\exp\Big\{i\int d^8z\Big[J^\dagger(z)\frac{1}{\partial^2}J(z)\Big]\Big\}. \tag{1.3.36}$$

The functional derivatives $\delta J(z_2)/\delta J(z_1)$ and $\delta J^\dagger(z_2)/\delta J^\dagger(z_1)$ are (cf. eqs.(1.3.13))

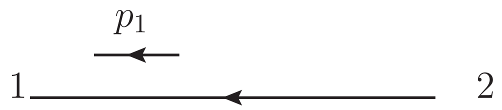
$$\begin{aligned} \frac{\delta J(z_2)}{\delta J(z_1)} &= -\frac{1}{4}\bar{\mathcal{D}}_1\bar{\mathcal{D}}_1\delta^8(z_1 - z_2), \\ \frac{\delta J^\dagger(z_2)}{\delta J^\dagger(z_1)} &= -\frac{1}{4}\mathcal{D}_1\mathcal{D}_1\delta^8(z_1 - z_2). \end{aligned} \tag{1.3.37}$$

Hence, substituting the eq.(1.3.36) obtained for Z_0 in eq.(1.3.31), the expression of the propagator of the chiral superfield is

$$\begin{aligned} < \Phi(z_1)\Phi^\dagger(z_2) >_{free} &\equiv -i\frac{\delta^2 W_0[J, J^\dagger]}{\delta J(z_1)\delta J^\dagger(z_2)}\Bigg|_{J=J^\dagger=0} = -\frac{\delta^2 \ln Z_0[J, J^\dagger]}{\delta J(z_1)\delta J^\dagger(z_2)}\Bigg|_{J=J^\dagger=0}, \\ &= -\frac{i}{16}\frac{\bar{\mathcal{D}}_1\bar{\mathcal{D}}_1\,\mathcal{D}_1\mathcal{D}_1}{\partial_1^2}\delta^8(z_1 - z_2) \end{aligned} \tag{1.3.38}$$

where the last step follows by partial integration of the operator $\mathcal{D}\mathcal{D}$ and by eq.(B.1.5c). In momentum space eq.(1.3.38) becomes

$$< \Phi(p_1,\theta_1,\bar{\theta}_1)\Phi^{\dagger}(-p_1,\theta_2,\bar{\theta}_2) >_{free}= \frac{i}{16p_1^2}\bar{\mathcal{D}}_1^{p_1}\bar{\mathcal{D}}_1^{p_1}\mathcal{D}_1^{p_1}\mathcal{D}_1^{p_1}\delta^{(4)}(\theta_{12}) \tag{1.3.39}$$



where p_1 is the ingoing momentum corresponding to the superspace point z_1 of the chiral superfield $\Phi(z_1)$. In the picture above, the convention for the direction of the arrows of the fields is that in a point with a chiral (antichiral) superfield one has an inward (outward) arrow.
The application of the supercovariant derivatives on $\delta^{(4)}(\theta_{12}) \equiv \theta_{12}^2\bar{\theta}_{12}^2$ in eq.(1.3.39) using the formulae in the Appendix A gives an alternative form for the chiral propagator

$$\begin{aligned}
&< \Phi(p_1,\theta_1,\bar{\theta}_1)\Phi^{\dagger}(-p_1,\theta_2,\bar{\theta}_2) >_{free}= \frac{i}{p_1^2} - \frac{i}{p_1^2}p_{1\mu}\theta_1\sigma^{\mu}\bar{\theta}_1 + \frac{i}{4}\theta_1^2\bar{\theta}_1^2 - \frac{i}{p_1^2}p_{1\mu}\theta_2\sigma^{\mu}\bar{\theta}_2 + \frac{i}{4}\theta_2^2\bar{\theta}_2^2 + \frac{2i}{p_1^2}p_{1\mu}\theta_1\sigma^{\mu}\bar{\theta}_2 \\
&- i\theta_1^2\bar{\theta}_1\bar{\theta}_2 - i\bar{\theta}_2^2\theta_1\theta_2 + i\theta_1^2\bar{\theta}_2^2 + \frac{i}{p_1^2}p_{1\mu}p_{1\nu}\theta_1\sigma^{\mu}\bar{\theta}_1\theta_2\sigma^{\nu}\bar{\theta}_2 - \frac{i}{4}p_{1\mu}\theta_1\sigma^{\mu}\bar{\theta}_1\theta_2^2\bar{\theta}_2^2 - \frac{i}{4}p_{1\mu}\theta_2\sigma^{\mu}\bar{\theta}_2\theta_1^2\bar{\theta}_1^2 - \frac{i}{2}p_{1\mu}\theta_2\sigma^{\mu}\bar{\theta}_1\theta_1^2\bar{\theta}_2^2 \\
&+ \frac{i}{16}p_1^2\theta_1^2\bar{\theta}_1^2\theta_2^2\bar{\theta}_2^2
\end{aligned} \tag{1.3.40}$$

From the propagator of the chiral superfield one can derive the propagator of its component fields, such as $< \phi(p_1)\phi^*(-p_1) >$, or two point mixed correlators between a chiral superfield and its components, such as $< \phi(p_1)\Phi^{\dagger}(-p_1,\theta_2,\bar{\theta}_2) >$.
The component fields ϕ, ψ and F and their complex conjugates can be obtained from $\Phi(\Phi^{\dagger})$ by differentiation in θ and then by evaluating at $\theta = \bar{\theta} = 0$

$$\phi(p_1) = \Phi(p_1,\theta_1,\bar{\theta}_1)\Big|_{\theta_1=\bar{\theta}_1=0} \tag{1.3.41a}$$

$$\psi_{\alpha}(p_1) = \frac{1}{\sqrt{2}}\partial_{1\alpha}\Phi(p_1,\theta_1,\bar{\theta}_1)\Big|_{\theta_1=\bar{\theta}_1=0} = \frac{1}{\sqrt{2}}\mathcal{D}_{1\alpha}^{p_1}\Phi(p_1,\theta_1,\bar{\theta}_1)\Big|_{\theta_1=\bar{\theta}_1=0} \tag{1.3.41b}$$

$$F(p_1) = \frac{1}{4}\partial_1\partial_1\Phi(p_1,\theta_1,\bar{\theta}_1)\Big|_{\theta_1=\bar{\theta}_1=0} = -\frac{1}{4}\mathcal{D}_1^{p_1}\mathcal{D}_1^{p_1}\Phi(p_1,\theta_1,\bar{\theta}_1)\Big|_{\theta_1=\bar{\theta}_1=0} \tag{1.3.41c}$$

$$\phi^*(p_1) = \Phi^{\dagger}(p_1,\theta_1,\bar{\theta}_1)\Big|_{\theta_1=\bar{\theta}_1=0} \tag{1.3.41d}$$

$$\bar{\psi}_{\dot{\alpha}}(p_1) = -\frac{1}{\sqrt{2}}\bar{\partial}_{1\dot{\alpha}}\Phi^{\dagger}(p_1,\theta_1,\bar{\theta}_1)\Big|_{\theta=\bar{\theta}=0} = \frac{1}{\sqrt{2}}\bar{\mathcal{D}}_{1\dot{\alpha}}^{p_1}\Phi^{\dagger}(p_1,\theta_1,\bar{\theta}_1)\Big|_{\theta=\bar{\theta}=0} \tag{1.3.41e}$$

$$F^*(p_1) = \frac{1}{4}\bar{\partial}_1\bar{\partial}_1\Phi^{\dagger}(p_1,\theta_1,\bar{\theta}_1)\Big|_{\theta_1=\bar{\theta}_1=0} = -\frac{1}{4}\bar{\mathcal{D}}_1^{p_1}\bar{\mathcal{D}}_1^{p_1}\Phi^{\dagger}(p_1,\theta_1,\bar{\theta}_1)\Big|_{\theta_1=\bar{\theta}_1=0} \tag{1.3.41f}$$

In eqs.(1.3.41) the Grassmanian derivatives $\partial_{\alpha}(\bar{\partial}_{\dot{\alpha}})$ can be substituted with the supercovariant derivatives $\mathcal{D}_{\alpha}(\bar{\mathcal{D}})$, because one is evaluating at $\theta = \bar{\theta} = 0$. The introduction of the superderivatives again allows the use of the $D-$algebra. We will call eqs.(1.3.41) **projection**.
From eqs.(1.3.41) and from the expression for the chiral propagator (1.3.39) or (1.3.40) one has

$$\begin{aligned}
< \phi(p_1)\phi^*(-p_1) >_{free} &=< \Phi(p_1,\theta_1,\bar{\theta}_1)\Phi^{\dagger}(-p_1,\theta_2,\bar{\theta}_2) >_{free}\Big|_{\theta_1=\bar{\theta}_1=\theta_2=\bar{\theta}_2=0} \\
&= \frac{i}{p_1^2}
\end{aligned} \tag{1.3.42a}$$

$$< \psi_{\alpha}(p_1)\bar{\psi}_{\dot{\alpha}}(-p_1) >_{free} = \frac{i}{p_1^2}p_{1\mu}\sigma^{\mu}_{\alpha\dot{\alpha}} \tag{1.3.42b}$$

$$< F(p_1)F^*(-p_1) >_{free} = i \tag{1.3.42c}$$

and

$$< \phi(p_1)\Phi^\dagger(-p_1,\theta_2,\bar\theta_2) > =< \Phi(p_1,\theta_1,\bar\theta_1)\Phi^\dagger(-p_1,\theta_2,\bar\theta_2) >_{free} \Big|_{\theta_1=\bar\theta_1=0}$$

$$= \frac{i}{p_1^2} - \frac{i}{p_1^2}p_{1\mu}\theta_2\sigma^\mu\bar\theta_2 + \frac{i}{4}\theta_2^2\bar\theta_2^2 \tag{1.3.43a}$$

$$< \Phi(p_1,\theta_1,\bar\theta_1), \phi^*(-p_1) > = \frac{i}{p_1^2} - \frac{i}{p_1^2}p_{1\mu}\theta_1\sigma^\mu\bar\theta_1 + \frac{i}{4}\theta_1^2\bar\theta_1^2 \tag{1.3.43b}$$

Until now the correlators of the component fields have been derived from the correlators of the superfields. But one could proceed in the opposite direction and construct the propagator of the chiral superfield from the propagators of the component fields [18]. The latter can be computed by expanding in components the free part of the action for chiral superfields which after neglecting surface terms is

$$S_{free} = \int d^8z \Phi^\dagger(z)\Phi(z) = \int d^4x \left(i\psi\sigma^\mu\partial_\mu\bar\psi - \phi^*\partial_\mu\partial^\mu\phi + F^*F\right). \tag{1.3.44}$$

By inverting these kinetic terms, one can obtain the component propagators (1.3.42). Then, the propagator of the chiral superfield is obtained by substituting in $< \Phi(p,\theta_1,\bar\theta_1)\Phi^\dagger(-p,\theta_2,\bar\theta_2 >$ the expansions (C.0.3)-(C.0.4) of $\Phi(p,\theta,\bar\theta)$ and $\Phi^\dagger(p,\theta,\bar\theta)$ and the expressions of the component propagators.

There exists a third form for the chiral propagator:

$$< \Phi(p_1,\theta_1,\bar\theta_1)\Phi^\dagger(-p_1,\theta_2,\bar\theta_2 >_{free} = \frac{i}{p_1^2}\exp\Big(- p_{1\mu}\big(\theta_1\sigma^\mu\bar\theta_1 + \theta_2\sigma^\mu\bar\theta_2 - 2\theta_1\sigma^\mu\bar\theta_2\big)\Big) \tag{1.3.45}$$

One can verify that the Taylor expansion of eq.(1.3.45) coincides with eq.(1.3.40).

1.3.2 The propagators of vector and superghosts superfields

In this section the expressions for the propagators of the vector V and of the ghost superfields will be obtained. Before dealing with the quantization of super Yang-Mills(SYM) theories, let us briefly review the case of ordinary Yang-Mills(YM) theories using the Faddeev-Popov method.

An infinitesimal gauge transformation is given by

$$\delta A_\mu^{a\,\omega} = A_\mu^{a\,\omega} - A_\mu^a = \partial_\mu\omega^a - g f_{abc}A_\mu^b\omega^c$$

or

$$\delta A_\mu^\omega = A_\mu^\omega - A_\mu = \partial_\mu\omega + ig[A_\mu,\omega], \tag{1.3.46}$$

where $\omega(x) = \omega^a(x)T^a$ is an element of the gauge algebra and $A_\mu(x) = A_\mu^a(x)T^a$
The YM action is gauge invariant by construction $S_{SYM}(A_\mu) = S_{SYM}(A_\mu^\omega)$, hence the naive expression for the path integral

$$Z = \int dA_\mu \,\exp iS_{YM} \tag{1.3.47}$$

is not well defined because the integration is extended over all A_μ's, even those related by gauge transformation. Another related problem is that the kinetic (quadratic) operator is not invertible over the set of all field configurations so that the propagator, needed for doing perturbation theory, cannot be defined unless one restricts this set, summing over a guage family only once.
To this end, one can introduce a guage invariant functional integral over the gauge group $\Delta_{\mathcal{G}}(A_\mu)$, also called Faddeev-Popov determinant

$$\Delta_{\mathcal{G}}(A_\mu) = \int d\omega\, \delta[\mathcal{G}(A_\mu^\omega) - f(x)] \tag{1.3.48}$$

where $d\omega$ is the Haar measure with the property that under a change of variables from $\omega \to \omega'' = \omega'\omega$, it remains invariant $d\omega'' = d\omega$, $\mathcal{G}(A_\mu) = \mathcal{G}^a(A_\mu)T^a$ is a guage variant function of A_μ while $f(x) = f^a(x)T^a$ is a field independent function. Introducing in the functional integral a factor $\mathbf{1} = \Delta_{\mathcal{G}}^{-1}\Delta_{\mathcal{G}}$ leads to

$$Z = N \int dA_\mu \, \Delta_{\mathcal{G}}^{-1} \int d\omega \, \delta[\mathcal{G}(A_\mu^\omega) - f] \exp(iS_{YM}) \tag{1.3.49}$$

One can make a change of variables, i.e. a gauge transformation from A_μ to A_μ^ω, under which $dA_\mu = dA_\mu^\omega$. From the gauge invariance of $\Delta_{\mathcal{G}}^{-1}$ and of the action and after renaming the variable of integration back to A_μ, the result is

$$Z = N \int dA_\mu \, \Delta_{\mathcal{G}}^{-1} \int d\omega \, \delta[\mathcal{G}(A_\mu) - f] \exp(iS_{YM}) \tag{1.3.50}$$

The integrand is independent of ω and the integral in $d\omega$ gives an (infinite) constant which can be absorbed in the normalization N. Therefore Z can be written as

$$Z = N \int dA_\mu \, \Delta_{\mathcal{G}}^{-1} \, \delta[\mathcal{G}(A_\mu) - f] \exp\big(iS_{YM}\big) \tag{1.3.51}$$

By construction Z is independent on both f and $\mathcal{G}$, so one can average Z over f by introducing another factor $\mathbf{1}$ in the functional integral in the form [1.9]

$$1 = N' \int df \, \exp\Big(- \frac{i}{\alpha\tau_{\mathcal{R}}} \int d^4x \mathrm{Tr} f^2 \Big), \tag{1.3.52}$$

where α is a parameter. By using $\delta(\mathcal{G}(A_\mu) - f))$ for the integration in df, one arrives at

$$Z = N \int dA_\mu \, \Delta_{\mathcal{G}}^{-1} \, \exp i \Big(S_{YM} + S_{GF} \Big) \tag{1.3.53}$$

where the constant N has absorbed N' and the gauge-fixing part S_{GF} of the action is

$$S_{GF} = -\frac{1}{\alpha\tau_{\mathcal{R}}} \int d^4x \mathrm{Tr} \, [\mathcal{G}(A_\mu)]^2 \tag{1.3.54}$$

Finally, expressing $\Delta_{\mathcal{G}}^{-1}$ in eq.(1.3.53) as a functional integral produces a new term in the action. This can be done by manipulating the $\delta-$ function of eq.(1.3.48). The gauge group can be parametrized by a gauge parameter $\omega(x)$ in such a way that $\mathcal{G}(A_\mu^\omega) = f(x)$ for $\omega = 0$. Then [1.10]

$$\begin{aligned} \Delta_{\mathcal{G}}(A_\mu) &= \int d\omega \, \delta[\mathcal{G}(A_\mu^\omega) - f] = \int d\omega \, \Big(\det \frac{\delta\mathcal{G}}{\delta\omega} \Big)^{-1} \delta(\omega) = \int d\omega \, \delta \Big(\frac{\delta\mathcal{G}}{\delta\omega}\omega \Big) \\ &= \int d\omega d\omega' \exp\Big[\frac{i}{\tau_{\mathcal{R}}} \mathrm{Tr} \Big(\omega' \frac{\delta\mathcal{G}}{\delta\omega}\omega \Big) \Big], \end{aligned} \tag{1.3.55}$$

where we have used the generalization to infinite dimensional spaces of the identities involving $\delta-$ functions in finite dimension ones.[1.11] In the last equation $\frac{\delta\mathcal{G}}{\delta\omega}$ has to be evaluated at $\omega = 0$ because of the presence of $\delta(\omega)$. To obtain $\Delta_{\mathcal{G}}^{-1}$ it is enough to replace ω^a and ω'^a by the Grassmann fields $c^a(x)$ and $c^{\dagger a}(x)$ called ghost fields. So one arrives at

$$Z[J] = \int dA_\mu \, \exp i \Big(S_{YM} + S_{GF} + S_{GH} + i \int d^4x J_\mu^a A_\mu^a \Big). \tag{1.3.56}$$

[1.9] see for the definition of $\tau_{\mathcal{R}}$ eq(1.1.17)

[1.10] Here a compact notation has been used. $\frac{\delta\mathcal{G}}{\delta w}$ stands for a matrix of functional derivatives $\frac{\delta\mathcal{G}^a(x)}{\delta\omega^b(y)}$, $\frac{\delta\mathcal{G}}{\delta\omega}\omega$ stands for the functional product $\int d^4y \, \frac{\delta\mathcal{G}^a(x)}{\delta\omega^b(y)}\omega^b(y)$ and $\frac{1}{\tau_{\mathcal{R}}}\mathrm{Tr}\,\omega' \, \frac{\delta\mathcal{G}}{\delta\omega}\omega$ stands for $\int d^4x d^4y \, \omega'^a(x) \frac{\delta\mathcal{G}^a(x)}{\delta\omega^b(y)}\omega^b(y)$

[1.11] In the second step of eq.(1.3.55) has been made use of the generalization of the identity $\delta\big(\vec{f}(\vec{x})\big) = \sum_r \delta(\vec{x} - \vec{x}_r)/\big|\det \frac{\delta f_i}{\delta x_j}\big|$, where $\vec{x}_r$ is a zero of $\vec{f}(\vec{x})$ which is a function from $\mathbf{R}^N$ to $\mathbf{R}^N$. The third step follows from the generalization of the identity involving the vector $\vec{x}$ and a matrix A: $\delta(A\vec{x}) = \delta(\vec{x})/\big|\det A\big|$. The last step is the generalization of the integral representation of the $\delta-$ function: $\delta(\vec{k}) = \int d\vec{x} \exp(i\vec{k} \cdot \vec{x})$. In eq.(1.3.55) the role of $\vec{x}$ is played by ω' and of $\vec{k}$ by $\frac{\delta\mathcal{G}}{\delta\omega}\omega$.

where a term with sources has been included and the action for the ghosts S_{ghost} is given by

$$S_{GH} = \frac{1}{\tau_{\mathcal{R}}}\mathrm{Tr}\Big(c^\dagger \frac{\delta \mathcal{G}}{\delta \omega} c\Big) = \int d^4x\, d^4y\, c^{\dagger a}(x) \frac{\delta \mathcal{G}^a(x)}{\delta \omega^b(y)} c^b(y) \tag{1.3.57}$$

with $c(x) = c^a(x)T^a$ and similarly for $c^\dagger(x)$.

This discussion can be generalized to superspace. It is sufficient to consider only the pure super Yang-Mills part of the action (1.1.29):

$$S_{SYM} = \frac{1}{128 g^2 \tau_R}\mathrm{Tr}\Big[\int d^6z W^\alpha W_\alpha + \int d^6\bar{z} \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}\Big] \tag{1.3.58}$$

The two integrals in eq.(1.3.58) are equal to each other up to the surface term $\mathrm{Tr}\,\epsilon^{\mu\nu\rho\sigma}\int d^4x F_{\mu\nu}F_{\rho\sigma}$ that can be neglected. Then, from the definition (1.1.26) of W_α, after integration by parts of the operator $\bar{\mathcal{D}}\bar{\mathcal{D}}$ and using eq.(1.2.2e), it follows that

$$\begin{aligned} S_{SYM} &= \frac{1}{64 g^2 \tau_R}\int d^6z \mathrm{Tr}\Big(W^\alpha W_\alpha\Big) = \frac{1}{64 g^2 \tau_R}\int d^6z \mathrm{Tr}\Big(\bar{\mathcal{D}}\bar{\mathcal{D}} e^{-gV}\mathcal{D}^\alpha e^{gV}\Big)\Big(\bar{\mathcal{D}}\bar{\mathcal{D}} e^{-gV}\mathcal{D}_\alpha e^{gV}\Big) \\ &= -\frac{1}{16 g^2 \tau_R}\int d^8z \mathrm{Tr}\Big(e^{-gV}\mathcal{D}^\alpha e^{gV}\Big)\bar{\mathcal{D}}\bar{\mathcal{D}}\left(e^{-gV}\mathcal{D}_\alpha e^{gV}\right) \end{aligned} \tag{1.3.59}$$

From the definition of the Lie derivative eq.(1.1.25), one can prove the relation $e^{\mathcal{L}_X}Y = e^X Y e^{-X}$ by expanding the exponentials. This allows to write the term $e^{-V}\mathcal{D}_\alpha e^V$ as power series in gV

$$\begin{aligned} (e^{-gV}\mathcal{D}_\alpha e^{gV}) &= [e^{-\mathcal{L}_{gV}}\mathcal{D}_\alpha]\cdot 1 \\ &= (\mathcal{D}_\alpha + [-gV,\mathcal{D}_\alpha] + \frac{1}{2}[-gV,[-gV,\mathcal{D}_\alpha]] + \frac{1}{3!}[-gV[-gV,[-gV,\mathcal{D}_\alpha]]] + \cdots)\cdot 1 \\ &= g\mathcal{D}_\alpha V + \frac{1}{2}g^2[(\mathcal{D}_\alpha V),V] + \frac{1}{3!}g^2[[(\mathcal{D}_\alpha V),V],V] + \cdots \end{aligned} \tag{1.3.60}$$

Note that in the first two lines $\mathcal{D}_\alpha$ acts on all the terms on its right. The third line, where $\mathcal{D}_\alpha$ acts only on V, is obtained by developing the expression in the second line.

Thus, inserting the expansion (1.3.60) in eq.(1.3.59) and integrating by parts, one can read the various terms in powers of V. The part quadratic in V, which is of zeroth order in g, is given by:

$$S^{(2)}_{SYM} = \frac{1}{16\tau_{\mathcal{R}}}\int d^8z \mathrm{Tr}(V\mathcal{D}^\alpha\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}_\alpha V) = \frac{1}{2\tau_{\mathcal{R}}}\int d^8z \mathrm{Tr}(V\Pi_0\partial^2 V), \tag{1.3.61}$$

where Π_0 is defined in eq.(1.3.16) . In terms of V^a, $S^{(2)}_{SYM}$ has the form

$$S^{(2)}_{SYM} = \frac{1}{2}\int d^8z V^a\Pi_0\partial^2 V^a. \tag{1.3.62}$$

The operator $\partial^2\Pi_0$ is not invertible because it annihilates the chiral and the antichiral parts of V, that is Π_+V and Π_-V, by applying the orthogonality of the projectors cf.(1.3.18). As in YM theory, it is necessary to introduce a gauge-fixing term to have a quadratic kinetic operator that is invertible in order to derive a propagator.

In SYM theory a gauge transformation (1.1.20) of the vector superfield involves two gauge functions which are two superfields with opposite chirality $\Lambda(z)$ and $\Lambda^\dagger(z)$. Hence, two gauge fixing superfunctions $\mathcal{K}(V^\Lambda)$ and $\mathcal{K}^\dagger(V^\Lambda)$, one chiral and the other antichiral , have to be introduced so as to define the Faddeev-Popov superdeterminant

$$\Delta_{\mathcal{K}} = \int d\Lambda d\Lambda^\dagger\, \delta[\mathcal{K}(V^\Lambda) - f(z)]\delta[\mathcal{K}^\dagger(V^\Lambda) - f^\dagger]. \tag{1.3.63}$$

Introducing a factor of $\mathbf{1} = \Delta_{\mathcal{K}}^{-1}\Delta_{\mathcal{K}}$ in the naive expression for the functional integral

$$Z[0] = N\int dV \exp i(S_{SYM}) \tag{1.3.64}$$

gives

$$Z[0] = N \int dV \exp i\Big(S_{SYM}\Big)\, \Delta_{\mathcal{K}}^{-1} \int d\Lambda d\Lambda^\dagger\, \delta[\mathcal{K}(V^\Lambda) - f(z)]\, \delta[\mathcal{K}^\dagger(V^\Lambda) - f^\dagger(z)]. \tag{1.3.65}$$

A convenient choice for $\mathcal{K}(V^\Lambda)$ and $\mathcal{K}^\dagger(V^\Lambda)$ is

$$\mathcal{K}(V) = -\frac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}V(z), \qquad \mathcal{K}^\dagger(V) = -\frac{1}{4}\mathcal{D}\mathcal{D}V(z). \tag{1.3.66}$$

Moreover, as in the YM case, averaging over f and $f^\dagger$ with the weighting factor

$$\int df df^\dagger \exp\Big(-\frac{i}{\alpha\tau_{\mathcal{R}}}\int d^8z\, \mathrm{Tr} f^\dagger(z) f(z)\Big) \tag{1.3.67}$$

leads to

$$Z[0] = N \int dV \exp i\Big(S_{SYM} + S_{GF}\Big)\, \Delta_{\mathcal{K}}^{-1}, \tag{1.3.68}$$

where

$$S_{GF} = -\frac{1}{16\alpha\tau_{\mathcal{R}}}\int d^8z \mathrm{Tr}\big(\bar{\mathcal{D}}\bar{\mathcal{D}}V\mathcal{D}\mathcal{D}V\big) = -\frac{1}{32\alpha\tau_{\mathcal{R}}}\int d^8z \mathrm{Tr}\Big(V\{\bar{\mathcal{D}}\bar{\mathcal{D}},\mathcal{D}\mathcal{D}\}V\Big). \tag{1.3.69}$$

The second equality follows by partial integration and ciclicity of the trace. The sum of eq.(1.3.61) and eq.(1.3.69), both quadratic in V, gives:

$$S^{(2)}_{SYM} + S_{GF} = \frac{1}{2\tau_{\mathcal{R}}}\int d^8z\, \mathrm{Tr}\Big[V\partial^2 V + \frac{1}{16}\Big(1-\frac{1}{\alpha}\Big)V\{\bar{\mathcal{D}}\bar{\mathcal{D}},\mathcal{D}\mathcal{D}\}V\Big], \tag{1.3.70}$$

where eq.(B.1.2k) has been used.
Choosing $\alpha = 1$ makes the terms with fourth order derivatives vanish and hence there is no term of the form $\frac{1}{k^4}$ in the vector propagator which would lead to an infrared divergence even off-shell. This is called the (supersymmetric) Fermi-Feynman gauge (SFF). Thus, one can write

$$S^{(2)}_{SYM} + S^{SFF}_{GF} = \frac{1}{2\tau_{\mathcal{R}}}\int d^8z\, \mathrm{Tr}\big(V\partial^2V\big) = \frac{1}{2}\int d^8z\, V^a\partial^2V^a. \tag{1.3.71}$$

Including a source term, the free generating superfunctional for V is

$$Z_0[j] = Z_0[0]\int dV \exp i\Big(S^{(2)}_{SYM} + S^{SFF}_{GF} + \frac{1}{\tau_{\mathcal{R}}}\mathrm{Tr}(j\cdot V)\Big) = \int dV \exp i\Big(\int d^8z\, \frac{1}{2}V^a(z)\partial^2V^a(z) + j^a(z)V^a(z)\Big). \tag{1.3.72}$$

After performing the Gaussian integral by means of eq.(1.3.33), one has for $W_0[j] = -i\ln Z_0[j]$ the result:

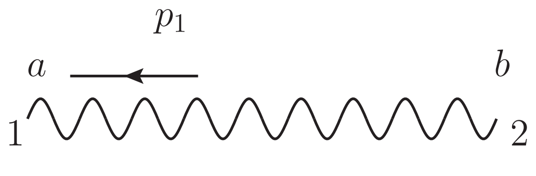
$$W_0[j]^{SFF} = -\frac{1}{2}\int d^8z\, j^a\frac{1}{\partial^2}j^a. \tag{1.3.73}$$

Hence the two point connected Green's function is

$$< V^a(z_1)V^b(z_2) >_{free} = -i\frac{\delta^2 W_0[j]}{\delta j^a(z_1)\delta j^b(z_2)} = \frac{i}{\partial^2}\delta^{ab}\delta^{(8)}(z_1 - z_2). \tag{1.3.74}$$

In momentum space eq.(1.3.74) becomes

$$< V^a(p_1,\theta_1,\bar\theta_1)V^b(-p_1,\theta_2,\bar\theta_2) > = -\frac{i}{p_1^2}\delta^{ab}\delta^{(4)}(\theta_{12}) \tag{1.3.75}$$



As already done for the chiral superfield (cf. 1.3.41), one can obtain the component fields of vector superfield by applying an appropriate θ derivative on the expression (1.1.13) of V. For example

$$A^{a\mu}(p_1) = \frac{1}{2}\bar{\partial}_{1\dot{\alpha}}\partial_{1\alpha}\bar{\sigma}^{\mu\,\dot{\alpha}\alpha}V^a(p,\theta_1,\bar{\theta}_1)\Big|_{\theta_1=\bar{\theta}_1=0} = -\frac{1}{2}\bar{\mathcal{D}}_{1\dot{\alpha}}\mathcal{D}_{1\alpha}\bar{\sigma}^{\mu\,\dot{\alpha}\alpha}V^a(p,\theta_1,\bar{\theta}_1)\Big|_{\theta_1=\bar{\theta}_1=0}. \tag{1.3.76}$$

From this kind of projection one can get the propagator for the component fields (cf. 1.3.42)

$$\begin{aligned} < A^{a\mu}(p_1)A^{b\nu}(-p_1) >_{free} &= \frac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\bar{\sigma}^{\nu\dot{\beta}\beta}\bar{\partial}_{1\dot{\alpha}}\partial_{1\alpha}\bar{\partial}_{2\dot{\beta}}\partial_{2\beta} < V^a(p_1,\theta_1,\bar{\theta}_1)V^b(-p_1,\theta_2,\bar{\theta}_2 >_{free}\Big|_{\theta_1=\bar{\theta}_1=\theta_2=\bar{\theta}_2=0} \\ &= -\frac{2i}{p_1^2}\eta^{\mu\nu}\delta^{ab}. \end{aligned} \tag{1.3.77}$$

Also, mixed correlators (cf. 1.3.43) can be derived

$$< A^{a\mu}(p_1)V^b(-p_1,\theta_2,\bar{\theta}_2 >_{free} = \frac{1}{2}\bar{\partial}_{1\dot{\alpha}}\partial_{1\alpha}\bar{\sigma}^{\mu\,\dot{\alpha}\alpha} < V^a(p_1,\theta_1,\bar{\theta}_1)V^b(-p_1,\theta_2,\bar{\theta}_2) >_{free}\Big|_{\theta_1=\bar{\theta}_1=0} = -\frac{2i}{p_1^2}\theta_2\sigma^\mu\bar{\theta}_2 \tag{1.3.78}$$

What remains to do is to find an expression for $\Delta_{\mathcal{K}}^{-1}$. One can replace the two $\delta-$ functions in eq.(1.3.63) by the integral representations eq.(1.3.55), introducing the chiral and antichiral parameters Λ' and $\Lambda'^\dagger$ respectively. Repeating the steps leading to eq.(1.3.55) with $\frac{\delta F}{\delta\omega}\omega$ replaced by $\frac{\delta\mathcal{K}}{\delta\Lambda}\Lambda+\frac{\delta\mathcal{K}}{\delta\Lambda^\dagger}\Lambda^\dagger$ and $\frac{\delta\mathcal{K}^\dagger}{\delta\Lambda}\Lambda+\frac{\delta\mathcal{K}^\dagger}{\delta\Lambda^\dagger}\Lambda^\dagger$, one can write [1.12]

$$\Delta_{\mathcal{K}}(V) = \int d\Lambda d\Lambda^\dagger d\Lambda' d\Lambda'^\dagger \exp\frac{i}{\tau_{\mathcal{R}}}\Big[\int d^6z\,\mathrm{Tr}\left(\Lambda'\Big(\frac{\delta\mathcal{K}}{\delta\Lambda}\Lambda + \frac{\delta\mathcal{K}}{\delta\Lambda^\dagger}\Lambda^\dagger\Big)\right) + \int d^6\bar{z}\,\mathrm{Tr}\left(\Lambda'^\dagger\Big(\frac{\delta\mathcal{K}^\dagger}{\delta\Lambda}\Lambda + \frac{\delta\mathcal{K}^\dagger}{\delta\Lambda^\dagger}\Lambda^\dagger\Big)\right)\Big], \tag{1.3.80}$$

To obtain $\Delta_{\mathcal{K}}^{-1}$, the parameters Λ and Λ' have to be replaced by the anticommuting chiral ghost fields $C(z)$ and $iA(z)$ where $C(z) = C^a(z)T^a$, $A(z) = A^a(z)T^a$ and the "i" in front of $A(z)$ is a convention.
Thus

$$\Delta_{\mathcal{K}}^{-1} = \int dCdAdC^\dagger dA^\dagger \exp i\Big(S_{GH}\Big), \tag{1.3.81}$$

where

$$S_{GH} = \frac{1}{\tau_{\mathcal{R}}}\Big[\mathrm{Tr}\Big(A\Big(\frac{\delta\mathcal{K}}{\delta\Lambda}C + \frac{\delta\mathcal{K}}{\delta\Lambda^\dagger}C^\dagger\Big)\Big) + \mathrm{Tr}\Big(A^\dagger\Big(\frac{\delta\mathcal{K}^\dagger}{\delta\Lambda}C + \frac{\delta\mathcal{K}^\dagger}{\delta\Lambda^\dagger}C^\dagger\Big)\Big)\Big]. \tag{1.3.82}$$

As in the YM case, the functional derivatives $\delta\mathcal{K}/\delta\Lambda$ and $\delta\mathcal{K}^\dagger/\delta\Lambda^\dagger$ have to be evaluated at $\Lambda = \Lambda^\dagger = 0$. Thus, in $\mathcal{K}(V^\Lambda)$, $\mathcal{K}^\dagger(V^\Lambda)$ and in the expression of δV^Λ (see eq.(1.1.22)), only terms linear in $\Lambda(\Lambda^\dagger)$ have to be kept

$$\begin{aligned} \mathcal{K}(V^\Lambda) &= -\frac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}V^\Lambda = -\frac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}\Big(V + \widehat{H}(V)\Lambda + \widehat{H}^\dagger(V)\Lambda^\dagger\Big) \\ \mathcal{K}^\dagger(V^\Lambda) &= -\frac{1}{4}\mathcal{D}\mathcal{D}V^\Lambda = -\frac{1}{4}\mathcal{D}\mathcal{D}\Big(V + \widehat{H}(V)\Lambda + \widehat{H}^\dagger(V)\Lambda^\dagger\Big), \end{aligned}$$

S_{GH} becomes

$$S_{GH} = \frac{i}{\tau_{\mathcal{R}}}\int d^8z\,\mathrm{Tr}\Big[A\,\widehat{H}(V)C + A\,\widehat{H}^\dagger(V)C^\dagger - A^\dagger\widehat{H}(V)C - A^\dagger\widehat{H}^\dagger(V)C^\dagger\Big]. \tag{1.3.83}$$

[1.12] Here $\frac{1}{\tau_{\mathcal{R}}}\mathrm{Tr}\left(\Lambda'\frac{\delta\mathcal{K}}{\delta\Lambda}\Lambda\right)$ stands for $\int d^6zd^6z'\,\Lambda'^a(z)\frac{\delta\mathcal{K}^a(z)}{\delta\Lambda^b(z')}\Lambda^b(z')$ and similarly for the other integrals. The terms inside the parenthesis multiplying Λ' are chiral, while those multiplying $\Lambda'^\dagger$ are antichiral.
If O is a matrix with bosonic components and M_1, M_2 are two column vectors then

$$\int dM_1dM_2\exp\left(M_1^TOM_2\right) \propto \left(\det O\right)^\alpha \tag{1.3.79}$$

where α is -1 or $+1$ if the components of M_1, M_2 are bosonic or fermionic respectively. In this case $M_1 = \begin{pmatrix}\Lambda' \\ \Lambda'^\dagger\end{pmatrix}$, $M_2 = \begin{pmatrix}\Lambda \\ \Lambda^\dagger\end{pmatrix}$ and $O = \begin{pmatrix}\frac{\delta\mathcal{K}}{\delta\Lambda} & \frac{\delta\mathcal{K}}{\delta\Lambda^\dagger} \\ \frac{\delta\mathcal{K}^\dagger}{\delta\Lambda} & \frac{\delta\mathcal{K}^\dagger}{\delta\Lambda^\dagger}\end{pmatrix}$

From eq.(1.1.24), S_{GH} can be written more compactly as

$$S_{GH} = \frac{1}{\tau_{\mathcal{R}}} \int d^8z\, (A^\dagger - A)\, \mathcal{L}_{\frac{gV}{2}} [C^\dagger + C + \coth \mathcal{L}_{\frac{gV}{2}} (C^\dagger + C)] \tag{1.3.84}$$

Substituting the expressions (1.1.23) for $\widehat{H}(V)$ and $\widehat{H}^\dagger(V)$ in eq.(1.3.83) and keeping only the linear and quadratic terms in V gives

$$S_{GH} = \frac{1}{\tau_{\mathcal{R}}} \int d^8z\, \mathrm{Tr}\Big(A^\dagger C + AC^\dagger + \frac{1}{2} g(A^\dagger - A)[V, C + C^\dagger] + \frac{1}{12} g^2 (A^\dagger - A)[V, [V, C - C^\dagger]] + O(V^4)\Big), \tag{1.3.85}$$

where the chiral and antichiral terms AC and $A^\dagger C^\dagger$ vanish because the integration is over the full superspace $\int d^8z$. The free generating superfunctional for superghosts with source terms is

$$Z_0[\eta, \eta', \eta^\dagger, \eta'^\dagger] = \int dC dC^\dagger dA dA^\dagger \exp i \Big\{ S_{GH}^{free}[C, A, C^\dagger, A^\dagger] + \frac{1}{\tau_{\mathcal{R}}} \mathrm{Tr}\Big(\eta \cdot C + \eta' \cdot A + C^\dagger \cdot \eta^\dagger + A^\dagger \cdot \eta'^\dagger\Big)\Big\} \tag{1.3.86}$$

where

$$S_{GH}^{free} = \frac{1}{\tau_{\mathcal{R}}} \int d^8z\, \mathrm{Tr}\Big(A^\dagger C + AC^\dagger\Big), \tag{1.3.87}$$

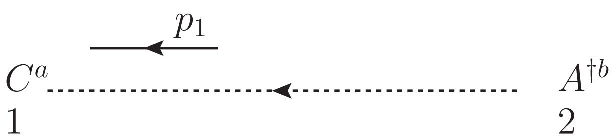
$$\eta \cdot C = \int d^6z\, \eta(z) C(z), \quad C^\dagger \cdot \eta^\dagger = \int d^6\bar{z}\, C^\dagger(z) \eta^\dagger(z) \tag{1.3.88}$$

and similarly for $A, A^\dagger, \eta', \eta'^\dagger$. Following the same steps as for the propagator of chiral superfield one finds

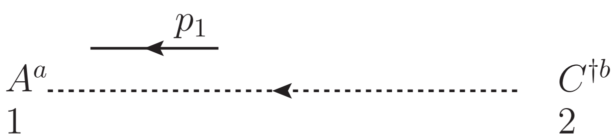
$$W_0[\eta, \eta', \eta^\dagger, \eta'^\dagger] = \int d^8z \Big(\eta'^{\dagger a} \frac{1}{\partial^2} \eta^a - \eta^{\dagger a} \frac{1}{\partial^2} \eta'^a\Big) \tag{1.3.89}$$

By functional differentiation of eq.(1.3.89) with respect to the sources of the superghosts, one can obtain for the propagators of the superghosts

$$< C^a(p_1, \theta_1, \bar{\theta}_1) A^{\dagger b}(-p_1, \theta_2, \bar{\theta}_2) >_{free} = \frac{i}{16 p_1^2} \delta^{ab}\, \bar{\mathcal{D}}_1^{p_1} \bar{\mathcal{D}}_1^{p_1} \mathcal{D}_1^{p_1} \mathcal{D}_1^{p_1} \delta^{(4)}(\theta_{12}) \tag{1.3.90}$$



$$< A^a(p_1, \theta_1, \bar{\theta}_1) C^{\dagger b}(-p_1, \theta_2, \bar{\theta}_2) >_{free} = -\frac{i}{16 p_1^2} \delta^{ab}\, \bar{\mathcal{D}}_1^{p_1} \bar{\mathcal{D}}_1^{p_1} \mathcal{D}_1^{p_1} \mathcal{D}_1^{p_1} \delta^{(4)}(\theta_{12}) \tag{1.3.91}$$



Let us stress the minus sign in the propagator $< AC^\dagger >$ with respect to the propagator $< CA^\dagger >$ and to the propagator of the chiral superfield (1.3.39).

1.4 Interaction vertices

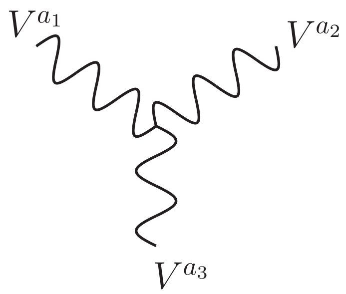
The complete action containing the ghost, gauge and matter superfields is given by:

$$S_{TOT} = S_{SYM} + S_{GF}^{SFF} + S_{GH} + S_{MAT} \tag{1.4.1}$$

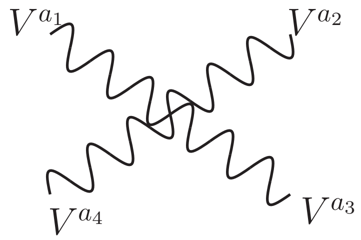
Expanding the sum of the super Yang-Mills part of the action S_{SYM} and the gauge-fixing term S_{GF}^{SFF} in the super-Feynman guage (SFF) up to the fifth order in V gives

$$S_{SYM}+S_{GH}^{SFF}=\frac{1}{\tau_{\mathcal{R}}}\int d^8z\mathrm{Tr}\Big(\frac{1}{2}V\partial^2V+\frac{g}{3!}[V,[V,\mathcal{D}_\alpha V]]-\frac{g^2}{4!}[V,[V,[V,\mathcal{D}_\alpha V]]]+\frac{g^3}{5!}[V,[V,[V,[V,\mathcal{D}_\alpha V]]]]+O(V^6 \tag{1.4.2}$$

By substituting $V=V^aT^a$ in eq.(1.4.2) and using eqs.(1.1.17), one obtains the self interaction vertices for V^a with explicit color indices



$$S_{SYM}^{(3)}=-\frac{i}{16}\,g\,f_{a_1a_2a_3}\int d^8z\Big(\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}^\alpha V^{a_1}\Big)\Big(\mathcal{D}_\alpha V^{a_2}\Big)V^{a_3}. \tag{1.4.3}$$



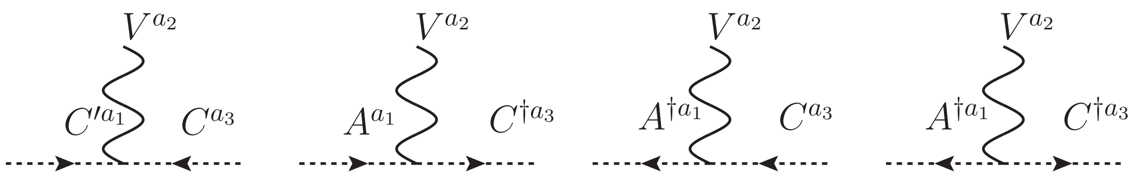
$$\begin{aligned}S_{SYM}^{(4)}&=g^2\,f_{a_1a_2b}f_{ba_3a_4}\int d^8z\Big[\frac{1}{64}\Big(V^{a_1}\mathcal{D}^\alpha V^{a_2}\Big)\Big(\bar{\mathcal{D}}\bar{\mathcal{D}}(V^{a_3}\mathcal{D}_\alpha V^{a_4})\Big)-\frac{1}{48}\Big(V^{a_1}(\mathcal{D}^\alpha V^{a_2})V^{a_3}\Big)\Big(\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}_\alpha V^{a_4}\Big)\Big]\\&=g^2f_{a_1a_2b}f_{ba_3a_4}\int d^8z\Big[\frac{1}{192}\Big(\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}^\alpha V^{a_1}\Big)V^{a_2}V^{a_3}\Big(\mathcal{D}_\alpha V^{a_4}\Big)-\frac{1}{32}\Big(\mathcal{D}^\alpha V^{a_1}\Big)V^{a_2}\Big(\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}_\alpha V^{a_3}\Big)\bar{\mathcal{D}}^{\dot\alpha}V_4^a\quad.\\&+\frac{1}{64}\Big(\mathcal{D}^\alpha V^{a_1}\Big)V^{a_2}\Big(\mathcal{D}_\alpha V^{a_3}\Big)\bar{\mathcal{D}}\bar{\mathcal{D}}V^{a_4}\Big]\end{aligned} \tag{1.4.4}$$

For the quintic vertex see Appendix D.

The interaction terms between the vector superfield and the superghosts is

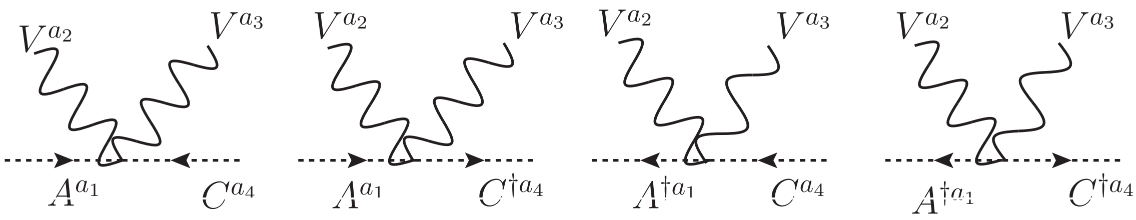
$$S_{GH}^{int}=S_{GH}-S_{GH}^{free}=\frac{1}{\tau_{\mathcal{R}}}\int d^8z\,\mathrm{Tr}\Big(\frac{1}{2}g(A^\dagger-A)[V,C+C^\dagger]+\frac{1}{12}g^2(A^\dagger-A)[V,[V,C-C^\dagger]]+O(V^4)\Big) \tag{1.4.5}$$

In terms of V^a, C^a, etc.. the V-ghost-ghost vertices are given by



$$S_{GH}^{(3)} = \frac{ig}{2} f_{a_1 a_2 a_3} \int d^8 z \Big[- A^{a_1} V^{a_2} C^{a_3} - A^{a_1} V^{a_2} C^{\dagger a_3} + A^{\dagger a_1} V^{a_2} C^{a_3} + A^{\dagger a_1} V^{a_2} C^{\dagger a_3} \Big] \tag{1.4.6}$$

while the V^2-ghost-ghost vertices are



$$S_{GH}^{(4)} = \frac{g^2}{12} f_{a_1 a_2 b} f_{a_3 a_4 b} \int d^8 z \Big(A^{a_1} V^{a_2} V^{a_3} C^{a_4} - A^{a_1} V^{a_2} V^{a_3} C^{\dagger a_4} - A^{\dagger a_1} V^{a_2} V^{a_3} C^{a_4} + A^{\dagger a_1} V^{a_2} V^{a_3} C^{\dagger a_4} \Big). \tag{1.4.7}$$

Since coth is an odd function, eq.(1.3.84) implies that the only (symbolically written) vertex AV^pC, where p is odd, different from zero is when $p = 1$.

Finally, if the chiral superfields belong to the representation $\mathcal{R}$ of the gauge group with type index i and also have flavor indices I, S_{MAT} is

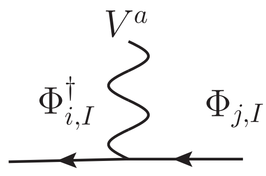
$$S_{MAT} = \int d^8 z \Phi^\dagger_{i,I} (e^V)_{ij} \Phi_{j,I} + \int d^6 z \mathcal{W}(\Phi_{i,I}) + \int d^6 \bar{z} \mathcal{W}^\dagger (\Phi^\dagger_{i,I}) \tag{1.4.8}$$

where $\mathcal{W}$ is the superpotential (see eq.(1.1.28)).

The interaction terms between the matter and gauge superfields is

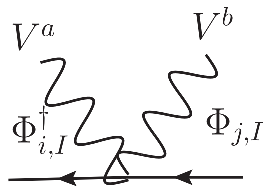
$$S_{int}^{(\Phi,V)} = \int d^8 z \Phi^\dagger_{i,I} \Big[\left(e^V\right)_{ij} - \delta_{ij} \Big] \Phi_{j,I}. \tag{1.4.9}$$

The gauge-matter vertex at order g is



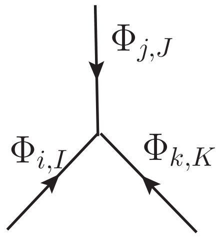
$$S_{int}^{\Phi V \Phi} = g \int d^8 z \, \Phi^\dagger_{i,I} (T^a)_{ij} \Phi_{j,I} V^a. \tag{1.4.10}$$

where T^a are the generators of the gauge group. The gauge-matter vertex at order g^2 is

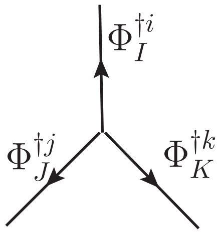


$$S_{int}^{\Phi V^2 \Phi} = \frac{g^2}{2} \int d^8 z \, \Phi^\dagger_{i,I} (T^a)_{ik} (T^b)_{kj} \, \Phi_{j,I} \, V^a V^b. \tag{1.4.11}$$

The self interaction vertex for the matter superfield is given by the cubic part of $\mathcal{W}(\mathcal{W}^\dagger)$:



$$S^{\Phi}_{int} = \frac{1}{3!}\lambda^{IJK}_{ijk} \int d^6 z \, \Phi^i_I \Phi^j_J \Phi^k_K = \frac{1}{3!}\lambda^{IJK}_{ijk} \int d^8 z \, \Phi^i_I \Phi^j_J \Phi^k_K \, \delta^2(\bar{\theta}). \tag{1.4.12}$$



$$S^{\Phi^\dagger}_{int} = \frac{1}{3!}\lambda^{*\,IJK}_{ijk} \int d^6 \bar{z} \, \Phi^{\dagger i}_I \Phi^{\dagger j}_J \Phi^{\dagger k}_K = \frac{1}{3!}\lambda^{*\,IJK}_{ijk} \int d^8 z \, \Phi^{\dagger i}_I \Phi^{\dagger j}_J \Phi^{\dagger k}_K \, \delta^2(\theta), \tag{1.4.13}$$

where the constants λ^{IJK}_{ijk} have to be invariant both under the gauge group and the flavor group.

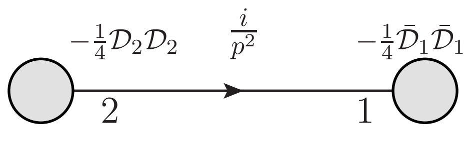
Let us note that in the gauge-matter vertices there is always an ingoing chiral line and an outgoing antichiral line. Instead in the gauge-ghost vertices the two lines can both be chiral or antichiral.

1.5 Improved super-Feynman rules

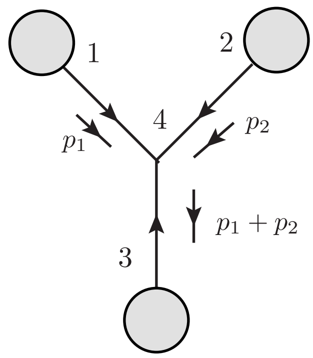
The form of the superpropagator and the interactions terms discussed above correspond to the original conventions of [7] and [8].
A modified set of super-Feynman rules was proposed in [19] and leads to a considerable simplification for chiral superfields. These new conventions adopt for all the superpropagators the expression $\pm i\delta_{12}/p^2$ and associate an integral over $\int d^4\theta$ to all the vertices.
In fact one can move the factors $-\frac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}$ and $-\frac{1}{4}\mathcal{D}\mathcal{D}$ in (1.3.13) from the chiral superpropagator, which now assumes the form

$$< \Phi(p)\Phi^\dagger(-p) >= \frac{i}{p^2}, \tag{1.5.1}$$

to the vertices for each chiral (or antichiral) superfield. Symbolically we have



In the presence of a Φ^3 (or $\Phi^{\dagger 3}$) vertex this rule has to be additionally modified in order to convert $\int d^2\theta$ (or $\int d^2\bar{\theta}$) into $\int d^4\theta$. In fact, let us consider the part of a superdiagram involving a chiral vertex (in z_4)



The internal chiral lines ending in z_4 give

$$\dots \int d^2\theta_4 (\frac{i\bar{\mathcal{D}}_4\bar{\mathcal{D}}_4\mathcal{D}_1\mathcal{D}_1\delta_{14}}{16p_1^2}) (\frac{i\bar{\mathcal{D}}_4\bar{\mathcal{D}}_4\mathcal{D}_2\mathcal{D}_2\delta_{24}}{16p_2^2}) (\frac{i\bar{\mathcal{D}}_4\bar{\mathcal{D}}_4\mathcal{D}_3\mathcal{D}_3\delta_{34}}{16(p_1+p_2)^2}) \dots \tag{1.5.2}$$

By partial integration of $\bar{\mathcal{D}}_4\bar{\mathcal{D}}_4$ in the first factor since $\bar{\mathcal{D}}_{4\dot{\alpha}}\bar{\mathcal{D}}_{4\dot{\beta}}\bar{\mathcal{D}}_{4\dot{\gamma}} = 0$, eq.(1.5.2) becomes

$$\dots \int d^2\theta_4 \Big[-\frac{\bar{\mathcal{D}}_4\bar{\mathcal{D}}_4}{4} \Big((-\frac{i\mathcal{D}_1\mathcal{D}_1\delta_{14}}{4p_1^2}) (\frac{i\bar{\mathcal{D}}_4\bar{\mathcal{D}}_4\mathcal{D}_2\mathcal{D}_2\delta_{24}}{16p_2^2}) (\frac{i\bar{\mathcal{D}}_4\bar{\mathcal{D}}_4\mathcal{D}_3\mathcal{D}_3\delta_{34}}{16(p_1+p_2)^2}) \Big) \Big] \dots \tag{1.5.3}$$

Then, from eq.(1.2.2e), it follows that

$$\dots \int d^4\theta_4 (-\frac{i\mathcal{D}_1\mathcal{D}_1\delta_{14}}{4p_1^2}) (\frac{i\bar{\mathcal{D}}_4\bar{\mathcal{D}}_4\mathcal{D}_2\mathcal{D}_2\delta_{24}}{16p_2^2}) (\frac{i\bar{\mathcal{D}}_4\bar{\mathcal{D}}_4\mathcal{D}_3\mathcal{D}_3\delta_{34}}{16(p_1+p_2)^2}) \dots \tag{1.5.4}$$

Thus, the result is that one of the three factors $-\frac{1}{4}\bar{\mathcal{D}}_4\bar{\mathcal{D}}_4$ has been absorbed to convert $\int d^2\theta_4$ in $\int d^4\theta_4$. A similar discussion is valid for the $\Phi^{\dagger 3}$ vertex and a factor $-\frac{1}{4}\mathcal{D}_4\mathcal{D}_4$. Moreover, if one is computing 1PI diagrams for the effective action, an appropriate superfield is associated to each amputated external line. If this external superfield is chiral (antichiral), one has to omit the factor $-\frac{1}{4}\bar{\mathcal{D}}_4\bar{\mathcal{D}}_4$ $(-\frac{1}{4}\mathcal{D}_4\mathcal{D}_4)$ on the corresponding amputated external line. We can summarize the so called improved super-Feynman rules for the computation of the effective action as follows[1.13]

- for chiral superfields the propagator is $< \Phi\Phi^\dagger >= \frac{i\delta_{12}}{p^2}$, for vectors superfields is $< VV >= -\frac{i\delta_{12}}{p^2}$, while for the super-ghosts $< CA^\dagger >= \frac{i\delta_{12}}{p^2}$ and $< AC^\dagger >= -\frac{i\delta_{12}}{p^2}$;
- $\int d^4\theta_{vert}$ is associated to each vertex with an extra $-\frac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}$ (or $-\frac{1}{4}\mathcal{D}\mathcal{D}$) for each internal chiral (or antichiral) superfield. In a Φ^3 ($\Phi^{\dagger 3}$) vertex one factor of $-\frac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}$ $(-\frac{1}{4}\mathcal{D}\mathcal{D})$ has to be omitted for converting $\int d^2\theta$ ($\int d^2\bar{\theta}$) into $\int d^4\theta$;
- an appropriate superfield must be associated to each external line. A factor $-\frac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}$ ($-\frac{1}{4}\mathcal{D}\mathcal{D}$) has to be omitted at a vertex for each external chiral (antichiral) superfield;
- $\int \frac{d^Dk}{(2\pi)^D}$ is associated to each independent loop[1.14] and $\int \prod_{p_{ext}} \frac{d^4p_{ext}}{(2\pi)^4} \Big[(2\pi)^4\delta^{(4)}\big(\sum_{ext} p_{ext}\big)\Big]$ for the external momenta.

The improved super Feynman rules and the use of $D-$ algebra are very useful if one has to compute the effective action (or 1PI diagrams) in superspace. In this case, the external lines have not the factor $-\frac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}$ ($-\frac{1}{4}\mathcal{D}\mathcal{D}$) with superderivatives. In fact the superderivatives have to be integrated by parts and this leads to a rapid increase of the terms to compute.
Instead, the conventional super Feynman rules are more appropriate to the computation of reducible diagrams. In fact one can use the form (1.3.40) for the chiral propagator which has no explicit superderivatives. Also, with this method, if one has to calculate Green's functions with external component fields, one can

[1.13] these rules give a 1PI superdiagram. To obtain the corresponding term of the effective action Γ an overall factor $-i$ is needed (see eq.(1.3.28))
[1.14] $\int d^Dk$ indicates supersymmetric dimensional regularization (see below)

project on them from the beginning (see eq.(1.3.43)) without computing before the effective action in superspace.
We have used the technique based on conventional super Feynman rules to obtain the results of chapter (**??**) after implementing it in a Maple program. Some of the super-diagrams are calculated also with the improved Feynman rules and $D-$algebra to have a check of the correctness of the results.

1.6 Examples of computation in superspace

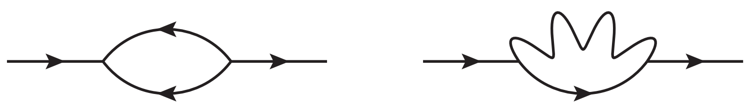
In this section we apply the formalism introduced before to compute some diagrams with external component fields. The calculations will employ both the conventional and the improved super Feynman rules.
To simplify the notation we consider a single chiral superfield Φ coupled to an abelian vector superfield

$$S = \int d^8 z \Phi^\dagger \Phi + \frac{f}{3!} \int d^6 z \Phi^3 + \frac{f^*}{3!} \int d^6 \bar{z} \Phi^\dagger + \frac{1}{64} \Big(\int d^6 z W^\alpha W_\alpha + \int d^6 \bar{z} \bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha} \Big) + \int d^8 z \Phi^\dagger e^{gV} \Phi. \quad (1.6.1)$$

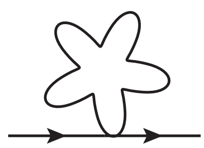
In the following discussion, ϕ indicates the scalar component of the chiral supermultiplet, while $< i\, j >^p_\Phi$, $< \phi\, i >^p$, $< i\, \phi^* >^p$ and $< i\, j >^p_V$ are shortcuts for $< \Phi(\theta_i, \bar\theta_i, p) \Phi^\dagger(\theta_j, \bar\theta_j, -p) >$, $< \phi(p) \Phi^\dagger(\theta_i, \bar\theta_i, -p) >$, $< \Phi(\theta_i, \bar\theta_i, p) \phi^*(-p) >$ and $< V(\theta_i, \bar\theta_i, p) V(\theta_j, \bar\theta_j, -p) >$ respectively.

1.6.1 $< \phi\phi^* >_{1Loop}$ using the conventional Feynman rules

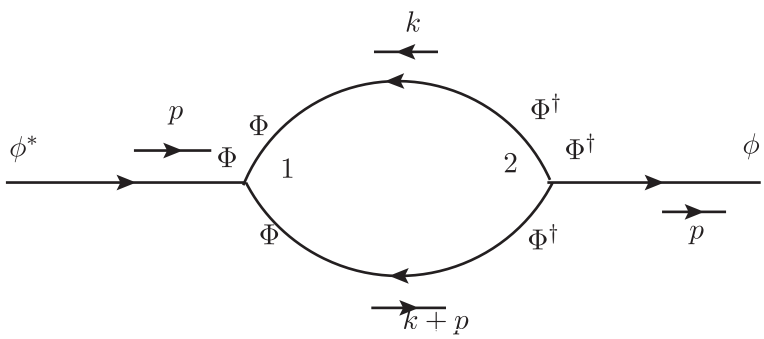
From the vertices of the lagrangian and by the Wick theorem , one can deduce that the super-diagrams contributing to the amplitude $< \phi\phi^* >_{1Loop}$ are



We are neglecting diagrams with self-contractions inside a vertex such as



which corresponds to the self-contraction $|\Phi^\dagger \overline{V V} \Phi|$. In fact in the Dyson formulation of S-matrix these contributions are automatically neglected.
In momentum space the first superdiagram



gives (see appendix E for the symmetry factor)

$$-\frac{|f|^2}{2}\int d^2\theta_1 d^2\bar\theta_2 \int \frac{d^D k}{(2\pi)^D} < 1\phi^* >^p < \phi 2 >^p < 12 >^k_\Phi < 12 >^{-k-p}_\Phi \,. \tag{1.6.2}$$

In the last expression, one has to substitute eqs.(1.3.40)-(1.3.43). The terms containing $\bar\theta_1$ or θ_2 can be neglected since chiral integrals $\int d^2\theta_1$ and $\int d^2\bar\theta_2$ can be written as $\int d^4\theta_1\,\delta^{(2)}(\bar\theta_1)$ and $\int d^4\theta_2\,\delta^{(2)}(\theta_2)$ respectively. Hence, the expression (1.6.2) becomes

$$-\frac{|f|^2}{2}\int d^2\theta_1 d^2\bar\theta_2 \int \frac{d^D k}{(2\pi)^D}\big(\frac{i}{p^2}\big)\big(\frac{i}{p^2}\big)\big(\frac{i}{k^2}+\frac{2i}{k^2}k_\mu\theta_1\sigma^\mu\bar\theta_2+i\theta_1^2\bar\theta_2^2\big)\big(\frac{i}{(k+p)^2}-\frac{2i}{(k+p)^2}(k+p)_\mu\theta_1\sigma^\mu\bar\theta_2+i\theta_1^2\bar\theta_2^2\big). \tag{1.6.3}$$

The only terms that survive are those involving $\theta_1^2\bar\theta_2^2$

$$-\frac{|f|^2}{2}\Big(\frac{i}{p^2}\Big)^2\int d^2\theta_1 d^2\bar\theta_2\int\frac{d^D k}{(2\pi)^D}\Big(\frac{i}{k^2}i\theta_1^2\bar\theta_2^2-\frac{(2i)^2}{k^2(k+p)^2}\frac{1}{2}\theta_1^2\bar\theta_2^2\,k\cdot(k+p)+\frac{i}{(k+p)^2}i\theta_1^2\bar\theta_2^2\Big). \tag{1.6.4}$$

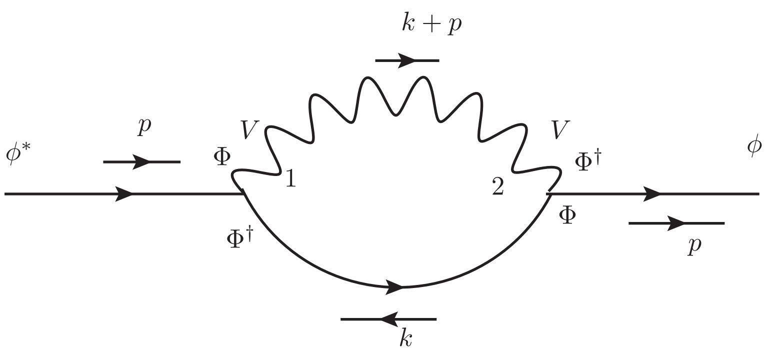
Making the integration, one obtains

$$\Big(\frac{i}{p^2}\Big)^2\Big(\frac{|f|^2}{2}p^2\,B_0(p)\Big) \tag{1.6.5}$$

where

$$B_0(p)=\int\frac{d^D k}{(2\pi)^D}\frac{1}{k^2}\frac{1}{(k+p)^2}. \tag{1.6.6}$$

The second superdiagram is



In momentum space it is equal to (see Appendix E for the symmetry factor)

$$-g^2\int d^4\theta_1 d^4\theta_2\int\frac{d^D k}{(2\pi)^D} < 1\phi^* >^p < \phi 2 >^p < 21 >^{-k}_\Phi < 21 >^{k+p}_V \tag{1.6.7}$$

The delta δ_{12} in the correlator $< 21 >^{k+p}_V$ drops one the two θ integrals. Hence, from eqs.(1.3.40)-(1.3.43), one can write

$$-g^2\int d^4\theta_1\int\frac{d^D k}{(2\pi)^D}\big(\frac{i}{p^2}-\frac{i}{p^2}p_\mu\theta_1\sigma^\mu\bar\theta_1+\frac{i}{4}\theta_1^2\bar\theta_1^2\big)\big(\frac{i}{p^2}-\frac{i}{p^2}p_\mu\theta_1\sigma^\mu\bar\theta_1+\frac{i}{4}\theta_1^2\bar\theta_1^2\big)\big(\frac{-i}{(k+p)^2}\big)\big(\frac{i}{k^2}\big). \tag{1.6.8}$$

Only terms involving $\theta_1^2\bar\theta_1^2$ survive, giving

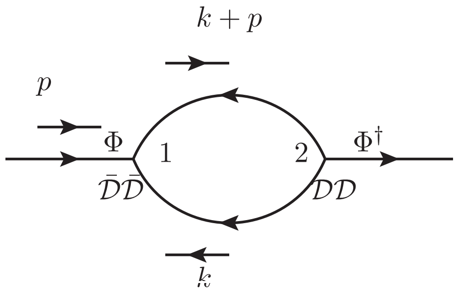
$$\Big(\frac{i}{p^2}\Big)^2\Big(-g^2p^2\,B_0(p)\Big). \tag{1.6.9}$$

Thus the total result is

$$<\phi(p)\phi^*(-p)>_{1Loop}=\Big(\frac{i}{p^2}\Big)^2\big(p^2\,B_0(p)\big)\big(\frac{|f|^2}{2}-g^2\big). \tag{1.6.10}$$

1.6.2 $<\phi\phi^*>_{1Loop}$ using the improved Feynman rules

Using the improved Feynman rules, the same results can be reproduced but in a little more indirect way. In fact, first the part of the effective action associated to a 1PI super-diagram is calculated. Thus, one has an effective vertex Γ_{int} as in eq.(1.3.3). After Wick contractions with the external component fields, one obtains the result.
In the first superdiagram



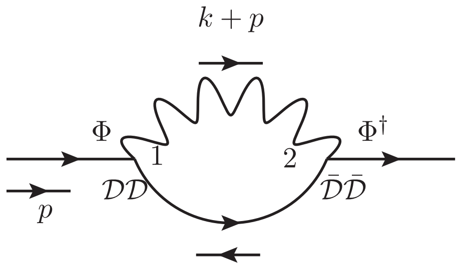
we have indicated the factors $-\frac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}(-\frac{1}{4}\mathcal{D}\mathcal{D})$ on the internal lines and the external superfields Φ ($\Phi^\dagger$) on the external lines. Thus, one has

$$-\frac{|f|^2}{2}\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}d^4\theta_2 d^4\theta_1\Phi^\dagger(-p,\theta_2)\frac{i\delta_{12}}{(k+p)^2}\left((-)\frac{\mathcal{D}_2^{-k}\mathcal{D}_2^{-k}}{4}(-)\frac{\bar{\mathcal{D}}_1^k\bar{\mathcal{D}}_1^k}{4}\frac{i\delta_{12}}{k^2}\right)\Phi(p,\theta_1) \tag{1.6.11}$$

where according to our conventions the momentum label on the super-derivative at 1 is k, while that on the super-derivative at 2 is $-k$, since the momentum k flows into vertex 1 and out of the vertex 2. The use of $D-$algebra allows to express a term in the effective action as an integral over a single $d^4\theta$. In fact, in our case by the identity (B.2.7d), one can replace $\bar{\mathcal{D}}_1^k\bar{\mathcal{D}}_1^k$ by $\bar{\mathcal{D}}_2^{-k}\bar{\mathcal{D}}_2^{-k}$; after applying the identity (B.2.7f), a δ_{12} remains and it allows to eliminate a θ integral giving

$$\frac{|f|^2}{2}\int\frac{d^Dp}{(2\pi)^D}B_0(p)\int d^4\theta\Phi^\dagger(-p,\theta)\Phi(p,\theta). \tag{1.6.12}$$

Wick contractions with the external ϕ,ϕ^* finally give again (1.6.5).
As for the second super-diagram,



the associated part of the effective action is

$$-g^2\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}d^4\theta_2 d^4\theta_1\Phi^\dagger(-p,\theta_2)\frac{-i\delta_{12}}{(k+p)^2}\left((-)\frac{\bar{\mathcal{D}}_2^{-k}\bar{\mathcal{D}}_2^{-k}}{4}(-)\frac{\mathcal{D}_1^k\mathcal{D}_1^k}{4}\frac{i\delta_{12}}{k^2}\right)\Phi(p,\theta_1). \tag{1.6.13}$$

By the same steps as before, one obtains

$$-g^2\int\frac{d^Dp}{(2\pi)^D}B_0(p)\Phi^\dagger(-p,\theta_2)\Phi(p,\theta_1) \tag{1.6.14}$$

and after Wick contractions, the result is eq.(1.6.9).

1.6.3 Supersymmetric dimensional regularization

The integral (1.6.6) is logarithmically divergent, so it has to be regularized. Dimensional regularization does not preserve supersymmetry, that is, it violates certain supersymmetric Ward identities. This is to due to the fact that a necessary condition for supersymmetry is the equality of fermionic and bosonic degrees of freedom. This equality in general is lost if the number of space-time dimensions is changed.

A modified version of dimensional regularization called dimensional reduction has been proposed to render it compatible with SUSY [23]-[25]. In this regularization scheme the momentum integrals are D-dimensional while the number of field components is kept fixed. Thus $\gamma-$matrix algebra and $\mathcal{D}-$algebra is done in four dimensions, while loop momentum integrals are done in $D-$dimension. Even dimensional reduction presents some problems related to the treatment of the Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$ when the number of loops gets large(> 4) [26]. The question whether there exists a supersymmetric regularization scheme valid for Super Yang-Mills theory to all orders in perturbation theory is open[25]

Another variant of dimensional regularization preserving supersymmetry is the four dimensional helicity scheme[27]-[29]. As in dimensional reduction, in this scheme all external momenta and polarization vectors are kept in four dimensions, only the loop momenta are continued to D dimensions.

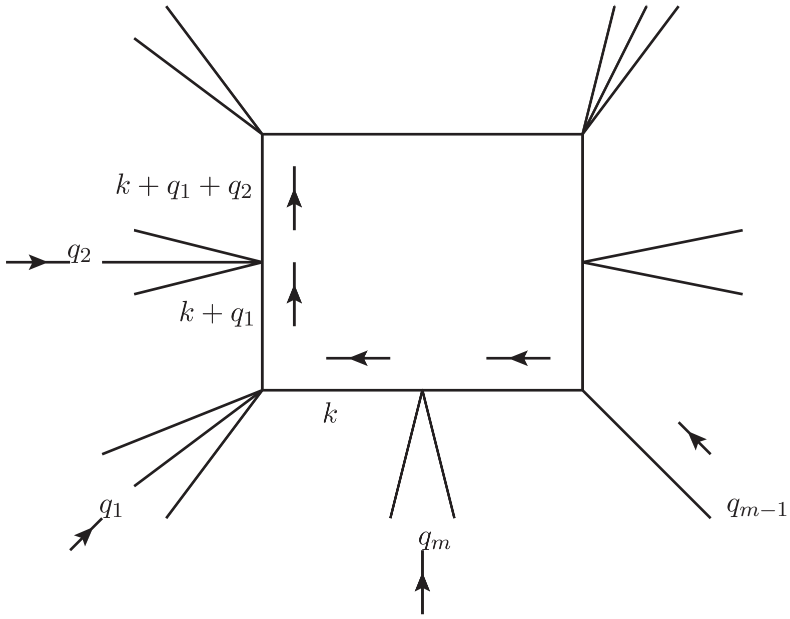
Chapter 2

Decomposition of one loop amplitudes

In the last years there has been a great progress in the computation of loop amplitudes by the so called on shell (or unitarity) methods [46]-[52]. In this approach one reconstructs scattering amplitudes from their singularities, which are determined by simpler (lower-point and lower loop) amplitudes. Hence all the information is extracted from on-shell amplitudes, i.e with external physical states without using Feynman diagrams (which require building blocks with off-shell states).
The power of these methods relies on the possibility to decompose loop amplitudes in a basis of known integrals. In fact at one loop an amplitude can be reduced to a linear combination of integrals plus a possible rational function. The coefficients multiplying the integrals can be determined only by the (branch cut) singularities of the amplitude.
The computation of the remaining rational function $\mathcal{R}$ is more complicated. The presence of $\mathcal{R}$ is related to the ultraviolet behavior of the theory under consideration. In fact, it is absent if the loop momentum integrals of the amplitude satisfy a certain power counting criterion and in this case the amplitude is called cut-constructible. This happens for the supersymmetric gauge theories which have an improved ultraviolet behavior with respect to non-SUSY theories because of cancellation between bosons and fermions in the loop.
In the first section of the chapter we present the notation and discuss a first decomposition of one loop integrals based on Lorentz covariance. A brief review of ultraviolet and infrared divergences for loop integrals is also given.
In the second section we discuss in detail the Passarino-Veltman method [42] to reduce one loop integrals in a basis of known scalar integrals.
The third section deals with the power-counting criterion for the absence of the rational function $\mathcal{R}$ in the decomposition. Moreover, there is a brief presentation of unitarity methods.
In section four we review the proof that gluon amplitudes in super Yang-Mills theories are cut-constructible.
In the last section we discuss another kind of decomposition for the amplitudes which deals with the color factors and in fact is called color decomposition.
In this chapter we have followed closely [44] and [48].

2.1 One loop integrals

In the computation of one-loop amplitudes or correlation functions, one is faced with a Feynman diagram with a given topology. For example, let us consider the diagram



It corresponds to an integral of the form (the following discussion deals only with massless internal lines)

$$I_m[P_r(k)] = \int \frac{d^D k}{(2\pi)^D} \frac{P_r(k)}{k^2(k+q_1)^2(k+q_1+q_2)^2 \dots (k+q_1+q_2 \dots q_{m-1})^2}, \tag{2.1.1}$$

where q_i, $i = 1..m$ are the momenta or sums of the momenta of the external particles which are all assumed to be incoming. Here m is the number of the interaction vertices (and of the internal propagators) in the given topology and of the denominators in the integral. $P_r(k)$ is a polynomial of degree r in the loop momentum k and is a tensor of rank r (for example for $r = 2$, $P_r(k)$ could be $P_2(k) = k^\mu k^\nu$). The dimension of space-time is set to $D = 4 - 2\epsilon$ to regularize the divergences of the integral (see the discussion below).
The integrals $I_m[P_r(k)]$ or in short I_m^r are called m-point tensor integrals of rank r. In the case $P_r(k) = 1$, the integrals $I_m[1]$ or in short I_m are called scalar integrals.
The loop momentum k^μ can be contracted with the external momenta and polarization vectors.
We will assume that the external vectors are purely four-dimensional as happens in dimensional reduction and in the four dimensional helicity scheme (see section (1.6.3)). As we will show below, any amplitude can be written as a linear combination of scalar integrals. In the limit $\epsilon \to 0$, one needs to include scalar integrals with up to four propagators

$$\mathcal{A}_{1\,loop} = \sum_j \Big(c_{4;j} I_{4;j} + c_{3;j} I_{3;j} + c_{2;j} I_{2;j} \Big) + \mathcal{R} + O(\epsilon), \tag{2.1.2}$$

where the coefficients $c_{2,j}, c_{3,j}, c_{4,j}$ and $\mathcal{R}$ are rational functions of the kinematical invariants and are evaluated in $D = 4$, i.e. are **independent** on ϵ. The symbol j specifies which combination of the external momenta enters the scalar integral.
The function $\mathcal{R}$ is called the rational part of the decomposition and the scalar integrals I_2, I_3, I_4 with two, three and four propagators are named bubbles, triangles and boxes respectively.
The decomposition (2.1.2) reduces the calculation of any one-loop amplitude to the determination of both the coefficients $c_{2,j}, c_{3,j}, c_{4,j}$ and the rational part $\mathcal{R}$, since the analytic expressions of the scalar integrals are known [30]-[33].

In literature there is another notation for the scalar integrals I_2, I_3, I_4 which are indicated as B_0, C_0, D_0 respectively.[2.1] According to this alternative notation (we will employ both the notations), the tensor integrals up to four-point and rank-four (which are relevant for the discussion below) are denoted as

$$
\begin{aligned}
&B_0; B^\mu; B^{\mu\nu}(p_1) = \int \frac{d^D k}{(2\pi)^D} \frac{1; k^\mu; k^\mu k^\nu}{k^2(k+p_1)^2}, \\
&C_0; C^\mu; C^{\mu\nu}; C^{\mu\nu\rho}(p_1, p_2) = \int \frac{d^D k}{(2\pi)^D} \frac{1; k^\mu; k^\mu k^\nu; k^\mu k^\nu k^\rho}{k^2(k+p_1)^2(k+p_1+p_2)^2} \\
&D_0; D^\mu; D^{\mu\nu}; D^{\mu\nu\rho}; D^{\mu\nu\rho\gamma}(p_1, p_2, p_3) = \int \frac{d^D k}{(2\pi)^D} \frac{1; k^\mu; k^\mu k^\nu; k^\mu k^\nu k^\rho; k^\mu k^\nu k^\rho k^\gamma}{k^2(k+p_1)^2(k+p_1+p_2)^2(k+p_1+p_2+p_3)^2}
\end{aligned}
\tag{2.1.6}
$$

Using Lorentz symmetry, one can write the bubble tensor integrals B^μ and $B^{\mu\nu}$ as

$$
B^\mu(p_1) = p_1^\mu\, b_1(p_1^2), \qquad B^{\mu\nu}(p_1) = \eta^{\mu\nu} b_{00}(p_1^2) + p_1^\mu p_1^\nu\, b_{11}(p_1^2). \tag{2.1.7}
$$

As in [44], we will refer to the coefficients $b_1(p_1^2), b_{00}(p_1^2), b_{11}(p_1^2)$ as forms factors or reduction coefficients. Moreover, we will say that a form factor has m points and rank r if it is associated to an m point tensor integral of rank r.

Similarly for the triangle tensors integrals, one has

$$
\begin{aligned}
C^\mu &= p_1^\mu c_1 + p_2^\mu c_2 \\
C^{\mu\nu} &= \eta^{\mu\nu} c_{00} + p_1^\mu p_1^\nu c_{11} + p_2^\mu p_2^\nu c_{22} + (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) c_{12} \\
&= \eta^{\mu\nu} c_{00} + \sum_{i \le j = 1}^{2} p_i^{\{\mu} p_j^{\nu\}} c_{ij} \\
C^{\mu\nu\rho} &= \sum_{i=1}^{2} \eta^{\{\mu\nu} p_i^{\rho\}} c_{00i} + \sum_{i \le j \le k = 1}^{2} p_i^{\{\mu} p_j^\nu p_k^{\rho\}} c_{ijk}
\end{aligned}
\tag{2.1.8}
$$

where the symbol $\{\ldots\}$ denotes completely symmetrization

$$
p_1^{\{\mu} p_1^{\nu\}} = p_1^\mu p_1^\nu, \quad p_1^{\{\mu} p_2^{\nu\}} = p_1^\mu p_2^\nu + p_2^\mu p_1^\nu, \tag{2.1.9}
$$

and we have omitted to indicate the dependence of the $c-$form factors on p_1, p_2. For the box tensors integrals one has

$$
\begin{aligned}
&D^\mu = p_1^\mu d_1 + p_2^\mu d_2 + p_3^\mu d_3, \qquad D^{\mu\nu} = \eta^{\mu\nu} d_{00} + \sum_{i \le j = 1}^{3} p_i^{\{\mu} p_j^{\nu\}} d_{ij} \\
&D^{\mu\nu\rho} = \sum_{i=1}^{3} \eta^{\{\mu\nu} p_i^{\rho\}} d_{00i} + \sum_{i \le j \le k = 1}^{3} p_i^{\{\mu} p_j^\nu p_k^{\rho\}} d_{ijk} \\
&D^{\mu\nu\rho\gamma} = \eta^{\{\mu\nu} \eta^{\rho\gamma\}} d_{0000} + \sum_{i \le j = 1}^{3} \eta^{\{\mu\nu} p_i^\rho p_j^{\gamma\}} d_{00ij} + + \sum_{i \le j \le k \le l = 1}^{3} p_i^{\{\mu} p_j^\nu p_k^\rho p_l^{\gamma\}} d_{ijkl},
\end{aligned}
\tag{2.1.10}
$$

where the dependence of the $d-$form factors on p_1, p_2, p_3 has not been indicated.

[2.1] In dimensional regularization and in its variants, one can neglect the 'tadpole' scalar integral, indicated in literature as I_1 or A_0, with a single massless internal propagator

$$
A_0 = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2}. \tag{2.1.3}
$$

In fact, requiring linearity, uniqueness of the result and analyticity in ϵ forces a regulated Feynman integral with a scaleless integrand

$$
\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2)^\alpha}. \tag{2.1.4}
$$

to vanish (see [80] and appendix A of [41]). For the same reason, the integrals

$$
B_0; B^\mu; B^{\mu\nu}(p_1) = \int \frac{d^D k}{(2\pi)^D} \frac{1; k^\mu; k^\mu k^\nu}{k^2(k+p_1)^2} \tag{2.1.5}
$$

with $p_1^2 = 0$ can be assumed to vanish. Infact, if $p_1^2 = 0$, after the Feynman parametrization these can be expressed as integrals of the form (2.1.4).

2.1.1 Divergences of the integrals

Ultraviolet divergences

The integrals I_m^r can have ultraviolet (UV) and/or infrared (IR) divergences.
As far as the UV divergences are concerned, from power counting one has that in four dimensions an integral I_m^r is ultraviolet divergent only if the rank r is higher than $r \geq 2m-4$. In a renormalizable quantum field theory the highest rank r for an m point integral is $r = m$. Thus, in four dimensions for $m \geq 5$ an m point integral is UV finite and the only UV divergent integrals are

$$(A_0), B_0, B^{\mu}, B^{\mu\nu}, C^{\mu\nu}, C^{\mu\nu\rho}, D^{\mu\nu\rho\gamma}, \tag{2.1.11}$$

while the scalar integrals C_0 and D_0 are UV finite .
There are various methods to regularize the UV divergences. One can introduce a cut-off Λ for the momentum of the virtual particles or use dimensional regularization by shifting the dimension of integration to $D = 4-2\epsilon$. In the former method, the divergences[2.2] appear as powers of $\ln\Lambda$, in the latter one as poles in $1/\epsilon$, i.e. there is the correspondence $\ln\Lambda_{UV} \leftrightarrow \frac{1}{\epsilon}$.
In a renormalizable theory at L loops, UV divergences give at most the pole $1/\epsilon^L$ (or the power $(\ln\Lambda_{UV})^L$). Also, the cancellation of UV infinities requires renormalization of the parameters in the Lagrangian.

Infrared divergences

In presence of massless particles, the integrals I_m^r can have also **infrared** divergences [36]-[40].
These arise in the integration over the phase-space when one computes the physical measurable cross section and in the integration over the loop momentum when one computes the loop contributions to an amplitude.
The IR divergences in the phase-space are due to a configuration with an external (on shell) massless particle which is soft, i.e. with vanishing momentum $p^\mu \to 0$, or with collinear massless external particles, i.e. with proportional momenta $p_i \propto p_j$. In a guage theory infrared divergences have an universal form [81]-[83].
As for the loop momentum integral, one can find IR divergences for example in a one-loop diagram with internal massless propagators and at least one of the external particles which is on shell and massless. The IR divergences arise in the region of integration over the loop momentum when a virtual particle is on-shell and soft or collinear to an external massless particle.
In contrast to UV divergences, the IR ones don't need renormalization, since in the computation of a physical cross section IR infinities coming from phase-space cancels those produced by loop integration [38]. If one regularizes IR divergences by dimensional regularization, at one loop a purely soft or a purely collinear virtual particle gives a pole $1/\epsilon$, while a soft and collinear virtual particle gives a pole $1/\epsilon^2$. At L loops one has at most a pole $1/\epsilon^{2L}$. Dimensional regularization allows to keep the massless external particles on shell and hence use the on-shell methods (see the introduction) for the computation of amplitudes.
In a massless gauge theory, an alternative regularization for IR infinities is obtained by giving a (small) mass to the external particles, i.e. to consider them off-shell. This off-shell regularization allows to keep the dimension of space-time equal to four (see section 3.2). In our results of chapter 3, we will use off-shell regularization for the IR divergences.
The correspondence between poles in $1/\epsilon$ and powers of $\ln m^2$ for IR infinities is the same as for UV ones, i.e. $\ln m_{IR}^2 \leftrightarrow \frac{1}{\epsilon}$.
The scalar integrals with massless lines, which are relevant for the following discussion, can have IR divergences as well. In fact the triangles C_0 and the boxes D_0 with internal massless lines have IR divergences if at least one of the external legs is massless, while are IR finite if all the external legs are massive. Note that one has a 'massive' leg , i.e. $q^2 \neq 0$, even in the case of more massless particles i_1, i_2,etc... that converge at the same vertex, since in this case the total inflowing momentum is $q^2 = (p_{i_1} + p_{i_2} + \ldots)^2 \neq 0$.
As for the bubble scalar integral $B_0(p)$, it is IR divergent if $p^2 = 0$. In dimensional regularization this kind of integral can be neglected (as the scalar tadpole A_0 integral, see footnote 2.1).
In ref. [47], one can find the explicit expressions in dimensional regularization of the scalar integrals B_0, C_0, D_0 in the case where one or more of the external lines are massless. An expression of C_0 and D_0 in the case where all the external lines are massive is given in [32]-[34].

[2.2] we are considering only logarithmic divergences and not power-law divergences

2.2 Passarino-Veltman decomposition

There are various techniques [42]-[44] to reduce tensor integrals I_m^r to scalar integrals according to eq.(2.1.2). We illustrate the technique developed by Passarino and Veltman [42] in the case of up to four point tensor integrals, since as discussed below an m-point tensor integral with $m > 4$ can be reduced to these.
In particular, let's consider $C^{\mu\nu}$ in (2.1.6).
Contracting the Lorentz decomposition of $C^{\mu\nu}$ in (2.1.8) with p_1 and p_2 one obtains

$$
\begin{aligned}
p_{1\,\mu} C^{\mu\nu} &= p_1^\nu (p_1^2 c_{11} + p_1 \cdot p_2\, c_{12} + c_{00}) + p_2^\nu (p_1^2 c_{12} + p_1 \cdot p_2\, c_{22}), \\
p_{2\,\mu} C^{\mu\nu} &= p_1^\nu (p_1 \cdot p_2\, c_{11} + p_2^2 c_{12}) + p_2^\nu (p_1 \cdot p_2\, c_{12} + p_2^2 c_{22} + c_{00}).
\end{aligned} \tag{2.2.1}
$$

Using the definition (2.1.6) of $C^{\mu\nu}$, the left-hand side of eqs.(2.2.1) can be expressed as

$$
\begin{aligned}
p_{1\,\mu} C^{\mu\nu} &= \int \frac{d^D k}{(2\pi)^D} \frac{k \cdot p_1\, k^\nu}{k^2 (k+p_1)^2 (k+p_1+p_2)^2} \\
p_{2\,\mu} C^{\mu\nu} &= \int \frac{d^D k}{(2\pi)^D} \frac{k \cdot p_2\, k^\nu}{k^2 (k+p_1)^2 (k+p_1+p_2)^2}
\end{aligned} \tag{2.2.2}
$$

The scalar products $k \cdot p_1$ and $k \cdot p_2$ in the numerators can be written in terms of the denominators, which are inverse Feynman propagators, by the identities

$$
\begin{aligned}
k \cdot p_1 &= \frac{1}{2}\big((k+p_1)^2 - k^2 - p_1^2\big) \\
k \cdot p_2 &= \frac{1}{2}\big((k+p_1+p_2)^2 - (k+p_1)^2 - p_2^2 - 2p_1 \cdot p_2\big).
\end{aligned} \tag{2.2.3}
$$

Shifting if necessary the variable of integration $k \to k - p_1$ to make always appear the term k^2 in the denominator, one can express the right-hand sides of (2.2.2) in terms of triangle tensors of rank one C^μ, bubble tensors of rank one B^μ and the bubble scalar integral B_0. Then one uses the Lorentz decompositions (2.1.7)-(2.1.8) to write B^μ and C^μ in terms of forms factors b_1 and c_i.
Finally, equating the expressions multiplying p_1^ν, p_2^ν in eqs.(2.2.1) with those obtained from eqs.(2.2.2) after these manipulations, one gets the linear algebraic systems

$$
G_2 \begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix} = \begin{pmatrix} R_1^{[c1]} \\ R_2^{[c1]} \end{pmatrix}, \quad G_2 \begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix} = \begin{pmatrix} R_1^{[c2]} \\ R_2^{[c2]} \end{pmatrix} \tag{2.2.4}
$$

where, following the notation of [44], G_2 is the 2×2 Gram matrix

$$
\begin{pmatrix} p_1^2 & p_1 \cdot p_2 \\ p_1 \cdot p_2 & p_2^2 \end{pmatrix}, \tag{2.2.5}
$$

the constants terms $R_1^{[c1]}, R_2^{[c1]}$ are

$$
\begin{aligned}
R_1^{[c1]} &= \frac{1}{2}\big(b_1(p_1+p_2) - p_1^2 c_1(p_1,p_2) + B_0(p_2) - 2c_{00}(p_1,p_2)\big) \\
R_2^{[c1]} &= \frac{1}{2}\big(b_1(p_1) - b_1(p_1+p_2) + (-p_2^2 - 2p_1 \cdot p_2) c_1(p_1,p_2)\big),
\end{aligned} \tag{2.2.6}
$$

while $R_1^{[c2]}, R_2^{[c2]}$ are

$$
\begin{aligned}
R_1^{[c2]} &= \frac{1}{2}\big(b_1(p_1+p_2) - b_1(p_2) - p_1^2 c_2(p_1,p_2)\big) \\
R_2^{[c2]} &= \frac{1}{2}\big(- b_1(p_1+p_2) + (-p_2^2 - 2p_1 \cdot p_2) c_2(p_1,p_2) - 2c_{00}(p_1,p_2)\big)
\end{aligned} \tag{2.2.7}
$$

In ref.[44], one can find the constants terms, like $R_1^{[c1]}, R_2^{[c1]}$,etc.., of the linear algebraic systems for all the form factors up to four point and four rank.
Solving eqs.(2.2.4), one can write all c_{ij} as a linear combination of c_{00}, c_i, b_i, B_0

$$
c_{ij} \to c_{00}, c_i, b_i, B_0, \tag{2.2.8}
$$

where the symbol $\to$ means that the term on the left is a linear combination of the terms on the right. Let's note that from eqs.(2.2.4) the coefficient c_{12} can be determined in two different ways. This gives a check for the computation.
The form factors c_{00}, d_{00}, d_{0000}[2.3] which multiply only $\eta^{\mu\nu}$ tensors in eqs.(2.1.8)-(2.1.10) have to be treated in a little different way.
For example, to determine c_{00}, one contracts with $\eta^{\mu\nu}$ the expression for $C^{\mu\nu}$ in eq.(2.1.6) and (2.1.8), obtaining respectively (after shifting $k \to k - p_1$ in the integral of eq.(2.1.6))

$$\eta_{\mu\nu} C^{\mu\nu} = B_0(p_2), \quad \eta_{\mu\nu} C^{\mu\nu} = D c_{00} + p_1^2 c_{11} + 2 p_1 \cdot p_2\, c_{12} + p_2^2 c_{22}. \tag{2.2.9}$$

One can equate the right-hand sides of the two equations and note from the two systems in (2.2.4) that the expression $p_1^2 c_{11} + 2 p_1 \cdot p_2\, c_{12} + p_2^2 c_{22}$ is the sum of $R_1^{[c1]}$ and $R_2^{[c2]}$. Hence, using eqs.(2.2.6)-(2.2.7) one can write

$$c_{00} = \frac{1}{2(D-2)} \Big(B_0(p_2) - \big(-p_2^2 - 2 p_1 \cdot p_2 \big)\, c_2(p_1, p_2) + p_1 c_1(p_1, p_2) \Big). \tag{2.2.10}$$

Note the presence of the dimension of the space-time D in the first factor of eq.(2.2.10).
All the form factors of the tensor integrals with $m \leq 4$ have been studied ([42],[44]).
One can write the reduction chain (taken from [44])[2.4]

$$\begin{aligned}
&d_{ijkl} \to d_{00ij}, d_{ijk}, c_{ijk}, c_{ij}, c_i, C_0,\\
&d_{00ij} \to d_{ijk}, d_{ij}, c_{ij}, c_i,\\
&d_{0000} \to d_{00i}, d_{00}, c_{00},\\
&d_{ijk} \to d_{00i}, d_{ij}, c_{ij}, c_i,\\
&d_{00i} \to d_{ij}, d_i, c_i, C_0,\\
&d_{ij} \to d_{00}, d_i, c_i, C_0,\\
&d_{00} \to d_i, D_0, C_0,\\
&d_i \to D_0, C_0,\\
&-----------\\
&c_{ijk} \to c_{00i}, c_{ij}, b_{ij}, b_i,\\
&c_{00i} \to c_{ij}, c_i, b_i, B_0,\\
&c_{ij} \to c_{00}, c_i, b_i, B_0,\\
&c_{00} \to c_i, C_0, B_0,\\
&c_i \to C_0, B_0,\\
&-----------\\
&b_{ij} \to b_{00}, b_i\\
&b_{00} \to b_i, B_0,\\
&b_i \to B_0.
\end{aligned} \tag{2.2.11}$$

For the discussion on the rational part $\mathcal{R}$ below, it is important to note that the external vectors are assumed to be purely four dimensional and consequently the contraction procedure just described does not introduce an explicit dependence on the space-time dimension D in the reduction equations.
This explicit dependence on D (or ϵ) comes from the contraction $\eta_{\mu\nu}\, \eta^{\mu\nu}$. So it appears in the equations determining those forms factors in (2.1.6) which multiply at least one $\eta^{\mu\nu}$ tensor

$$D \in b_{00}, c_{00}, c_{00i}, d_{00}, d_{00i}, d_{00ii}, d_{0000}. \tag{2.2.12}$$

Hence, from the reduction chain (2.2.11), one has that all the form factors of rank $r \geq 2$ are dependent on D (or ϵ), since in their reduction path there is at least one of the reduction coefficient of (2.2.12).
Also, for the form factors $b_1(p_1), b_{11}(p_1)$, and consequently for all the other form factors depending on them,

[2.3] the form factors with 'mixed' indices, i.e. $c_{00i}, d_{00i}, d_{00ij}$ can be computed in both ways. The computation of b_{00} is trivial

[2.4] Unlike our discussion, in [44] the internal propagators are supposed to be massive one has to distinguish the case $p_1^2 = 0$ from the case $p_1^2 \neq 0$. In the case $p_1^2 \neq 0$, which is relevant for the off shell amplitudes of chapter 3, they can be written as (see appendix A of [44])

$$b_1(p_1) = -\frac{1}{2}B_0(p_1), \quad b_{11}(p_1) = \frac{1}{4}\frac{D}{D-1}B_0(p_1), \tag{2.2.13}$$

while, in the case $p_1^2 = 0$, they vanish (see footnote 2.1).

Following the reduction chain (2.2.11), finally one can express all the form factors for $m \leq 4$ as linear combination of B_0, C_0, D_0.

$$\begin{aligned}
&b_{00}, b_{11}, b_1 \to B_0 \\
&c_i, c_{00}, c_{ij}, c_{00i}, c_{ijk} \to C_0, B_0 \\
&d_i, d_{00} \to D_0, C_0 \\
&d_{ij}, d_{00i}, d_{ijk}, d_{0000}, d_{00ij}, d_{ijkl} \to D_0, C_0, B_0. \qquad (2.2.14)
\end{aligned}$$
$$(2.2.15)$$

In particular, all the form factors in (2.2.12) are reduced to a linear combination of scalar integrals multiplied by a coefficient depending on D, for example b_{00} is

$$b_{00}(p) = -\frac{1}{4(D-1)}p^2 B_0(p) \tag{2.2.16}$$

Another important consideration is that even if the form factors $c_i, d_{ij}, d_{ijk}, d_{00i}$ are UV finite, in their decomposition there is B_0 which is UV divergent. Thus expanding in powers of ϵ the B_0's[2.5] present in the reduction, the sum of all the UV poles $1/\epsilon$ has to give zero. This happens if the sum of all the coefficients multiplying the B_0's in the decomposition vanishes.
The discussion above can be easily generalized to the case of an m-point tensor integral with $m > 4$. In fact, one can express the form factors of an m point tensor integral of rank r in terms of those associated to an m point integral of rank $r-1$ and of those associated to $m-1$ point integrals of rank $r-1$ or less. Thus, the iteration of this procedure reduces an $m-$point tensor integral with $m > 4$ to box tensor integrals ($m = 4$) and scalar m point integrals. As shown above, tensor integrals with $m \leq 4$ can be reduced to bubble, triangle and box scalar integrals. Finally, as proven in [43],[45], if one neglects, in dimensional regularization, terms of order ϵ, scalar integrals with $m > 4$ can be reduced to box integrals, i.e.

$$I_m = \sum c_{4;j} I_{4;j} + O(\epsilon), \quad m > 4. \tag{2.2.18}$$

2.3 Rational parts and cut constructibility

From the considerations above, it follows that any one-loop amplitude can be decomposed as follows

$$\mathcal{A}_{1\,loop} = \sum_j \Big(c_{4;j}(\epsilon) I_{4;j} + c_{3;j}(\epsilon) I_{3;j} + c_{2;j}(\epsilon) I_{2;j} \Big) + O(\epsilon), \tag{2.3.1}$$

where the coefficients of eq.(2.3.1) depend on ϵ unlike the coefficients of eq.(2.1.2) which can be read as $c_{2;j}(\epsilon)|_{\epsilon=0} \equiv c_{2;j}$, etc...
Rational terms, which are related to ultraviolet singularities, arise if one expands the coefficients of eq.(2.3.1) in ϵ. In fact, terms of order $O(\epsilon)$ cancel the UV poles $1/\epsilon$ giving a finite result[2.6].
For an amplitude with massless internal propagators, the only UV divergent scalar integral is B_0 (in the massive case there is also the tadpole integral A_0). Therefore, in order to have a rational part $\mathcal{R}$, the decomposition of a tensor integral has to contain terms of the form

$$\epsilon B_0(p) = 1 + O(\epsilon). \tag{2.3.2}$$

[2.5] In dimensional regularization B_0 is [47]

$$B_0(p) = \frac{1}{\epsilon} - \ln p^2 + 2 - \gamma, \tag{2.2.17}$$

where γ is the Euler's constant.

[2.6] here UV divergence are assumed to be regularized by dimensional regularization while IR divergence by off-shell regularization

Let us consider first the case of bubble, triangle and box tensor integrals ($m \leq 4$). From the previous discussion, one can conclude that the tensor integrals of rank $r < 2$ have $\mathcal{R} = 0$, since all their form factors are independent of ϵ.
From (2.2.12), (2.2.14) and from the consideration after eq.(2.2.14), it follows that all the reduction coefficients $b_{00}, c_{00}, c_{00i}, d_{00i}, d_{00ii}, d_{0000}$ contain $\epsilon B_0(p)$. Thus, $B^{\mu\nu}, C^{\mu\nu}, C^{\mu\nu\rho}, D^{\mu\nu\rho\gamma}$, which are UV divergent and $D^{\mu\nu\rho}$, which is UV finite, have $\mathcal{R} \neq 0$ because have one of these form factors. For example

$$B^{\mu\nu}(p) = \Big(\frac{p^\mu p^\nu}{3} - \frac{\eta^{\mu\nu}\, p^2}{12}\Big) B_0(p) + \frac{1}{18}\Big(p^\mu p^\nu - \eta^{\mu\nu}\, p^2\Big) + O(\epsilon). \tag{2.3.3}$$

On the contrary, $D^{\mu\nu}$ has $\mathcal{R} = 0$, because the dependence on ϵ of its form factors d_{ij} is due to d_{00} which does not contain B_0. Thus, by direct inspection, one has that for m point integrals with $m \leq 4$, if the rank r is $r \leq m-2$ then their decomposition has not rational parts. The only exception to this rule is for $r = 1$ and $m = 2$, i.e. for B^μ, which is UV divergent. In fact, its form factor b_1 is independent on ϵ and so $\mathcal{R} = 0$. So one can summarize the condition to have $\mathcal{R} = 0$ as

$$r \leq \max\{m-2, 1\}. \tag{2.3.4}$$

Higher point-integrals ($m > 4$) in four dimension are all ultraviolet finite. Also, the Passarino-Veltman reduction maintains the difference $m - r$, since at each step the most ultraviolet-singular term has both m and r reduced by one unit with respect to the previous step. Hence, their reduction path cannot generate a rational part if the rank is $r \leq m-2$ as stated in (2.3.4). .

Unitarity methods

The importance of having a decomposition (2.1.2) with $\mathcal{R} = 0$ is related to the possibility to construct an amplitude only by its branch cut singularities [46]-[50].
In fact an amplitude can have singularities. At tree level, these are represented by poles as kinematic invariants vanish due to an almost on-shell internal propagator. At the loop level, amplitudes can have poles as well as branch cuts when more than one internal propagators are on shell. The scalar integrals appearing in (2.1.2) are expressed by logarithms and dilogaritms, which indeed have branch cuts. On the contrary rational functions can't have this kind of singularities.
If in the expansion (2.1.2) the rational part is absent $\mathcal{R} = 0$, then the amplitude can be determined only by the discontinuities across its branch cuts and, as already said in the introduction, if this happens is called **cut-constructible**.
In a given channel, one can compute the branch cut discontinuity for both sides of (2.1.2). Since the scalar bubble, triangle and box integrals I_m are all known, their discontinuity ΔI_m are known as well. A color-ordered or planar amplitude (see section 2.5) receives contribution only from diagrams with a particular cyclic order of the external legs and so can have singularities only in kinematic invariants made out of squares of sums of cyclically adjacent momenta. This implies [46]-[47] that a planar amplitude can be decomposed in scalar integrals containing logarithms and dilogarithms which produce cuts that are independent of those produced by other integrals. Thus, choosing the appropriate channel one can pick up a single term in (2.1.2)

$$\Delta\mathcal{A} = c_i \Delta I_i \tag{2.3.5}$$

To determine the coefficient c_i from (2.3.5), one has to compute the discontinuity $\Delta\mathcal{A}$ in a given channel in a different way. In fact, the unitarity of the S matrix (from this the name of unitarity methods),

$$SS^\dagger = 1 \tag{2.3.6}$$

implies that the interaction matrix T defined by $S = 1 + iT$ obeys

$$2ImT = T^\dagger T. \tag{2.3.7}$$

This equation relates terms of different order in perturbation theory. Expanding it in the coupling constant, one has that the imaginary part of loop amplitudes can be determined from (four dimensional) phase-space integrals of products of lower-order **on shell** amplitudes (from this the name on-shell methods) without the

need of computing all the off-shell Feynman diagrams of a given order in the coupling constant.
In particular, the imaginary part of one-loop amplitudes is related to the product of two tree amplitudes (this is equivalent to putting on-shell two internal propagators of the one-loop amplitude, i.e. to doing a 'double cut').
This imaginary part is related to the discontinuity $\Delta\mathcal{A}_{1\,loop}$ across a branch cut in a given channel [54].
A possible rational part of the amplitude has no branch cut singularities and cannot be found by the unitarity method just described (where the cut is evaluated strictly in four dimensions). If one evaluated the cut in D dimension, then had information even on the rational part [51], [85]. Hence, amplitudes for which the condition (2.3.4) is satisfied are cut-constructible.
A generalization of the method just described consists in putting on shell (cutting) three or more propagators [52]. This technique has been extended also to (on-shell) superspace [53].

2.4 Decomposition of Super Yang-Mills Amplitudes

One-loop color-ordered gluon amplitudes in massless supersymmetric gauge theories satisfy the power-counting criterion (2.3.4), i.e. they are cut-constructible [46]-[48]. To show this, it is enough to study only the effective action $\Gamma(A^a_\mu)$ at one loop since the presence of trees attached to the loop does not change the power-counting of the loop integrand.
The reason why super Yang-Mills theories satisfy the power-counting criterion (2.3.4) is that in the loop diagrams of SYM theories there are cancellations between fermionic and bosonic fields which lower the degree of divergence of the loop integral.

Background field method

We will study the effective action using the background-field method [54]-[60]. It allows to quantize a gauge field theory without losing explicit gauge invariance.
In fact in the conventional formulation, one derives Feynman rules from a total Lagrangian which is not gauge invariant because is the sum of the classical Lagrangian and of gauge-fixing and ghosts terms. Any physical quantity will be gauge-invariant but quantities with no direct physical interpretation like off-shell Green functions or counterterms may not be gauge invariant.
Let's consider a renormalizable gauge theory with vector, spinor and scalar fields (in this section we are dealing with conventional (component) fields not superfields). Ghosts fields are also included.
In the background-field method one has to split the gauge field in a 'classical' background field $A^a_{B\mu}$ and in a 'quantum' field $A^a_{Q\mu}$

$$A^a_\mu \to A^a_{B\,\mu} + A^a_{Q\,\mu} \tag{2.4.1}$$

One can compute the effective action $\Gamma(A^a_{B\mu})$ treating $A^a_{B\mu}$ as an external fixed field, while $A^a_{Q\mu}$ can appear only in the internal lines of 1PI diagrams and is the variable of integration in the functional integral.
To find the propagator for A^a_Q, one has to choose a gauge-fixing function. Let's consider the covariant derivative D_μ with respect to the background gauge field

$$D_\mu = \partial_\mu - iA^a_{B\,\mu}T^a_R, \tag{2.4.2}$$

where the T^a_R are the generators of the gauge group in the representation R. One can choose a gauge-fixing function $G(A_B)$ dependent on A^a_B

$$G(A_B) = D^\mu A_{Q\,\mu}. \tag{2.4.3}$$

This allows to write a total Lagrangian L_{tot}, which includes gauge-fixing and ghosts terms, which is gauge-invariant with respect to the background gauge transformation

$$A^a_{B\mu} \to A^a_{B\,\mu} + D_\mu \alpha^a, \tag{2.4.4}$$

where α^a is the gauge parameter. Under this transformation A^a_Q transform as a matter field in the adjoint representation.
To compute the effective action $\Gamma(A^a_B)$ at one loop, one has to drop the terms linear in $A^a_{Q\,\mu}$ (which are

associated to reducible diagrams) and consider only terms quadratic in A^a_Q and in the other fields (since these are the terms that produce the vertices for the 1PI diagrams at one loop). After some manipulation this quadratic part of the lagrangian L_{quad} can be written as a sum of terms of the form [54]

$$\Phi \Delta_{R,(m,n)} \Phi \tag{2.4.5}$$

where here Φ is one of the fields in the Lagrangian belonging to the representation R of the gauge group and to the representation (m, n) of the Lorentz group (with a notation that uses the isomorphism between the Lie algebra of the Lorentz group and that of $SU(2) \times SU(2)$).
The symbol $\Delta_{R,(m,n)}$ stands for the operator

$$\Delta_{R,(m,n)} = D_\mu D^\mu \mathbf{1}_{(m,n)} - F^a_{\mu\nu} S^{\mu\nu}_{(m,n)} T^a_R \tag{2.4.6}$$

where $F^a_{\mu\nu}$ is the background tensor field strength associated to $A^a_{B\mu}$, $S^{\mu\nu}_{(m,n)}$ are the generators of the Lorentz group for the representation (m, n). Thus, assuming to have a guage theory with a Weyl fermion and a complex scalar both in the representation R of the gauge group, the effective action for the background field A^a_B at one loop is obtained from

$$\begin{aligned} e^{\Gamma(A^a_B)_{1\,loop}} &= \int \mathcal{D}\,\Phi e^{i\int d^4x \sum \Phi\, \Delta_{R,(m,n)} \Phi} \\ &= (\det \Delta_{R,(0,0)})^{-1} (\det \Delta_{R,(1/2,0)})^{1/2} (\det \Delta_{Adj,(1/2,1/2)})^{-1/2} (\det \Delta_{Adj,(0,0)})^{+1}. \end{aligned} \tag{2.4.7}$$

where the first term in the right-hand side of eq.(2.4.7) comes from the complex scalar, the second from the Weyl fermion, the third from the vector and the last one from the ghosts (Adj stands for adjoint representation). For a supersymmetric gauge theory with a vector supermultiplet $N = 1$ SUSY (which has a Weyl fermion and a vector) and n_c chiral supermultiplets $N = 1$ SUSY[2.7](each with a Weyl fermion and a complex scalar), eq. (2.4.7) implies that

$$\Gamma(A^a_B)_{1\,loop} = -n_c \ln(\det \Delta_{R,(0,0)}) + \frac{(n_c+1)}{2} \ln(\det \Delta_{R,(1/2,0)}) - \frac{1}{2} \ln(\det \Delta_{Adj,(1/2,1/2)}) + \ln(\det \Delta_{Adj,(0,0)}) \tag{2.4.8}$$

From eq.(2.4.6), by factorizing $D^\mu D_\mu \equiv D^2$ and by using the identity

$$\ln\det(1 + M) = \mathrm{Tr}\ln(1 + M) = \mathrm{Tr}(M) - \frac{1}{2}\mathrm{Tr}(M^2) + \frac{1}{3}\mathrm{Tr}(M^3) + \dots \tag{2.4.9}$$

where M is an operator, one has schematically

$$\begin{aligned} \ln\det(\Delta_{R,(m,n)}) &= \ln\det(-D^2)\,\mathrm{Tr}_{(m,n)}(\mathbf{1}) + O(F^2)\,\mathrm{Tr}_{(m,n)}(S^{\mu_1\nu_1} S^{\mu_2\nu_2}) \\ &\quad + O(F^3)\,\mathrm{Tr}_{(m,n)}(S^{\mu_1\nu_1} S^{\mu_2\nu_2} S^{\mu_3\nu_3}) \dots , \end{aligned} \tag{2.4.10}$$

where F^2 stands for the quadratic term $F_{\mu_1\nu_1} F_{\mu_2\nu_2}$, etc.. and we have explicitly written the operator D^2 only in the first term. Also, the symbol of trace over color indices has been omitted.
In eq.(2.4.10) the term linear in $F^{\mu\nu}$ is absent since the Lorentz generators are traceless $\mathrm{Tr}_{(m,n)} S^{\mu\nu} = 0$.
As for the first term in eq.(2.4.10) with no $F^{\mu\nu}$, one has that

$$\mathrm{Tr}_{(0,0)}(\mathbf{1}) = 1,\ \mathrm{Tr}_{(\frac{1}{2},0)}(\mathbf{1}) = 2,\ \mathrm{Tr}_{(\frac{1}{2},\frac{1}{2})}(\mathbf{1}) = 4, \tag{2.4.11}$$

As said above, in a renormalizable theory for an m point 1PI diagram the power of the loop momentum k is at most m.
In the Lagrangian the derivative ∂_μ inside D_μ acts on A^a_Q, while that inside $F^{\mu\nu}$ acts on A^a_B. It follows that D^2 contains the loop momentum k, while $F^{\mu\nu}$ contains only the external momenta. Thus the leading behavior in k is given by the first term of eq.(2.4.10). But after substituting eq. (2.4.10) in eq.(2.4.8), from (2.4.11) one has that the coefficient in front of this term is zero for every value of n_c. Therefore, for

[2.7] a vector supermultiplet $N = 2$ SUSY can be decomposed in a vector supermultiplet $N = 1$ plus a chiral supermultiplet, while a vector supermultiplet $N = 4$ is formed by a vector supermultiplet $N = 1$ and three chiral supermultiplet $N = 1$

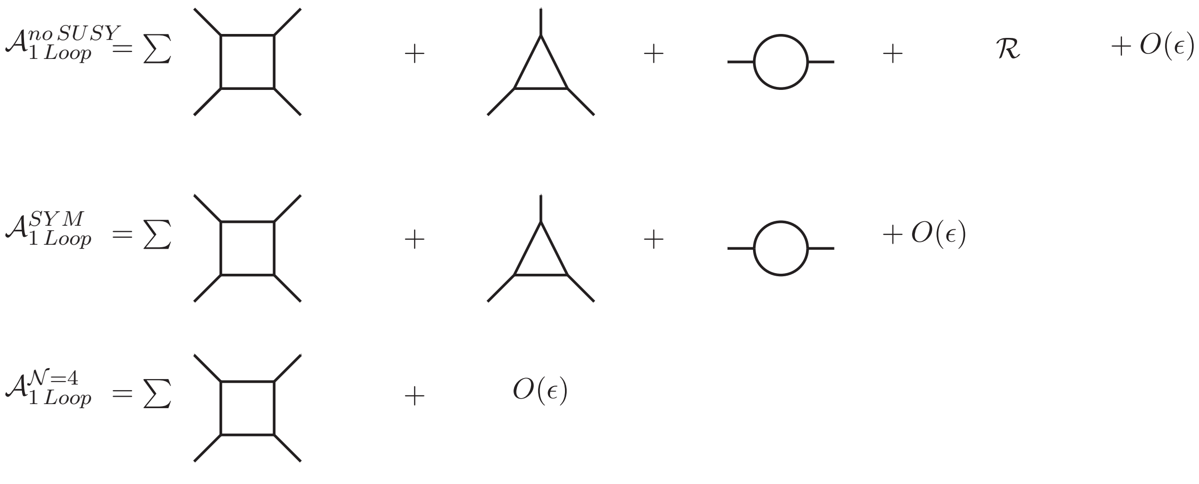
super Yang-Mills theories the leading term in k contains two $F^{\mu\nu}$. Each $F^{\mu\nu}$ reduces by one the number of powers of k in the numerator of the loop integrand. It follows that for the effective action for external gluons and hence for the amplitudes whose external gluons are all gluons, the maximum degree of k in the loop integrand is $m-2$. So these amplitudes verify the criterion (2.3.4) and have no a rational part in their decomposition, $\mathcal{R}=0$.
Up to now we have discussed only amplitudes with external gluons. In ref.[47] an argument is given that extends this result to any amplitude (with any external particle) in a generic supersymmetric gauge theory with a vector supermultiplet $\mathcal{N}=1$ coupled to n_c chiral supermultiplets $\mathcal{N}=1$ with no superpotential. This result is conjectured to be valid even in presence of a superpotential [47].
Moreover, on-shell amplitudes with particles belonging to the same supermultiplet are related by linear relations due to the supersymmetric Ward identities [61]-[65].
Let us consider the supermultiplet $\mathcal{N}=4$ SYM which is composed of a vector supermultiplet $\mathcal{N}=1$ and three chiral supermultiplets $\mathcal{N}=1$, i.e. it has one gluon, four Weyl fermions and six real (or three complex) scalars (see section 3.1). This combination of fields implies that in (2.4.8) the terms with traces of products of two $S^{\mu\nu}_{(1/2,0)}$ cancel those with two $S^{\mu\nu}_{(1/2,1/2)}$. In fact the trace of two $S^{\mu\nu}_{(1/2,0)}$ gives

$$\mathrm{Tr}_{(1/2,0)}\Big(S^{\mu_1\nu_1}S^{\mu_2\nu_2}\Big)=\frac{1}{2}\Big(\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}-\eta^{\mu_1\nu_2}\eta^{\nu_1\mu_2}+i\epsilon^{\mu_1\nu_1\mu_2\nu_2}\Big). \tag{2.4.12}$$

The Levi-Civita tensor can be neglected since gives a term of the form $F\widetilde{F}$. The trace of two $S^{\mu\nu}_{(1/2,1/2)}$ is

$$\mathrm{Tr}_{(1/2,1/2)}\Big(S^{\mu_1\nu_1}S^{\mu_2\nu_2}\Big)=2\Big(\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}-\eta^{\mu_1\nu_2}\eta^{\nu_1\mu_2}\Big). \tag{2.4.13}$$

By considering the coefficients appearing in eq.(2.4.8) with $n_c=3$, one obtains that terms with two $F^{\mu\nu}$ vanish. The same cancellation happens for terms with three $F^{\mu\nu}$ and hence the first non vanishing term has four $F^{\mu\nu}$.
Thus for one loop gluon amplitudes in $\mathcal{N}=4$ SYM the maximum degree of k in the loop integrand is $m-4$. This implies that the decomposition of these amplitudes contains only boxes, but neither triangles nor bubbles. Super Ward identities, which for the $\mathcal{N}=4$ theory have been solved [64]-[65], allow to extend this result to amplitudes containing also the other particles of the supermultiplet.
In ref.[66] it is shown that other gauge theories have gluon scattering amplitudes free of bubbles and triangles if the representation of the matter fields satisfy certain conditions.
Also, supergravity with $\mathcal{N}=8$ SUSY is believed to be closely related to the $\mathcal{N}=4$ theory and to have an S-matrix which is free of bubbles and triangles [67].
The discussion above is summarized by the picture below



2.5 Color decomposition

In the following, we will deal with $\mathcal{N}=4$ SYM whose fields belong to the adjoint representation of the gauge group $SU(N)$. In the case of amplitudes with particles in the adjoint representation, one can use group

theory to decompose them in color structures given by single traces or product of traces of the generators of the gauge group [68]-[71] .
The fact that the generators T^a of $SU(N)$ form a complete set of traceless hermitian $N \times N$ matrices implies the 'Fierz identities'

$$(T^a)_{i_1}^{\ j_1}(T^a)_{i_2}^{\ j_2} = \delta_{i_1}^{j_2}\delta_{i_2}^{j_1} - \frac{1}{N}\delta_{i_1}^{j_1}\delta_{i_2}^{j_2}. \tag{2.5.1}$$

As can be seen by contracting both the members of the previous equation by $\delta_{j_1}^{i_1}$, the $-\frac{1}{N}$ term guarantees the tracelessness condition.
By contracting appropriately both sides of eq.(2.5.1) with the matrix elements of two generic matrices X and Y, one has the two identities

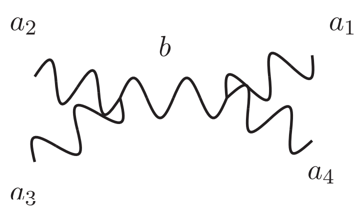
$$\begin{aligned} \mathrm{Tr}(T^a X)\mathrm{Tr}(T^a Y) &= \mathrm{Tr}(XY) - \frac{1}{N}\mathrm{Tr}(X)\mathrm{Tr}(Y) \\ \mathrm{Tr}(T^a X T^a Y) &= \mathrm{Tr}(X)\mathrm{Tr}(Y) - \frac{1}{N}\mathrm{Tr}(XY). \end{aligned} \tag{2.5.2}$$

In the following, we will be interested in the study of the large N limit, which is also called planar since in this limit the leading contribute is given by planar Feynman diagrams that can be drawn in a plane without self-intersections. Hence we will neglect the terms with $-\frac{1}{N}$ in the eqs.(2.5.2).

If one is dealing with a Feynman diagram with particles in the adjoint representation, the vertices provide products of structure constants of the gauge group f^{abc}, which using (1.1.17) can be written as

$$f^{abc} = -\frac{i}{\tau_{\mathcal{R}}}\mathrm{Tr}([T^a, T^b]T^c), \tag{2.5.3}$$

while the propagators give δ^{ab} factors which allow to contract the color indices coming from different vertices.
Let's consider a tree diagram like



After using eq.(2.5.3) to express the structure constants in terms of the generators T^a, one obtains products of traces of the generators, like

$$\mathrm{Tr}(T^{a_1}T^{a_2}T^b)\mathrm{Tr}(T^bT^{a_3}T^{a_4}) \tag{2.5.4}$$

where the index b is contracted.
In the case of tree diagrams, the contracted indices belong always to different traces and using eq.(2.5.2), one can reduce (2.5.4) to

$$\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}). \tag{2.5.5}$$

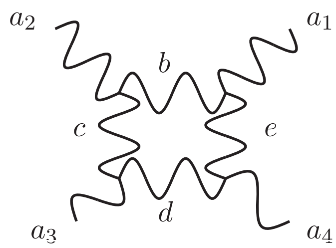
So any tree diagram with n external states in the adjoint can be reduced to a sum of single trace terms terms $\mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma}(2)}\dots T^{a_{\sigma}(n)})$, for some permutation σ of the n particles.
This leads to the following decomposition for an amplitude at tree level with n external particle in the adjoint

$$\mathcal{A}_n^{tree}(\{p_i, h_i, a_i\}) = g^{n-2}\sum_{\sigma\in S_n/Z_n}\mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}\dots T^{a_{\sigma(n)}})A_n^{tree}(\sigma(p_1,h_1),\dots,\sigma(p_n,h_n)). \tag{2.5.6}$$

In this equation $\mathcal{A}^{tree}$ represents the total amplitude of n particles with momenta $p_i, i = 1\dots n$, helicities h_i and adjoint indices a_i. The factors A_n^{tree} multiplying the traces are called partial or color-ordered amplitudes and contain the kinematic information. S_n is the set of all permutations of n objects, while Z_n is the subset of the cyclic permutations which leave invariant the trace and hence also the associated partial amplitudes. In fact the sum is over all the non-equivalent orderings of the n particles which are $(n-1)!$ and are denoted by $\sigma \in S_n/Z_n \equiv S_{n-1}$.

This decomposition is useful since the partial amplitudes receive contribution only from planar Feynman diagrams whose external legs follow the ordering of the color trace associated to the partial amplitude under consideration. Hence, the singularities of the partial amplitudes can occur only in kinematic invariants made out of squares of sums of cyclically adjacent momenta.
Moreover, since these color structures are independent, each color ordered partial amplitude has to be gauge-invariant.

The decomposition (2.5.10) of tree amplitudes is valid for each value of N [48]. At loop level, one has a similar decomposition, but it is valid only in the large N limit. For example, let's consider the diagram



The iterated use of the first of eqs.(2.5.2) finally gives two kind of terms depending on whether generators with contracted indices are next to each other inside a trace or are separated by other generators

$$\begin{aligned}&\mathrm{Tr}(T^{a_1}T^{a_2}T^{b}T^{b}T^{a_3}T^{a_4})\\&\mathrm{Tr}(T^{a_1}T^{a_2}T^{b}T^{a_3}T^{b}T^{a_4}).\end{aligned} \tag{2.5.7}$$

In the first case, since $\mathrm{Tr}(1) = N$, the second of eqs.(2.5.2) gives

$$N\,\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}), \tag{2.5.8}$$

while in the second case, it gives the product of two traces

$$\mathrm{Tr}(T^{a_1}T^{a_2})\mathrm{Tr}(T^{a_3}T^{a_4}). \tag{2.5.9}$$

This discussion can be generalized to L loops. As before, one can split traces or generate additional powers of N keeping the number of traces fixed. The terms with T traces at L loops will have an explicit coefficient N^{L+1-T}.
Hence in the large N limit, the leading term has single-trace color structures $N^L\mathrm{Tr}(T^{a_1}\dots T^{a_n})$, while the terms with two or more traces have a lower power in N. In the large N limit, at L loops the leading term can be written as

$$\mathcal{A}_n^{L\,loop}(\{p_i,h_i,a_i\})|_{planar} = g^{n-2}(g^2N)^L\sum_{\sigma\in S_n/Z_n}\mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma}(2)}\dots T^{a_{\sigma(n)}})A_n^L(\sigma(p_1,h_1),\dots,\sigma(p_n,h_n)). \tag{2.5.10}$$

't Hooft suggested that in the planar limit guage theories simplify considerably and can have a stringy description [72]. The Maldacena's conjecture gives a concrete realization of this idea [2].

Chapter 3

Off shell amplitudes in $\mathcal{N} = 4$ SYM

This chapter is devoted to the presentation of our results. In particular, we discuss our computation of the off shell planar amplitudes (or equivalently the Green's functions) with four external scalars and with three and four external gluons in the $\mathcal{N} = 4$ SYM theory. Explicit expressions for the three and four gluon amplitudes can be found in the appendix G.
Moreover, we discuss our results concerning the $n-$point correlation functions with $n \geq 4$ in the $\beta-$ deformation of $\mathcal{N} = 4$ SYM theory.
For completeness we also review some known results. In fact we discuss the two point function both for scalars and gluons and the three point function for scalars in $\mathcal{N} = 4$ SYM, which are all vanishing. We give a discussion of the off-shell planar four gluon amplitude at tree level whose explicit expression can be found in the appendix G.
Moreover, we discuss the on-shell limit for all the off-shell amplitudes. This chapter is organized as follows.
The first section treats the formulation of $\mathcal{N} = 4$ SYM in terms of $\mathcal{N} = 1$ superfields.
The second section deals with the dual conformal symmetry and the box scalar integral which is covariant under this symmetry.
In the third section we present our conventions and a brief description of the Maple program which we have developed to do the computations.
The forth section is devoted to the motivations and the summary of our results.
The fifth and the sixth sections respectively deal with the scalar and the gluon amplitudes.
In the seventh section we discuss the correlation functions in the $\beta-$ deformation of $\mathcal{N} = 4$ SYM.
In the last section there are our conclusions.
For the first section we have followed [73], while for the second section we have followed [76], [78].

3.1 Formulations of $\mathcal{N} = 4$ SYM with $\mathcal{N} = 1$ superfields

The action of $\mathcal{N} = 4$ super Yang-Mills theory in four dimension was first obtained by dimensional reduction of $\mathcal{N} = 1$ SYM theory in ten dimensions [84] and has the form (taken from [78])

$$\begin{aligned}\mathcal{L} =& \frac{1}{\tau_{\mathcal{R}}}\mathrm{Tr}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda_i\sigma^\mu D_\mu\bar{\lambda}^i - \frac{1}{2}D_\mu\phi_{ij}D^\mu\phi^{ij} \\ &+ ig\lambda_i[\lambda_j,\phi^{ij}] + ig\bar{\lambda}^i[\bar{\lambda}^j,\phi_{ij}] + \frac{g^2}{4}[\phi_{ij},\phi_{kl}][\phi^{ij},\phi^{kl}]\Big).\end{aligned} \tag{3.1.1}$$

This form of the lagrangian has no manifest supersymmetry, i.e. it cannot be expressed in terms of superfields. In fact, one refers to (3.1.1) as the $\mathcal{N} = 0$ formulation of the $\mathcal{N} = 4$ theory. One can also refer to (3.1.1) as the $\mathcal{N} = 0$ gauge since for the quantization one introduces a gauge-fixing term which is not supersymmetric.
One can verify that the lagrangian (3.1.1) is invariant under $\mathcal{N} = 4$ SUSY transformations. Because of not manifest SUSY, these transformation close only on-shell, i.e. by using the equations of motion, and are nonlinear in the fields.
Instead, (3.1.1) has manifest $R-$symmetry $SU(4)$. Indeed, the theory has a vector field A_μ, which is a singlet under $SU(4)$, four Weyl fermions $\lambda^i_\alpha (i = 1..4)$ which transform in the fundamental $\underline{4}$ of $SU(4)$ and six real

scalar fields $\phi_{ij} = -\phi_{ji}, (i = 1..4)$ transforming in the $\underline{6}$ of $SU(4)$. The $\underline{6}$ correspond to an antisymmetric rank two tensor ϕ_{ij} of $SU(4)$[3.1]. This representation is real since for an antisymmetric tensor ϕ_{ij}, one can define the following $SU(4)$ invariant reality condition

$$\phi^{\dagger}_{ij} = \phi^{ij} \equiv \frac{1}{2}\epsilon^{ijkl}\phi_{kl}. \tag{3.1.2}$$

Since all these fields belong to the adjoint representation of the gauge group $SU(N)$, one can express them with matrices as in (3.1.1) by contraction with the generators of the gauge group, i.e. $A_\mu = A^a_\mu T^a$,etc..
There are other formulations or gauges in which $\mathcal{N} = 4$ SYM theory can be studied. In our results we have used a formulation in terms of $\mathcal{N} = 1$ superfields. The field content of the theory is given by one $\mathcal{N} = 1$ vector superfield $V = V^a T^a$ and three $\mathcal{N} = 1$ chiral superfields $\Phi_i = \Phi^a_i T^a$, all in the adjoint representation of the gauge group.
In this formalism the six real scalars are grouped in three complex scalar fields ϕ_i which are the scalar components of the chiral superfields Φ_i. Three of the Weyl fermions belong to the Φ_i, while the fourth fermion belongs to the vector superfield V.
Thus, unlike the $\mathcal{N} = 0$ formulation where all the four fermions are in the same representation of $SU(4)$, in this formulation the fermions are no longer all explicitly related to each other and only the subgroup $SU(3) \times U(1)$ of the original $SU(4)$ symmetry is manifest. In the $\mathcal{N} = 1$ formulation , the remaining three supersymmetry transformations and the global $SU(4)/SU(3) \times U(1)$ are realized non linearly [20].
The representations of $SU(4)$ decompose in representations of $SU(3)$ according to $\underline{6} \to \underline{3} + \underline{3}^*$, $\underline{4} \to \underline{3} + \underline{1}$. The chiral superfields Φ^i transform in the $\underline{3}$ (and the antichiral ones $\Phi^{\dagger}_i$ in the $\underline{3}^*$), while the vector superfield V is a singlet of $SU(3)$.
The condition of having manifest $\mathcal{N} = 1$ SUSY and the superfield content leads to the lagrangian to have the form (1.1.29) where now the chiral superfields have a flavor index in the representation $\underline{3}$ of $SU(3)$. The only term which remains to be fixed is the superpotential

$$\mathcal{W} \approx \frac{1}{3!}\lambda^{ijk}_{abc}\Phi^a_i\Phi^b_j\Phi^c_k. \tag{3.1.3}$$

But this has to be invariant under the flavor group $SU(3)$ and under the gauge group and so the tensor λ^{ijk}_{abc} has to be a singlet under these to groups . The only singlet with three indices in the fundamental of $SU(3)$ is ϵ^{ijk} which is completely antisymmetric and so there has to be complete antisymmetry of the adjoint indices a, b, c as well. Since the only singlet under the gauge group with three indices in the adjoint and completely antisymmetric is f_{abc}, the superpotential has to be

$$\begin{aligned}\mathcal{W} &\approx \frac{1}{3!}\epsilon^{ijk}f_{abc}\Phi^a_i\Phi^b_j\Phi^c_k \\ &\equiv f_{abc}\Phi^a_1\Phi^b_2\Phi^c_3 \\ &\equiv -\frac{i}{\tau_{\mathcal{R}}}\mathrm{Tr}(\Phi_1, [\Phi_2, \Phi_3]).\end{aligned} \tag{3.1.4}$$

Hence, the action for the $\mathcal{N} = 4$ theory in the gauge $\mathcal{N} = 1$ has the form [19]-[22]

$$\begin{aligned}S = \frac{1}{\tau_{\mathcal{R}}}\mathrm{Tr}\Big[\int d^4x\Big(d^4\theta e^{-gV}\Phi^{\dagger}_i e^{gV}\Phi_i + \frac{1}{64g^2}\int d^2\theta W^\alpha W_\alpha + ig\int d^2\theta\Phi_1, [\Phi_2, \Phi_3] \\ + ig\int d^2\bar{\theta}\Phi^{\dagger}_1, [\Phi^{\dagger}_2, \Phi^{\dagger}_3] - \frac{1}{16\alpha}\int d^4\theta\mathcal{D}\mathcal{D}V\bar{\mathcal{D}}\bar{\mathcal{D}}V\Big)\Big],\end{aligned} \tag{3.1.5}$$

where we have included the gauge-fixing term but not the superghosts terms and $\mathcal{N} = 4$ SUSY imposes to have only one coupling constant g. $\mathcal{N} = 4$ SUSY also fixes the numerical factor in front of the superpotential in eq.(3.1.5).
Expanding this action up to order g^2, one can write

$$S = \int d^4x d^4\theta\Big(\frac{1}{2}V^a\partial^2 V^a + \Phi^{\dagger a}_i\Phi^a_i + igf_{a_1a_2a_3}\Phi^{\dagger a_1}_i V^{a_2}\Phi^{a_3}_i - \frac{g^2}{2}f_{a_1a_2a_3}f_{a_3b_2b_3}\Phi^{\dagger a_1}_i V^{a_2}V^{b_2}\Phi^{b_3}_i + O(g^3)\Big)$$

[3.1] in fact, the antisymmetric part of the product of two $\underline{4}$ is equivalent to a $\underline{6}$, i.e. $\underline{4} \otimes \underline{4}|_{anti} \approx \underline{6}$

$$- g f_{a_1 a_2 a_3} \int d^4x d^2\theta \, \Phi_1^{a_1} \Phi_2^{a_2} \Phi_3^{a_3} - g f_{a_1 a_2 a_3} \int d^4x d^2\bar{\theta} \, \Phi_1^{\dagger a_1} \Phi_2^{\dagger a_2} \Phi_3^{\dagger a_3} \tag{3.1.6}$$

If one integrates eq.(3.1.5) over the θ variables, eliminate the auxiliary fields by the equations of motion and chooses the Wess-Zumino gauge, one obtains a component field formulation with only symmetry $SU(3)\times U(1)$ manifest. This formulation can be seen to be perfectly equivalent to eq.(3.1.1)[3.2].
There exists also a manifestly $\mathcal{N}=2$ formulation which uses harmonic superspace [74] and an infinite number of auxiliary fields. On the contrary, a formulation with manifest $\mathcal{N}=4$ is not available. It is important to remark that even if the $\mathcal{N}=4$ theory is finite, in a non-supersymmetric gauge such as that of eq.(3.1.1), the gauge dependent propagators do get divergent corrections. On the contrary the propagators have no or only finite corrections in a supersymmetric formalism [73].
Only the β function, which is gauge independent, vanishes in all the gauges.

3.2 Conformal integrals

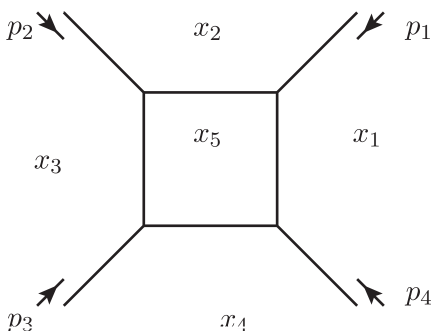
The $\mathcal{N}=4$ theory is conformal invariant even at the quantum level since its β-function vanishes to all orders of perturbation theory. The conformal properties are manifest when one considers gauge independent quantities such as correlation functions of gauge invariant composite operators.
It has been observed [98]-[101] that the four-gluon planar amplitude in $\mathcal{N}=4$ theory has another kind of conformal symmetry called dual since it acts in momentum space [75]-[78]. This hidden symmetry is not related, at least not in an obvious way, to the conventional conformal symmetry of $\mathcal{N}=4$ SYM theory.
The four gluon planar amplitude has been calculated with the generalized unitarity method which employs the dimensional regularization scheme to regularize the infrared divergences and assumes that the external legs are on shell, $p_i^2=0$.
The integrals appearing in the on-shell dimensional regularized amplitude up to four loops have a special property. In fact if one puts their external legs off-shell and keeps the dimension of space-time equal to four, these integrals are finite and covariant under dual conformal symmetry.
Let's consider the one-loop amplitude. In this case, there is the one-loop scalar box integral

$$D_0(p_1,p_2,p_3) = \int \frac{d^4k}{k^2(k+p_1)^2(k+p_1+p_2)^2(k+p_1+p_2+p_3)^2}, \tag{3.2.1}$$

where we have followed the notation of section 2.1[3.3] and the space-time dimension is equal to four. To study its conformal properties, one has to pass to the dual variables x_i

$$p_1 = x_{12}, \quad p_2 = x_{23}, \quad p_3 = x_{34}, \quad p_4 = x_{41}, \quad k = x_{51}, \tag{3.2.2}$$

where $x_{ij} = x_i - x_j$.
This choice of variables, which are not related to the coordinates of the original coordinate space, automatically satisfies the constraint given by the conservation of momentum $\sum_i p_i = 0$.
Also, one can introduce a dual diagram by associating a dual coordinate to each face delimited by the lines of the original diagram



[3.2] If one only integrates eq.(3.1.5) over the θ variables, one obtains an action in which all the component fields of the superfields appear (even F, C, S, D, etc.. see eqs.(1.1.11)-(1.1.13). Obviously this action is completely equivalent to eq.(3.1.5) [73].
[3.3] One can find the on-shell dimensional regularized version of the box scalar integral with infrared poles for example in [47], while its off-shell finite version in [31]-[32].

By using these variables, the integral (3.2.1) becomes

$$D_0(x_1,x_2,x_3,x_4)=\int\frac{d^4x_5}{x_{15}^2x_{25}^2x_{35}^2x_{45}^2}. \tag{3.2.3}$$

It is manifestly invariant under translations and rotations of the x coordinate. To check the covariance under special conformal transformation, it is convenient to introduce the conformal inversion operator I

$$I: x^\mu\to\frac{x^\mu}{x^2}. \tag{3.2.4}$$

In fact, a special conformal transformation is given by composing an inversion, a translation and another inversion. Under I, one has

$$I:\ \frac{1}{x_{ij}^2}\to\frac{x_i^2x_j^2}{x_{ij}^2},\ d^4x\to\frac{d^4x}{(x^2)^4} \tag{3.2.5}$$

Hence, the integral (3.2.3) is covariant under inversion only if the space-time dimension is equal to four

$$I:\ D_0\to\int\frac{d^4x_5}{(x_5^2)^4}\frac{(x_1^2x_5^2)(x_2^2x_5^2)(x_3^2x_5^2)(x_4^2x_5^2)}{x_{15}^2x_{25}^2x_{35}^2x_{45}^2}=x_1^2x_2^2x_3^2x_4^2\,D_0. \tag{3.2.6}$$

In fact, in four dimension, the factors x_5^2 provided by the propagators are canceled by those provided by the measure. Thus, to be unbroken dual conformal symmetry requires that the space-time dimension is kept equal to four. On the other hand, if one regularizes the infrared divergences of the one loop amplitude with dimensional regularization, which shifts the dimension of the space-time from four to $D=4-2\epsilon_{IR}$ (with $\epsilon_{IR}<0$), then dual conformal symmetry is broken.
On the contrary, dual conformal symmetry is preserved, if one uses the off-shell regularization[3.4], i.e. if one lets the external particles to have $p^2\neq 0$.

Moreover, one can also say that under inversion D_0 transforms homogeneously with weight +2 for each of the coordinates x_i. Under dilatations D_0 transforms homogeneously as well

$$x^\mu\to\lambda x^\mu:\ D_0\to\lambda^{-4}D_0. \tag{3.2.7}$$

In the four-gluon amplitude at one loop, D_0 is multiplied by $(p_1+p_2)^2(p_2+p_3)^2$. Expressing this product with dual variables x_i, one obtains

$$\mathcal{M}_{1\,loop}(x_1,x_2,x_3,x_4)=\int d^4x_5\frac{x_{13}^2x_{24}^2}{x_{15}^2x_{25}^2x_{35}^2x_{45}^2} \tag{3.2.8}$$

which is invariant under all the generators of the dual conformal symmetry.
The covariance of D_0 under conformal transformation implies that it can be expressed as a conformally covariant factor multiplied by a function $\Phi^{(1)}(s,t)$ [3.5] of the conformally invariant cross-ratios s and t

$$D_0=\frac{1}{x_{13}^2x_{24}^2}\Phi^{(1)}(s,t), \tag{3.2.9}$$

where

$$s=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\ t=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2} \tag{3.2.10}$$

and the function $\Phi^{(1)}(s,t)$ is expressed in terms of logarithms and dilogarithms [3.6] [31]-[32]

$$\Phi^{(1)}(s,t)=\frac{1}{\lambda}\Big[2\mathrm{Li}_2(-\rho s)+2\mathrm{Li}_2(-\rho t)+\ln\frac{t}{s}\ln\frac{1+\rho t}{1+\rho s}+\ln(\rho s)\ln(\rho t)+\frac{\pi^2}{3}\Big], \tag{3.2.11}$$

[3.4] In this off-shell regularization, the internal propagators $\frac{1}{k^2}$ remain massless. One could have a different IR regulator by replacing the massless internal propagators with massive propagators $\frac{1}{k^2+m^2}$, but in this case dual conformal symmetry would be broken.

[3.5] the superscript (1) refers to the fact that the one loop scalar box (or ladder) integral is the first of the series of L loop ladder integrals whose expression with off-shell external legs is known [31]-[32],[79].

[3.6] this formula is valid for $s,t>0$. See [103] for a discussion of the analytic continuation where

$$\lambda(s,t) = \sqrt{(1-s-t)^2 - 4st}, \qquad \rho(s,t) = 2(1-s-t+\lambda)^{-1}. \tag{3.2.12}$$

The same function $\Phi^{(1)}(s,t)$ appears in the triangle scalar integral C_0. In fact due to conformal covariance one can multiply (3.2.9) by x_{14}^2 and take the limit $x_4 \to \infty$. Thus, one gets

$$C_0(x_1,x_2,x_3) \equiv \int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2} = \lim_{x_4\to\infty} x_{14}^2 D_0(x_1,x_2,x_3,x_4) = \frac{1}{x_{13}^2}\Phi^{(1)}(\hat{s},\hat{t}) \tag{3.2.13}$$

where $\hat{s}$ and $\hat{t}$ are the limit for $x_4 \to \infty$ of s and t respectively

$$\hat{s} = \frac{x_{12}^2}{x_{13}^2}, \ \hat{t} = \frac{x_{23}^2}{x_{13}^2} \tag{3.2.14}$$

Triangle scalar integrals (as well as bubbles) are not conformal integrals.
The infrared limit corresponds to set $p_i^2 \equiv x_{ii+1}^2 \equiv m^2$ and take $m^2 \to 0$. In this limit $s \to 0$, $t \to 0$ and $\Phi^{(1)}(s,t)$ behaves like $(\ln(m^2))^2$. Thus, in the infrared limit triangles and box scalar integrals have a logarithmic divergence, as expected (see section (2.1.1)). Finally, it is important to note that dual conformal invariance is a property of planar amplitudes only. In fact, a non-planar box diagram expressed in terms of the dual variables (3.2.2) doesn't transform homogeneously under the conformal inversion I.

3.3 Conventions and sketch of the computation

- In the following, we will consider only **planar** amplitudes, that is only the part of the total amplitudes containing single trace terms $\mathbf{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$.
- We recall that we use the following normalization for the generators of the gauge group

$$\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}, \tag{3.3.1}$$

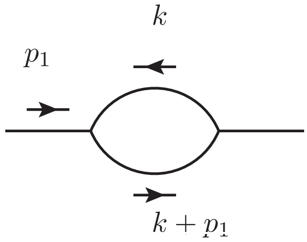
  i.e. $\tau_{\mathcal{R}} = \frac{1}{2}$ in eqs.(1.1.17).

- The computations are done with super Feynman diagrams in the $\mathcal{N}=1$ formulation of the $\mathcal{N}=4$ SYM theory. In particular we have employed the supersymmetric Fermi-Feynman gauge, i.e. $\alpha = 1$ in eq.(1.3.69), since for $\alpha \neq 1$ one has infrared divergences even in the off-shell regime (see section 1.3.2) .
- All the **external momenta** are assumed to be **incoming**.
- Moreover, the complex scalars belong to the fundamental representation $\underline{3}$ of $SU(3)$, thus they can have three flavors. For simplicity, in the amplitudes with external **scalars** we will always assume that they have **the same flavor**[3.7] and so will omit the flavor index
- We have computed Green's functions (or correlation functions or n-point functions). The corresponding off-shell amplitudes are obtained simply by neglecting (amputating) the external propagators. For example the relation between the Green's function with four scalars and the corresponding off shell amplitude is

$$<\phi^{a_1}(p_1)\,\phi^{\dagger\,a_2}(p_2)\,\phi^{a_3}(p_3)\,\phi^{\dagger\,a_4}(p_4)>= \Big(\frac{i}{p_1^2}\frac{i}{p_2^2}\frac{i}{p_3^2}\frac{i}{p_4^2}\Big) \times \mathcal{A}^{off\ shell}\Big(\phi^{a_1}(p_1)\,\phi^{\dagger\,a_2}(p_2)\,\phi^{a_3}(p_3)\,\phi^{\dagger\,a_4}(p_4)\Big). \tag{3.3.2}$$

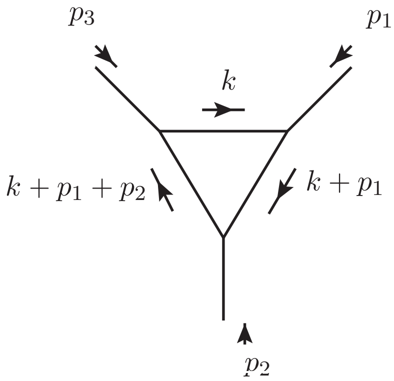
For the scalar integral B_0, C_0, D_0, we will follow the notation of eq.(2.1.6). Here, we give their definitions and their properties under the dual conformal symmetry when the external legs are **off-shell**, i.e. $p^2 \neq 0$.

[3.7] The only exception is given by the three point function



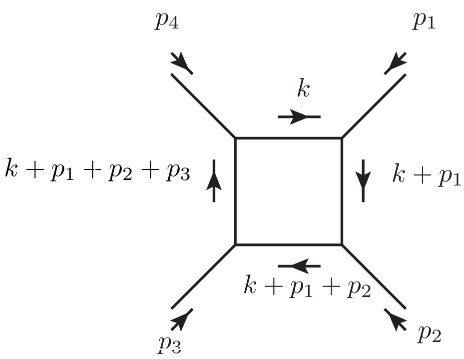
$$B_0(p_1) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 (k+p_1)^2} \tag{3.3.3}$$

The bubble scalar integral B_0 is **divergent** in the ultraviolet and is **not** covariant under the dual conformal symmetry.



$$C_0(p_1, p_2) = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 (k+p_1)^2 (k+p_1+p_2)^2}. \tag{3.3.4}$$

The triangle scalar integral C_0 is finite in the ultraviolet and is **not** covariant under the dual conformal symmetry.



$$D_0(p_1, p_2, p_3) = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 (k+p_1)^2 (k+p_1+p_2)^2 (k+p_1+p_2+p_3)^2} \tag{3.3.5}$$

The box scalar integral D_0 is finite in the ultraviolet and is covariant under the dual conformal symmetry.

sketch of the computation

In order to perform the computation we developed a Maple program which is based on the conventional super Feynman rules (see sections 1.3 and 1.4) and carries out the following operations

- it makes Wick contractions,

- computes color factors and extracts the planar terms,
- integrates over the θ variables of the superspace with the chiral propagator as in eq.(1.3.40)
- computes traces of σ matrices and products of $Levi-Civita$ tensors $\epsilon^{\mu_1\mu_2\mu_3\mu_4}$,
- performs the Passarino-Veltman reduction.

For a check of the correctness of the results, we have computed the irreducible diagrams of the amplitude with four scalars also with the improved super Feynman rules and the $\mathcal{D}-$algebra (see sections 1.2 and 1.5).

3.4 Motivations and summary of the results

In this chapter we have computed the off shell planar amplitudes (or equivalently the Green's functions) with four external scalars, and with three and four external gluons in the $\mathcal{N}=4$ SYM theory.
The reasons for these computations are the following

- There has been considerable effort motivated by the AdS-CFT correspondence to compute the correlation functions of gauge invariant composite operators in the $\mathcal{N}=4$ SYM theory , while correlation functions of elementary fields have received much less attention. In [17] one can find the off-shell four gluon amplitude computed in the background field gauge, but other four point functions are not known.

- The dimensional regularization breaks dual conformal symmetry (see section (3.2)). To preserve this symmetry one has to keep the number of space-time dimensions equal to four. To do so, one can use off shell regularization to regularize IR divergences even if one looses manifest gauge invariance.[3.8]

  In the off shell four gluon amplitude computed in [17] with the background field method, only the box scalar integral appears. This scalar integral is covariant under dual conformal symmetry (see section (3.2)) and so dual conformal symmetry is present even in the off shell regime in this gauge.
  Hence, it is important to know if in a different (supersymmetric) gauge, the decomposition of an off shell four point amplitude gives only a box scalar integral or also triangle and bubble scalar integrals, which lack this symmetry.
  In our computation in a $\mathcal{N}=1$ supersymmetric gauge, we have found that the off shell four scalar amplitude $\mathcal{A}^{off\,shell}_{1\,loop}\left(\phi\phi^\dagger\phi\phi^\dagger\right)$ is built out of a box integral as well as triangles, while the off shell four gluon amplitude $\mathcal{A}^{off\,shell}_{1\,loop}\left(A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}\right)$ is built out of a box as well as triangles and bubbles.
  Therefore, in the decomposition of these amplitudes appear integrals which are not dual conformal covariant. In other words, the presence of the dual conformal symmetry for the off shell amplitudes depends on the choice of the gauge.

- While triangle and box are finite in the ultraviolet, bubbles are UV divergent integrals. We have found that both the off-shell one loop amplitude with three gluons and that with four gluons have a decomposition containing bubbles. In spite of the presence of these UV divergent integrals, these amplitudes are finite since the sum of all the divergent terms arising from the bubbles vanishes.

- In the off-shell regularization, the regulator is obtained by giving a (small) mass m to the external particles. To remove this regulator one has to set $p_i^2=m^2$ and take $m^2\to 0$. If one removes the regulator in the expression of observable quantities, such as cross sections, all the divergent terms cancel out and the result is finite.
  On the other hand, removing the regulator in the amplitudes, which are not observables, produces

[3.8] In [109], another kind of regularization, called Higgs or massive regularization, is introduced. It regularizes IR divergences by giving an expectation value to some of the scalar fields. Even if this regularization allows to work in four dimensions, it breaks dual conformal symmetry. But one can 'deform' the generators of the symmetry in such a way that Higgs regularization preserves this extended version of the dual conformal symmetry

a divergent result. In the following, we will consider the 'on shell limit' of the off shell amplitude $\lim_{p^2\to 0}\mathcal{A}^{off\,shell}$, that is those parts of the off shell amplitude which don't vanish (but are finite or divergent) when one removes the regulator ($p^2 \to 0$). One can compare this on-shell limit of the off shell amplitude with the on-shell dimensional regularized version of the amplitude $\mathcal{A}^{on\,shell}_{dim.\,reg.}$ where the on-shell condition $p^2 = 0$ is imposed from the beginning. It is important to stress that in $\lim_{p^2\to 0}\mathcal{A}^{off\,shell}$ the integrals are off-shell regularized, while in $\mathcal{A}^{on\,shell}_{dim.\,reg.}$ they are dimensionally regularized.
It may happen that this on-shell limit $\lim_{p^2\to 0}\mathcal{A}^{off\,shell}$ contains more terms with respect to $\mathcal{A}^{on\,shell}_{dim.\,reg.}$. In fact, in the off-shell amplitudes there could be contributions such as

$$p^2 I(p), \tag{3.4.1}$$

where $I(p)$ is some term. These contributions cannot be present if one imposes from the beginning the on shell condition $p^2 = 0$, as happens in the generalized unitarity method which has been employed in the computation of the four gluon on shell amplitude up to four loops [98]-[101].
If in the on shell limit $p^2 \to 0$, the term $I(p)$ behaves like $\frac{1}{p^2}$ then contributions like (3.4.1) give a non vanishing result. Hence, in this case one has that $\lim_{p^2\to 0}\mathcal{A}^{off\,shell}$ doesn't match $\mathcal{A}^{on\,shell}_{dim.\,reg.}$, i.e.

$$\lim_{p^2\to 0}\mathcal{A}^{off\,shell} \not\equiv \mathcal{A}^{on\,shell}_{dim.\,reg.}. \tag{3.4.2}$$

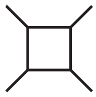
In our case, this is due to the form factors of the Passarino-Veltman decomposition which are different depending on whether $p^2 = 0$ or $p^2 \neq 0$ (see eqs.2.2.13). Therefore, it is important to compute the off shell amplitude and then make the on shell limit.
We have found that for the four scalar and four gluon off shell amplitudes, in the on shell limit terms like (3.4.1) are all vanishing and hence the on-shell limit of the off-shell amplitudes matches the on-shell dimensional regularized version of the amplitudes, i.e.

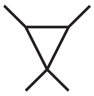
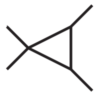
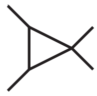
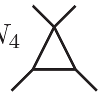
$$\begin{aligned}
&\lim_{p^2\to 0}\mathcal{A}^{off\,shell}_{1\,loop}\left(\phi\phi^\dagger\phi\phi^\dagger\right) \equiv \mathcal{A}^{on\,shell}_{dim.\,reg.}\left(\phi\phi^\dagger\phi\phi^\dagger\right),\\
&\lim_{p^2\to 0}\mathcal{A}^{off\,shell}_{1\,loop}\left(A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}\right) \equiv \mathcal{A}^{on\,shell}_{dim.\,reg.}\left(A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}\right)
\end{aligned} \tag{3.4.3}$$

- We have also studied the so called β deformation of $\mathcal{N} = 4$ SYM. It is a theory obtained by modifying the superpotential of $\mathcal{N} = 4$ SYM in such a way to break SUSY down to $\mathcal{N} = 1$ but maintaining the property of conformal invariance and finiteness. The superpotential of the β deformation depends on two complex parameters, i.e. h and β which gives the name of the theory.
In [96], it has been shown that in the planar limit and with β real, all the amplitudes of the β deformation coincide with the corresponding amplitudes of $\mathcal{N} = 4$ up to phase factors. In [94] three point functions of elementary (super)fields have been studied in the case of complex β. It has been observed that the three point functions involving vector superfields are equal to their value in $\mathcal{N} = 4$ SYM up to two loops.
We have studied some $n-$point correlation functions with $n \geq 4$ (or equivalently off shell amplitudes) in the case of complex β. In particular, we have considered the correlation functions with four and six vector superfields $< V^{a_1}V^{a_2}V^{a_3}V^{a_4} >$ and $< V^{a_1}V^{a_2}V^{a_3}V^{a_4}V^{a_5}V^{a_6} >$ respectively. We have found that at two loops they are different from to their value in $\mathcal{N} = 4$ SYM, since they receive non planar corrections.
We have also considered the 'mixed' chiral-vector correlation functions with a chiral, an antichiral and two or three vector superfields $< \Phi^{a_1}\Phi^{\dagger\,a_2}V^{a_3}V^{a_4} >$ and $< \Phi^{a_1}\Phi^{\dagger\,a_2}V^{a_3}V^{a_4}V^{a_5} >$ respectively (the flavor indices are omitted). Already at one loop they receive a non planar correction with respect to the corresponding correlation functions of $\mathcal{N} = 4$ SYM.

Summary of the results in $\mathcal{N} = 4$ SYM

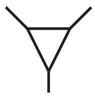
The decomposition in scalar integrals of the off shell amplitude with four scalars contains a box and triangles as well.

$$\mathcal{A}^{offshell}_{1loop}(\phi\phi^\dagger\phi\phi^\dagger) = Q$$ 

$$+ N_1$$  $$+ N_2$$  $$+ N_3$$  $$+ N_4$$ 

All the coefficients Q, N_1, N_2, N_3, N_4 multiplying the scalar integrals are functions of the external momenta. In the on shell limit all the terms with triangles vanish. In fact, taking $p_i^2 = m^2$ and considering the limit $m^2 \to 0$, the scalar triangles C_0 behave like $C_0 \approx (\ln(m^2))^2$ (see sections 2.1.1, 3.2), while the coefficients N_i like $N_i \approx m^2$. Hence, the product $N_i C_0$ vanishes in the on shell limit.

As for the off shell amplitude with three gluons, its decomposition contains a triangle and bubbles as well

$$\mathcal{A}^{offshell}_{1\,loop}(A_{\mu_1}A_{\mu_2}A_{\mu_3}) = N_{\mu_1\mu_2\mu_3}$$ 

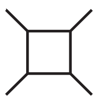
$$+ M_{1,\mu_1\mu_2\mu_3}$$  $$+ M_{2,\mu_1\mu_2\mu_3}$$  $$+ M_{3,\mu_1\mu_2\mu_3}$$ 

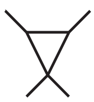
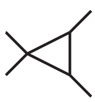
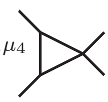
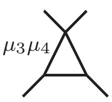
All the coefficients $N_{\mu_1\mu_2\mu_3}, M_{1,\mu_1\mu_2\mu_3}, M_{2,\mu_1\mu_2\mu_3}, M_{3,\mu_1\mu_2\mu_3}$ multiplying the scalar integrals are functions of the external momenta.
Even if the bubble scalar integrals are divergent in UV, the amplitude remains UV finite since the sum of the coefficients multiplying the bubbles is zero

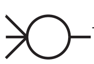
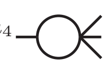
$$M_{1,\mu_1\mu_2\mu_3} + M_{2,\mu_1\mu_2\mu_3} + M_{3,\mu_1\mu_2\mu_3} = 0, \tag{3.4.4}$$

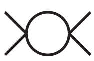
and hence the sum of all the UV divergent terms vanishes. On shell all the terms with both bubbles and triangle vanish.

The decomposition of the off shell amplitude with four gluons contains a box as well as triangles and bubbles

$$\mathcal{A}^{offshell}_{1\,loop}(A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}) = Q_{\mu_1\mu_2\mu_3\mu_4}$$ 

$$+ N_{1,\mu_1\mu_2\mu_3\mu_4}$$  $$+ N_{2,\mu_1\mu_2\mu_3\mu_4}$$  $$+ N_{3,\mu_1\mu_2\mu_3\mu_4}$$  $$+ N_{4,\mu_1\mu_2\mu_3\mu_4}$$ 

$$+ M_{1,\mu_1\mu_2\mu_3\mu_4}$$  $$+ M_{2,\mu_1\mu_2\mu_3\mu_4}$$  $$+ M_{3,\mu_1\mu_2\mu_3\mu_4}$$  $$+ M_{4,\mu_1\mu_2\mu_3\mu_4}$$ 

$$+ M_{5,\mu_1\mu_2\mu_3\mu_4}$$  $$+ M_{6,\mu_1\mu_2\mu_3\mu_4}$$ 

All the coefficients $Q_{\mu_1\mu_2\mu_3\mu_4}, N_{1,\mu_1\mu_2\mu_3\mu_4}, \ldots N_{4,\mu_1\mu_2\mu_3\mu_4}, M_{1,\mu_1\mu_2\mu_3\mu_4}, \ldots M_{6,\mu_1\mu_2\mu_3\mu_4}$ multiplying the scalar integrals are functions of the external momenta.

As in the case of the amplitude with three gluons, the finiteness in the off-shell regime is guaranteed by the vanishing of the sum of all the coefficients multiplying the bubbles

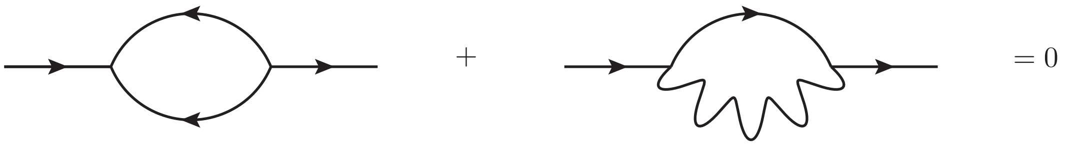
$$\sum_{i=1..6} M_{i,\mu_1\mu_2\mu_3\mu_4} = 0. \tag{3.4.5}$$

On shell the terms with bubbles and triangles vanish and only the box remains. Hence, as in the case of four scalars, in the on-shell limit no new integrals appear with respect to the dimensional regularization scheme, where the on-shell condition is applied from the beginning.
Thus, in this gauge which preserves $\mathcal{N}=1$ SUSY, the off-shell amplitudes do contain non-conformal integrals, but in the on-shell limit they vanish.

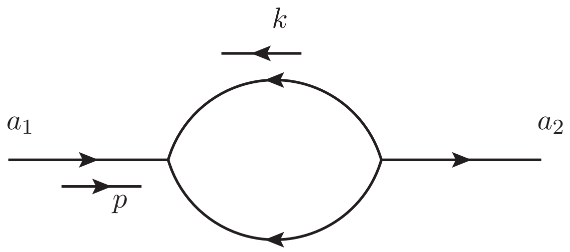
3.5 Amplitudes with scalars

3.5.1 The two-point function at one loop

The two point function at one loop is zero [19]



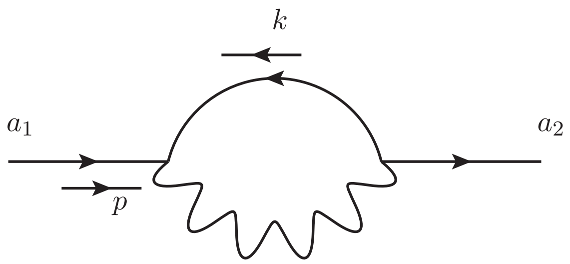
In fact, the first superdiagram



gives

$$N\mathrm{Tr}(T^{a_1}T^{a_2})\left(\frac{i}{p}\right)^2 (2g^2p^2)\, B_0(p). \tag{3.5.1}$$

The second superdiagram



gives

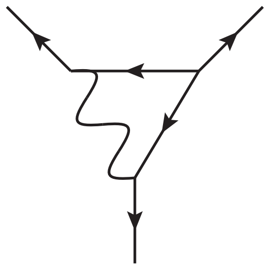
$$N\mathrm{Tr}(T^{a_1}T^{a_2})\left(\frac{i}{p}\right)^2 (-2g^2p^2)\, B_0(p). \tag{3.5.2}$$

Hence,

$$< \phi^{\dagger\, a_1}(p)\, \phi^{a_2}(-p) >_{1\, Loop} = 0 \tag{3.5.3}$$

3.5.2 The three-point function at one loop

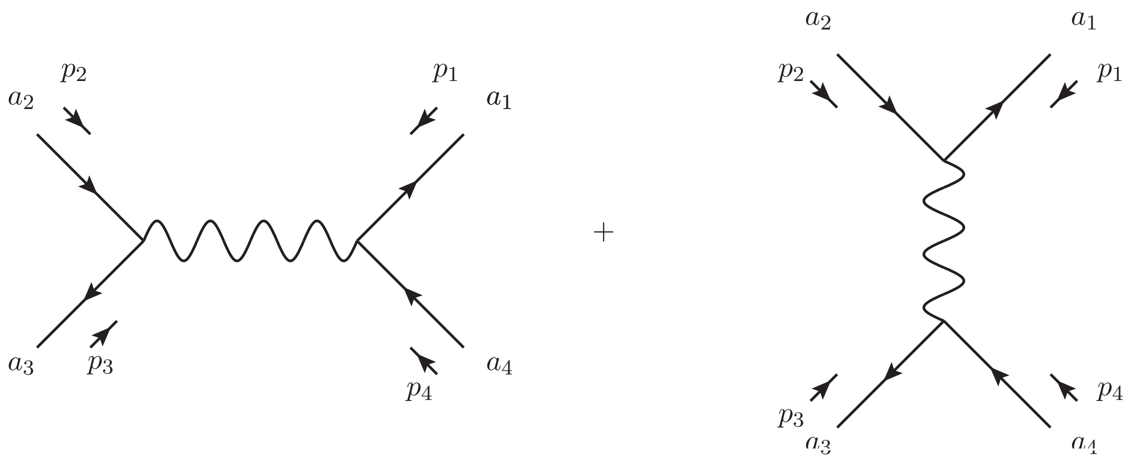
At one loop, the three point function $< \phi_1^{a_1}(p_1)\, \phi_2^{a_2}(p_2)\, \phi_3^{a_3}(p_3) >_{1\,Loop}$[3.9] gives zero. In fact its only superdiagram



vanishes.

3.5.3 Four scalar planar amplitude at tree level

At tree level, the planar four-point function $< \phi^{a_1}(p_1)\, \phi^{\dagger\, a_2}(p_2)\, \phi^{a_3}(p_3)\, \phi^{\dagger\, a_4}(p_4) >_{tree}^{planar}$ is given by the two superdiagrams



One can impose the conservation of momenta $\sum_i p_i = 0$ to express one momentum in favor of the others i.e $p_2 = -p_1 - p_3 - p_4$. The result for the planar off-shell amplitude is

$$\mathcal{A}_{tree}^{off\,shell}\Big(\phi^{a_1}(p_1)\, \phi^{\dagger\, a_2}(p_2)\, \phi^{a_3}(p_3)\, \phi^{\dagger\, a_4}(p_4)\Big) = g^2 \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\times \\ 2i\Big(\frac{(p_1+p_3)^2}{(p_1+p_4)^2} + \frac{(p_1+p_3)^2}{(p_3+p_4)^2}\Big) \tag{3.5.4}$$

On shell one can impose further kinematic conditions

$$\begin{aligned} &p_1^2 = p_3^2 = p_4^2 = 0, \\ &p_2^2 = (p_1+p_3+p_4)^2 = 0 \Rightarrow p_1\cdot p_3 = -p_1\cdot p_4 - p_3\cdot p_4 \end{aligned} \tag{3.5.5}$$

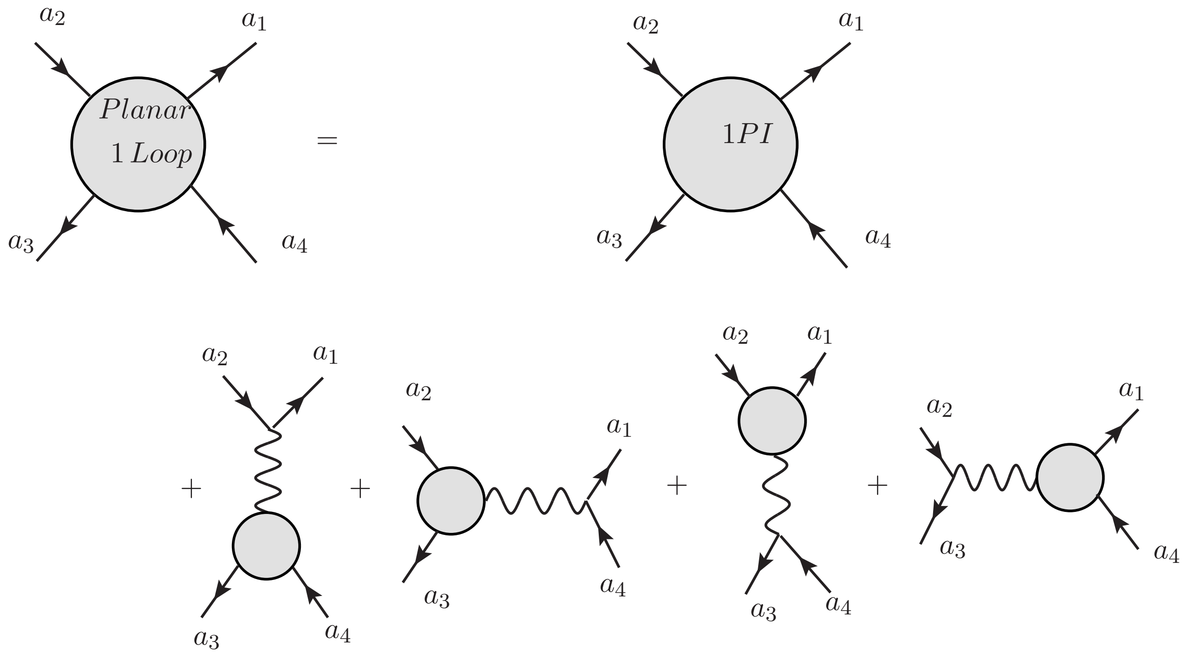
Hence, the on-shell amplitude is given by

$$\begin{aligned} \mathcal{A}_{tree}^{on\,shell}\Big(\phi^{a_1}\,\phi^{\dagger\, a_2}\,\phi^{a_3}\,\phi^{\dagger\, a_4}\Big) &= -2i\, g^2 \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) \times \frac{(p_1\cdot p_3)^2}{p_1\cdot p_4\, p_3\cdot p_4} \\ &\equiv -2i\, g^2 \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) \times \frac{(p_1\cdot p_4 + p_3\cdot p_4)^2}{p_1\cdot p_4\, p_3\cdot p_4} \end{aligned} \tag{3.5.6}$$

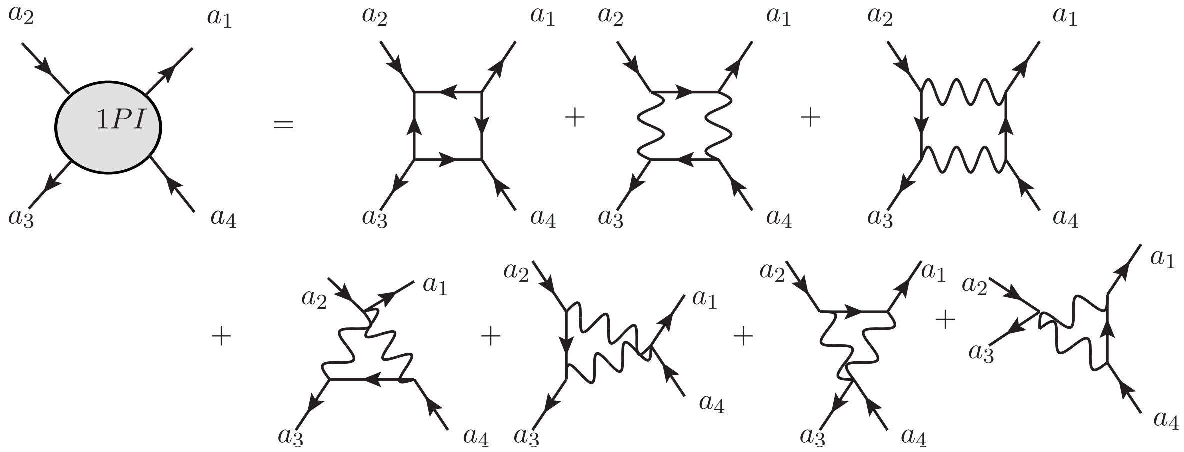
[3.9] In this case the flavor indices have to be all different. In fact, because of the $R-$symmetry $SU(3)$, the correlation functions have to be singlet under $SU(3)$. The complex scalars ϕ_i^a belong to the fundamental $\underline{3}$ of $SU(3)$. Since the only singlet with three indices in the fundamental of $SU(3)$ is ϵ_{ijk} which is completely antisymmetric, all the flavor indices have to be different.

3.5.4 Four scalar planar amplitude at one loop

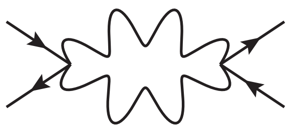
At one loop, the planar four-point function is given by one-particle irreducible and reducible superdiagrams



The 1PI part is composed by the superdiagrams



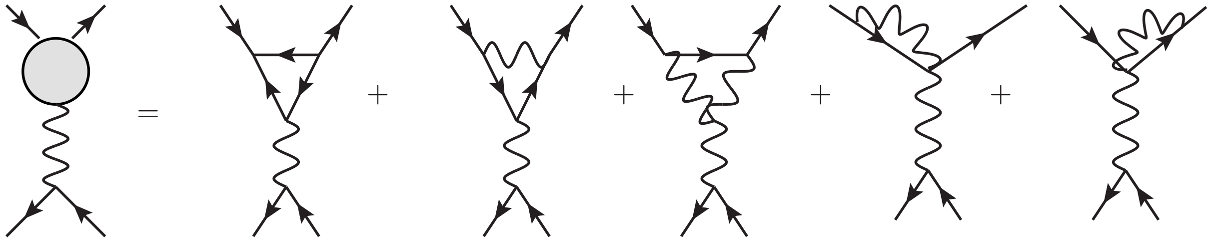
The superdiagram



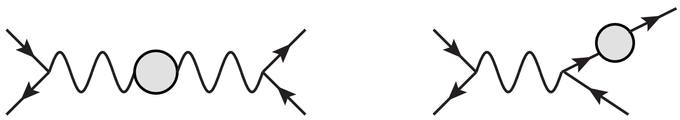
vanishes, since involves the product of two propagators of the vector superfield and so the product of two fermionic delta functions $\delta_{12}\delta_{12}$ which is zero. The 1PI part of the four-point function is given by the expression

$$
\begin{aligned}
&< \phi^{a_1}\,\phi^{\dagger\,a_2}\,\phi^{a_3}\,\phi^{\dagger\,a_4} >^{planar}_{1PI} = Ng^4 \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\Big(\frac{i}{p_1^2}\frac{i}{p_2^2}\frac{i}{p_3^2}\frac{i}{p_4^2}\Big)\times \\
&\Big(4p_1\cdot p_3\, p_2\cdot p_4\; D_0(p_1,p_4,p_3) + \frac{(p_2+p_4)^2}{2}\big(C_0(p_1,p_3+p_4) + C_0(p_1,p_4) + C_0(p_1+p_4,p_3) + C_0(p_4,p_3)\big)\Big).
\end{aligned}
\tag{3.5.7}
$$

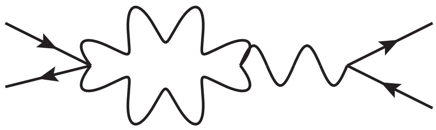
The reducible superdiagrams that contribute are



while the superdiagrams

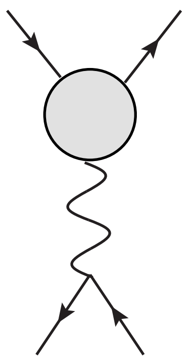


don't contribute, because contain the one-loop two-point functions which vanish. Neither contributes the superdiagram



because of the color factor which gives zero.

The reducible part with the color indices as indicated in the figure



has the expression

$$
\begin{aligned}
&Ng^4 \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\Big(\frac{i}{p_1^2}\frac{i}{p_2^2}\frac{i}{p_3^2}\frac{i}{p_4^2}\Big)\,\frac{1}{(p_1+p_2)^2}\,C_0(p_1,p_3+p_4)\times\\
&\frac{1}{2}\Big(p_1^2p_2^2 - p_1^2p_4^2 - p_2^2p_2^2 - p_2^2p_4^2 - 4\,p_1^2\,p_2\cdot p_4 - 2\,p_2^2 p_2\cdot p_4 - 2\,p_4^2\,p_1\cdot p_2 - 4\,p_1\cdot p_2\,p_2\cdot p_4 + 2\,p_2^2\,p_1\cdot p_4\Big).
\end{aligned}
$$

The other reducible parts can be obtained from this one by symmetry. As one can see neither the irreducible part neither the reducible part have bubbles in their decomposition.

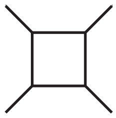
The one loop planar off shell amplitude with four scalars has the expression

$$
\begin{aligned}
\mathcal{A}_{1\,loop}^{off\,shell}\Big(\phi^{a_1}\,\phi^{\dagger\,a_2}\,\phi^{a_3}\,\phi^{\dagger\,a_4}\Big) =& Ng^4\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\times\\
&\Big(4p_1\cdot p_3\,p_2\cdot p_4\,D_0(p_1,p_4,p_3)\\
&-\frac{p_1^2\,p_2\cdot p_4 + p_2^2\,p_1\cdot p_3}{(p_1+p_2)^2}C_0(p_1,p_3+p_4)
\end{aligned}
$$

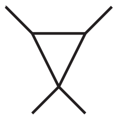
$$
\begin{aligned}
&- \frac{p_1^2\, p_2 \cdot p_4 + p_4^2\, p_1 \cdot p_3}{(p_1+p_4)^2} C_0(p_1, p_4) \\
&- \frac{p_3^2\, p_2 \cdot p_4 + p_2^2\, p_1 \cdot p_3}{(p_3+p_2)^2} C_0(p_1+p_4, p_3) \\
&- \frac{p_3^2\, p_2 \cdot p_4 + p_4^2\, p_1 \cdot p_3}{(p_3+p_4)^2} C_0(p_4, p_3)\Big). \qquad (3.5.8)
\end{aligned}
$$

or pictorially

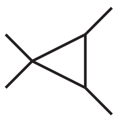
$$
\mathcal{A}_{1\,loop}^{off\,shell} = N g^4 \mathrm{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) \times \qquad (3.5.9)
$$



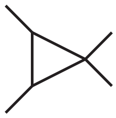
$$4 p_1 \cdot p_3\, p_2 \cdot p_4$$



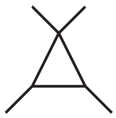
$$-\frac{1}{(p_1+p_2)^2}(p_1^2\, p_2 \cdot p_4 + p_2^2\, p_1 \cdot p_3)$$



$$-\frac{1}{(p_1+p_4)^2}(p_1^2\, p_2 \cdot p_4 + p_4^2\, p_1 \cdot p_3)$$



$$-\frac{1}{(p_3+p_2)^2}(p_3^2\, p_2 \cdot p_4 + p_2^2\, p_1 \cdot p_3)$$



$$-\frac{1}{(p_3+p_4)^2}(p_3^2\, p_2 \cdot p_4 + p_4^2\, p_1 \cdot p_3)$$

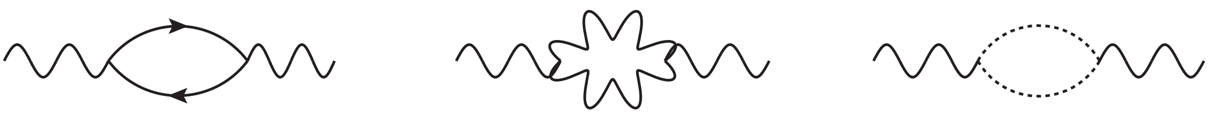
where we have not eliminated the momentum variable $p_2 = -p_1 - p_3 - p_4$ to present the result in a more symmetric form.
Using the conservation of momenta, one obtains for the on-shell limit of the one loop amplitude the expression

$$
\begin{aligned}
\mathcal{A}_{1\,loop}^{on\,shell} &= N g^4 \mathrm{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) \times 4 (p_1 \cdot p_3)^2\, D_0(p_1, p_4, p_3) \\
&= 2 i N g^2\, p_1 \cdot p_4\, p_3 \cdot p_4\, \mathcal{A}_{tree}^{on\,shell}\, D_0(p_1, p_4, p_3) \qquad (3.5.10)
\end{aligned}
$$

3.6 Amplitudes with gluons

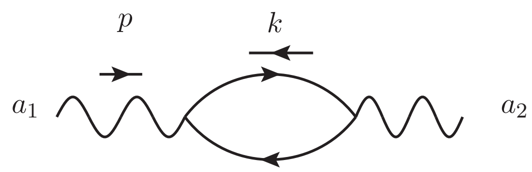
3.6.1 The two-point function at one loop

The two point function $< A_{\mu_1}^{a_1}(p) A_{\mu_2}^{a_2}(-p) >_{1\,loop}$ at one loop is given by the superdiagrams



where around the loop chiral superfields, vector superfields and ghost superfields propagate respectively.

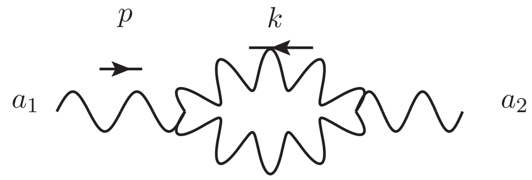
The superdiagram with the chiral loop



gives

$$N\mathrm{Tr}(T^{a_1}T^{a_2})\Big(\frac{-2i}{p}\Big)^2\times$$
$$\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(k+p)^2}\,3\,(\frac{1}{2}p_{\mu_1}p_{\mu_2}+\eta_{\mu_1\mu_2}k^2+\eta_{\mu_1\mu_2}\,p\cdot k+ik^\nu p^\tau\epsilon_{\nu\tau\mu_1\mu_2}) \tag{3.6.1}$$

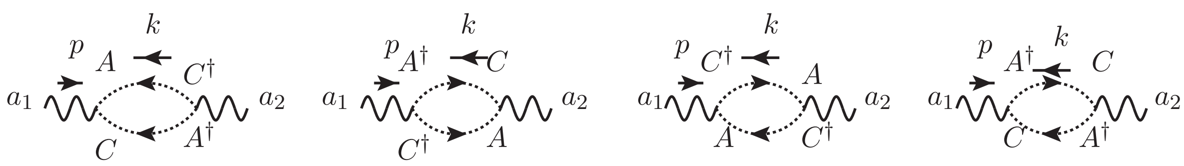
The superdiagram with the vector loop



gives

$$N\mathrm{Tr}(T^{a_1}T^{a_2})\Big(\frac{-2i}{p}\Big)^2\times$$
$$\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(k+p)^2}(\frac{3}{2}p_{\mu_1}p_{\mu_2}-\frac{5}{4}\eta_{\mu_1\mu_2}p^2) \tag{3.6.2}$$

Since we have various vertices involving the superghosts (see section 1.4), actually there are four superdiagrams with the ghost loop



which give the total result

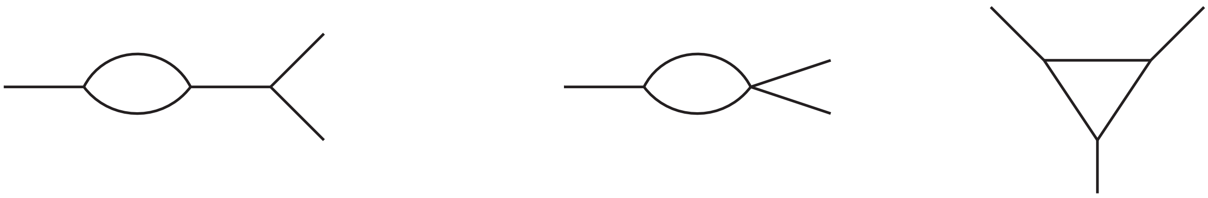
$$N\mathrm{Tr}(T^{a_1}T^{a_2})\Big(\frac{-2i}{p}\Big)^2\times$$
$$\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(k+p)^2}\,(\frac{5}{2}\eta_{\mu_1\mu_2}k^2+\frac{5}{2}\eta_{\mu_1\mu_2}\,p\cdot k+\frac{5}{4}\eta_{\mu_1\mu_2}p^2+\frac{5}{2}ik^\nu p^\tau\epsilon_{\nu\tau\mu_1\mu_2}). \tag{3.6.3}$$

After summing all the contributions and making the Passarino-Veltman reduction (neglecting the tadpole scalar integral), one has that the two-point function vanishes [19]

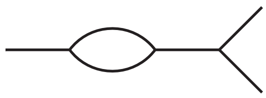
$$< A^{a_1}_{\mu_1}(p)A^{a_2}_{\mu_2}(-p) >_{1\,loop}= 0. \tag{3.6.4}$$

3.6.2 Three gluon planar amplitude at one loop

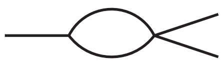
The one loop planar three-point function $< A^{a_1}_{\mu_1}(p_1)A^{a_2}_{\mu_2}(p_2)A^{a_3}_{\mu_3}(p_3) >^{planar}_{1\,loop}$ has superdiagrams with three possible topologies (now the pictures don't indicate bubble or triangle scalar integrals but topologies)

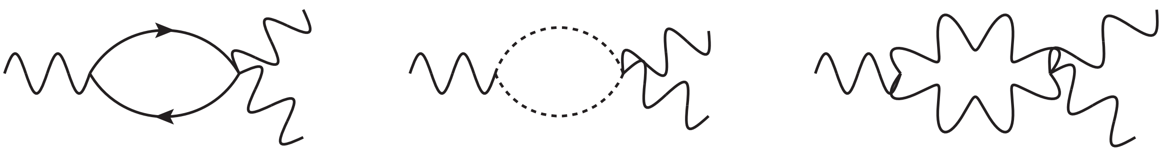


The topology

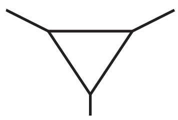


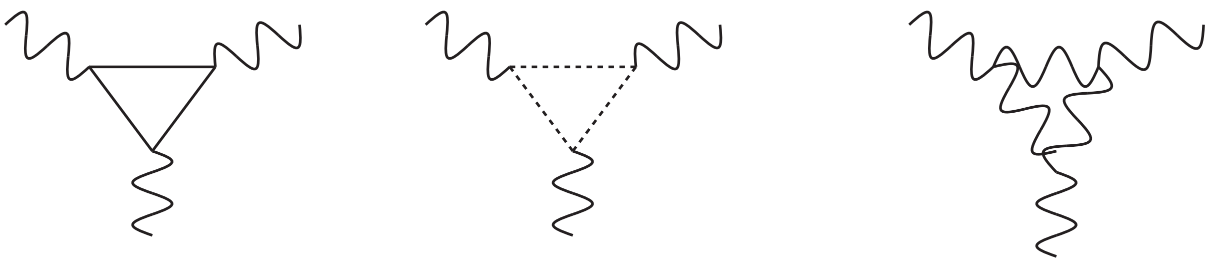
doesn't contribute since involves the one loop two point-function which is zero.
The superdiagrams associated to the second topology are





Only the superdiagram with the vector loop contributes since both the superdiagram with the chiral loop and the superdiagram with the ghost loop give zero since their color factor vanishes.
The superdiagrams of the third topology are

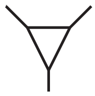




After doing the Passarino Veltman decomposition and substituting p_2 with $p_2 = -p_1 - p_3$, the final result for the planar off shell amplitude with three gluons is

$$\mathcal{A}^{off\,shell}_{1\,loop}\Big(A^{a_1}_{\mu_1}(p_1)A^{a_2}_{\mu_2}(p_2)A^{a_3}_{\mu_3}(p_3)\Big) = Ng^3\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3})\times$$
$$\Big(M_{1\,,\mu_1\mu_2\mu_3}\,B_0(p_1) + M_{2\,,\mu_1\mu_2\mu_3}\,B_0(p_1+p_3) + M_{3\,,\mu_1\mu_2\mu_3}\,B_0(p_3) + N_{\mu_1\mu_2\mu_3}\,C_0(p_1,p_3)\Big), \tag{3.6.5}$$

or pictorially (omitting the color indices and the color factor)

$$\mathcal{A}^{offshell}_{1\,loop}(A_{\mu_1}A_{\mu_2}A_{\mu_3}) = \quad N_{\mu_1\mu_2\mu_3}$$ 

$$+ \quad M_{1,\mu_1\mu_2\mu_3}$$  $$+ \quad M_{2,\mu_1\mu_2\mu_3}$$  $$+ \quad M_{3,\mu_1\mu_2\mu_3}$$ 

where the coefficients $N_{\mu_1\mu_2\mu_3}, M_{1,\mu_1\mu_2\mu_3}, M_{2,\mu_1\mu_2\mu_3}, M_{3,\mu_1\mu_2\mu_3}$ multiplying the scalar integrals are functions of the external momenta and are given in the appendix G.1.

As already stated, even if in the decomposition there are bubbles, the amplitude is finite since

$$M_{1\,,\mu_1\mu_2\mu_3} + M_{2\,,\mu_1\mu_2\mu_3} + M_{3\,,\mu_1\mu_2\mu_3} = 0, \tag{3.6.6}$$

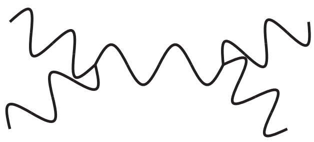
and hence the sum of all the UV terms vanishes.
On shell, one has to impose the conditions

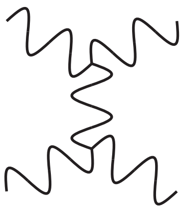
$$p_1^2 = 0,\; p_3^2 = 0,\; p_2^2 = (p_1 + p_3)^2 = 0 \Rightarrow \; p_1 \cdot p_3 = 0. \tag{3.6.7}$$

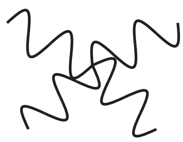
Hence, in the on-shell limit all the terms of the amplitude vanish (see the appendix G.1).

3.6.3 Four gluon planar amplitude at tree level

The planar off-shell amplitude $\mathcal{A}^{off\,shell}_{tree}\Big(A^{a_1}_{\mu_1}(p_1)A^{a_2}_{\mu_2}(p_2)A^{a_3}_{\mu_3}(p_3)A^{a_4}_{\mu_4}(p_4)\Big)$ is obtained by computing the superdiagrams







The result for $\mathcal{A}^{off\,shell}_{tree}\Big(A^{a_1}_{\mu_1}(p_1)A^{a_2}_{\mu_2}(p_2)A^{a_3}_{\mu_3}(p_3)A^{a_4}_{\mu_4}(p_4)\Big)$ is given in the appendix G.2.

3.6.4 The on-shell limit

To do the on shell limit, one can choose the spinor representation of the polarization vectors as made in the spinor helicity formalism (see Appendix F).
Also, one can employ the notation i^+, i^- to indicate the external gluon i with helicity $+1$ or -1 respectively.
Using super Ward identities , one can prove that in a SYM theory the on-shell $n-$gluon amplitudes with all helicities the same, like $\mathcal{A}^{on\,shell}(1^+, 2^+, \ldots, n^+)$, vanish [48].
The amplitudes with all helicites the same but one, like $\mathcal{A}^{on\,shell}(1^-, 2^+, \ldots, n^+)$ vanish as well. Thus, the first non-zero $n-$ gluon amplitude ($n \geq 4$) has two gluons with helicity -1 and the remaining with helicity $+1$ or vice-versa and is usually referred to as the maximally helicity violating (MHV) amplitude.
Let's consider the amplitude with the configuration $(1^-, 2^-, 3^+, 4^+)$ and eliminate one of the momenta, i.e $p_2 = -p_1 - p_3 - p_4$, in favor of the others.
The polarization vectors λ_i have to satisfy the condition of trasversality

$$\lambda_i \cdot p_i = 0. \tag{3.6.8}$$

(We don't indicate the helicities explicitly. Thus, in the configuration $(1^-, 2^-, 3^+, 4^+)$, λ_1 stands for λ_1^-, λ_3 stands for λ_3^+, etc..). Choosing as reference momenta (see Appendix F) $q_1 = q_2 = p_4, \;\; q_3 = q_4 = p_1$ or in brief (p_4, p_4, p_1, p_1), one has that the polarization vectors satisfy

$$\begin{aligned} &\lambda_i \cdot \lambda_j = 0 \quad \text{except} \quad \lambda_2 \cdot \lambda_3 \neq 0 \\ &\lambda_1 \cdot p_4 = \lambda_2 \cdot p_4 = \lambda_3 \cdot p_1 = \lambda_4 \cdot p_1. = 0 \end{aligned} \tag{3.6.9}$$

The amplitude with four gluons has four vector indices and can be decomposed in 138 Lorentz structures which have three forms (see eqs.(2.1.10)). The first form $(\eta\eta)$ has two η tensors, like $\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}$, the second form (ηpp) has one η tensor and two momenta, such as $\eta_{\mu_1\mu_2} p_{1\,\mu_3} p_{3\,\mu_4}$ and the third form $(pppp)$ has four momenta, like $p_{1\,\mu_1} p_{3\,\mu_2} p_{1\,\mu_3} p_{4\,\mu_4}$.
Off shell all the 138 Lorentz structures could contribute, but on shell if one chooses the reference momenta appropriately only a few of them contribute.
In fact, with the choice (p_4, p_4, p_1, p_1) for the reference momenta, from eqs.(3.6.8),(3.6.9) one has that after contracting with the external polarization vectors, only the structures $\eta_{\mu_2\mu_3} p_{3\,\mu_1} p_{3\,\mu_4}, \;\; p_{3\,\mu_1} p_{1\,\mu_2} p_{4\,\mu_3} p_{3\,\mu_4}$ and $p_{3\,\mu_1} p_{3\,\mu_2} p_{4\,\mu_3} p_{3\,\mu_4}$, corresponding to $\lambda_2 \cdot \lambda_3 \, \lambda_1 \cdot p_3 \, \lambda_4 \cdot p_3, \;\; \lambda_1 \cdot p_3 \, \lambda_2 \cdot p_1 \, \lambda_3 \cdot p_4 \, \lambda_4 \cdot p_3$ and $\lambda_1 \cdot p_3 \, \lambda_2 \cdot p_3 \, \lambda_3 \cdot p_4 \, \lambda_4 \cdot p_3$ respectively, contribute.
Moreover, while off-shell all these structures are independent, on-shell they can be dependent. In fact, for example from $\lambda_2 \cdot p_2 = 0$ and $\lambda_2 \cdot p_4 = 0$ it follows that

$$\lambda_2 \cdot (p_1 + p_3) = -\lambda_2 \cdot (p_2 + p_4) = 0 \Rightarrow \lambda_2 \cdot p_1 = -\lambda_2 \cdot p_3. \tag{3.6.10}$$

Thus

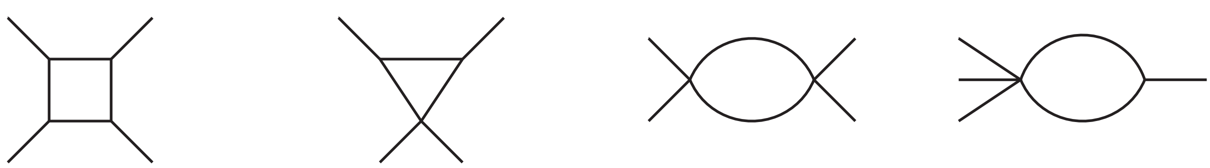
$$\lambda_1 \cdot p_3 \, \lambda_2 \cdot p_1 \, \lambda_3 \cdot p_4 \, \lambda_4 \cdot p_3 = -\lambda_1 \cdot p_3 \, \lambda_2 \cdot p_3 \, \lambda_3 \cdot p_4 \, \lambda_4 \cdot p_3. \tag{3.6.11}$$

As for the off shell tree level amplitude $\mathcal{A}_{tree}^{off\,shell}\Big(A_{\mu_1}^{a_1}(p_1) A_{\mu_2}^{a_2}(p_2) A_{\mu_3}^{a_3}(p_3) A_{\mu_4}^{a_4}(p_4)\Big)$, one has (see the appendix G.2) that the structures $p_{3\,\mu_1} p_{1\,\mu_2} p_{4\,\mu_3} p_{3\,\mu_4}$ and $p_{3\,\mu_1} p_{3\,\mu_2} p_{4\,\mu_3} p_{3\,\mu_4}$ are not present in the result. Thus, at tree level the on shell planar amplitude with the configuration $(1^-, 2^-, 3^+, 4^+)$ has the expression

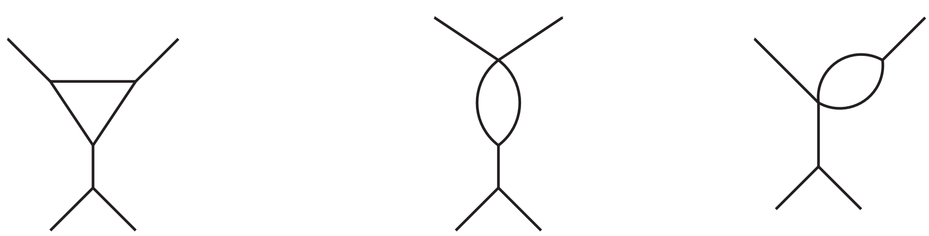
$$\mathcal{A}_{tree}^{on\,shell}(1^-, 2^-, 3^+, 4^+) = g^2 \, \mathrm{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) \, \frac{i}{2\, p_3 \cdot p_4} \lambda_2 \cdot \lambda_3 \, \lambda_1 \cdot p_3 \, \lambda_4 \cdot p_3. \tag{3.6.12}$$

3.6.5 Four gluon planar amplitude at one loop

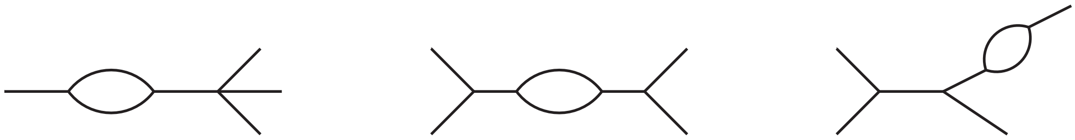
The topologies associated to the $1PI$ superdiagrams which contribute to the four gluon planar amplitude at one loop $\mathcal{A}_{1\,Loop}^{off\,shell}\Big(A_{\mu_1}^{a_1}(p_1) A_{\mu_2}^{a_2}(p_2) A_{\mu_3}^{a_3}(p_3) A_{\mu_4}^{a_4}(p_4)\Big)$ are



The topologies associated to the reducible superdiagrams are



The topologies

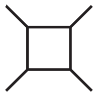


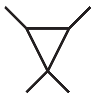
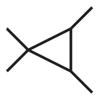
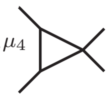
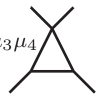
don't contribute since they contain the two-point function which is vanishing.

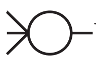
Summing all the superdiagrams and making the Passarino-Veltman procedure, one obtains for the off shell four gluon planar amplitude at one loop the decomposition

$$
\begin{aligned}
&\mathcal{A}^{off\,shell}_{1\,loop}\Big(A^{a_1}_{\mu_1}(p_1)A^{a_2}_{\mu_2}(p_2)A^{a_3}_{\mu_3}(p_3)A^{a_4}_{\mu_4}(p_4)\Big) = Ng^4\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\times \\
&\Big(Q_{\mu_1\mu_2\mu_3\mu_4}\,D_0(p_1,p_4,p_3) \\
&+N_{1,\mu_1\mu_2\mu_3\mu_4}\,C_0(p_1,p_3+p_4)+N_{2,\mu_1\mu_2\mu_3\mu_4}\,C_0(p_1,p_4)+N_{3,\mu_1\mu_2\mu_3\mu_4}\,C_0(p_1+p_4,p_3)+N_{4,\mu_1\mu_2\mu_3\mu_4}\,C_0(p_4,p_3) \\
&+M_{1,\mu_1\mu_2\mu_3\mu_4}\,B_0(p_1)+M_{2,\mu_1\mu_2\mu_3\mu_4}\,B_0(p_4)+M_{3,\mu_1\mu_2\mu_3\mu_4}\,B_0(p_1+p_3+p_4)+M_{4,\mu_1\mu_2\mu_3\mu_4}\,B_0(p_3) \\
&+M_{5,\mu_1\mu_2\mu_3\mu_4}\,B_0(p_1+p_4)+M_{6,\mu_1\mu_2\mu_3\mu_4}\,B_0(p_3+p_4)\Big),
\end{aligned}
\tag{3.6.13}
$$

or pictorially (omitting the color indices and the color factor)

$\mathcal{A}^{offshell}_{1\,loop}(A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}) =\ Q_{\mu_1\mu_2\mu_3\mu_4}$ 

$+\ N_{1,\mu_1\mu_2\mu_3\mu_4}$  $+\ N_{2,\mu_1\mu_2\mu_3\mu_4}$  $+N_{3,\mu_1\mu_2\mu_3\mu_4}$  $+\ N_{4,\mu_1\mu_2\mu_3\mu_4}$ 

$+\ M_{1,\mu_1\mu_2\mu_3\mu_4}$  $+\ M_{2,\mu_1\mu_2\mu_3\mu_4}$  $+M_{3,\mu_1\mu_2\mu_3\mu_4}$  $+\ M_{4,\mu_1\mu_2\mu_3\mu_4}$ 

$+\ M_{5,\mu_1\mu_2\mu_3\mu_4}$  $+\ M_{6,\mu_1\mu_2\mu_3\mu_4}$ 

This off-shell amplitude is finite since the sum of the coefficients of the bubbles vanish

$$\sum_{i=1..6} M_{i,\mu_1\mu_2\mu_3\mu_4} = 0 \tag{3.6.14}$$

and all the UV divergent terms cancel out.
We have found that all the possible (138) Lorentz structures (see eqs.(2.1.10) and section 3.6.4) contribute to the one loop off-shell four gluon amplitude. In the off-shell regime there cannot be cancellations between terms containing different Lorentz structures, since off-shell all these Lorentz structures are independent of each other. Hence, the vanishing of the sum of all the UV terms has to happen for each Lorentz structure independently of the others. In fact, let's consider the Lorentz structure $\eta_{\mu_2\mu_3}p_{3\,\mu_1}p_{3\,\mu_4}$ which is relevant for the on-shell amplitude $\mathcal{A}^{on\,shell}_{1\,loop}(1^-,2^-,3^+,4^+)$ (see section 3.6.4 and the discussion below).
Let's define as

$$Q_V, N_{V1}, \dots N_{V4}, M_1 \dots M_6 \tag{3.6.15}$$

those parts of the coefficients

$$Q_{\mu_1\mu_2\mu_3\mu_4}, N_{1,\mu_1\mu_2\mu_3\mu_4}, \dots N_{4,\mu_1\mu_2\mu_3\mu_4}, M_{1,\mu_1\mu_2\mu_3\mu_4} \dots M_{6,\mu_1\mu_2\mu_3\mu_4} \tag{3.6.16}$$

which multiply this Lorentz structure $\eta_{\mu_2\mu_3}p_{3\,\mu_1}p_{3\,\mu_4}$. In the appendix G.3 we give the expressions of $Q_V, N_{V1}, \dots N_{V4}, M_1 \dots M_6$. One can verify that for this Lorentz structure the sum of all the coefficients multiplying the bubbles vanishes, i.e.

$$\sum_{i=1\dots 6} M_i = 0.$$

Moreover, since the N_{Vi} are different from zero, one has that, in the decomposition of the off-shell four gluon amplitude, appear triangles as well.

The on-shell limit

As in section 3.6.4, for the on shell limit let's consider the configuration $(1^-, 2^-, 3^+, 4^+)$ with reference momenta (p_4, p_4, p_1, p_1). In this case only the structures $\eta_{\mu_2\mu_3}p_{3\,\mu_1}p_{3\,\mu_4}$, $p_{3\,\mu_1}p_{1\,\mu_2}p_{4\,\mu_3}p_{3\,\mu_4}$ and $p_{3\,\mu_1}p_{3\,\mu_2}p_{4\,\mu_3}p_{3\,\mu_4}$ contribute.
We have found that in the off-shell one loop amplitude the coefficient multiplying $p_{3\,\mu_1}p_{1\,\mu_2}p_{4\,\mu_3}p_{3\,\mu_4}$ is equal to the coefficient multiplying $p_{3\,\mu_1}p_{3\,\mu_2}p_{4\,\mu_3}p_{3\,\mu_4}$. Since one has that $\lambda_1 \cdot p_3\, \lambda_2 \cdot p_1\, \lambda_3 \cdot p_4\, \lambda_4 \cdot p_3 = -\lambda_1 \cdot p_3\, \lambda_2 \cdot p_3\, \lambda_3 \cdot p_4\, \lambda_4 \cdot p_3$ (see eq.(3.6.11)), when these two Lorentz structures are contracted with the external polarization vectors, their sum vanish. Hence, as in the case of the tree level amplitude, for the one loop amplitude in the on shell limit contributes only the Lorentz structure $\eta_{\mu_2\mu_3}p_{3\,\mu_1}p_{3\,\mu_4}$.
In the on shell limit, where $p_i^2 = 0, i = 1 \dots 4$ and $p_1 \cdot p_3 = -p_1 \cdot p_4 - p_3 \cdot p_4$ (see eqs.3.5.5), all the coefficients multiplying the triangles and those multiplying the bubbles vanish (see the appendix G.3) , i.e.

$$N_{V1}^{on\,shell} = \dots N_{V4}^{on\,shell} = 0, \quad M_1^{on\,shell} = \dots M_6^{on\,shell} = 0.$$

The only non vanishing coefficient in the on shell limit is Q_V which gives

$$Q^{on\,shell} = -p_1 \cdot p_4.$$

Hence, one has

$$\begin{aligned} \mathcal{A}_{1\,loop}^{on\,shell}(1^-, 2^-, 3^+, 4^+) &= -N\, g^4\, \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\, \lambda_2 \cdot \lambda_3\, \lambda_1 \cdot p_3\, \lambda_4 \cdot p_3\, p_1 \cdot p_4\, D_0(p_1, p_4, p_3) \\ &\equiv 2iNg^2\, p_1 \cdot p_4\, p_3 \cdot p_4\, \mathcal{A}_{tree}^{on\,shell}(1^-, 2^-, 3^+, 4^+)\, D_0(p_1, p_4, p_3). \end{aligned} \tag{3.6.17}$$

This result agrees with the expected result as computed in [110].

3.7 Correlation functions in the β-deformed N=4 theory

In [89] Leigh and Strassler found that it is possible to deform the $\mathcal{N} = 4$ SYM, i.e. to modify its superpotential $\mathcal{W}_{\mathcal{N}=4}$, obtaining a class of $\mathcal{N} = 1$ SYM theories which has the same superfield content as $\mathcal{N} = 4$ SYM and which maintains conformal invariance and finiteness.
An important example of these theories is represented by the so called β-deformation. It is obtained by replacing the superpotential of the $\mathcal{N} = 4$ SYM by

$$\begin{aligned} \mathcal{W}_{\mathcal{N}=4} &= 2ig \int d^6 z \mathrm{Tr}\Phi_1[\Phi_2, \Phi_3] \rightarrow \\ \mathcal{W}_{\beta} &= 2ih \int d^6 z \mathrm{Tr}\big(e^{i\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\beta}\Phi_1\Phi_3\Phi_2\big), \end{aligned} \tag{3.7.1}$$

where the parameters h and β, which gives the name to the theory, can be considered as complex functions of the gauge coupling g and expanded in power series in g. The coefficients of the expansions can be determined by requiring to have conformal invariance.
The gravitational dual of the β deformation in the case of real β has been found in [90].
The superpotential of the β deformation breaks $\mathcal{N} = 4$ SUSY to $\mathcal{N} = 1$ and the original $SU(4)$ $R-$ symmetry to $U(1)_R$. However, a global $U(1) \times U(1)$ survives. Its generators can be chosen to act as follows

$$U(1)_1 : \quad (\Phi_1, \Phi_2, \Phi_3) \rightarrow (\Phi_1, e^{i\varphi_1}\Phi_2, e^{-i\varphi_1}\Phi_3)$$

$$U(1)_2: \quad (\Phi_1, \Phi_2, \Phi_3) \to (e^{-i\varphi_2}\Phi_1, e^{i\varphi_2}\Phi_2, \Phi_3). \tag{3.7.2}$$

Also, there is a discrete symmetry $Z_3 \times Z_3$ which is given by the transformations

$$\begin{aligned}(\Phi_1, \Phi_2, \Phi_3) &\to (\Phi_2, \Phi_3, \Phi_1)\\ (\Phi_1, \Phi_2, \Phi_3) &\to (\Phi_1, \omega\Phi_2, \omega^2\Phi_3),\end{aligned} \tag{3.7.3}$$

with $\omega^3 = 1$.
In the case of a theory with $\mathcal{N} = 1$ SUSY and superfields in the adjoint representation, as the β deformation, to guarantee conformal invariance it is sufficient to impose the finiteness of the propagator of the chiral superfield $< \Phi_i^a \Phi_j^{\dagger\, b} >$.
In any $\mathcal{N} = 1$ supersymmetric gauge theory the only potential divergences are those of the chiral and the vector propagators $< \Phi\Phi^\dagger >$ and $< VV >$. In fact,

- from the non renormalization theorem of the superpotential $\mathcal{W}$ [19], one has that in the chiral sector the only divergence is given by the chiral propagator $< \Phi\Phi^\dagger >$, while the chiral vertex can have only a finite renormalization. In other words, the beta function of the Yukawa coupling β_Y depends only on the anomalous dimension of the chiral superfield γ_Φ.

- In the gauge sector, one can choose the background field gauge where the renormalization of the vertices coincides with that of the vector propagator. In any gauge there are Ward identities which relate β_g to γ_V.

- Moreover, if the chiral superfields belong to the adjoint representation, the divergence of the vector propagator is related to that of the chiral propagator . This can be seen in the background field gauge [91] or from the exact form of the $\beta_{g,NSVZ}$ function for SYM theories found in [92](we recall that its vanishing is scheme-independent).

- Further, in the β-deformation, because of the discrete $Z_3 \times Z_3$ symmetry of the action, the matrix $(\gamma_\Phi)^I_J$ of γ-functions of the chiral superfields is proportional to the unit matrix in the flavor space

  $$(\gamma_\Phi)^I_J = \gamma\delta^I_J, \tag{3.7.4}$$

  and so it is enough to require a single condition $\gamma = 0$ to ensure conformal invariance.
  In other words, in the β-deformation to assure that both the gauge and Yukawa functions β_g and β_Y vanish and so the theory is conformal invariant, it is sufficient to impose that the chiral propagator $< \Phi_1^a \Phi_1^{\dagger\, b} >$ is finite. The choice of flavor indices is a mere convention since they are all on the same footing because of the $Z_3 \times Z_3$ symmetry (in the following we will omit to indicate them).

In the β-deformed theories one can make one further simplification [93],[94].
To find the condition of conformal invariance at a certain order g^{2n} in perturbation theory[3.10] instead of computing the chiral propagator $< \Phi\Phi^\dagger >_\beta$ in the β-deformed theory, it is more convenient to compute the difference[3.11]

$$< \Phi(z_1)\Phi^\dagger(z_2) >_{\mathcal{N}=4} - < \Phi(z_1)\Phi^\dagger(z_2) >_\beta . \tag{3.7.5}$$

We already know that the chiral propagator in $\mathcal{N} = 4$ SYM $< \Phi\Phi^\dagger >_{\mathcal{N}=4}$ is finite, thus if the difference (3.7.5) is finite, the chiral propagator of the β deformation $< \Phi\Phi^\dagger >_\beta$ if finite as well. In the difference, all the superdiagrams without the chiral vertex, which comes from the superpotential, cancel out. In fact, the (anti)chiral vertex is the only vertex which is different in the two theories.
Actually, it is only the color factor associated to the vertex which differs in the two theories. In fact, for the $\mathcal{N} = 4$, one has

$$\mathcal{W}_{\mathcal{N}=4} = 2ig \int d^6z\, \mathrm{Tr}\Phi_1[\Phi_2, \Phi_3]$$

[3.10] since the couplings h, β are dependent on g, a diagram of order g^{2n} could have a number of loops lower than n, so in this case by a calculation to n loops one means a calculation up to order g^{2n} in the coupling constant

[3.11] in this section it doesn't make difference whether we work in coordinate or in momentum space since we are interested essentially in the computation of color factors

$$\equiv -g f_{a_1 a_2 a_3} \int d^6 z \, \Phi_1^{a_1} \Phi_2^{a_2} \Phi_3^{a_3}, \tag{3.7.6}$$

while for the β-deformation

$$\begin{aligned} \mathcal{W}_\beta &= 2ih \int d^6 z \, \mathrm{Tr}\big(e^{i\pi\beta} \Phi_1 \Phi_2 \Phi_3 - e^{-i\pi\beta} \Phi_1 \Phi_3 \Phi_2\big) \\ &\equiv -h\big(f_{a_1 a_2 a_3} \cos \pi\beta + d_{a_1 a_2 a_3} \sin \pi\beta\big) \int d^6 z \Phi_1^{a_1} \Phi_2^{a_2} \Phi_3^{a_3}, \end{aligned} \tag{3.7.7}$$

where we have introduced the completely symmetric invariant tensor $d_{a_1 a_2 a_3}$ which (using the normalization (3.3.1) for the generators of the gauge group) is given by

$$d_{a_1 a_2 a_3} = 2 \, \mathrm{Tr}\big(T^{a_1} \{T^{a_2}, T^{a_3}\}\big). \tag{3.7.8}$$

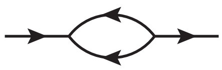
An alternative way to parametrize the superpotential (3.7.1) is the following

$$\begin{aligned} \mathcal{W}_\beta &= 2f \int d^6 z \, \mathrm{Tr}\Phi_1 [\Phi_2, \Phi_3] + 2d \int d^6 z \, \mathrm{Tr}\Phi_1 \{\Phi_2, \Phi_3\} \\ &= \big(if \, f_{a_1 a_2 a_3} + d \, d_{a_1 a_2 a_3}\big) \int d^6 z \, \Phi_1^{a_1} \Phi_2^{a_2} \Phi_3^{a_3} \end{aligned} \tag{3.7.9}$$

where the complex parameters f and d are related to h and β

$$f = ih \cos \pi\beta \qquad d = -h \sin \pi\beta. \tag{3.7.10}$$

At order g^2 the only superdiagram contributing to (3.7.5) is



As stated above, this superdiagram differs in the $\mathcal{N} = 4$ SYM and in the β-deformation only by the color factor which factors out from the rest of the diagram which is the same in the two theories (and so its divergent part is the same). Hence, the condition for the finiteness of the chiral propagator at order g^2 is given by the vanishing of the difference $\Delta^{a,b}(g, h, \beta)$ of the color factors in the two theories, $\mathcal{F}^{a,b}_{\mathcal{N}=4}$ and $\mathcal{F}^{a,b}_\beta$ respectively

$$\Delta^{a,b}(g, h, \beta) \equiv \mathcal{F}^{a,b}_{\mathcal{N}=4} - \mathcal{F}^{a,b}_\beta = 0. \tag{3.7.11}$$

After computing the colour factors and using that

$$f_{a_1 b_1 b_2} f_{a_2 b_1 b_2} = N \delta_{a_1 a_2} \qquad d_{a_1 b_1 b_2} d_{a_2 b_1 b_2} = \frac{N^2 - 4}{N} \delta_{a_1 a_2}, \tag{3.7.12}$$

one has that the finiteness condition at order g^2 is given by

$$|h|^2 \Big(\cos \pi\beta \cos \pi\bar{\beta} + \frac{N^2 - 4}{N^2} \sin \pi\beta \sin \pi\bar{\beta} \Big) = g^2, \tag{3.7.13}$$

or in terms of f and d

$$|f|^2 + \frac{N^2 - 4}{N^2} |d|^2 = g^2. \tag{3.7.14}$$

In the planar limit and with real β, the condition (3.7.13) reduces to

$$N \to \infty : \quad g^2 = |h|^2, \tag{3.7.15}$$

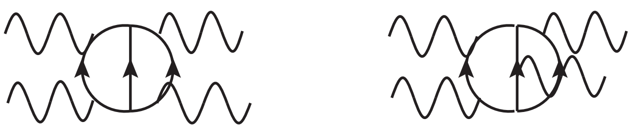
independently of the value of β. In [95], it has been shown that in the planar limit and with β real the condition (3.7.15) guarantees the conformal invariance to all orders in perturbation theory.
Moreover, all the amplitudes of the β-deformed theory with $\beta \in R$ coincide with the ones of $\mathcal{N} = 4$ up to phase factors [96].
If β is complex the order g^2 condition which is $g^2 = |h|^2 \cosh(2\pi \mathrm{Im}\beta)$ is not more sufficient to assure

conformal invariance at higher orders [93]-[94].
The considerations made above for the chiral propagator are valid for any n-point correlation function $\langle \mathcal{O}_1(z_1) \dots \mathcal{O}_n(z_n) \rangle$ where $\mathcal{O}_i(z_i)$ is an elementary field or a composite operator (present in both theories). In fact, to study if a correlation function in the $\beta-$ deformed theory receives a correction with respect to $\mathcal{N}=4$ SYM, one has to compute the difference

$$< \mathcal{O}_1(z_1) \dots \mathcal{O}_n(z_n) >_{\mathcal{N}=4} - < \mathcal{O}_1(z_1) \dots \mathcal{O}_n(z_n) >_{\beta} . \tag{3.7.16}$$

As before, in the difference all the superdiagrams without chiral vertices don't contribute and the difference of each superdiagram in the two theories is given by the difference of the color factors multiplying the rest of the diagram which is the same in the two theories.
In [94], three point functions, that is the triple chiral (antichiral) vertex, the chiral-antichiral-vector vertex, the triple vector vertex and the ghost-ghost-vector vertex, have been studied.

At order g^3 and order g^5 all three vertices with external vector lines are exactly equal to the corresponding ones in $\mathcal{N}=4$ SYM. Only the triple chiral vertex has at order g^5 a finite non-planar correction. At order g^7 the ghost-ghost-vector vertex is equal to the corresponding vertex in $\mathcal{N}=4$ SYM, while the other three vertex receive corrections from non-planar diagrams. Only the triple chiral vertex receives also finite planar corrections at order g^7.
Here we have studied some $n-$point functions with $n \geq 4$. In fact we have considered the correlation function with four and six vector superfields $< V^{a_1}V^{a_2}V^{a_3}V^{a_4} >$ and $< V^{a_1}V^{a_2}V^{a_3}V^{a_4}V^{a_5}V^{a_6} >$ respectively.
At one loop[3.12], there are no superdiagrams with (anti)chiral vertices contributing to these correlation functions. So, at one loop these are equal to their $\mathcal{N}=4$ value.
At two loops, all receive non-planar corrections. Hence, unlike the three point functions, $n-$point functions with $n \geq 4$ differ from their $\mathcal{N}=4$ value already at the first order in which there are diagrams with chiral vertices.
In fact, let's consider the four point function $< V^{a_1}V^{a_2}V^{a_3}V^{a_4} >$. The difference between its value in the $\mathcal{N}=4$ SYM and in the β-deformation is given by the two superdiagrams



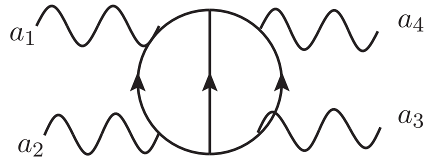
The superdiagram on the left is planar, that on the right is non planar. For each of these two superdiagrams we have computed the difference between the color factors

$$\Delta^{a_1a_2a_3a_4}(g,f,d) \equiv \mathcal{F}^{a_1a_2a_3a_4}_{\mathcal{N}=4} - \mathcal{F}^{a_1a_2a_3a_4}_{\beta} , \tag{3.7.17}$$

which is a function of g, f and d (we have considered the parametrization of (3.7.9)).
After substituting in (3.7.17) the condition of conformal invariance (3.7.14), we have contracted $\Delta^{a_1a_2a_3a_4}$ with all the non-cyclically equivalent traces of four generators $\mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}})$ with $\sigma \in S_n/Z_n \equiv S_{n-1}$ (see section 2.5).
As already stated in section 2.5, in the large N limit, at two loops, the leading contribution in the colour factor is given by single-trace terms like $N^2\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$.
Also, the contraction of $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$ with $\mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}})$ is at most of order $O(N^4)$ (for n generators it would be of order $O(N^n)$).
Hence, if the contraction

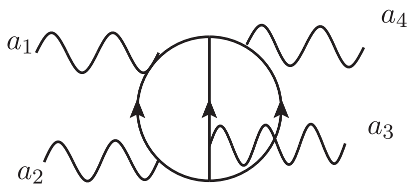
$$\mathcal{C} \equiv \Delta^{a_1a_2a_3a_4}\mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}) \tag{3.7.18}$$

[3.12] here there is no mismatch between the number of the loops in the diagrams and the order of perturbation theory is of order $O(N^6)$, then one has a planar correction, while if it is of lower order, we have a non-planar correction. With the aid of a Maple program, we have obtained the following results for the contractions $\mathcal{C}$ of all the various non-cyclically equivalent traces $\mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}})$ with the $\Delta^{a_1a_2a_3a_4}$ associated to each of the two superdiagrams



$$\{Tr\,(a_1,a_2,a_3,a_4)\,,\,Tr\,(a_1,a_2,a_4,a_3)\,,\,Tr\,(a_1,a_3,a_4,a_2)\,,\,Tr\,(a_1,a_4,a_3,a_2)\}$$
$$\mathcal{C} = -\frac{3}{4}\,d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.19}$$

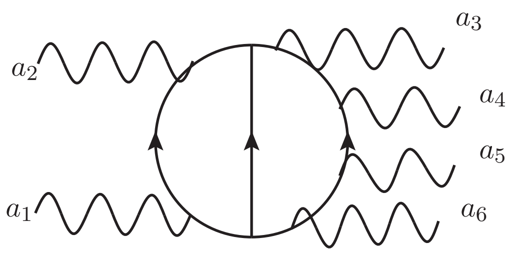
$$\{Tr\,(a_1,a_3,a_2,a_4)\,,\,Tr\,(a_1,a_4,a_2,a_3)\}$$
$$\mathcal{C} = \frac{3}{2}\,d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.20}$$



$$\{Tr\,(a_1,a_2,a_3,a_4)\,,\,Tr\,(a_1,a_2,a_4,a_3)\,,\,Tr\,(a_1,a_3,a_4,a_2)\,,\,Tr\,(a_1,a_4,a_3,a_2)\}$$
$$\mathcal{C} = \frac{3}{4}\,d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.21}$$

$$\{Tr\,(a_1,a_3,a_2,a_4)\,,\,Tr\,(a_1,a_4,a_2,a_3)\}$$
$$\mathcal{C} = -\frac{3}{2}\,d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.22}$$

Here, we have grouped together the traces giving the same contraction $\mathcal{C}$ and we have used for the traces a notation such that, for example, $Tr(a_1,a_2,a_3,a_4)$ stands for $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$. Since the contractions $\mathcal{C}$ are of order $O(N^4)$, one has that the four point function $< V^{a_1}V^{a_2}V^{a_3}V^{a_4} >$ receives a non-planar correction with respect to its value in the $\mathcal{N}=4$ SYM.
As for the six point function $< V^{a_1}V^{a_2}V^{a_3}V^{a_4}V^{a_5}V^{a_6} >$, we have that the superdiagrams appearing in the difference $< V^{a_1}V^{a_2}V^{a_3}V^{a_4}V^{a_5}V^{a_6} >_{\mathcal{N}=4} - < V^{a_1}V^{a_2}V^{a_3}V^{a_4}V^{a_5}V^{a_6} >_\beta$ and the associated contractions $\mathcal{C}$ are



$$\{Tr\,(a_1,a_2,a_3,a_4,a_5,a_6)\,,Tr\,(a_1,a_2,a_6,a_5,a_4,a_3)\,,Tr\,(a_1,a_3,a_4,a_2,a_6,a_5)\,,Tr\,(a_1,a_3,a_4,a_5,a_6,a_2)\,,$$
$$Tr\,(a_1,a_3,a_5,a_2,a_6,a_4)\,,Tr\,(a_1,a_3,a_6,a_2,a_5,a_4)\,,Tr\,(a_1,a_4,a_3,a_2,a_5,a_6)\,,Tr\,(a_1,a_4,a_5,a_2,a_6,a_3)\,,$$
$$Tr\,(a_1,a_4,a_6,a_2,a_5,a_3)\,,Tr\,(a_1,a_5,a_3,a_2,a_4,a_6)\,,Tr\,(a_1,a_5,a_4,a_2,a_3,a_6)\,,Tr\,(a_1,a_5,a_6,a_2,a_4,a_3)\,,$$
$$Tr\,(a_1,a_6,a_3,a_2,a_4,a_5)\,,Tr\,(a_1,a_6,a_4,a_2,a_3,a_5)\,,Tr\,(a_1,a_6,a_5,a_2,a_3,a_4)\,,Tr\,(a_1,a_6,a_5,a_4,a_3,a_2)\}$$

$$\mathcal{C}=\frac{3}{16}\,d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1)\,\left(N^2+1\right) \tag{3.7.23}$$

$$\{Tr\,(a_1,a_2,a_3,a_4,a_6,a_5)\,,Tr\,(a_1,a_2,a_3,a_5,a_4,a_6)\,,Tr\,(a_1,a_2,a_3,a_5,a_6,a_4)\,,Tr\,(a_1,a_2,a_3,a_6,a_4,a_5)\,,$$
$$Tr\,(a_1,a_2,a_3,a_6,a_5,a_4)\,,Tr\,(a_1,a_2,a_4,a_3,a_5,a_6)\,,Tr\,(a_1,a_2,a_4,a_3,a_6,a_5)\,,Tr\,(a_1,a_2,a_4,a_5,a_3,a_6)\,,$$
$$Tr\,(a_1,a_2,a_4,a_5,a_6,a_3)\,,Tr\,(a_1,a_2,a_4,a_6,a_3,a_5)\,,Tr\,(a_1,a_2,a_4,a_6,a_5,a_3)\,,Tr\,(a_1,a_2,a_5,a_3,a_4,a_6)\,,$$
$$Tr\,(a_1,a_2,a_5,a_3,a_6,a_4)\,,Tr\,(a_1,a_2,a_5,a_4,a_3,a_6)\,,Tr\,(a_1,a_2,a_5,a_4,a_6,a_3)\,,Tr\,(a_1,a_2,a_5,a_6,a_3,a_4)\,,$$
$$Tr\,(a_1,a_2,a_5,a_6,a_4,a_3)\,,Tr\,(a_1,a_2,a_6,a_3,a_4,a_5)\,,Tr\,(a_1,a_2,a_6,a_3,a_5,a_4)\,,Tr\,(a_1,a_2,a_6,a_4,a_3,a_5)\,,$$
$$Tr\,(a_1,a_2,a_6,a_4,a_5,a_3)\,,Tr\,(a_1,a_2,a_6,a_5,a_3,a_4)\,,Tr\,(a_1,a_3,a_4,a_2,a_5,a_6)\,,Tr\,(a_1,a_3,a_4,a_6,a_5,a_2)\,,$$
$$Tr\,(a_1,a_3,a_5,a_4,a_6,a_2)\,,Tr\,(a_1,a_3,a_5,a_6,a_4,a_2)\,,Tr\,(a_1,a_3,a_6,a_2,a_4,a_5)\,,Tr\,(a_1,a_3,a_6,a_4,a_2,a_5)\,,$$
$$Tr\,(a_1,a_3,a_6,a_4,a_5,a_2)\,,Tr\,(a_1,a_3,a_6,a_5,a_4,a_2)\,,Tr\,(a_1,a_4,a_2,a_5,a_3,a_6)\,,Tr\,(a_1,a_4,a_2,a_6,a_3,a_5)\,,$$
$$Tr\,(a_1,a_4,a_3,a_2,a_6,a_5)\,,Tr\,(a_1,a_4,a_3,a_5,a_6,a_2)\,,Tr\,(a_1,a_4,a_3,a_6,a_5,a_2)\,,Tr\,(a_1,a_4,a_5,a_2,a_3,a_6)\,,$$
$$Tr\,(a_1,a_4,a_5,a_3,a_6,a_2)\,,Tr\,(a_1,a_4,a_5,a_6,a_3,a_2)\,,Tr\,(a_1,a_4,a_6,a_3,a_2,a_5)\,,Tr\,(a_1,a_4,a_6,a_3,a_5,a_2)\,,$$
$$Tr\,(a_1,a_4,a_6,a_5,a_3,a_2)\,,Tr\,(a_1,a_5,a_2,a_3,a_6,a_4)\,,Tr\,(a_1,a_5,a_2,a_4,a_6,a_3)\,,Tr\,(a_1,a_5,a_3,a_4,a_6,a_2)\,,$$
$$Tr\,(a_1,a_5,a_3,a_6,a_2,a_4)\,,Tr\,(a_1,a_5,a_3,a_6,a_4,a_2)\,,Tr\,(a_1,a_5,a_4,a_2,a_6,a_3)\,,Tr\,(a_1,a_5,a_4,a_3,a_6,a_2)\,,$$
$$Tr\,(a_1,a_5,a_4,a_6,a_3,a_2)\,,Tr\,(a_1,a_5,a_6,a_2,a_3,a_4)\,,Tr\,(a_1,a_5,a_6,a_3,a_4,a_2)\,,Tr\,(a_1,a_5,a_6,a_4,a_3,a_2)\,,$$
$$Tr\,(a_1,a_6,a_3,a_2,a_5,a_4)\,,Tr\,(a_1,a_6,a_3,a_4,a_5,a_2)\,,Tr\,(a_1,a_6,a_3,a_5,a_2,a_4)\,,Tr\,(a_1,a_6,a_3,a_5,a_4,a_2)\,,$$
$$Tr\,(a_1,a_6,a_4,a_3,a_5,a_2)\,,Tr\,(a_1,a_6,a_4,a_5,a_3,a_2)\,,Tr\,(a_1,a_6,a_5,a_2,a_4,a_3)\,,Tr\,(a_1,a_6,a_5,a_3,a_4,a_2)\}$$

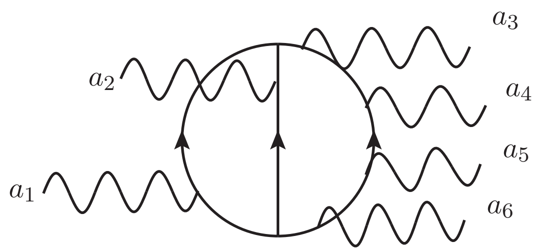
$$\mathcal{C}=\frac{3}{16}\,d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.24}$$

$$\{Tr\,(a_1,a_3,a_2,a_4,a_5,a_6)\,,Tr\,(a_1,a_3,a_2,a_6,a_5,a_4)\,,Tr\,(a_1,a_3,a_4,a_5,a_2,a_6)\,,Tr\,(a_1,a_3,a_4,a_6,a_2,a_5)\,,$$
$$Tr\,(a_1,a_3,a_5,a_6,a_2,a_4)\,,Tr\,(a_1,a_4,a_2,a_3,a_5,a_6)\,,Tr\,(a_1,a_4,a_2,a_6,a_5,a_3)\,,Tr\,(a_1,a_4,a_5,a_6,a_2,a_3)\,,$$
$$Tr\,(a_1,a_5,a_2,a_3,a_4,a_6)\,,Tr\,(a_1,a_5,a_2,a_6,a_4,a_3)\,,Tr\,(a_1,a_5,a_4,a_3,a_2,a_6)\,,Tr\,(a_1,a_6,a_2,a_3,a_4,a_5)\,,$$
$$Tr\,(a_1,a_6,a_2,a_5,a_4,a_3)\,,Tr\,(a_1,a_6,a_4,a_3,a_2,a_5)\,,Tr\,(a_1,a_6,a_5,a_3,a_2,a_4)\,,Tr\,(a_1,a_6,a_5,a_4,a_2,a_3)\}$$

$$\mathcal{C}=-\frac{3}{16}\,d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1)\,\left(N^2+2\right) \tag{3.7.25}$$

$$\{Tr\,(a_1,a_3,a_2,a_4,a_6,a_5)\,,Tr\,(a_1,a_3,a_2,a_5,a_4,a_6)\,,Tr\,(a_1,a_3,a_2,a_5,a_6,a_4)\,,Tr\,(a_1,a_3,a_2,a_6,a_4,a_5)\,,$$
$$Tr\,(a_1,a_3,a_5,a_2,a_4,a_6)\,,Tr\,(a_1,a_3,a_5,a_4,a_2,a_6)\,,Tr\,(a_1,a_3,a_6,a_5,a_2,a_4)\,,Tr\,(a_1,a_4,a_2,a_3,a_6,a_5)\,,$$
$$Tr\,(a_1,a_4,a_2,a_5,a_6,a_3)\,,Tr\,(a_1,a_4,a_3,a_5,a_2,a_6)\,,Tr\,(a_1,a_4,a_3,a_6,a_2,a_5)\,,Tr\,(a_1,a_4,a_5,a_3,a_2,a_6)\,,$$
$$Tr\,(a_1,a_4,a_6,a_2,a_3,a_5)\,,Tr\,(a_1,a_4,a_6,a_5,a_2,a_3)\,,Tr\,(a_1,a_5,a_2,a_4,a_3,a_6)\,,Tr\,(a_1,a_5,a_2,a_6,a_3,a_4)\,,$$
$$Tr\,(a_1,a_5,a_3,a_2,a_6,a_4)\,,Tr\,(a_1,a_5,a_3,a_4,a_2,a_6)\,,Tr\,(a_1,a_5,a_4,a_6,a_2,a_3)\,,Tr\,(a_1,a_5,a_6,a_3,a_2,a_4)\,,$$
$$Tr\,(a_1,a_5,a_6,a_4,a_2,a_3)\,,Tr\,(a_1,a_6,a_2,a_3,a_5,a_4)\,,Tr\,(a_1,a_6,a_2,a_4,a_3,a_5)\,,Tr\,(a_1,a_6,a_2,a_4,a_5,a_3)\,,$$
$$Tr\,(a_1,a_6,a_2,a_5,a_3,a_4)\,,Tr\,(a_1,a_6,a_3,a_4,a_2,a_5)\,,Tr\,(a_1,a_6,a_4,a_2,a_5,a_3)\,,Tr\,(a_1,a_6,a_4,a_5,a_2,a_3)\}$$

$$\mathcal{C} = -\frac{3}{8}\, d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.26}$$



$$\{Tr\,(a_1,a_2,a_3,a_4,a_5,a_6)\,,\ Tr\,(a_1,a_2,a_6,a_5,a_4,a_3)\,,\ Tr\,(a_1,a_3,a_4,a_2,a_6,a_5)\,,\ Tr\,(a_1,a_3,a_4,a_5,a_6,a_2)\,,$$
$$Tr\,(a_1,a_3,a_5,a_2,a_6,a_4)\,,\ Tr\,(a_1,a_3,a_6,a_2,a_5,a_4)\,,\ Tr\,(a_1,a_4,a_3,a_2,a_5,a_6)\,,\ Tr\,(a_1,a_4,a_5,a_2,a_6,a_3)\,,$$
$$Tr\,(a_1,a_4,a_6,a_2,a_5,a_3)\,,\ Tr\,(a_1,a_5,a_3,a_2,a_4,a_6)\,,\ Tr\,(a_1,a_5,a_4,a_2,a_3,a_6)\,,\ Tr\,(a_1,a_5,a_6,a_2,a_4,a_3)\,,$$
$$Tr\,(a_1,a_6,a_3,a_2,a_4,a_5)\,,\ Tr\,(a_1,a_6,a_4,a_2,a_3,a_5)\,,\ Tr\,(a_1,a_6,a_5,a_2,a_3,a_4)\,,\ Tr\,(a_1,a_6,a_5,a_4,a_3,a_2)\}$$

$$\mathcal{C} = -\frac{3}{16}\, d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1)\left(N^2+1\right) \tag{3.7.27}$$

$$\{Tr\,(a_1,a_2,a_3,a_4,a_6,a_5)\,,\ Tr\,(a_1,a_2,a_3,a_5,a_4,a_6)\,,\ Tr\,(a_1,a_2,a_3,a_5,a_6,a_4)\,,\ Tr\,(a_1,a_2,a_3,a_6,a_4,a_5)\,,$$
$$Tr\,(a_1,a_2,a_3,a_6,a_5,a_4)\,,\ Tr\,(a_1,a_2,a_4,a_3,a_5,a_6)\,,\ Tr\,(a_1,a_2,a_4,a_3,a_6,a_5)\,,\ Tr\,(a_1,a_2,a_4,a_5,a_3,a_6)\,,$$
$$Tr\,(a_1,a_2,a_4,a_5,a_6,a_3)\,,\ Tr\,(a_1,a_2,a_4,a_6,a_3,a_5)\,,\ Tr\,(a_1,a_2,a_4,a_6,a_5,a_3)\,,\ Tr\,(a_1,a_2,a_5,a_3,a_4,a_6)\,,$$
$$Tr\,(a_1,a_2,a_5,a_3,a_6,a_4)\,,\ Tr\,(a_1,a_2,a_5,a_4,a_3,a_6)\,,\ Tr\,(a_1,a_2,a_5,a_4,a_6,a_3)\,,\ Tr\,(a_1,a_2,a_5,a_6,a_3,a_4)\,,$$
$$Tr\,(a_1,a_2,a_5,a_6,a_4,a_3)\,,\ Tr\,(a_1,a_2,a_6,a_3,a_4,a_5)\,,\ Tr\,(a_1,a_2,a_6,a_3,a_5,a_4)\,,\ Tr\,(a_1,a_2,a_6,a_4,a_3,a_5)\,,$$
$$Tr\,(a_1,a_2,a_6,a_4,a_5,a_3)\,,\ Tr\,(a_1,a_2,a_6,a_5,a_3,a_4)\,,\ Tr\,(a_1,a_3,a_4,a_2,a_5,a_6)\,,\ Tr\,(a_1,a_3,a_4,a_6,a_5,a_2)\,,$$
$$Tr\,(a_1,a_3,a_5,a_4,a_6,a_2)\,,\ Tr\,(a_1,a_3,a_5,a_6,a_4,a_2)\,,\ Tr\,(a_1,a_3,a_6,a_2,a_4,a_5)\,,\ Tr\,(a_1,a_3,a_6,a_4,a_2,a_5)\,,$$
$$Tr\,(a_1,a_3,a_6,a_4,a_5,a_2)\,,\ Tr\,(a_1,a_3,a_6,a_5,a_4,a_2)\,,\ Tr\,(a_1,a_4,a_2,a_5,a_3,a_6)\,,\ Tr\,(a_1,a_4,a_2,a_6,a_3,a_5)\,,$$
$$Tr\,(a_1,a_4,a_3,a_2,a_6,a_5)\,,\ Tr\,(a_1,a_4,a_3,a_5,a_6,a_2)\,,\ Tr\,(a_1,a_4,a_3,a_6,a_5,a_2)\,,\ Tr\,(a_1,a_4,a_5,a_2,a_3,a_6)\,,$$
$$Tr\,(a_1,a_4,a_5,a_3,a_6,a_2)\,,\ Tr\,(a_1,a_4,a_5,a_6,a_3,a_2)\,,\ Tr\,(a_1,a_4,a_6,a_3,a_2,a_5)\,,\ Tr\,(a_1,a_4,a_6,a_3,a_5,a_2)\,,$$
$$Tr\,(a_1,a_4,a_6,a_5,a_3,a_2)\,,\ Tr\,(a_1,a_5,a_2,a_3,a_6,a_4)\,,\ Tr\,(a_1,a_5,a_2,a_4,a_6,a_3)\,,\ Tr\,(a_1,a_5,a_3,a_4,a_6,a_2)\,,$$
$$Tr\,(a_1,a_5,a_3,a_6,a_2,a_4)\,,\ Tr\,(a_1,a_5,a_3,a_6,a_4,a_2)\,,\ Tr\,(a_1,a_5,a_4,a_2,a_6,a_3)\,,\ Tr\,(a_1,a_5,a_4,a_3,a_6,a_2)\,,$$
$$Tr\,(a_1,a_5,a_4,a_6,a_3,a_2)\,,\ Tr\,(a_1,a_5,a_6,a_2,a_3,a_4)\,,\ Tr\,(a_1,a_5,a_6,a_3,a_4,a_2)\,,\ Tr\,(a_1,a_5,a_6,a_4,a_3,a_2)\,,$$
$$Tr\,(a_1,a_6,a_3,a_2,a_5,a_4)\,,\ Tr\,(a_1,a_6,a_3,a_4,a_5,a_2)\,,\ Tr\,(a_1,a_6,a_3,a_5,a_2,a_4)\,,\ Tr\,(a_1,a_6,a_3,a_5,a_4,a_2)\,,$$
$$Tr\,(a_1,a_6,a_4,a_3,a_5,a_2)\,,\ Tr\,(a_1,a_6,a_4,a_5,a_3,a_2)\,,\ Tr\,(a_1,a_6,a_5,a_2,a_4,a_3)\,,\ Tr\,(a_1,a_6,a_5,a_3,a_4,a_2)\}$$

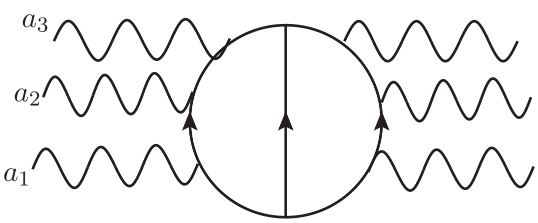
$$\mathcal{C} = -\frac{3}{16}\, d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.28}$$

$$\{Tr\,(a_1,a_3,a_2,a_4,a_5,a_6)\,,\ Tr\,(a_1,a_3,a_2,a_6,a_5,a_4)\,,\ Tr\,(a_1,a_3,a_4,a_5,a_2,a_6)\,,\ Tr\,(a_1,a_3,a_4,a_6,a_2,a_5)\,,$$
$$Tr\,(a_1,a_3,a_5,a_6,a_2,a_4)\,,\ Tr\,(a_1,a_4,a_2,a_3,a_5,a_6)\,,\ Tr\,(a_1,a_4,a_2,a_6,a_5,a_3)\,,\ Tr\,(a_1,a_4,a_5,a_6,a_2,a_3)\,,$$
$$Tr\,(a_1,a_5,a_2,a_3,a_4,a_6)\,,\ Tr\,(a_1,a_5,a_2,a_6,a_4,a_3)\,,\ Tr\,(a_1,a_5,a_4,a_3,a_2,a_6)\,,\ Tr\,(a_1,a_6,a_2,a_3,a_4,a_5)\,,$$
$$Tr\,(a_1,a_6,a_2,a_5,a_4,a_3)\,,\ Tr\,(a_1,a_6,a_4,a_3,a_2,a_5)\,,\ Tr\,(a_1,a_6,a_5,a_3,a_2,a_4)\,,\ Tr\,(a_1,a_6,a_5,a_4,a_2,a_3)\}$$

$$\mathcal{C} = \frac{3}{16}\, d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1)\left(N^2+2\right) \tag{3.7.29}$$

$\{Tr\,(a_1,a_3,a_2,a_4,a_6,a_5)\,,\, Tr\,(a_1,a_3,a_2,a_5,a_4,a_6)\,,\, Tr\,(a_1,a_3,a_2,a_5,a_6,a_4)\,,\, Tr\,(a_1,a_3,a_2,a_6,a_4,a_5)\,,$
$Tr\,(a_1,a_3,a_5,a_2,a_4,a_6)\,,\, Tr\,(a_1,a_3,a_5,a_4,a_2,a_6)\,,\, Tr\,(a_1,a_3,a_6,a_5,a_2,a_4)\,,\, Tr\,(a_1,a_4,a_2,a_3,a_6,a_5)\,,$
$Tr\,(a_1,a_4,a_2,a_5,a_6,a_3)\,,\, Tr\,(a_1,a_4,a_3,a_5,a_2,a_6)\,,\, Tr\,(a_1,a_4,a_3,a_6,a_2,a_5)\,,\, Tr\,(a_1,a_4,a_5,a_3,a_2,a_6)\,,$
$Tr\,(a_1,a_4,a_6,a_2,a_3,a_5)\,,\, Tr\,(a_1,a_4,a_6,a_5,a_2,a_3)\,,\, Tr\,(a_1,a_5,a_2,a_4,a_3,a_6)\,,\, Tr\,(a_1,a_5,a_2,a_6,a_3,a_4)\,,$
$Tr\,(a_1,a_5,a_3,a_2,a_6,a_4)\,,\, Tr\,(a_1,a_5,a_3,a_4,a_2,a_6)\,,\, Tr\,(a_1,a_5,a_4,a_6,a_2,a_3)\,,\, Tr\,(a_1,a_5,a_6,a_3,a_2,a_4)\,,$
$Tr\,(a_1,a_5,a_6,a_4,a_2,a_3)\,,\, Tr\,(a_1,a_6,a_2,a_3,a_5,a_4)\,,\, Tr\,(a_1,a_6,a_2,a_4,a_3,a_5)\,,\, Tr\,(a_1,a_6,a_2,a_4,a_5,a_3)\,,$
$Tr\,(a_1,a_6,a_2,a_5,a_3,a_4)\,,\, Tr\,(a_1,a_6,a_3,a_4,a_2,a_5)\,,\, Tr\,(a_1,a_6,a_4,a_2,a_5,a_3)\,,\, Tr\,(a_1,a_6,a_4,a_5,a_2,a_3)\}$

$$\mathcal{C} = \frac{3}{8}\, d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.30}$$



$$\{Tr\,(a_1,a_2,a_3,a_4,a_5,a_6)\,,\, Tr\,(a_1,a_6,a_5,a_4,a_3,a_2)\}$$

$$\mathcal{C} = -\frac{3}{16}\, d\bar{d}N^2\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.31}$$

$\{Tr\,(a_1,a_2,a_3,a_4,a_6,a_5)\,,\, Tr\,(a_1,a_2,a_3,a_5,a_4,a_6)\,,\, Tr\,(a_1,a_2,a_3,a_5,a_6,a_4)\,,\, Tr\,(a_1,a_2,a_3,a_6,a_4,a_5)\,,$
$Tr\,(a_1,a_2,a_4,a_5,a_6,a_3)\,,\, Tr\,(a_1,a_2,a_4,a_6,a_5,a_3)\,,\, Tr\,(a_1,a_2,a_5,a_4,a_6,a_3)\,,\, Tr\,(a_1,a_2,a_5,a_6,a_4,a_3)\,,$
$Tr\,(a_1,a_2,a_6,a_4,a_5,a_3)\,,\, Tr\,(a_1,a_2,a_6,a_5,a_4,a_3)\,,\, Tr\,(a_1,a_3,a_2,a_4,a_5,a_6)\,,\, Tr\,(a_1,a_3,a_2,a_4,a_6,a_5)\,,$
$Tr\,(a_1,a_3,a_2,a_5,a_4,a_6)\,,\, Tr\,(a_1,a_3,a_2,a_5,a_6,a_4)\,,\, Tr\,(a_1,a_3,a_2,a_6,a_4,a_5)\,,\, Tr\,(a_1,a_3,a_2,a_6,a_5,a_4)\,,$
$Tr\,(a_1,a_3,a_4,a_5,a_6,a_2)\,,\, Tr\,(a_1,a_3,a_4,a_6,a_5,a_2)\,,\, Tr\,(a_1,a_3,a_5,a_4,a_6,a_2)\,,\, Tr\,(a_1,a_3,a_5,a_6,a_4,a_2)\,,$
$Tr\,(a_1,a_3,a_6,a_4,a_5,a_2)\,,\, Tr\,(a_1,a_3,a_6,a_5,a_4,a_2)\,,\, Tr\,(a_1,a_4,a_2,a_5,a_3,a_6)\,,\, Tr\,(a_1,a_4,a_2,a_6,a_3,a_5)\,,$
$Tr\,(a_1,a_4,a_3,a_6,a_2,a_5)\,,\, Tr\,(a_1,a_4,a_5,a_6,a_2,a_3)\,,\, Tr\,(a_1,a_4,a_6,a_5,a_2,a_3)\,,\, Tr\,(a_1,a_4,a_6,a_5,a_3,a_2)\,,$
$Tr\,(a_1,a_5,a_2,a_6,a_3,a_4)\,,\, Tr\,(a_1,a_5,a_3,a_6,a_2,a_4)\,,\, Tr\,(a_1,a_5,a_4,a_6,a_2,a_3)\,,\, Tr\,(a_1,a_5,a_4,a_6,a_3,a_2)\,,$
$Tr\,(a_1,a_5,a_6,a_4,a_2,a_3)\,,\, Tr\,(a_1,a_5,a_6,a_4,a_3,a_2)\,,\, Tr\,(a_1,a_6,a_3,a_5,a_2,a_4)\,,\, Tr\,(a_1,a_6,a_4,a_5,a_2,a_3)\,,$
$Tr\,(a_1,a_6,a_4,a_5,a_3,a_2)\,,\, Tr\,(a_1,a_6,a_5,a_4,a_2,a_3)\}$

$$\mathcal{C} = 0 \tag{3.7.32}$$

$$\{Tr\,(a_1,a_2,a_3,a_6,a_5,a_4)\,,\, Tr\,(a_1,a_4,a_5,a_6,a_3,a_2)\}$$

$$\mathcal{C} = \frac{3}{16}\, d\bar{d}N^2\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.33}$$

$$\{Tr\,(a_1,a_2,a_4,a_3,a_5,a_6)\,,\,Tr\,(a_1,a_2,a_4,a_5,a_3,a_6)\,,\,Tr\,(a_1,a_2,a_4,a_6,a_3,a_5)\,,\,Tr\,(a_1,a_2,a_5,a_3,a_4,a_6)\,,$$
$$Tr\,(a_1,a_2,a_5,a_6,a_3,a_4)\,,\,Tr\,(a_1,a_2,a_6,a_3,a_4,a_5)\,,\,Tr\,(a_1,a_3,a_4,a_2,a_5,a_6)\,,\,Tr\,(a_1,a_3,a_4,a_5,a_2,a_6)\,,$$
$$Tr\,(a_1,a_3,a_5,a_6,a_2,a_4)\,,\,Tr\,(a_1,a_3,a_6,a_2,a_4,a_5)\,,\,Tr\,(a_1,a_4,a_2,a_3,a_5,a_6)\,,\,Tr\,(a_1,a_4,a_2,a_6,a_5,a_3)\,,$$
$$Tr\,(a_1,a_4,a_3,a_2,a_6,a_5)\,,\,Tr\,(a_1,a_4,a_3,a_6,a_5,a_2)\,,\,Tr\,(a_1,a_4,a_5,a_2,a_3,a_6)\,,\,Tr\,(a_1,a_4,a_6,a_2,a_3,a_5)\,,$$
$$Tr\,(a_1,a_5,a_2,a_3,a_4,a_6)\,,\,Tr\,(a_1,a_5,a_3,a_2,a_6,a_4)\,,\,Tr\,(a_1,a_5,a_3,a_6,a_4,a_2)\,,\,Tr\,(a_1,a_5,a_4,a_2,a_6,a_3)\,,$$
$$Tr\,(a_1,a_5,a_4,a_3,a_2,a_6)\,,\,Tr\,(a_1,a_5,a_4,a_3,a_6,a_2)\,,\,Tr\,(a_1,a_5,a_6,a_2,a_3,a_4)\,,\,Tr\,(a_1,a_6,a_2,a_3,a_4,a_5)\,,$$
$$Tr\,(a_1,a_6,a_2,a_5,a_4,a_3)\,,\,Tr\,(a_1,a_6,a_3,a_2,a_5,a_4)\,,\,Tr\,(a_1,a_6,a_3,a_5,a_4,a_2)\,,\,Tr\,(a_1,a_6,a_4,a_3,a_2,a_5)\,,$$
$$Tr\,(a_1,a_6,a_4,a_3,a_5,a_2)\,,\,Tr\,(a_1,a_6,a_5,a_2,a_4,a_3)\,,\,Tr\,(a_1,a_6,a_5,a_3,a_2,a_4)\,,\,Tr\,(a_1,a_6,a_5,a_3,a_4,a_2)\}$$

$$\mathcal{C} = \frac{3}{32}\,d\bar{d}N^2\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.34}$$

$$\{Tr\,(a_1,a_2,a_4,a_3,a_6,a_5)\,,\,Tr\,(a_1,a_2,a_5,a_3,a_6,a_4)\,,\,Tr\,(a_1,a_2,a_5,a_4,a_3,a_6)\,,\,Tr\,(a_1,a_2,a_6,a_3,a_5,a_4)\,,$$
$$Tr\,(a_1,a_2,a_6,a_4,a_3,a_5)\,,\,Tr\,(a_1,a_2,a_6,a_5,a_3,a_4)\,,\,Tr\,(a_1,a_3,a_4,a_2,a_6,a_5)\,,\,Tr\,(a_1,a_3,a_5,a_4,a_2,a_6)\,,$$
$$Tr\,(a_1,a_3,a_6,a_2,a_5,a_4)\,,\,Tr\,(a_1,a_3,a_6,a_5,a_2,a_4)\,,\,Tr\,(a_1,a_4,a_2,a_3,a_6,a_5)\,,\,Tr\,(a_1,a_4,a_2,a_5,a_6,a_3)\,,$$
$$Tr\,(a_1,a_4,a_3,a_2,a_5,a_6)\,,\,Tr\,(a_1,a_4,a_3,a_5,a_6,a_2)\,,\,Tr\,(a_1,a_4,a_5,a_2,a_6,a_3)\,,\,Tr\,(a_1,a_4,a_5,a_3,a_2,a_6)\,,$$
$$Tr\,(a_1,a_4,a_5,a_3,a_6,a_2)\,,\,Tr\,(a_1,a_4,a_6,a_3,a_2,a_5)\,,\,Tr\,(a_1,a_4,a_6,a_3,a_5,a_2)\,,\,Tr\,(a_1,a_5,a_2,a_3,a_6,a_4)\,,$$
$$Tr\,(a_1,a_5,a_3,a_2,a_4,a_6)\,,\,Tr\,(a_1,a_5,a_3,a_4,a_6,a_2)\,,\,Tr\,(a_1,a_5,a_4,a_2,a_3,a_6)\,,\,Tr\,(a_1,a_5,a_6,a_2,a_4,a_3)\,,$$
$$Tr\,(a_1,a_5,a_6,a_3,a_2,a_4)\,,\,Tr\,(a_1,a_5,a_6,a_3,a_4,a_2)\,,\,Tr\,(a_1,a_6,a_2,a_3,a_5,a_4)\,,\,Tr\,(a_1,a_6,a_2,a_4,a_5,a_3)\,,$$
$$Tr\,(a_1,a_6,a_3,a_2,a_4,a_5)\,,\,Tr\,(a_1,a_6,a_3,a_4,a_5,a_2)\,,\,Tr\,(a_1,a_6,a_4,a_2,a_3,a_5)\,,\,Tr\,(a_1,a_6,a_5,a_2,a_3,a_4)\}$$

$$\mathcal{C} = -\frac{3}{32}\,d\bar{d}N^2\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.35}$$

$$\{Tr\,(a_1,a_3,a_4,a_6,a_2,a_5)\,,\,Tr\,(a_1,a_3,a_5,a_2,a_4,a_6)\,,\,Tr\,(a_1,a_5,a_2,a_6,a_4,a_3)\,,\,Tr\,(a_1,a_6,a_4,a_2,a_5,a_3)\}$$

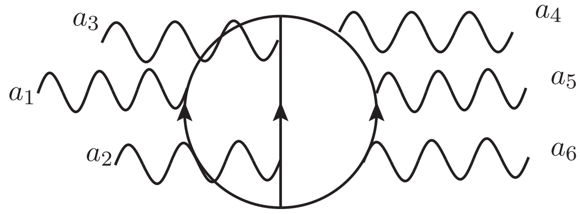
$$\mathcal{C} = \frac{3}{32}\,d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1)\left(N^2+6\right) \tag{3.7.36}$$

$$\{Tr\,(a_1,a_3,a_5,a_2,a_6,a_4)\,,\,Tr\,(a_1,a_3,a_6,a_4,a_2,a_5)\,,\,Tr\,(a_1,a_4,a_6,a_2,a_5,a_3)\,,\,Tr\,(a_1,a_5,a_2,a_4,a_6,a_3)\}$$

$$\mathcal{C} = -\frac{3}{32}\,d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1)\left(N^2+6\right) \tag{3.7.37}$$

$$\{Tr\,(a_1,a_4,a_3,a_5,a_2,a_6)\,,\,Tr\,(a_1,a_5,a_2,a_4,a_3,a_6)\,,\,Tr\,(a_1,a_5,a_3,a_4,a_2,a_6)\,,\,Tr\,(a_1,a_6,a_2,a_4,a_3,a_5)\,,$$
$$Tr\,(a_1,a_6,a_2,a_5,a_3,a_4)\,,\,Tr\,(a_1,a_6,a_3,a_4,a_2,a_5)\}$$

$$\mathcal{C} = \frac{9}{16}\,d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.38}$$



$\{Tr\,(a_1,a_2,a_3,a_4,a_5,a_6)\,,\,Tr\,(a_1,a_2,a_3,a_4,a_6,a_5)\,,\,Tr\,(a_1,a_2,a_3,a_5,a_4,a_6)\,,\,Tr\,(a_1,a_2,a_3,a_5,a_6,a_4)\,,$
$Tr\,(a_1,a_2,a_3,a_6,a_4,a_5)\,,\,Tr\,(a_1,a_3,a_2,a_4,a_6,a_5)\,,\,Tr\,(a_1,a_3,a_2,a_5,a_4,a_6)\,,\,Tr\,(a_1,a_3,a_2,a_5,a_6,a_4)\,,$
$Tr\,(a_1,a_3,a_2,a_6,a_4,a_5)\,,\,Tr\,(a_1,a_3,a_2,a_6,a_5,a_4)\,,\,Tr\,(a_1,a_4,a_2,a_6,a_3,a_5)\,,\,Tr\,(a_1,a_4,a_3,a_5,a_2,a_6)\,,$
$Tr\,(a_1,a_4,a_3,a_6,a_2,a_5)\,,\,Tr\,(a_1,a_4,a_5,a_6,a_2,a_3)\,,\,Tr\,(a_1,a_4,a_6,a_5,a_2,a_3)\,,\,Tr\,(a_1,a_4,a_6,a_5,a_3,a_2)\,,$
$Tr\,(a_1,a_5,a_2,a_4,a_3,a_6)\,,\,Tr\,(a_1,a_5,a_2,a_6,a_3,a_4)\,,\,Tr\,(a_1,a_5,a_3,a_4,a_2,a_6)\,,\,Tr\,(a_1,a_5,a_3,a_6,a_2,a_4)\,,$
$Tr\,(a_1,a_5,a_4,a_6,a_2,a_3)\,,\,Tr\,(a_1,a_5,a_4,a_6,a_3,a_2)\,,\,Tr\,(a_1,a_5,a_6,a_4,a_2,a_3)\,,\,Tr\,(a_1,a_5,a_6,a_4,a_3,a_2)\,,$
$Tr\,(a_1,a_6,a_2,a_4,a_3,a_5)\,,\,Tr\,(a_1,a_6,a_2,a_5,a_3,a_4)\,,\,Tr\,(a_1,a_6,a_3,a_4,a_2,a_5)\,,\,Tr\,(a_1,a_6,a_4,a_5,a_2,a_3)\,,$
$Tr\,(a_1,a_6,a_4,a_5,a_3,a_2)\,,\,Tr\,(a_1,a_6,a_5,a_4,a_3,a_2)\}$

$$\mathcal{C} = -\frac{3}{16}\,d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.39}$$

$\{Tr\,(a_1,a_2,a_3,a_6,a_5,a_4)\,,\,Tr\,(a_1,a_3,a_2,a_4,a_5,a_6)\,,\,Tr\,(a_1,a_4,a_5,a_6,a_3,a_2)\,,\,Tr\,(a_1,a_6,a_5,a_4,a_2,a_3)\}$

$$\mathcal{C} = -\frac{3}{16}\,d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1)\,\left(N^2+1\right) \tag{3.7.40}$$

$\{Tr\,(a_1,a_2,a_4,a_3,a_5,a_6)\,,\,Tr\,(a_1,a_2,a_4,a_3,a_6,a_5)\,,\,Tr\,(a_1,a_2,a_5,a_3,a_4,a_6)\,,\,Tr\,(a_1,a_2,a_6,a_3,a_4,a_5)\,,$
$Tr\,(a_1,a_2,a_6,a_3,a_5,a_4)\,,\,Tr\,(a_1,a_3,a_4,a_2,a_5,a_6)\,,\,Tr\,(a_1,a_3,a_4,a_2,a_6,a_5)\,,\,Tr\,(a_1,a_3,a_5,a_2,a_6,a_4)\,,$
$Tr\,(a_1,a_3,a_6,a_2,a_4,a_5)\,,\,Tr\,(a_1,a_3,a_6,a_2,a_5,a_4)\,,\,Tr\,(a_1,a_4,a_5,a_2,a_6,a_3)\,,\,Tr\,(a_1,a_4,a_5,a_3,a_6,a_2)\,,$
$Tr\,(a_1,a_4,a_6,a_2,a_5,a_3)\,,\,Tr\,(a_1,a_5,a_4,a_2,a_6,a_3)\,,\,Tr\,(a_1,a_5,a_4,a_3,a_6,a_2)\,,\,Tr\,(a_1,a_5,a_6,a_2,a_4,a_3)\,,$
$Tr\,(a_1,a_5,a_6,a_3,a_4,a_2)\,,\,Tr\,(a_1,a_6,a_4,a_3,a_5,a_2)\,,\,Tr\,(a_1,a_6,a_5,a_2,a_4,a_3)\,,\,Tr\,(a_1,a_6,a_5,a_3,a_4,a_2)\}$

$$\mathcal{C} = \frac{3}{32}\,d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1)\,\left(N^2+4\right) \tag{3.7.41}$$

$\{Tr\,(a_1,a_2,a_4,a_5,a_3,a_6)\,,\,Tr\,(a_1,a_2,a_5,a_4,a_3,a_6)\,,\,Tr\,(a_1,a_2,a_5,a_6,a_3,a_4)\,,\,Tr\,(a_1,a_2,a_6,a_4,a_3,a_5)\,,$
$Tr\,(a_1,a_2,a_6,a_5,a_3,a_4)\,,\,Tr\,(a_1,a_3,a_4,a_5,a_2,a_6)\,,\,Tr\,(a_1,a_3,a_4,a_6,a_2,a_5)\,,\,Tr\,(a_1,a_3,a_5,a_4,a_2,a_6)\,,$
$Tr\,(a_1,a_3,a_5,a_6,a_2,a_4)\,,\,Tr\,(a_1,a_3,a_6,a_5,a_2,a_4)\,,\,Tr\,(a_1,a_4,a_2,a_3,a_5,a_6)\,,\,Tr\,(a_1,a_4,a_2,a_5,a_6,a_3)\,,$
$Tr\,(a_1,a_4,a_2,a_6,a_5,a_3)\,,\,Tr\,(a_1,a_4,a_3,a_2,a_6,a_5)\,,\,Tr\,(a_1,a_4,a_3,a_5,a_6,a_2)\,,\,Tr\,(a_1,a_4,a_3,a_6,a_5,a_2)\,,$
$Tr\,(a_1,a_4,a_5,a_2,a_3,a_6)\,,\,Tr\,(a_1,a_4,a_6,a_2,a_3,a_5)\,,\,Tr\,(a_1,a_5,a_2,a_3,a_4,a_6)\,,\,Tr\,(a_1,a_5,a_2,a_6,a_4,a_3)\,,$
$Tr\,(a_1,a_5,a_3,a_2,a_6,a_4)\,,\,Tr\,(a_1,a_5,a_3,a_4,a_6,a_2)\,,\,Tr\,(a_1,a_5,a_4,a_3,a_2,a_6)\,,\,Tr\,(a_1,a_5,a_6,a_2,a_3,a_4)\,,$
$Tr\,(a_1,a_6,a_2,a_3,a_4,a_5)\,,\,Tr\,(a_1,a_6,a_2,a_4,a_5,a_3)\,,\,Tr\,(a_1,a_6,a_2,a_5,a_4,a_3)\,,\,Tr\,(a_1,a_6,a_3,a_2,a_5,a_4)\,,$
$Tr\,(a_1,a_6,a_3,a_4,a_5,a_2)\,,\,Tr\,(a_1,a_6,a_3,a_5,a_4,a_2)\,,\,Tr\,(a_1,a_6,a_4,a_3,a_2,a_5)\,,\,Tr\,(a_1,a_6,a_5,a_3,a_2,a_4)\}$

$$\mathcal{C} = -\frac{3}{32}\,d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1)\,\left(N^2+2\right) \tag{3.7.42}$$

$\{Tr\,(a_1,a_2,a_4,a_5,a_6,a_3)\,,\,Tr\,(a_1,a_2,a_6,a_5,a_4,a_3)\,,\,Tr\,(a_1,a_3,a_4,a_5,a_6,a_2)\,,\,Tr\,(a_1,a_3,a_6,a_5,a_4,a_2)\}$

$$\mathcal{C} = \frac{3}{16}\,d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1)\,\left(N^2+2\right) \tag{3.7.43}$$

$\{Tr\,(a_1,a_2,a_4,a_6,a_3,a_5)\,,\,Tr\,(a_1,a_3,a_6,a_4,a_2,a_5)\,,\,Tr\,(a_1,a_5,a_2,a_4,a_6,a_3)\,,\,Tr\,(a_1,a_5,a_3,a_6,a_4,a_2)\}$

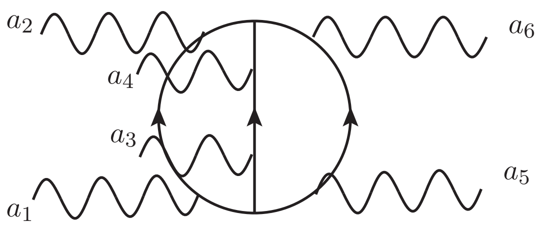
$$\mathcal{C} = -\frac{3}{32}\,d\bar{d}\,(N-1)\,(N+1)\,(N-2)^2\,(N+2)^2 \tag{3.7.44}$$

$\{Tr\,(a_1,a_2,a_4,a_6,a_5,a_3)\,,\,Tr\,(a_1,a_2,a_5,a_4,a_6,a_3)\,,\,Tr\,(a_1,a_2,a_5,a_6,a_4,a_3)\,,\,Tr\,(a_1,a_2,a_6,a_4,a_5,a_3)\,,$
$Tr\,(a_1,a_3,a_4,a_6,a_5,a_2)\,,\,Tr\,(a_1,a_3,a_5,a_4,a_6,a_2)\,,\,Tr\,(a_1,a_3,a_5,a_6,a_4,a_2)\,,\,Tr\,(a_1,a_3,a_6,a_4,a_5,a_2)\,,$
$Tr\,(a_1,a_4,a_2,a_5,a_3,a_6)\,,\,Tr\,(a_1,a_6,a_3,a_5,a_2,a_4)\}$

$$\mathcal{C} = \frac{3}{8}\,d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.45}$$

$\{Tr\,(a_1,a_2,a_5,a_3,a_6,a_4)\,,\,Tr\,(a_1,a_3,a_5,a_2,a_4,a_6)\,,\,Tr\,(a_1,a_4,a_2,a_3,a_6,a_5)\,,\,Tr\,(a_1,a_4,a_3,a_2,a_5,a_6)\,,$
$Tr\,(a_1,a_4,a_5,a_3,a_2,a_6)\,,\,Tr\,(a_1,a_4,a_6,a_3,a_2,a_5)\,,\,Tr\,(a_1,a_4,a_6,a_3,a_5,a_2)\,,\,Tr\,(a_1,a_5,a_2,a_3,a_6,a_4)\,,$
$Tr\,(a_1,a_5,a_3,a_2,a_4,a_6)\,,\,Tr\,(a_1,a_5,a_4,a_2,a_3,a_6)\,,\,Tr\,(a_1,a_5,a_6,a_3,a_2,a_4)\,,\,Tr\,(a_1,a_6,a_2,a_3,a_5,a_4)\,,$
$Tr\,(a_1,a_6,a_3,a_2,a_4,a_5)\,,\,Tr\,(a_1,a_6,a_4,a_2,a_3,a_5)\,,\,Tr\,(a_1,a_6,a_4,a_2,a_5,a_3)\,,\,Tr\,(a_1,a_6,a_5,a_2,a_3,a_4)\}$

$$\mathcal{C} = \frac{3}{32}\,d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1)\,\left(N^2-2\right) \tag{3.7.46}$$



$\{Tr\,(a_1,a_2,a_3,a_4,a_5,a_6)\,,\,Tr\,(a_1,a_2,a_5,a_6,a_3,a_4)\,,\,Tr\,(a_1,a_4,a_3,a_6,a_5,a_2)\,,\,Tr\,(a_1,a_6,a_5,a_4,a_3,a_2)\}$

$$\mathcal{C} = -\frac{3}{32}\,d\bar{d}\,(N-1)\,(N+1)\,(N-2)^2\,(N+2)^2 \tag{3.7.47}$$

$\{Tr\,(a_1,a_2,a_3,a_4,a_6,a_5)\,,\,Tr\,(a_1,a_2,a_4,a_3,a_5,a_6)\,,\,Tr\,(a_1,a_2,a_4,a_3,a_6,a_5)\,,\,Tr\,(a_1,a_2,a_5,a_6,a_4,a_3)\,,$
$Tr\,(a_1,a_2,a_6,a_5,a_3,a_4)\,,\,Tr\,(a_1,a_2,a_6,a_5,a_4,a_3)\,,\,Tr\,(a_1,a_3,a_4,a_5,a_6,a_2)\,,\,Tr\,(a_1,a_3,a_4,a_6,a_5,a_2)\,,$
$Tr\,(a_1,a_4,a_3,a_5,a_6,a_2)\,,\,Tr\,(a_1,a_5,a_6,a_3,a_4,a_2)\,,\,Tr\,(a_1,a_5,a_6,a_4,a_3,a_2)\,,\,Tr\,(a_1,a_6,a_5,a_3,a_4,a_2)\}$

$$\mathcal{C} = \frac{3}{32}\, d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1)\,\left(N^2+4\right) \tag{3.7.48}$$

$\{Tr\,(a_1,a_2,a_3,a_5,a_4,a_6)\,,\,Tr\,(a_1,a_2,a_3,a_6,a_4,a_5)\,,\,Tr\,(a_1,a_2,a_4,a_5,a_3,a_6)\,,\,Tr\,(a_1,a_2,a_4,a_6,a_3,a_5)\,,$
$Tr\,(a_1,a_2,a_5,a_3,a_6,a_4)\,,\,Tr\,(a_1,a_2,a_5,a_4,a_6,a_3)\,,\,Tr\,(a_1,a_2,a_6,a_3,a_5,a_4)\,,\,Tr\,(a_1,a_2,a_6,a_4,a_5,a_3)\,,$
$Tr\,(a_1,a_3,a_2,a_4,a_5,a_6)\,,\,Tr\,(a_1,a_3,a_2,a_4,a_6,a_5)\,,\,Tr\,(a_1,a_3,a_2,a_5,a_6,a_4)\,,\,Tr\,(a_1,a_3,a_2,a_6,a_5,a_4)\,,$
$Tr\,(a_1,a_3,a_4,a_5,a_2,a_6)\,,\,Tr\,(a_1,a_3,a_4,a_6,a_2,a_5)\,,\,Tr\,(a_1,a_3,a_5,a_4,a_6,a_2)\,,\,Tr\,(a_1,a_3,a_5,a_6,a_2,a_4)\,,$
$Tr\,(a_1,a_3,a_6,a_4,a_5,a_2)\,,\,Tr\,(a_1,a_3,a_6,a_5,a_2,a_4)\,,\,Tr\,(a_1,a_4,a_2,a_3,a_5,a_6)\,,\,Tr\,(a_1,a_4,a_2,a_3,a_6,a_5)\,,$
$Tr\,(a_1,a_4,a_2,a_5,a_6,a_3)\,,\,Tr\,(a_1,a_4,a_2,a_6,a_5,a_3)\,,\,Tr\,(a_1,a_4,a_3,a_5,a_2,a_6)\,,\,Tr\,(a_1,a_4,a_3,a_6,a_2,a_5)\,,$
$Tr\,(a_1,a_4,a_5,a_3,a_6,a_2)\,,\,Tr\,(a_1,a_4,a_5,a_6,a_2,a_3)\,,\,Tr\,(a_1,a_4,a_6,a_3,a_5,a_2)\,,\,Tr\,(a_1,a_4,a_6,a_5,a_2,a_3)\,,$
$Tr\,(a_1,a_5,a_2,a_3,a_4,a_6)\,,\,Tr\,(a_1,a_5,a_2,a_4,a_3,a_6)\,,\,Tr\,(a_1,a_5,a_2,a_6,a_3,a_4)\,,\,Tr\,(a_1,a_5,a_2,a_6,a_4,a_3)\,,$
$Tr\,(a_1,a_5,a_3,a_4,a_2,a_6)\,,\,Tr\,(a_1,a_5,a_3,a_6,a_4,a_2)\,,\,Tr\,(a_1,a_5,a_4,a_3,a_2,a_6)\,,\,Tr\,(a_1,a_5,a_4,a_6,a_3,a_2)\,,$
$Tr\,(a_1,a_5,a_6,a_3,a_2,a_4)\,,\,Tr\,(a_1,a_5,a_6,a_4,a_2,a_3)\,,\,Tr\,(a_1,a_6,a_2,a_3,a_4,a_5)\,,\,Tr\,(a_1,a_6,a_2,a_4,a_3,a_5)\,,$
$Tr\,(a_1,a_6,a_2,a_5,a_3,a_4)\,,\,Tr\,(a_1,a_6,a_2,a_5,a_4,a_3)\,,\,Tr\,(a_1,a_6,a_3,a_4,a_2,a_5)\,,\,Tr\,(a_1,a_6,a_3,a_5,a_4,a_2)\,,$
$Tr\,(a_1,a_6,a_4,a_3,a_2,a_5)\,,\,Tr\,(a_1,a_6,a_4,a_5,a_3,a_2)\,,\,Tr\,(a_1,a_6,a_5,a_3,a_2,a_4)\,,\,Tr\,(a_1,a_6,a_5,a_4,a_2,a_3)\}$

$$\mathcal{C} = -\frac{3}{32}\, d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1)\,\left(N^2+2\right) \tag{3.7.49}$$

$\{Tr\,(a_1,a_2,a_3,a_5,a_6,a_4)\,,\,Tr\,(a_1,a_2,a_4,a_5,a_6,a_3)\,,\,Tr\,(a_1,a_2,a_5,a_3,a_4,a_6)\,,\,Tr\,(a_1,a_2,a_6,a_3,a_4,a_5)\,,$
$Tr\,(a_1,a_3,a_4,a_2,a_5,a_6)\,,\,Tr\,(a_1,a_3,a_5,a_2,a_4,a_6)\,,\,Tr\,(a_1,a_3,a_6,a_2,a_4,a_5)\,,\,Tr\,(a_1,a_3,a_6,a_5,a_4,a_2)\,,$
$Tr\,(a_1,a_4,a_3,a_2,a_6,a_5)\,,\,Tr\,(a_1,a_4,a_6,a_2,a_3,a_5)\,,\,Tr\,(a_1,a_4,a_6,a_5,a_3,a_2)\,,\,Tr\,(a_1,a_5,a_3,a_2,a_6,a_4)\,,$
$Tr\,(a_1,a_5,a_4,a_2,a_6,a_3)\,,\,Tr\,(a_1,a_5,a_4,a_3,a_6,a_2)\,,\,Tr\,(a_1,a_5,a_6,a_2,a_3,a_4)\,,\,Tr\,(a_1,a_6,a_4,a_2,a_5,a_3)\,,$
$Tr\,(a_1,a_6,a_4,a_3,a_5,a_2)\,,\,Tr\,(a_1,a_6,a_5,a_2,a_4,a_3)\}$

$$\mathcal{C} = \frac{3}{16}\, d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1)\,\left(N^2+2\right) \tag{3.7.50}$$

$\{Tr\,(a_1,a_2,a_3,a_6,a_5,a_4)\,,\,Tr\,(a_1,a_2,a_4,a_6,a_5,a_3)\,,\,Tr\,(a_1,a_2,a_5,a_4,a_3,a_6)\,,\,Tr\,(a_1,a_2,a_6,a_4,a_3,a_5)\,,$
$Tr\,(a_1,a_3,a_4,a_2,a_6,a_5)\,,\,Tr\,(a_1,a_3,a_5,a_6,a_4,a_2)\,,\,Tr\,(a_1,a_4,a_3,a_2,a_5,a_6)\,,\,Tr\,(a_1,a_4,a_5,a_6,a_3,a_2)\,,$
$Tr\,(a_1,a_5,a_3,a_4,a_6,a_2)\,,\,Tr\,(a_1,a_5,a_6,a_2,a_4,a_3)\,,\,Tr\,(a_1,a_6,a_3,a_4,a_5,a_2)\,,\,Tr\,(a_1,a_6,a_5,a_2,a_3,a_4)\}$

$$\mathcal{C} = \frac{3}{8}\, d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.51}$$

$\{Tr\,(a_1,a_3,a_2,a_5,a_4,a_6)\,,\,Tr\,(a_1,a_3,a_2,a_6,a_4,a_5)\,,\,Tr\,(a_1,a_3,a_5,a_2,a_6,a_4)\,,\,Tr\,(a_1,a_3,a_5,a_4,a_2,a_6)\,,$
$Tr\,(a_1,a_3,a_6,a_2,a_5,a_4)\,,\,Tr\,(a_1,a_3,a_6,a_4,a_2,a_5)\,,\,Tr\,(a_1,a_4,a_2,a_5,a_3,a_6)\,,\,Tr\,(a_1,a_4,a_2,a_6,a_3,a_5)\,,$
$Tr\,(a_1,a_4,a_5,a_2,a_6,a_3)\,,\,Tr\,(a_1,a_4,a_5,a_3,a_2,a_6)\,,\,Tr\,(a_1,a_4,a_6,a_2,a_5,a_3)\,,\,Tr\,(a_1,a_4,a_6,a_3,a_2,a_5)\,,$
$Tr\,(a_1,a_5,a_2,a_3,a_6,a_4)\,,\,Tr\,(a_1,a_5,a_2,a_4,a_6,a_3)\,,\,Tr\,(a_1,a_5,a_3,a_2,a_4,a_6)\,,\,Tr\,(a_1,a_5,a_3,a_6,a_2,a_4)\,,$
$Tr\,(a_1,a_5,a_4,a_2,a_3,a_6)\,,\,Tr\,(a_1,a_5,a_4,a_6,a_2,a_3)\,,\,Tr\,(a_1,a_6,a_2,a_3,a_5,a_4)\,,\,Tr\,(a_1,a_6,a_2,a_4,a_5,a_3)\,,$
$Tr\,(a_1,a_6,a_3,a_2,a_4,a_5)\,,\,Tr\,(a_1,a_6,a_3,a_5,a_2,a_4)\,,\,Tr\,(a_1,a_6,a_4,a_2,a_3,a_5)\,,\,Tr\,(a_1,a_6,a_4,a_5,a_2,a_3)\}$

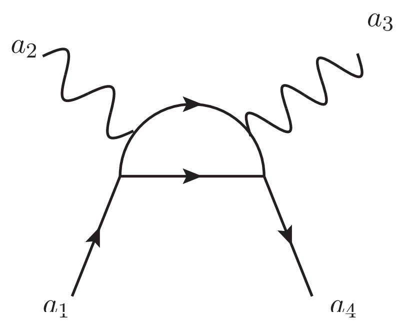
$$\mathcal{C} = -\frac{3}{16}\, d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.52}$$

$$\{Tr\,(a_1, a_4, a_5, a_2, a_3, a_6)\,,\, Tr\,(a_1, a_6, a_3, a_2, a_5, a_4)\}$$

$$\mathcal{C} = \frac{3}{16}\, d\bar{d}\,(N-1)\,(N+1)\,(N-2)^2\,(N+2)^2 \tag{3.7.53}$$

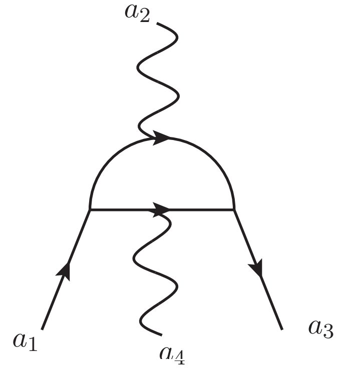
Since the contractions $\mathcal{C}$ are at most of order $O(N^6)$, the six point function $< V^{a_1}V^{a_2}V^{a_3}V^{a_4}V^{a_5}V^{a_6} >$ has a non-planar correction with respect to its value in the $\mathcal{N}=4$ SYM (a planar correction would be of order $O(N^8)$).
We have also studied the 'mixed' chiral-vector correlation functions with a chiral, an antichiral and two or three vector superfields $< \Phi^{a_1}\Phi^{\dagger\,a_2}V^{a_3}V^{a_4} >$ and $< \Phi^{a_1}\Phi^{\dagger\,a_2}V^{a_3}V^{a_4}V^{a_5} >$ respectively (the flavor indices are omitted)

They have a non planar correction already at one loop, which is in this case the first order with diagrams containing chiral vertices.
In fact, for the correlation function $< \Phi^{a_1}\Phi^{\dagger\,a_2}V^{a_3}V^{a_4} >$, the superdiagrams contributing to the difference between the β deformation and $\mathcal{N}=4$ SYM and the associated color contractions $\mathcal{C}$ are



$$\{Tr\,(a_1, a_2, a_3, a_4)\,,\, Tr\,(a_1, a_2, a_4, a_3)\,,\, Tr\,(a_1, a_3, a_2, a_4)\,,\, Tr\,(a_1, a_3, a_4, a_2)\,,$$
$$Tr\,(a_1, a_4, a_2, a_3)\,,\, Tr\,(a_1, a_4, a_3, a_2)\}$$

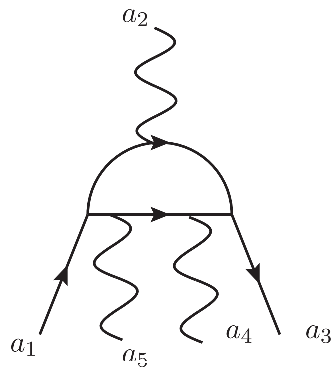
$$\mathcal{C} = \frac{1}{2}\, d\bar{d}\frac{(N-1)\,(N-2)\,(N+2)\,(N+1)}{N} \tag{3.7.54}$$



$$\{ Tr\,(a_1,a_2,a_3,a_4)\,,\, Tr\,(a_1,a_2,a_4,a_3)\,,\, Tr\,(a_1,a_3,a_2,a_4)\,,\, Tr\,(a_1,a_3,a_4,a_2)\,,$$
$$Tr\,(a_1,a_4,a_2,a_3)\,,\, Tr\,(a_1,a_4,a_3,a_2)\} \tag{3.7.55}$$

$$\mathcal{C} = -\frac{1}{2}\,d\bar{d}\frac{(N-1)\,(N-2)\,(N+2)\,(N+1)}{N} \tag{3.7.56}$$

Since, the color contractions $\mathcal{C}$ are at most of order $O(N^3)$, there is a non-planar deviation (a planar correction would be of order $O(N^5)$). As for the five point correlation function $< \Phi^{a_1}\Phi^{\dagger\, a_2}V^{a_3}V^{a_4}V^{a_5} >$, the superdiagrams contributing to the difference between the β deformation and $\mathcal{N}=4$ SYM and the associated color contractions $\mathcal{C}$ are



$$\{ Tr\,(a_1,a_2,a_3,a_4,a_5)\,,\, Tr\,(a_1,a_2,a_4,a_5,a_3)\,,\, Tr\,(a_1,a_2,a_5,a_3,a_4)\,,\, Tr\,(a_1,a_3,a_4,a_2,a_5)\,,$$
$$Tr\,(a_1,a_3,a_5,a_2,a_4)\,,\, Tr\,(a_1,a_4,a_2,a_3,a_5)\,,\, Tr\,(a_1,a_4,a_5,a_2,a_3)\}$$

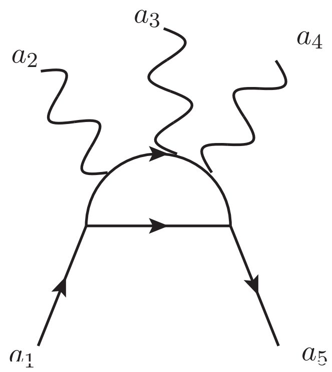
$$\mathcal{C} = \frac{i}{4}\,d\bar{d}\,(N-2)\,(N+2)\,(N+1) \tag{3.7.57}$$

$$\{ Tr\,(a_1,a_2,a_3,a_5,a_4)\,,\, Tr\,(a_1,a_2,a_4,a_3,a_5)\,,\, Tr\,(a_1,a_2,a_5,a_4,a_3)\,,\, Tr\,(a_1,a_3,a_2,a_4,a_5)\,,$$
$$Tr\,(a_1,a_3,a_4,a_5,a_2)\,,\, Tr\,(a_1,a_4,a_3,a_2,a_5)\,,\, Tr\,(a_1,a_4,a_5,a_3,a_2)\,,\, Tr\,(a_1,a_5,a_2,a_3,a_4)\,,$$
$$Tr\,(a_1,a_5,a_3,a_4,a_2)\,,\, Tr\,(a_1,a_5,a_4,a_2,a_3)\}$$

$$\mathcal{C} = 0 \tag{3.7.58}$$

$$\{ Tr\,(a_1,a_3,a_2,a_5,a_4)\,,\, Tr\,(a_1,a_3,a_5,a_4,a_2)\,,\, Tr\,(a_1,a_4,a_2,a_5,a_3)\,,\, Tr\,(a_1,a_4,a_3,a_5,a_2)\,,$$
$$Tr\,(a_1,a_5,a_2,a_4,a_3)\,,\, Tr\,(a_1,a_5,a_3,a_2,a_4)\,,\, Tr\,(a_1,a_5,a_4,a_3,a_2)\}$$

$$\mathcal{C} = -\frac{i}{4}\,d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.59}$$



$\{Tr\,(a_1,a_2,a_3,a_4,a_5)\,,\,Tr\,(a_1,a_2,a_4,a_5,a_3)\,,\,Tr\,(a_1,a_2,a_5,a_3,a_4)\,,\,Tr\,(a_1,a_4,a_5,a_2,a_3)\,,$
$Tr\,(a_1,a_5,a_2,a_3,a_4)\}$

$$\mathcal{C} = \frac{i}{4}\, d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.60}$$

$\{Tr\,(a_1,a_2,a_3,a_5,a_4)\,,\,Tr\,(a_1,a_2,a_4,a_3,a_5)\,,\,Tr\,(a_1,a_2,a_5,a_4,a_3)\,,\,Tr\,(a_1,a_3,a_2,a_4,a_5)\,,$
$Tr\,(a_1,a_3,a_4,a_2,a_5)\,,\,Tr\,(a_1,a_3,a_4,a_5,a_2)\,,\,Tr\,(a_1,a_4,a_2,a_3,a_5)\,,\,Tr\,(a_1,a_4,a_5,a_3,a_2)\,,$
$Tr\,(a_1,a_5,a_2,a_4,a_3)\,,\,Tr\,(a_1,a_5,a_3,a_2,a_4)\,,\,Tr\,(a_1,a_5,a_3,a_4,a_2)\,,\,Tr\,(a_1,a_5,a_4,a_2,a_3)\}$

$$\mathcal{C} = 0 \tag{3.7.61}$$

$\{Tr\,(a_1,a_3,a_2,a_5,a_4)\,,\,Tr\,(a_1,a_3,a_5,a_4,a_2)\,,\,Tr\,(a_1,a_4,a_3,a_2,a_5)\,,\,Tr\,(a_1,a_4,a_3,a_5,a_2)\,,$
$Tr\,(a_1,a_5,a_4,a_3,a_2)\}$

$$\mathcal{C} = -\frac{i}{4}\, d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.62}$$

$$\{Tr\,(a_1,a_3,a_5,a_2,a_4)\}$$

$$\mathcal{C} = \frac{i}{2}\, d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.63}$$

$$\{Tr\,(a_1,a_4,a_2,a_5,a_3)\}$$

$$\mathcal{C} = -\frac{i}{2}\, d\bar{d}\,(N-1)\,(N-2)\,(N+2)\,(N+1) \tag{3.7.64}$$

Since, the color contractions $\mathcal{C}$ are at most of order $O(N^4)$, there is a non-planar deviation (a planar correction would be of order $O(N^6)$).

3.8 Conclusions and Outlook

As for the $\mathcal{N}=4$ SYM theory, we can summarize our results as follows

- In the off-shell regime dual conformal symmetry depends on the choice of the gauge. In fact, we have computed in a $\mathcal{N}=1$ supersymmetric gauge the off-shell planar amplitude with four external scalars $\mathcal{A}^{off\,shell}_{1\,loop}\Big(\phi\phi^\dagger\phi\phi^\dagger\Big)$ and with four external gluons $\mathcal{A}^{off\,shell}_{1\,loop}\Big(A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}\Big)$ at one loop. The decomposition of these amplitudes contains non-conformal scalar integrals.
- The on-shell limit of the off-shell amplitudes $\mathcal{A}^{off\,shell}_{1\,loop}\Big(\phi\phi^\dagger\phi\phi^\dagger\Big)$ and $\mathcal{A}^{off\,shell}_{1\,loop}\Big(A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}\Big)$ matches the on-shell dimensional regularized version of the amplitudes, i.e.

$$\lim_{p^2\to 0}\mathcal{A}^{off\,shell}_{1\,loop}\Big(\phi\phi^\dagger\phi\phi^\dagger\Big)\equiv\mathcal{A}^{on\,shell}_{dim.\,reg.}\Big(\phi\phi^\dagger\phi\phi^\dagger\Big),$$
$$\lim_{p^2\to 0}\mathcal{A}^{off\,shell}_{1\,loop}\Big(A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}\Big)\equiv\mathcal{A}^{on\,shell}_{dim.\,reg.}\Big(A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}\Big). \tag{3.8.1}$$

- The decomposition of the one loop off-shell planar amplitudes with three and four gluons contains bubble scalar integrals which are divergent in the ultraviolet. In spite of the presence of these UV divergent integrals, these gluon amplitudes are UV finite since the sum of all the divergent terms arising from the bubbles vanishes

As for the β deformation of the $\mathcal{N}=4$ SYM, we have found that at two loops the correlation functions with four and six vector superfields $< V^{a_1}V^{a_2}V^{a_3}V^{a_4} >$ and $< V^{a_1}V^{a_2}V^{a_3}V^{a_4}V^{a_5}V^{a_6} >$ receive non planar corrections with respect to their value in $\mathcal{N}=4$ SYM.
The 'mixed' chiral-vector correlation functions with a chiral, an antichiral and two or three vector superfields $< \Phi^{a_1}\Phi^{\dagger\,a_2}V^{a_3}V^{a_4} >$ and $< \Phi^{a_1}\Phi^{\dagger\,a_2}V^{a_3}V^{a_4}V^{a_5} >$ receive non planar corrections at one loop.

Outlook

As for the $\mathcal{N}=4$ SYM theory, one could compute off-shell planar four-point amplitudes at one loop in a manifestly $\mathcal{N}=2$ formulation using harmonic superspace. Thus one could see whether in this gauge the decomposition of the amplitudes gives only box scalar integrals or in other words whether in this gauge dual conformal symmetry is present or not.
As noted in [76], one should compute at two loop the off-shell planar four gluon amplitude (in a $\mathcal{N}=1$ supersymmetric gauge) to see whether the on-shell limit of the off-shell amplitude differs from the on-shell dimensional regularized version of the amplitude where the on shell condition $p^2=0$ is imposed from the beginning.

Appendix A

Conventions and identities

A.1 Metric

We use the space-time metric tensor

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1).$$

Thus the mass shell condition is $p^2 = m^2$

A.1.1 Weyl spinors

The two component Weyl spinor ψ_α (left-handed) and $\bar{\psi}_{\dot{\alpha}}$ (right handed) belong to the representations (1/2, 0) and (0, 1/2) of the Lorentz group $SO(3,1)$ respectively.
They can be also defined as the objects carrying the fundamental representations of the group of complex 2×2 matrices with determinant equal to one, $Sl(2, \mathbf{C})$, which is the universal covering group of $SO(1,3)$. In fact ψ_α and $\bar{\psi}_{\dot{\alpha}}$ transform as

$$\psi'_\alpha = M_\alpha^{\ \beta}\, \psi_\beta, \quad \bar{\psi}'_{\dot{\alpha}} = (M^*)_{\dot{\alpha}}^{\ \dot{\beta}}\, \bar{\psi}_{\dot{\beta}},$$

where M is an element of $Sl(2, \mathbf{C})$. Note that ψ_α and $\bar{\psi}_{\dot{\alpha}}$ give inequivalent representations and that $\bar{\psi}_{\dot{\alpha}}$ is identified with $(\psi_\alpha)^\dagger$.

A.2 Epsilon Tensors

The antisymmetric epsilon tensors $\epsilon^{\alpha\beta}, \epsilon^{\dot{\alpha}\dot{\beta}}$ and their inverse $\epsilon_{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}}$ have components $\epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = -\epsilon^{21} = -\epsilon^{\dot{2}\dot{1}} = 1$ and $\epsilon_{12} = \epsilon_{\dot{1}\dot{2}} = -\epsilon_{21} = -\epsilon_{\dot{2}\dot{1}} = -1$. Thus:

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{A.2.1a}$$

$$\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \epsilon^{\gamma\beta}\epsilon_{\beta\alpha} = \delta_\alpha^\gamma, \quad \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}} = \epsilon^{\dot{\gamma}\dot{\beta}}\epsilon_{\dot{\beta}\dot{\alpha}} = \delta_{\dot{\alpha}}^{\dot{\gamma}}. \tag{A.2.1b}$$

They are used to lower and raise spinorial indices

$$\psi^\alpha \equiv \epsilon^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$$

,

$$\bar{\psi}^{\dot{\alpha}} \equiv \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}, \quad \bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}.$$

One can verify [15] that the transformation of the spinors with upper indices under an element of $Sl(2, \mathbf{C})$ is $\psi'^\alpha = (M^{-1^T})^\alpha_{\ \beta}\psi^\beta$ and
$\bar{\psi}'^{\dot{\alpha}} = (M^{*-1^T})^{\dot{\alpha}}_{\ \dot{\beta}}\bar{\psi}^{\dot{\beta}}$

A.2.1 Spinor contractions

Spinors anticommute. Spinors in the (1/2, 0) representation are contracted 'in the $\searrow$ direction'

$$\psi\chi \equiv \psi^\alpha \chi_\alpha = \epsilon^{\alpha\beta}\psi_\beta\chi_\alpha = -\epsilon^{\beta\alpha}\psi_\beta\chi_\alpha = -\psi_\beta\chi^\beta = \chi^\beta\psi_\beta = \chi\psi \tag{A.2.2}$$

Spinors in the (0, 1/2) representation are contracted in 'in the $\nearrow$ direction'

$$\bar\psi\bar\chi \equiv \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\alpha}\bar\chi_{\dot\beta} = -\epsilon^{\dot\beta\dot\alpha}\bar\psi_{\dot\alpha}\bar\chi_{\dot\beta} = -\bar\psi^{\dot\beta}\bar\chi_{\dot\beta} = \bar\chi_{\dot\beta}\bar\psi^{\dot\beta} = \bar\chi\bar\psi \tag{A.2.3}$$

$$(\chi\psi)^\dagger = (\psi\chi)^\dagger = \bar\chi\bar\psi = \bar\psi\bar\chi \tag{A.2.4}$$

The product of spinor components are proportional to the ϵ tensor

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta \tag{A.2.5a}$$
$$\theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta \tag{A.2.5b}$$
$$\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta \tag{A.2.5c}$$
$$\bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta \tag{A.2.5d}$$

We also use the notation θ^2 and $\bar\theta^2$ for $\theta^\alpha\theta_\alpha$ and $\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}$ respectively.

A.3 Sigma matrices

The sigma matrices σ^μ are defined as

$$\sigma^\mu = (\mathbf{1}, \sigma^1, \sigma^2, \sigma^3) \tag{A.3.1}$$

with

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.3.2}$$

Their natural spinor indices are undotted and dotted lower indices $\sigma^\mu_{\alpha\dot\alpha}$.
The barred sigma matrices with their natural (dotted and undotted upper) indices are

$$\bar\sigma^\mu = (\mathbf{1}, -\sigma^1, -\sigma^2, -\sigma^3) \tag{A.3.3a}$$
$$\bar\sigma^{\mu\,\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\,\sigma^\mu_{\beta\dot\beta} \tag{A.3.3b}$$
$$\sigma^\mu_{\alpha\dot\alpha} = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\bar\sigma^{\mu\,\beta\dot\beta} \tag{A.3.3c}$$

$$\begin{aligned} \sigma^{\mu\nu} &\equiv \frac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu) \\ \bar\sigma^{\mu\nu} &\equiv \frac{i}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu) \end{aligned} \tag{A.3.4}$$

A.4 Identities for the sigma matrices

Useful identities for the the sigma matrices are

$$(\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu)_\alpha^{\ \beta} = 2\eta^{\mu\nu}\delta_\alpha^{\ \beta} \quad \text{i.e.} \quad \sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = 2\eta^{\mu\nu}\mathbf{1} \tag{A.4.1a}$$
$$(\bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu)^{\dot\alpha}_{\ \dot\beta} = 2\eta^{\mu\nu}\delta^{\dot\alpha}_{\ \dot\beta} \quad \text{i.e.} \quad \bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu = 2\eta^{\mu\nu}\mathbf{1} \tag{A.4.1b}$$
$$\sigma^\mu_{\alpha\dot\alpha}\bar\sigma_\mu^{\beta\dot\beta} = 2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta} \tag{A.4.1c}$$
$$\sigma^\mu\bar\sigma^\nu\sigma^\rho = \eta^{\mu\nu}\sigma^\rho - \eta^{\mu\rho}\sigma^\nu + \eta^{\nu\rho}\sigma^\mu - i\epsilon^{\mu\nu\rho\tau}\sigma_\tau \tag{A.4.1d}$$

$$\mathrm{Tr}\mathbf{1} = \delta_\alpha^\alpha = \delta_{\dot\alpha}^{\dot\alpha} = 2 \tag{A.4.1e}$$
$$\mathrm{Tr}(\sigma^\mu \bar\sigma^\nu) = \sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\alpha} = 2\eta^{\mu\nu} \tag{A.4.1f}$$
$$\mathrm{Tr}(\sigma^\mu\bar\sigma^\nu\sigma^\rho\bar\sigma^\tau) = 2(\eta^{\mu\nu}\eta^{\rho\tau} + \eta^{\mu\tau}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\tau} - i\epsilon^{\mu\nu\rho\tau}) \tag{A.4.1g}$$
$$\theta\sigma^\mu\bar\xi\,\theta\sigma^\nu\bar\xi = \frac{1}{2}\eta^{\mu\nu}\theta\theta\,\bar\xi\bar\xi \tag{A.4.1h}$$
$$(\chi\sigma^\mu\bar\psi)^\dagger = \psi\sigma^{\mu\,\dagger}\bar\chi = \psi\sigma^\mu\bar\chi \tag{A.4.1i}$$
$$\chi\sigma^\mu\bar\psi = -\bar\psi\bar\sigma^\mu\chi \tag{A.4.1j}$$
$$\bar\chi\bar\sigma^\mu\sigma^\nu\bar\psi = \bar\psi\bar\sigma^\nu\sigma^\mu\bar\chi \tag{A.4.1k}$$
$$\chi\sigma^\mu\bar\sigma^\nu\psi = \psi\sigma^\nu\bar\sigma^\mu\chi \tag{A.4.1l}$$
$$\chi\sigma^\mu\bar\sigma^\nu\sigma^\rho\bar\psi = -\bar\psi\bar\sigma^\rho\sigma^\nu\bar\sigma^\mu\chi \tag{A.4.1m}$$
$$\bar\chi\bar\sigma^\mu\sigma^\nu\bar\sigma^\rho\psi = -\psi\sigma^\rho\bar\sigma^\nu\sigma^\mu\bar\chi \tag{A.4.1n}$$
$$\bar\chi\bar\sigma^\mu\sigma^\nu\bar\sigma^\rho\sigma^\tau\bar\psi = \bar\psi\bar\sigma^\tau\sigma^\rho\bar\sigma^\nu\bar\sigma^\mu\bar\chi \tag{A.4.1o}$$
$$\chi\sigma^\mu\bar\sigma^\nu\sigma^\rho\bar\sigma^\tau\psi = \psi\sigma^\tau\bar\sigma^\rho\sigma^\nu\bar\sigma^\mu\chi \tag{A.4.1p}$$

The totally antisymmetric pseudo-tensor $\epsilon^{\mu\nu\rho\tau}$ satisfies $\epsilon^{0123} = -\epsilon_{0123} = -1$ The identities (A.4.1j)-(A.4.1p) can be easily generalized to an arbitrary even or odd number of matrices σ^μ and $\bar\sigma^\mu$.
For example the derivation of (A.4.1l) is [A.1]

$$\begin{aligned}
\chi\sigma^\mu\bar\sigma^\nu\psi &\equiv \chi^{\alpha_1}\sigma^\mu_{\alpha_1\dot\alpha_1}\bar\sigma^{\nu\,\dot\alpha_1\alpha_2}\psi_{\alpha_2}\\
&= \epsilon^{\alpha_1\alpha_3}\chi_{\alpha_3}(\epsilon_{\alpha_1\alpha_4}\epsilon_{\dot\alpha_1,\dot\alpha_2}\bar\sigma^{\mu\,\dot\alpha_2\alpha_4})(\epsilon^{\dot\alpha_1\dot\alpha_3}\epsilon^{\alpha_2\alpha_5}\sigma^\nu_{\alpha_5\dot\alpha_3})\epsilon_{\alpha_2\alpha_6}\psi^{\alpha_6}\\
&= (-\epsilon^{\alpha_3\alpha_1}\epsilon_{\alpha_1\alpha_4}\chi_{\alpha_3})\bar\sigma^{\mu\,\dot\alpha_2\ \ \alpha_4}\sigma^\nu_{\alpha_5\dot\alpha_3}(-\epsilon_{\dot\alpha_2\dot\alpha_1}\epsilon^{\dot\alpha_1\dot\alpha_3})(-\epsilon^{\alpha_5\alpha_2}\epsilon_{\alpha_2\alpha_6})\psi^{\alpha_6},\\
&= -\chi_{\alpha_4}\bar\sigma^{\mu\,\dot\alpha_3\ \ \alpha_4}\sigma^\nu_{\alpha_5\dot\alpha_3}\psi^{\alpha_5}\\
&= \psi^{\alpha_5}\sigma^\nu_{\alpha_5\dot\alpha_3}\bar\sigma^{\mu\,\dot\alpha_3\alpha_4}\chi_{\alpha_4}\\
&= \psi\sigma^\nu\bar\sigma^\mu\chi.
\end{aligned} \tag{A.4.2}$$

where in the second step eqs.(A.3.3b)-(A.3.3c) have been employed and the third one follows from antisymmetry of ϵ. In the forth step eq.(A.2.1b) has been used, while in the fifth one spinors have been swapped. Other identities can be found in [97].

A.5 Grassmann differentiation

Let's consider differentiation with respect Grassmann variables $\partial_\alpha \equiv \partial/\partial\theta^\alpha, \partial^\alpha \equiv \partial/\partial\theta_\alpha, \bar\partial^{\dot\alpha} \equiv \partial/\partial\bar\theta_{\dot\alpha}$ and $\bar\partial_{\dot\alpha} \equiv \partial/\partial\bar\theta^{\dot\alpha}$
By definition,

$$\partial_\alpha\theta^\beta = \delta_\alpha^\beta \tag{A.5.1a}$$
$$\partial^\alpha\theta_\beta = \delta^\alpha_\beta \tag{A.5.1b}$$
$$\bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} = \delta_{\dot\alpha}^{\dot\beta} \tag{A.5.1c}$$
$$\bar\partial^{\dot\alpha}\bar\theta_{\dot\beta} = \delta^{\dot\alpha}_{\dot\beta} \tag{A.5.1d}$$
$$\partial_\alpha\theta_\beta = -\epsilon_{\alpha\beta} \tag{A.5.1e}$$
$$\partial^\alpha\theta^\beta = -\epsilon^{\alpha\beta} \tag{A.5.1f}$$
$$\bar\partial_{\dot\alpha}\bar\theta_{\dot\beta} = -\epsilon_{\dot\alpha\dot\beta} \tag{A.5.1g}$$
$$\bar\partial^{\dot\alpha}\bar\theta^{\dot\beta} = -\epsilon^{\dot\alpha\dot\beta} \tag{A.5.1h}$$

[A.1] A faster way to derive the same result is obtained by observing that in a generic expression one can convert a contraction of indices in one direction, i.e ${}^\alpha \ldots {}_\alpha$, to another direction by putting a minus sign, i.e. ${}^\alpha \ldots {}_\alpha = -{}_\alpha \ldots {}^\alpha$. If these indices belong to sigma matrices, then one have to convert σ matrices in $\bar\sigma$ and vice versa.
For example $\chi\sigma^\mu\bar\sigma^\nu\psi = \chi^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\beta}\psi_\beta = (-)^3\,\chi_\alpha\bar\sigma^{\dot\alpha\alpha}\sigma^\nu_{\beta\dot\alpha}\psi^\beta = \psi\sigma^\nu\bar\sigma^\mu\chi$

Obviously $\partial_\alpha \bar\theta_{\dot\beta} = \partial_\alpha \bar\theta^{\dot\beta} = \partial^\alpha \bar\theta^{\dot\beta} = \partial^\alpha \bar\theta_{\dot\beta} = \bar\partial_{\dot\alpha}\theta_\beta = \bar\partial_{\dot\alpha}\theta^\beta = \bar\partial^{\dot\alpha}\theta^\beta = \bar\partial^{\dot\alpha}\theta_\beta = 0$

Indicating $\partial/\partial\theta_1^\alpha$ with $\partial_{1\alpha}$, $\theta_1 - \theta_2$ with θ_{12} and $(\theta_1-\theta_2)^2$ with θ_{12}^2, there are other useful identities

$$\partial_{1\alpha}\,\theta_{12}^2 = 2\theta_{12\alpha} \tag{A.5.2a}$$
$$\bar\partial_{1\dot\alpha}\,\bar\theta_{12}^2 = -2\bar\theta_{12\dot\alpha} \tag{A.5.2b}$$
$$\partial_1^\alpha\,\theta_{12}^2 = -2\theta_{12}^\alpha \tag{A.5.2c}$$
$$\bar\partial_1^{\dot\alpha}\,\bar\theta_{12}^2 = 2\bar\theta_{12}^{\dot\alpha} \tag{A.5.2d}$$
$$\partial_{1\alpha}\,\theta_1\theta_2 = \theta_{2\alpha} \tag{A.5.2e}$$
$$\partial_1^\alpha\,\theta_1\theta_2 = -\theta_2^\alpha \tag{A.5.2f}$$
$$\bar\partial_{1\dot\alpha}\,\bar\theta_1\bar\theta_2 = -\bar\theta_{2\dot\alpha} \tag{A.5.2g}$$
$$\bar\partial_1^{\dot\alpha}\bar\theta_1\bar\theta_2 = \bar\theta_2^{\dot\alpha} \tag{A.5.2h}$$

All components of ∂, $\bar\partial$ anticommute with one another

$$0 = \{\partial_\alpha, \partial_\beta\} = \{\bar\partial_{\dot\alpha}, \bar\partial_{\dot\beta}\} = \{\partial_\alpha, \bar\partial_{\dot\beta}\} \tag{A.5.3}$$

When ∂_α and $\bar\partial_{\dot\alpha}$ act on a product, they satisfy the Leibniz rules. If in this product there are fermionic fields and/or Grassmann coordinates, there can be a minus sign. For example, if ψ and χ are fermionic fields, then

$$\partial_\alpha(\psi\chi) = (\partial_\alpha\psi)\chi - \psi(\partial_\alpha\chi) \tag{A.5.4a}$$
$$\bar\partial_{\dot\alpha}(\psi\chi) = (\bar\partial_{\dot\alpha}\psi)\chi - \psi(\bar\partial_{\dot\alpha}\chi), \tag{A.5.4b}$$

Applying the Leibnitz rules (A.5.4a)-(A.5.4b) and eqs.(A.5.1a)-(A.5.1h), one obtains two useful identities involving $\partial\partial \equiv \partial^\alpha\partial_\alpha$ and $\bar\partial\bar\partial \equiv \bar\partial_{\dot\alpha}\bar\partial^{\dot\alpha}$

$$\partial\partial(\theta\theta) = \bar\partial\bar\partial(\bar\theta\bar\theta) = 4 \tag{A.5.5}$$

In fact $\partial^\alpha\partial_\alpha(\theta^\beta\theta_\beta) = \partial^\alpha(\delta_\alpha^\beta\,\theta_\beta - \theta^\beta(-\epsilon_{\alpha\beta})) = \delta_\alpha^\beta\delta_\beta^\alpha - \epsilon^{\alpha\beta}\epsilon_{\alpha\beta} = 4$, where we have used eqs.(A.4.1e) and (A.2.1b). Acting on ∂ or $\bar\partial$ with the epsilon tensor gives a minus sign

$$\epsilon^{\alpha\beta}\partial_\beta = -\partial^\alpha \tag{A.5.6a}$$
$$\epsilon_{\alpha\beta}\partial^\beta = -\partial_\alpha \tag{A.5.6b}$$
$$\epsilon_{\dot\alpha\dot\beta}\bar\partial^{\dot\beta} = -\bar\partial_{\dot\alpha} \tag{A.5.6c}$$
$$\epsilon^{\dot\alpha\dot\beta}\bar\partial_{\dot\beta} = -\bar\partial^{\dot\alpha} \tag{A.5.6d}$$

A.6 Grassmann integration

Grassmann integration is defined using the rules

$$d^2\theta = -\frac{1}{4}d\theta^\alpha d\theta_\alpha \tag{A.6.1a}$$
$$d^2\bar\theta = -\frac{1}{4}d\bar\theta_{\dot\alpha} d\bar\theta^{\dot\alpha} \tag{A.6.1b}$$
$$d^4\theta = d^2\bar\theta d^2\theta \tag{A.6.1c}$$
$$\int d^2\theta = \int d^2\bar\theta = \int d^2\theta\,\theta^\alpha = \int d^2\bar\theta\,\bar\theta^{\dot\alpha} = 0 \tag{A.6.1d}$$
$$\int d^2\theta\,\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta} \tag{A.6.1e}$$
$$\int d^2\bar\theta\,\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \frac{1}{2}\epsilon^{\dot\alpha\dot\beta} \tag{A.6.1f}$$
$$\int d^2\theta\,\theta\theta = \int d^2\bar\theta\,\bar\theta\bar\theta = 1 \tag{A.6.1g}$$

$$\int d^4\theta\, \theta\theta\, \bar\theta\bar\theta = 1 \tag{A.6.1h}$$

The integration measures (A.6.1a)-(A.6.1c) are defined in such a way that (A.6.1g) and (A.6.1h) hold. Grassmann integration and differentiation are equivalent. In fact

$$\int d^2\theta f(\theta,\bar\theta) = \frac{1}{4}\partial\partial f(\theta,\bar\theta) \tag{A.6.2a}$$

$$\int d^2\bar\theta f(\theta,\bar\theta) = \frac{1}{4}\bar\partial\bar\partial f(\theta,\bar\theta) \tag{A.6.2b}$$

$$\int d^2\theta \partial_\alpha f(\theta,\bar\theta) = \int d^2\bar\theta \bar\partial_{\dot\alpha} f(\theta,\bar\theta) = 0 \tag{A.6.2c}$$

$$\int d^4\theta f(\theta,\bar\theta) = \frac{1}{16}\partial\partial\, \bar\partial\bar\partial f(\theta,\bar\theta) \tag{A.6.2d}$$

where $f(\theta\bar\theta)$ is a generic function of θ and $\bar\theta$.

A.7 Fermionic delta function

The fermionic delta functions are defined as

$$\delta^{(2)}(\theta) = \theta\theta, \quad \delta^{(2)}(\bar\theta) = \bar\theta\bar\theta \tag{A.7.1a}$$

$$\text{and satisfy} \int d^2\theta\, \delta^{(2)}(\theta) = \int d^2\bar\theta\, \delta^{(2)}(\bar\theta) = 1 \tag{A.7.1b}$$

$$\delta^{(4)}(\theta) = \delta^{(2)}(\theta)\delta^{(2)}(\bar\theta) = \theta\theta\, \bar\theta\bar\theta \tag{A.7.1c}$$

$$\int d^4\theta\, \delta^{(4)}(\theta) = 1. \tag{A.7.1d}$$

(A.7.1e)

We use the symbol δ_{12} to indicate

$$\delta_{12} \equiv \delta^{(4)}(\theta_1 - \theta_2) = \delta^{(4)}(\theta_{12}) = \theta_{12}^2\, \bar\theta_{12}^2 \tag{A.7.1f}$$

$$\int d^4\theta_1\, \delta_{12} = \int d^4\theta_2\, \delta_{12} = 1. \tag{A.7.1g}$$

Choosing two points in the superspace with $z_1 = (x_1, \theta_1, \bar\theta_1)$ and $z_2 = (x_2, \theta_2, \bar\theta_2)$, and denoting $d^8z \equiv d^4x d^4\theta$ one can define

$$\delta^{(8)}(z_1 - z_2) = \delta^{(4)}(\theta_1 - \theta_2)\delta^{(4)}(x_1 - x_2) \tag{A.7.1h}$$

$$\int d^8z_1\, \delta^{(8)}(z_1 - z_2) = \int d^8z_2\, \delta^{(8)}(z_1 - z_2) = 1. \tag{A.7.1i}$$

Appendix B

D-algebra

In this appendix we list some useful identities for the superderivatives.

B.1 D-algebra in coordinate space

B.1.1 Definitions

The covariant superderivatives are defined as

$$\mathcal{D}_\alpha = \partial_\alpha - i\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu \tag{B.1.1a}$$

$$\mathcal{D}^\alpha = -\partial^\alpha + i\bar\theta_{\dot\beta}\bar\sigma^{\mu\,\dot\beta\alpha}\partial_\mu \tag{B.1.1b}$$

$$\bar{\mathcal{D}}_{\dot\alpha} = -\bar\partial_{\dot\alpha} + i\theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu \tag{B.1.1c}$$

$$\bar{\mathcal{D}}^{\dot\alpha} = \bar\partial^{\dot\alpha} - i\bar\sigma^{\mu\,\dot\alpha\beta}\theta_\beta\partial_\mu \tag{B.1.1d}$$

They are spinors and hence satisfy $\mathcal{D}^\alpha = \epsilon^{\alpha\beta}\mathcal{D}_\beta$ and $\bar{\mathcal{D}}^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar{\mathcal{D}}_{\dot\beta}$.
If the theta variables $\theta_1, \bar\theta_1$ have a further label, such as 1 in this case, then we will also use the notation $\mathcal{D}_{1\,\alpha}, \mathcal{D}_1^\alpha$, etc..., to indicate $\mathcal{D}_{1\,\alpha} = \partial_{1\,\alpha} - i\sigma^\mu_{\alpha\dot\beta}\bar\theta_1^{\dot\beta}\partial_\mu$, $\quad \mathcal{D}_1^\alpha = -\partial_1^\alpha + i\bar\theta_{1\,\dot\beta}\bar\sigma^{\mu\,\dot\beta\alpha}\partial_\mu$, etc... respectively.

B.1.2 Anticommutation and commutation relations

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = 0 \tag{B.1.2a}$$

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\beta}\} = 2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu \tag{B.1.2b}$$

$$\{\mathcal{D}^\alpha, \bar{\mathcal{D}}^{\dot\beta}\} = 2i\bar\sigma^{\mu\dot\beta\alpha}\partial_\mu \tag{B.1.2c}$$

$$\mathcal{D}\sigma^\mu\bar{\mathcal{D}} + \bar{\mathcal{D}}\bar\sigma^\mu\mathcal{D} = 4i\partial^\mu \tag{B.1.2d}$$

$$[\mathcal{D}_\alpha, \bar{\mathcal{D}}\bar{\mathcal{D}}] = 4i\sigma^\mu_{\alpha\dot\beta}\bar{\mathcal{D}}^{\dot\beta}\,\partial_\mu \tag{B.1.2e}$$

$$[\mathcal{D}^\alpha, \bar{\mathcal{D}}\bar{\mathcal{D}}] = -4i\bar{\mathcal{D}}_{\dot\beta}\bar\sigma^{\mu\dot\beta\alpha}\,\partial_\mu \tag{B.1.2f}$$

$$[\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{D}\mathcal{D}] = -4i\mathcal{D}^\beta\sigma^\mu_{\beta\dot\alpha}\,\partial_\mu \tag{B.1.2g}$$

$$[\bar{\mathcal{D}}^{\dot\alpha}, \mathcal{D}\mathcal{D}] = 4i\bar\sigma^{\mu\dot\alpha\beta}\mathcal{D}_\beta\,\partial_\mu, \tag{B.1.2h}$$

$$[\mathcal{D}\mathcal{D}, \bar{\mathcal{D}}\bar{\mathcal{D}}] = 8i\mathcal{D}\sigma^\mu\bar{\mathcal{D}}\,\partial_\mu + 16\partial\partial \tag{B.1.2i}$$

$$[\bar{\mathcal{D}}\bar{\mathcal{D}}, \mathcal{D}\mathcal{D}] = 8i\bar{\mathcal{D}}\bar\sigma^\mu\mathcal{D}\,\partial_\mu + 16\partial\partial \tag{B.1.2j}$$

$$\mathcal{D}^\alpha\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}_\alpha = \bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}\mathcal{D}\bar{\mathcal{D}}^{\dot\alpha} = 8\partial^2 + \frac{1}{2}\{\mathcal{D}\mathcal{D}, \bar{\mathcal{D}}\bar{\mathcal{D}}\} \tag{B.1.2k}$$

$$\mathcal{D}\mathcal{D}\,\bar{\mathcal{D}}\bar{\mathcal{D}}\,\mathcal{D}\mathcal{D} = -16\partial\partial\,\mathcal{D}\mathcal{D} \tag{B.1.2l}$$

$$\bar{\mathcal{D}}\bar{\mathcal{D}}\,\mathcal{D}\mathcal{D}\,\bar{\mathcal{D}}\bar{\mathcal{D}} = -16\partial\partial\,\bar{\mathcal{D}}\bar{\mathcal{D}} \tag{B.1.2m}$$

B.1.3 Leibnitz rules

$$\mathcal{D}_\alpha(\mathcal{B}_1\mathcal{B}_2) = (\mathcal{D}_\alpha\mathcal{B}_1)\mathcal{B}_2 + \mathcal{B}_1\mathcal{D}_\alpha(\mathcal{B}_2) \tag{B.1.3a}$$
$$\mathcal{D}_\alpha(\mathcal{F}_1\mathcal{B}_2) = (\mathcal{D}_\alpha\mathcal{F}_1)\mathcal{B}_2 - \mathcal{F}_1\mathcal{D}_\alpha(\mathcal{B}_2), \tag{B.1.3b}$$
$$\mathcal{D}_\alpha(\mathcal{B}_1\mathcal{F}_2) = (\mathcal{D}_\alpha\mathcal{B}_1)\mathcal{F}_2 + \mathcal{B}_1\mathcal{D}_\alpha(\mathcal{F}_2) \tag{B.1.3c}$$
$$\mathcal{D}_\alpha(\mathcal{F}_1\mathcal{F}_2) = (\mathcal{D}_\alpha\mathcal{F}_1)\mathcal{F}_2 - \mathcal{F}_1\mathcal{D}_\alpha(\mathcal{F}_2) \tag{B.1.3d}$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are bosonic superfields, while $\mathcal{F}_1$ and $\mathcal{F}_2$ are fermionic ones.
Note the minus sign in eqs.(B.1.3b) and (B.1.3d).

B.1.4 Integrations by parts

$$\int d^4xd^4\theta\, \mathcal{B}_1(\mathcal{D}_\alpha\mathcal{B}_2)\mathcal{B}_3 = -\int d^4xd^4\theta\, (\mathcal{D}_\alpha\mathcal{B}_1)\mathcal{B}_2\mathcal{B}_3 - \int d^4xd^4\theta\, \mathcal{B}_1\mathcal{B}_2(\mathcal{D}_\alpha\mathcal{B}_3) \tag{B.1.4a}$$
$$\int d^4xd^4\theta\, \mathcal{B}_1(\bar{\mathcal{D}}_{\dot\alpha}\mathcal{B}_2)\mathcal{B}_3 = -\int d^4xd^4\theta\, (\bar{\mathcal{D}}_{\dot\alpha}\mathcal{B}_1)\mathcal{B}_2\mathcal{B}_3 - \int d^4xd^4\theta\, \mathcal{B}_1\mathcal{B}_2(\bar{\mathcal{D}}_{\dot\alpha}\mathcal{B}_3) \tag{B.1.4b}$$
$$\int d^4xd^4\theta\, \mathcal{B}_1(\mathcal{D}\mathcal{D}\mathcal{B}_2)\mathcal{B}_3 = \int d^4xd^4\theta\, (\mathcal{D}\mathcal{D}\mathcal{B}_1)\mathcal{B}_2\mathcal{B}_3 + 2\int d^4xd^4\theta\, (\mathcal{D}^\alpha\mathcal{B}_1)\mathcal{B}_2(\mathcal{D}_\alpha\mathcal{B}_3) + \int d^4xd^4\theta\, \mathcal{B}_1\mathcal{B}_2(\mathcal{D}\mathcal{D}\mathcal{B}_3) \tag{B.1.4c}$$
$$\int d^4xd^4\theta\, \mathcal{B}_1(\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{B}_2)\mathcal{B}_3 = \int d^4xd^4\theta\, (\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{B}_1)\mathcal{B}_2\mathcal{B}_3 + 2\int d^4xd^4\theta(\bar{\mathcal{D}}_{\dot\alpha}\mathcal{B}_1)\mathcal{B}_2(\bar{\mathcal{D}}^{\dot\alpha}\mathcal{B}_3) + \int d^4xd^4\theta\, \mathcal{B}_1\mathcal{B}_2(\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{B}_3) \tag{B.1.4d}$$
$$\int d^4xd^4\theta\, \mathcal{F}_1(\mathcal{D}\mathcal{D}\mathcal{B}_2)\mathcal{B}_3 = \int d^4xd^4\theta\, (\mathcal{D}\mathcal{D}\mathcal{F}_1)\mathcal{B}_2\mathcal{B}_3 - 2\int d^4xd^4\theta(\mathcal{D}^\alpha\mathcal{F}_1)\mathcal{B}_2(\mathcal{D}_\alpha\mathcal{B}_3) + \int d^4xd^4\theta\, \mathcal{F}_1\mathcal{B}_2(\mathcal{D}\mathcal{D}\mathcal{B}_3) \tag{B.1.4e}$$

where $\mathcal{B}_1$,$\mathcal{B}_2$,$\mathcal{B}_3$ are bosonic superfields, while $\mathcal{F}_1$ is a fermionic one.
(Note the minus sign in the second term of the r.h.s of eq.(B.1.4e) with respect to eq.(B.1.4c) since $\mathcal{F}_1$ is fermionic while $\mathcal{B}_1$ is bosonic)

B.1.5 $\mathcal{D}$ and fermionic delta δ

The action of the superderivatives on the fermionic δ functions is

$$\mathcal{D}_{1\alpha}\,\delta^{(8)}(z_1 - z_2) = -\mathcal{D}_{2\alpha}\,\delta^{(8)}(z_1 - z_2), \tag{B.1.5a}$$
$$\bar{\mathcal{D}}_{1\dot\alpha}\,\delta^{(8)}(z_1 - z_2) = -\bar{\mathcal{D}}_{2\dot\alpha}\,\delta^{(8)}(z_1 - z_2), \tag{B.1.5b}$$
$$\mathcal{D}_1\mathcal{D}_1\,\delta^{(8)}(z_1 - z_2) = \mathcal{D}_2\mathcal{D}_2\,\delta^{(8)}(z_1 - z_2), \tag{B.1.5c}$$
$$\bar{\mathcal{D}}_1\bar{\mathcal{D}}_1\,\delta^{(8)}(z_1 - z_2) = \bar{\mathcal{D}}_2\bar{\mathcal{D}}_2\,\delta^{(8)}(z_1 - z_2). \tag{B.1.5d}$$

For the action on products of δ functions one finds

$$\delta_{12}\delta_{12} = \delta_{12}\mathcal{D}_\alpha\delta_{12} = \delta_{12}\bar{\mathcal{D}}_{\dot\alpha}\delta_{12} = \delta_{12}\mathcal{D}\mathcal{D}\delta_{12} = \delta_{12}\bar{\mathcal{D}}\bar{\mathcal{D}}\delta_{12} = \delta_{12}\mathcal{D}_\alpha\bar{\mathcal{D}}\bar{\mathcal{D}}\delta_{12} = \delta_{12}\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}\mathcal{D}\delta_{12} = 0 \tag{B.1.5e}$$

$$\delta_{12}\mathcal{D}\mathcal{D}\,\bar{\mathcal{D}}\bar{\mathcal{D}}\delta_{12} = 16\delta_{12} \tag{B.1.5f}$$
$$\delta_{12}\bar{\mathcal{D}}\bar{\mathcal{D}}\,\mathcal{D}\mathcal{D}\delta_{12} = 16\delta_{12} \tag{B.1.5g}$$

B.2 D-algebra in momentum space

The momentum-space counterpart of the above relations is immediate.

B.2.1 Definitions

$$\mathcal{D}^p_\alpha = \partial_\alpha - \sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta} p_\mu \tag{B.2.1a}$$
$$\mathcal{D}^{p,\alpha} = -\partial^\alpha + \bar\theta_{\dot\beta}\bar\sigma^{\mu\,\dot\beta\alpha} p_\mu \tag{B.2.1b}$$
$$\bar{\mathcal{D}}^p_{\dot\alpha} = -\bar\partial_{\dot\alpha} + \theta^\beta\sigma^\mu_{\beta\dot\alpha} p_\mu \tag{B.2.1c}$$
$$\bar{\mathcal{D}}^{p,\dot\alpha} = \bar\partial^{\dot\alpha} - \bar\sigma^{\mu\,\dot\alpha\beta}\theta_\beta p_\mu \tag{B.2.1d}$$

$\mathcal{D}^{p,\alpha} = \epsilon^{\alpha\beta}\mathcal{D}^p_\beta$ and $\bar{\mathcal{D}}^{p,\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar{\mathcal{D}}^p_{\dot\beta}$.
The superscript p on $\mathcal{D}^p$ and $\bar{\mathcal{D}}^p$ indicates the ingoing momentum p corresponding to the superspace point $z \equiv (x, \theta, \bar\theta)$. Also, the momentum in the supercovariant derivative can always be taken as the momentum of the superfield on which it act.
If the theta variables $\theta_1, \bar\theta_1$ have a further label, such as 1 in this case, then we will also use the notation $\mathcal{D}^p_{1\alpha}, \mathcal{D}^{p,\alpha}_1$, etc..., to indicate $\mathcal{D}^p_{1\alpha} = \partial_{1\alpha} - \sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}_1 p_\mu, \quad \mathcal{D}^{p,\alpha}_1 = -\partial^\alpha_1 + \bar\theta_{1\,\dot\beta}\bar\sigma^{\mu\,\dot\beta\alpha} p_\mu$, etc... respectively.

B.2.2 Anticommutation and commutation relations

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = 0 \tag{B.2.2a}$$
$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} p_\mu \tag{B.2.2b}$$
$$\{\mathcal{D}^\alpha, \bar{\mathcal{D}}^{\dot\beta}\} = 2\bar\sigma^{\mu\dot\beta\alpha} p_\mu \tag{B.2.2c}$$
$$\mathcal{D}\sigma^\mu\bar{\mathcal{D}} + \bar{\mathcal{D}}\bar\sigma^\mu\mathcal{D} = 4p^\mu \tag{B.2.2d}$$
$$[\mathcal{D}_\alpha, \bar{\mathcal{D}}\bar{\mathcal{D}}] = 4\sigma^\mu_{\alpha\dot\beta}\bar{\mathcal{D}}^{\dot\beta}\, p_\mu \tag{B.2.2e}$$
$$[\mathcal{D}^\alpha, \bar{\mathcal{D}}\bar{\mathcal{D}}] = -4\bar{\mathcal{D}}_{\dot\beta}\bar\sigma^{\mu\dot\beta\alpha}\, p_\mu \tag{B.2.2f}$$
$$[\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{D}\mathcal{D}] = -4\mathcal{D}^\beta\sigma^\mu_{\beta\dot\alpha}\, p_\mu \tag{B.2.2g}$$
$$[\bar{\mathcal{D}}^{\dot\alpha}, \mathcal{D}\mathcal{D}] = 4\bar\sigma^{\mu\dot\alpha\beta}\mathcal{D}_\beta\, p_\mu, \tag{B.2.2h}$$
$$[\mathcal{D}\mathcal{D}, \bar{\mathcal{D}}\bar{\mathcal{D}}] = 8\mathcal{D}\sigma^\mu\bar{\mathcal{D}}\, p_\mu - 16p^2 \tag{B.2.2i}$$
$$[\bar{\mathcal{D}}\bar{\mathcal{D}}, \mathcal{D}\mathcal{D}] = 8\bar{\mathcal{D}}\bar\sigma^\mu\mathcal{D}\, p_\mu - 16p^2 \tag{B.2.2j}$$
$$\mathcal{D}^\alpha\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}_\alpha = \bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}\mathcal{D}\bar{\mathcal{D}}^{\dot\alpha} = -8p^2 + \frac{1}{2}\{\mathcal{D}\mathcal{D}, \bar{\mathcal{D}}\bar{\mathcal{D}}\} \tag{B.2.2k}$$
$$\mathcal{D}\mathcal{D}\,\bar{\mathcal{D}}\bar{\mathcal{D}}\,\mathcal{D}\mathcal{D} = 16p^2\mathcal{D}\mathcal{D} \tag{B.2.2l}$$
$$\bar{\mathcal{D}}\bar{\mathcal{D}}\,\mathcal{D}\mathcal{D}\,\bar{\mathcal{D}}\bar{\mathcal{D}} = 16p^2\bar{\mathcal{D}}\bar{\mathcal{D}} \tag{B.2.2m}$$

For instance, the derivation of eq.(B.2.2e) is

$$\begin{aligned}
[\mathcal{D}_\alpha, \bar{\mathcal{D}}\bar{\mathcal{D}}] &= \mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot\beta}\bar{\mathcal{D}}^{\dot\beta} - \bar{\mathcal{D}}_{\dot\beta}\bar{\mathcal{D}}^{\dot\beta}\mathcal{D}_\alpha \\
&= -\bar{\mathcal{D}}_{\dot\beta}\mathcal{D}_\alpha\bar{\mathcal{D}}^{\dot\beta} + 2\sigma^\mu_{\alpha\dot\beta}\bar{\mathcal{D}}^{\dot\beta} p_\mu - \bar{\mathcal{D}}_{\dot\beta}\bar{\mathcal{D}}^{\dot\beta}\mathcal{D}_\alpha \\
&= \bar{\mathcal{D}}^{\dot\beta}\mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot\beta} + 2\sigma^\mu_{\alpha\dot\beta}\bar{\mathcal{D}}^{\dot\beta} p_\mu - \bar{\mathcal{D}}_{\dot\beta}\bar{\mathcal{D}}^{\dot\beta}\mathcal{D}_\alpha \\
&= -\bar{\mathcal{D}}^{\dot\beta}\bar{\mathcal{D}}_{\dot\beta}\mathcal{D}_\alpha + 2\sigma^\mu_{\alpha\dot\beta}\bar{\mathcal{D}}^{\dot\beta} p_\mu + 2\sigma^\mu_{\alpha\dot\beta}\bar{\mathcal{D}}^{\dot\beta} p_\mu - \bar{\mathcal{D}}_{\dot\beta}\bar{\mathcal{D}}^{\dot\beta}\mathcal{D}_\alpha \\
&= 4\sigma^\mu_{\alpha\dot\beta}\bar{\mathcal{D}}^{\dot\beta} p_\mu.
\end{aligned} \tag{B.2.3}$$

In the second step $\mathcal{D}_\alpha$ has been anticommutated with $\bar{\mathcal{D}}_{\dot\beta}$ (eq.(B.2.2b)), while the third one follows from $\bar{\mathcal{D}}_{\dot\beta}\mathcal{D}_\alpha\bar{\mathcal{D}}^{\dot\beta} = -\bar{\mathcal{D}}^{\dot\beta}\mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot\beta}$. In the forth step $\mathcal{D}_\alpha$ has been anticommutated again with $\bar{\mathcal{D}}_{\dot\beta}$ (eq.B.2.2b) and the last one follows from $\bar{\mathcal{D}}_{\dot\beta}\bar{\mathcal{D}}^{\dot\beta}\mathcal{D}_\alpha = -\bar{\mathcal{D}}^{\dot\beta}\bar{\mathcal{D}}_{\dot\beta}\mathcal{D}_\alpha$

B.2.3 Leibnitz rules

$$\mathcal{D}^p_\alpha(\mathcal{B}_1\mathcal{B}_2) = (\mathcal{D}^q_\alpha\mathcal{B}_1)\mathcal{B}_2 + \mathcal{B}_1\mathcal{D}^{p-q}_\alpha(\mathcal{B}_2) \tag{B.2.4a}$$
$$\mathcal{D}^p_\alpha(\mathcal{F}_1\mathcal{B}_2) = (\mathcal{D}^q_\alpha\mathcal{F}_1)\mathcal{B}_2 - \mathcal{F}_1\mathcal{D}^{p-q}_\alpha(\mathcal{B}_2), \tag{B.2.4b}$$
$$\mathcal{D}^p_\alpha(\mathcal{B}_1\mathcal{F}_2) = (\mathcal{D}^q_\alpha\mathcal{B}_1)\mathcal{F}_2 + \mathcal{B}_1\mathcal{D}^{p-q}_\alpha(\mathcal{F}_2) \tag{B.2.4c}$$
$$\mathcal{D}^p_\alpha(\mathcal{F}_1\mathcal{F}_2) = (\mathcal{D}^q_\alpha\mathcal{F}_1)\mathcal{F}_2 - \mathcal{F}_1\mathcal{D}^{p-q}_\alpha(\mathcal{F}_2), \tag{B.2.4d}$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are the Fourier transform of bosonic superfields , while $\mathcal{F}_1$ and $\mathcal{F}_2$ of fermionic ones.

B.2.4 Integrations by parts

$$\int d^4\theta\,\mathcal{B}_1(\mathcal{D}^{p_2}_\alpha\mathcal{B}_2)\mathcal{B}_3 = -\int d^4\theta\,(\mathcal{D}^{p_1}_\alpha\mathcal{B}_1)\mathcal{B}_2\mathcal{B}_3 - \int d^4\theta\,\mathcal{B}_1\mathcal{B}_2(\mathcal{D}^{p_3}_\alpha\mathcal{B}_3) \tag{B.2.5a}$$
$$\int d^4\theta\,\mathcal{B}_1(\bar{\mathcal{D}}^{p_2}_{\dot\alpha}\mathcal{B}_2)\mathcal{B}_3 = -\int d^4\theta\,(\bar{\mathcal{D}}^{p_1}_{\dot\alpha}\mathcal{B}_1)\mathcal{B}_2\mathcal{B}_3 - \int d^4\theta\,\mathcal{B}_1\mathcal{B}_2(\bar{\mathcal{D}}^{p_3}_{\dot\alpha}\mathcal{B}_3), \tag{B.2.5b}$$
$$\int d^4\theta\,\mathcal{B}_1(\mathcal{D}^{p_2}\mathcal{D}^{p_2}\mathcal{B}_2)\mathcal{B}_3 = \int d^4\theta\,(\mathcal{D}^{p_1}\mathcal{D}^{p_1}\mathcal{B}_1)\mathcal{B}_2\mathcal{B}_3\ + 2\int d^4\theta\,(\mathcal{D}^{p_1\alpha}\mathcal{B}_1)\mathcal{B}_2(\mathcal{D}^{p_3}_\alpha\mathcal{B}_3) + \int d^4\theta\,\mathcal{B}_1\mathcal{B}_2(\mathcal{D}^{p_3}\mathcal{D}^{p_3}\mathcal{B}_3) \tag{B.2.5c}$$
$$\int d^4\theta\,\mathcal{B}_1(\bar{\mathcal{D}}^{p_2}\bar{\mathcal{D}}^{p_2}\mathcal{B}_2)\mathcal{B}_3 = \int d^4\theta\,(\bar{\mathcal{D}}^{p_1}\bar{\mathcal{D}}^{p_1}\mathcal{B}_1)\mathcal{B}_2\mathcal{B}_3\ + 2\int d^4\theta(\bar{\mathcal{D}}^{p_1}_{\dot\alpha}\mathcal{B}_1)\mathcal{B}_2(\bar{\mathcal{D}}^{p_3,\dot\alpha}\mathcal{B}_3)\ + \int d^4\theta 4\,\mathcal{B}_1\mathcal{B}_2(\mathcal{D}^{p_3}\mathcal{D}^{p_3}\mathcal{B}_3) \tag{B.2.5d}$$
$$\int d^4\theta\,\mathcal{F}_1(\mathcal{D}^{p_2}\mathcal{D}^{p_2}\mathcal{B}_2)\mathcal{B}_3 = \int d^4\theta\,(\mathcal{D}^{p_1}\mathcal{D}^{p_1}\mathcal{F}_1)\mathcal{B}_2\mathcal{B}_3\ - 2\int d^4\theta(\mathcal{D}^{p_1,\alpha}\mathcal{F}_1)\mathcal{B}_2(\mathcal{D}^{p_3}_\alpha\mathcal{B}_3) + \int d^4\theta\,\mathcal{F}_1\mathcal{B}_2(\mathcal{D}^{p_3}\mathcal{D}^{p_3}\mathcal{B}_3) \tag{B.2.5e}$$

where $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ are the Fourier transform of bosonic superfields, while $\mathcal{F}_1$ of a fermionic one and $p_1{+}p_2{+}p_3 = 0$. We have assumed that $\mathcal{B}_1(\mathcal{F}_1)$, $\mathcal{B}_2$ and $\mathcal{B}_3$ depend on the momenta p_1, p_2 and p_3 respectively.

For example, Eqs.(B.2.5a) can be derived simply by using the definition of supercovariant derivatives (B.2.1a) and the Leibnitz rules for the Grassmann derivatives ∂_α and $\bar\partial_{\dot\alpha}$.
Eq.(B.2.5c) follows from:

$$\begin{aligned}\int d^4\theta\,\mathcal{B}_1(\mathcal{D}^{p_2}\mathcal{D}^{p_2}\mathcal{B}_2)\mathcal{B}_3 &= \int d^4\theta\,\mathcal{B}_1\big(\mathcal{D}^{p_2\alpha}(\mathcal{D}^{p_2}_\alpha\mathcal{B}_2)\big)\mathcal{B}_3 = -\int d^4\theta\,(\mathcal{D}^{p_1\alpha}\mathcal{B}_1)(\mathcal{D}^{p_2}_\alpha\mathcal{B}_2)\mathcal{B}_3 + \int d^4\theta\,\mathcal{B}_1(\mathcal{D}^{p_2}_\alpha\mathcal{B}_2)(\mathcal{D}^{p_3\alpha}\mathcal{B}_3)\\
&= -\int d^4\theta\,(\mathcal{D}^{p_1}_\alpha\mathcal{D}^{p_1\alpha}\mathcal{B}_1)\mathcal{B}_2\mathcal{B}_3 + \int d^4\theta\,(\mathcal{D}^{p_1\alpha}\mathcal{B}_1)\mathcal{B}_2(\mathcal{D}^{p_3}_\alpha\mathcal{B}_3) - \int d^4\theta\,(\mathcal{D}^{p_1}_\alpha\mathcal{B}_1)\mathcal{B}_2(\mathcal{D}^{p_3\alpha}\mathcal{B}_3) - \int d^4\theta\,\mathcal{B}_1\mathcal{B}_2(\mathcal{D}^{p_3}_\alpha\mathcal{D}^{p_3\alpha}\mathcal{B}_3)\\
&= \int d^4\theta\,(\mathcal{D}^{p_1}\mathcal{D}^{p_1}\mathcal{B}_1)\mathcal{B}_2\mathcal{B}_3\ + 2\int d^4\theta\,(\mathcal{D}^{p_1\alpha}\mathcal{B}_1)\mathcal{B}_2(\mathcal{D}^{p_3}_\alpha\mathcal{B}_3) + \int d^4\theta\,\mathcal{B}_1\mathcal{B}_2(\mathcal{D}^{p_3}\mathcal{D}^{p_3}\mathcal{B}_3)\end{aligned} \tag{B.2.6}$$

In the second and third step we used repeatedly the integration by part (B.2.5a) making attention of the fact that $\mathcal{D}^{p_2}_\alpha\mathcal{B}_2$ and $\mathcal{D}^{p_1\alpha}\mathcal{B}_1$ are fermionic superfield. In the last step the identity ${}_\alpha\ldots{}^\alpha = -{}^\alpha\ldots{}_\alpha$ has been used.

B.2.5 $\mathcal{D}$ and fermionic delta function δ

The action of the superderivatives on the fermionic δ functions in the momentum space is

$$\mathcal{D}^p_{1\alpha}\,\delta_{12} = -\mathcal{D}^{-p}_{2\alpha}\,\delta_{12} \tag{B.2.7a}$$
$$\bar{\mathcal{D}}^p_{1\dot\alpha}\,\delta_{12} = -\bar{\mathcal{D}}^{-p}_{2\dot\alpha}\,\delta_{12} \tag{B.2.7b}$$
$$\mathcal{D}^p_1\mathcal{D}^p_1\,\delta_{12} = \mathcal{D}^{-p}_2\mathcal{D}^{-p}_2\,\delta_{12} \tag{B.2.7c}$$

$$\bar{\mathcal{D}}_1^p \bar{\mathcal{D}}_1^p \, \delta_{12} = \bar{\mathcal{D}}_2^{-p} \bar{\mathcal{D}}_2^{-p} \, \delta_{12} \tag{B.2.7d}$$

$$\delta_{12}\delta_{12} = \delta_{12}\mathcal{D}_\alpha\delta_{12} = \delta_{12}\bar{\mathcal{D}}_{\dot\alpha}\delta_{12} = \delta_{12}\mathcal{D}\mathcal{D}\delta_{12} = \delta_{12}\bar{\mathcal{D}}\bar{\mathcal{D}}\delta_{12} = \delta_{12}\mathcal{D}_\alpha\bar{\mathcal{D}}\bar{\mathcal{D}}\delta_{12} = \delta_{12}\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}\mathcal{D}\delta_{12} = 0 \tag{B.2.7e}$$

$$\delta_{12}\mathcal{D}\mathcal{D}\,\bar{\mathcal{D}}\bar{\mathcal{D}}\delta_{12} = 16\delta_{12} \tag{B.2.7f}$$
$$\delta_{12}\bar{\mathcal{D}}\bar{\mathcal{D}}\,\mathcal{D}\mathcal{D}\delta_{12} = 16\delta_{12} \tag{B.2.7g}$$

Eq. (B.2.7a) can be derived by observing that $\partial_{1\alpha}\delta_{12} = \partial_{1\alpha}\theta_{12}^2 = -\partial_{2\alpha}\theta_{12}^2 = -\partial_{2\alpha}\delta_{12}$. and that $p_\mu\delta_{12} = -(-p_\mu)\delta_{12}$.

Eq. (B.2.7c) follows from

$$\begin{aligned} &\mathcal{D}_1^p \mathcal{D}_1^p \, \delta_{12} = \mathcal{D}_1^{p\alpha} \mathcal{D}_{1\alpha}^p \, \delta_{12} = -\mathcal{D}_1^{p\alpha} \mathcal{D}_{2\alpha}^{-p} \, \delta_{12} = \\ &= \mathcal{D}_{2\alpha}^{-p} \mathcal{D}_1^{p\,\alpha} \, \delta_{12} = -\mathcal{D}_{2\alpha}^{-p} \mathcal{D}_2^{-p\,\alpha} \, \delta_{12} = \mathcal{D}_2^{-p} \mathcal{D}_2^{-p} \, \delta_{12}. \end{aligned} \tag{B.2.8}$$

In the second and forth steps eq.(B.2.7a) has been used and the third one follows from $\{\mathcal{D}_{2\alpha}, \mathcal{D}_1^\alpha\} = 0$. The last one is a consequence of $\mathcal{D}_{2\alpha}\mathcal{D}_2^\alpha = -\mathcal{D}_2^\alpha\mathcal{D}_{2\alpha} = \mathcal{D}_2\mathcal{D}_2$.

In eq. (B.2.7e), for example $\delta_{12}\mathcal{D}_\alpha\delta_{12} = 0$ follows from

$$\begin{aligned} \delta_{12}\mathcal{D}_{1\alpha}\delta_{12} &= \theta_{12}^2\bar\theta_{12}^2\,(\partial_{1\alpha} - \sigma^\mu_{\alpha\dot\beta}\bar\theta_1^{\dot\beta} p_\mu)\theta_{12}^2\bar\theta_{12}^2 \\ &= \theta_{12}^2\bar\theta_{12}^2(2\theta_{12\alpha})\bar\theta_{12}^2 - \sigma^\mu_{\alpha\dot\beta}\bar\theta_1^{\dot\beta} p_\mu \theta_{12}^2\bar\theta_{12}^2\,\theta_{12}^2\bar\theta_{12}^2 \\ &= 0 \end{aligned} \tag{B.2.9}$$

In the second step the relation $\partial_{1\alpha}\theta_{12}^2 = 2\theta_{12\alpha}$ has been employed, while the third step is a consequence of the fact that the product of three or more component of θ_{12} or of $\bar\theta_{12}$ gives zero.
In fact, the only way to have a non zero result is to act with two chiral derivatives ∂_α on θ_{12}^2 and with two antichiral derivatives $\bar\partial_{\dot\alpha}$ on $\bar\theta_{12}^2$. Doing so, one can write $\delta_{12}\mathcal{D}_1\mathcal{D}_1\bar{\mathcal{D}}_1\bar{\mathcal{D}}_1\delta_{12} = \delta_{12}\partial_1\partial_1\bar\partial_1\bar\partial_1\delta_{12} = \delta_{12}(\partial_1\partial_1\theta_{12}^2)(\bar\partial_1\bar\partial_1\bar\theta_{12}^2) = 16\delta_{12}$ because $\mathcal{D}\mathcal{D} = \partial\partial + \ldots$, $\bar{\mathcal{D}}\bar{\mathcal{D}} = \bar\partial\bar\partial + \ldots$ and $\partial_1\partial_1\theta_{12}^2 = \bar\partial_1\bar\partial_1\bar\theta_{12}^2 = 4$ (see eq.(A.5.5).

Appendix C

Propagators

The expansion in components of a chiral (antichiral) superfield $\Phi(x,\theta,\bar\theta)$ ($\Phi^\dagger(x,\theta,\bar\theta)$) is given by

$$\Phi(x,\theta,\bar\theta) = \phi(x) - i\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^\mu\partial_\mu\phi(x) + \sqrt{2}\theta\psi(x) + \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi\sigma^\mu\bar\theta + \theta\theta F(x), \tag{C.0.1}$$

$$\Phi^\dagger(x,\theta,\bar\theta) = \phi^*(x) + i\theta\sigma^\mu\bar\theta\partial_\mu\phi^*(x) - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^\mu\partial_\mu\phi^*(x) + \sqrt{2}\bar\theta\bar\psi(x) - \frac{i}{\sqrt{2}}\bar\theta\bar\theta\theta\sigma^\mu\partial_\mu\bar\psi + \bar\theta\bar\theta F^*(x). \tag{C.0.2}$$

The expansion in components of the Fourier transform of a chiral (antichiral) superfield $\Phi(p,\theta,\bar\theta)$ ($\Phi^\dagger(p,\theta,\bar\theta)$) is:

$$\Phi(p,\theta,\bar\theta) = \phi(p) - p_\mu\theta\sigma^\mu\bar\theta\phi(p) + \frac{1}{4}p^2\theta\theta\bar\theta\bar\theta\phi(p) + \sqrt{2}\theta\psi(p) + \frac{p_\mu}{\sqrt{2}}\theta\theta\,\psi(p)\sigma^\mu\bar\theta + \theta\theta F(p) \tag{C.0.3}$$

$$\Phi^\dagger(p,\theta,\bar\theta) = \phi^*(p) + p_\mu\,\theta\sigma^\mu\bar\theta\phi^*(p) + \frac{1}{4}p^2\,\theta\theta\bar\theta\bar\theta\phi^*(p) + \sqrt{2}\bar\theta\bar\psi(p) - \frac{p_\mu}{\sqrt{2}}\,\bar\theta\bar\theta\theta\sigma^\mu\bar\psi(p) + \bar\theta\bar\theta F^*(p). \tag{C.0.4}$$

The expansion in components of a vector superfield $V(x,\theta,\bar\theta)$ is

$$\begin{aligned} V(x,\theta,\bar\theta) &= C(x) + \sqrt{2}\theta\chi(x) + \sqrt{2}\bar\theta\bar\chi(x) + \theta\theta S(x) + \bar\theta\bar\theta S^*(x) + \theta\sigma^\mu\bar\theta A_\mu(x) \\ &+ \theta\theta\,\bar\theta\Big(\bar\lambda - \frac{i}{\sqrt{2}}\bar\sigma^\mu\partial_\mu\chi(x)\Big) + \bar\theta\bar\theta\,\theta\Big(\lambda(x) - \frac{i}{\sqrt{2}}\sigma^\mu\partial_\mu\bar\chi(x)\Big) + \frac{1}{2}\theta\theta\,\bar\theta\bar\theta\Big(D(x) - \frac{1}{2}\partial^\mu\partial_\mu C(x)\Big). \end{aligned} \tag{C.0.5}$$

The expansion in components of the Fourier transform of a vector superfield $V(p,\theta,\bar\theta)$ is

$$\begin{aligned} V(p,\theta,\bar\theta) &= C(p) + \sqrt{2}\theta\chi(p) + \sqrt{2}\bar\theta\bar\chi(p) + \theta\theta S(p) + \bar\theta\bar\theta S^*(p) + \theta\sigma^\mu\bar\theta A_\mu(p) \\ &+ \theta\theta\,\bar\theta\Big(\bar\lambda(p) - \frac{1}{\sqrt{2}}\bar\sigma^\mu p_\mu\chi(p)\Big) + \bar\theta\bar\theta\,\theta\Big(\lambda(p) - \frac{1}{\sqrt{2}}\sigma^\mu p_\mu\bar\chi(p)\Big) + \frac{1}{2}\theta\theta\,\bar\theta\bar\theta\Big(D(p) + \frac{1}{2}p^2 C(p)\Big). \end{aligned} \tag{C.0.6}$$

C.1 Propagator of the chiral superfield

$$< \Phi(p_1,\theta_1,\bar\theta_1)\Phi^\dagger(-p_1,\theta_2,\bar\theta_2) >_{free} = \frac{i}{16p_1^2}\,\bar{\mathcal{D}}_1^{p_1}\bar{\mathcal{D}}_1^{p_1}\mathcal{D}_1^{p_1}\mathcal{D}_1^{p_1}\delta^{(4)}(\theta_{12}) \tag{C.1.1}$$

$$< \Phi(p_1,\theta_1,\bar\theta_1)\Phi^\dagger(-p_1,\theta_2,\bar\theta_2) >_{free} = \frac{i}{p_1^2} - \frac{i}{p_1^2}p_{1\mu}\theta_1\sigma^\mu\bar\theta_1 + \frac{i}{4}\theta_1^2\bar\theta_1^2 - \frac{i}{p_1^2}p_{1\mu}\theta_2\sigma^\mu\bar\theta_2 + \frac{i}{4}\theta_2^2\bar\theta_2^2 + \frac{2i}{p_1^2}p_{1\mu}\theta_1\sigma^\mu\bar\theta_2 \tag{C.1.2}$$

$$- i\theta_1^2\bar\theta_1\bar\theta_2 - i\bar\theta_2^2\theta_1\theta_2 + i\theta_1^2\bar\theta_2^2 + \frac{i}{p_1^2}p_{1\mu}p_{1\nu}\theta_1\sigma^\mu\bar\theta_1\theta_2\sigma^\nu\bar\theta_2 - \frac{i}{4}p_{1\mu}\theta_1\sigma^\mu\bar\theta_1\theta_2^2\bar\theta_2^2 - \frac{i}{4}p_{1\mu}\theta_2\sigma^\mu\bar\theta_2\theta_1^2\bar\theta_1^2 - \frac{i}{2}p_{1\mu}\theta_2\sigma^\mu\bar\theta_1\theta_1^2\bar\theta_2^2 \tag{C.1.3}$$

$$+ \frac{i}{16}p_1^2\theta_1^2\bar\theta_1^2\theta_2^2\bar\theta_2^2 \tag{C.1.4}$$

$$< \Phi(p_1,\theta_1,\bar\theta_1)\Phi^\dagger(-p_1,\theta_2,\bar\theta_2 >_{free} = \frac{i}{p_1^2}\exp\Big(- p_\mu\big(\theta_1\sigma^\mu\bar\theta_1 + \theta_2\sigma^\mu\bar\theta_2 - 2\theta_1\sigma^\mu\bar\theta_2\big)\Big) \tag{C.1.5}$$

C.2 Propagator of the vector superfield

$$< V^a(p_1,\theta_1,\bar\theta_1)V^b(-p_1,\theta_2,\bar\theta_2) >_{free} = -\frac{i}{p_1^2}\delta^{ab}\delta^{(4)}(\theta_{12}) \tag{C.2.1}$$

C.3 Mixed correlators

$$\begin{aligned} < \phi(p_1)\Phi^\dagger(-p_1,\theta_2,\bar\theta_2) >_{free} &= < \Phi(p_1,\theta_1,\bar\theta_1)\Phi^\dagger(-p_1,\theta_2,\bar\theta_2) >_{free}\Big|_{\theta_1=\bar\theta_1=0} \\ &= \frac{i}{p_1^2} - \frac{i}{p_1^2}p_{1\mu}\theta_2\sigma^\mu\bar\theta_2 + \frac{i}{4}\theta_2^2\bar\theta_2^2 \qquad \text{(C.3.1a)} \\ < \Phi(p_1,\theta_1,\bar\theta_1), \phi^*(-p_1) >_{free} &= \frac{i}{p_1^2} - \frac{i}{p_1^2}p_{1\mu}\theta_1\sigma^\mu\bar\theta_1 + \frac{i}{4}\theta_1^2\bar\theta_1^2 \qquad \text{(C.3.1b)} \end{aligned}$$

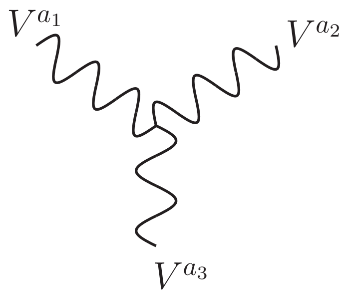
$$< A^{a\mu}(p_1)V^b(-p_1,\theta_2,\bar\theta_2 >_{free} = \frac{1}{2}\bar\partial_{1\dot\alpha}\partial_{1\alpha}\bar\sigma^{\mu\,\dot\alpha\alpha} < V^a(p_1,\theta_1,\bar\theta_1)V^b(-p_1,\theta_2,\bar\theta_2) >_{free}\Big|_{\theta_1=\bar\theta_1=0} = -\frac{2i}{p_1^2}\theta_2\sigma^\mu\bar\theta_2 \tag{C.3.2}$$

Appendix D

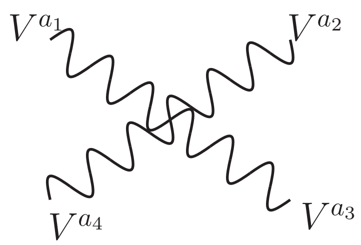
Interaction vertices for $\mathcal{N} = 4$ SYM

In this appendix we give the expression of the interaction vertices for $\mathcal{N} = 4$ SYM.

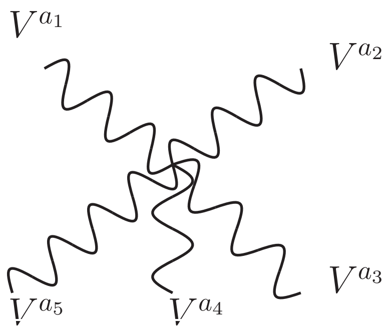
The self interaction vertices for the vector superfield V^a up to order g^3 are



$$S^{(3)}_{SYM} = -\frac{i}{16}\, g\, f_{a_1 a_2 a_3} \int d^8 z \Big(\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}^\alpha V^{a_1}\Big)\Big(\mathcal{D}_\alpha V^{a_2}\Big) V^{a_3}. \tag{D.0.1}$$

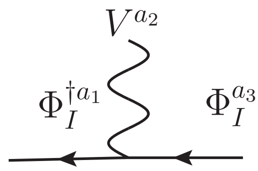


$$\begin{aligned} S^{(4)}_{SYM} &= g^2\, f_{a_1 a_2 b} f_{b a_3 a_4} \int d^8 z \Big[\frac{1}{64}\Big(V^{a_1}\mathcal{D}^\alpha V^{a_2}\Big)\Big(\bar{\mathcal{D}}\bar{\mathcal{D}}(V^{a_3}\mathcal{D}_\alpha V^{a_4})\Big) - \frac{1}{48}\Big(V^{a_1}(\mathcal{D}^\alpha V^{a_2})V^{a_3}\Big)\Big(\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}_\alpha V^{a_4}\Big)\Big] \\ &= g^2 f_{a_1 a_2 b} f_{b a_3 a_4} \int d^8 z \Big[\frac{1}{192}\Big(\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}^\alpha V^{a_1}\Big) V^{a_2} V^{a_3}\Big(\mathcal{D}_\alpha V^{a_4}\Big) - \frac{1}{32}\Big(\mathcal{D}^\alpha V^{a_1}\Big) V^{a_2}\Big(\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}_\alpha V^{a_3}\Big)\bar{\mathcal{D}}^{\dot\alpha} V^a_4 \quad . \\ &+ \frac{1}{64}\Big(\mathcal{D}^\alpha V^{a_1}\Big) V^{a_2}\Big(\mathcal{D}_\alpha V^{a_3}\Big)\bar{\mathcal{D}}\bar{\mathcal{D}} V^{a_4}\Big] \end{aligned} \tag{D.0.2}$$



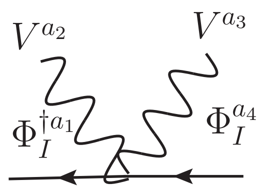
$$
\begin{aligned}
S^{(5)}_{SYM} &= i\, g^3\, f_{a_1 a_2 b_1} f_{a_3 b_1 b_2} f_{a_4 a_5 b_2} \cdot \\
&\int d^8 z \Big[\frac{1}{192} \Big(V^{a_1} \mathcal{D}^\alpha V^{a_2} \Big) V^{a_3} V^{a_4} \Big(\bar{\mathcal{D}} \bar{\mathcal{D}} \mathcal{D}_\alpha V^{a_5} \Big) - \frac{1}{96} \Big(V^{a_1} \big(\mathcal{D}^\alpha V^{a_2} \Big) V^{a_3} \Big(\bar{\mathcal{D}} \bar{\mathcal{D}} \big(V^{a_4} \mathcal{D}_\alpha V^{a_5} \big) \Big) \Big] \\
&= -i\, g^3 f_{a_1 a_2 b_1} f_{a_3 b_1 b_2} f_{a_4 a_5 b_2} \int d^8 z \Big[\frac{1}{192} \Big(V^{a_1} \mathcal{D}^\alpha V^{a_2} \Big) V^{a_3} V^{a_4} \Big(\bar{\mathcal{D}} \bar{\mathcal{D}} \mathcal{D}_\alpha V^{a_5} \Big) \\
&+ \frac{1}{96} \Big(V^{a_1} \mathcal{D}^\alpha V^{a_2} \Big) V^{a_3} \Big(\bar{\mathcal{D}} \bar{\mathcal{D}} V^{a_4} \Big) \Big(\mathcal{D}_\alpha V^{a_5} \Big) + \frac{1}{48} \Big(V^{a_1} \mathcal{D}^\alpha V^{a_2} \Big) V^{a_3} \Big(\bar{\mathcal{D}}_{\dot\alpha} V^{a_4} \Big) \Big(\mathcal{D}^{\dot\alpha} V^{a_5} \Big) \Big].
\end{aligned}
\tag{D.0.3}
$$

The gauge-matter vertex at order g is



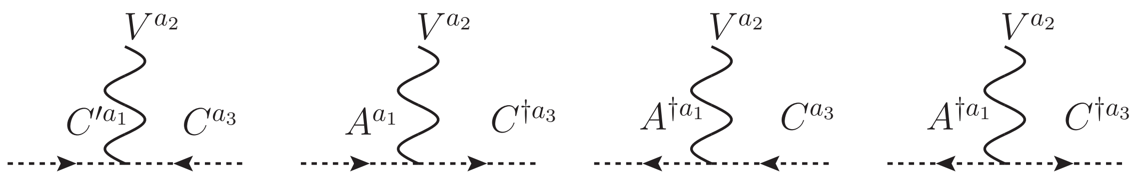
$$
S^{\Phi V \Phi}_{int} = i g f_{a_1 a_2 a_3} \int d^8 z\, \Phi_I^{\dagger a_1} V^{a_2} \Phi_I^{a_3}. \tag{D.0.4}
$$

where I is the flavor index.
The gauge-matter vertex at order g^2 is

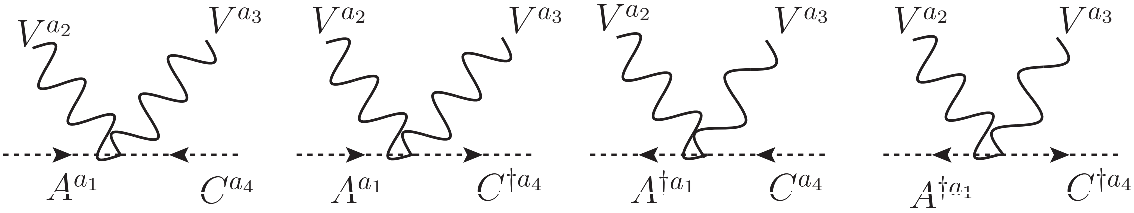


$$
S^{\Phi V^2 \Phi}_{int} = -\frac{g^2}{2} f_{a_1 a_2 b} f_{b a_3 a_4} \int d^8 z\, \Phi_I^{\dagger a_1} V^{a_2} V^{a_3} \Phi_I^{a_4}. \tag{D.0.5}
$$

The ghost-vector vertices up to order g^2 are

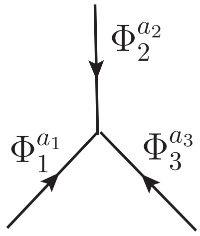


$$S_{GH}^{(3)} = \frac{ig}{2} f_{a_1a_2a_3} \int d^8z \Big[- A^{a_1} V^{a_2} C^{a_3} - A^{a_1} V^{a_2} C^{\dagger a_3} + A^{\dagger a_1} V^{a_2} C^{a_3} + A^{\dagger a_1} V^{a_2} C^{\dagger a_3} \Big] \tag{D.0.6}$$

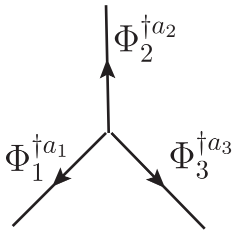


$$S_{GH}^{(4)} = \frac{g^2}{12} f_{a_1a_2b} f_{ba_3a_4} \int d^8z \Big(A^{a_1} V^{a_2} V^{a_3} C^{a_4} - A^{a_1} V^{a_2} V^{a_3} C^{\dagger a_4} - A^{\dagger a_1} V^{a_2} V^{a_3} C^{a_4} + A^{\dagger a_1} V^{a_2} V^{a_3} C^{\dagger a_4} \Big) \tag{D.0.7}$$

The self interaction vertices for the matter superfields are



$$S_{int}^{\Phi} = -g\, f_{a_1a_2a_3} \int d^6z\, \Phi_1^{a_1} \Phi_2^{a_2} \Phi_3^{a_3} \tag{D.0.8}$$

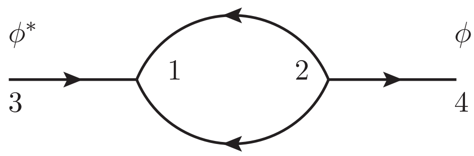


$$S_{int}^{\Phi^\dagger} = -g\, f_{a_1a_2a_3} \int d^6\bar{z}\, \Phi_1^{\dagger a_1} \Phi_2^{\dagger a_2} \Phi_3^{\dagger a_3}. \tag{D.0.9}$$

Appendix E

Symmetry factors

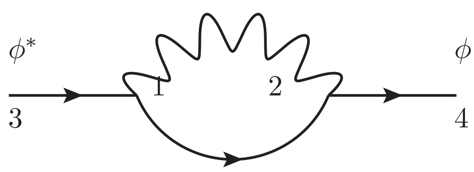
Feynman rules give the rules for associating analytic expressions to each part of a Feynman diagram, i.e. propagators, vertices and external points. However, the overall numerical factor has to be determined separately. This factor is called the symmetry factor of the diagram (see for example [54],[102]).
Usually the coefficients of the interaction terms in the Lagrangian are chosen so that the symmetry factor is one. One can determine the symmetry factor by the symmetry of the diagram, that is by the number of ways of interchanging components without changing the diagram, but there can be the possibility to be in doubt. Another way is to count the equivalent Wick contractions which give the same analytic expression. We have chosen this second way and in this appendix we give some examples.
Here, to simplify the notation, we work in (super)coordinate space. Moreover, below ϕ is the scalar component of the chiral supermultiplet and $< i\,j >_\Phi$, $< ij >_\phi$ and $< ij >_{\phi^*}$ are shortcuts for $< \Phi(z_i), \Phi^\dagger(z_j) >$, $< \phi(x_i), \Phi^\dagger(z_j) >$ and $< \Phi(z_i), \phi^*(x_j) >$ respectively.



This diagram corresponds to the Wick contraction

$$\int d^6 z_1 d^6 \bar{z}_2 < \phi^*(x_3)\phi(x_4)|\Phi(z_1)\Phi(z_1)\Phi(z_1)|\Phi^\dagger(z_2)\Phi^\dagger(z_2)\Phi^\dagger(z_2)| > \qquad \text{(E.0.1)}$$

It is formed by the two vertices Φ^3 and $\Phi^{\dagger 3}$, hence it is obtained considering the second order term $\frac{i^2 S_{int}^2}{2}$ in the exponential $e^{iS_{int}}$ (see eq.(1.3.3)). Inside $S^2 = \big(\ldots + \Phi^3 + \Phi^{\dagger 3}\ldots\big)^2 = \ldots + 2\Phi^3\Phi^{\dagger 3} + \ldots$, the coefficient in front of $\Phi^3\Phi^{\dagger 3}$ is 2. Thus the Taylor expansion gives a factor $\frac{i^2\times 2}{2} = -1$.
The other combinatorial factor arises from the Wick contractions. Starting from the left, the first contraction $\phi^*(x_3)\Phi(z_1) =< 13 >_{\phi^*}$ can be done in three different ways because there are three superfields Φ at z_1, and similarly the second contraction $\phi(x_4)\Phi^\dagger(z_2) =< 42 >_\phi$ gives another factor of 3. Then one of the two remaining superfields Φ at z_1 can have two contractions $\Phi(z_1)\Phi^\dagger(z_2)$ with the two remaining superfield at z_2. Finally what remains is a superfield at z_1 and one at z_2 that gives one obliged contraction $\Phi(z_1)\Phi^\dagger(z_2)$. So the factor from the contractions is $3\times 3\times 2$ which has to be multiplied by $\frac{1}{3!}\frac{1}{3!}$ coming from the numerical constants in front of the superpotential in the action. Thus the total symmetry factor is $\frac{-1\times 3\times 3\times 2}{3!3!} = -\frac{1}{2}$.



This diagram correspond to the Wick contraction

$$\int d^8z_1 d^8z_2 < \phi^*(x_3)\phi(x_4)|\Phi^\dagger(z_1)V(z_1)\Phi(z_1)|\Phi^\dagger(z_2)V(z_2)\Phi(z_2)| > \tag{E.0.2}$$

It has two identical vertices $(\Phi^\dagger V\Phi)(\Phi^\dagger V\Phi)$, hence it comes from the second order term $\frac{i^2 S_{int}^2}{2}$ in the exponential $e^{iS_{int}}$. Inside $S^2 = \left(\ldots+\Phi^\dagger V\Phi+\ldots\right)^2 = \ldots+(\Phi^\dagger V\Phi)^2+\ldots$, the coefficient in front of $(\Phi^\dagger V\Phi)$ is 1. Thus the Taylor expansion gives a factor $\frac{i^2}{2} = -\frac{1}{2}$. From the Wick contractions, one has two ways to connect $\phi^*(x_3)$ to a $\Phi^\dagger V\Phi$ vertex since we have two identical vertices of this kind. After choosing this vertex connected to $\phi^*(x_3)$, then all the contractions are obliged and don't provide any other combinatorial factor. Thus, the total symmetry factor is $-\frac{1}{2}\times 2 = -1$.

Appendix F

Spinor helicity formalism

The spinor helicity formalism [104]-[108] allows to have extremely compact representations of amplitudes involving massless particles. In this formalism one expresses the momentum and the polarization vector of a massless particle of spin one in terms of spinor variables.
Any momentum vector p_μ can be represented by a 2×2 matrix $p_{\alpha\dot{\alpha}}$ by contracting p_μ with the sigma matrices $\sigma^\mu_{\alpha\dot{\alpha}}$

$$p_{\alpha\dot{\alpha}} = p_\mu \sigma^\mu_{\alpha\dot{\alpha}}. \tag{F.0.1}$$

Any 2×2 matrix has rank at most two, so it can be written as [104]

$$p_{\alpha\dot{\alpha}} = \mu_\alpha \tilde{\mu}_{\dot{\alpha}} + \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}, \tag{F.0.2}$$

where λ, μ are some Weyl spinors belonging to the representation $(1/2, 0)$ of the Lorentz group, while $\tilde{\lambda}, \tilde{\mu}$ belong to the representation $(0, 1/2)$.
For a light-like momentum of a massless particle, one has that the determinant of the matrix $p_{\alpha\dot{\alpha}}$ vanishes since

$$\det(p_{\alpha\dot{\alpha}}) = p^2 = 0. \tag{F.0.3}$$

In this case the matrix $p_{\alpha\dot{\alpha}}$ has rank which is at most equal to one and can be written as $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$, that is

$$p^2 = 0 \;\Leftrightarrow\; p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}. \tag{F.0.4}$$

If the momentum p^μ is real then λ and $\tilde{\lambda}$ are the complex conjugate of each other, i.e. $\tilde{\lambda} = \lambda^*$. On the contrary, if one assumes that p^μ is complex as in the generalized unitarity methods then $\tilde{\lambda}$ and λ are not related by complex conjugation, but are independent.
Let's consider the case of more than one particle and let's denote as $p_{i\mu}$ the momentum of the $i-$th particle having the matrix representation $p_{i\alpha\dot{\alpha}} = \lambda_{i\alpha} \tilde{\lambda}_{i\dot{\alpha}}$. One can introduce the spinor products

$$< ij > \equiv < \lambda_i \lambda_j > \equiv \epsilon^{\beta\alpha} \lambda_{i\,\alpha} \lambda_{j\,\beta}, \quad [ij] \equiv [\tilde{\lambda}_i \tilde{\lambda}_j] \equiv \epsilon_{\dot{\beta}\dot{\alpha}} \tilde{\lambda}_i^{\dot{\alpha}} \tilde{\lambda}_i^{\dot{\beta}}, \tag{F.0.5}$$

which are antisymmetric and hence $< ii >= [ii] = 0$. With these definitions a scalar product between the momenta of two particles can be written as

$$2p_i \cdot p_j =< ij > [ij]. \tag{F.0.6}$$

Polarization vectors

The polarization vectors $\epsilon^\pm_\mu$ of a massless vector particle with helicities $+1$ or -1 respectively can also be expressed in terms spinor variables. They have to satisfy the conditions

$$p \cdot \epsilon^\pm_\mu = 0, \quad (\epsilon^\pm)^2 = 0, \quad \epsilon^+ \cdot \epsilon^- = -1, \tag{F.0.7}$$

where p_μ is the momentum of the particle and as above p_μ has the representation $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$.
Let's consider an arbitrary reference light-like momentum q^μ, i.e. $q^2 = 0$, satisfying the condition $q \cdot p \neq 0$. Since $q^2 = 0$, from eq.(F.0.4) one has that q^μ can be represented as $q_{\alpha\dot{\alpha}} = \mu_\alpha \tilde{\mu}_{\dot{\alpha}}$ for some spinors μ and $\tilde{\mu}$,

where $q_{\alpha\dot{\alpha}} = q_\mu \sigma^\mu_{\alpha\dot{\alpha}}$.
Then, one can choose for $\epsilon^\pm_\mu$ the representation

$$\epsilon^+_{\alpha\dot{\alpha}} = \frac{\mu_\alpha \tilde{\lambda}_{\dot{\alpha}}}{< \lambda\mu >} \quad \epsilon^-_{\alpha\dot{\alpha}} = \frac{\lambda_\alpha \tilde{\mu}_{\dot{\alpha}}}{[\tilde{\lambda}\tilde{\mu}]}, \tag{F.0.8}$$

where $\epsilon^\pm_{\alpha\dot{\alpha}} \equiv \epsilon^\pm_\mu \sigma^\mu_{\alpha\dot{\alpha}}$. In fact, the representation (F.0.8) satisfies the eqs.(F.0.7).
Changing the reference momentum corresponds to making a gauge transformation. Hence, the on-shell amplitudes are independent of the choice of the reference momentum, since they are gauge-invariant.
Let's consider the case of more than one massless vector particle and let's introduce for the polarization vector of the $i-$th particle the notation $\epsilon^\pm_{i\mu}(p_i, q_i)$ where the first argument indicates the momentum of the particle and the second argument the reference momentum.
One can prove the identities

$$\begin{aligned} & q_i \cdot \epsilon^\pm_i(p_i, q_i) = 0 \\ & \epsilon^\pm_i(p_i, q_i) \cdot \epsilon^\pm_j(p_j, q_i) = 0 \\ & \epsilon^\pm_i(p_i, q_i) \cdot \epsilon^\mp_j(p_j, p_i) = 0. \end{aligned} \tag{F.0.9}$$

From these identities, it follows that it is convenient to choose the reference momenta of like-helicity particles to be the same and to coincide with the external momenta of some of the particles with the opposite helicity.

Appendix G

Off-shell planar amplitudes

In this appendix we give the explicit expressions for the planar three gluon amplitude at one loop $\mathcal{A}^{off\,shell}_{1\,loop}\Big(A^{a_1}_{\mu_1}(p_1)A^{a_2}_{\mu_2}(p_2)A^{a_3}_{\mu_3}(p_3)\Big)$ and for the planar four gluon amplitude at tree level $\mathcal{A}^{off\,shell}_{tree}\Big(A^{a_1}_{\mu_1}(p_1)A^{a_2}_{\mu_2}(p_2)A^{a_3}_{\mu_3}(p_3)A^{a_4}_{\mu_4}(p_4)\Big)$. We also give the explicit expression of the part of the four gluon planar amplitude at one loop $\mathcal{A}^{off\,shell}_{1\,Loop}\Big(A^{a_1}_{\mu_1}(p_1)A^{a_2}_{\mu_2}(p_2)A^{a_3}_{\mu_3}(p_3)A^{a_4}_{\mu_4}(p_4)\Big)$ associated to the Lorentz structure $\eta_{\mu_2\mu_3}p_{3\,\mu_1}p_{3\,\mu_4}$ which is relevant for the on-shell amplitude $\mathcal{A}^{on\,shell}_{1\,loop}(1^-,2^-,3^+,4^+)$ (see section 3.6.4).

G.1 Three gluon planar amplitude at one loop

After doing the Passarino Veltman decomposition, substituting p_2 with $p_2=-p_1-p_3$ and introducing the variable $u\equiv p_1\cdot p_3$, the final result for the off-shell planar three gluon amplitude is

$$\begin{aligned}&\mathcal{A}^{off\,shell}_{1\,loop}\Big(A^{a_1}_{\mu_1}(p_1)A^{a_2}_{\mu_2}(p_2)A^{a_3}_{\mu_3}(p_3)\Big)=Ng^3\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3})\times\\&\Big(M_{1\,,\mu_1\mu_2\mu_3}\,B_0(p_1)+M_{2\,,\mu_1\mu_2\mu_3}\,B_0(p_1+p_3)+M_{3\,,\mu_1\mu_2\mu_3}\,B_0(p_3)+N_{\mu_1\mu_2\mu_3}\,C_0(p_1,p_3)\Big),\end{aligned}\tag{G.1.1}$$

where

$$\begin{aligned}&M_{1\,\mu_1\mu_2\mu_3}=\\&-2\,{p_1}^4\eta_{\mu_1\mu_2}p_{3,\mu_3}-4\,p_{1,\mu_2}p_{1,\mu_3}p_{1,\mu_1}u+u^2\eta_{\mu_3\mu_1}p_{3,\mu_2}+p_{3,\mu_3}\eta_{\mu_1\mu_2}u^2-2\,\eta_{\mu_1\mu_2}up_{3,\mu_3}{p_1}^2\\&-p_{1,\mu_3}p_{3,\mu_2}p_{3,\mu_1}u+2\,{p_1}^2\eta_{\mu_1\mu_2}p_{1,\mu_3}u+2\,p_{1,\mu_3}\eta_{\mu_1\mu_2}u^2-p_{1,\mu_2}p_{3,\mu_1}p_{3,\mu_3}u-2\,p_{3,\mu_3}p_{3,\mu_2}p_{1,\mu_1}u\\&+2\,u\eta_{\mu_3\mu_1}p_{3,\mu_2}{p_1}^2+4\,p_{3,\mu_2}p_{3,\mu_1}p_{3,\mu_3}{p_1}^2+p_{1,\mu_3}p_{3,\mu_2}p_{1,\mu_1}{p_1}^2+p_{3,\mu_3}p_{3,\mu_2}p_{1,\mu_1}{p_1}^2\\&-p_{3,\mu_3}p_{1,\mu_1}p_{1,\mu_2}u+2\,p_{3,\mu_3}p_{1,\mu_1}p_{1,\mu_2}{p_1}^2-2\,p_{1,\mu_3}p_{3,\mu_2}p_{1,\mu_1}u-p_{1,\mu_2}p_{3,\mu_1}p_{1,\mu_3}u\\&+2\,p_{1,\mu_2}p_{3,\mu_1}p_{3,\mu_3}{p_1}^2-2\,\eta_{\mu_3\mu_2}up_{3,\mu_1}{p_1}^2+p_{1,\mu_3}p_{3,\mu_2}p_{3,\mu_1}{p_1}^2-p_{1,\mu_2}\eta_{\mu_3\mu_1}{p_1}^2{p_3}^2\\&-p_{3,\mu_3}\eta_{\mu_1\mu_2}{p_1}^2{p_3}^2+2\,{p_3}^2\eta_{\mu_3\mu_2}p_{1,\mu_1}u-p_{3,\mu_2}\eta_{\mu_3\mu_1}{p_1}^2{p_3}^2-u^2\eta_{\mu_3\mu_1}p_{1,\mu_2}\\&+p_{1,\mu_2}p_{3,\mu_1}p_{1,\mu_3}{p_1}^2+2\,p_{1,\mu_1}\eta_{\mu_3\mu_2}u^2-2\,p_{3,\mu_1}\eta_{\mu_3\mu_2}{p_1}^2{p_3}^2\ ,\end{aligned}\tag{G.1.2}$$

$$\begin{aligned}&M_{2\,,\mu_1\mu_2\mu_3}=\\&2\,{p_1}^4\eta_{\mu_1\mu_2}p_{3,\mu_3}+2\,\eta_{\mu_3\mu_2}u^2p_{3,\mu_1}+4\,p_{1,\mu_2}p_{1,\mu_3}p_{1,\mu_1}u-2\,u^2\eta_{\mu_3\mu_1}p_{3,\mu_2}-2\,\eta_{\mu_3\mu_2}{p_3}^4p_{1,\mu_1}\\&+p_{3,\mu_3}\eta_{\mu_1\mu_2}u^2+4\,\eta_{\mu_1\mu_2}up_{3,\mu_3}{p_1}^2-2\,{p_1}^2\eta_{\mu_1\mu_2}p_{1,\mu_3}u-2\,p_{1,\mu_3}\eta_{\mu_1\mu_2}u^2-p_{1,\mu_2}p_{3,\mu_1}p_{3,\mu_3}u-4\,p_{3,\mu_2}p_{3,\mu_1}p_{3,\mu_3}u\\&+2\,\eta_{\mu_3\mu_2}p_{3,\mu_1}{p_3}^2u+p_{3,\mu_3}p_{3,\mu_2}p_{1,\mu_1}u-2\,u\eta_{\mu_3\mu_1}p_{3,\mu_2}{p_1}^2-4\,p_{3,\mu_2}p_{3,\mu_1}p_{3,\mu_3}{p_1}^2-p_{1,\mu_3}p_{3,\mu_2}p_{1,\mu_1}{p_1}^2\\&-p_{3,\mu_3}p_{3,\mu_2}p_{1,\mu_1}{p_1}^2-p_{3,\mu_3}p_{1,\mu_1}p_{1,\mu_2}u-2\,p_{3,\mu_3}p_{1,\mu_1}p_{1,\mu_2}{p_1}^2+p_{1,\mu_3}p_{3,\mu_2}p_{1,\mu_1}u-2\,p_{1,\mu_2}p_{3,\mu_1}p_{3,\mu_3}{p_1}^2\\&+2\,\eta_{\mu_3\mu_2}up_{3,\mu_1}{p_1}^2-p_{1,\mu_3}p_{3,\mu_2}p_{3,\mu_1}{p_1}^2+p_{3,\mu_3}\eta_{\mu_1\mu_2}{p_1}^2{p_3}^2-4\,{p_3}^2\eta_{\mu_3\mu_2}p_{1,\mu_1}u\\&+p_{3,\mu_3}p_{1,\mu_1}p_{1,\mu_2}{p_3}^2+4\,p_{1,\mu_1}p_{1,\mu_2}p_{1,\mu_3}{p_3}^2-p_{1,\mu_1}\eta_{\mu_3\mu_2}{p_1}^2{p_3}^2+p_{1,\mu_2}p_{3,\mu_1}p_{1,\mu_3}{p_3}^2\\&+p_{1,\mu_2}p_{3,\mu_1}p_{3,\mu_3}{p_3}^2-2\,u\eta_{\mu_1\mu_2}p_{1,\mu_3}{p_3}^2+p_{1,\mu_3}p_{3,\mu_2}p_{3,\mu_1}{p_3}^2+2\,p_{1,\mu_3}p_{3,\mu_2}p_{1,\mu_1}{p_3}^2\\&+2\,u\eta_{\mu_3\mu_1}p_{1,\mu_2}{p_3}^2-2\,p_{1,\mu_3}\eta_{\mu_1\mu_2}{p_1}^2{p_3}^2+2\,p_{3,\mu_3}p_{3,\mu_2}p_{1,\mu_1}{p_3}^2+2\,u^2\eta_{\mu_3\mu_1}p_{1,\mu_2}-p_{1,\mu_2}p_{3,\mu_1}p_{1,\mu_3}{p_1}^2\end{aligned}$$

$$
- p_{1,\mu_1}\eta_{\mu_3\mu_2}u^2 + 2\,p_{3,\mu_1}\eta_{\mu_3\mu_2}{p_1}^2{p_3}^2\;, \tag{G.1.3}
$$

$$
\begin{aligned}
&M_{3\,,\mu_1\mu_2\mu_3} = \\
&- 2\,\eta_{\mu_3\mu_2}u^2p_{3,\mu_1} + u^2\eta_{\mu_3\mu_1}p_{3,\mu_2} + 2\,\eta_{\mu_3\mu_2}{p_3}^4p_{1,\mu_1} - 2\,p_{3,\mu_3}\eta_{\mu_1\mu_2}u^2 - 2\,\eta_{\mu_1\mu_2}up_{3,\mu_3}{p_1}^2 + p_{1,\mu_3}p_{3,\mu_2}p_{3,\mu_1}u \\
&+ 2\,p_{1,\mu_2}p_{3,\mu_1}p_{3,\mu_3}u + 4\,p_{3,\mu_2}p_{3,\mu_1}p_{3,\mu_3}u - 2\,\eta_{\mu_3\mu_2}p_{3,\mu_1}{p_3}^2u + p_{3,\mu_3}p_{3,\mu_2}p_{1,\mu_1}u + 2\,p_{3,\mu_3}p_{1,\mu_1}p_{1,\mu_2}u \\
&+ p_{1,\mu_3}p_{3,\mu_2}p_{1,\mu_1}u + p_{1,\mu_2}p_{3,\mu_1}p_{1,\mu_3}u + p_{1,\mu_2}\eta_{\mu_3\mu_1}{p_1}^2{p_3}^2 + 2\,{p_3}^2\eta_{\mu_3\mu_2}p_{1,\mu_1}u + p_{3,\mu_2}\eta_{\mu_3\mu_1}{p_1}^2{p_3}^2 \\
&- p_{3,\mu_3}p_{1,\mu_1}p_{1,\mu_2}{p_3}^2 - 4\,p_{1,\mu_1}p_{1,\mu_2}p_{1,\mu_3}{p_3}^2 + p_{1,\mu_1}\eta_{\mu_3\mu_2}{p_1}^2{p_3}^2 - p_{1,\mu_2}p_{3,\mu_1}p_{1,\mu_3}{p_3}^2 \\
&- p_{1,\mu_2}p_{3,\mu_1}p_{3,\mu_3}{p_3}^2 + 2\,u\eta_{\mu_1\mu_2}p_{1,\mu_3}{p_3}^2 - p_{1,\mu_3}p_{3,\mu_2}p_{3,\mu_1}{p_3}^2 - 2\,p_{1,\mu_3}p_{3,\mu_2}p_{1,\mu_1}{p_3}^2 \\
&- 2\,u\eta_{\mu_3\mu_1}p_{1,\mu_2}{p_3}^2 + 2\,p_{1,\mu_3}\eta_{\mu_1\mu_2}{p_1}^2{p_3}^2 - 2\,p_{3,\mu_3}p_{3,\mu_2}p_{1,\mu_1}{p_3}^2 - u^2\eta_{\mu_3\mu_1}p_{1,\mu_2} - p_{1,\mu_1}\eta_{\mu_3\mu_2}u^2\;,
\end{aligned} \tag{G.1.4}
$$

$$
\begin{aligned}
&N_{\mu_1\mu_2\mu_3} = \\
&2\,u^3p_{3,\mu_2}\eta_{\mu_3\mu_1} + 4\,\eta_{\mu_1\mu_2}u^3p_{1,\mu_3} + 2\,p_{1,\mu_2}p_{3,\mu_1}p_{3,\mu_3}u^2 + 4\,{p_1}^2\eta_{\mu_1\mu_2}p_{1,\mu_3}u^2 \\
&- p_{1,\mu_2}{p_1}^2\eta_{\mu_3\mu_1}u^2 + p_{1,\mu_2}{p_1}^4\eta_{\mu_3\mu_1}{p_3}^2 + p_{1,\mu_1}{p_1}^4\eta_{\mu_3\mu_2}{p_3}^2 - 6\,\eta_{\mu_3\mu_2}{p_3}^2p_{1,\mu_1}u^2 \\
&+ 4\,\eta_{\mu_1\mu_2}u^2p_{1,\mu_3}{p_3}^2 + {p_3}^2p_{3,\mu_3}\eta_{\mu_1\mu_2}u^2 + {p_3}^2p_{3,\mu_2}\eta_{\mu_3\mu_1}u^2 + 6\,u^2\eta_{\mu_1\mu_2}p_{3,\mu_3}{p_1}^2 \\
&- {p_3}^4p_{3,\mu_3}\eta_{\mu_1\mu_2}{p_1}^2 - 3\,u^2\eta_{\mu_3\mu_1}p_{1,\mu_2}{p_3}^2 + 6\,\eta_{\mu_3\mu_2}{p_3}^4p_{3,\mu_1}{p_1}^2 - 4\,\eta_{\mu_3\mu_2}p_{3,\mu_1}{p_3}^2u^2 \\
&+ 6\,{p_1}^4p_{3,\mu_1}\eta_{\mu_3\mu_2}{p_3}^2 + 5\,{p_3}^4p_{1,\mu_2}\eta_{\mu_3\mu_1}{p_1}^2 - 4\,\eta_{\mu_3\mu_2}u^2p_{3,\mu_1}{p_1}^2 + 2\,{p_1}^4\eta_{\mu_1\mu_2}p_{3,\mu_3}u \\
&- 6\,{p_1}^4\eta_{\mu_1\mu_2}p_{1,\mu_3}{p_3}^2 - 2\,p_{1,\mu_3}p_{3,\mu_2}p_{1,\mu_1}u^2 - 2\,{p_1}^4\eta_{\mu_1\mu_2}p_{3,\mu_3}{p_3}^2 - 5\,{p_1}^4p_{3,\mu_2}\eta_{\mu_3\mu_1}{p_3}^2 \\
&- 6\,p_{1,\mu_3}{p_3}^4\eta_{\mu_1\mu_2}{p_1}^2 + 3\,u^2\eta_{\mu_3\mu_1}p_{3,\mu_2}{p_1}^2 + 2\,\eta_{\mu_3\mu_2}{p_3}^4p_{1,\mu_1}{p_1}^2 - p_{1,\mu_1}{p_1}^2\eta_{\mu_3\mu_2}u^2 \\
&- 2\,\eta_{\mu_3\mu_2}{p_3}^4p_{1,\mu_1}u - {p_3}^4p_{3,\mu_2}\eta_{\mu_3\mu_1}{p_1}^2 + 2\,p_{3,\mu_3}p_{3,\mu_2}p_{1,\mu_1}u^2 - 2\,p_{3,\mu_3}p_{1,\mu_1}p_{1,\mu_2}u^2 \\
&- 4\,u\eta_{\mu_3\mu_1}p_{3,\mu_2}{p_1}^2{p_3}^2 - 4\,p_{3,\mu_1}u^3\eta_{\mu_3\mu_2} - 4\,p_{3,\mu_2}p_{3,\mu_1}p_{3,\mu_3}{p_1}^2{p_3}^2 \\
&+ 2\,p_{3,\mu_3}p_{3,\mu_2}p_{1,\mu_1}u{p_3}^2 + 4\,u\eta_{\mu_3\mu_1}p_{1,\mu_2}{p_1}^2{p_3}^2 + 4\,p_{1,\mu_2}p_{1,\mu_3}p_{1,\mu_1}{p_1}^2{p_3}^2 - 2\,p_{1,\mu_2}p_{3,\mu_1}p_{3,\mu_3}u{p_1}^2 \\
&- 4\,p_{3,\mu_2}p_{3,\mu_1}p_{3,\mu_3}u{p_1}^2 + 8\,\eta_{\mu_3\mu_2}{p_3}^2p_{3,\mu_1}u{p_1}^2 - 2\,\eta_{\mu_3\mu_2}up_{1,\mu_1}{p_1}^2{p_3}^2 - p_{3,\mu_3}p_{3,\mu_2}p_{1,\mu_1}u{p_1}^2 \\
&+ p_{1,\mu_3}p_{3,\mu_2}p_{3,\mu_1}u{p_3}^2 - 8\,{p_1}^2\eta_{\mu_1\mu_2}p_{1,\mu_3}u{p_3}^2 + 2\,\eta_{\mu_1\mu_2}up_{3,\mu_3}{p_1}^2{p_3}^2 + 4\,p_{1,\mu_2}p_{1,\mu_3}p_{1,\mu_1}u{p_3}^2 \\
&- 2\,p_{3,\mu_3}p_{1,\mu_1}p_{1,\mu_2}u{p_1}^2 + p_{1,\mu_2}p_{3,\mu_1}p_{3,\mu_3}u{p_3}^2 - p_{3,\mu_3}p_{3,\mu_2}p_{1,\mu_1}{p_1}^2{p_3}^2 + 3\,p_{1,\mu_3}p_{3,\mu_2}p_{1,\mu_1}{p_1}^2{p_3}^2 \\
&+ 2\,p_{1,\mu_3}p_{3,\mu_2}p_{1,\mu_1}u{p_3}^2 - p_{1,\mu_3}p_{3,\mu_2}p_{3,\mu_1}u{p_1}^2 + p_{3,\mu_3}p_{1,\mu_1}p_{1,\mu_2}{p_1}^2{p_3}^2 \\
&- 3\,p_{3,\mu_1}p_{1,\mu_2}p_{3,\mu_3}{p_1}^2{p_3}^2 - p_{1,\mu_3}p_{3,\mu_2}p_{1,\mu_1}u{p_1}^2 + p_{1,\mu_2}p_{3,\mu_1}p_{1,\mu_3}u{p_3}^2 \\
&- p_{1,\mu_2}p_{3,\mu_1}p_{1,\mu_3}u{p_1}^2 + p_{3,\mu_3}p_{1,\mu_1}p_{1,\mu_2}u{p_3}^2 - 2\,u^3\eta_{\mu_3\mu_1}p_{1,\mu_2}\;.
\end{aligned} \tag{G.1.5}
$$

On shell, one has to impose the conditions

$$
p_1^2 = 0,\; p_3^2 = 0,\; p_2^2 = (p_1+p_3)^2 = 0 \Rightarrow\; u \equiv p_1\cdot p_3 = 0. \tag{G.1.6}
$$

Hence, in the on-shell limit all the terms of the amplitude vanish.

G.2 Four gluon planar amplitude at tree level

After substituting p_2 with $p_2 = -p_1 - p_3 - p_4$, introducing the variables $u \equiv p_1\cdot p_3$, $v \equiv p_1\cdot p_4$ and $w \equiv p_3\cdot p_4$, one obtains for the off-shell planar four gluon amplitude at tree level the result

$$
\begin{aligned}
&\mathcal{A}_{tree}^{off\,shell}\Big(A_{\mu_1}^{a_1}(p_1)A_{\mu_2}^{a_2}(p_2)A_{\mu_3}^{a_3}(p_3)A_{\mu_4}^{a_4}(p_4)\Big) = g^2\,\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\,\frac{i}{(p_1+p_4)^2(p_3+p_4)^2}\times \\
&\Big(- 1/2\,{p_4}^2p_{1,\mu_3}\eta_{\mu_2\mu_4}p_{1,\mu_1} - 1/2\,{p_1}^2\eta_{\mu_2\mu_4}p_{3,\mu_3}p_{3,\mu_1} \\
&+ 3/4\,{p_4}^2p_{4,\mu_2}\eta_{\mu_3\mu_4}p_{3,\mu_1} - p_{1,\mu_3}p_{4,\mu_1}\eta_{\mu_2\mu_4}{p_4}^2 + 2\,\eta_{\mu_2\mu_3}p_{4,\mu_1}p_{3,\mu_4}v \\
&+ 2\,\eta_{\mu_2\mu_3}p_{4,\mu_1}p_{3,\mu_4}w + 1/2\,p_{4,\mu_4}\eta_{\mu_1\mu_3}p_{1,\mu_2}v + 1/2\,\eta_{\mu_2\mu_3}p_{1,\mu_1}vp_{4,\mu_4} \\
&+ p_{1,\mu_4}\eta_{\mu_1\mu_2}{p_4}^2p_{1,\mu_3} + 1/4\,p_{4,\mu_2}\eta_{\mu_1\mu_4}{p_3}^2p_{3,\mu_3} + 1/2\,p_{4,\mu_4}\eta_{\mu_1\mu_2}{p_3}^2p_{1,\mu_3}
\end{aligned}
$$

$$
\begin{aligned}
&- p_{1,\mu_4}\eta_{\mu_1\mu_3}p_{1,\mu_2}w - 1/4\,p_{4,\mu_2}p_{4,\mu_4}\eta_{\mu_1\mu_3}{p_3}^2 - 1/2\,\eta_{\mu_2\mu_3}w\eta_{\mu_1\mu_4}{p_3}^2\\
&+ 3/4\,{p_4}^2\eta_{\mu_1\mu_4}p_{4,\mu_3}p_{4,\mu_2} + 1/4\,\eta_{\mu_2\mu_3}p_{1,\mu_1}{p_4}^2p_{4,\mu_4}\\
&- 1/2\,\eta_{\mu_2\mu_4}p_{1,\mu_1}wp_{3,\mu_3} + 3/4\,p_{1,\mu_3}p_{4,\mu_2}\eta_{\mu_1\mu_4}{p_4}^2 + 1/2\,p_{4,\mu_4}\eta_{\mu_2\mu_3}wp_{4,\mu_1}\\
&+ 1/2\,p_{4,\mu_2}\eta_{\mu_1\mu_4}wp_{4,\mu_3} - 2\,{p_4}^2p_{4,\mu_3}\eta_{\mu_2\mu_4}p_{4,\mu_1} + 1/2\,p_{4,\mu_4}\eta_{\mu_1\mu_2}wp_{3,\mu_3}\\
&- 1/2\,p_{1,\mu_4}\eta_{\mu_1\mu_3}p_{1,\mu_2}{p_3}^2 + 1/4\,p_{4,\mu_2}\eta_{\mu_1\mu_4}{p_1}^2p_{3,\mu_3} - 1/4\,{p_4}^2p_{3,\mu_1}\eta_{\mu_3\mu_4}p_{1,\mu_2}\\
&- 1/2\,p_{4,\mu_4}\eta_{\mu_1\mu_2}{p_1}^2p_{1,\mu_3} - 2\,\eta_{\mu_2\mu_4}p_{4,\mu_1}p_{4,\mu_3}v + 1/2\,\eta_{\mu_2\mu_3}\eta_{\mu_1\mu_4}{p_4}^2u - 1/2\,\eta_{\mu_1\mu_4}p_{1,\mu_2}wp_{3,\mu_3}\\
&- 1/2\,\eta_{\mu_1\mu_4}p_{1,\mu_2}vp_{3,\mu_3} - {p_4}^2\eta_{\mu_2\mu_3}p_{3,\mu_1}p_{1,\mu_4} + p_{4,\mu_1}p_{4,\mu_2}\eta_{\mu_3\mu_4}w\\
&- 1/4\,{p_1}^2\eta_{\mu_1\mu_4}p_{3,\mu_3}p_{1,\mu_2} + 1/2\,{p_1}^2\eta_{\mu_2\mu_3}p_{3,\mu_1}p_{4,\mu_4}\\
&- 1/4\,\eta_{\mu_1\mu_4}p_{1,\mu_2}{p_3}^2p_{3,\mu_3} - 1/4\,\eta_{\mu_1\mu_4}p_{1,\mu_2}{p_3}^2p_{1,\mu_3} + 1/2\,{p_1}^2\eta_{\mu_2\mu_3}p_{4,\mu_1}p_{4,\mu_4}\\
&+ 1/2\,p_{4,\mu_2}\eta_{\mu_3\mu_4}p_{1,\mu_1}v - \eta_{\mu_2\mu_4}p_{4,\mu_1}p_{4,\mu_3}{p_1}^2 - {p_4}^2p_{1,\mu_3}\eta_{\mu_1\mu_2}p_{3,\mu_4}\\
&- 1/4\,{p_1}^2p_{3,\mu_1}\eta_{\mu_3\mu_4}p_{3,\mu_2} - 1/4\,{p_1}^2p_{3,\mu_1}\eta_{\mu_3\mu_4}p_{1,\mu_2}\\
&- 1/2\,\eta_{\mu_1\mu_4}p_{1,\mu_2}wp_{1,\mu_3} - 1/4\,{p_1}^2p_{4,\mu_2}p_{4,\mu_4}\eta_{\mu_1\mu_3} + vp_{4,\mu_3}\eta_{\mu_1\mu_4}p_{4,\mu_2}\\
&+ {p_4}^2\eta_{\mu_2\mu_4}\eta_{\mu_1\mu_3}v - vp_{1,\mu_3}\eta_{\mu_1\mu_2}p_{4,\mu_4} - 1/2\,p_{4,\mu_4}\eta_{\mu_1\mu_3}p_{1,\mu_2}w\\
&- 1/4\,p_{4,\mu_4}\eta_{\mu_1\mu_3}p_{1,\mu_2}{p_3}^2 + 1/2\,\eta_{\mu_2\mu_3}p_{1,\mu_1}{p_3}^2p_{3,\mu_4}\\
&- 1/2\,{p_4}^2\eta_{\mu_1\mu_4}p_{1,\mu_2}p_{3,\mu_3} + 1/4\,{p_1}^2\eta_{\mu_1\mu_4}p_{3,\mu_3}p_{3,\mu_2} - 1/2\,\eta_{\mu_2\mu_4}p_{1,\mu_1}{p_3}^2p_{4,\mu_3}\\
&- 1/4\,{p_4}^2p_{3,\mu_1}\eta_{\mu_3\mu_4}p_{3,\mu_2} + 1/2\,p_{4,\mu_1}p_{4,\mu_2}\eta_{\mu_3\mu_4}{p_3}^2 + 1/2\,p_{4,\mu_1}p_{4,\mu_2}\eta_{\mu_3\mu_4}v\\
&+ 1/2\,\eta_{\mu_2\mu_3}p_{1,\mu_1}{p_1}^2p_{3,\mu_4} - 1/2\,{p_4}^2\eta_{\mu_3\mu_4}p_{3,\mu_2}p_{1,\mu_1}\\
&+ 1/2\,\eta_{\mu_3\mu_4}p_{1,\mu_1}p_{1,\mu_2}w - 1/2\,{p_4}^2p_{1,\mu_4}\eta_{\mu_1\mu_3}p_{1,\mu_2} + 1/4\,\eta_{\mu_3\mu_4}p_{1,\mu_1}p_{1,\mu_2}{p_3}^2\\
&+ 1/4\,{p_1}^2p_{4,\mu_1}\eta_{\mu_3\mu_4}p_{4,\mu_2} + 1/2\,p_{4,\mu_3}p_{3,\mu_2}\eta_{\mu_1\mu_4}{p_1}^2 + 1/2\,p_{4,\mu_2}\eta_{\mu_3\mu_4}p_{1,\mu_1}w\\
&+ 1/4\,p_{4,\mu_2}\eta_{\mu_3\mu_4}p_{1,\mu_1}{p_3}^2 + 1/2\,p_{1,\mu_4}p_{3,\mu_2}\eta_{\mu_1\mu_3}{p_3}^2 - 1/4\,\eta_{\mu_2\mu_4}p_{1,\mu_1}{p_3}^2p_{3,\mu_3}\\
&- 1/2\,\eta_{\mu_2\mu_4}p_{1,\mu_1}vp_{3,\mu_3} + 1/4\,\eta_{\mu_3\mu_4}p_{1,\mu_1}p_{1,\mu_2}{p_4}^2 - 1/2\,vp_{3,\mu_1}\eta_{\mu_3\mu_4}p_{1,\mu_2}\\
&- 1/2\,v\eta_{\mu_1\mu_3}p_{3,\mu_2}p_{4,\mu_4} - 1/2\,{p_4}^2\eta_{\mu_3\mu_4}\eta_{\mu_1\mu_2}{p_1}^2 - 1/2\,vp_{3,\mu_1}\eta_{\mu_3\mu_4}p_{3,\mu_2}\\
&+ 1/4\,p_{4,\mu_4}\eta_{\mu_2\mu_3}{p_3}^2p_{4,\mu_1} - 1/4\,\eta_{\mu_2\mu_4}p_{1,\mu_1}{p_1}^2p_{3,\mu_3} + p_{4,\mu_4}\eta_{\mu_1\mu_2}p_{4,\mu_3}w\\
&+ v\eta_{\mu_3\mu_4}\eta_{\mu_1\mu_2}u - 1/2\,{p_4}^2\eta_{\mu_3\mu_4}\eta_{\mu_1\mu_2}w - 2\,v\eta_{\mu_2\mu_4}p_{4,\mu_3}p_{3,\mu_1}\\
&- v\eta_{\mu_3\mu_4}\eta_{\mu_1\mu_2}w + 1/2\,{p_4}^2\eta_{\mu_3\mu_4}\eta_{\mu_1\mu_2}u - 1/2\,p_{4,\mu_2}p_{4,\mu_4}\eta_{\mu_1\mu_3}{p_4}^2\\
&- 1/2\,{p_4}^2p_{3,\mu_4}\eta_{\mu_1\mu_3}p_{4,\mu_2} + 1/2\,p_{4,\mu_3}p_{3,\mu_2}\eta_{\mu_1\mu_4}w + 1/4\,p_{4,\mu_3}p_{3,\mu_2}\eta_{\mu_1\mu_4}{p_3}^2\\
&- 2\,\eta_{\mu_2\mu_3}p_{1,\mu_4}wp_{3,\mu_1} - 1/2\,{p_1}^2\eta_{\mu_3\mu_4}\eta_{\mu_1\mu_2}w - \eta_{\mu_2\mu_3}p_{1,\mu_4}{p_3}^2p_{3,\mu_1}\\
&- 1/4\,\eta_{\mu_3\mu_4}p_{1,\mu_1}p_{3,\mu_2}{p_3}^2 - 1/4\,\eta_{\mu_3\mu_4}p_{1,\mu_1}p_{3,\mu_2}{p_1}^2\\
&+ 1/2\,p_{4,\mu_4}\eta_{\mu_1\mu_2}p_{4,\mu_3}{p_3}^2 + 2\,p_{1,\mu_4}\eta_{\mu_1\mu_2}wp_{1,\mu_3} - 1/4\,{p_4}^2p_{1,\mu_3}\eta_{\mu_1\mu_4}p_{3,\mu_2}\\
&- p_{4,\mu_1}p_{3,\mu_2}\eta_{\mu_3\mu_4}w - 1/2\,p_{4,\mu_1}p_{3,\mu_2}\eta_{\mu_3\mu_4}{p_3}^2 - 1/4\,{p_4}^2p_{1,\mu_3}\eta_{\mu_1\mu_4}p_{1,\mu_2}\\
&- 1/2\,\eta_{\mu_2\mu_4}p_{1,\mu_1}p_{1,\mu_3}{p_3}^2 + 1/2\,p_{1,\mu_4}p_{3,\mu_2}\eta_{\mu_1\mu_3}{p_4}^2\\
&+ 1/2\,{p_4}^2p_{4,\mu_2}\eta_{\mu_3\mu_4}p_{1,\mu_1} - 1/2\,{p_1}^2\eta_{\mu_3\mu_4}\eta_{\mu_1\mu_2}v + 1/2\,{p_1}^2\eta_{\mu_3\mu_4}\eta_{\mu_1\mu_2}u\\
&- 1/2\,p_{3,\mu_2}p_{1,\mu_3}\eta_{\mu_1\mu_4}w - 1/4\,p_{3,\mu_2}p_{1,\mu_3}\eta_{\mu_1\mu_4}{p_3}^2 - 1/4\,{p_1}^2p_{3,\mu_4}\eta_{\mu_1\mu_2}p_{3,\mu_3}\\
&+ 1/2\,{p_4}^2\eta_{\mu_2\mu_4}\eta_{\mu_1\mu_3}{p_3}^2 - v\eta_{\mu_2\mu_4}p_{4,\mu_3}p_{1,\mu_1} - v\eta_{\mu_2\mu_4}p_{3,\mu_3}p_{3,\mu_1}\\
&+ 1/2\,p_{1,\mu_4}\eta_{\mu_1\mu_2}{p_1}^2p_{3,\mu_3} - p_{4,\mu_1}\eta_{\mu_2\mu_4}wp_{3,\mu_3} + 1/2\,{p_4}^2\eta_{\mu_1\mu_3}p_{1,\mu_2}p_{3,\mu_4}\\
&- p_{4,\mu_1}\eta_{\mu_2\mu_4}vp_{3,\mu_3} - 1/4\,{p_1}^2\eta_{\mu_1\mu_3}p_{3,\mu_2}p_{4,\mu_4} + 1/2\,vp_{4,\mu_3}\eta_{\mu_1\mu_2}p_{4,\mu_4}\\
&- 1/2\,vp_{3,\mu_4}\eta_{\mu_1\mu_2}p_{3,\mu_3} + 1/4\,{p_1}^2p_{4,\mu_1}\eta_{\mu_3\mu_4}p_{1,\mu_2} - 1/2\,p_{4,\mu_1}\eta_{\mu_2\mu_4}{p_3}^2p_{3,\mu_3}\\
&+ 1/4\,p_{4,\mu_2}\eta_{\mu_3\mu_4}p_{1,\mu_1}{p_1}^2 + 1/2\,p_{1,\mu_4}\eta_{\mu_1\mu_2}{p_3}^2p_{3,\mu_3} + 2\,v\eta_{\mu_2\mu_4}\eta_{\mu_1\mu_3}w\\
&- 1/2\,\eta_{\mu_3\mu_4}p_{1,\mu_1}p_{3,\mu_2}w + 1/2\,{p_1}^2\eta_{\mu_2\mu_4}\eta_{\mu_1\mu_3}{p_3}^2 + 2\,p_{1,\mu_4}\eta_{\mu_1\mu_2}wp_{4,\mu_3}\\
&+ 1/4\,{p_1}^2p_{4,\mu_4}\eta_{\mu_1\mu_2}p_{4,\mu_3} + 3/2\,vp_{3,\mu_1}\eta_{\mu_3\mu_4}p_{4,\mu_2} - 3/4\,{p_1}^2p_{4,\mu_1}\eta_{\mu_3\mu_4}p_{3,\mu_2}\\
&- 3/2\,v\eta_{\mu_3\mu_4}\eta_{\mu_1\mu_2}{p_4}^2 + 2\,vp_{4,\mu_3}\eta_{\mu_1\mu_2}p_{1,\mu_4} + 1/2\,{p_4}^2\eta_{\mu_1\mu_4}p_{3,\mu_3}p_{4,\mu_2}
\end{aligned}
$$

$$
\begin{aligned}
&+ 1/2\, {p_1}^2 {p_4}^2 \eta_{\mu_2\mu_4} \eta_{\mu_1\mu_3} + 1/2\, v \eta_{\mu_1\mu_4} p_{3,\mu_3} p_{3,\mu_2} \\
&- 1/2\, {p_4}^2 \eta_{\mu_2\mu_4} p_{3,\mu_3} p_{3,\mu_1} - 1/2\, {p_4}^2 \eta_{\mu_2\mu_4} p_{3,\mu_3} p_{1,\mu_1} - 1/2\, \eta_{\mu_3\mu_4} p_{1,\mu_1} p_{3,\mu_2} v \\
&+ 3/4\, p_{1,\mu_3} p_{4,\mu_2} \eta_{\mu_1\mu_4} {p_3}^2 - v \eta_{\mu_1\mu_4} \eta_{\mu_2\mu_3} w - 1/2\, {p_1}^2 p_{3,\mu_4} \eta_{\mu_1\mu_3} p_{3,\mu_2} \\
&+ \eta_{\mu_2\mu_3} p_{1,\mu_1} w p_{3,\mu_4} - {p_1}^2 p_{1,\mu_3} \eta_{\mu_1\mu_2} p_{3,\mu_4} - 1/2\, {p_1}^2 \eta_{\mu_2\mu_4} p_{4,\mu_3} p_{1,\mu_1} \\
&- 1/2\, v \eta_{\mu_1\mu_4} \eta_{\mu_2\mu_3} {p_3}^2 - 3/2\, p_{4,\mu_3} \eta_{\mu_1\mu_4} p_{1,\mu_2} w + 1/2\, {p_1}^2 \eta_{\mu_1\mu_3} p_{1,\mu_2} p_{3,\mu_4} \\
&+ 3/4\, {p_4}^2 \eta_{\mu_2\mu_3} p_{4,\mu_1} p_{4,\mu_4} - 1/2\, p_{4,\mu_3} \eta_{\mu_1\mu_4} p_{1,\mu_2} {p_1}^2 \\
&+ 3/4\, {p_4}^2 p_{4,\mu_1} \eta_{\mu_3\mu_4} p_{1,\mu_2} - 1/2\, {p_1}^2 p_{3,\mu_4} \eta_{\mu_1\mu_3} p_{4,\mu_2} - 1/4\, {p_4}^2 p_{3,\mu_4} \eta_{\mu_1\mu_2} p_{3,\mu_3} \\
&+ v p_{3,\mu_4} \eta_{\mu_2\mu_3} p_{1,\mu_1} + 1/2\, p_{4,\mu_4} \eta_{\mu_1\mu_3} p_{3,\mu_2} w + 1/4\, p_{4,\mu_4} \eta_{\mu_1\mu_3} p_{3,\mu_2} {p_3}^2 \\
&- \eta_{\mu_2\mu_4} p_{1,\mu_1} w p_{4,\mu_3} + 1/4\, \eta_{\mu_2\mu_3} p_{1,\mu_1} {p_1}^2 p_{4,\mu_4} - 1/4\, \eta_{\mu_2\mu_3} p_{1,\mu_1} {p_4}^2 p_{1,\mu_4} \\
&+ 3/2\, p_{4,\mu_2} \eta_{\mu_1\mu_4} w p_{1,\mu_3} - 1/2\, p_{4,\mu_2} p_{4,\mu_4} \eta_{\mu_1\mu_3} v - 5/4\, {p_4}^2 p_{4,\mu_3} \eta_{\mu_1\mu_4} p_{1,\mu_2} \\
&- 3/2\, \eta_{\mu_2\mu_3} \eta_{\mu_1\mu_4} w {p_4}^2 - p_{4,\mu_2} \eta_{\mu_1\mu_3} p_{1,\mu_4} w + 1/4\, p_{4,\mu_2} \eta_{\mu_1\mu_4} {p_3}^2 p_{4,\mu_3} \\
&+ 1/2\, p_{4,\mu_2} \eta_{\mu_1\mu_4} w p_{3,\mu_3} + 3/4\, p_{4,\mu_1} p_{4,\mu_2} \eta_{\mu_3\mu_4} {p_4}^2 + v \eta_{\mu_2\mu_3} p_{4,\mu_1} p_{4,\mu_4} \\
&+ 3/4\, {p_4}^2 \eta_{\mu_1\mu_4} p_{4,\mu_3} p_{3,\mu_2} - p_{1,\mu_3} p_{4,\mu_1} \eta_{\mu_2\mu_4} {p_3}^2 \\
&- {p_4}^2 p_{4,\mu_3} \eta_{\mu_2\mu_4} p_{1,\mu_1} - 1/2\, p_{4,\mu_2} p_{4,\mu_4} \eta_{\mu_1\mu_3} w + 2\, {p_4}^2 p_{3,\mu_4} \eta_{\mu_2\mu_3} p_{4,\mu_1} \\
&- 1/4\, \eta_{\mu_2\mu_3} p_{1,\mu_1} {p_3}^2 p_{1,\mu_4} + {p_4}^2 \eta_{\mu_2\mu_3} p_{1,\mu_1} p_{3,\mu_4} + v \eta_{\mu_2\mu_3} p_{3,\mu_1} p_{4,\mu_4} \\
&- 1/2\, p_{4,\mu_2} \eta_{\mu_1\mu_3} p_{1,\mu_4} {p_4}^2 - \eta_{\mu_2\mu_3} w^2 \eta_{\mu_1\mu_4} + \eta_{\mu_2\mu_3} p_{4,\mu_1} p_{3,\mu_4} {p_1}^2 \\
&+ \eta_{\mu_2\mu_3} p_{4,\mu_1} p_{3,\mu_4} {p_3}^2 + p_{1,\mu_4} \eta_{\mu_1\mu_2} {p_4}^2 p_{3,\mu_3} + p_{1,\mu_4} \eta_{\mu_1\mu_2} {p_3}^2 p_{4,\mu_3} \\
&- 2\, v p_{1,\mu_3} \eta_{\mu_1\mu_2} p_{3,\mu_4} + p_{1,\mu_4} \eta_{\mu_1\mu_2} {p_1}^2 p_{4,\mu_3} + 2\, v p_{3,\mu_4} \eta_{\mu_2\mu_3} p_{3,\mu_1} \\
&- 1/2\, p_{4,\mu_2} \eta_{\mu_1\mu_3} p_{1,\mu_4} {p_3}^2 + 1/2\, p_{4,\mu_2} \eta_{\mu_1\mu_4} v p_{3,\mu_3} - 2\, p_{1,\mu_3} p_{4,\mu_1} \eta_{\mu_2\mu_4} w \\
&- 3/2\, v \eta_{\mu_3\mu_4} p_{3,\mu_2} p_{4,\mu_1} + v \eta_{\mu_1\mu_3} p_{1,\mu_2} p_{3,\mu_4} + p_{4,\mu_4} \eta_{\mu_1\mu_2} w p_{1,\mu_3} \\
&+ {p_4}^2 \eta_{\mu_2\mu_4} \eta_{\mu_1\mu_3} w + p_{1,\mu_4} \eta_{\mu_1\mu_2} w p_{3,\mu_3} + 2\, {p_4}^2 p_{4,\mu_3} \eta_{\mu_1\mu_2} p_{1,\mu_4} \\
&+ {p_1}^2 \eta_{\mu_2\mu_4} \eta_{\mu_1\mu_3} w + p_{1,\mu_4} \eta_{\mu_1\mu_2} v p_{3,\mu_3} - p_{4,\mu_1} \eta_{\mu_2\mu_4} {p_4}^2 p_{3,\mu_3} \\
&+ v \eta_{\mu_2\mu_4} \eta_{\mu_1\mu_3} {p_3}^2 - 1/2\, {p_4}^2 \eta_{\mu_1\mu_4} \eta_{\mu_2\mu_3} {p_3}^2 - 1/2\, v \eta_{\mu_1\mu_4} \eta_{\mu_2\mu_3} {p_4}^2 \\
&- 1/2\, \eta_{\mu_2\mu_3} p_{1,\mu_1} w p_{1,\mu_4} - v p_{3,\mu_4} \eta_{\mu_1\mu_3} p_{3,\mu_2} + 1/2\, {p_4}^4 \eta_{\mu_2\mu_4} \eta_{\mu_1\mu_3} \\
&- v p_{3,\mu_4} \eta_{\mu_1\mu_3} p_{4,\mu_2} + 1/4\, p_{4,\mu_4} \eta_{\mu_1\mu_2} {p_3}^2 p_{3,\mu_3} - p_{4,\mu_1} \eta_{\mu_2\mu_4} {p_3}^2 p_{4,\mu_3} - p_{4,\mu_4} \eta_{\mu_2\mu_3} w p_{3,\mu_1} \\
&+ 1/2\, p_{4,\mu_1} \eta_{\mu_3\mu_4} p_{1,\mu_2} {p_3}^2 - 5/4\, {p_4}^2 \eta_{\mu_3\mu_4} p_{3,\mu_2} p_{4,\mu_1} \\
&- {p_4}^2 p_{4,\mu_3} \eta_{\mu_2\mu_4} p_{3,\mu_1} - \eta_{\mu_2\mu_4} p_{1,\mu_1} p_{1,\mu_3} w + p_{1,\mu_4} \eta_{\mu_1\mu_2} {p_3}^2 p_{1,\mu_3} \\
&- v p_{4,\mu_3} \eta_{\mu_1\mu_4} p_{1,\mu_2} + 1/2\, p_{4,\mu_1} \eta_{\mu_3\mu_4} p_{1,\mu_2} v + 1/2\, p_{4,\mu_2} \eta_{\mu_1\mu_4} {p_1}^2 p_{4,\mu_3} \\
&- {p_1}^2 \eta_{\mu_2\mu_4} p_{4,\mu_3} p_{3,\mu_1} - 1/2\, {p_1}^2 \eta_{\mu_2\mu_4} p_{3,\mu_3} p_{4,\mu_1} + 1/4\, p_{4,\mu_4} \eta_{\mu_1\mu_2} {p_4}^2 p_{3,\mu_3} \\
&+ 3/4\, {p_1}^2 p_{4,\mu_2} \eta_{\mu_3\mu_4} p_{3,\mu_1} + 1/4\, {p_4}^2 \eta_{\mu_1\mu_4} p_{3,\mu_3} p_{3,\mu_2} - 1/2\, {p_4}^4 \eta_{\mu_3\mu_4} \eta_{\mu_1\mu_2} \\
&- 3/4\, \eta_{\mu_1\mu_4} p_{1,\mu_2} {p_3}^2 p_{4,\mu_3} + 1/2\, u \eta_{\mu_2\mu_3} \eta_{\mu_1\mu_4} {p_3}^2 + 1/4\, {p_1}^2 p_{4,\mu_4} \eta_{\mu_1\mu_3} p_{1,\mu_2} \\
&- 1/2\, {p_4}^4 \eta_{\mu_1\mu_4} \eta_{\mu_2\mu_3} + p_{1,\mu_4} p_{3,\mu_2} \eta_{\mu_1\mu_3} w + v p_{4,\mu_3} \eta_{\mu_1\mu_4} p_{3,\mu_2} \\
&+ u \eta_{\mu_2\mu_3} \eta_{\mu_1\mu_4} w + {p_1}^2 p_{3,\mu_4} \eta_{\mu_2\mu_3} p_{3,\mu_1} - 2\, p_{4,\mu_1} \eta_{\mu_2\mu_4} w p_{4,\mu_3} + 3/4\, p_{4,\mu_4} \eta_{\mu_1\mu_2} p_{4,\mu_3} {p_4}^2 \\
&- v^2 \eta_{\mu_3\mu_4} \eta_{\mu_1\mu_2} - 1/2\, {p_4}^2 p_{3,\mu_4} \eta_{\mu_1\mu_3} p_{3,\mu_2} + p_{4,\mu_1} \eta_{\mu_3\mu_4} p_{1,\mu_2} w \\
&+ {p_4}^2 p_{3,\mu_4} \eta_{\mu_2\mu_3} p_{3,\mu_1} - 1/2\, p_{4,\mu_4} \eta_{\mu_2\mu_3} {p_3}^2 p_{3,\mu_1} \Big)
\end{aligned}
\tag{G.2.1}
$$

G.3 Four gluon planar amplitude at one loop

Summing all the superdiagrams and making the Passarino-Veltman procedure, one obtains for the off shell planar four gluon amplitude at one loop the decomposition

$$\mathcal{A}_{1\,loop}^{off\,shell}\Big(A_{\mu_1}^{a_1}(p_1) A_{\mu_2}^{a_2}(p_2) A_{\mu_3}^{a_3}(p_3) A_{\mu_4}^{a_4}(p_4)\Big) = N g^4 \mathrm{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) \times$$

$$\Big(Q_{\mu_1\mu_2\mu_3\mu_4}\, D_0(p_1, p_4, p_3)$$
$$+ N_{1,\mu_1\mu_2\mu_3\mu_4}\, C_0(p_1, p_3 + p_4) + N_{2,\mu_1\mu_2\mu_3\mu_4}\, C_0(p_1, p_4) + N_{3,\mu_1\mu_2\mu_3\mu_4}\, C_0(p_1 + p_4, p_3) + N_{4,\mu_1\mu_2\mu_3\mu_4}\, C_0(p_4, p_3)$$
$$+ M_{1,\mu_1\mu_2\mu_3\mu_4}\, B_0(p_1) + M_{2,\mu_1\mu_2\mu_3\mu_4}\, B_0(p_4) + M_{3,\mu_1\mu_2\mu_3\mu_4}\, B_0(p_1 + p_3 + p_4) + M_{4,\mu_1\mu_2\mu_3\mu_4}\, B_0(p_3)$$
$$+ M_{5,\mu_1\mu_2\mu_3\mu_4}\, B_0(p_1 + p_4) + M_{6,\mu_1\mu_2\mu_3\mu_4}\, B_0(p_3 + p_4)\Big) \tag{G.3.1}$$

Let's define as

$$Q_V, N_{V1}, \ldots N_{V4}, M_1 \ldots M_6 \tag{G.3.2}$$

those parts of the coefficients

$$Q_{\mu_1\mu_2\mu_3\mu_4}, N_{1,\mu_1\mu_2\mu_3\mu_4}, \ldots N_{4,\mu_1\mu_2\mu_3\mu_4}, M_{1,\mu_1\mu_2\mu_3\mu_4} \ldots M_{6,\mu_1\mu_2\mu_3\mu_4} \tag{G.3.3}$$

which multiply the Lorentz structure $\eta_{\mu_2\mu_3} p_{3\,\mu_1} p_{3\,\mu_4}$.
Defining as before, $u = p_1 \cdot p_3$, $v = p_1 \cdot p_4$ and $w = p_3 \cdot p_4$, we can write

$$Q_V = \frac{Num_{Q_V}}{Den_{Q_V}},$$
$$Den_{Q_V} = 64\,\left(v^2 {p_3}^2 - {p_4}^2 {p_1}^2 {p_3}^2 + {p_4}^2 u^2 + w^2 {p_1}^2 - 2\,wvu\right)^2$$
$$Num_{Q_V} = -88\,v^5 {p_3}^4 - 32\,v^4 w^3 - 24\,{p_3}^6 v^4 - 32\,v^5 w^2 - {p_4}^8 {p_1}^4 {p_3}^2 + {p_4}^8 {p_1}^2 u^2 + {p_1}^8 {p_4}^2 w^2 - 2\,{p_1}^6 w^2 v^2$$
$$- {p_1}^6 u^2 {p_4}^4 - 24\,{p_4}^4 {p_3}^6 {p_1}^4 - 2\,{p_4}^6 v^2 u^2 - 34\,{p_4}^6 {p_3}^4 {p_1}^4 - {p_4}^6 {p_1}^4 w^2 - 16\,v^3 {p_1}^4 w^2$$
$$- 88\,{p_3}^4 v^4 w - 48\,{p_1}^4 v^2 w^3 - 6\,{p_4}^6 {p_1}^2 wvu - 20\,v{p_1}^4 wu{p_4}^4 + 40\,v^3 {p_1}^2 wu{p_4}^2 + 16\,{p_4}^6 {p_1}^2 u^3$$
$$- 16\,v^2 {p_1}^2 wu{p_4}^4 + 2\,{p_1}^6 wvu{p_4}^2 + 16\,v{p_1}^6 w{p_4}^2 {p_3}^2 - 16\,vu{p_4}^6 {p_1}^2 {p_3}^2 + 8\,v^4 {p_1}^2 {p_3}^2 {p_4}^2$$
$$+ 2\,v{p_1}^2 {p_4}^6 u^2 - 10\,v^2 {p_1}^2 u^2 {p_4}^4 + 16\,v^3 u{p_3}^2 {p_4}^4 + 2\,v{p_1}^4 w^2 {p_4}^4 + 32\,v^4 u{p_4}^2 w$$
$$- 6\,v{p_1}^6 {p_4}^4 {p_3}^2 + 8\,{p_4}^4 v^3 uw + 6\,v^3 {p_1}^4 {p_4}^2 {p_3}^2 - 32\,v^4 w{p_3}^2 {p_4}^2 + 8\,{p_4}^4 {p_1}^2 w^2 v^2 + {p_4}^6 v^2 {p_1}^2 {p_3}^2$$
$$- 4\,{p_4}^6 {p_1}^4 wu + 30\,v^2 {p_1}^4 w^2 {p_4}^2 + {p_1}^6 v^2 {p_4}^2 {p_3}^2 - 6\,v{p_1}^4 u^2 {p_4}^4 - 6\,{p_4}^6 {p_1}^4 v{p_3}^2 - 36\,{p_4}^4 {p_1}^2 uvw^2 - 32\,v^3 u^2 w{p_4}^2$$
$$+ 8\,{p_4}^4 u^2 {p_1}^4 {p_3}^2 - 48\,{p_1}^4 {p_4}^2 uvw{p_3}^2 + 48\,{p_1}^2 v^3 uw^2 - 64\,{p_1}^2 v^2 u^2 w{p_4}^2 - 16\,{p_1}^4 wv^3 {p_3}^2 + 32\,{p_3}^4 v^3 u{p_4}^2$$
$$+ 8\,{p_4}^4 {p_1}^4 w^3 - 64\,v^5 w{p_3}^2 + 16\,{p_1}^4 v^2 uw^2 - 96\,v^4 w^2 {p_3}^2 + 32\,v^4 uw^2 - 64\,{p_1}^2 v^3 w^3$$
$$- 24\,uv^4 {p_3}^4 - 2\,{p_1}^6 {p_4}^6 {p_3}^2 - 8\,v^3 u^2 {p_4}^4 - 40\,v^4 {p_1}^2 w^2 - 8\,{p_4}^2 v^4 w^2 - 30\,{p_1}^2 v^4 {p_3}^4 + 8\,v^2 u^3 {p_4}^4$$
$$- 34\,{p_4}^4 {p_3}^4 {p_1}^6 + 16\,{p_1}^6 {p_4}^2 w^3 + 64\,{p_4}^4 {p_1}^2 v^2 {p_3}^4 + 48\,{p_4}^2 {p_3}^2 {p_1}^6 w^2 + 36\,{p_1}^4 {p_4}^2 uvw^2$$
$$- 8\,{p_1}^4 {p_4}^4 u^2 w + 20\,{p_1}^2 vu^3 {p_4}^4 + 28\,{p_1}^4 {p_4}^2 vw^3 - 52\,{p_4}^4 {p_1}^2 vwu^2 + 80\,{p_4}^2 {p_1}^2 {p_3}^4 v^2 u + 32\,v^3 {p_3}^4 wu$$
$$+ 8\,{p_4}^4 {p_1}^4 {p_3}^2 w^2 - 20\,{p_1}^6 u{p_4}^4 {p_3}^2 + 6\,{p_4}^4 v^3 {p_1}^2 {p_3}^2 - 144\,{p_4}^4 {p_3}^2 {p_1}^2 vwu$$
$$+ 24\,v^3 {p_1}^2 w^2 {p_4}^2 + 10\,v{p_1}^6 {p_4}^2 w^2 - 6\,v^2 {p_1}^4 {p_4}^4 {p_3}^2 + 48\,{p_4}^2 {p_3}^6 {p_1}^2 v^2 - 4\,{p_1}^6 wu{p_4}^4$$
$$+ 88\,v^3 {p_1}^2 {p_4}^2 w{p_3}^2 - 16\,v^2 {p_3}^4 w^2 {p_1}^2 + 16\,v^2 {p_1}^4 wu{p_4}^2 - 16\,v^2 {p_3}^4 {p_4}^2 u^2 + 16\,{p_1}^6 w^2 {p_4}^2 u$$
$$- 32\,{p_4}^4 {p_3}^4 {p_1}^2 vu + 48\,{p_4}^6 u^2 {p_1}^2 {p_3}^2 - 56\,u{p_3}^4 {p_1}^4 {p_4}^4 + 84\,{p_4}^2 {p_3}^2 {p_1}^2 v^2 w^2$$
$$+ 144\,{p_4}^2 {p_3}^4 {p_1}^2 wv^2 - 96\,{p_4}^4 {p_3}^4 {p_1}^4 v - 208\,v^3 {p_3}^2 {p_1}^2 w^2 + 64\,{p_4}^2 {p_1}^4 {p_3}^4 v^2 + 16\,{p_4}^6 {p_1}^2 u^2 w$$
$$+ 52\,{p_4}^4 {p_1}^2 wv^2 {p_3}^2 - 12\,{p_4}^4 {p_1}^2 uv^2 {p_3}^2 - 20\,{p_1}^6 {p_4}^4 w{p_3}^2 - 128\,{p_4}^2 {p_1}^2 {p_3}^2 v^2 wu$$
$$- 32\,v^3 {p_3}^4 {p_1}^2 w - 12\,v^2 u^2 {p_1}^2 {p_3}^2 {p_4}^2 - 20\,{p_4}^6 {p_1}^4 w{p_3}^2 + 56\,{p_1}^2 v^3 u{p_3}^2 {p_4}^2 + 184\,{p_1}^2 v^3 {p_3}^4 {p_4}^2$$
$$- 64\,{p_1}^2 v^4 w{p_3}^2 - 8\,{p_1}^4 {p_4}^4 uw^2 + 200\,{p_4}^2 {p_3}^2 {p_1}^4 vw^2 - 52\,{p_1}^4 {p_3}^2 v^2 w^2 + 32\,v^4 u{p_3}^2 {p_4}^2 - 56\,{p_4}^4 v^2 u^2 w$$
$$- 52\,{p_4}^4 v^2 u^2 {p_3}^2 + 104\,{p_4}^4 u^2 {p_1}^2 v{p_3}^2 + 72\,{p_4}^2 v^2 u{p_1}^2 w^2 - {p_1}^8 {p_4}^4 {p_3}^2 + 56\,v^3 {p_1}^2 {p_3}^2 wu$$
$$- 56\,w{p_3}^4 {p_1}^4 {p_4}^4 + 80\,{p_4}^2 v^3 uw^2 + 20\,{p_1}^4 {p_4}^2 uv^2 {p_3}^2 - 24\,v{p_1}^4 {p_4}^4 w{p_3}^2 - 128\,w^2 v^2 {p_4}^2 u^2 - 28\,{p_1}^4 {p_4}^2 vwu^2$$
$$+ 152\,{p_4}^2 v^3 uw{p_3}^2 - 16\,{p_4}^6 u^4 - 16\,{p_1}^6 w^4 - 80\,{p_1}^4 {p_4}^4 wu{p_3}^2 + 32\,{p_1}^4 {p_4}^2 vw{p_3}^4 + 8\,{p_1}^4 {p_4}^4 u^3$$
$$- 112\,v^3 {p_4}^2 u^2 {p_3}^2 + 224\,v^4 wu{p_3}^2 - 20\,{p_1}^4 u{p_4}^6 {p_3}^2 + 8\,{p_4}^2 v^2 {p_1}^2 w^3 + 84\,{p_1}^4 {p_4}^2 wv^2 {p_3}^2 - 56\,{p_1}^4 {p_4}^4 uv{p_3}^2$$
$$- 30\,v^4 {p_3}^4 {p_4}^2 - 32\,w^4 v^2 {p_1}^2 + 64\,w^3 v^3 u - 48\,v{p_4}^4 u^4 - 80\,vw^4 {p_1}^4 - 192\,w^2 v^3 u^2 + 192\,{p_4}^2 u^3 wv^2$$
$$+ 16\,{p_1}^2 {p_4}^2 u^3 wv - 12\,{p_1}^6 w^3 v + 12\,{p_4}^6 vu^3 + 48\,{p_4}^2 w^3 {p_1}^2 vu - 16\,{p_1}^4 {p_4}^2 u^2 w^2 + 48\,{p_1}^4 w^3 vu - 16\,v^2 u^3 {p_3}^2 {p_4}^2$$
$$+ 32\,v^3 u^2 {p_3}^2 w + 16\,{p_3}^4 {p_1}^2 {p_4}^4 u^2 + 16\,{p_3}^4 {p_1}^4 {p_4}^2 w^2 + 16\,{p_4}^4 vu^3 {p_3}^2 - 192\,v{p_4}^2 u^2 w^2 {p_1}^2 + 256\,w^3 {p_1}^2 v^2 u$$

$$
\begin{aligned}
&- 16\,{p_4}^4u^2w^2{p_1}^2 + 80\,{p_4}^4u^3wv - 32\,{p_1}^2w^2v^2u^2 + 32\,w{p_1}^2{p_4}^4{p_3}^2u^2 - 48\,w^3v^2{p_3}^2{p_1}^2 \\
&+ 96\,w^2v^3{p_3}^2u + 32\,w^3{p_1}^4{p_4}^2{p_3}^2 + 32\,u{p_1}^4{p_4}^2{p_3}^2w^2 - 80\,u^2{p_1}^2{p_4}^2{p_3}^2wv + 32\,{p_4}^4w{p_1}^2u^3 \\
&+ 32\,{p_4}^2w^3{p_1}^4u + 32\,u^3{p_1}^2{p_4}^4{p_3}^2 - 16\,{p_1}^4vw^3{p_3}^2 + 16\,v^2u{p_3}^2w^2{p_1}^2 \\
&- 32\,{p_3}^4{p_1}^2{p_4}^2wvu - 48\,{p_4}^2vu{p_3}^2w^2{p_1}^2 - 80\,{p_4}^2v^2u^2{p_3}^2w,
\end{aligned}
\tag{G.3.4}
$$

$$
N_{V1} := \frac{Num_{N_{V1}}}{Den_{N_{V1}}}
$$

$$
\begin{aligned}
&Den_{N_{V1}} = 64\,(p_4+p_3)^2\left(-{p_1}^2{p_4}^2 - 2\,{p_1}^2w - {p_3}^2{p_1}^2 + v^2 + 2\,vu + u^2\right)\times \\
&\left(v^2{p_3}^2 - {p_4}^2{p_1}^2{p_3}^2 + {p_4}^2u^2 + w^2{p_1}^2 - 2\,wvu\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&Num_{N_{V1}} = 218\,{p_4}^4{p_3}^2{p_1}^2wvu^3 - 140\,{p_4}^2{p_3}^2{p_1}^2w^2vu^3 - 248\,w^3{p_1}^2vu^3{p_4}^2 + 15\,{p_1}^8{p_4}^2w{p_3}^6 \\
&+ 4\,v^3{p_3}^8{p_1}^4 - 48\,w^5{p_1}^6v + 60\,w^2v^4{p_1}^2{p_4}^4 - 104\,{p_3}^2{p_1}^2v^4w^2u + 72\,v^3{p_1}^2{p_3}^2u^3w + 56\,v^4{p_1}^2{p_3}^2u^2w - 40\,v^5{p_1}^2{p_3}^2uw \\
&+ 128\,{p_3}^2{p_1}^4v^3w^3 + 47\,{p_3}^6{p_1}^6v{p_4}^4 + 75\,{p_3}^4{p_1}^6v{p_4}^6 + 200\,{p_3}^4{p_1}^2v^5w - 20\,w^3{p_1}^6{p_3}^6 - 24\,w^2v^6{p_4}^2 \\
&+ 160\,{p_3}^4{p_1}^2v^5{p_4}^2 - 74\,v^3{p_1}^4{p_4}^4w^2 - 72\,v^3{p_1}^4{p_4}^6{p_3}^2 + 12\,v^3{p_1}^4{p_4}^2w^3 \\
&- 5\,{p_1}^8{p_4}^2w{p_3}^4v - 64\,v^7w{p_3}^2 + 48\,v^5w^3{p_1}^2 - 28\,{p_3}^8v^2w{p_1}^4 + 16\,{p_3}^4{p_1}^4v^3w^2 \\
&- 195\,{p_3}^4{p_1}^4v^3{p_4}^4 - 150\,v{p_1}^6{p_4}^2w^4 + 24\,v{p_1}^6{p_4}^8{p_3}^2 + 38\,v^5{p_4}^4w{p_1}^2 + 100\,v^5{p_4}^2w^2{p_1}^2 - 42\,{p_4}^6{p_1}^4w^2v^2 \\
&- 28\,{p_4}^6{p_1}^4v^3w + 72\,v^5{p_1}^2{p_4}^4{p_3}^2 - 103\,{p_3}^6{p_1}^4v^3{p_4}^2 - 8\,{p_4}^8{p_1}^4wv^2 \\
&- 42\,{p_1}^6{p_4}^6{p_3}^2v^2 + 4\,{p_1}^6{p_3}^6{p_4}^2wv + 6\,v{p_1}^6{p_4}^6w^2 - 72\,{p_4}^2{p_3}^2v^6w + 80\,v^3w^4{p_1}^4 \\
&+ 52\,{p_1}^8{p_4}^6{p_3}^4 + 10\,{p_4}^6v^4w{p_1}^2 - 102\,{p_1}^8w^3{p_4}^2{p_3}^2 + 2\,{p_1}^8w^2{p_4}^2{p_3}^4 - 96\,w^2v^6{p_3}^2 \\
&+ 146\,v^4{p_1}^4{p_3}^4w + 117\,v^4{p_1}^4{p_3}^4{p_4}^2 - 40\,v^6{p_1}^2{p_3}^2w - 44\,{p_1}^6{p_3}^4w^3v + 254\,{p_4}^4{p_3}^2{p_1}^2v^4w \\
&+ 14\,{p_1}^8{p_4}^8{p_3}^2 + 38\,{p_1}^8{p_4}^4{p_3}^6 - 289\,{p_4}^6{p_3}^2{p_1}^4v^2w - 14\,v^6{p_1}^2{p_3}^2{p_4}^2 + 112\,v^4{p_1}^4{p_3}^2w^2 \\
&+ 42\,v^4{p_1}^4{p_3}^2{p_4}^4 - 139\,{p_1}^6{p_4}^4{p_3}^4v^2 - 12\,u^2{p_4}^{10}{p_1}^4 + 6\,{p_4}^8{p_1}^6vw + 64\,w^4{p_1}^2v^4 - {p_1}^8w^2v{p_3}^4 \\
&+ 100\,{p_4}^2{p_3}^4{p_1}^2wvu^3 - 32\,w^4{p_1}^2vu^3 - 32\,w^4{p_1}^2v^2u^2 - 112\,w^3{p_1}^2vu^3{p_3}^2 - 264\,w^3{p_1}^2v^2u^2{p_4}^2 \\
&- 336\,w^3{p_1}^2v^2u^2{p_3}^2 - 180\,{p_4}^2{p_3}^2{p_1}^2w^2v^2u^2 + 324\,{p_4}^2{p_3}^4{p_1}^2wv^2u^2 \\
&- 288\,w^3{p_1}^2v^3u{p_4}^2 - 108\,{p_4}^2{p_3}^6{p_1}^4wvu + 258\,{p_4}^4{p_3}^2{p_1}^2wv^2u^2 - 128\,w^4{p_1}^2v^3u - 352\,w^3{p_1}^2v^3u{p_3}^2 \\
&+ 14\,{p_4}^{10}{p_3}^2{p_1}^6 - 14\,{p_4}^4{p_3}^2v^6 + 804\,{p_4}^2{p_3}^2{p_1}^4w^3vu + 368\,w^4{p_1}^4vu{p_3}^2 \\
&+ 183\,{p_4}^2{p_3}^4{p_1}^4w^2vu + 150\,{p_4}^4{p_3}^2{p_1}^2wv^3u + 80\,w^5{p_1}^4vu - 112\,{p_4}^2{p_3}^2{p_1}^2w^2v^3u + 490\,{p_4}^2{p_3}^4{p_1}^2wv^3u \\
&+ 412\,w^4{p_1}^4vu{p_4}^2 - 44\,{p_4}^6{p_3}^2{p_1}^4wvu + 336\,{p_4}^4{p_3}^2{p_1}^4w^2vu + 538\,v^4{p_3}^4{p_1}^2w{p_4}^2 \\
&- 248\,{p_4}^4{p_3}^4{p_1}^4wvu + 63\,{p_4}^6u^4{p_1}^2{p_3}^2 + 408\,v^4{p_3}^4{p_1}^2w^2 + 157\,v^4{p_3}^4{p_1}^2{p_4}^4 - 21\,{p_4}^6w^3{p_1}^6 \\
&+ 6\,{p_4}^8w^2{p_1}^6 - 6\,{p_1}^8w^4v + 56\,{p_3}^6{p_1}^2v^5 - 88\,v^5{p_3}^6u + 40\,v^4{p_3}^8{p_1}^2 - 28\,{p_4}^6u^5v \\
&- 42\,{p_4}^6u^4v^2 + 23\,{p_4}^8u^4{p_1}^2 - 44\,{p_4}^4u^5v{p_3}^2 - 36\,{p_4}^4u^4v^2w - 98\,{p_4}^4u^4v^2{p_3}^2 + 94\,{p_4}^6u^4{p_1}^2w \\
&- 387\,{p_3}^4{p_1}^4v^3{p_4}^2w - 189\,{p_3}^2{p_1}^6v{p_4}^2w^3 + 95\,{p_3}^2{p_1}^6v{p_4}^4w^2 + 434\,{p_4}^4{p_1}^6{p_3}^4w^2 - 133\,{p_4}^6{p_1}^4{p_3}^4u^2 \\
&- 268\,{p_3}^2{p_1}^4v^3w^2{p_4}^2 - 25\,{p_3}^4{p_1}^6v{p_4}^2w^2 + 187\,{p_3}^4{p_1}^6v{p_4}^4w - 15\,{p_1}^8w^3v{p_4}^2 + 43\,v^4{p_1}^4{p_3}^6 - 30\,v^6{p_1}^2{p_3}^4 \\
&- 5\,{p_1}^8w^3v{p_3}^2 + 170\,v^4{p_1}^4{p_3}^2{p_4}^2w - 13\,{p_1}^8w^2v{p_3}^2{p_4}^2 - 81\,{p_1}^6{p_4}^2{p_3}^6v^2 - 333\,v^3{p_1}^4{p_4}^4{p_3}^2w - 232\,v^5{p_3}^4uw \\
&+ 155\,v^4{p_3}^6{p_1}^2{p_4}^2 + 228\,v^4{p_3}^6{p_1}^2w - 40\,{p_4}^4u^5vw - 16\,v^7{p_4}^2w - 24\,v^7{p_4}^2{p_3}^2 + 96\,w^2v^4u^2{p_3}^2 \\
&- 130\,v^4{p_3}^4u^2{p_4}^2 - 120\,v^4{p_3}^4u^2w - 122\,v^5{p_3}^4u{p_4}^2 - 32\,w^3v^6 + 127\,v{p_1}^6{p_4}^6{p_3}^2w + 270\,v^5{p_1}^2{p_4}^2w{p_3}^2 \\
&+ 64\,w^2v^2u^4{p_3}^2 + 168\,w^2v^3u^3{p_4}^2 + 192\,w^2v^3u^3{p_3}^2 + 152\,w^2v^4u^2{p_4}^2 + 32\,w^3v^3u^3 + 32\,w^3v^4u^2 - 80\,v^4{p_3}^6u^2 - 4\,{p_4}^4v^6w \\
&+ {p_1}^6v^3{p_3}^6 + 247\,{p_4}^4{p_1}^6{p_3}^6w + 324\,{p_4}^6{p_1}^6{p_3}^4w + 96\,w^2v^2u^4{p_4}^2 - 100\,w^4{p_1}^6{p_4}^4 \\
&- 116\,w^5{p_1}^6{p_4}^2 + 86\,{p_4}^6{p_1}^6{p_3}^6 + 36\,{p_4}^4{p_1}^6{p_3}^8 + 64\,{p_4}^8{p_1}^6{p_3}^4 + 56\,w^4{p_1}^4u^2{p_4}^2 \\
&+ 40\,w^4{p_1}^4u^2{p_3}^2 - 4\,w^4{p_1}^4v^2{p_4}^2 + 56\,w^4{p_1}^4v^2{p_3}^2 - 196\,w^4{p_1}^6{p_3}^2{p_4}^2 \\
&- 351\,{p_4}^4{p_1}^4{p_3}^4u^2w - 68\,{p_4}^4{p_1}^4{p_3}^6u^2 - 175\,{p_4}^6{p_1}^4{p_3}^4v^2 + 7\,{p_1}^6v^3{p_3}^2w{p_4}^2 \\
&- 758\,{p_4}^4{p_1}^4{p_3}^4v^2w - 209\,{p_4}^4{p_1}^4{p_3}^6v^2 + 96\,{p_4}^4u^4{p_1}^2w^2 + 40\,{p_4}^4u^4{p_3}^4{p_1}^2 - 77\,{p_4}^8{p_3}^2{p_1}^4u^2
\end{aligned}
$$

$$+\,144\,p_4^4u^4p_1^2wp_3^2-356\,p_4^6p_3^2p_1^4u^2w+102\,p_4^4p_3^2p_1^6w^3+35\,p_4^2p_3^4p_1^6w^3-5\,p_1^8p_4^4p_3^2vw$$
$$+\,244\,p_4^6p_3^2p_1^6w^2+108\,p_4^2p_3^6p_1^6w^2-76\,p_4^2p_3^8p_1^4v^2-475\,p_4^2p_3^6p_1^4v^2w-392\,w^3p_3^2p_1^4v^2p_4^2$$
$$-\,92\,w^3p_3^4p_1^4v^2-56\,p_4^2u^4p_3^4v^2-100\,p_4^4u^3p_3^2v^3-32\,p_4^2u^4p_3^2v^2w-8\,p_4^6u^6+128\,p_1^4w^4v^2u$$
$$+\,24\,p_4^2u^3p_3^2v^3w-94\,p_4^2u^3p_3^4v^3-56\,p_4^4u^2p_3^2v^4+8\,p_4^2u^2p_3^2v^4w+96\,p_1^4w^4vu^2-128\,uv^5w^2p_3^2$$
$$+\,16\,w^5p_1^4u^2+32\,w^5p_1^4v^2-112\,w^5p_1^6p_3^2-88\,w^4p_1^6p_3^4-104\,uv^5wp_3^2p_4^2+256\,w^3p_3^2p_1^2v^4+472\,w^2p_3^2p_1^2v^4p$$
$$-\,847\,w^2p_3^4p_1^4v^2p_4^2-112\,w^2p_3^6p_1^4v^2+40\,u^3v^3wp_3^4-163\,p_4^6u^2w^2p_1^4-24\,u^4v^3p_3^4+32\,u^4v^3w^2-80\,u^4v^3p_4^4$$
$$+\,23\,u^4p_4^6p_1^4-112\,u^3v^4p_3^4-30\,u^3p_4^6v^3-16\,u^3v^4p_4^4-3\,u^3p_1^6p_4^6-19\,u^3p_4^8p_1^4$$
$$-\,24\,u^3p_3^6v^3+64\,u^3v^4w^2-6\,u^2p_4^6v^4-45\,p_1^8w^3p_4^4+2\,p_1^8wp_4^8-2\,p_1^8w^2p_4^6+2\,p_4^{10}p_1^6w$$
$$-\,82\,p_4^4u^2w^3p_1^4-16\,p_4^4u^7-435\,p_4^4u^2w^2p_1^4p_3^2-40\,p_4^2u^2w^3p_1^4p_3^2+32\,p_4^2u^5w^2v+32\,u^4p_3^4v^2w$$
$$+\,128\,u^4v^3wp_3^2+78\,u^4vp_1^2p_4^6+176\,u^4v^3p_4^2w-184\,u^4v^3p_4^2p_3^2-32\,u^4vw^3p_1^2-32\,u^4p_4^2w^2p_1^4$$
$$+\,55\,u^4p_4^4p_1^4p_3^2-18\,u^4p_4^4v^2p_1^2-32\,u^4v^2w^2p_1^2+2\,u^4vp_1^4p_4^4+226\,u^4p_3^2p_1^2vp_4^4+116\,u^4vp_1^2p_4^4w$$
$$+\,16\,u^4wp_1^2p_3^4p_4^2-205\,p_1^6wp_4^4v^2p_3^2+2\,u^4p_4^2p_1^4p_3^2v-34\,u^4p_4^2v^2p_1^2p_3^2+4\,u^4p_4^2wp_1^4v$$
$$+\,52\,u^4vp_4^2p_1^2p_3^4+46\,u^4p_4^4wp_1^4-3\,u^3p_1^4p_4^6v-142\,u^3p_4^4p_1^4w^2-6\,u^3p_1^6w^2p_4^2+44\,u^3p_3^6vp_4^2p_1^2$$
$$-\,48\,u^3p_3^4vw^2p_1^2-16\,p_4^2u^5wvp_3^2+16\,p_4^4u^3wv^3-113\,u^3p_4^6p_1^4w-208\,u^3p_4^2v^4p_3^2-9\,u^3p_1^6wp_4^4$$
$$+\,33\,u^3p_4^8p_1^2v+18\,u^3v^3p_1^2p_4^4+24\,u^3vw^3p_1^4-123\,u^3p_4^6p_1^4p_3^2-3\,u^3p_1^6p_4^4p_3^2$$
$$+\,16\,u^3p_3^2w^3p_1^4-14\,u^3v^3p_3^4p_1^2-120\,u^3p_4^4p_1^4p_3^4+50\,u^3v^2p_1^2p_4^6+160\,u^3p_4^2v^4w$$
$$+\,6\,u^3p_1^4v^2p_4^4+32\,u^3p_4^2w^3p_1^4-4\,u^3p_1^4v^2w^2-56\,u^3p_1^2v^3w^2+128\,u^3v^4wp_3^2-48\,u^3p_1^2w^3v^2$$
$$+\,6\,u^2p_1^4v^3p_4^4+28\,u^2v^3p_3^6p_1^2-22\,u^2p_1^4p_4^6v^2+16\,u^2w^3p_1^4p_3^4+26\,u^2v^4p_1^2p_4^4+56\,u^2p_4^4v^4w$$
$$-\,79\,u^2p_4^8p_1^4w+18\,u^2p_4^8p_1^2v^2+13\,u^2w^2p_1^6p_4^4+174\,u^3p_4^2p_3^2p_1^2wv^2-110\,u^3p_4^2p_3^2p_1^4vw-24\,u^3p_3^4p_1^2wv^2$$
$$-\,72\,u^3p_4^2p_3^2p_1^4w^2+260\,u^3p_4^2p_3^4p_1^2v^2-20\,u^3vp_3^2p_1^4w^2+8\,u^2w^3p_1^6p_3^2-4\,p_1^4w^2v^5+12\,p_1^6w^3v^3$$
$$+\,32\,p_4^2u^4w^2p_1^2p_3^2-16\,p_4^4u^6w-8\,p_4^4u^6p_3^2-2\,p_1^4v^5p_3^2w+12\,p_1^6v^3p_3^2w^2-32\,w^6p_1^6-32\,v^6p_3^6$$
$$-\,200\,p_1^2p_3^2u^3w^2v^2-2\,u^3p_1^4p_3^2wv^2-339\,u^3p_1^4p_4^4p_3^2w+23\,u^3p_4^2p_3^4p_1^4v-268\,u^3p_1^4p_4^2vw^2$$
$$-\,3\,u^3p_1^6wp_4^2p_3^2-56\,u^3p_4^2p_3^4p_1^4w+374\,u^3p_4^4p_1^2p_3^2v^2-60\,u^3v^3p_1^2p_4^2p_3^2+84\,u^3vp_1^4p_4^4p_3^2$$
$$-\,98\,u^3vp_1^4wp_4^4+6\,u^3v^2p_1^4p_4^2p_3^2+34\,u^3p_4^4p_1^2wv^2+10\,u^3v^2p_1^4wp_4^2+6\,u^2p_1^4wv^3p_4^2-46\,v^6p_3^4p_4^2$$
$$-\,104\,v^6p_3^4w-16\,u^2p_3^6v^2wp_1^2-83\,u^2vp_1^4p_4^6w-u^2p_1^6p_4^2vp_3^4+14\,u^2p_4^6p_1^2wv^2-16\,u^2v^2p_3^4p_1^4w$$
$$-\,15\,u^2p_4^2p_1^6p_3^4w+4\,u^2p_1^6w^2vp_3^2-2\,u^2v^3p_1^2p_4^6-14\,u^2vp_1^4p_4^8+64\,u^2v^5p_4^2w-160\,u^2v^5p_4^2p_3^2$$
$$-\,6\,u^2p_1^6p_4^6v-23\,u^2p_1^6wp_4^6+16\,u^2v^5p_4^4-192\,u^2v^5p_3^4-4\,u^2p_1^6p_4^8-64\,u^6vp_4^4-16\,u^6p_4^4p_1^2+40\,u^5p_1^2p_4^6$$
$$-\,104\,u^5v^2p_4^4-32\,uw^3v^5+8\,uv^6p_4^4-2\,up_1^8w^4+2\,up_1^8p_4^8+2\,up_4^6v^5$$
$$+\,2\,up_4^{10}p_1^6-64\,uv^6w^2-32\,uw^5p_1^6-144\,uv^6p_3^4+460\,w^3p_1^4vup_4^4$$
$$-\,2\,p_1^4wv^5p_4^2+5\,p_1^6v^3p_3^4w+p_1^6v^3p_3^4p_4^2-348\,u^3v^2p_1^2p_4^2w^2-86\,u^2v^2wp_1^4p_4^4$$
$$-\,32\,u^2p_4^2p_3^6p_1^4v-12\,u^2v^3p_1^4w^2+136\,u^2p_1^4v^2w^3-64\,u^2v^5wp_3^2-176\,u^2p_1^2v^3w^3-65\,u^2p_1^6p_4^6p_3^2$$
$$-\,40\,u^2v^4p_1^2w^2-58\,u^2p_1^2v^4p_3^4-77\,u^2p_4^4p_3^4p_1^6+94\,u^2p_1^6p_4^2w^3-48\,u^2v^4p_1^2p_3^2p_4^2+284\,w^3p_1^4vup_3^4$$
$$-\,5\,u^2vp_1^6wp_4^2p_3^2+34\,u^2vp_1^4w^2p_4^4-7\,u^2vp_1^6p_4^4p_3^2+6\,u^2v^3p_1^4p_4^2p_3^2-112\,u^2v^2p_1^4w^2p_4^2$$
$$-\,191\,u^2p_4^6p_1^4vp_3^2-6\,u^2p_1^4wv^3p_3^2+45\,u^2p_4^2p_3^2p_1^6w^2+360\,u^2p_1^4p_4^2vw^3+346\,u^2p_4^4v^3p_1^2p_3^2$$
$$-\,168\,w^2v^2u^2p_1^2p_4^4-p_1^8p_4^2vp_3^6-24\,v^6p_1^2w^2-68\,p_1^8w^4p_4^2-173\,p_1^6v^2w^2p_4^2p_3^2-4\,p_4^2p_3^8p_1^6v$$
$$-\,2\,u^2vp_1^6p_4^2w^2+108\,u^2v^2p_1^4p_4^4p_3^2+346\,u^2v^3p_1^2p_4^2wp_3^2-257\,u^2p_4^4p_3^4p_1^4v+130\,u^2p_4^2p_1^4p_3^4v^2$$
$$-\,152\,u^2p_1^6p_4^4wp_3^2-16\,u^2v^3p_3^4p_1^2w-192\,w^2v^2u^2p_3^4p_1^2-122\,p_4^6p_1^4p_3^4vu-119\,p_4^4p_1^4p_3^6vu$$
$$+\,440\,u^2p_1^2v^3p_3^4p_4^2+276\,u^2p_4^2p_3^2p_1^4vw^2-60\,u^2p_1^4p_3^2v^2w^2-429\,u^2vp_1^4p_4^4wp_3^2-84\,u^2p_1^4p_4^2vwp_3^4$$
$$+\,10\,u^2p_1^4p_4^2wv^2p_3^2+8\,u^2p_1^6w^3v+32\,u^6vp_4^2w-16\,u^6vp_4^2p_3^2+132\,p_4^6p_3^2p_1^2u^2v^2+258\,p_4^4p_3^4p_1^2u^2v^2$$
$$+\,128\,p_4^6p_3^2p_1^2u^3v+139\,p_4^4p_3^4p_1^2u^3v-621\,p_4^4p_3^2p_1^4w^2v^2-128\,p_4^2p_3^4p_1^4w^2u^2+16\,p_1^2p_3^2wv^2u^4$$
$$+\,136\,v^3wp_1^2u^3p_4^2+26\,v^4p_1^4wp_4^4-2\,p_1^8p_4^6vw+244\,p_4^4p_3^4p_1^2v^3u+207\,p_4^2p_3^6p_1^2v^3u+224\,u^2p_1^4vw^3p_3^2$$

$$
\begin{aligned}
&-15\,u^2{p_1}^6{p_4}^4vw+76\,u^2vw^2{p_1}^4{p_3}^4-26\,u^2v^3{p_1}^2{p_4}^4w-44\,u^2wp_1{}^4{p_3}^6{p_4}^2\\
&+32\,u^5v^2w{p_3}^2+112\,u^5v^2{p_4}^2w+96\,u^5{p_1}^2{p_4}^4w+72\,u^5{p_3}^2{p_1}^2{p_4}^4\\
&-36\,u^5{p_1}^2{p_4}^4v-88\,u^5v^2{p_4}^2{p_3}^2-16\,u^5{p_3}^4v{p_4}^2+10\,u{p_1}^6{p_4}^8v+5\,u{p_1}^8{p_4}^2w^3-u{p_1}^8w^3{p_3}^2\\
&+24\,uw^2v^5{p_4}^2-32\,u{p_4}^4{p_3}^2v^5+144\,{p_4}^2{p_3}^6{p_1}^2v^2u^2+140\,v^2w{p_1}^2{p_4}^2u^4+64\,v^4{p_1}^2w{p_4}^2u^2-24\,w^2{p_3}^4{p_1}^2v^3u\\
&+116\,uv^3w{p_3}^6{p_1}^2+82\,{p_4}^6u^3wv{p_1}^2-84\,{p_4}^4u^3w^2v{p_1}^2+16\,u^5w{p_1}^2{p_4}^2{p_3}^2-4\,u^5{p_1}^2v{p_4}^2{p_3}^2-u{p_1}^6v^2{p_3}^4w\\
&-28\,u{p_3}^8v{p_4}^2{p_1}^4+176\,u{p_1}^6vw^3{p_4}^2+60\,u{p_3}^6vw^2{p_1}^4-2\,uv^4{p_1}^2{p_4}^4w+9\,u{p_1}^8{p_4}^2w^2{p_3}^2-216\,uw^2v^3{p_1}^2{p_4}^4\\
&-{p_1}^8{p_4}^4{p_3}^4v-12\,v^6{p_1}^2w{p_4}^2-10\,{p_1}^8{p_4}^4w^2v+4\,{p_1}^6{p_4}^4v^3w+16\,{p_1}^4w^4u^3-260\,u^2v^3{p_1}^2{p_4}^2w^2\\
&+3\,u{p_1}^8{p_4}^2w{p_3}^4-17\,u{p_1}^6{p_4}^6{p_3}^2v-78\,u{p_1}^6{p_4}^4{p_3}^4v-216\,uv^3{p_1}^4{p_3}^2w^2+49\,uv^3{p_1}^4{p_3}^2{p_4}^4+24\,u{p_1}^6{p_4}^2{p_3}^4wv\\
&+6\,u{p_1}^6w^2v^2{p_4}^2+4\,u{p_1}^6w^2v^2{p_3}^2-15\,{p_1}^6{p_3}^6wv^2-224\,u^4v{p_1}^2{p_4}^2w^2+152\,uv^3{p_1}^4{p_3}^4{p_4}^2-14\,uv^3{p_1}^4{p_3}^4w\\
&-32\,uv^5{p_1}^2{p_3}^2{p_4}^2+24\,u{p_4}^4wv^5+12\,u{p_1}^6w^3v^2-64\,uw^4{p_1}^6{p_3}^2-66\,uw^4{p_1}^6{p_4}^2+64\,u{p_4}^4{p_1}^6{p_3}^6+99\,u{p_4}^6{p_1}^6{p_3}^4\\
&+76\,v^4{p_1}^4w^2{p_4}^2+112\,w^3{p_1}^2v^4{p_4}^2+84\,uv^4{p_3}^6{p_1}^2+19\,uv^3{p_1}^4{p_3}^6-74\,uv^5{p_1}^2{p_3}^4+10\,uv^5{p_1}^2{p_4}^4-16\,uv^6{p_4}^2w\\
&-14\,uv^4{p_4}^6{p_1}^2+2\,u{p_4}^{10}{p_1}^4v+15\,u{p_1}^8{p_4}^4w^2+3\,u{p_1}^8{p_4}^4{p_3}^4-4\,u{p_1}^6{p_4}^6v^2+5\,u{p_1}^8{p_4}^6{p_3}^2+10\,u{p_1}^8{p_4}^6w\\
&+79\,uw^2{p_1}^6{p_4}^6+45\,uw^3{p_1}^6{p_4}^4-24\,uw^3{p_1}^6{p_3}^4-20\,uv^3{p_1}^4{p_4}^6-88\,uv^6{p_3}^2{p_4}^2+4\,u{p_4}^8{p_1}^4v^2+2\,uv^3{p_1}^4{p_3}^2w{p_4}^2\\
&-32\,uv^2{p_3}^6{p_1}^4w+40\,u{p_4}^2{p_1}^6{p_3}^6w+208\,uv^4{p_3}^4{p_1}^2w+392\,uv^4{p_3}^4{p_1}^2{p_4}^2+291\,u{p_4}^4{p_1}^6{p_3}^4w-347\,u{p_4}^4{p_1}^4{p_3}^4v^2\\
&-17\,u{p_4}^2{p_3}^2{p_1}^6w^3+297\,u{p_4}^4{p_3}^2{p_1}^6w^2-48\,u^4vw^2{p_1}^2{p_3}^2+96\,u{p_4}^2{p_3}^4{p_1}^6w^2-148\,u{p_4}^2{p_3}^6{p_1}^4v^2\\
&+256\,uw^3{p_3}^2{p_1}^4v^2+20\,uw^2{p_3}^4{p_1}^4v^2+48\,u{p_1}^6w{p_4}^4v{p_3}^2+26\,u{p_4}^8{p_1}^6w-160\,uv^6{p_3}^2w+37\,u{p_1}^6{p_4}^8{p_3}^2-4\,u{p_4}^8v^3{p_1}^2\\
&-12\,u{p_1}^4w^2v^4+2\,u{p_1}^4v^4{p_4}^4-112\,uv^4{p_1}^2w^3-40\,uv^5{p_1}^2w^2-32\,u{p_1}^4v^3w^3\\
&-52\,u{p_1}^6vw^4+28\,u{p_3}^8v^3{p_1}^2+284\,u{p_1}^4v^2w^3{p_4}^2+198\,uv^4{p_1}^2{p_3}^2{p_4}^4\\
&+426\,uv^4{p_1}^2{p_3}^2{p_4}^2w+66\,uv^2{p_1}^4w^2{p_4}^4-6\,u{p_1}^4v^4{p_3}^2w+2\,u{p_1}^4v^4{p_3}^2{p_4}^2\\
&-60\,uv^3w{p_1}^4{p_4}^4-3\,u{p_1}^6v^2{p_3}^4{p_4}^2+215\,u{p_1}^6{p_4}^6{p_3}^2w-147\,uv^2{p_1}^4{p_4}^6{p_3}^2+316\,u{p_1}^6vw^2{p_3}^2{p_4}^2\\
&-471\,uv^2{p_1}^4{p_4}^4{p_3}^2w+201\,u{p_4}^6w^2{p_1}^4v+36\,u{p_4}^8wv{p_1}^4-282\,v^2w{p_1}^6{p_3}^4{p_4}^2+42\,{p_4}^6{p_3}^2{p_1}^2v^4\\
&+107\,{p_4}^8{p_3}^2{p_1}^6w-28\,{p_1}^8w^4{p_3}^2+20\,{p_1}^6v^2w^4-36\,v^4u{p_4}^2w^2{p_1}^2+136\,{p_1}^8w{p_4}^4{p_3}^4-4\,vw^2{p_1}^6{p_3}^6+64\,{p_1}^6v^2w^3{p_3}^2\\
&-50\,{p_1}^6w^2{p_4}^4v^2-16\,{p_1}^6w{p_4}^6v^2-28\,u{p_4}^2{p_3}^2{p_1}^4w^2v^2-481\,u{p_4}^2{p_3}^4{p_1}^4v^2w-60\,u{p_4}^6wv^3{p_1}^2\\
&-7\,u{p_1}^6{p_4}^4v^2{p_3}^2-8\,u{p_1}^6{p_4}^4v^2w+15\,u{p_1}^8{p_4}^4{p_3}^2w-7\,u{p_1}^6v^2{p_3}^2{p_4}^2w+173\,u{p_1}^6w^2{p_4}^4v+60\,u{p_1}^6w{p_4}^6v\\
&-19\,u{p_1}^6v{p_4}^2{p_3}^6-8\,uv^2{p_1}^4{p_4}^6w+61\,u{p_4}^6{p_3}^2{p_1}^2v^3+35\,u{p_1}^6vw^2{p_3}^4-18\,{p_1}^6v^2w^3{p_4}^2+56\,{p_1}^8w^2{p_4}^4{p_3}^2\\
&-42\,{p_4}^8{p_3}^2{p_1}^4v^2+75\,{p_1}^8w{p_4}^6{p_3}^2-2\,u{p_1}^4wv^4{p_4}^2-88\,u{p_1}^4v^3w^2{p_4}^2-29\,u{p_4}^8{p_3}^2{p_1}^4v+60\,u{p_1}^6vw^3{p_3}^2-32\,v^7w^2\\
&+48\,v^4{p_1}^4w^3-40\,v^7{p_3}^4-7\,{p_1}^6v^2w^2{p_3}^4-4\,v^3w{p_1}^4{p_3}^6-96\,{p_3}^2{p_1}^6vw^4-75\,v{p_1}^6{p_4}^4w^3+28\,w{p_1}^6{p_3}^8{p_4}^2\\
&-62\,w^3{p_1}^4v^2{p_4}^4-13\,{p_1}^8w^3{p_3}^4+168\,{p_3}^2{p_1}^2v^5w^2-424\,{p_3}^2{p_1}^2v^3w^2u^2+56\,{p_1}^2{p_4}^2u^5wv\\
&-4\,{p_1}^8w^5+14\,{p_1}^6w^2v^3{p_4}^2, \qquad\qquad \text{(G.3.5)}
\end{aligned}
$$

$$
\begin{aligned}
&N_{V2}=\frac{Num_{N_{V2}}}{Den_{N_{V2}}},\\
&Den_{N_{V2}}=32\left(v^2{p_3}^2-{p_4}^2{p_1}^2{p_3}^2+{p_4}^2u^2+w^2{p_1}^2-2\,wvu\right)^2\\
&Num_{N_{V2}}=\left(v^2-{p_1}^2{p_4}^2\right)\times\big(6\,{p_4}^4u^2+12\,v^3{p_3}^2-{p_1}^4{p_4}^2u+12\,{p_3}^2v^2u-16\,vuw^2-8\,uwv^2-7\,{p_1}^2{p_4}^4{p_3}^2\\
&\;+{p_1}^4wv-uv{p_4}^4-{p_4}^4u{p_1}^2-16\,{p_3}^2u{p_1}^2{p_4}^2-16\,uwv{p_4}^2-4\,{p_3}^2uv{p_4}^2-8\,{p_1}^2wuv-16\,{p_3}^2vuw\\
&\;-5\,{p_1}^2{p_4}^2uv+4\,{p_1}^2wv{p_3}^2-16\,w{p_1}^2{p_4}^2{p_3}^2-3\,{p_1}^2v{p_4}^2w-12\,{p_1}^2v{p_4}^2{p_3}^2-8\,{p_1}^2{p_4}^2wu\\
&\;+8\,w^2v^2+6\,{p_1}^4w^2+8\,v^3w+8\,w^3{p_1}^2-16\,vu^2w+2\,{p_1}^2{p_4}^2u^2+8\,{p_1}^2uw^2+4\,v{p_4}^2u^2-{p_1}^4w{p_4}^2\\
&\;-12\,{p_3}^4{p_1}^2{p_4}^2-7\,{p_1}^4{p_3}^2{p_4}^2+7\,{p_1}^2{p_3}^2v^2+8\,{p_4}^2u^3+20\,v^2w{p_3}^2+7\,v^2{p_4}^2{p_3}^2\\
&\;+2\,v^2{p_4}^2w+8\,{p_4}^2u^2w+8\,w^2{p_1}^2{p_3}^2+12\,{p_1}^2w^2v-4\,{p_4}^2v^2u+12\,{p_3}^4v^2\\
&\;+2\,{p_4}^2w^2{p_1}^2+8\,{p_3}^2{p_4}^2u^2-{p_1}^2{p_4}^4w+6\,{p_1}^2wv^2\big), \qquad\qquad \text{(G.3.6)}
\end{aligned}
$$

$$N_{V3} = \frac{Num_{N_{V3}}}{Den_{N_{V3}}},$$

$$Den_{N_{V3}} = 64\,\left(-{p_3}^2{p_1}^2 - 2\,{p_3}^2v - {p_4}^2{p_3}^2 + u^2 + 2\,uw + w^2\right)\left(v^2{p_3}^2 - {p_4}^2{p_1}^2{p_3}^2 + {p_4}^2u^2 + w^2{p_1}^2 - 2\,wvu\right)^2$$

$$\begin{aligned}
Num_{N_{V3}} &= 15\,{p_1}^6v^2{p_3}^4w + 13\,{p_1}^8w^3{p_3}^2 - 20\,v^5{p_3}^4{p_4}^2 + 40\,v^5{p_3}^4w + 66\,w^3v^3{p_1}^2{p_3}^2 - 3\,{p_1}^8{p_4}^2w^2{p_3}^2\\
&+ 148\,w^2v^3{p_1}^2{p_3}^4 - 15\,{p_1}^8{p_4}^2w{p_3}^4 + 3\,{p_1}^6{p_4}^4{p_3}^4v + 8\,{p_4}^4{p_1}^4{p_3}^6v + 8\,{p_4}^2{p_3}^6{p_1}^2v^3\\
&+ 44\,v^3{p_1}^4{p_3}^2w^2 - 69\,{p_1}^6{p_4}^2{p_3}^4wv + 48\,w^2v^5{p_3}^2 + 40\,{p_4}^2{p_3}^4{p_1}^2v^3w + 2\,w^2v^3{p_1}^2{p_4}^2{p_3}^2\\
&- 80\,{p_4}^4{p_1}^4{p_3}^4vw + 11\,{p_1}^6w^2v^2{p_3}^2 + 17\,v^3{p_1}^4{p_3}^4{p_4}^2 + 69\,v^3{p_1}^4{p_3}^4w - 20\,w^5{p_1}^4v - 12\,w^4{p_1}^2v^3\\
&+ 70\,{p_4}^2{p_3}^2{p_1}^4w^3v - 27\,{p_4}^4{p_3}^2{p_1}^4w^2v - 124\,{p_4}^2{p_3}^4{p_1}^4w^2v - 84\,{p_4}^2{p_3}^6{p_1}^4wv\\
&+ u^2v{p_1}^6{p_4}^2{p_3}^2 + 3\,u^2{p_1}^6{p_4}^4{p_3}^2 + 116\,{p_4}^4{p_3}^4{p_1}^2u^2v - 96\,{p_4}^4{p_3}^2{p_1}^2u^3w\\
&- 96\,{p_4}^4{p_3}^2{p_1}^2u^2w^2 + 96\,{p_4}^2{p_3}^2{p_1}^4w^3u - 48\,{p_4}^2{p_3}^4{p_1}^2v^2w^2 + 232\,{p_4}^2u^3{p_3}^2v^2w + 8\,w^4{p_3}^2{p_1}^2v^2\\
&- 88\,uv^3w{p_3}^4{p_4}^2 - 176\,uv^3w^3{p_3}^2 + 32\,{p_4}^2u^2w^4{p_1}^2 - 96\,{p_4}^2u^4w^2v - 128\,{p_4}^2u^3w^3v - 8\,{p_4}^4u^3wv{p_3}^2 - 24\,{p_3}^4v^3w^3\\
&- 4\,{p_3}^6v^3{p_4}^4 + 28\,{p_3}^6v^3w{p_4}^2 + 4\,{p_3}^6v{p_4}^6{p_1}^2 - 48\,w^4{p_1}^4u^2 + 6\,w^4{p_1}^4v^2 + 24\,w^4{p_1}^6{p_3}^2\\
&+ 4\,w^4{p_1}^6{p_4}^2 - 64\,w^2v^2u^4 - 32\,w^2v^3u^3 - 16\,w^2v^4u^2 - 56\,v^4{p_3}^4u^2 - 72\,v^5{p_3}^4u - 16\,v^4{p_3}^6{p_1}^2 + 12\,{p_4}^4u^5v\\
&- 18\,{p_4}^4u^4v^2 - 4\,{p_4}^6u^4{p_1}^2 + 21\,u^2v{p_1}^4{p_4}^4{p_3}^2 + 32\,u^2{p_1}^4{p_4}^2v{p_3}^4 + 5\,u^2{p_1}^4{p_4}^2v^2{p_3}^2\\
&- 4\,v^3{p_1}^4{p_3}^6 - 20\,v^5{p_1}^2{p_3}^4 - 28\,{p_3}^6{p_4}^4{p_1}^4w + 24\,v^3{p_3}^4u^3 - v^3{p_4}^6{p_3}^4 - 16\,v^5{p_3}^6 + 16\,v^2{p_3}^4{p_1}^2u^2w\\
&+ 28\,v^2{p_3}^6{p_1}^4w - 32\,v^2{p_3}^2u^4w + 56\,{p_3}^6v^4w - 28\,v^3{p_3}^6{p_1}^2u - 28\,{p_4}^2{p_1}^6{p_3}^6w + 8\,v^2{p_4}^6{p_3}^4{p_1}^2 + 6\,v^3{p_4}^4{p_3}^2w^2\\
&+ {p_4}^8{p_1}^2{p_3}^4v + 44\,{p_3}^4{p_4}^2{p_1}^4w^3 - 44\,{p_4}^2{p_1}^2{p_3}^4u^3v + 44\,{p_4}^2{p_1}^4{p_3}^4u^2w + 14\,{p_4}^2v^2w^4{p_1}^2\\
&+ 4\,{p_1}^2{p_4}^4vw^4 - 25\,v^2{p_4}^4w^2{p_1}^2{p_3}^2 - 5\,{p_4}^6{p_1}^2{p_3}^2vw^2 - 4\,v^3{p_4}^2w^4 - 10\,v^2{p_1}^2{p_4}^2u^4\\
&- 16\,{p_4}^2{p_1}^2{p_3}^2u^4w + 19\,u{p_1}^6{p_4}^2{p_3}^4v + {p_1}^8w^4 + 28\,{p_4}^2{p_1}^4{p_3}^6uv + 16\,{p_4}^2u^5v{p_3}^2 - 19\,uv^3{p_1}^4{p_3}^4\\
&+ 15\,u{p_4}^4{p_1}^6{p_3}^4 - 160\,{p_1}^4w^2v^2{p_3}^2u - 16\,{p_1}^2{p_4}^2u^5w + 4\,{p_1}^2{p_4}^2u^5v + 59\,u{p_4}^2{p_3}^4{p_1}^4v^2\\
&+ 24\,u^3v^2{p_1}^2{p_3}^2w - 76\,{p_1}^4w^2v{p_3}^2u^2 - 12\,{p_4}^4{p_3}^2{p_1}^4wvu + 98\,v^4{p_3}^4{p_1}^2w - 150\,{p_4}^2{p_3}^2{p_1}^4w^2vu\\
&- 68\,{p_4}^2{p_3}^2{p_1}^2wv^3u - 104\,{p_4}^2{p_3}^4{p_1}^4wvu + 160\,{p_4}^2{p_3}^2{p_1}^2wv^2u^2 - 68\,w^3{p_1}^2v^3u\\
&- 20\,w^4{p_1}^4vu + 56\,{p_4}^2{p_3}^2{p_1}^2wvu^3 + 128\,w^3{p_1}^2vu^3 + 8\,w^3{p_1}^2v^2u^2 - 30\,{p_4}^4{p_1}^6{p_3}^4w + 40\,{p_4}^4{p_1}^4{p_3}^4u^2\\
&+ 16\,{p_4}^4{p_1}^4{p_3}^4v^2 - 24\,{p_4}^4u^4{p_1}^2w - 56\,{p_4}^4u^4{p_3}^2{p_1}^2 + 6\,{p_4}^6{p_3}^2{p_1}^4u^2\\
&+ 28\,{p_4}^2{p_3}^2{p_1}^6w^3 - 6\,{p_4}^4{p_3}^2{p_1}^6w^2 - 40\,{p_4}^2{p_3}^4{p_1}^6w^2 + 16\,{p_4}^2{p_3}^6{p_1}^4v^2 + 100\,w^3{p_3}^2{p_1}^4v^2\\
&- 24\,{p_4}^2u^4{p_3}^2v^2 - 10\,{p_4}^2u^3{p_3}^2v^3 - 8\,{p_4}^2u^2{p_3}^2v^4 + 120\,u^2v^4w{p_3}^2 + 16\,uv^5w{p_3}^2 + 76\,w^2{p_3}^2{p_1}^2v^4\\
&+ 56\,w^2{p_3}^4{p_1}^4v^2 + 144\,u^3v^3w{p_3}^2 + 48\,{p_4}^2u^2w^3{p_1}^4 - 8\,{p_4}^2u^4wv^2 + 44\,{p_4}^2u^3wv^3 - 32\,{p_4}^2u^4w^2{p_1}^2 + 20\,w^3{p_1}^6{p_3}^4\\
&- 16\,w^3{p_1}^4{p_3}^2u^2 + 48\,w^2{p_1}^2u^3v{p_3}^2 - 24\,w^5{p_1}^2v^2 - 60\,w^2{p_1}^4{p_3}^4uv + 42\,v^4{p_3}^4{p_4}^2w\\
&+ 24\,v^4{p_3}^2w^3 - 16\,vw^6{p_1}^2 - 18\,u^4v{p_1}^2{p_4}^4 - 16\,w^5v{p_3}^2{p_1}^2 + 4\,w^5{p_1}^2v{p_4}^2 + 24\,{p_1}^4u^3{p_4}^2w^2 + 8\,u^5{p_4}^6\\
&- u^4{p_4}^8 + 16\,u^5{p_4}^4{p_3}^2 - 32\,u^5wv^2 - 16\,u^4v^3w - 8\,u^4{p_4}^6v + 16\,u^4{p_4}^6w - u^3{p_4}^8w - 13\,u^3{p_3}^2{p_4}^8 - 4\,u^4{p_4}^4vw\\
&+ 32\,u^4{p_3}^2w{p_4}^4 + 48\,u^4vw^2{p_1}^2 - 16\,u^4{p_1}^2wv^2 - 12\,u^4{p_1}^2{p_4}^2wv - 16\,u^3{p_4}^2w^2{p_1}^2{p_3}^2 - 8\,u^3w^2v^2{p_4}^2\\
&- 32\,u^3w^2v^2{p_3}^2 - 28\,u^3{p_4}^6{p_1}^2{p_3}^2 - 7\,u^3{p_4}^6{p_1}^2w + 2\,u^3{p_4}^4v^2w - 68\,u^3{p_4}^4v^2{p_3}^2 - 48\,u^3{p_4}^4{p_1}^2w^2\\
&- 44\,u^3{p_4}^4{p_1}^2{p_3}^4 - 16\,u^3{p_4}^2{p_3}^4v^2 - 24\,u^3{p_4}^6w{p_3}^2 + 16\,u^3{p_4}^4{p_3}^2w^2 - 6\,u^3{p_4}^6vw - 60\,u^3{p_3}^4v{p_4}^4\\
&+ 3\,u^3{p_1}^6w{p_4}^2 - 44\,u^3{p_4}^4w^2v + 56\,u^3{p_4}^2{p_1}^4{p_3}^2w + 36\,u^3{p_4}^2v^2w{p_1}^2 + 18\,u^3{p_4}^2w{p_1}^4v - 20\,u^3{p_4}^6{p_3}^4\\
&+ 16\,u^3v^4w - 32\,u^2w^3v^3 - 8\,u^2w^3{p_1}^6 - 8\,u^3v^2w^2{p_1}^2 + 12\,u^3{p_1}^2v^3w + 2\,u^3v^2w{p_1}^4 - 55\,u^3{p_4}^6v{p_3}^2 + 20\,u^3vw^2{p_1}^4\\
&- 3\,u^3{p_4}^4w{p_1}^4 + 52\,u^2w^2v^3{p_4}^2 + 2\,u^2{p_4}^4v^3{p_3}^2 + 6\,u^2{p_1}^6w^2{p_4}^2 + 104\,u^2v^3w{p_3}^4 + 4\,u^2{p_3}^4v{p_4}^6 + 20\,u^2vw^3{p_1}^4\\
&+ 43\,u^2{p_3}^2{p_4}^6wv + 9\,u^2{p_4}^6{p_1}^2{p_3}^2w + 94\,u^2{p_4}^4{p_1}^2{p_3}^2vw + 76\,u^2{p_4}^2{p_3}^4{p_1}^2wv + 208\,u^2{p_3}^4v^2w{p_4}^2\\
&+ 5\,u^2v^2{p_4}^6{p_3}^2 + 22\,u^2v^2{p_4}^4w^2 + 3\,u^2{p_4}^8{p_1}^2{p_3}^2 - 6\,u^2{p_4}^6{p_1}^2w^2 - 4\,u^2{p_1}^6w^2v + u^2{p_4}^8v{p_3}^2 + 2\,u^2{p_4}^6w^2v\\
&- 48\,u^2{p_4}^2w^4v + 32\,u^2w^3v^2{p_3}^2 - 26\,u^2{p_1}^4v^2w^2 - 24\,u^2{p_4}^4w^3{p_1}^2 + 16\,u^2{p_1}^4{p_3}^2wv^2 + 15\,u^2{p_1}^6w{p_4}^2{p_3}^2\\
&- 44\,u^2{p_1}^2v^3w^2 + 16\,u^2{p_4}^2{p_3}^2w^3{p_1}^2 - 28\,u^2{p_3}^4{p_4}^4{p_1}^2w + 51\,u^2{p_4}^2{p_3}^2{p_1}^4vw + 60\,u^2wv{p_3}^4{p_4}^4
\end{aligned}$$

$$- 12\, u^2p_4^4w^2p_1^2v + 104\, u^2w^3p_1^2vp_4^2 + 68\, u^2p_4^4vp_3^2w^2 + 21\, u^2p_4^6p_1^2p_3^2v + 176\, u^2p_4^4v^2p_3^2w$$
$$- 28\, u^2p_4^4w^3v + 40\, u^2v^2w^3p_4^2 + 24\, uw^3p_1^6p_3^2 + 2\, uv^2p_4^4w^3 - 3\, up_4^6p_1^2w^3 - 4\, uv^3p_4^2w^3 + 28\, up_4^4p_3^6p_1^4$$
$$+ 30\, up_4^6p_3^4p_1^4 - 22\, up_1^4v^2w^3 + 3\, up_4^4p_1^4w^3 - 40\, uv^4w^2p_3^2 - 15\, up_3^2p_4^6w^2p_1^2 + 77\, up_4^6vp_1^2p_3^4$$
$$+ 32\, uv^2p_3^4p_1^4w - 115\, up_4^4vw^2p_1^2p_3^2 - 40\, up_4^2p_1^6p_3^4w - 35\, up_1^6w^2vp_3^2 - 56\, up_4^4w^3p_3^2p_1^2 - 48\, up_4^4wp_3^4v^2$$
$$+ 40\, up_4^6wp_3^4p_1^2 - 5\, uv^2p_4^6p_3^2w - 28\, uv^3p_4^4p_3^2w + 3\, up_4^8p_1^2p_3^2w + 12\, up_4^6p_1^2p_3^2vw + up_1^6wv^2p_3^2$$
$$- 6\, up_1^2p_4^4vw^3 - 3\, up_1^8p_4^2wp_3^2 + 68\, uw^4p_1^2vp_4^2 + up_1^8w^3 - 48\, uv^4w^3 - 32\, uv^3w^4 - 56\, up_3^6v^4 - 16\, up_1^6w^4$$
$$+ 32\, uv^2w^5 - 24\, uvp_1^6wp_4^2p_3^2 + 8\, up_1^4wv^3p_3^2 + 7\, up_1^6p_4^2w^3$$
$$- 130\, uv^4p_3^4p_4^2 - 24\, uw^4v^2p_1^2 + 2\, up_1^6w^3v - 15\, up_4^6v^2p_3^4$$
$$- 77\, up_4^4v^3p_3^4 + 32\, uv^2w^4p_4^2 - 84\, uv^3p_3^6p_4^2 - 104\, uv^3p_3^4w^2 + 32\, uv^2p_3^2w^4 - 28\, uv^2p_3^6p_4^4 + 28\, up_4^6p_1^2p_3^6$$
$$+ 15\, up_3^4p_4^8p_1^2 + 100\, up_4^4p_1^2v^2p_3^4 - 9\, up_4^2p_3^2p_1^6w^2 - 24\, up_4^4p_1^4p_3^2w^2 + 28\, up_4^2p_3^6p_1^2v^2$$
$$- 208\, uv^2p_3^4w^2p_1^2 - 304\, up_4^2p_3^2p_1^2v^2w^2 - 21\, up_4^4p_1^2wv^2p_3^2 - 3\, up_1^6p_4^4wp_3^2 + 3\, up_4^6p_1^4wp_3^2$$
$$+ 28\, up_3^4p_1^4p_4^2w^2 - 32\, up_4^4v^2p_3^2w^2 - 16\, uv^2p_3^4p_4^2w^2 + 12\, up_4^2v^2p_1^2w^3 - 63\, up_1^4p_4^2wv^2p_3^2$$
$$+ 84\, up_4^4p_1^2p_3^6v + 16\, up_4^2p_1^2p_3^2w^4 - 44\, up_4^4p_1^2p_3^4w^2 - 76\, up_4^2p_1^2p_3^4vw^2 + 84\, v^3wp_3^6p_1^2 + 8\, p_1^6v^2p_3^4p_4^2$$
$$+ 16\, p_4^4u^6 - 16\, p_1^4w^3u^3 + 16\, p_4^4u^3w^3 - 8\, w^5p_1^6 - 16\, v^6p_3^4 + 7\, w^3p_1^2vp_4^4p_3^2 + 36\, w^3p_1^2v^2p_4^2p_3^2$$
$$- 21\, p_1^6vw^2p_3^2p_4^2 + 28\, w^3p_1^4vup_4^2 - 16\, v^3w^5 - 8\, w^3p_1^4vup_3^2 + 44\, w^2v^2u^2p_1^2p_4^2 + 24\, p_1^4p_4^2w^4u$$
$$- 232\, w^2v^2u^2p_3^2p_1^2 + 96\, p_4^4p_1^4p_3^4vu + 24\, p_4^4p_3^2p_1^4u^2w - 16\, w^5p_1^4p_3^2 + 34\, p_4^4p_3^2p_1^2u^2v^2$$
$$- 78\, p_4^4p_3^2p_1^2u^3v - 38\, p_4^2p_3^2p_1^4w^2v^2 + 96\, p_4^2p_3^2p_1^4w^2u^2 - 68\, p_4^2p_3^4p_1^4v^2w - 24\, p_4^2p_3^4p_1^2v^3u$$
$$+ 30\, p_1^4u^2p_4^2w^2v - 16\, v^4w^4 + 32\, p_4^2p_3^4p_1^2v^2u^2 - 16\, wvp_3^2p_4^2u^4 - 194\, w^2p_3^2p_1^2v^3u + 72\, uv^3wp_3^4p_1^2$$
$$- 48\, w^3vp_3^2p_4^2u^2 - 74\, v^4p_3^4up_1^2 + 44\, w^3vp_3^4p_4^2p_1^2 + 16\, p_3^4v^2p_1^2w^3 - 20\, p_4^4u^3wvp_1^2$$
$$+ 24\, p_4^2u^3w^2p_1^2v - 16\, w^6p_1^4 + 8\, u^3p_4^6w^2 - 80\, w^2vp_3^2p_4^2u^3 - 32\, p_1^4p_3^2uw^4 - 28\, p_3^6v^2p_1^2p_4^2w$$
$$- 15\, p_4^6p_1^4p_3^4w + 16\, p_3^2vup_1^2w^4 - 8\, v^4p_1^4p_3^4 - 28\, wvp_3^6p_1^2p_4^4 + 80\, p_3^2vu^2p_1^2w^3 + 98\, u^2v^3wp_1^2p_3^2$$
$$+ 6\, p_1^6vw^4 + 52\, p_4^2v^3p_3^4w^2 - 8\, p_4^4v^4p_3^4 + 3\, p_4^4p_1^4w^4 + 60\, p_4^2vp_3^2w^4p_1^2 - 48\, p_4^4vp_3^4w^2p_1^2$$
$$- 27\, v^2wp_1^2p_3^4p_4^4 + 36\, p_4^2v^4p_3^2w^2 + 15\, p_4^4w^3p_1^4p_3^2 + 10\, p_4^2v^3p_3^2w^3 - 3\, p_4^6p_1^4w^2p_3^2 - 8\, p_4^2v^2p_3^2w^3u$$
$$+ 11\, p_4^4v^3p_3^4w + p_1^8w^2vp_3^2 + 24\, u^4v^3p_3^2 - 8\, u^4v^3p_4^2 - 3\, u^4p_4^4p_1^4 - 2\, u^4p_4^2p_1^4v - 15\, u^3p_1^4p_4^4p_3^2$$
$$+ 24\, u^3v^4p_3^2 - p_1^6v^3p_3^4 + 4\, p_1^6vp_4^2p_3^6 + 20\, p_1^2v^4p_3^2uw$$
$$- 11\, p_3^4p_4^6p_1^2wv - 28\, p_4^2v^4p_3^2uw + 16\, p_4^2p_1^2w^5u - 28\, v^3p_3^4p_1^2u^2$$
$$- 23\, u^3p_4^2p_3^2p_1^4v + 234\, p_4^2u^2v^3p_3^2w - 90\, p_4^2uv^3p_3^2w^2 + 60\, w^3p_1^4vp_3^4 + 68\, w^4p_1^4vp_3^2 + 4\, p_1^6vw^2p_3^4 + 16\, u^6vp_4^2$$
$$+ 8\, u^5v^2p_4^2 - 40\, p_3^2v^3w^4 + 16\, v^4p_3^4uw + 14\, v^3p_1^2p_3^2u^3 + p_1^8p_4^2p_3^4v + 104\, v^4p_3^4w^2$$
$$+ 64\, w^4v^2u^2 - 48\, w^5p_1^4u - 24\, p_4^6u^4p_3^2 + 48\, p_4^4u^5w + 48\, p_4^4u^4w^2 - 64\, p_4^2p_3^4p_1^2wv^2u$$
$$+ 56\, p_4^4p_3^4p_1^2wvu + 40\, p_4^2p_3^2p_1^2w^2vu^2 - 8\, p_4^2p_3^2p_1^2w^3vu - 248\, w^3p_1^2v^2up_3^2 + 96\, w^4p_1^2vu^2 - 16\, w^2v^3u^2p_3^2$$
$$- 40\, p_4^4p_1^4p_3^4w^2 - 16\, v^4p_3^6p_4^2 + 248\, w^2v^2u^2p_4^2p_3^2 + 56\, w^4p_1^4p_4^2p_3^2 - 76\, p_4^4u^4p_3^2v + 40\, p_4^6p_3^4p_1^2u^2$$
$$+ 16\, p_4^4p_3^6p_1^2v^2 - 92\, p_4^2u^2p_3^4v^3 - 24\, p_4^4u^2p_3^4v^2 - 52\, p_3^2p_1^2vp_4^2u^4 + 2\, p_3^2p_1^2v^3p_4^2u^2$$
$$- 52\, p_3^2p_1^2v^2p_4^2u^3 + 17\, p_4^4p_3^4p_1^2v^3 + 18\, w^4p_1^4vp_4^2 + 3\, p_4^6p_3^4p_1^4v + 63\, p_1^6vw^3p_3^2, \tag{G.3.7}$$

$$N_{V4} = \frac{Num_{N_{V4}}}{Den_{N_{V4}}},$$
$$Den_{N_{V4}} = 64\,(p_4+p_3)^2\left(-p_4^2p_3^2+w^2\right)\left(v^2p_3^2-p_4^2p_1^2p_3^2+p_4^2u^2+w^2p_1^2-2\,wvu\right)^2$$
$$Num_{N_{V4}} = 104\, p_4^4w^4vu^2 + 55\, p_3^2p_4^4w^4p_1^4 - 38\, p_3^4p_4^6p_1^4w^2 - 20\, v^2p_3^8p_4^4p_1^2 + 24\, p_4^2p_3^2v^3w^4$$
$$+ 96\, u^3vp_4^2w^4 + 144\, p_3^2w^4vp_4^2u^2 - 48\, p_3^2w^4v^2p_4^2u - 2\, p_4^2p_3^2p_1^4w^3vu - 4\, w^4p_1^4vup_4^2$$
$$+ 2\, p_4^6p_3^2p_1^4wvu + 4\, p_4^4p_3^2p_1^4w^2vu + 2\, p_4^4p_3^4p_1^4wvu + 8\, p_3^8p_4^6p_1^4 - 8\, w^6p_1^4p_3^2$$
$$+ 88\, p_3^4v^3w^4 - 108\, p_3^6v^3w^2p_4^2 - 8\, p_3^4v^2w^4p_1^2 + 2\, p_4^4p_1^6p_3^4w^2$$

$$
\begin{aligned}
&- {p_4}^6{p_1}^4{p_3}^4u^2 - 24\,{p_3}^6{p_4}^4{p_1}^4w^2 + 28\,{p_3}^6v^2w^2{p_4}^2{p_1}^2 - 31\,{p_4}^4w{p_3}^6v^3 + 16\,{p_4}^4w^3{p_3}^4{p_1}^4\\
&- 2\,{p_4}^6{p_1}^6{p_3}^4w - w^4{p_1}^6{p_4}^4 - 2\,w^5{p_1}^6{p_4}^2 - {p_4}^6{p_1}^6{p_3}^6 - {p_4}^8{p_1}^6{p_3}^4\\
&+ 2\,w^4{p_1}^4v^2{p_4}^2 + 2\,w^4{p_1}^4v^2{p_3}^2 - w^4{p_1}^6{p_3}^2{p_4}^2 + 31\,{p_4}^6w{p_3}^6v{p_1}^2 - 8\,{p_4}^6w{p_3}^6{p_1}^4\\
&+ 24\,{p_3}^4{p_4}^2{p_1}^4w^4 + {p_4}^6{p_1}^4{p_3}^4v^2 - 24\,{p_4}^2v^4{p_3}^6w + 14\,{p_3}^4v^3w^3{p_1}^2\\
&+ 12\,{p_3}^2vw^5{p_1}^4 - 4\,{p_4}^2v^2w^5{p_1}^2 + 32\,{p_4}^4w{p_3}^6v^2{p_1}^2 - 22\,{p_1}^2v^3w^3{p_3}^2{p_4}^2 + 4\,{p_4}^4v^3{p_3}^8\\
&+ 24\,v^3{p_4}^2w^5 + 2\,{p_4}^4{p_1}^4{p_3}^4v^2w + {p_4}^4{p_1}^4{p_3}^6v^2 - {p_4}^8{p_3}^2{p_1}^4u^2 - 2\,{p_4}^6{p_3}^2{p_1}^4u^2w\\
&- 32\,{p_1}^2v^2w^5u - 16\,v^2u^2{p_4}^2w^4 + 4\,{p_4}^4{p_3}^2{p_1}^6w^3 + 2\,{p_4}^6{p_3}^2{p_1}^6w^2 - 6\,w^3{p_3}^2{p_1}^4v^2{p_4}^2 - 10\,{p_1}^4{p_4}^8w{p_3}^4\\
&+ 4\,w^5{p_1}^4v^2 - 15\,{p_4}^2v^3{p_3}^6w{p_1}^2 - 27\,{p_3}^4vw^3{p_1}^4{p_4}^2 + 15\,{p_3}^6vw{p_1}^4{p_4}^4 - 3\,w^2{p_3}^4{p_1}^4v^2{p_4}^2\\
&+ {p_4}^6u^2w^2{p_1}^4 + 24\,w^5{p_1}^2v^3 + 2\,{p_4}^4u^2w^3{p_1}^4 + {p_4}^4u^2w^2{p_1}^4{p_3}^2 - 30\,{p_4}^4v^2w^3{p_1}^2{p_3}^2\\
&- 22\,v^3{p_4}^4w^3{p_3}^2 - 9\,v^3{p_4}^6w{p_3}^4 - 4\,{p_1}^4{p_4}^2vw^5 - 12\,{p_4}^4{p_1}^2vw^5 - 5\,v^3{p_4}^4w{p_3}^4{p_1}^2\\
&- 24\,{p_4}^2w^6{p_1}^4 + 34\,{p_4}^6{p_1}^2{p_3}^4v^2w + 9\,{p_4}^8{p_1}^2vw{p_3}^4 + 3\,{p_4}^6{p_1}^2vw^3{p_3}^2 - 8\,{p_4}^2w^5{p_1}^4{p_3}^2\\
&+ 5\,{p_4}^6{p_1}^4{p_3}^4vw + 48\,w^6v{p_3}^2{p_1}^2 + 8\,w^6{p_1}^2v{p_4}^2 + 32\,{p_3}^2v^3w^4u + 40\,{p_3}^2v^2w^5{p_1}^2\\
&+ 32\,w^7{p_1}^2v - 2\,w^3{p_1}^4vu{p_4}^4 - 62\,w^4{p_1}^4v{p_4}^2{p_3}^2 - 78\,w^4{p_1}^2v{p_4}^4{p_3}^2 - 150\,w^4{p_1}^2v^2{p_4}^2{p_3}^2\\
&+ 104\,w^2v{p_3}^6{p_4}^4{p_1}^2 - 148\,w^4v{p_3}^4{p_4}^2{p_1}^2 - 4\,{p_3}^8v{p_4}^6{p_1}^2 + 32\,{p_4}^4u{p_3}^6v^3\\
&+ 16\,{p_4}^6u^3{p_3}^4v - 19\,{p_4}^8{p_1}^4u{p_3}^4 - 19\,{p_4}^6{p_1}^4u{p_3}^6 + 32\,v^3u{p_4}^6{p_3}^4 + 24\,vw^6{p_1}^4\\
&+ 19\,{p_4}^4{p_1}^2{p_3}^6v^2u + 19\,{p_4}^6{p_1}^2{p_3}^4v^2u - 32\,{p_4}^6u{p_3}^6v{p_1}^2 + 4\,{p_4}^6v^4{p_3}^4 - 32\,{p_4}^8{p_1}^2vu{p_3}^4\\
&- 3\,{p_4}^8v^2{p_1}^2{p_3}^4 - 5\,{p_1}^4{p_4}^8v{p_3}^4 - 5\,{p_1}^2u^2{p_4}^8v{p_3}^2 - 5\,{p_1}^2u^2{p_4}^6v{p_3}^4\\
&+ 5\,v^3{p_1}^2{p_4}^6{p_3}^4 - 4\,{p_4}^8v^2u^2{p_3}^2 - 3\,{p_4}^4{p_3}^2{p_1}^4w^2v^2 - {p_4}^{10}{p_1}^2u^2{p_3}^2\\
&+ 16\,{p_4}^8u^3v{p_3}^2 + 9\,{p_1}^2u^3{p_4}^8{p_3}^2 + {p_4}^8v^3{p_3}^4 - {p_4}^{10}vu^2{p_3}^2 - 5\,{p_4}^8vu^2{p_3}^4 - 4\,{p_3}^6v{p_4}^6u^2\\
&+ 43\,{p_3}^6v^2{p_4}^6u + 28\,{p_3}^8v^2{p_4}^4u - 28\,{p_3}^8u{p_4}^6{p_1}^2 - 43\,{p_3}^6u{p_4}^8{p_1}^2 + 15\,{p_4}^8v^2{p_3}^4u\\
&+ 33\,{p_3}^4u^3{p_4}^8 + 20\,{p_3}^6u^3{p_4}^6 + 13\,{p_4}^{10}u^3{p_3}^2 - 15\,{p_4}^{10}u{p_3}^4{p_1}^2 + 9\,{p_4}^6u^3{p_1}^2{p_3}^4\\
&- 64\,w^6v^2u - 96\,w^5{p_3}^2v^2u - 48\,w^5{p_3}^2u{p_1}^2{p_4}^2 + 16\,w^5v{p_1}^2{p_3}^4 - 16\,w^5{p_4}^4u{p_1}^2\\
&- 32\,w^5{p_1}^4{p_4}^2u - 32\,w^4uv^3{p_4}^2 - 36\,w^4{p_4}^4v^2u + 6\,w^4{p_4}^6u{p_1}^2 + 56\,w^4u^2v{p_1}^2{p_4}^2 - 16\,w^4u{p_1}^4{p_3}^2{p_4}^2\\
&- 4\,w^4{p_4}^4vu{p_1}^2 - 16\,w^4{p_3}^2v^2u{p_1}^2 - 36\,w^4{p_4}^2v^2u{p_1}^2 - 14\,w^4{p_4}^4v^2{p_1}^2 - 32\,w^5uv^3\\
&+ 16\,w^4v^4{p_4}^2 + 4\,w^4v^3{p_4}^4 - 32\,w^3{p_4}^6u^3 + 24\,w^3{p_3}^6v^3 - 6\,w^2{p_4}^8u^3 + w{p_4}^{10}u^3\\
&- 16\,w^4{p_3}^4u{p_4}^2{p_1}^2 + 96\,w^4{p_4}^4u{p_1}^2{p_3}^2 + 12\,w^4{p_1}^2v^3{p_4}^2 - 10\,w^4{p_1}^4{p_4}^4u - 8\,w^4v{p_1}^4{p_4}^4\\
&- 32\,w^4uv^2{p_3}^4 - 8\,w^3{p_4}^4v^3u + 3\,w^3{p_4}^6{p_1}^4u - 44\,w^3{p_3}^6v{p_4}^2{p_1}^2 + 24\,w^3{p_4}^6vu^2 - 2\,w^3{p_4}^6v^2u\\
&+ 3\,w^3{p_4}^8u{p_1}^2 - 48\,w^3u^3{p_4}^4{p_3}^2 + 28\,w^3u^2{p_3}^2{p_1}^2{p_4}^2v - 10\,w^3{p_4}^2{p_1}^2{p_3}^2v^2u\\
&+ 89\,w^3{p_4}^6{p_1}^2u{p_3}^2 + 144\,w^3{p_4}^4{p_3}^4{p_1}^2u + 40\,w^3{p_4}^2v^2u{p_3}^4 + 48\,w^3u^2{p_4}^2{p_3}^4v\\
&- 60\,w^3{p_4}^4u^2v{p_3}^2 + 8\,w^3v^2u^2{p_4}^4 + 10\,w^3{p_4}^6{p_1}^2u^2 - 16\,w^3{p_4}^4u^3{p_1}^2 - 24\,w^2{p_4}^4v^4{p_3}^2\\
&+ 73\,w^3{p_4}^4{p_3}^2{p_1}^4u + 32\,w^3{p_1}^2u^2{p_4}^4v + 10\,w^3{p_4}^6{p_1}^2vu - 10\,w^3v^2{p_1}^2{p_4}^4u + 70\,w^3{p_4}^4uv^2{p_3}^2\\
&+ 144\,w^2{p_3}^4u{p_4}^4v^2 - 156\,w^2{p_4}^6vu^2{p_3}^2 + 42\,w^2{p_4}^6v^2{p_3}^2u - 2\,w^2{p_4}^8vu^2 - 6\,w^2{p_4}^6v^3{p_3}^2\\
&- 16\,w^2{p_3}^4u^3{p_4}^4 + 24\,w^2{p_4}^6u^3{p_3}^2 + w^2{p_4}^8{p_1}^2u^2 + 16\,w^2{p_3}^6v^2{p_4}^2u\\
&- 21\,w^2{p_3}^4u{p_4}^6{p_1}^2 + 44\,w^2{p_3}^6u{p_4}^4{p_1}^2 + 9\,w^2{p_4}^8u{p_3}^2{p_1}^2 - 8\,w^2{p_4}^6v^2u^2 - 6\,w^2{p_4}^6{p_1}^2u^3\\
&+ 51\,wu^3{p_3}^2{p_4}^8 + 64\,wu^3{p_4}^6{p_3}^4 + 4\,w{p_4}^8vu^3 + w{p_1}^2u^3{p_4}^8 - 28\,w{p_3}^8v^3{p_4}^2 + 32\,v^4w^5\\
&+ 36\,w^2vu{p_4}^6{p_1}^2{p_3}^2 + 2\,w^2v{p_1}^2{p_4}^6u^2 + 48\,w^2v^3u{p_3}^2{p_4}^4 + 17\,w^2{p_4}^6v^2{p_1}^2{p_3}^2\\
&+ 13\,w^2{p_4}^6{p_1}^4v{p_3}^2 - 18\,w^2{p_4}^4v^3{p_1}^2{p_3}^2 + 35\,w^2u{p_3}^4{p_1}^4{p_4}^4 + 26\,w^2{p_4}^4{p_1}^2uv^2{p_3}^2\\
&- 60\,w^2{p_4}^4u^2{p_1}^2v{p_3}^2 + 29\,w^2{p_1}^4u{p_4}^6{p_3}^2 - 8\,w^2u^3{p_1}^2{p_4}^4{p_3}^2 - 188\,w^2{p_4}^4u^2{p_3}^4v\\
&- 31\,w{p_4}^4{p_3}^4{p_1}^2u^2v + 73\,{p_4}^6{p_3}^2{p_1}^2u^2w^2 + 56\,uv^3w^3{p_3}^4 - 16\,{p_4}^4u^2w^4{p_1}^2 - 32\,{p_4}^2u^2w^5{p_1}^2\\
&+ 64\,{p_4}^4u^3w^3v - 64\,w^4v^2u^2{p_3}^2 + 48\,{p_4}^6u^4{p_3}^2w - 24\,{p_4}^4u^4w^2{p_3}^2 + 120\,{p_4}^4{p_3}^2{p_1}^2u^2w^3\\
&+ 88\,{p_4}^4{p_3}^4{p_1}^2u^2w^2 + 8\,uv^3w{p_3}^4{p_4}^4 - 80\,uv^3w^2{p_3}^4{p_4}^2 - 76\,uv^3w{p_3}^6{p_4}^2
\end{aligned}
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\begin{aligned}
&+ 64\,uv^3w^3{p_3}^2{p_4}^2 - 32\,{p_4}^2u^2w^4{p_1}^2{p_3}^2 + 80\,{p_4}^2u^3w^3v{p_3}^2 - 72\,{p_4}^6u^3wv{p_3}^2 - 160\,{p_4}^4u^3w^2v{p_3}^2 \\
&- 108\,{p_4}^4u^3wv{p_3}^4 - 64\,w^5v^2u^2 + 32\,{p_4}^8u^4{p_3}^2 + 32\,{p_4}^6u^4{p_3}^4 - 24\,{p_4}^6u^4w^2 - 32\,{p_4}^4u^4w^3 \\
&+ 10\,{p_4}^6{p_3}^4{p_1}^2wvu + 160\,{p_4}^4{p_3}^4{p_1}^2w^2vu + 84\,{p_4}^4{p_3}^6{p_1}^2wvu - 10\,{p_4}^4{p_3}^2{p_1}^2w^3vu \\
&- 160\,{p_4}^2{p_3}^2{p_1}^2w^4vu - 132\,{p_4}^2{p_3}^4{p_1}^2w^3vu + 48\,w^5{p_1}^2vu{p_3}^2 - 56\,w^2v^2u^2{p_4}^4{p_3}^2 \\
&+ 88\,w^3v^2u^2{p_4}^2{p_3}^2 + 48\,w^2v^2u^2{p_4}^2{p_3}^4 - 57\,{p_4}^8{p_3}^4{p_1}^2u^2 - 56\,{p_4}^6{p_3}^6{p_1}^2u^2 \\
&- 88\,{p_4}^6{p_3}^4{p_1}^2u^2w + 48\,{p_4}^6u^2{p_3}^4v^2 + 52\,{p_4}^4u^2{p_3}^6v^2 + 24\,{p_4}^4u^2{p_3}^4v^2w - 96\,{p_4}^2w^5v^2u \\
&+ 19\,wu^3{p_4}^6{p_1}^2{p_3}^2 - 111\,wu^2{p_3}^4v{p_4}^6 - 56\,wu^2v^2{p_4}^6{p_3}^2 - 10\,wu^2{p_4}^8{p_1}^2{p_3}^2 \\
&- 45\,wu^2{p_4}^8v{p_3}^2 - 41\,wu^2{p_4}^6{p_1}^2{p_3}^2v - 41\,wu{p_4}^6{p_3}^4{p_1}^4 + 83\,wu{p_4}^6v^2{p_3}^4 + 104\,wuv^2{p_3}^6{p_4}^4 \\
&- 96\,wu{p_4}^6{p_1}^2{p_3}^6 - 73\,wu{p_3}^4{p_4}^8{p_1}^2 + 51\,wu{p_4}^4{p_1}^2v^2{p_3}^4 - 3\,w{p_1}^4{p_4}^8u{p_3}^2 \\
&+ 20\,wv^3u{p_4}^6{p_3}^2 - 10\,w{p_4}^8{p_1}^2uv{p_3}^2 + 13\,w{p_1}^2u{p_4}^6v^2{p_3}^2 + 5\,w{p_4}^8v^2{p_3}^2u - 60\,w{p_3}^6v{p_4}^4u^2 \\
&+ 28\,w{p_3}^8v{p_4}^4{p_1}^2 - 3\,w{p_4}^{10}u{p_3}^2{p_1}^2 + 96\,{p_4}^2u^2w^5v - 32\,{p_4}^4u^3w^4 - 32\,{p_4}^2w^6{p_1}^2u + 32\,v^3w^6 \\
&+ 75\,{p_4}^6{p_1}^2w^2{p_3}^4v - 5\,{p_4}^8{p_1}^2{p_3}^6v - 5\,{p_4}^6{p_1}^4{p_3}^6v - 90\,{p_4}^2v^3{p_3}^4w^3 \\
&+ 40\,v^3w^4{p_1}^2{p_3}^2 - 24\,{p_4}^4v^4{p_3}^4w + 96\,w^5v^3{p_3}^2 + 48\,v^2w^6{p_1}^2 + 5\,{p_4}^6v^3{p_3}^6 + 16\,{p_4}^4v^4{p_3}^6 \\
&- 10\,{p_4}^4{p_1}^4w^5 + 64\,v^4w^4{p_3}^2 - 72\,{p_4}^2v^2{p_3}^4w^3{p_1}^2 + 45\,{p_4}^4v{p_3}^4w^3{p_1}^2 - 108\,{p_4}^2v{p_3}^2w^5{p_1}^2 \\
&- 48\,v^3w^2{p_1}^2{p_3}^4{p_4}^2 - 24\,{p_4}^2v^4{p_3}^2w^3 - 23\,{p_4}^6v^2{p_3}^6{p_1}^2 - 96\,{p_4}^2v^4{p_3}^4w^2 \\
&+ 43\,{p_4}^4w^2{p_1}^4v{p_3}^4 + 20\,{p_4}^6{p_1}^4w^3{p_3}^2 + 5\,{p_4}^4v^3{p_3}^6{p_1}^2 - 80\,{p_4}^4v^3{p_3}^4w^2 \\
&- {p_4}^4{p_1}^4w^3{p_3}^2v + 32\,vw^6{p_1}^2u + 125\,{p_4}^4v^2{p_3}^4w^2{p_1}^2 + 7\,{p_3}^6{p_4}^8{p_1}^4 + 12\,v^4{p_3}^8{p_4}^2 \\
&- {p_1}^2{p_4}^{10}v{p_3}^4 - 4\,{p_1}^2{p_4}^6vw^4 + 5\,{p_1}^2{p_4}^8v{p_3}^2w^2 + 2\,{p_1}^4{p_4}^8{p_3}^2w^2 - {p_1}^4{p_4}^{10}{p_3}^4 \\
&- {p_1}^4{p_4}^6w^4 + 24\,{p_3}^4v^4w^3 - 8\,{p_3}^6v^4w^2, \qquad \text{(G.3.8)}
\end{aligned}
$$

$$
\begin{aligned}
&M_1 = \frac{Num_{M_1}}{Den_{M_1}}, \\
&Den_{M_1} = 64\,\left(-{p_1}^2{p_4}^2 - 2\,{p_1}^2w - {p_3}^2{p_1}^2 + v^2 + 2\,vu + u^2\right)\left(v^2{p_3}^2 - {p_4}^2{p_1}^2{p_3}^2 + {p_4}^2u^2 + w^2{p_1}^2 - 2\,wvu\right)\times \\
&(p_4 + p_3)^2, \\
&Num_{M_1} = -\Big(- {p_1}^4{p_4}^4 - 2\,{p_1}^4w^2 - 12\,{p_1}^2v{p_4}^2{p_3}^2 - 7\,{p_1}^2{p_4}^4w - 9\,{p_3}^2{p_1}^2{p_4}^4 - 10\,{p_4}^2w^2{p_1}^2 + 12\,v^3{p_3}^2 \\
&+ 8\,v^3w + 8\,{p_3}^4v^2 - 3\,{p_1}^4w{p_4}^2 + {p_1}^2{p_4}^2v^2 - 8\,{p_3}^4{p_1}^2{p_4}^2 + 4\,{p_3}^2{p_4}^2u^2 - 12\,{p_3}^2u{p_1}^2{p_4}^2 \\
&+ 4\,v^3{p_4}^2 + 8\,w^2v^2 + 5\,{p_3}^2{p_4}^2vu + 16\,{p_3}^2v^2u - 8\,w^2{p_1}^2{p_3}^2 - 4\,{p_3}^2{p_1}^2wv - 4\,w{p_1}^2{p_3}^4 \\
&+ 6\,v^2{p_4}^2w - 25\,w{p_1}^2{p_4}^2{p_3}^2 + 4\,u{p_3}^4v + uv{p_4}^4 + 8\,{p_4}^2v^2u - 4\,{p_4}^4u{p_1}^2 - 12\,{p_1}^2{p_4}^2wv \\
&- 12\,{p_1}^2{p_4}^2wu + 12\,v{p_4}^2u^2 - 4\,{p_3}^2{p_1}^2uw + {p_1}^2{p_3}^2v^2 + 8\,{p_4}^2u^3 + 4\,{p_3}^2vu^2 + v^2{p_4}^4 + 4\,{p_3}^2vuw \\
&+ 20\,v^2w{p_3}^2 + 4\,u^2{p_4}^4 - {p_1}^2{p_4}^6 + 9\,v^2{p_4}^2{p_3}^2 - 4\,{p_1}^2{p_4}^4v + {p_1}^2{p_3}^2uv + 2\,{p_1}^2wuv - 8\,vuw^2 \\
&+ {p_1}^2{p_4}^2uv - 8\,vu^2w + 8\,{p_4}^2u^2w - 2\,v{p_4}^2uw - {p_1}^4{p_3}^2w - {p_1}^4{p_3}^2{p_4}^2 + 2\,{p_1}^2wv^2\Big){p_1}^2 \qquad \text{(G.3.9)}
\end{aligned}
$$

$$
\begin{aligned}
&M_2 = \frac{Num_{M_2}}{Den_{M_2}} \\
&Den_{M_2} = 64\,\left(-{p_4}^2{p_3}^2 + w^2\right)\left(v^2{p_3}^2 - {p_4}^2{p_1}^2{p_3}^2 + {p_4}^2u^2 + w^2{p_1}^2 - 2\,wvu\right)(p_4 + p_3)^2 \\
&Num_{M_2} = -\Big(- 5\,{p_4}^2wv{p_3}^2 + 4\,{p_3}^2{p_1}^2{p_4}^4 - 4\,{p_4}^2w^2{p_1}^2 - {p_4}^4vw + 4\,u{p_4}^2{p_3}^4 + {p_4}^6u - 6\,{p_4}^2w^2v + 4\,{p_3}^4v^2 \\
&- 4\,{p_3}^4{p_1}^2{p_4}^2 + 8\,{p_3}^2{p_4}^2u^2 + {p_3}^2u{p_1}^2{p_4}^2 - 8\,w^2v^2 + 4\,{p_3}^2{p_4}^2vu + 4\,w^2{p_1}^2{p_3}^2 \\
&- {p_3}^2{p_1}^2wv - 8\,vw^3 - 2\,vw^2{p_1}^2 - 4\,v^2{p_4}^2w + 4\,uv{p_4}^4 + {p_4}^4u{p_1}^2 - {p_1}^2{p_4}^2wv + 2\,{p_1}^2{p_4}^2wu - 12\,{p_3}^2vw^2 \\
&+ 5\,{p_3}^2{p_4}^4u + 12\,{p_3}^2{p_4}^2uw - 12\,{p_3}^2vuw - 4\,v^2w{p_3}^2 - 4\,v^2{p_4}^2{p_3}^2 + 6\,{p_4}^4uw - 8\,vuw^2 + 8\,{p_4}^2uw^2 \\
&+ 8\,{p_4}^2u^2w + 12\,v{p_4}^2uw - 4\,{p_3}^4vw\Big){p_4}^2 \qquad \text{(G.3.10)}
\end{aligned}
$$

$$M_3 = \frac{Num_{M_3}}{Den_{M_3}}$$

$$Den_{M_3} = 64\,(p_4+p_3)^2\left(-p_1^2p_4^2 - 2\,p_1^2w - p_3^2p_1^2 + v^2 + 2\,vu + u^2\right)\times$$
$$\left(v^2p_3^2 - p_4^2p_1^2p_3^2 + p_4^2u^2 + w^2p_1^2 - 2\,wvu\right)\left(-p_3^2p_1^2 - 2\,p_3^2v - p_4^2p_3^2 + u^2 + 2\,uw + w^2\right)$$

$$\begin{aligned}
Num_{M_3} ={}& -24\,v^3p_3^6p_1^2 + 7\,p_1^4wv^3p_4^2 - 24\,w^3p_1^4p_3^4 - 8\,vp_1^4p_4^6w + 14\,w^2p_1^2u^3p_4^2 + 20\,w^2uv^2p_3^4\\
&+ 24\,wu^2v^2p_3^4 + 9\,p_4^6vp_1^2p_3^4 - 16\,p_3^4p_4^2v^3w + 15\,p_1^6p_4^2vp_3^4 + 6\,p_4^4v^4w - 6\,p_4^4v^4p_3^2\\
&- p_4^8p_1^4w - 5\,p_4^6p_1^2wv^2 - 8\,v^3p_3^2u^3 - 16\,v^4p_3^2u^2 - u^2p_1^4p_3^2v^2 + 41\,wp_4^6p_1^2up_3^2\\
&- 18\,v^2p_3^4p_1^2u^2 - 16\,v^2p_3^4p_1^4w - 18\,p_3^4up_4^4v^2 - 33\,p_4^4vw^2p_1^2p_3^2 + 146\,v^2p_3^2p_1^2wu^2\\
&- 60\,v^3p_3^4p_1^2u - 8\,v^2p_3^6p_1^4 - 8\,v^5p_3^2u + 8\,p_4^2p_1^6p_3^6 - 69\,p_4^2p_1^2p_3^2u^3v\\
&+ 5\,p_4^2p_1^4p_3^4u^2 - 29\,p_4^2p_1^4p_3^2wu^2 - 21\,p_4^2p_1^2p_3^2u^4 + 16\,p_4^2p_1^6p_3^4w\\
&+ 60\,p_4^2p_1^4p_3^4uv + 16\,p_4^2u^5v - 24\,p_4^2u^3v^3 + 6\,p_4^2u^4p_1^2w + 4\,p_4^6u^2p_1^4 - 24\,p_4^2u^2v^4\\
&- 5\,p_4^4u^4p_1^2 - 14\,w^2p_1^6p_4^4 + 36\,w^2p_1^2u^3v - 21\,w^2p_1^4p_3^2u^2 - 10\,w^3p_1^4u^2 + 8\,w^2p_1^2u^4\\
&- 3\,w^2p_1^4p_3^2uv - 13\,w^2p_1^6p_3^4 - 15\,p_4^6vu^2p_3^2 - 37\,w^3p_1^6p_3^2 - 48\,wv^2u^4 - 64\,wv^3u^3 - 22\,p_1^6w^2vp_3^2\\
&- 16\,wv^5u - 16\,wvu^5 - 48\,wv^4u^2 + 8\,v^5p_4^2w - 8\,v^5p_4^2p_3^2 + 42\,wvu^3p_1^2p_3^2 + 26\,wvup_1^4p_3^4\\
&- v^3p_4^4p_3^2w + p_4^6p_1^2p_3^2vw + 20\,v^3p_4^2w^3 - 12\,u^2v^3p_1^2p_4^2 - uv^3p_1^4p_3^2 + 8\,u^4vp_1^2p_4^2\\
&- 10\,w^2p_4^4u^3 + 14\,up_4^2p_3^4p_1^6 - 14\,up_3^4p_1^4v^2 + 8\,up_4^6p_1^4v - 8\,vu^4p_4^2w + up_1^6p_4^6\\
&- p_1^6v^2p_3^4 - p_1^6wv^2p_3^2 + up_1^6vp_3^2p_4^2 - 6\,v^4up_4^2p_1^2 + up_1^6p_4^4v - up_1^4v^3p_4^2\\
&- 4\,p_3^2p_1^2v^3u^2 - 2\,p_1^6wp_4^6 - 16\,w^5p_1^2v - 35\,p_1^2p_4^4vw^3 + p_1^8p_4^2wp_3^2 + 18\,v^3p_3^2p_4^2w^2\\
&- 24\,vp_3^4w^3p_1^2 - 15\,p_1^4v^3p_3^4 - 46\,w^4p_1^2vp_4^2 - 22\,p_1^4u^3p_4^2p_3^2 - 12\,u^3v^2p_3^4\\
&- 32\,u^3v^2w^2 - 24\,up_3^4w^2p_1^4 - 12\,up_3^6v^2p_1^2 - 8\,u^2v^2p_3^6 + 8\,u^3vwp_3^4 + 8\,p_3^6vuwp_1^2\\
&+ 48\,u^2vp_3^4p_1^2w - 24\,uv^3p_3^6 + 8\,p_4^2u^6 - 46\,w^4p_1^4p_4^2 - 40\,w^4p_1^4p_3^2 - 10\,p_4^4u^4w - 17\,p_4^4u^4p_3^2\\
&+ 32\,w^3v^2u^2 + 5\,p_1^4up_4^4v^2 + 32\,p_3^2v^3w^3 + 120\,w^2v^2u^2p_3^2 - 8\,w^4p_1^2vu - 16\,w^5p_1^4\\
&+ 22\,p_4^2u^2w^3p_1^2 - 20\,p_4^2u^3w^2v - 12\,v^2wp_1^4p_4^4 - p_4^8u^3 - 35\,p_4^4u^2p_3^2vw - 2\,p_1^8w^2p_3^2\\
&- p_1^8p_4^4w - 7\,p_4^6v^2p_3^2u + p_4^8vp_3^2p_1^2 - 2\,p_4^8vu^2 - p_4^6v^3p_3^2 + 8\,v^3p_4^4w^2 - 10\,vp_1^2p_4^6w^2\\
&+ p_4^6wv^3 - p_4^8wp_1^2v + p_1^6v^2p_4^2w - 12\,p_3^6v^2p_4^2u + 18\,p_3^4up_4^6p_1^2 + 12\,p_3^6up_4^4p_1^2\\
&+ 7\,p_4^8up_3^2p_1^2 + 24\,p_4^2p_3^6p_1^4v - 4\,p_1^8w^3 + 29\,w^3p_4^4up_1^2 - 16\,w^2p_4^4vu^2\\
&- 8\,w^2p_3^2u^3p_4^2 - 18\,p_1^4u^2p_4^2p_3^2v + 36\,w^2p_3^2up_4^2v^2 + 37\,p_1^4wup_3^2v^2 - 8\,p_3^4u^3p_4^4\\
&- 9\,p_4^6u^3p_3^2 + 8\,u^5p_1^2p_4^2 - 14\,up_4^4v^4 - 11\,p_4^4uwp_3^2v^2 - 24\,v^5p_3^4 + 32\,v^4w^3 + 16\,v^3w^4 - 16\,p_3^6v^4\\
&+ 16\,v^5w^2 + p_4^8p_1^4p_3^2 + 5\,p_4^8p_1^2u^2 - 4\,p_1^8p_4^2w^2 + 2\,p_1^6w^2v^2 - p_1^6u^2p_4^4\\
&+ 8\,p_4^4p_3^6p_1^4 - 19\,p_4^6v^2u^2 + 9\,p_4^6p_3^4p_1^4 - 10\,p_4^6p_1^4w^2 + 14\,v^3p_1^4w^2 - 32\,p_3^4v^4w\\
&- 44\,p_1^4v^2w^3 + 40\,p_4^6p_1^2wvu + 14\,vp_1^4wup_4^4 + 66\,v^3p_1^2wup_4^2 + p_4^6p_1^2u^3 + 62\,v^2p_1^2wup_4^4\\
&+ 6\,p_1^6wvup_4^2 + 7\,vp_1^6wp_4^2p_3^2 + 35\,vup_4^6p_1^2p_3^2 - 14\,v^4p_1^2p_3^2p_4^2 + 12\,vp_1^2p_4^6u^2\\
&- 12\,v^2p_1^2u^2p_4^4 - 35\,v^3up_3^2p_4^4 - 54\,vp_1^4w^2p_4^4 - 4\,v^4up_4^2w + 7\,vp_1^6p_4^4p_3^2\\
&- 12\,p_4^4v^3uw - 7\,v^3p_1^4p_4^2p_3^2 - 6\,v^4wp_3^2p_4^2 - 32\,p_4^4p_1^2w^2v^2 + 5\,p_4^6v^2p_1^2p_3^2\\
&+ 6\,p_4^6p_1^4wu - 46\,v^2p_1^4w^2p_4^2 - p_1^6v^2p_4^2p_3^2 - 2\,vp_1^4u^2p_4^4 + 8\,p_4^6p_1^4vp_3^2\\
&+ 65\,p_4^4p_1^2uvw^2 - 56\,v^3u^2wp_4^2 + p_4^4u^2p_1^4p_3^2 + 64\,p_1^4p_4^2uvwp_3^2 + 132\,p_1^2v^3uw^2\\
&+ 62\,p_1^2v^2u^2wp_4^2 - 7\,p_1^4wv^3p_3^2 - 68\,p_3^4v^3up_4^2 - 35\,p_4^4p_1^4w^3 - 8\,v^5wp_3^2 + 66\,p_1^4v^2uw^2\\
&+ 24\,v^4w^2p_3^2 + 48\,v^4uw^2 + 12\,p_1^2v^3w^3 - 64\,uv^4p_3^4 + 2\,p_1^6p_4^6p_3^2 - 44\,v^3u^2p_4^4 + 28\,v^4p_1^2w^2\\
&+ 28\,p_4^2v^4w^2 - 38\,p_1^2v^4p_3^4 - 50\,v^2u^3p_4^4\\
&+ 10\,p_4^4p_3^4p_1^6 - 31\,p_1^6p_4^2w^3 + 21\,p_4^4p_1^2v^2p_3^4 - 23\,p_4^2p_3^2p_1^6w^2 - 27\,p_1^4p_4^2uvw^2\\
&+ p_1^4p_4^4u^2w - 21\,p_1^2vu^3p_4^4 - 115\,p_1^4p_4^2vw^3 + 51\,p_4^4p_1^2vwu^2 + 32\,p_4^2p_1^2p_3^4v^2u - 16\,v^3p_3^4wu
\end{aligned}$$

$$- 33\, {p_4}^4 {p_1}^4 {p_3}^2 w^2 + 7\, {p_1}^6 u {p_4}^4 {p_3}^2 + 132\, {p_4}^4 {p_3}^2 {p_1}^2 vwu + 6\, v^3 {p_1}^2 w^2 {p_4}^2$$
$$- 28\, v {p_1}^6 {p_4}^2 w^2 + 12\, v^2 {p_1}^4 {p_4}^4 {p_3}^2 + 8\, {p_4}^2 {p_3}^6 {p_1}^2 v^2 - 3\, {p_1}^6 wu {p_4}^4 - 16\, v^2 {p_3}^4 w^2 {p_1}^2$$
$$+ 43\, v^2 {p_1}^4 wu {p_4}^2 - 66\, v^2 {p_3}^4 {p_4}^2 u^2 - 22\, {p_1}^6 w^2 {p_4}^2 u + 68\, {p_4}^4 {p_3}^4 {p_1}^2 vu + 14\, {p_4}^6 u^2 {p_1}^2 {p_3}^2$$
$$+ 32\, u {p_3}^4 {p_1}^4 {p_4}^4 - 54\, {p_4}^2 {p_3}^2 {p_1}^2 v^2 w^2 + 16\, {p_4}^2 {p_3}^4 {p_1}^2 w v^2 + 40\, {p_4}^4 {p_3}^4 {p_1}^4 v$$
$$- 2\, v^3 {p_3}^2 {p_1}^2 w^2 + 28\, {p_4}^2 {p_1}^4 {p_3}^4 v^2 + 27\, {p_4}^6 {p_1}^2 u^2 w + 5\, {p_4}^4 {p_1}^2 w v^2 {p_3}^2$$
$$+ 32\, {p_4}^4 {p_1}^2 u v^2 {p_3}^2 + 2\, {p_1}^6 {p_4}^4 w {p_3}^2 + 162\, {p_4}^2 {p_1}^2 {p_3}^2 v^2 wu - 48\, v^3 {p_3}^4 {p_1}^2 w - 54\, v^2 u^2 {p_1}^2 {p_3}^2 {p_4}^2$$
$$+ {p_4}^6 {p_1}^4 w {p_3}^2 - 28\, {p_1}^2 v^3 u {p_3}^2 {p_4}^2 - 16\, {p_1}^2 v^3 {p_3}^4 {p_4}^2 - 14\, {p_1}^2 v^4 w {p_3}^2$$
$$- 19\, {p_1}^4 {p_4}^4 u w^2 - 95\, {p_4}^2 {p_3}^2 {p_1}^4 v w^2 - 46\, {p_1}^4 {p_3}^2 v^2 w^2 - 46\, v^4 u {p_3}^2 {p_4}^2 - 60\, {p_4}^4 v^2 u^2 w$$
$$- 77\, {p_4}^4 v^2 u^2 {p_3}^2 - {p_4}^4 u^2 {p_1}^2 v {p_3}^2 + 128\, {p_4}^2 v^2 u {p_1}^2 w^2 + {p_1}^8 {p_4}^4 {p_3}^2 + 74\, v^3 {p_1}^2 {p_3}^2 wu$$
$$+ 16\, w {p_3}^4 {p_1}^4 {p_4}^4 + 52\, {p_4}^2 v^3 u w^2 - {p_1}^4 {p_4}^2 u v^2 {p_3}^2 + 8\, v {p_1}^4 {p_4}^4 w {p_3}^2 - 4\, w^2 v^2 {p_4}^2 u^2$$
$$+ 20\, {p_1}^4 {p_4}^2 v w u^2 - 6\, {p_4}^2 v^3 u w {p_3}^2 - 5\, {p_4}^6 u^4 - 22\, {p_1}^6 w^4 + 24\, {p_1}^4 {p_4}^4 w u {p_3}^2 + 48\, {p_1}^4 {p_4}^2 v w {p_3}^4$$
$$- 6\, {p_1}^4 {p_4}^4 u^3 - 96\, v^3 {p_4}^2 u^2 {p_3}^2 + 8\, v^4 w u {p_3}^2 + 14\, {p_1}^4 u {p_4}^6 {p_3}^2 - 64\, {p_4}^2 v^2 {p_1}^2 w^3$$
$$+ 12\, {p_1}^4 {p_4}^2 w v^2 {p_3}^2 + 36\, {p_1}^4 {p_4}^4 u v {p_3}^2 - 30\, v^4 {p_3}^4 {p_4}^2 - 32\, w^4 v^2 {p_1}^2 + 80\, w^3 v^3 u - 20\, v {p_4}^4 u^4$$
$$- 70\, v w^4 {p_1}^4 + 32\, w^2 v^3 u^2 - 68\, {p_4}^2 u^3 w v^2 - 28\, {p_1}^6 w^3 v - 17\, {p_4}^6 v u^3 + 14\, {p_4}^2 w^3 {p_1}^2 v u - 27\, {p_1}^4 {p_4}^2 u^2 w^2$$
$$- 30\, {p_1}^4 w^3 v u - 94\, v^2 u^3 {p_3}^2 {p_4}^2 + 56\, v^3 u^2 {p_3}^2 w + 21\, {p_3}^4 {p_1}^2 {p_4}^4 u^2 - 29\, {p_3}^4 {p_1}^4 {p_4}^2 w^2$$
$$- 65\, {p_4}^4 v u^3 {p_3}^2 + 84\, v {p_4}^2 u^2 w^2 {p_1}^2 + 104\, w^3 {p_1}^2 v^2 u + 45\, {p_4}^4 u^2 w^2 {p_1}^2 - 52\, {p_4}^4 u^3 w v + 148\, {p_1}^2 w^2 v^2 u^2$$
$$+ 43\, w {p_1}^2 {p_4}^4 {p_3}^2 u^2 - 52\, w^3 v^2 {p_3}^2 {p_1}^2 + 128\, w^2 v^3 {p_3}^2 u - 85\, w^3 {p_1}^4 {p_4}^2 {p_3}^2$$
$$- 67\, u {p_1}^4 {p_4}^2 {p_3}^2 w^2 + 49\, u^2 {p_1}^2 {p_4}^2 {p_3}^2 w v + 5\, {p_4}^4 w {p_1}^2 u^3 - 67\, {p_4}^2 w^3 {p_1}^4 u - 19\, u^3 {p_1}^2 {p_4}^4 {p_3}^2$$
$$- 121\, {p_1}^4 v w^3 {p_3}^2 + 196\, v^2 u {p_3}^2 w^2 {p_1}^2 + 106\, {p_3}^4 {p_1}^2 {p_4}^2 w v u + 81\, {p_4}^2 v u {p_3}^2 w^2 {p_1}^2$$
$$- 50\, {p_4}^2 v^2 u^2 {p_3}^2 w - 7\, {p_1}^6 {p_4}^4 v w + 14\, v^4 {p_1}^2 w {p_4}^2 + {p_1}^8 {p_4}^2 {p_3}^4 - 16\, u^4 w {p_1}^2 v + 8\, u^3 {p_1}^4 w^2$$
$$- 12\, u^3 {p_1}^4 w {p_4}^2 - 12\, u^3 {p_3}^4 {p_1}^2 {p_4}^2 - 7\, u {p_4}^6 v^3 + 2\, u {p_4}^8 {p_1}^4 - u {p_4}^8 v^2 + u {p_4}^{10} {p_1}^2$$
$$- 24\, u w^3 {p_1}^6 + 40\, u^2 {p_3}^2 w {p_1}^4 v - 28\, u^3 {p_1}^2 w v^2 - 13\, u^2 {p_4}^6 v w - 2\, u^2 {p_1}^6 w {p_4}^2 - 24\, u^2 {p_1}^2 v^3 w - 2\, u^2 v^2 w {p_1}^4$$
$$+ 56\, u^2 v w^2 {p_1}^4 - 2\, u {p_1}^4 w v^3 + 7\, u {p_4}^8 {p_1}^2 v - 12\, u v^4 {p_1}^2 w + 12\, u {p_4}^2 {p_1}^4 {p_3}^6 + 12\, u v^2 {p_1}^2 {p_4}^6$$
$$+ 2\, u v^2 {p_4}^4 w^2 + 27\, u {p_4}^6 {p_1}^2 w^2 + 8\, u {p_1}^6 w^2 v - 20\, u {p_1}^6 w^2 {p_3}^2 - 5\, u {p_4}^6 w v^2 + 9\, u {p_4}^8 w {p_1}^2$$
$$+ 4\, u {p_1}^6 v {p_3}^2 w + 7\, u {p_1}^6 w {p_4}^2 {p_3}^2 - 4\, u^4 {p_4}^2 {p_3}^4 - 4\, {p_3}^6 w^2 {p_1}^4 - 4\, {p_3}^6 v w^2 {p_1}^2 - 8\, u^4 {p_4}^2 {p_3}^2 w$$
$$- 8\, {p_1}^2 w^3 u^2 {p_3}^2 - 4\, {p_1}^2 w^2 u^2 {p_3}^4 - 41\, v w^2 {p_1}^4 {p_3}^4 + 4\, {p_1}^2 {p_3}^6 u^2 {p_4}^2$$
$$- 4\, u^2 {p_4}^2 {p_3}^6 v - 8\, {p_3}^2 {p_1}^2 u w^4 - 4\, {p_3}^4 {p_1}^2 u w^3 + 16\, {p_3}^2 v u^2 w^3 + 8\, {p_3}^4 v u^2 w^2 - 4\, u^3 w {p_4}^2 {p_3}^4$$
$$+ 16\, {p_1}^2 {p_3}^4 u^2 {p_4}^2 w - 8\, u^2 {p_4}^2 {p_3}^4 v w + 20\, {p_3}^4 {p_1}^2 u w^2 {p_4}^2 + 8\, {p_3}^6 {p_1}^2 u w {p_4}^2 - 9\, {p_4}^4 v^3 {p_3}^4$$
$$+ 61\, {p_4}^4 w^2 {p_1}^2 u {p_3}^2 + 16\, {p_3}^4 {p_4}^4 {p_1}^2 w v - 85\, {p_3}^2 {p_4}^2 w^3 {p_1}^2 v - 8\, v^3 {p_3}^6 {p_4}^2 + 12\, v^3 {p_3}^4 w^2$$
$$+ 40\, v^2 {p_3}^2 u^3 w + 48\, v^2 {p_3}^2 u w^3 + 8\, {p_4}^4 {p_1}^2 {p_3}^6 v - 17\, {p_4}^4 u^2 {p_3}^4 v + 2\, {p_1}^2 {p_3}^2 u^3 v^2$$
$$+ 60\, v^2 {p_3}^4 {p_1}^2 u w - 52\, v^3 {p_3}^4 u^2 + 16\, {p_4}^2 u^5 w + 8\, {p_4}^2 u^4 w^2 + 16\, w^3 {p_1}^2 u^3 + 8\, w^4 {p_1}^2 u^2 - 32\, w^2 v u^4$$
$$- 16\, w^3 v u^3 - 20\, v^2 {p_3}^4 {p_4}^2 u w - 29\, {p_4}^2 {p_1}^2 {p_3}^4 v w^2 + 22\, {p_4}^2 {p_1}^4 {p_3}^4 u w + 11\, {p_4}^2 {p_1}^2 {p_3}^4 v u^2$$
$$+ 54\, {p_4}^4 {p_1}^2 {p_3}^4 u w - 37\, {p_4}^2 {p_1}^2 {p_3}^2 u^3 w + 7\, {p_4}^2 {p_1}^2 {p_3}^2 u w^3 - 36\, {p_4}^2 u^4 {p_3}^2 v - 25\, {p_4}^4 u^3 {p_3}^2 w$$
$$+ 10\, {p_4}^2 u^2 {p_3}^2 v w^2 - 40\, w^4 {p_1}^2 {p_3}^2 v - 69\, w^3 {p_1}^4 {p_3}^2 u + 102\, w^2 {p_1}^2 {p_3}^2 v u^2 - 58\, {p_4}^2 u^3 w v {p_3}^2$$
$$+ 7\, {p_4}^2 u^2 w^2 {p_1}^2 {p_3}^2 - 32\, u^3 {p_3}^4 v {p_4}^2 + 24\, u {p_4}^2 {p_1}^2 {p_3}^6 v - 6\, u^3 v^2 {p_1}^2 {p_4}^2$$
$$- u^2 v^2 {p_1}^4 {p_4}^2 - u^2 {p_4}^2 {p_3}^2 {p_1}^6 - 12\, w^3 {p_1}^2 v u {p_3}^2 - 8\, u v^5 {p_4}^2 + 16\, w^4 v^2 u - 34\, w^4 u {p_1}^4$$
$$+ 6\, w^4 u {p_1}^2 {p_4}^2 + 60\, w^3 {p_1}^2 u^2 v + 24\, w^3 {p_4}^2 v^2 u - 7\, w {p_4}^6 u^3 - 6\, {p_3}^2 {p_1}^2 v^4 u + 4\, w^3 v {p_4}^2 u^2 + 16\, w^2 u^3 v {p_3}^2$$
$$+ 16\, w^2 {p_3}^4 v u {p_1}^2, \qquad \text{(G.3.11)}$$

$$M_4 = \frac{Num_{M_4}}{Den_{M_4}}$$
$$Den_{M_4} = 64\ (p_4 + p_3)^2 \times \left(v^2 {p_3}^2 - {p_4}^2 {p_1}^2 {p_3}^2 + {p_4}^2 u^2 + w^2 {p_1}^2 - 2\, wvu\right) \times \left(-{p_4}^2 {p_3}^2 + w^2\right) \times$$

$$
\begin{aligned}
&\left(-{p_3}^2{p_1}^2-2\,{p_3}^2v-{p_4}^2{p_3}^2+u^2+2\,uw+w^2\right)\\
&Num_{M_4}=4\,w^3{p_1}^4{p_3}^4+4\,{p_3}^6v^2w{p_1}^2+20\,w^2{p_4}^4vu{p_3}^2-2\,w^2{p_1}^2u^3{p_4}^2-32\,w^2uv^2{p_3}^4-4\,wu^2v^2{p_3}^4\\
&+{p_3}^2{p_4}^6w^2{p_1}^2-8\,{p_3}^4{p_4}^2v^3w-10\,w{p_4}^6u^2{p_3}^2+2\,w{p_4}^6{p_1}^2u{p_3}^2+2\,w{p_4}^6{p_3}^2uv\\
&+3\,{p_4}^4vw^2{p_1}^2{p_3}^2-5\,{p_4}^6vu^2{p_3}^2-2\,{p_4}^4w^3{p_3}^2{p_1}^2-4\,{p_4}^4w{p_3}^4v^2+4\,{p_4}^6w{p_3}^4{p_1}^2\\
&-2\,w^2{p_4}^4u^3-16\,w^5{p_3}^2{p_1}^2-8\,w^5{p_1}^2v-8\,v^3{p_3}^2{p_4}^2w^2-2\,{p_4}^2w^5{p_1}^2-4\,{p_3}^4{p_1}^2w^4\\
&-8\,w^2v^2{p_3}^6-28\,v^2{p_3}^4w^3-4\,v{p_3}^4w^3{p_1}^2-4\,w^4{p_1}^2v{p_4}^2-w^4{p_1}^4{p_4}^2-w^4{p_1}^4{p_3}^2\\
&+16\,w^3v^2u^2-16\,{p_3}^2v^3w^3+8\,w^2v^2u^2{p_3}^2-4\,w^4{p_1}^2vu-2\,w^5{p_1}^4-2\,{p_4}^2u^2w^3{p_1}^2-8\,{p_4}^2u^3w^2v\\
&-14\,{p_4}^4u^2{p_3}^2vw-w^3{p_4}^4u{p_1}^2-2\,w^2{p_4}^4vu^2-24\,w^2{p_3}^2u^3{p_4}^2+32\,w^2{p_3}^2u{p_4}^2v^2\\
&-4\,{p_3}^4u^3{p_4}^4-4\,{p_4}^6u^3{p_3}^2+8\,{p_4}^4uw{p_3}^2v^2+{p_4}^4{p_1}^4{p_3}^2w^2+10\,{p_4}^4{p_3}^2{p_1}^2vwu\\
&-6\,v^2{p_3}^4w^2{p_1}^2-4\,v^2{p_3}^4{p_4}^2u^2-{p_4}^6u^2{p_1}^2{p_3}^2-6\,{p_4}^2{p_3}^2{p_1}^2v^2w^2-4\,{p_4}^2{p_3}^4{p_1}^2wv^2\\
&-{p_4}^2{p_3}^2{p_1}^4vw^2-4\,{p_4}^4v^2u^2{p_3}^2-{p_4}^4u^2{p_1}^2v{p_3}^2+4\,w{p_3}^4{p_1}^4{p_4}^4+8\,w^2v^2{p_4}^2u^2\\
&+2\,{p_1}^4{p_4}^4wu{p_3}^2-2\,{p_4}^2w^3{p_1}^2vu-{p_3}^4{p_1}^2{p_4}^4u^2+{p_3}^4{p_1}^4{p_4}^2w^2-4\,{p_4}^4vu^3{p_3}^2\\
&+2\,v{p_4}^2u^2w^2{p_1}^2-{p_4}^4u^2w^2{p_1}^2-4\,{p_4}^4u^3wv-2\,w{p_1}^2{p_4}^4{p_3}^2u^2-12\,w^3v^2{p_3}^2{p_1}^2\\
&-2\,w^3{p_1}^4{p_4}^2{p_3}^2+4\,u{p_1}^4{p_4}^2{p_3}^2w^2-2\,u^2{p_1}^2{p_4}^2{p_3}^2wv-{p_4}^4w{p_1}^2u^3\\
&-{p_4}^2w^3{p_1}^4u-2\,{p_1}^4vw^3{p_3}^2+10\,{p_3}^4{p_1}^2{p_4}^2wvu+24\,{p_4}^2vu{p_3}^2w^2{p_1}^2-4\,{p_4}^2v^2u^2{p_3}^2w\\
&-8\,{p_3}^6v{p_4}^2{p_1}^2w-4\,{p_3}^6vw^2{p_1}^2-8\,u^4{p_4}^2{p_3}^2w-8\,{p_1}^2w^3u^2{p_3}^2-vw^2{p_1}^4{p_3}^4\\
&-4\,u^2{p_4}^2{p_3}^6v-24\,{p_3}^2{p_1}^2uw^4-4\,{p_3}^4{p_1}^2uw^3+48\,{p_3}^2vu^2w^3+8\,{p_3}^4vu^2w^2-4\,u^3w{p_4}^2{p_3}^4\\
&+4\,{p_4}^2{p_3}^6v^2w+16\,{p_1}^2{p_3}^4u^2{p_4}^2w-4\,{p_4}^4u^2{p_3}^6-8\,{p_4}^2v^2{p_3}^2w^3+4\,u^2{p_4}^2{p_3}^4vw\\
&+36\,{p_3}^4{p_1}^2uw^2{p_4}^2+8\,{p_3}^6{p_1}^2uw{p_4}^2-4\,{p_4}^4{p_1}^2{p_3}^6w-8\,{p_4}^4u^2{p_3}^4w\\
&-16\,{p_4}^2u^2w^3{p_3}^2-4\,{p_4}^2u^2w^2{p_3}^4+32\,w^4vu{p_3}^2+8\,w^3vu{p_3}^4+20\,{p_4}^2w^3{p_1}^2{p_3}^4+4\,{p_4}^2w^2{p_1}^2{p_3}^6\\
&+16\,{p_4}^2v{p_3}^4w^2u+8\,{p_4}^2v{p_3}^6wu+8\,{p_4}^4w^2{p_1}^2u{p_3}^2+8\,{p_3}^4{p_4}^4{p_1}^2wv-6\,{p_3}^2{p_4}^2w^3{p_1}^2v\\
&-2\,{p_4}^4v^2{p_3}^2w^2-8\,v^3{p_3}^4w^2-16\,v^2{p_3}^2w^4-8\,w^6{p_1}^2-10\,v^2{p_3}^4{p_4}^2w^2-8\,v^2{p_3}^2uw^3\\
&+3\,{p_4}^2{p_1}^2{p_3}^2w^4+5\,{p_4}^4{p_1}^2{p_3}^4w^2-9\,{p_4}^4u^2{p_3}^4v-8\,{p_4}^2u^2w^4-5\,{p_4}^6u^2{p_3}^4\\
&-8\,{p_4}^2u^4w^2-16\,{p_4}^2u^3w^3-8\,w^4{p_1}^2u^2-16\,w^5{p_1}^2u+16\,w^5vu+16\,w^3vu^3+8\,v^2{p_3}^4{p_4}^2uw\\
&-{p_4}^2{p_1}^2{p_3}^4vw^2+2\,{p_4}^2{p_1}^4{p_3}^4uw-{p_4}^2{p_1}^2{p_3}^4vu^2+10\,{p_4}^4{p_1}^2{p_3}^4uw\\
&-{p_4}^2{p_1}^2{p_3}^2u^3w+19\,{p_4}^2{p_1}^2{p_3}^2uw^3-13\,{p_4}^4u^2{p_3}^2w^2-13\,{p_4}^4u^3{p_3}^2w-4\,w{p_1}^4{p_3}^6{p_4}^2\\
&+14\,{p_4}^2u^2{p_3}^2vw^2-12\,w^4{p_1}^2{p_3}^2v-w^3{p_1}^4{p_3}^2u-{p_4}^4w^4{p_1}^2+2\,w^2{p_1}^2{p_3}^2vu^2\\
&+26\,w^3vu{p_3}^2{p_4}^2+10\,wvu{p_3}^4{p_4}^4+32\,w^4vu^2-12\,{p_4}^2u^3wv{p_3}^2+15\,{p_4}^2u^2w^2{p_1}^2{p_3}^2-4\,u^3{p_3}^4v{p_4}^2\\
&-10\,w^3{p_1}^2vu{p_3}^2-{p_4}^8{p_3}^2u^2+16\,w^4v^2u-2\,w^4u{p_1}^4-2\,w^4u{p_1}^2{p_4}^2+4\,w^4{p_4}^2uv+4\,w^3{p_1}^2u^2v\\
&+2\,w^3uv{p_4}^4+8\,w^3{p_4}^2v^2u-2\,w^3u^2{p_4}^4-w^2{p_4}^6u^2-w{p_4}^6u^3+8\,w{p_3}^6v^3-4\,w^3v{p_4}^2u^2\\
&+16\,w^2u^3v{p_3}^2-12\,w^2{p_3}^4vu{p_1}^2,
\end{aligned}
\qquad (G.3.12)
$$

$$
\begin{aligned}
&M_5=\frac{Num_{M_5}}{Den_{M_5}},\\
&Den_{M_5}=64\,\left(v^2{p_3}^2-{p_4}^2{p_1}^2{p_3}^2+{p_4}^2u^2+w^2{p_1}^2-2\,wvu\right)\times\\
&\left(-{p_3}^2{p_1}^2-2\,{p_3}^2v-{p_4}^2{p_3}^2+u^2+2\,uw+w^2\right)\\
&Num_{M_5}=\left(2\,v+{p_1}^2+{p_4}^2\right)\left(vu-wv+u{p_4}^2-{p_1}^2w\right)\left(4\,u+4\,v+{p_4}^2+4\,w+{p_1}^2+4\,{p_3}^2\right),
\end{aligned}
\qquad (G.3.13)
$$

$$
M_6=0 \qquad (G.3.14)
$$

In the on shell limit, where $p_1^2=0, i=1\ldots 4$ and $p_1\cdot p_3=-p_1\cdot p_4-p_3\cdot p_4$, i.e. $u=-v-w$, all the coefficients multiplying triangles and those multiplying bubbles , vanish

$$
N_{V1}^{on\,shell}=\ldots N_{V4}^{on\,shell}=0,\quad M_1^{on\,shell}=\ldots M_6^{on\,shell}=0.
$$

The only non vanishing coefficient in the on shell limit is Q_V which gives

$$Q_V^{on\,shell} = -v \equiv -p_1 \cdot p_4.$$

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