Published for SISSA by Springer

Received: October 12, 2019
Revised: November 2, 2019
Accepted: November 17, 2019
Published: December 5, 2019

Off-diagonal Bethe Ansatz for the $D_3^{(1)}$ model

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Abstract: The exact solutions of the $D_3^{(1)}$ model (or the $so(6)$ quantum spin chain) with either periodic or general integrable open boundary conditions are obtained by using the off-diagonal Bethe Ansatz. From the fusion, the complete operator product identities are obtained, which are sufficient to enable us to determine spectrum of the system. Eigenvalues of the fused transfer matrices are constructed by the $T-Q$ relations for the periodic case and by the inhomogeneous $T-Q$ one for the non-diagonal boundary reflection case. The present method can be generalized to deal with the $D_n^{(1)}$ model directly.

Keywords: Bethe Ansatz, Lattice Integrable Models

ArXiv ePrint: 1909.08534

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https://doi.org/10.1007/JHEP12(2019)051

JHEP12(2019)051

Contents

1 Introduction

Quantum integrable models play an important role in the condensed matter, cold atoms, theoretical and mathematical physics [1, 2]. Recently, the models with general integrable non-diagonal boundary reflections attract many attentions. Due to the U(1)-symmetry broken, the traditional Bethe Ansatz does not work. People developed many interesting methods, such as q-Onsager algebra [3–6], separation of variables [7–11], modified algebraic Bethe Ansatz [13–16], off-diagonal Bethe Ansatz (ODBA) [17, 18] and others [19].

The exact solution of integrable models associated with high rank Lie algebra is a very interesting issue. The case that the models with periodic or diagonal boundary reflection, where the U(1)-charge is conserved, has been studied extensively. For example, Reshetikhin derived the energy spectrum of the periodic quantum integrable models associated with B_n, C_n, D_n and other Lie algebras by using the analytic Bethe Ansatz [20, 21]. Martins and Ramos studied this kind of models with periodic boundary condition by using the algebraic Bethe Ansatz [22]. Li, Shi and Yue investigated the open boundary cases where the reflection matrices only have the diagonal elements [23–25].

The ODBA is a universal method to treat the quantum integrable models with or without U(1)-symmetry, even for the high rank cases. The nested ODBA was proposed

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to study the spin chain associated with A_n Lie algebra with generic non-diagonal boundary reflections [26–28]. The ODBA also has been applied to the models associated with $A_2^{(2)}$ [29], B_2 [30] and C_2 [31] Lie algebras.

In this paper, we develop a nested ODBA method to approach the quantum integrable $D_3^{(1)}$ (simplest case of $D_n^{(1)}$) model with either periodic or non-diagonal open boundary conditions. We have obtained the complete operator product identities based on the fusion [32–36] and the eigenvalues of the system based on the (inhomogeneous) $T-Q$ relations.

The paper is organized as follows. In section 2, we review the R-matrix associated with the $D_3^{(1)}$ model, which is the starting point. In section 3, we construct the $D_3^{(1)}$ model with periodic boundary condition. The Hamiltonian, closed functional relations among the eigenvalues of the fused transfer matrices, and the spectrum of the system are obtained with fusion techniques. In section 4, we study the $D_3^{(1)}$ model with an off-diagonal open boundary reflection. By constructing some operator product identities, we derive the exact eigenvalues of the transfer matrix in terms of an inhomogeneous $T-Q$ relation. Section 5 is attributed to concluding remarks.

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2 The R-matrix

Throughout, $\mathbf{V}$ denotes a six-dimensional linear space which endows the fundamental vectorial representation of $so(6)$ (or the D_3) algebra, and let $\{|i\rangle, i=1,2,\cdots,6\}$ be an orthogonal basis of it. We adopt the standard notations: for any matrix $A \in \mathrm{End}(\mathbf{V})$, A_j is an embedding operator in the tensor space $\mathbf{V}\otimes\mathbf{V}\otimes\cdots$, which acts as A on the j-th space and as identity on the other factor spaces; for $B \in \mathrm{End}(\mathbf{V}\otimes\mathbf{V})$, B_{ij} is an embedding operator of B in the tensor space, which acts as identity on the factor spaces except for the i-th and j-th ones.

The R-matrix $R(u) \in \mathrm{End}(\mathbf{V} \otimes \mathbf{V})$ of the $D_3^{(1)}$ model is

$$R_{ij}^{vv} = \left(\begin{array}{cccccc|cccccc|cccccc|cccccc|cccccc|cccccc}
a&&&&& & &&&&& & &&&&& & &&&&& & &&&&& & &&&&& \\
&b&&&& & g&&&&& & &&&&& & &&&&& & &&&&& & &&&&& \\
&&b&&& & &&&&& & g&&&&& & &&&&& & &&&&& & &&&&& \\
&&&b&& & &&&&& & &&&&& & g&&&&& & &&&&& & &&&&& \\
&&&&b& & &&&&& & &&&&& & &&&&& & g&&&&& & &&&&& \\
&&&&&e & &&&&d& & &&&d&& & &&d&&& & &d&&&& & f&&&&& \\
\hline
&g&&&& & b&&&&& & &&&&& & &&&&& & &&&&& & &&&&& \\
&&&&& & &a&&&& & &&&&& & &&&&& & &&&&& & &&&&& \\
&&&&& & &&b&&& & &g&&&& & &&&&& & &&&&& & &&&&& \\
&&&&& & &&&b&& & &&&&& & &g&&&& & &&&&& & &&&&& \\
&&&&&d & &&&&e& & &&&d&& & &&d&&& & &f&&&& & d&&&&& \\
&&&&& & &&&&&b & &&&&& & &&&&& & &&&&& & &g&&&& \\
\hline
&&g&&& & &&&&& & b&&&&& & &&&&& & &&&&& & &&&&& \\
&&&&& & &&g&&& & &b&&&& & &&&&& & &&&&& & &&&&& \\
&&&&& & &&&&& & &&a&&& & &&&&& & &&&&& & &&&&& \\
&&&&&d & &&&&d& & &&&e&& & &&f&&& & &d&&&& & d&&&&& \\
&&&&& & &&&&& & &&&&b& & &&&&& & &&g&&& & &&&&& \\
&&&&& & &&&&& & &&&&&b & &&&&& & &&&&& & &&g&&& \\
\hline
&&&g&& & &&&&& & &&&&& & b&&&&& & &&&&& & &&&&& \\
&&&&& & &&&g&& & &&&&& & &b&&&& & &&&&& & &&&&& \\
&&&&&d & &&&&d& & &&&f&& & &&e&&& & &d&&&& & d&&&&& \\
&&&&& & &&&&& & &&&&& & &&&a&& & &&&&& & &&&&& \\
&&&&& & &&&&& & &&&&& & &&&&b& & &&&g&& & &&&&& \\
&&&&& & &&&&& & &&&&& & &&&&&b & &&&&& & &&&g&& \\
\hline
&&&&g& & &&&&& & &&&&& & &&&&& & b&&&&& & &&&&& \\
&&&&&d & &&&&f& & &&&d&& & &&d&&& & &e&&&& & d&&&&& \\
&&&&& & &&&&& & &&&&g& & &&&&& & &&b&&& & &&&&& \\
&&&&& & &&&&& & &&&&& & &&&&g& & &&&b&& & &&&&& \\
&&&&& & &&&&& & &&&&& & &&&&& & &&&&a& & &&&&& \\
&&&&& & &&&&& & &&&&& & &&&&& & &&&&&b & &&&&g& \\
\hline
&&&&&f & &&&&d& & &&&d&& & &&d&&& & &d&&&& & e&&&&& \\
&&&&& & &&&&&g & &&&&& & &&&&& & &&&&& & &b&&&& \\
&&&&& & &&&&& & &&&&&g & &&&&& & &&&&& & &&b&&& \\
&&&&& & &&&&& & &&&&& & &&&&&g & &&&&& & &&&b&& \\
&&&&& & &&&&& & &&&&& & &&&&& & &&&&&g & &&&&b& \\
&&&&& & &&&&& & &&&&& & &&&&& & &&&&& & &&&&&a
\end{array}\right), \tag{2.1}$$

where the matrix elements are

$$a = (u+1)(u+2), \qquad b = u(u+2), \qquad f = 2,$$
$$d = -u, \qquad e = u(u+1), \qquad g = u+2,$$

and u is the spectral parameter.

The R-matrix (2.1) satisfies the properties

$$\begin{aligned}
\text{regularity}: &\quad R_{12}^{vv}(0) = \rho_1(0)^{\frac{1}{2}}\mathcal{P}_{12}, \\
\text{unitarity}: &\quad R_{12}^{vv}(u)R_{21}^{vv}(-u) = \rho_1(u) = a(u)a(-u), \\
\text{crossing symmetry}: &\quad R_{12}^{vv}(u) = V_1\left\{R_{12}^{vv}(-u-2)\right\}^{t_2} V_1 = V_2\left\{R_{12}^{vv}(-u-2)\right\}^{t_1} V_2,
\end{aligned} \tag{2.2}$$

where $\mathcal{P}_{12}$ is the permutation operator with the matrix elements $[\mathcal{P}_{12}]_{kl}^{ij} = \delta_{il}\delta_{jk}$, t_i denotes the transposition in the i-th space, and $R_{21} = \mathcal{P}_{12}R_{12}\mathcal{P}_{12}$, and the crossing-matrix V is

$$V = \begin{pmatrix} &&&&&1 \\ &&&&1& \\ &&&1&& \\ &&1&&& \\ &1&&&& \\ 1&&&&& \end{pmatrix}, \quad V^2 = \mathrm{id}. \tag{2.3}$$

Combining the crossing-symmetry and the unitarity of the R-matrix, one can derive the crossing-unitarity relation

$$\text{crossing unitarity :} \qquad R^{vv}_{12}(u)^{t_1} R^{vv}_{21}(-u-4)^{t_1} = \rho_1(u+2). \tag{2.4}$$

The R-matrix (2.1) satisfies the Yang-Baxter equation

$$R^{vv}_{12}(u-v)R^{vv}_{13}(u)R^{vv}_{23}(v) = R^{vv}_{23}(v)R^{vv}_{13}(u)R^{vv}_{12}(u-v). \tag{2.5}$$

3 $D^{(1)}_3$ model with periodic boundary condition

For the periodic boundary condition, we introduce the monodromy matrix

$$T^v_0(u) = R^{vv}_{01}(u-\theta_1)R^{vv}_{02}(u-\theta_2)\cdots R^{vv}_{0N}(u-\theta_N), \tag{3.1}$$

where the index 0 indicates the auxiliary space and the other tensor space $\mathbf{V}^{\otimes N}$ is the physical or quantum space, N is the number of sites and $\{\theta_j\}$ are the inhomogeneous parameters. The monodromy matrix satisfies the Yang-Baxter relation

$$R^{vv}_{12}(u-v)T^v_1(u)T^v_2(v) = T^v_2(v)T^v_1(u)R^{vv}_{12}(u-v). \tag{3.2}$$

The transfer matrix is given by the trace of monodromy matrix in the auxiliary space

$$t^{(p)}(u) = tr_0 T^v_0(u). \tag{3.3}$$

From the Yang-Baxter relation (3.2), one can prove that the transfer matrices with different spectral parameters commute with each other, $[t^{(p)}(u), t^{(p)}(v)] = 0$. Therefore, $t^{(p)}(u)$ serves as the generating function of the conserved quantities of the system. The Hamiltonian is given in terms of the transfer matrix (3.3) as

$$H_p = \frac{\partial \ln t^{(p)}(u)}{\partial u}|_{u=0,\{\theta_j\}=0} = \sum_{k=1}^{N} H_{k\,k+1}, \tag{3.4}$$

where

$$H_{k\,k+1} = \mathcal{P}_{k\,k+1}\frac{\partial}{\partial u}R_{k\,k+1}(u)\,|_{u=0}\,, \tag{3.5}$$

and the periodic boundary condition reads

$$H_{N\,N+1} = H_{N\,1}. \tag{3.6}$$

3.1 The fusion

At some special points, the R-matrix (2.1) degenerates into the projector operators which enables us to do the fusion. For example, at the point of $u=-2$, we have

$$R^{vv}_{12}(-2) = P^{vv(1)}_{12}S^{(1)}_{12}, \tag{3.7}$$

where $S^{(1)}_{12}$ is a constant matrix $\in \mathrm{End}(\mathbf{V}\otimes\mathbf{V})$ omitted here, $P^{vv(1)}_{12}$ is the one-dimensional projector

$$P^{vv(1)}_{12} = |\psi_0\rangle\langle\psi_0|, \quad P^{vv(1)}_{21} = P^{vv(1)}_{12}, \tag{3.8}$$

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and the basis vector reads

$$|\psi_0\rangle = \frac{1}{\sqrt{6}}(|16\rangle + |25\rangle + |34\rangle + |43\rangle + |52\rangle + |61\rangle).$$

From the Yang-Baxter equation (2.5), we obtain the following fusion identities

$$\begin{aligned} P_{21}^{vv(1)} R_{13}^{vv}(u) R_{23}^{vv}(u-2) P_{21}^{vv(1)} &= a(u)e(u-2) P_{21}^{vv(1)}, \\ P_{12}^{vv(1)} R_{31}^{vv}(u) R_{32}^{vv}(u-2) P_{12}^{vv(1)} &= a(u)e(u-2) P_{12}^{vv(1)}. \end{aligned} \tag{3.9}$$

At the point of $u = -1$, we have

$$R_{12}^{vv}(-1) = P_{12}^{vv(16)} S_{12}^{(16)}, \tag{3.10}$$

where $S_{12}^{(16)}$ is a constant matrix omitted here and $P_{12}^{vv(16)}$ is a 16-dimensional projector

$$P_{12}^{vv(16)} = \sum_{i=1}^{16} |\phi_i^{(16)}\rangle\langle\phi_i^{(16)}|, \quad P_{21}^{vv(16)} = P_{12}^{vv(16)},$$

with the basis vectors:

$$\begin{aligned}
|\phi_1^{(16)}\rangle &= \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle), & |\phi_2^{(16)}\rangle &= \frac{1}{\sqrt{2}}(|13\rangle - |31\rangle), & |\phi_3^{(16)}\rangle &= \frac{1}{\sqrt{2}}(|14\rangle - |41\rangle), \\
|\phi_4^{(16)}\rangle &= \frac{1}{\sqrt{2}}(|15\rangle - |51\rangle), & |\phi_5^{(16)}\rangle &= \frac{1}{\sqrt{2}}(|16\rangle - |61\rangle), & |\phi_6^{(16)}\rangle &= \frac{1}{\sqrt{2}}(|23\rangle - |32\rangle), \\
|\phi_7^{(16)}\rangle &= \frac{1}{\sqrt{2}}(|24\rangle - |42\rangle), & |\phi_8^{(16)}\rangle &= \frac{1}{\sqrt{2}}(|25\rangle - |52\rangle), & |\phi_9^{(16)}\rangle &= \frac{1}{\sqrt{2}}(|26\rangle - |62\rangle), \\
|\phi_{10}^{(16)}\rangle &= \frac{1}{\sqrt{2}}(|34\rangle - |43\rangle), & |\phi_{11}^{(16)}\rangle &= \frac{1}{\sqrt{2}}(|35\rangle - |53\rangle), & |\phi_{12}^{(16)}\rangle &= \frac{1}{\sqrt{2}}(|36\rangle - |63\rangle), \\
|\phi_{13}^{(16)}\rangle &= \frac{1}{\sqrt{6}}(|34\rangle + |43\rangle & |\phi_{14}^{(16)}\rangle &= \frac{1}{\sqrt{2}}(|45\rangle - |54\rangle), \\
&\quad + |25\rangle + |52\rangle \\
&\quad + |16\rangle + |61\rangle), \\
|\phi_{15}^{(16)}\rangle &= \frac{1}{\sqrt{2}}(|46\rangle - |64\rangle), & |\phi_{16}^{(16)}\rangle &= \frac{1}{\sqrt{2}}(|56\rangle - |65\rangle).
\end{aligned}$$

From the Yang-Baxter equation (2.5) and the 16-dimensional projector $P_{12}^{vv(16)}$, we obtain the fusion identities

$$\begin{aligned} P_{21}^{vv(16)} R_{13}^{vv}(u) R_{23}^{vv}(u-1) P_{21}^{vv(16)} &= (u-1)(u+2) S_{1'2'} R_{1'3}^{s_+v}\left(u-\frac{1}{2}\right) R_{2'3}^{s_-v}\left(u-\frac{1}{2}\right) S_{1'2'}^{-1}, \\ P_{12}^{vv(16)} R_{31}^{vv}(u) R_{32}^{vv}(u-1) P_{12}^{vv(16)} &= (u-1)(u+2) S_{1'2'} R_{31'}^{vs_+}\left(u-\frac{1}{2}\right) R_{32'}^{vs_-}\left(u-\frac{1}{2}\right) S_{1'2'}^{-1}. \end{aligned} \tag{3.11}$$

In eq. (3.11), we take the fusion in the auxiliary spaces and obtain a 16-dimensional auxiliary space $\mathbf{V}_{\langle 12\rangle}$. We can show that the resulting 16-dimensional auxiliary space is exact a direct tensor-product of two 4-dimensional auxiliary spaces $\mathbf{V}^{(\mathbf{s}_+)}$ and $\mathbf{V}^{(\mathbf{s}_-)}$, which endows the (anti) spinor representation of $so(6)$ respectively. The matrix $S_{1'2'}$ is defined in the direct product space $\mathbf{V}^{(\mathbf{s}_+)}\otimes\mathbf{V}^{(\mathbf{s}_-)}$ with the matrix form

$$S_{1'2'}=\left(\begin{array}{cccc|cccc|cccc|cccc}
1 & & & & & & & & & & & & & & & \\
 & & & & -1 & & & & & & & & & & & \\
 & -1 & & & & & & & & & & & & & & \\
 & & & & & -1 & & & & & & & & & & \\
\hline
 & & & b_2 & & & -b_2 & & & -b_2 & & & b_2 & & & \\
 & & & & & & & & 1 & & & & & & & \\
 & & 1 & & & & & & & & & & & & & \\
 & & & b_2 & & & b_2 & & & b_2 & & & b_2 & & & \\
\hline
 & & & & & & & & & & 1 & & & & & \\
 & & & -b_2 & & & -b_2 & & & b_2 & & & b_2 & & & \\
 & & & & & & & -1 & & & & & & & & \\
 & & & & & & & & & & & -1 & & & & \\
\hline
 & & & -a_2 & & & a_2 & & & -a_2 & & & a_2 & & & \\
 & & & & & & & & & & & & & -1 & & \\
 & & & & & & & & & & & & & & -1 & \\
 & & & & & & & & & & & & & & & -1
\end{array}\right),$$

where

$$a_2=\frac{\sqrt{3}}{2},\qquad b_2=\frac{1}{2}.$$

In eq. (3.11), the matrices $R^{s\pm v}_{1'2}(u)$ are defined in the $\mathbf{V}^{(\mathbf{s}_\pm)}\otimes\mathbf{V}$ space and take the forms of

$$R^{s+v}_{1'2}=\left(\begin{array}{cccccc|cccccc|cccccc|cccccc}
a_1 & & & & & & & & & & & & & & & & & & & & & & & \\
 & a_1 & & & & & & & & & & & & & & & & & & & & & & \\
 & & a_1 & & & & & & & & & & & & & & & & & & & & & \\
 & & & b_1 & & & & -1 & & & & & -1 & & & & & & & & & & & \\
 & & & & b_1 & & & & 1 & & & & & & & & & & -1 & & & & & \\
 & & & & & b_1 & & & & & & & & & 1 & & & & & 1 & & & & \\
\hline
 & & & & & & a_1 & & & & & & & & & & & & & & & & & \\
 & & & -1 & & & & b_1 & & & & & -1 & & & & & & & & & & & \\
 & & & & 1 & & & & b_1 & & & & & & & & & & 1 & & & & & \\
 & & & & & & & & & a_1 & & & & & & & & & & & & & & \\
 & & & & & & & & & & a_1 & & & & & & & & & & & & & \\
 & & & & & & & & & & & b_1 & & & & & 1 & & & & & -1 & & \\
\hline
 & & & -1 & & & & -1 & & & & & b_1 & & & & & & & & & & & \\
 & & & & & & & & & & & & & a_1 & & & & & & & & & & \\
 & & & & & 1 & & & & & & & & & b_1 & & & & & -1 & & & & \\
 & & & & & & & & & & & & & & & a_1 & & & & & & & & \\
 & & & & & & & & & & & 1 & & & & & b_1 & & & & & 1 & & \\
 & & & & & & & & & & & & & & & & & a_1 & & & & & & \\
\hline
 & & & & -1 & & & & 1 & & & & & & & & & & b_1 & & & & & \\
 & & & & & 1 & & & & & & & & & -1 & & & & & b_1 & & & & \\
 & & & & & & & & & & & & & & & & & & & & a_1 & & & \\
 & & & & & & & & & & & -1 & & & & & 1 & & & & & b_1 & & \\
 & & & & & & & & & & & & & & & & & & & & & & a_1 & \\
 & & & & & & & & & & & & & & & & & & & & & & & a_1
\end{array}\right),$$

$$R^{s-v}_{1'2}=\left(\begin{array}{cccccc|cccccc|cccccc|cccccc}
a_1 & & & & & & & & & & & & & & & & & & & & & & & \\
 & a_1 & & & & & & & & & & & & & & & & & & & & & & \\
 & & b_1 & & & & & -1 & & & & & -1 & & & & & & & & & & & \\
 & & & a_1 & & & & & & & & & & & & & & & & & & & & \\
 & & & & b_1 & & & & & 1 & & & & & & & & & -1 & & & & & \\
 & & & & & b_1 & & & & & & & & & & 1 & & & & 1 & & & & \\
\hline
 & & & & & & a_1 & & & & & & & & & & & & & & & & & \\
 & & -1 & & & & & b_1 & & & & & -1 & & & & & & & & & & & \\
 & & & & & & & & a_1 & & & & & & & & & & & & & & & \\
 & & & & 1 & & & & & b_1 & & & & & & & & & 1 & & & & & \\
 & & & & & & & & & & a_1 & & & & & & & & & & & & & \\
 & & & & & & & & & & & b_1 & & & & & 1 & & & & -1 & & & \\
\hline
 & & -1 & & & & & -1 & & & & & b_1 & & & & & & & & & & & \\
 & & & & & & & & & & & & & a_1 & & & & & & & & & & \\
 & & & & & & & & & & & & & & a_1 & & & & & & & & & \\
 & & & & & 1 & & & & & & & & & & b_1 & & & & -1 & & & & \\
 & & & & & & & & & & & 1 & & & & & b_1 & & & & 1 & & & \\
 & & & & & & & & & & & & & & & & & a_1 & & & & & & \\
\hline
 & & & & -1 & & & & & 1 & & & & & & & & & b_1 & & & & & \\
 & & & & & 1 & & & & & & & & & & 1 & & & & b_1 & & & & \\
 & & & & & & & & & & & -1 & & & & & 1 & & & & b_1 & & & \\
 & & & & & & & & & & & & & & & & & & & & & a_1 & & \\
 & & & & & & & & & & & & & & & & & & & & & & a_1 & \\
 & & & & & & & & & & & & & & & & & & & & & & & a_1
\end{array}\right),$$

where the matrix elements are

$$a_1 = u + \frac{3}{2}, \quad b_1 = u + \frac{1}{2}.$$

The matrices $R^{s\pm v}_{1'2}(u)$ have the properties

$$\begin{aligned} \text{unitarity}: &\qquad R^{s\pm v}_{1'2}(u) R^{vs\pm}_{21'}(-u) = \rho_s(u) = a_1(u)a_1(-u), \\ \text{crossing unitarity}: &\qquad R^{s\pm v}_{1'2}(u)^{t_{1'}} R^{vs\pm}_{21'}(-u-4)^{t_{1'}} = \rho_s(u+2), \end{aligned} \tag{3.12}$$

and satisfy the Yang-Baxter equations

$$R^{s\pm v}_{1'2}(u_1-u_2)R^{s\pm v}_{1'3}(u_1-u_3)R^{vv}_{23}(u_2-u_3) = R^{vv}_{23}(u_2-u_3)R^{s\pm v}_{1'3}(u_1-u_3)R^{s\pm v}_{1'2}(u_1-u_2). \tag{3.13}$$

Now, we consider the degenerate case of matrix $R^{s+v}_{1'2}(u)$. At the point of $u = -\frac{3}{2}$, we have

$$R^{s+v}_{1'2}\left(-\frac{3}{2}\right) = P^{(+)}_{1'2} S^{(+)}_{1'2}, \tag{3.14}$$

where $S^{(+)}_{1'2}$ is a constant matrix omitted here and $P^{(+)}_{1'2}$ is a 4-dimensional projector

$$P^{(+)}_{1'2} = \sum_{i=1}^{4} |\phi^{(+)}_i\rangle\langle\phi^{(+)}_i|,$$

with the basis vectors

$$\begin{aligned} |\phi^{(+)}_1\rangle &= \frac{1}{\sqrt{3}}(|14\rangle + |22\rangle + |31\rangle), & |\phi^{(+)}_2\rangle &= \frac{1}{\sqrt{3}}(|15\rangle - |23\rangle + |41\rangle), \\ |\phi^{(+)}_3\rangle &= \frac{1}{\sqrt{3}}(|16\rangle - |33\rangle - |42\rangle), & |\phi^{(+)}_4\rangle &= \frac{1}{\sqrt{3}}(|26\rangle - |35\rangle + |44\rangle). \end{aligned}$$

Taking the fusion by using the projector $P^{(+)}_{1'2}$, we obtain

$$\begin{aligned} P^{(+)}_{1'2} R^{vv}_{23}(u) R^{s+v}_{1'3}\left(u-\frac{3}{2}\right) P^{(+)}_{1'2} &= (u-1)(u+2) R^{s-v}_{\langle 1'2\rangle 3}\left(u-\frac{1}{2}\right), \\ P^{(+)}_{21'} R^{vv}_{32}(u) R^{vs+}_{31'}\left(u-\frac{3}{2}\right) P^{(+)}_{21'} &= (u-1)(u+2) R^{vs-}_{3\langle 1'2\rangle}\left(u-\frac{1}{2}\right). \end{aligned} \tag{3.15}$$

We note that the dimension of the fused auxiliary space $\mathbf{V}_{\langle 1'2\rangle}$ is 4.

At the point of $u = -\frac{3}{2}$, we also have

$$R^{s-v}_{1'2}\left(-\frac{3}{2}\right) = P^{(-)}_{1'2} S^{(-)}_{1'2}, \tag{3.16}$$

where $P^{(-)}_{1'2}$ is a 4-dimensional projector

$$P^{(-)}_{1'2} = \sum_{i=1}^{4} |\phi^{(-)}_i\rangle\langle\phi^{(-)}_i|,$$

with the basis vectors

$$|\phi_1^{(-)}\rangle = \frac{1}{\sqrt{3}}(|13\rangle + |22\rangle + |31\rangle), \qquad |\phi_2^{(-)}\rangle = \frac{1}{\sqrt{3}}(|15\rangle - |24\rangle + |41\rangle),$$
$$|\phi_3^{(-)}\rangle = \frac{1}{\sqrt{3}}(|16\rangle - |34\rangle - |42\rangle), \qquad |\phi_4^{(-)}\rangle = \frac{1}{\sqrt{3}}(|26\rangle - |35\rangle + |43\rangle),$$

and the $S_{1'2}^{(-)}$ is a constant matrix omitted here. Taking the fusion by using the projector $P_{1'2}^{(-)}$, we obtain

$$\begin{aligned} P_{1'2}^{(-)} R_{23}^{vv}(u) R_{1'3}^{s-v}\left(u-\frac{3}{2}\right) P_{1'2}^{(-)} &= (u-1)(u+2) R_{\langle 1'2\rangle 3}^{s+v}\left(u-\frac{1}{2}\right), \\ P_{21'}^{(-)} R_{32}^{vv}(u) R_{31'}^{vs-}\left(u-\frac{3}{2}\right) P_{21'}^{(-)} &= (u-1)(u+2) R_{3\langle 1'2\rangle}^{vs+}\left(u-\frac{1}{2}\right). \end{aligned} \tag{3.17}$$

3.2 The operator product identities

From the fused R-matrices $R_{0'j}^{s\pm v}$, we can define the fused monodromy matrices

$$T_{0'}^{\pm}(u) = R_{0'1}^{s\pm v}(u-\theta_1) R_{0'2}^{s\pm v}(u-\theta_2) \cdots R_{0'N}^{s\pm v}(u-\theta_N), \tag{3.18}$$

which satisfy the Yang-Baxter relations

$$R_{00'}^{vs\pm}(u-v) T_0(u) T_{0'}^{\pm}(v) = T_{0'}^{\pm}(v) T_0(u) R_{00'}^{vs\pm}(u-v). \tag{3.19}$$

Taking the partial trace in the auxiliary space, we obtain the fused transfer matrices

$$t_{\pm}^{(p)}(u) = tr_{0'} T_{0'}^{\pm}(u). \tag{3.20}$$

From the Yang-Baxter relations (3.2) and (3.19) at certain points and using the properties of projectors, we obtain

$$\begin{aligned} T_1(\theta_j)\, T_2(\theta_j-2) &= P_{21}^{vv(1)}\, T_1(\theta_j)\, T_2(\theta_j-2), \\ T_1(\theta_j)\, T_2(\theta_j-1) &= P_{21}^{vv(16)}\, T_1(\theta_j)\, T_2(\theta_j-1), \\ T_2(\theta_j)\, T_{1'}^{+}\left(\theta_j-\frac{3}{2}\right) &= P_{1'2}^{(+)}\, T_2(\theta_j)\, T_{1'}^{+}\left(\theta_j-\frac{3}{2}\right), \\ T_2(\theta_j)\, T_{1'}^{-}\left(\theta_j-\frac{3}{2}\right) &= P_{1'2}^{(-)}\, T_2(\theta_j)\, T_{1'}^{-}\left(\theta_j-\frac{3}{2}\right). \end{aligned} \tag{3.21}$$

By using the fusion identities (3.9), (3.11), (3.15) and (3.17), we have following fusion identities

$$\begin{aligned} P_{21}^{vv(1)} T_1(u)\, T_2(u-2) P_{21}^{vv(1)} &= P_{21}^{vv(1)} \prod_{i=1}^{N} a(u-\theta_i) e(u-\theta_i-2) \times \mathrm{id}, \\ P_{21}^{vv(16)} T_1(u)\, T_2(u-1) P_{21}^{vv(16)} &= \prod_{i=1}^{N} \tilde{\rho}_0(u-\theta_i) S_{1'2'}\, T_{1'}^{+}\left(u-\frac{1}{2}\right) T_{2'}^{-}\left(u-\frac{1}{2}\right) S_{1'2'}^{-1}, \\ P_{1'2}^{(+)} T_2(u)\, T_{1'}^{+}\left(u-\frac{3}{2}\right) P_{1'2}^{(+)} &= \prod_{i=1}^{N} \tilde{\rho}_0(u-\theta_i)\, T_{\langle 1'2\rangle}^{-}\left(u-\frac{1}{2}\right), \\ P_{1'2}^{(-)} T_2(u)\, T_{1'}^{-}\left(u-\frac{3}{2}\right) P_{1'2}^{(-)} &= \prod_{i=1}^{N} \tilde{\rho}_0(u-\theta_i)\, T_{\langle 1'2\rangle}^{+}\left(u-\frac{1}{2}\right), \end{aligned} \tag{3.22}$$

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where

$$\tilde{\rho}_0(u) = (u-1)(u+2).$$

Taking the partial trace in the auxiliary spaces and using the relations (3.21) and (3.22), we obtain following operator product identities

$$\begin{aligned}
t^{(p)}(\theta_j)\, t^{(p)}(\theta_j-2) &= \prod_{i=1}^{N} a(\theta_j-\theta_i)e(\theta_j-\theta_i-2)\times \mathrm{id}, \\
t^{(p)}(\theta_j)\, t^{(p)}(\theta_j-1) &= \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j-\theta_i)\, t_+^{(p)}\left(\theta_j-\frac{1}{2}\right)\, t_-^{(p)}\left(\theta_j-\frac{1}{2}\right), \\
t^{(p)}(\theta_j)\, t_+^{(p)}\left(\theta_j-\frac{3}{2}\right) &= \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j-\theta_i)\, t_-^{(p)}\left(\theta_j-\frac{1}{2}\right), \\
t^{(p)}(\theta_j)\, t_-^{(p)}\left(\theta_j-\frac{3}{2}\right) &= \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j-\theta_i)\, t_+^{(p)}\left(\theta_j-\frac{1}{2}\right).
\end{aligned} \tag{3.23}$$

From the definitions, the asymptotic behaviors of the transfer matrices can be calculated directly as

$$\begin{aligned}
t^{(p)}(u)|_{u\to\pm\infty} &= 6u^{2N}\times \mathrm{id} + \cdots, \\
t_\pm^{(p)}(u)|_{u\to\pm\infty} &= 4u^{N}\times \mathrm{id} + \cdots.
\end{aligned} \tag{3.24}$$

Let us denote the eigenvalues of the transfer matrices $t^{(p)}(u)$ and $t_\pm^{(p)}(u)$ as $\Lambda^{(p)}(u)$ and $\Lambda_\pm^{(p)}(u)$, respectively. We note that the eigenvalues $\Lambda^{(p)}(u)$ and $\Lambda_\pm^{(p)}(u)$ are the polynomials of u with degrees $2N$ and N, respectively. Thus we need $4N+3$ conditions to determine the polynomials $\Lambda^{(p)}(u)$ and $\Lambda_\pm^{(p)}(u)$.

From the operator product identities (3.23), we have the functional relations among the eigenvalues

$$\begin{aligned}
\Lambda^{(p)}(\theta_j)\, \Lambda^{(p)}(\theta_j-2) &= \prod_{i=1}^{N} a(\theta_j-\theta_i)e(\theta_j-\theta_i-2), \\
\Lambda^{(p)}(\theta_j)\, \Lambda^{(p)}(\theta_j-1) &= \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j-\theta_i)\, \Lambda_+^{(p)}\left(\theta_j-\frac{1}{2}\right)\, \Lambda_-^{(p)}\left(\theta_j-\frac{1}{2}\right), \\
\Lambda^{(p)}(\theta_j)\, \Lambda_+^{(p)}\left(\theta_j-\frac{3}{2}\right) &= \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j-\theta_i)\, \Lambda_-^{(p)}\left(\theta_j-\frac{1}{2}\right), \\
\Lambda^{(p)}(\theta_j)\, \Lambda_-^{(p)}\left(\theta_j-\frac{3}{2}\right) &= \prod_{i=1}^{N} \tilde{\rho}_0(\theta_j-\theta_i)\, \Lambda_+^{(p)}\left(\theta_j-\frac{1}{2}\right).
\end{aligned} \tag{3.25}$$

And the corresponding asymptotic behaviors read

$$\begin{aligned}
\Lambda^{(p)}(u)|_{u\to\pm\infty} &= 6u^{2N} + \cdots, \\
\Lambda_\pm^{(p)}(u)|_{u\to\pm\infty} &= 4u^{N} + \cdots.
\end{aligned} \tag{3.26}$$

Then we arrive at that $4N$ functional relations (3.25) together with 3 asymptotic behaviors (3.26) give us sufficient conditions to determine the eigenvalues.

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3.3 The $T - Q$ relations

Let us introduce some functions

$$
\begin{aligned}
Z_1^{(p)}(u) &= \prod_{j=1}^{N} a(u-\theta_j)\,\frac{Q_p^{(1)}(u-1)}{Q_p^{(1)}(u)},\\
Z_2^{(p)}(u) &= \prod_{j=1}^{N} b(u-\theta_j)\,\frac{Q_p^{(1)}(u+1)Q_p^{(2)}(u-1)Q_p^{(3)}(u-1)}{Q_p^{(1)}(u)Q_p^{(2)}(u)Q_p^{(3)}(u)},\\
Z_3^{(p)}(u) &= \prod_{j=1}^{N} b(u-\theta_j)\,\frac{Q_p^{(2)}(u-1)Q_p^{(3)}(u+1)}{Q_p^{(2)}(u)Q_p^{(3)}(u)},\\
Z_4^{(p)}(u) &= \prod_{j=1}^{N} b(u-\theta_j)\,\frac{Q_p^{(2)}(u+1)Q_p^{(3)}(u-1)}{Q_p^{(2)}(u)Q_p^{(3)}(u)},\\
Z_5^{(p)}(u) &= \prod_{j=1}^{N} b(u-\theta_j)\,\frac{Q_p^{(1)}(u)Q_p^{(2)}(u+1)Q_p^{(3)}(u+1)}{Q_p^{(1)}(u+1)Q_p^{(2)}(u)Q_p^{(3)}(u)},\\
Z_6^{(p)}(u) &= \prod_{j=1}^{N} e(u-\theta_j)\,\frac{Q_p^{(1)}(u+2)}{Q_p^{(1)}(u+1)},\\
Q_p^{(1)}(u) &= \prod_{k=1}^{L_1}\left(u-\mu_k^{(1)}+\frac{1}{2}\right), \qquad Q_p^{(2)}(u)=\prod_{k=1}^{L_2}(u-\mu_k^{(2)}+1),\\
Q_p^{(3)}(u) &= \prod_{k=1}^{L_3}(u-\mu_k^{(3)}+1).
\end{aligned}
\tag{3.27}
$$

The relations (3.25) and (3.26) enable us to parameterize the eigenvalues of the transfer matrices in terms of some homogeneous $T - Q$ relations as

$$
\begin{aligned}
\Lambda^{(p)}(u) &= Z_1^{(p)}(u)+Z_2^{(p)}(u)+Z_3^{(p)}(u)+Z_4^{(p)}(u)+Z_5^{(p)}(u)+Z_6^{(p)}(u),\\
\Lambda_+^{(p)}(u) &= \prod_{i=1}^{N} a_1(u-\theta_i)\left[\frac{Q_p^{(2)}(u-\frac{3}{2})}{Q_p^{(2)}(u-\frac{1}{2})}+\frac{Q_p^{(1)}(u-\frac{1}{2})Q_p^{(2)}(u+\frac{1}{2})}{Q_p^{(1)}(u+\frac{1}{2})Q_p^{(2)}(u-\frac{1}{2})}\right]\\
&\quad+\prod_{i=1}^{N} b_1(u-\theta_i)\left[\frac{Q_p^{(3)}(u+\frac{3}{2})}{Q_p^{(3)}(u+\frac{1}{2})}+\frac{Q_p^{(1)}(u+\frac{3}{2})Q_p^{(3)}(u-\frac{1}{2})}{Q_p^{(1)}(u+\frac{1}{2})Q_p^{(3)}(u+\frac{1}{2})}\right],\\
\Lambda_-^{(p)}(u) &= \prod_{i=1}^{N} a_1(u-\theta_i)\left[\frac{Q_p^{(3)}(u-\frac{3}{2})}{Q_p^{(3)}(u-\frac{1}{2})}+\frac{Q_p^{(1)}(u-\frac{1}{2})Q_p^{(3)}(u+\frac{1}{2})}{Q_p^{(1)}(u+\frac{1}{2})Q_p^{(3)}(u-\frac{1}{2})}\right]\\
&\quad+\prod_{i=1}^{N} b_1(u-\theta_i)\left[\frac{Q_p^{(2)}(u+\frac{3}{2})}{Q_p^{(2)}(u+\frac{1}{2})}+\frac{Q_p^{(1)}(u+\frac{3}{2})Q_p^{(2)}(u-\frac{1}{2})}{Q_p^{(1)}(u+\frac{1}{2})Q_p^{(2)}(u+\frac{1}{2})}\right].
\end{aligned}
\tag{3.28}
$$

The regularities of the eigenvalues $\Lambda^{(p)}(u)$ and $\Lambda_{\pm}^{(p)}(u)$ lead to that the Bethe roots $\{\mu_k^{(m)}\}$ should satisfy the Bethe ansatz equations (BAEs):

$$\begin{aligned}\frac{Q_p^{(1)}(\mu_k^{(1)}-\frac{3}{2})Q_p^{(2)}(\mu_k^{(1)}-\frac{1}{2})Q_p^{(3)}(\mu_k^{(1)}-\frac{1}{2})}{Q_p^{(1)}(\mu_k^{(1)}+\frac{1}{2})Q_p^{(2)}(\mu_k^{(1)}-\frac{3}{2})Q_p^{(3)}(\mu_k^{(1)}-\frac{3}{2})}&=-\prod_{j=1}^{N}\frac{\mu_k^{(1)}-\frac{1}{2}-\theta_j}{\mu_k^{(1)}+\frac{1}{2}-\theta_j}, && k=1,\cdots,L_1,\\ \frac{Q_p^{(1)}(\mu_l^{(2)})Q_p^{(2)}(\mu_l^{(2)}-2)}{Q_p^{(1)}(\mu_l^{(2)}-1)Q_p^{(2)}(\mu_l^{(2)})}&=-1, && l=1,\cdots,L_2,\\ \frac{Q_p^{(1)}(\mu_l^{(3)})Q_p^{(3)}(\mu_l^{(3)}-2)}{Q_p^{(1)}(\mu_l^{(3)}-1)Q_p^{(3)}(\mu_l^{(3)})}&=-1, && l=1,\cdots,L_3.\end{aligned} \tag{3.29}$$

We note that the BAEs obtained from the regularity of $\Lambda^{(p)}(u)$ are the same as those obtained from the regularity of $\Lambda_{\pm}^{(p)}(u)$. It is easy to check that $\Lambda^{(p)}(u)$ and $\Lambda_{\pm}^{(p)}(u)$ satisfy the functional relations (3.25) and the asymptotic behaviors (3.26). Therefore, we conclude that $\Lambda^{(p)}(u)$ and $\Lambda_{\pm}^{(p)}(u)$ are the eigenvalues of the transfer matrices $t^{(p)}(u)$ and $t_{\pm}^{(p)}(u)$, respectively. It is easy to check that the homogeneous $T-Q$ relations (3.28) and associated BAEs (3.29) coincide with those obtained previously by others methods [20–22].

Then the eigenvalues of the Hamiltonian (3.4) can be obtained by the $\Lambda^{(p)}(u)$ as

$$E_p=\frac{\partial\ln\Lambda^{(p)}(u)}{\partial u}|_{u=0,\{\theta_j\}=0}. \tag{3.30}$$

4 $D_3^{(1)}$ model with non-diagonal open boundary condition

In this section, we consider the system with general integrable open boundary condition. Let us introduce a pair of reflection matrices $K^v(u)$ and $\bar{K}^v(u)$. The former satisfies the reflection equation (RE)

$$R_{12}^{vv}(u-v)K_1^v(u)R_{21}^{vv}(u+v)K_2^v(v)=K_2^v(v)R_{12}^{vv}(u+v)K_1^v(u)R_{21}^{vv}(u-v), \tag{4.1}$$

and the latter satisfies the dual RE

$$R_{12}^{vv}(-u+v)\bar{K}_1^v(u)R_{21}^{vv}(-u-v-4)\bar{K}_2^v(v)=\bar{K}_2^v(v)R_{12}^{vv}(-u-v-4)\bar{K}_1^v(u)R_{21}^{vv}(-u+v). \tag{4.2}$$

For the open case, instead of the "row-to-row" monodromy matrix $T_0^v(u)$ given by (3.1), one needs consider the "double-row" monodromy matrix to describe the reflection process. Let us introduce another "row-to-row" monodromy matrix

$$\hat{T}_0^v(u)=R_{N0}^{vv}(u+\theta_N)\cdots R_{20}^{vv}(u+\theta_2)R_{10}^{vv}(u+\theta_1), \tag{4.3}$$

which satisfies the Yang-Baxter relation

$$R_{12}^{vv}(u-v)\hat{T}_1^v(u)\hat{T}_2^v(v)=\hat{T}_2^v(v)\hat{T}_1^v(u)R_{12}^{vv}(u-v). \tag{4.4}$$

The transfer matrix $t(u)$ is defined as

$$t(u)=tr_0\{\bar{K}_0^v(u)T_0^v(u)K_0^v(u)\hat{T}_0^v(u)\}. \tag{4.5}$$

From the Yang-Baxter relation, reflection equation and its dual, one can prove that the transfer matrices with different spectral parameters commute with each other, $[t(u), t(v)] = 0$. Therefore, $t(u)$ serves as the generating function of all the conserved quantities of the system. The Hamiltonian is constructed by taking the derivative of the logarithm of the transfer matrix

$$H = \frac{\partial \ln t(u)}{\partial u}|_{u=0,\{\theta_j\}=0} = \sum_{k=1}^{N-1} H_{kk+1} + \frac{K_1^v(0)'}{2\xi} + \frac{tr_0\{\bar{K}_0^v(0)H_{N0}\}}{tr_0\bar{K}_0^v(0)} + \text{constant}, \tag{4.6}$$

where $H_{k\,k+1}$ is given by (3.5) and $K^v(0) = \xi \times \text{id}$.

4.1 Reflection matrix

In this paper, we consider the general integrable open boundary condition where the reflection matrix has the non-diagonal elements which breaks the U(1)-symmetry of the system. Here we chose the reflection matrix $K^{v-}(u)$, which is the solution of reflection equation (4.1), as

$$K^v(u) = \begin{pmatrix} K_{11}^v(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{22}^v(u) & 0 & K_{24}^v(u) & 0 & 0 \\ 0 & 0 & K_{33}^v(u) & 0 & K_{35}^v(u) & 0 \\ 0 & K_{42}^v(u) & 0 & K_{44}^v(u) & 0 & 0 \\ 0 & 0 & K_{53}^v(u) & 0 & K_{55}^v(u) & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{66}^v(u) \end{pmatrix}, \tag{4.7}$$

where the non-vanishing matrix elements are

$$\begin{aligned} K_{11}^v(u) &= \frac{(2 - c_2 - 2c_2u)(2 + c_2 + 2c_2u)}{2}, \\ K_{22}^v(u) &= \frac{(2 - c_2 - 2cu)(2 + c_2 + 2c_2u)}{2}, \\ K_{24}^v(u) &= -c_1u(2 + c_2 + 2c_2u), \\ K_{33}^v(u) &= \frac{(2 + c_2 - 2cu)(2 - c_2 - 2c_2u)}{2}, \\ K_{35}^v(u) &= c_1u(2 - c_2 - 2c_2u), \\ K_{42}^v(u) &= -c_3u(2 + c_2 + 2c_2u), \\ K_{44}^v(u) &= \frac{(2 - c_2 + 2cu)(2 + c_2 + 2c_2u)}{2}, \\ K_{53}^v(u) &= -c_3u(-2 + c_2 + 2c_2u), \\ K_{55}^v(u) &= \frac{(2 + c_2 + 2cu)(2 - c_2 - 2c_2u)}{2}, \\ K_{66}^v(u) &= \frac{(2 - c_2 - 2c_2u)(2 + c_2 + 2c_2u)}{2}. \end{aligned} \tag{4.8}$$

Here c_1, c_2 and c_3 are free boundary parameters, where c is expressed in terms of them as

$$c^2 = c_2^2 - c_1c_3.$$

The general off-diagonal reflection matrix for the trigonometric $D_n^{(1)}$ vertex model has been constructed by Lima-Santos and Malara [37, 38]. The reflection matrix (4.7) adopted in this paper can be obtained from that of the $D_3^{(1)}$ with 3-free parameters case in [37, 38] by taking the rational limit.

The dual reflection matrix $\bar{K}^v(u)$ is also a non-diagonal one and given by

$$\bar{K}^v(u) = K^v(-u-2)|_{(c_1,c_2,c_3)\to(c_1',c_2',c_3')}, \tag{4.9}$$

where c_1', c_2' and c_3' are the boundary parameters at the other side. For a generic choice of the three boundary parameters $\{c_i,\, c_i'|i=1,2,3\}$, it is easily to check that $[K^v(u),\, \bar{K}^v(v)] \neq 0$. This implies that the matrices $K^v(u)$ and $\bar{K}^v(u)$ cannot be diagonalized simultaneously.

Following the method developed in [39] and using the crossing-symmetry (2.2) of the R-matrix, the explicit expressions (4.7) and (4.9) of the reflection matrices, we find that the transfer matrix (4.5) possesses the crossing symmetry

$$t(-u-2) = t(u). \tag{4.10}$$

4.2 Fusion of the reflection matrix

In order to seek the relations that the transfer matrix (4.5) satisfies, we first consider the fusion of the reflection matrix. The fusion of the one-dimensional projector $P_{12}^{vv(1)}$ gives

$$\begin{aligned} P_{12}^{vv(1)} K_2^v(u) R_{12}^{vv}(2u-2) K_2^v(u-2) P_{21}^{vv(1)} &= (u-2)\left(u-\frac{3}{2}\right) h(u)h(-u) P_{12}^{vv(1)}, \\ P_{21}^{vv(1)} \bar{K}_1^v(u-2) R_{21}^{vv}(-2u-2) \bar{K}_2^v(u) P_{12}^{vv(1)} &= (u+2)\left(u+\frac{3}{2}\right) \tilde{h}(u)\tilde{h}(-u) P_{21}^{vv(1)}, \end{aligned} \tag{4.11}$$

where

$$h(u) = (2-c_2-2c_2u)(2+c_2+2c_2u), \quad \tilde{h}(u) = (2-c_2'-2c_2'u)(2+c_2'+2c_2'u). \tag{4.12}$$

The fusion of the 16-dimensional projector $P_{12}^{vv(16)}$ gives

$$\begin{aligned} &P_{21}^{vv(16)} K_1^v(u) R_{21}^{vv}(2u-1) K_2^v(u-1) P_{12}^{vv(16)} \\ &\quad = 2(u-1)h(u) S_{1'2'} K_{1'}^{s_+}\left(u-\frac{1}{2}\right) R_{2'1'}^{s_-s_+}(2u-1) K_{2'}^{s_-}\left(u-\frac{1}{2}\right) S_{1'2'}^{-1}, \\ &P_{12}^{vv(16)} \bar{K}_2^v(u-1) R_{12}^{vv}(-2u-3) \bar{K}_1^v(u) P_{21}^{vv(16)} \\ &\quad = -2(u+2)\tilde{h}(u) S_{1'2'} \bar{K}_{1'}^{s_-}\left(u-\frac{1}{2}\right) R_{1'2'}^{s_+s_-}(-2u-3) \bar{K}_{2'}^{s_+}\left(u-\frac{1}{2}\right) S_{1'2'}^{-1}, \end{aligned} \tag{4.13}$$

where

$$
R^{s_+s_-}_{1'2'} = \left(\begin{array}{cccc|cccc|cccc|cccc}
a_3 & & & & & & & & & & & & & & & \\
& a_3 & & & & & & & & & & & & & & \\
& & a_3 & & & & & & & & & & & & & \\
& & & b_3 & & & 1 & & & -1 & & & 1 & & & \\
\hline
& & & & a_3 & & & & & & & & & & & \\
& & & & & a_3 & & & & & & & & & & \\
& & & 1 & & & b_3 & & & 1 & & & -1 & & & \\
& & & & & & & a_3 & & & & & & & & \\
\hline
& & & & & & & & a_3 & & & & & & & \\
& & & -1 & & & 1 & & & b_3 & & & 1 & & & \\
& & & & & & & & & & a_3 & & & & & \\
& & & & & & & & & & & a_3 & & & & \\
\hline
& & & 1 & & & -1 & & & 1 & & & b_3 & & & \\
& & & & & & & & & & & & & a_3 & & \\
& & & & & & & & & & & & & & a_3 & \\
& & & & & & & & & & & & & & & a_3
\end{array}\right),
$$

$$
K^{s_+}(u) = \begin{pmatrix} 1-cu & c_1u & 0 & 0 \\ c_3u & 1+cu & 0 & 0 \\ 0 & 0 & 1+c_2u & 0 \\ 0 & 0 & 0 & 1-c_2u \end{pmatrix}, \tag{4.14}
$$

$$
K^{s_-}(u) = \begin{pmatrix} 1+c_2u & 0 & 0 & 0 \\ 0 & 1-c_2u & 0 & 0 \\ 0 & 0 & 1-cu & c_1u \\ 0 & 0 & c_3u & 1+cu \end{pmatrix}, \tag{4.15}
$$

and

$$
a_3 = u+2, \quad b_3 = u+1.
$$

The matrix $R^{s_+s_-}_{1'2'}(u)$ has properties

$$
\text{unitarity}: \qquad R^{s_+s_-}_{1'2'}(u)R^{s_-s_+}_{2'1'}(-u) = (2+u)(2-u), \tag{4.16}
$$

$$
\text{crossing unitarity}: \qquad R^{s_+s_-}_{1'2'}(u)^{t_{1'}}R^{s_-s_+}_{2'1'}(-u-4)^{t_{1'}} = \rho_{ss}(u) = -(u+1)(u+3),
$$

and satisfies the Yang-Baxter equations

$$
R^{s_+s_-}_{1'2'}(u_1-u_2)R^{s_+v}_{1'3}(u_1-u_3)R^{s_-v}_{2'3}(u_2-u_3) = R^{s_-v}_{2'3}(u_2-u_3)R^{s_+v}_{1'3}(u_1-u_3)R^{s_+s_-}_{1'2'}(u_1-u_2). \tag{4.17}
$$

The matrices $K^{s_\pm}(u)$ satisfy the reflection equations

$$
R^{s_\pm v}_{12}(u-v)K^{s_\pm}_1(u)R^{vs_\pm}_{21}(u+v)K^v_2(v) = K^v_2(v)R^{s_\pm v}_{12}(u+v)K^{s_\pm}_1(u)R^{vs_\pm}_{21}(u-v). \tag{4.18}
$$

The dual reflection matrices $\bar{K}^{s_\pm}(u)$ are constructed as

$$
\bar{K}^{s_\pm}(u) = K^{s_\pm}(-u-2)\Big|_{(c,c_1,c_2,c_3)\to(c',c'_1,c'_2,c'_3)}, \tag{4.19}
$$

which satisfy the dual REs

$$R_{12}^{s_\pm v}(-u+v)\bar{K}_1^{s\pm}(u)R_{21}^{vs\pm}(-u-v-4)\bar{K}_2^v(v)$$
$$=\bar{K}_2^v(v)R_{12}^{s_\pm v}(-u-v-4)\bar{K}_1^{s\pm}(u)R_{21}^{vs\pm}(-u+v). \tag{4.20}$$

The fusion of the 4-dimensional projector $P_{1'2}^{(\pm)}$ gives

$$P_{1'2}^{(\pm)}K_2^v(u)R_{1'2}^{s_\pm v}\left(2u-\frac{3}{2}\right)K_{1'}^{s_\pm}\left(u-\frac{3}{2}\right)P_{21'}^{(\pm)}=\left(u-\frac{3}{2}\right)h(u)K_{\langle 1'2\rangle}^{s_\mp}\left(u-\frac{1}{2}\right),$$
$$P_{21'}^{(\pm)}\bar{K}_{1'}^{s_\pm}\left(u-\frac{3}{2}\right)R_{21'}^{vs_\pm}\left(-2u-\frac{5}{2}\right)\bar{K}_2^v(u)P_{1'2}^{(\pm)}=-(u+2)\tilde{h}(u)\bar{K}_{\langle 1'2\rangle}^{s_\mp}\left(u-\frac{1}{2}\right). \tag{4.21}$$

4.3 Operators product relations

Define the fused monodromy matrices $\hat{T}_{0'}^{\pm}(u)$ by using the fused R-matrix $R^{vs\pm}$ as

$$\hat{T}_{0'}^{\pm}(u)=R_{N0'}^{vs\pm}(u+\theta_N)\cdots R_{20'}^{vs\pm}(u+\theta_2)R_{10'}^{vs\pm}(u+\theta_1), \tag{4.22}$$

which satisfy the Yang-Baxter relation

$$R_{00'}^{vs\pm}(u-v)\hat{T}_0(u)\hat{T}_{0'}^{\pm}(v)=\hat{T}_{0'}^{\pm}(v)\hat{T}_0(u)R_{00'}^{vs\pm}(u-v). \tag{4.23}$$

From the Yang-Baxter relations (4.4) and (4.23) at certain points and using the properties of projectors, we obtain

$$\hat{T}_1(-\theta_j)\,\hat{T}_2(-\theta_j-2)=P_{12}^{vv(1)}\,\hat{T}_1(-\theta_j)\,\hat{T}_2(-\theta_j-2),$$
$$\hat{T}_1(-\theta_j)\,\hat{T}_2(-\theta_j-1)=P_{12}^{vv(16)}\,\hat{T}_1(-\theta_j)\,\hat{T}_2(-\theta_j-1),$$
$$\hat{T}_2(-\theta_j)\,\hat{T}_{1'}^{+}\left(-\theta_j-\frac{3}{2}\right)=P_{21'}^{(+)}\,\hat{T}_2(-\theta_j)\,\hat{T}_{1'}^{+}\left(-\theta_j-\frac{3}{2}\right),$$
$$\hat{T}_2(-\theta_j)\,\hat{T}_{1'}^{-}\left(-\theta_j-\frac{3}{2}\right)=P_{21'}^{(-)}\,\hat{T}_2(-\theta_j)\,\hat{T}_{1'}^{-}\left(-\theta_j-\frac{3}{2}\right). \tag{4.24}$$

By using the fusion identities (3.9), (3.11), (3.15) and (3.17), we also obtain the fusion identities

$$P_{12}^{vv(1)}\hat{T}_1(u)\,\hat{T}_2(u-2)P_{12}^{vv(1)}=P_{12}^{vv(1)}\prod_{i=1}^{N}a(u+\theta_i)e(u+\theta_i-2)\times\mathrm{id},$$
$$P_{12}^{vv(16)}\hat{T}_1(u)\,\hat{T}_2(u-1)P_{12}^{vv(16)}=\prod_{i=1}^{N}\tilde{\rho}_0(u+\theta_i)S_{1'2'}\,\hat{T}_{1'}^{+}\left(u-\frac{1}{2}\right)\,\hat{T}_{2'}^{-}\left(u-\frac{1}{2}\right)S_{1'2'}^{-1},$$
$$P_{21'}^{(+)}\hat{T}_2(u)\,\hat{T}_{1'}^{+}\left(u-\frac{3}{2}\right)P_{21'}^{(+)}=\prod_{i=1}^{N}\tilde{\rho}_0(u+\theta_i)\,\hat{T}_{\langle 1'2\rangle}^{-}\left(u-\frac{1}{2}\right),$$
$$P_{21'}^{(-)}\hat{T}_2(u)\,\hat{T}_{1'}^{-}\left(u-\frac{3}{2}\right)P_{21'}^{(-)}=\prod_{i=1}^{N}\tilde{\rho}_0(u+\theta_i)\,\hat{T}_{\langle 1'2\rangle}^{+}\left(u-\frac{1}{2}\right). \tag{4.25}$$

The fused transfer matrices are defined as

$$t_\pm(u)=tr_{0'}\{\bar{K}_{0'}^{s\pm}(u)T_{0'}^{\pm}(u)K_{0'}^{s\pm}(u)\hat{T}_{0'}^{\pm}(u)\}. \tag{4.26}$$

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Direct calculation shows

$$t(u)t(u+\Delta) = [\rho_1(2u+\Delta+2)]^{-1} tr_{12}\{\bar{K}_2^v(u+\Delta)R_{12}^{vv}(-2u-4-\Delta) \times \bar{K}_1^v(u)T_1(u)T_2(u+\Delta)K_1^v(u)R_{21}^{vv}(2u+\Delta)K_2^v(u+\Delta)\hat{T}_1(u)\hat{T}_2(u+\Delta)\}, \tag{4.27}$$

$$t(u)t_\pm(u+\Delta) = [\rho_s(2u+\Delta+2)]^{-1} tr_{12}\{\bar{K}_2^{s\pm}(u+\Delta)R_{12}^{vs\pm}(-2u-4-\Delta) \times \bar{K}_1^v(u)T_1(u)T_2^\pm(u+\Delta)K_1^v(u)R_{21}^{s\pm v}(2u+\Delta)K_2^{s\pm}(u+\Delta)\hat{T}_1(u)\hat{T}_2^\pm(u+\Delta)\}, \tag{4.28}$$

$$t_+(u)t_-(u+\Delta) = [\rho_{ss}(2u+\Delta)]^{-1} tr_{12}\{\bar{K}_2^{s-}(u+\Delta)R_{12}^{s+s-}(-2u-4-\Delta) \times \bar{K}_1^{s+}(u)T_1^+(u)T_2^-(u+\Delta)K_1^{s+}(u) \times R_{21}^{s-s+}(2u+\Delta)K_2^{s-}(u+\Delta)\hat{T}_1^+(u)\hat{T}_2^-(u+\Delta)\}. \tag{4.29}$$

Here Δ choose $-2, -1, -\frac{3}{2}, 0$ for eq. (4.27),eq. (4.27),eq. (4.28) and eq. (4.29), respectively.

With the help of relations (3.21), (3.22), (4.11), (4.13), (4.21), (4.24), (4.25) and considering eqs. (4.27)–(4.29) at certain points, we arrive at

$$\begin{aligned}
t(\pm\theta_j)\,t(\pm\theta_j-2) &= \frac{1}{2^4}\frac{(\pm\theta_j-2)(\pm\theta_j+2)(\pm\theta_j-\frac{3}{2})(\pm\theta_j+\frac{3}{2})}{(\pm\theta_j-1)(\pm\theta_j+1)(\pm\theta_j-\frac{1}{2})(\pm\theta_j+\frac{1}{2})}h(\pm\theta_j)h(\mp\theta_j)\tilde{h}(\pm\theta_j) \\
&\quad\times\tilde{h}(\mp\theta_j)\prod_{i=1}^N a(\pm\theta_j-\pm\theta_i)e(\pm\theta_j-\pm\theta_i-2) \\
&\quad\times a(\pm\theta_j+\pm\theta_i)e(\pm\theta_j+\pm\theta_i-2)\times \mathrm{id}, \\
t(\pm\theta_j)\,t(\pm\theta_j-1) &= \frac{(\pm\theta_j-1)(\pm\theta_j+2)}{(\pm\theta_j-\frac{1}{2})(\pm\theta_j+\frac{3}{2})}\prod_{i=1}^N\tilde{\rho}_0(\pm\theta_j-\pm\theta_i)\tilde{\rho}_0(\pm\theta_j+\pm\theta_i) \\
&\quad\times h(\pm\theta_j)\tilde{h}(\pm\theta_j)t_+\left(\pm\theta_j-\frac{1}{2}\right)\,t_-\left(\pm\theta_j-\frac{1}{2}\right), \\
t(\pm\theta_j)\,t_+\left(\pm\theta_j-\frac{3}{2}\right) &= \frac{1}{2^2}\frac{(\pm\theta_j-\frac{3}{2})(\pm\theta_j+2)}{(\pm\theta_j-\frac{1}{2})(\pm\theta_j+1)}\prod_{i=1}^N\tilde{\rho}_0(\pm\theta_j-\pm\theta_i)\tilde{\rho}_0(\pm\theta_j+\pm\theta_i) \\
&\quad\times h(\pm\theta_j)\tilde{h}(\pm\theta_j)t_-\left(\pm\theta_j-\frac{1}{2}\right), \\
t(\pm\theta_j)\,t_-\left(\pm\theta_j-\frac{3}{2}\right) &= \frac{1}{2^2}\frac{(\pm\theta_j-\frac{3}{2})(\pm\theta_j+2)}{(\pm\theta_j-\frac{1}{2})(\pm\theta_j+1)}\prod_{i=1}^N\tilde{\rho}_0(\pm\theta_j-\pm\theta_i)\tilde{\rho}_0(\pm\theta_j+\pm\theta_i) \\
&\quad\times h(\pm\theta_j)\tilde{h}(\pm\theta_j)t_+\left(\pm\theta_j-\frac{1}{2}\right).
\end{aligned} \tag{4.30}$$

Meanwhile, from the definitions we also know the values of transfer matrices at some special points

$$\begin{aligned}
t(0) &= \frac{3}{2}(2-c_2)(2+c_2)(2-c_2')(2+c_2')\prod_{l=1}^N\rho_1(-\theta_l), \\
t\left(-\frac{1}{2}\right) &= 6\prod_{l=1}^N\rho_1(-\theta_l)t_\pm(-1), \\
t_+(0) &= t_-(0) = 4\prod_{l=1}^N\rho_s(-\theta_l).
\end{aligned} \tag{4.31}$$

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The corresponding asymptotic behaviors read

$$
\begin{aligned}
t(u)|_{u\to\pm\infty} &= -\frac{8c_2c_2'((c_3')^2c^2+c'^2c_3^2-c_3^2(c_2')^2-c_2^2(c_3')^2-2cc'c_3c_3'-c_2c_3c_2'c_3')}{c_3c_3'}u^{4N+4}\times \mathrm{id} \\
&\quad +\cdots, \\
t_+(u)|_{u\to\pm\infty} &= \frac{(c'c_3-c_3'c-c_2'c_3-c_3'c_2)(c'c_3-c_3'c+c_2'c_3+c_3'c_2)}{c_3c_3'}u^{2N+2}\times \mathrm{id}+\cdots, \\
t_-(u)|_{u\to\pm\infty} &= \frac{(c'c_3-c_3'c-c_2'c_3-c_3'c_2)(c'c_3-c_3'c+c_2'c_3+c_3'c_2)}{c_3c_3'}u^{2N+2}\times \mathrm{id}+\cdots. \qquad (4.32)
\end{aligned}
$$

Denote the eigenvalues of the fused transfer matrices $t(u)$ and $t_\pm(u)$ as $\Lambda(u)$ and $\Lambda_\pm(u)$, respectively. From eq. (4.30), we obtain the functional relations among the eigenvalues of the transfer matrices

$$
\begin{aligned}
\Lambda(\pm\theta_j)\,\Lambda(\pm\theta_j-2) &= \frac{1}{2^4}\frac{(\pm\theta_j-2)(\pm\theta_j+2)(\pm\theta_j-\frac{3}{2})(\pm\theta_j+\frac{3}{2})}{(\pm\theta_j-1)(\pm\theta_j+1)(\pm\theta_j-\frac{1}{2})(\pm\theta_j+\frac{1}{2})}h(\pm\theta_j)h(\mp\theta_j)\tilde{h}(\pm\theta_j) \\
&\quad\times\tilde{h}(\mp\theta_j)\prod_{i=1}^{N}a(\pm\theta_j-\pm\theta_i)e(\pm\theta_j-\pm\theta_i-2) \\
&\quad\times a(\pm\theta_j+\pm\theta_i)e(\pm\theta_j+\pm\theta_i-2), \\
\Lambda(\pm\theta_j)\,\Lambda(\pm\theta_j-1) &= \frac{(\pm\theta_j-1)(\pm\theta_j+2)}{(\pm\theta_j-\frac{1}{2})(\pm\theta_j+\frac{3}{2})}\prod_{i=1}^{N}\tilde{\rho}_0(\pm\theta_j-\pm\theta_i)\tilde{\rho}_0(\pm\theta_j+\pm\theta_i) \\
&\quad\times h(\pm\theta_j)\tilde{h}(\pm\theta_j)\Lambda_+\left(\pm\theta_j-\frac{1}{2}\right)\,\Lambda_-\left(\pm\theta_j-\frac{1}{2}\right), \\
\Lambda(\pm\theta_j)\,\Lambda_+\left(\pm\theta_j-\frac{3}{2}\right) &= \frac{1}{2^2}\frac{(\pm\theta_j-\frac{3}{2})(\pm\theta_j+2)}{(\pm\theta_j-\frac{1}{2})(\pm\theta_j+1)}\prod_{i=1}^{N}\tilde{\rho}_0(\pm\theta_j-\pm\theta_i)\tilde{\rho}_0(\pm\theta_j+\pm\theta_i) \\
&\quad\times h(\pm\theta_j)\tilde{h}(\pm\theta_j)\Lambda_-\left(\pm\theta_j-\frac{1}{2}\right), \\
\Lambda(\pm\theta_j)\,\Lambda_-\left(\pm\theta_j-\frac{3}{2}\right) &= \frac{1}{2^2}\frac{(\pm\theta_j-\frac{3}{2})(\pm\theta_j+2)}{(\pm\theta_j-\frac{1}{2})(\pm\theta_j+1)}\prod_{i=1}^{N}\tilde{\rho}_0(\pm\theta_j-\pm\theta_i)\tilde{\rho}_0(\pm\theta_j+\pm\theta_i) \\
&\quad\times h(\pm\theta_j)\tilde{h}(\pm\theta_j)\Lambda_+\left(\pm\theta_j-\frac{1}{2}\right). \qquad (4.33)
\end{aligned}
$$

Eqs. (4.31) and (4.32) give rise to the relations

$$
\begin{aligned}
\Lambda(0) &= \frac{3}{2}(2-c_2)(2+c_2)(2-c_2')(2+c_2')\prod_{l=1}^{N}\rho_1(-\theta_l), \\
\Lambda\left(-\frac{1}{2}\right) &= 6\prod_{l=1}^{N}\rho_1(-\theta_l)\Lambda_\pm(-1), \\
\Lambda_+(0) &= \Lambda_-(0) = 4\prod_{l=1}^{N}\rho_s(-\theta_l), \qquad (4.34)
\end{aligned}
$$

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and

$$\begin{aligned}
\Lambda(u)|_{u\to\pm\infty} &= -\frac{8c_2c_2'((c_3')^2c^2+c'^2c_3^2-c_3^2(c_2')^2-c_2^2(c_3')^2-2cc'c_3c_3'-c_2c_3c_2'c_3')}{c_3c_3'}u^{4N+4}\\
&\quad+\cdots,\\
\Lambda_+(u)|_{u\to\pm\infty} &= \frac{(c'c_3-c_3'c-c_2'c_3-c_3'c_2)(c'c_3-c_3'c+c_2'c_3+c_3'c_2)}{c_3c_3'}u^{2N+2}+\cdots,\\
\Lambda_-(u)|_{u\to\pm\infty} &= \frac{(c'c_3-c_3'c-c_2'c_3-c_3'c_2)(c'c_3-c_3'c+c_2'c_3+c_3'c_2)}{c_3c_3'}u^{2N+2}+\cdots. \qquad (4.35)
\end{aligned}$$

From the definitions, we know that the eigenvalues $\Lambda(u)$ and $\Lambda_\pm(u)$ are the polynomials of u with degrees $4N+4$ and $2N+2$, respectively. Meanwhile, $\Lambda(u)$ and $\Lambda_\pm(u)$ enjoy the crossing symmetries

$$\Lambda(-u-2)=\Lambda(u), \quad \Lambda_\pm(-u-2)=\Lambda_\pm(u). \qquad (4.36)$$

Therefore, in order to determine the explicit expressions of the polynomials, we need $4N+7$ conditions, which are all listed by eqs. (4.33)–(4.35).

4.4 Inhomogeneous T-Q relations

Let us introduce some functions

$$\begin{aligned}
Z_1(u) &= 2^2\frac{(u+2)(u+\frac{3}{2})}{(u+1)(u+\frac{1}{2})}\prod_{j=1}^N a(u-\theta_j)a(u+\theta_j)\\
&\quad\times h_1\left(u+\frac{1}{2}\right)h_1\left(u-\frac{1}{2}\right)\tilde{h}_1\left(u+\frac{1}{2}\right)\tilde{h}_1\left(u-\frac{1}{2}\right)\frac{Q^{(1)}(u-1)}{Q^{(1)}(u)},\\
Z_2(u) &= 2^2\frac{u(u+2)(u+\frac{3}{2})}{(u+1)(u+1)(u+\frac{1}{2})}\prod_{j=1}^N b(u-\theta_j)b(u+\theta_j)\\
&\quad\times h_1\left(u+\frac{1}{2}\right)h_2\left(u+\frac{3}{2}\right)\tilde{h}_1\left(u+\frac{1}{2}\right)\tilde{h}_2\left(u+\frac{3}{2}\right)\frac{Q^{(1)}(u+1)Q^{(2)}(u-1)Q^{(3)}(u-1)}{Q^{(1)}(u)Q^{(2)}(u)Q^{(3)}(u)},\\
Z_3(u) &= 2^2\frac{u(u+2)}{(u+1)(u+1)}\prod_{j=1}^N b(u-\theta_j)b(u+\theta_j)\\
&\quad\times h_1\left(u+\frac{1}{2}\right)h_2\left(u+\frac{3}{2}\right)\tilde{h}_1\left(u+\frac{1}{2}\right)\tilde{h}_2\left(u+\frac{3}{2}\right)\frac{Q^{(2)}(u-1)Q^{(3)}(u+1)}{Q^{(2)}(u)Q^{(3)}(u)},\\
Z_4(u) &= 2^2\frac{u(u+2)}{(u+1)(u+1)}\prod_{j=1}^N b(u-\theta_j)b(u+\theta_j)\\
&\quad\times h_1\left(u+\frac{1}{2}\right)h_2\left(u+\frac{3}{2}\right)\tilde{h}_1\left(u+\frac{1}{2}\right)\tilde{h}_2\left(u+\frac{3}{2}\right)\frac{Q^{(2)}(u+1)Q^{(3)}(u-1)}{Q^{(2)}(u)Q^{(3)}(u)},\\
Z_5(u) &= 2^2\frac{u(u+2)(u+\frac{1}{2})}{(u+1)(u+1)(u+\frac{3}{2})}\prod_{j=1}^N b(u-\theta_j)b(u+\theta_j)\\
&\quad\times h_1\left(u+\frac{1}{2}\right)h_2\left(u+\frac{3}{2}\right)\tilde{h}_1\left(u+\frac{1}{2}\right)\tilde{h}_2\left(u+\frac{3}{2}\right)\frac{Q^{(1)}(u)Q^{(2)}(u+1)Q^{(3)}(u+1)}{Q^{(1)}(u+1)Q^{(2)}(u)Q^{(3)}(u)},
\end{aligned}$$

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$$
\begin{aligned}
Z_6(u) &= 2^2\frac{u(u+\frac{1}{2})}{(u+1)(u+\frac{3}{2})}\prod_{j=1}^{N}e(u-\theta_j)e(u+\theta_j)\\
&\quad\times h_1\left(u+\frac{5}{2}\right)h_2\left(u+\frac{3}{2}\right)\tilde{h}_1\left(u+\frac{5}{2}\right)\tilde{h}_2\left(u+\frac{3}{2}\right)\frac{Q^{(1)}(u+2)}{Q^{(1)}(u+1)},\\
f_1(u) &= 2^2x\frac{u(u+2)(u+\frac{3}{2})}{u+1}\prod_{j=1}^{N}a(u-\theta_j)a(u+\theta_j)(u-\theta_j)(u+\theta_j)\\
&\quad\times h_1\left(u+\frac{1}{2}\right)\tilde{h}_1\left(u+\frac{1}{2}\right)\frac{Q^{(2)}(u-1)Q^{(3)}(u-1)}{Q^{(1)}(u)},\\
f_2(u) &= 2^2x\frac{u(u+2)(u+\frac{1}{2})}{u+1}\prod_{j=1}^{N}a(u-\theta_j)a(u+\theta_j)(u-\theta_j)(u+\theta_j)\\
&\quad\times h_2\left(u+\frac{3}{2}\right)\tilde{h}_2\left(u+\frac{3}{2}\right)\frac{Q^{(2)}(u+1)Q^{(3)}(u+1)}{Q^{(1)}(u+1)},
\end{aligned}
$$

where

$$
\begin{aligned}
Q^{(1)}(u) &= \prod_{k=1}^{L_1}\left(u-\mu_k^{(1)}+\frac{1}{2}\right)\left(u+\mu_k^{(1)}+\frac{1}{2}\right),\\
Q^{(2)}(u) &= \prod_{k=1}^{L_2}(u-\mu_k^{(2)}+1)(u+\mu_k^{(2)}+1),\\
Q^{(3)}(u) &= \prod_{k=1}^{L_3}(u-\mu_k^{(3)}+1)(u+\mu_k^{(3)}+1),\\
h_1(u) &= 1+c_2u,\quad h_2(u)=1-c_2u,\quad \tilde{h}_1(u)=1-c_2'u,\quad \tilde{h}_2(u)=1+c_2'u.
\end{aligned}
$$

The constraints (4.33)–(4.35) enables us to parameterize the eigenvalues of the transfer matrices $\Lambda(u)$ and $\Lambda_\pm(u)$ in terms of the inhomogeneous $T-Q$ relations

$$
\begin{aligned}
\Lambda(u) &= Z_1(u)+Z_2(u)+Z_3(u)+Z_4(u)+Z_5(u)+Z_6(u)+f_1(u)+f_2(u),\\
\Lambda_+(u) &= \prod_{i=1}^{N}a_1(u-\theta_i)a_1(u+\theta_i)h_1(u)\tilde{h}_1(u)\\
&\quad\times\left[\frac{u+2}{u+\frac{1}{2}}\frac{Q^{(2)}(u-\frac{3}{2})}{Q^{(2)}(u-\frac{1}{2})}+\frac{u(u+2)}{(u+1)(u+\frac{1}{2})}\frac{Q^{(1)}(u-\frac{1}{2})Q^{(2)}(u+\frac{1}{2})}{Q^{(1)}(u+\frac{1}{2})Q^{(2)}(u-\frac{1}{2})}\right]\\
&\quad+\prod_{i=1}^{N}b_1(u-\theta_i)b_1(u+\theta_i)h_2(u+2)\tilde{h}_2(u+2)\\
&\quad\times\left[\frac{u}{u+\frac{3}{2}}\frac{Q^{(3)}(u+\frac{3}{2})}{Q^{(3)}(u+\frac{1}{2})}+\frac{u(u+2)}{(u+1)(u+\frac{3}{2})}\frac{Q^{(1)}(u+\frac{3}{2})Q^{(3)}(u-\frac{1}{2})}{Q^{(1)}(u+\frac{1}{2})Q^{(3)}(u+\frac{1}{2})}\right]\\
&\quad+x\,u(u+2)\prod_{i=1}^{N}a_1(u-\theta_i)a_1(u+\theta_i)b_1(u-\theta_i)b_1(u+\theta_i)\frac{Q^{(2)}(u+\frac{1}{2})Q^{(3)}(u-\frac{1}{2})}{Q^{(1)}(u+\frac{1}{2})},
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_-(u) = &\prod_{i=1}^{N} a_1(u-\theta_i)a_1(u+\theta_i)h_1(u)\tilde{h}_1(u) \\
&\times\left[\frac{u+2}{u+\frac{1}{2}}\frac{Q^{(3)}(u-\frac{3}{2})}{Q^{(3)}(u-\frac{1}{2})}+\frac{u(u+2)}{(u+1)(u+\frac{1}{2})}\frac{Q^{(1)}(u-\frac{1}{2})Q^{(3)}(u+\frac{1}{2})}{Q^{(1)}(u+\frac{1}{2})Q^{(3)}(u-\frac{1}{2})}\right] \\
&+\prod_{i=1}^{N} b_1(u-\theta_i)b_1(u+\theta_i)h_2(u+2)\tilde{h}_2(u+2) \\
&\times\left[\frac{u}{u+\frac{3}{2}}\frac{Q^{(2)}(u+\frac{3}{2})}{Q^{(2)}(u+\frac{1}{2})}+\frac{u(u+2)}{(u+1)(u+\frac{3}{2})}\frac{Q^{(1)}(u+\frac{3}{2})Q^{(2)}(u-\frac{1}{2})}{Q^{(1)}(u+\frac{1}{2})Q^{(2)}(u+\frac{1}{2})}\right] \\
&+x\,u(u+2)\prod_{i=1}^{N} a_1(u-\theta_i)a_1(u+\theta_i)b_1(u-\theta_i)b_1(u+\theta_i)\frac{Q^{(2)}(u-\frac{1}{2})Q^{(3)}(u+\frac{1}{2})}{Q^{(1)}(u+\frac{1}{2})}.
\end{aligned}
$$

The regularities of the eigenvalues $\Lambda(u)$ and $\Lambda_\pm(u)$ lead to that the Bethe roots $\{\mu_k^{(m)}\}$ should satisfy the BAEs

$$
\begin{aligned}
&\frac{(\mu_k^{(1)}+\frac{1}{2})h_1(\mu_k^{(1)}-1)\tilde{h}_1(\mu_k^{(1)}-1)}{\prod_{j=1}^{N}(\mu_k^{(1)}-\frac{1}{2}-\theta_j)(\mu_k^{(1)}-\frac{1}{2}+\theta_j)}\frac{Q^{(1)}(\mu_k^{(1)}-\frac{3}{2})}{Q^{(2)}(\mu_k^{(1)}-\frac{3}{2})Q^{(3)}(\mu_k^{(1)}-\frac{3}{2})} \\
&+\frac{(\mu_k^{(1)}-\frac{1}{2})h_2(\mu_k^{(1)}+1)\tilde{h}_2(\mu_k^{(1)}+1)}{\prod_{j=1}^{N}(\mu_k^{(1)}+\frac{1}{2}-\theta_j)(\mu_k^{(1)}+\frac{1}{2}+\theta_j)}\frac{Q^{(1)}(\mu_k^{(1)}+\frac{1}{2})}{Q^{(2)}(\mu_k^{(1)}-\frac{1}{2})Q^{(3)}(\mu_k^{(1)}-\frac{1}{2})} \\
&\qquad=-x\,\mu_k^{(1)}\left(\mu_k^{(1)}-\frac{1}{2}\right)\left(\mu_k^{(1)}+\frac{1}{2}\right),\qquad k=1,\cdots,L_1, \\
&\frac{Q^{(1)}(\mu_l^{(2)})Q^{(2)}(\mu_l^{(2)}-2)}{Q^{(1)}(\mu_l^{(2)}-1)Q^{(2)}(\mu_l^{(2)})}=-\frac{\mu_l^{(2)}+\frac{1}{2}}{\mu_l^{(2)}-\frac{1}{2}},\qquad l=1,\cdots,L_2, \\
&\frac{Q^{(1)}(\mu_l^{(3)})Q^{(3)}(\mu_l^{(3)}-2)}{Q^{(1)}(\mu_l^{(3)}-1)Q^{(3)}(\mu_l^{(3)})}=-\frac{\mu_l^{(3)}+\frac{1}{2}}{\mu_l^{(3)}-\frac{1}{2}},\qquad l=1,\cdots,L_3,
\end{aligned}
\tag{4.37}
$$

where the numbers of Bethe roots should satisfy the constraint

$$
L_1 = L_2 + L_3 + N, \tag{4.38}
$$

and the parameter x is given by

$$
x=\frac{(c'c_3-c_3'c-c_2'c_3-c_3'c_2)(c'c_3-c_3'c+c_2'c_3+c_3'c_2)}{c_3c_3'}+4c_2c_2'. \tag{4.39}
$$

Again, the BAEs obtained from the regularity of $\Lambda(u)$ are the same as those obtained from the regularity of $\Lambda_\pm(u)$. The function $Q^{(m)}(u)$ has two zero points, and the BAEs obtained from these two points are the same. It is easy to check that $\Lambda(u)$ and $\Lambda_\pm(u)$ satisfy the functional relations (4.33), the values at the special points (4.34) and the asymptotic behaviors (4.35). Therefore, we conclude that $\Lambda(u)$ and $\Lambda_\pm(u)$ are the eigenvalues of the transfer matrices $t(u)$ and $t_\pm(u)$, respectively.

Finally, the eigenvalue E of Hamiltonian (4.6) can be obtained by the $\Lambda(u)$ as

$$
E=\frac{\partial \ln\Lambda(u)}{\partial u}|_{u=0,\{\theta_j\}=0}. \tag{4.40}
$$

5 Discussion

In this paper, we study the exact solution of $D_3^{(1)}$ model, with various boundary conditions including the periodic one and the non-diagonal reflection one. By using the fusion technique, we obtain the complete operator product identities of the fused transfer matrices. Based on them and the asymptotic behaviors as well as the special values at certain points, we obtain the Bethe Ansatz solutions of the system. The method and the results in this paper could be generalized to the $D_n^{(1)}$ case directly.

Acknowledgments

The financial supports from National Program for Basic Research of MOST (Grant Nos. 2016 YFA0300600 and 2016YFA0302104), National Natural Science Foundation of China (Grant Nos. 11934015, 11975183, 11547045, 11774397, 11775178, 11775177 and 91536115), Major Basic Research Program of Natural Science of Shaanxi Province (Grant Nos. 2017KCT-12, 2017ZDJC-32), Australian Research Council (Grant No. DP 190101529), Strategic Priority Research Program of the Chinese Academy of Sciences, and Double First-Class University Construction Project of Northwest University are gratefully acknowledged. J Cao, Y Wang and WL Yang would like to thank Prof. Yao-Zhong Zhang at the school of Mathematics and Physics of The University of Queensland for his hospitality, where some part work of the paper has been done. GL Li acknowledges the support from Shaanxi Province Key Laboratory of Quantum Information and Quantum Optoelectronic Devices, Xi'an Jiaotong University.

A Spinorial R-matrix

In this appendix, we show that the vectorial R-matrix $R^{vv}(u)$ (2.1) and the fused ones $R^{s\pm v}(u)$ (see above (3.12)) can be obtained from the spinorial R-matrix of $D_3^{(1)}$ model [40] by using the fusion. The spinorial R-matrix $R^{ss}_{1'2'}(u)$ of the $D_3^{(1)}$ model is the fundamental

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R-matrix of $su(4)$ one, and is a 16×16 matrix with the form

$$R^{ss}_{1'2'}(u)=\left(\begin{array}{cccc|cccc|cccc|cccc}
u+1 &&&&&&&&&&&&&&& \\
& u &&& 1 &&&&&&&&&&& \\
&& u &&&&&& 1 &&&&&&& \\
&&& u &&&&&&&&& 1 &&& \\ \hline
& 1 &&& u &&&&&&&&&&& \\
&&&&& u+1 &&&&&&&&&& \\
&&&&&& u &&& 1 &&&&&& \\
&&&&&&& u &&&&&& 1 && \\ \hline
&& 1 &&&&&& u &&&&&&& \\
&&&&&& 1 &&& u &&&&&& \\
&&&&&&&&&& u+1 &&&&& \\
&&&&&&&&&&& u &&& 1 & \\ \hline
&&& 1 &&&&&&&&& u &&& \\
&&&&&&& 1 &&&&&& u && \\
&&&&&&&&&&& 1 &&& u & \\
&&&&&&&&&&&&&&& u+1
\end{array}\right).$$

The spinorial R-matrix has following properties

$$\begin{aligned}
&\text{unitarity}: && R^{ss}_{1'2'}(u)R^{ss}_{2'1'}(-u)=(1+u)(1-u),\\
&\text{crossing}-\text{unitarity}: && R^{ss}_{1'2'}(u)^{t_{1'}}R^{ss}_{2'1'}(-u-4)^{t_{1'}}=-u(u+4),
\end{aligned}\tag{A.1}$$

and satisfies the Yang-Baxter equation

$$R^{ss}_{1'2'}(u_1-u_2)R^{ss}_{1'3'}(u_1-u_3)R^{ss}_{2'3'}(u_2-u_3)=R^{ss}_{2'3'}(u_2-u_3)R^{ss}_{1'3'}(u_1-u_3)R^{ss}_{1'2'}(u_1-u_2).\tag{A.2}$$

At the point of $u=-1$, we have

$$R^{ss}_{1'2'}(-1)=P^{(6)}_{1'2'}S^{(6)}_{1'2'},\tag{A.3}$$

where $P^{(6)}_{1'2'}$ is a 6-dimensional projector

$$P^{(6)}_{1'2'}=\sum_{i=1}^{6}|\phi^{(6)}_i\rangle\langle\phi^{(6)}_i|,\tag{A.4}$$

and the corresponding basis vectors are

$$\begin{aligned}
&|\phi^{(6)}_1\rangle=\frac{1}{\sqrt{2}}(|12\rangle-|21\rangle), && |\phi^{(6)}_2\rangle=\frac{1}{\sqrt{2}}(|13\rangle-|31\rangle), && |\phi^{(6)}_3\rangle=\frac{1}{\sqrt{2}}(|14\rangle-|41\rangle),\\
&|\phi^{(6)}_4\rangle=\frac{1}{\sqrt{2}}(|23\rangle-|32\rangle), && |\phi^{(6)}_5\rangle=\frac{1}{\sqrt{2}}(|24\rangle-|42\rangle), && |\phi^{(6)}_6\rangle=\frac{1}{\sqrt{2}}(|34\rangle-|43\rangle).
\end{aligned}$$

The $S^{(6)}_{1'2'}$ is a 6×6 constant matrix omitted here. The fusion of the 6-dimensional projector $P^{(6)}_{1'2'}$ gives

$$P^{(6)}_{2'3'}R^{ss}_{1'2'}\left(u+\frac{1}{2}\right)R^{ss}_{1'3'}\left(u-\frac{1}{2}\right)P^{(6)}_{2'3'}=\left(u-\frac{1}{2}\right)R^{s+v}_{1'\langle 2'3'\rangle}(u),\tag{A.5}$$

$$P^{(6)}_{1'2'}R^{s+v}_{2'3}\left(u+\frac{1}{2}\right)R^{s+v}_{1'3}\left(u-\frac{1}{2}\right)P^{(6)}_{1'2'}=R^{vv}_{\langle 1'2'\rangle 3}(u).\tag{A.6}$$

The dimension of the fused space $\mathbf{V}_{\langle 1'2'\rangle}$ is 6. From eq. (A.5), we obtain the fused R-matrix $R^{s+v}(u)$. The $R^{s-v}(u)$ can be obtained via eq. (3.15). From eq. (A.6), we obtain the vectorial R-matrix $R^{vv}(u)$ (2.1).

For the open case, the spinorial R-matrix $R^{ss}(u)$ and the spinorial reflection matrix $K^s(u)$ satisfy the reflection equation

$$R^{ss}_{1'2'}(u-v)K^s_{1'}(u)R^{ss}_{2'1'}(u+v)K^s_{2'}(v) = K^s_{2'}(v)R^{ss}_{1'2'}(u+v)K^s_{1'}(u)R^{ss}_{2'1'}(u-v). \tag{A.7}$$

One can check that matrix (4.14) is a solution of eq. (A.7), $K^{s+}(u) = K^s(u)$. By using eq. (4.21), we arrive at $K^{s-}(u)$ (4.15). The vectorial reflection matrix $K^v(u)$ (4.8) is obtained from $K^{s+}(u)$ by using the fusion with 6-dimensional projector $P^{(6)}_{1'2'}$

$$P^{(6)}_{1'2'}K^{s+}_{2'}\left(u+\frac{1}{2}\right)R^{ss}_{1'2'}(2u)K^{s+}_{1'}\left(u-\frac{1}{2}\right)P^{(6)}_{2'1'} = \left(u-\frac{1}{2}\right)K^v_{\langle 1'2'\rangle}(u). \tag{A.8}$$

References

[1] R.J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, U.S.A. (1982).

[2] V.E. Korepin, N.M. Bogoliubov and A.G. Izergin, *Quantum inverse scattering method and correlation function*, Cambridge University Press, Cambridge, U.K. (1993).

[3] P. Baseilhac, *The q-deformed analogue of the Onsager algebra: beyond the Bethe ansatz approach*, Nucl. Phys. **B 754** (2006) 309 [math-ph/0604036] [INSPIRE].

[4] P. Baseilhac and K. Koizumi, *Exact spectrum of the XXZ open spin chain from the q-Onsager algebra representation theory*, J. Stat. Mech. **0709** (2007) P09006 [hep-th/0703106] [INSPIRE].

[5] P. Baseilhac and S. Belliard, *Generalized q-Onsager algebras and boundary affine Toda field theories*, Lett. Math. Phys. **93** (2010) 213 [arXiv:0906.1215] [INSPIRE].

[6] P. Baseilhac and S. Belliard, *The half-infinite XXZ chain in Onsager's approach*, Nucl. Phys. **B 873** (2013) 550 [arXiv:1211.6304] [INSPIRE].

[7] H. Frahm, A. Seel and T. Wirth, *Separation of variables in the open XXX chain*, Nucl. Phys. **B 802** (2008) 351 [arXiv:0803.1776] [INSPIRE].

[8] H. Frahm, J.H. Grelik, A. Seel and T. Wirth, *Functional Bethe ansatz methods for the open XXX chain*, J. Phys. **A 44** (2011) 015001 [arXiv:1009.1081] [INSPIRE].

[9] G. Niccoli, *Non-diagonal open spin-1/2 XXZ quantum chains by separation of variables: complete spectrum and matrix elements of some quasi-local operators*, J. Stat. Mech. **1210** (2012) P10025 [arXiv:1206.0646] [INSPIRE].

[10] G. Niccoli, *Antiperiodic spin-1/2 XXZ quantum chains by separation of variables: complete spectrum and form factors*, Nucl. Phys. **B 870** (2013) 397 [arXiv:1205.4537] [INSPIRE].

[11] G. Niccoli, *Antiperiodic dynamical 6-vertex and periodic 8-vertex models I: complete spectrum by SOV and matrix elements of the identity on separate states*, J. Phys. **A 46** (2013) 075003 [arXiv:1207.1928] [INSPIRE].

[12] N. Kitanine, J.-M. Maillet and G. Niccoli, *Open spin chains with generic integrable boundaries: Baxter equation and Bethe ansatz completeness from separation of variables*, *J. Stat. Mech.* **1405** (2014) P05015 [arXiv:1401.4901] [INSPIRE].

[13] S. Belliard and N. Crampé, *Heisenberg XXX model with general boundaries: eigenvectors from algebraic Bethe ansatz*, *SIGMA* **9** (2013) 072 [arXiv:1309.6165] [INSPIRE].

[14] S. Belliard, *Modified algebraic Bethe ansatz for XXZ chain on the segment — I: triangular cases*, *Nucl. Phys.* **B 892** (2015) 1 [arXiv:1408.4840] [INSPIRE].

[15] S. Belliard and R.A. Pimenta, *Modified algebraic Bethe ansatz for XXZ chain on the segment — II: general cases*, *Nucl. Phys.* **B 894** (2015) 527 [arXiv:1412.7511] [INSPIRE].

[16] J. Avan, S. Belliard, N. Grosjean and R.A. Pimenta, *Modified algebraic Bethe ansatz for XXZ chain on the segment — III: proof*, *Nucl. Phys.* **B 899** (2015) 229 [arXiv:1506.02147] [INSPIRE].

[17] J. Cao, W.-L. Yang, K. Shi and Y. Wang, *Off-diagonal Bethe ansatz and exact solution of a topological spin ring*, *Phys. Rev. Lett.* **111** (2013) 137201 [arXiv:1305.7328] [INSPIRE].

[18] Y. Wang, W.-L. Yang, J. Cao and K. Shi, *Off-diagonal Bethe ansatz for exactly solvable models*, Springer, Berlin, Heidelberg, Germany (2015).

[19] R.I. Nepomechie, *An inhomogeneous T-Q equation for the open XXX chain with general boundary terms: completeness and arbitrary spin*, *J. Phys.* **A 46** (2013) 442002 [arXiv:1307.5049] [INSPIRE].

[20] N.Y. Reshetikhin, *The functional equation method in the theory of exactly soluble quantum systems*, *Sov. Phys. JETP* **57** (1983) 691 [*Zh. Eksp. Teor. Fiz.* **84** (1983) 1190].

[21] N.Y. Reshetikhin, *The spectrum of the transfer matrices connected with Kac-Moody algebras*, *Lett. Math. Phys.* **14** (1987) 235.

[22] M.J. Martins and P.B. Ramos, *The algebraic Bethe ansatz for rational braid-monoid lattice models*, *Nucl. Phys.* **B 500** (1997) 579 [hep-th/9703023] [INSPIRE].

[23] G.-L. Li, K.-J. Shi and R.-H. Yue, *Nested Bethe ansatz for the B_N vertex model with open boundary conditions*, *Nucl. Phys.* **B 696** (2004) 381 [INSPIRE].

[24] Y.-H. Tian, H.-L. Chen and X.-Q. Liu, *Periodic wave solutions to dispersive long-wave equations in $(2+1)$-dimensional space*, *Commun. Theor. Phys.* **44** (2005) 8.

[25] G.-L. Li and K.-J. Shi, *The algebraic Bethe ansatz for open vertex models*, *J. Stat. Mech.* **0701** (2007) P01018 [hep-th/0611127] [INSPIRE].

[26] J. Cao, W.-L. Yang, K. Shi and Y. Wang, *Nested off-diagonal Bethe ansatz and exact solutions of the* SU(N) *spin chain with generic integrable boundaries*, *JHEP* **04** (2014) 143 [arXiv:1312.4770] [INSPIRE].

[27] J. Cao, S. Cui, W.-L. Yang, K. Shi and Y. Wang, *Exact spectrum of the spin-s Heisenberg chain with generic non-diagonal boundaries*, *JHEP* **02** (2015) 036 [arXiv:1405.2692] [INSPIRE].

[28] K. Hao, J. Cao, G.-L. Li, W.-L. Yang, K. Shi and Y. Wang, *Exact solution of an* SU(N) *spin torus*, *J. Stat. Mech.* **1607** (2016) 073104 [arXiv:1601.04389] [INSPIRE].

[29] K. Hao, J. Cao, G.-L. Li, W.-L. Yang, K. Shi and Y. Wang, *Exact solution of the Izergin-Korepin model with general non-diagonal boundary terms*, *JHEP* **06** (2014) 128 [arXiv:1403.7915] [INSPIRE].

JHEP12(2019)051

[30] G.-L. Li et al., *Off-diagonal Bethe ansatz on the* SO(5) *spin chain*, *Nucl. Phys.* **B 946** (2019) 114719 [arXiv:1902.08891] [INSPIRE].

[31] G.-L. Li et al., *Exact solution of the* $sp(4)$ *integrable spin chain with generic boundaries*, *JHEP* **05** (2019) 067 [arXiv:1812.03618] [INSPIRE].

[32] M. Karowski, *On the bound state problem in* $(1+1)$*-dimensional field theories*, *Nucl. Phys.* **B 153** (1979) 244 [INSPIRE].

[33] P.P. Kulish, N. Yu. Reshetikhin and E.K. Sklyanin, *Yang-Baxter equation and representation theory*. 1, *Lett. Math. Phys.* **5** (1981) 393 [INSPIRE].

[34] P.P. Kulish and E.K. Sklyanin, *Quantum spectral transform method. Recent developments*, *Lect. Notes Phys.* **151** (1982) 61 [INSPIRE].

[35] A.N. Kirillov and N.Y. Reshetikhin, *Exact solution of the Heisenberg XXZ model of spin* s, *J. Sov. Math.* **35** (1986) 2627.

[36] A.N. Kirillov and N.Y. Reshetikhin, *Exact solution of the integrable XXZ Heisenberg model with arbitrary spin. I. The ground state and the excitation spectrum*, *J. Phys.* **A 20** (1987) 1565 [INSPIRE].

[37] A. Lima-Santos and R. Malara, $C_n^{(1)}$*,* $D_n^{(1)}$ *and* $A_{2n-1}^{(2)}$ *reflection* K*-matrices*, *Nucl. Phys.* **B 675** (2003) 661 [nlin/0307046] [INSPIRE].

[38] R. Malara and A. Lima-Santos, *On* $A_{n-1}^{(1)}$*,* $B_n^{(1)}$*,* $C_n^{(1)}$*,* $D_n^{(1)}$*,* $A_{2n}^{(2)}$*,* $A_{2n-1}^{(2)}$ *and* $D_{n+1}^{(2)}$ *reflection* K*-matrices*, *J. Stat. Mech.* **0609** (2006) P09013 [nlin/0412058] [INSPIRE].

[39] W.-L. Yang and Y.-Z. Zhang, T*-*Q *relation and exact solution for the XYZ chain with general nondiagonal boundary terms*, *Nucl. Phys.* **B 744** (2006) 312 [hep-th/0512154] [INSPIRE].

[40] D. Chicherin, S. Derkachov and A.P. Isaev, *The spinorial* R*-matrix*, *J. Phys.* **A 46** (2013) 485201 [arXiv:1303.4929] [INSPIRE].