



Randomized box–ball systems, limit shape of rigged configurations and thermodynamic Bethe ansatz

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Abstract

We introduce a probability distribution on the set of states in a generalized box–ball system associated with Kirillov–Reshetikhin (KR) crystals of type $A_n^{(1)}$. Their conserved quantities induce n -tuple of random Young diagrams in the rigged configurations. We determine their limit shape as the system gets large by analyzing the Fermionic formula by thermodynamic Bethe ansatz. The result is expressed as a logarithmic derivative of a deformed character of the KR modules and agrees with the stationary local energy of the associated Markov process of carriers.

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1. Background and main results

1.1. Box–ball systems

The box–ball system (BBS) [40] is an integrable cellular automaton in $1 + 1$ dimension. By now it has been generalized widely, and numerous aspects have been explored connected to quantum groups, crystal base theory (theory of quantum groups at $q = 0$), solvable lattice

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For the n -color BBS in general, there are an n -tuple of Young diagrams μ_1, \dots, μ_n in which each row is assigned with an integer. The n -tuple of Young diagrams and the assigned integers are called *configuration* and *rigging*, respectively. Thus in short, the action-angle variables of the BBS are *rigged configurations* [26]. It is the configuration that is preserved and the riggings that change linearly in time. One indeed sees that the first Young diagram $\mu_1 = (4, 3, 2)$ gives the list of amplitude of solitons which remains invariant under the time evolution. The other ones μ_2, \dots, μ_n are “higher” conserved quantities reflecting the internal degrees of freedom of solitons.² The n -color BBS is endowed with the higher time evolutions besides the simplest one $\mathcal{K}_2 \circ \mathcal{K}_3 \circ \dots \circ \mathcal{K}_{n+1}$ mentioned before. They are all commutative and change the riggings attached to μ_2, \dots, μ_n linearly.

Rigged configurations for type A_n has been formulated most generally in [22] extending the invention [19,20] in the 1980’s. These works were devoted to a proof of the *Fermionic formula* for a generalized Kostka–Foulkes polynomials (cf. [30]) by establishing an elaborate bijection between rigged configurations and other combinatorial objects. Roughly speaking in the context of BBS, the bijection provides the direct and inverse scattering maps [25,26]

$$\{\text{BBS states}\} \longleftrightarrow \{\text{action-angle variables}\}, \quad (1)$$

which transform the nonlinear dynamics in BBS to a straight motion. The n -tuple of Young diagrams form a label of the *iso-level sets* of BBS. The Fermionic formula tells the *multiplicity* of a given iso-level set.

1.3. Randomized BBS and main result

Now let us embark on a randomized version of the story. We assume that some probability distribution on the set of BBS states has been introduced. Then it is natural to ask;

- (i) What is the probability measure on the n -tuple of Young diagrams induced by (1)?
- (ii) What is the limit shape of them when the system size L of the BBS tends to infinity?

In this paper we answer (i) and (ii) for the most general BBS associated with the Kirillov–Reshetikhin (KR) crystals [17] of the quantum affine algebra $U_q(A_n^{(1)})$. The randomness of the BBS states we will be concerned with is the product of the one at each site. The latter is the probability distribution on a single KR crystal which is proportional to the exponential weight e^{wt} . (See (16).) Under this simple choice, the answer to (i) is given by the Fermionic form itself multiplied with the Boltzmann factor accounting for the e^{wt} contribution as a chemical potential term. (See (43).)

The Fermionic form measure is quite distinct in nature from the well studied ones like the Plancherel measure for the symmetric group and/or its Poissonized versions. It fits an asymptotic analysis by the *thermodynamic Bethe ansatz* (TBA) [42]. The method employs the idea of the grand canonical ensemble and captures the equilibrium characteristics of the system by a variational principle. The equilibrium condition shows up as the so called TBA equation. It plays a central role together with the equation of state connecting the density of balls with fugacity. Our TBA analysis is essentially the spectral parameter free version of [24, sec.14]. In particular the

² Unfortunately to extract them is not so simple and requires a nested Bethe ansatz (Gelfand-Zetlin) type resolution [26].

Y-system and the Q-system $(Q_i^{(a)})^2 = Q_{i-1}^{(a)} Q_{i+1}^{(a)} + \prod_{b \sim a} Q_i^{(b)}$ for the character $Q_i^{(a)}$ of the KR module come into the game naturally.

It turns out that a proper scaling of the Young diagrams is to shrink them vertically by $1/L$. This feature will be established in [23] by invoking the large deviation principle. The resulting limit shape is described by the logarithmic derivative of the deformed character of the KR modules as

$$\lim_{L \rightarrow \infty} \frac{1}{L} (\# \text{ of boxes in the left } i \text{ columns of } \mu_a) = \left. \frac{\partial \log(Q_i^{(a)} * Q_s^{(r)})}{\partial w} \right|_{w=1}. \tag{2}$$

See (72) and (24) for the definition of the deformed character $Q_i^{(a)} * Q_s^{(r)}$ for $(a, i), (r, s) \in [1, n] \times \mathbb{Z}_{\geq 1}$.³ The data (r, s) is specified according to the choice of the set of local states in the BBS.⁴ The quantity (2) coincides with the *stationary local energy* of a carrier in the relevant KR crystal derived in (21). Independent variables in the deformed characters are linked to the prescribed fugacity of the BBS by the equation of state (63) or equivalently (66). This general and intrinsic answer to the above question (ii) is our main result in this paper. Further concrete formulas are available for the simplest n -color BBS [38] in terms of Schur functions in (93) and (94). It will be interesting to investigate the results in this paper further in the light of recent results on the BBS from probabilistic viewpoints [3,5,23,29].

1.4. Outline of the paper

In Section 2 we recall basic facts on generalized BBS necessary in this paper. In Section 3 we consider the BBS in a randomized setting. It amounts to introducing a Markov process of carriers associated to each time evolution $T_i^{(a)}$. We construct a stationary measure of the process quite generally by the character of the relevant KR modules (Proposition 3.2). It leads to the stationary local energy (21) or equivalently (22). In Section 4 we recall the Fermionic formula based on [10,22] as a preparation for subsequent sections. The deformed character in (37), (72) and its logarithmic derivative will be the building blocks in describing the limit shape. Section 5 is the main part of the paper. We identify the Fermionic form with the probability measure on the n -tuple of Young diagrams induced from the randomized BBS via its conserved quantities. By a TBA analysis, a difference equation characterizing the limit shape of the Young diagrams is derived. Our main result is Theorem 5.1, which identifies the solution to the difference equation with the stationary local energy obtained in Section 3. It reveals a new connection between TBA and crystal theory via the limit shape problem. In Section 6 we deal with the simplest example corresponding to the n -color BBS in [38]. The scaled column length of the Young diagrams are given explicitly in terms of the Schur function involving the ball densities. We check the result against the stationary local energy of a randomly generated BBS states numerically and confirm a good agreement. Section 7 contains a summary and discussions. We conjecturally describe the difference equation and its solution like Theorem 5.1 uniformly for the BBS associated with the simply-laced quantum affine algebras $U_q(\hat{\mathfrak{g}})$ with $\hat{\mathfrak{g}} = A_n^{(1)}, D_n^{(1)}, E_{6,7,8}^{(1)}$. We also suggest some future problems as concluding remarks.

Throughout the paper we use the notation $\theta(\text{true}) = 1, \theta(\text{false}) = 0$.

³ In general $Q_{l_1}^{(k_1)} * \dots * Q_{l_m}^{(k_m)}$ is a w -deformation of the product $Q_{l_1}^{(k_1)} \dots Q_{l_m}^{(k_m)}$. Representation theoretically, it is the character of a generalized Demazure module [31].

⁴ The original n -color BBS [38] corresponds to the choice $(r, s) = (1, 1)$.

2. Box–ball systems

2.1. KR crystals

Consider the classical simple Lie algebra of type A_n . We denote its Cartan matrix by $(C_{ab})_{a,b=1}^n$, where $C_{ab} = C_{ba} = 2\delta_{ab} - \theta(a \sim b)$ and $a \sim b$ means that the two nodes a and b are connected by a bond in the Dynkin diagram, i.e. $|a - b| = 1$. Let $\varpi_1, \dots, \varpi_n$ be the fundamental weights and $\alpha_1, \dots, \alpha_n$ be the simple roots. They are related by $\alpha_a = \sum_{b=1}^n C_{ab}\varpi_b$. We use the set of positive roots Δ_+ , the weight lattice $P = \bigoplus_{a=1}^n \mathbb{Z}\varpi_a$, the root lattice $Q = \bigoplus_{a=1}^n \mathbb{Z}\alpha_a$ and their subsets $P_+ = \sum_{a=1}^n \mathbb{Z}_{\geq 0}\varpi_a$, $Q_+ = \sum_{a=1}^n \mathbb{Z}_{\geq 0}\alpha_a$. Denote the irreducible module with highest weight $\lambda \in P_+$ by $V(\lambda)$ and its character by $\text{ch } V(\lambda)$. The latter belongs to $\mathbb{Z}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ where $z_a = e^{\varpi_a}$.

Let $A_n^{(1)}$ be the non-twisted affinization of A_n [16] and $U_q = U_q(A_n^{(1)})$ be the quantum affine algebra (without derivation operator) [4,15]. There is a family of irreducible finite-dimensional representations $\{W_s^{(r)} \mid (r, s) \in [1, n] \times \mathbb{Z}_{\geq 0}\}$ of U_q called Kirillov–Reshetikhin (KR) module⁵ named after the related work on the Yangian [21]. As a representation of A_n , $W_s^{(r)}$ is isomorphic to $V(s\varpi_r)$. $W_s^{(r)}$ is known to have a crystal base $B_s^{(r)}$ [18,17]. Roughly speaking, it is a set of basis vectors of a U_q -module at $q = 0$. $B_s^{(r)}$ is called a KR crystal. It is identified with the set of semistandard tableaux of rectangular shape (s^r) with letters from $\{1, 2, \dots, n + 1\}$. The highest weight element of $B_s^{(r)}$, which is the tableau whose j -th row is all j , is denoted by $u_s^{(r)}$. For two crystals B_1, B_2 their tensor product $B_1 \otimes B_2$ is well defined, and as a set $B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}$.

Before explaining necessary ingredients related to KR crystals, we review a notion of tableau product $S \cdot T$ for two tableaux S, T . Let $\text{row}(T)$ be a row word of a tableau T . It is obtained by reading letters from bottom to top, left to right in each row. Let $\text{row}(T) = u_1 u_2 \cdots u_l$ and we apply to S the row bumping algorithm [8] successively as

$$(\cdots((S \leftarrow u_1) \leftarrow u_2) \leftarrow \cdots) \leftarrow u_l.$$

The resulting tableau is nothing but $S \cdot T$. Alternatively, let $\text{row}(S) = v_1 v_2 \cdots v_m$ and apply to T the column bumping algorithm successively as

$$v_1 \rightarrow (v_2 \rightarrow (\cdots(v_m \rightarrow T) \cdots)).$$

The result also gives $S \cdot T$.

We are ready to review the combinatorial R and the (local) energy H . Let b, c be elements of $B_i^{(a)}, B_s^{(r)}$ represented as tableaux. The combinatorial R is a bijection $R : B_i^{(a)} \otimes B_s^{(r)} \rightarrow B_s^{(r)} \otimes B_i^{(a)}$ and the energy H is a \mathbb{Z} -valued function on $B_i^{(a)} \otimes B_s^{(r)}$ determined by the following combinatorial rule [36]. The image of R is given in such a way that $R(b \otimes c) = \tilde{c} \otimes \tilde{b}$ is equivalent to $c \cdot b = \tilde{b} \cdot \tilde{c}$. The fact that for each $c \cdot b$ there is a unique such pair (\tilde{b}, \tilde{c}) is assured since the decomposition of the tensor product module $V(i\varpi_a) \otimes V(s\varpi_r)$ is multiplicity free. The value $H(b \otimes c) (= H(\tilde{c} \otimes \tilde{b}))$ is defined to be the number of nodes strictly below the $\max(a, r)$ -th row of the tableau $c \cdot b$. By definition, H is nonnegative and $H(u_i^{(a)} \otimes u_s^{(r)}) = 0$. The combinatorial R satisfies the Yang–Baxter equation (cf. [1]).

⁵ The actual KR modules carries a spectral parameter. In this paper it is irrelevant and hence suppressed.

Example 2.1. Consider the $A_3^{(1)}$ case. If $b = 1\ 2\ 2\ 3\ 4\ 4 \in B_6^{(1)}, c = 1\ 2\ 2\ 4 \in B_4^{(1)}$, the tableau product $c \cdot b$, the image of R and the value $H(b \otimes c)$ are given by

$$\begin{array}{c} 1\ 1\ 2\ 2\ 2\ 3\ 4\ 4 \\ 2\ 4 \end{array}, \quad 1\ 3\ 4\ 4 \otimes 1\ 2\ 2\ 2\ 2\ 4, \quad H = 2.$$

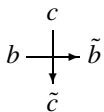
If $b = \begin{array}{c} 1\ 2\ 3 \\ 2\ 4\ 4 \end{array} \in B_3^{(2)}, c = \begin{array}{c} 1\ 1 \\ 2\ 3 \\ 3\ 4 \end{array} \in B_2^{(3)}$, then $row(b) = 214243, row(c) = 321431$ and they are

$$\begin{array}{c} 1\ 1\ 1\ 2\ 3 \\ 2\ 2\ 4\ 4 \\ 3\ 3 \\ 4 \end{array}, \quad \begin{array}{c} 1\ 1 \\ 2\ 3 \\ 3\ 4 \end{array} \otimes \begin{array}{c} 1\ 2\ 2 \\ 3\ 4\ 4 \end{array}, \quad H = 1.$$

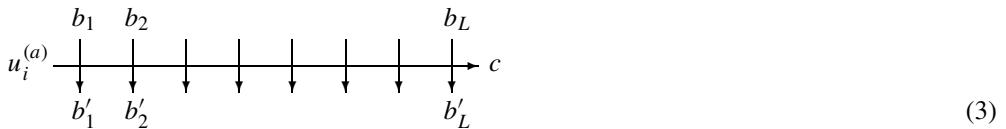
2.2. Deterministic box–ball system

The original BBS was introduced in [40]. Since then it has been generalized from various viewpoints. One of such generalizations was done by using KR crystals as we describe below.

Suppose for $b \otimes c \in B_i^{(a)} \otimes B_s^{(r)}$ we have $R(b \otimes c) = \tilde{c} \otimes \tilde{b}$. We illustrate it by



Take a sufficiently large integer L and consider $B_i^{(a)} \otimes (B_s^{(r)})^{\otimes L}$. Apply the combinatorial R on the j -th and $(j + 1)$ -th component (from the left) successively for $j = 1, 2, \dots, L$ to the element $u_i^{(a)} \otimes b_1 \otimes b_2 \otimes \dots \otimes b_L$ of $B_i^{(a)} \otimes (B_s^{(r)})^{\otimes L}$. Let the output be $b'_1 \otimes b'_2 \otimes \dots \otimes b'_L \otimes c$ of $(B_s^{(r)})^{\otimes L} \otimes B_i^{(a)}$. Graphically, it can be shown as below.



We call $B_i^{(a)}$ (and its elements) the *carrier* and $(B_s^{(r)})^{\otimes L}$ the *quantum space*. The map $T_i^{(a)} : (B_s^{(r)})^{\otimes L} \rightarrow (B_s^{(r)})^{\otimes L}$ defined by $b_1 \otimes b_2 \otimes \dots \otimes b_L \mapsto b'_1 \otimes b'_2 \otimes \dots \otimes b'_L$ is called the *time evolution operator*. Since $R(u_i^{(a)} \otimes u_s^{(r)}) = u_s^{(r)} \otimes u_i^{(a)}$, we have $T_i^{(a)}((u_s^{(r)})^{\otimes L}) = (u_s^{(r)})^{\otimes L}$ and $c = u_i^{(a)}$. Hence, $(u_s^{(r)})^{\otimes L}$ can be thought of as the vacuum state. If $a = r$ and $b_j = u_s^{(r)}$ for $j \geq J$ with such J that $L - J$ is sufficiently large, one can conclude $c = u_i^{(a)}$. However, if $a \neq r$, c is not always $u_i^{(a)}$, which means that some particles may be snatched away by $T_i^{(a)}$. To prevent such situations, we extend the quantum space $(B_s^{(r)})^{\otimes L}$ by tensoring $(B_1^{(a)})^{\otimes L'}$ for sufficiently large L' from right. Then one can assume c is always $u_i^{(a)}$. We call this extra tensor factor the *barrier*.

Next we recall the conserved quantity under the time evolution $T_i^{(a)}$ introduced for $a = 1$ in [7] for the n -color BBS [38]. In order to make c in (3) to be $u_i^{(a)}$ we attach the barrier if necessary and assume the number of the tensor factors in the quantum space to be L . Define c_j ($j = 1, 2, \dots, L$) by $c_0 = u_i^{(a)}, R(c_{j-1} \otimes b_j) = b'_j \otimes c_j$. The definition corresponds to setting the j -th vertex in the previous diagram as follows.

[9,7]. As we will see in Section 3, carriers also play a fundamental role in the randomized BBS via their *Markov processes*.

2.3. Rigged configuration as action angle variables

Here we review a combinatorial object called rigged configuration and see how it is used to linearize the BBS dynamics. Rigged configurations are defined based on data $\{(k_j, l_j)\}_{1 \leq j \leq L}$ such that $(k_j, l_j) \in [1, n] \times \mathbb{Z}_{\geq 1}$. Through the Kirillov–Schilling–Shimozono (KSS) bijection which we discuss later, it is related to the tensor product of KR crystals $B_{l_1}^{(k_1)} \otimes \cdots \otimes B_{l_L}^{(k_L)}$. A rigged configuration consists of a configuration, an n -tuple of Young diagrams μ_1, \dots, μ_n , and riggings, sequence of nonnegative integers attached to each row of μ_a for $a \in [1, n]$. Let $m_i^{(a)}$ be the number of rows of length i in μ_a . Define $e_i^{(a)}$ and *vacancy* $p_i^{(a)}$ by

$$p_i^{(a)} = \sum_{j=1}^L \delta_{a,k_j} \min(i, l_j) - \sum_{b=1}^n C_{ab} e_i^{(b)}, \tag{5}$$

$$e_i^{(a)} = \sum_{j \geq 1} \min(i, j) m_j^{(a)}. \tag{6}$$

A configuration is required to satisfy $p_i^{(a)} \geq 0$ for any $(a, i) \in [1, n] \times \mathbb{Z}_{\geq 1}$ and riggings of the rows of length i in μ_a must not exceed $p_i^{(a)}$. Among riggings of the rows of the same length in μ_a , the order does not matter. So we label riggings in non increasing order when going downwards. From these definitions one can immediately write down the number of the rigged configurations with the prescribed configuration μ_1, \dots, μ_n as

$$\prod_{1 \leq a \leq n, i \geq 1} \binom{p_i^{(a)} + m_i^{(a)}}{m_i^{(a)}}. \tag{7}$$

This is an ultimate generalization of the celebrated Bethe formula [2, eq.(45)] due to [19,20,22] for type $A_n^{(1)}$. See [11, sec.1] for a historical account and [24, sec.13] for a concise review. We will come back to this *Fermionic form* as the main object of the TBA analysis in Section 5.

The KSS bijection [22] gives an algorithm to construct an element of the tensor product of KR crystals $B = B_{l_1}^{(k_1)} \otimes \cdots \otimes B_{l_L}^{(k_L)}$ from a rigged configuration. The image of this bijection consists of special elements which we call *highest states*. Representation theoretically, they correspond to highest weight vectors of B . It is equivalent to saying that the tableau product $b_L \cdots b_1$ ($b_j \in B_{l_j}^{(k_j)}$) is a tableau such that letters in the i -th row are all i .

The KSS bijection separates the BBS states into action and angle variables. It is known [26] that if b is a highest state, then the application of $T_i^{(a)}$ causes, in the rigged configuration side, the increase of riggings by $\delta_{ac} \min(i, l)$ when they are attached to the length l row of μ_c .

Identifying rigged configurations originating in the Bethe ansatz [2] with action-angle variables of BBS implies a correspondence between Bethe strings in the former and solitons in the latter. This is natural as we will also comment in the end of Section 5.1. As far as the action variables are concerned, this *soliton/string correspondence* [25,26] is quantified most generally as [35]

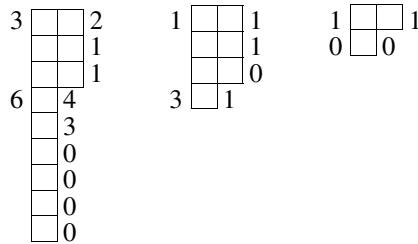
$$E_i^{(a)} = e_i^{(a)}. \tag{8}$$

Remember that the LHS is the row transfer matrix energy in (4), which was indeed known (for $a = 1$) to measure the amplitude of solitons [7] for the original n -color BBS [38]. The RHS is defined by (6) from the rigged configuration which is essentially an assembly of Bethe strings [2,19,20]. Thus the LHS and the RHS in (8) are referring to solitons and strings, respectively. Our main result Theorem 5.1 in this paper may be regarded as a generalization of (8) to a randomized situation.

Example 2.3. We give examples of the KSS bijection for $A_3^{(1)}$. An element of $(B_1^{(1)})^{\otimes 20}$

$$1 \otimes 2 \otimes 1 \otimes 2 \otimes 3 \otimes 4 \otimes 1 \otimes 1 \otimes 3 \otimes 2 \otimes 1 \otimes 1 \otimes 2 \otimes 3 \otimes 2 \otimes 1 \otimes 3 \otimes 4 \otimes 4 \otimes 1$$

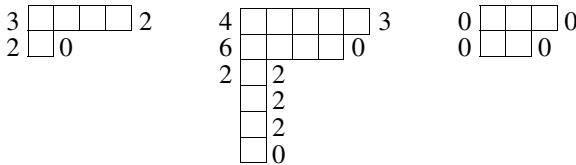
is a highest state which corresponds to the rigged configuration below. The numbers left to the Young diagram are vacancies. $p_2^{(1)} = 3, p_1^{(1)} = 6, p_2^{(2)} = 1, p_1^{(2)} = 3, p_2^{(3)} = 1, p_1^{(3)} = 0$.



Similarly,

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 1 & 2 & 1 & 3 \\ 2 & 2 & 2 & 3 & 4 & 3 & 3 & 2 & 2 & 2 & 4 & 4 & 3 & 3 & 2 & 4 \end{matrix} \otimes \dots \otimes$$

is a highest state of $(B_2^{(2)})^{\otimes 10}$ which corresponds to the following one.



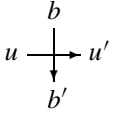
3. Randomized box–ball system

3.1. Markov process of carrier

Now we introduce a randomized version of BBS. Let $\pi_s^{(r)} : B_s^{(r)} \rightarrow \mathbb{R}_{>0}$ be a probability distribution meaning that $\sum_{b \in B_s^{(r)}} \pi_s^{(r)}(b) = 1$. We consider the ensemble of the states of the BBS on $(B_s^{(r)})^{\otimes L}$ in which the local states are independent and identically distributed (i.i.d.) according to $\pi_s^{(r)}$. Taking them as the initial condition, we apply a time evolution $T_i^{(a)}$. The carrier from $B_i^{(a)}$ proceeds to the right interacting with the random local states successively by the combinatorial R on $B_i^{(a)} \otimes B_s^{(r)}$. The bombardment by the random local states induces a stochastic process of the carrier. It is the Markov process on $B_i^{(a)}$ whose transition rate is specified as

$$\text{Rate}(u \rightarrow u') = \sum_{b, b' \in B_s^{(r)}, R(u \otimes b) = b' \otimes u'} \pi_s^{(r)}(b) \quad (u, u' \in B_i^{(a)}). \tag{9}$$

The condition on the sum is depicted as a vertex in (3) as



We have $\sum_{u' \in B_i^{(a)}} \text{Rate}(u \rightarrow u') = \sum_{b \in B_s^{(r)}} \pi_s^{(r)}(b) = 1$ indeed. In [23] it has been shown that this Markov process is irreducible and has the unique stationary measure for $r = s = 1$. We conjecture and assume the irreducibility for general r, s in what follows. Denote the resulting stationary measure by $\tilde{\pi}_i^{(a)} : B_i^{(a)} \rightarrow \mathbb{R}_{>0}$.

Example 3.1. Consider $A_2^{(1)}$ BBS with local states from $B_s^{(r)} = B_1^{(1)} = \{1, 2, 3\}$ and carrier from $B_i^{(a)} = B_2^{(1)} = \{11, 12, 13, 22, 23, 33\}$. Take $\pi_s^{(r)} : B_s^{(r)} \rightarrow \mathbb{R}_{>0}$ as $\pi_1^{(1)}(b) = p_b$ ($b = 1, 2, 3$) with some p_b obeying $p_1 + p_2 + p_3 = 1$. The combinatorial R is given by

$$\begin{array}{lll} 11 \otimes 1 \mapsto 1 \otimes 11 & 11 \otimes 2 \mapsto 1 \otimes 12 & 11 \otimes 3 \mapsto 1 \otimes 13, \\ 12 \otimes 1 \mapsto 2 \otimes 11 & 12 \otimes 2 \mapsto 1 \otimes 22 & 12 \otimes 3 \mapsto 2 \otimes 13, \\ 13 \otimes 1 \mapsto 3 \otimes 11 & 13 \otimes 2 \mapsto 1 \otimes 23 & 13 \otimes 3 \mapsto 1 \otimes 33, \\ 22 \otimes 1 \mapsto 2 \otimes 12 & 22 \otimes 2 \mapsto 2 \otimes 22 & 22 \otimes 3 \mapsto 2 \otimes 23, \\ 23 \otimes 1 \mapsto 3 \otimes 12 & 23 \otimes 2 \mapsto 3 \otimes 22 & 23 \otimes 3 \mapsto 2 \otimes 33, \\ 33 \otimes 1 \mapsto 3 \otimes 13 & 33 \otimes 2 \mapsto 3 \otimes 23 & 33 \otimes 3 \mapsto 3 \otimes 33. \end{array}$$

By the definition the transitions in the first, second and the third column happen with probabilities p_1, p_2, p_3 , respectively. Thus denoting $\tilde{\pi}_2^{(1)}(23) = \tilde{\pi}_{23}$ etc for short, the stationary condition reads

$$\begin{aligned} \tilde{\pi}_{11} &= p_1 \tilde{\pi}_{11} + p_1 \tilde{\pi}_{12} + p_1 \tilde{\pi}_{13}, \\ \tilde{\pi}_{12} &= p_1 \tilde{\pi}_{22} + p_1 \tilde{\pi}_{23} + p_2 \tilde{\pi}_{11}, \\ \tilde{\pi}_{13} &= p_1 \tilde{\pi}_{33} + p_3 \tilde{\pi}_{11} + p_3 \tilde{\pi}_{12}, \\ \tilde{\pi}_{22} &= p_2 \tilde{\pi}_{12} + p_2 \tilde{\pi}_{22} + p_2 \tilde{\pi}_{23}, \\ \tilde{\pi}_{23} &= p_2 \tilde{\pi}_{13} + p_2 \tilde{\pi}_{33} + p_3 \tilde{\pi}_{22}, \\ \tilde{\pi}_{33} &= p_3 \tilde{\pi}_{13} + p_3 \tilde{\pi}_{23} + p_3 \tilde{\pi}_{33}. \end{aligned}$$

Under the assumption $p_1 + p_2 + p_3 = 1$, these equations admit a unique solution $(\tilde{\pi}_{ij})$ such that $\sum_{1 \leq i \leq j \leq 3} \tilde{\pi}_{ij} = 1$. For instance let us parametrize p_i as

$$p_1 = \frac{z_1^2 z_2}{z_1 + z_1^2 z_2 + z_2^2}, \quad p_2 = \frac{z_2^2}{z_1 + z_1^2 z_2 + z_2^2}, \quad p_3 = \frac{z_1}{z_1 + z_1^2 z_2 + z_2^2}. \tag{10}$$

Then $\tilde{\pi}_{ij}$ is given by

$$\tilde{\pi}_{11} = \frac{z_1^4 z_2^2}{Q}, \quad \tilde{\pi}_{12} = \frac{z_1^2 z_2^3}{Q}, \quad \tilde{\pi}_{13} = \frac{z_1^3 z_2}{Q}, \quad \tilde{\pi}_{22} = \frac{z_2^4}{Q}, \quad \tilde{\pi}_{23} = \frac{z_1 z_2^2}{Q}, \quad \tilde{\pi}_{33} = \frac{z_1^2}{Q} \tag{11}$$

with $Q = z_1^2 + z_1^3 z_2 + z_1 z_2^2 + z_1^4 z_2^2 + z_1^2 z_2^3 + z_2^4$.

3.2. Stationary measure of carrier

To generalize Example 3.1 is straightforward. The stationary measure $\tilde{\pi}_i^{(a)}$ is determined by the following stationary condition of the carrier process:

$$\tilde{\pi}_i^{(a)}(u') = \sum_{u \in B_i^{(a)}} \tilde{\pi}_i^{(a)}(u) \text{Rate}(u \rightarrow u') = \sum_{u \otimes b \in B_i^{(a)} \otimes B_s^{(r)}, R(u \otimes b) \in B_s^{(r)} \otimes u'} \tilde{\pi}_i^{(a)}(u) \pi_s^{(r)}(b), \tag{12}$$

where the latter equality follows from (9). The carrier and local states are taken from $B_i^{(a)}$ and $B_s^{(r)}$, respectively.

For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{n+1})$, let $s_\lambda(w_1, \dots, w_{n+1})$ denote the associated Schur polynomial [30]:

$$s_\lambda(w_1, \dots, w_{n+1}) = \frac{\det(w_k^{\lambda_j + n + 1 - j})_{j,k=1}^{n+1}}{\det(w_k^{n+1-j})_{j,k=1}^{n+1}}. \tag{13}$$

This is the well-known Weyl formula for the character $\text{ch } V(\lambda)$ under the identification of λ with $\sum_{i=1}^n (\lambda_i - \lambda_{i+1}) \varpi_i \in P_+$. We use a special notation when λ is a rectangle.

$$Q_i^{(a)} = \text{ch } V(i\varpi_a) = \sum_{b \in B_i^{(a)}} e^{\text{wt}(b)} = s_{(i^a)}(w_1, \dots, w_{n+1}), \tag{14}$$

$$w_j = z_{j-1}^{-1} z_j \quad (1 \leq j \leq n + 1), \quad z_0 = z_{n+1} = 1. \tag{15}$$

The following proposition gives an explicit expression for the stationary measure $\tilde{\pi}_i^{(a)}$ under a particular choice of the measure $\pi_s^{(r)}$ for the local states.

Proposition 3.2. *The choice*

$$\tilde{\pi}_i^{(a)}(u) = \pi_i^{(a)}(u) = \frac{\exp \text{wt}(u)}{Q_i^{(a)}} \in \mathbb{R}(z_1, \dots, z_n) \quad (u \in B_i^{(a)}) \tag{16}$$

for any $(a, i) \in [1, n] \times \mathbb{Z}_{\geq 1}$ satisfies the stationary condition (12) and the normalization condition $\sum_{u \in B_i^{(a)}} \pi_i^{(a)}(u) = 1$.

Proof. Consider the combinatorial R

$$R: B_i^{(a)} \otimes B_s^{(r)} \rightarrow B_s^{(r)} \otimes B_i^{(a)} \tag{17}$$

$$u \otimes b \mapsto b' \otimes u'. \tag{18}$$

The combinatorial R preserves the weight wt . Take the $\exp \text{wt}$ of the both sides of (18) and sum over $b' \in B_s^{(r)}$ fixing $u' \in B_i^{(a)}$. The result reads

$$Q_i^{(a)} Q_s^{(r)} \sum_{u \otimes b \in B_i^{(a)} \otimes B_s^{(r)}, R(u \otimes b) \in B_s^{(r)} \otimes u'} \pi_i^{(a)}(u) \pi_s^{(r)}(b) = Q_s^{(r)} \exp \text{wt}(u'). \tag{19}$$

This yields (12) with $\tilde{\pi}_i^{(a)} = \pi_i^{(a)}$ by dividing by $Q_i^{(a)} Q_s^{(r)}$. The normalization condition is obvious from (14). \square

Proposition 3.2 tells that as long as the randomness $\pi_s^{(r)}$ of the local states are taken to be proportional to e^{wt} as in (16), the stationary measure $\tilde{\pi}_i^{(a)}$ of the carrier for $T_i^{(a)}$ is independent of the choice of the KR crystal $B_s^{(r)}$, and it is also given by the same formula. This is reminiscent of the integrability of the original (deterministic) BBS. For $A_2^{(1)}$, the KR module $W_s^{(1)}$ is isomorphic to the degree s symmetric tensor representation of sl_3 , and (16) indeed reproduces Example 3.1 if z_a is identified with e^{ϖ_a} .

3.3. Stationary local energy

In view of Proposition 3.2 we write the probability distribution $\tilde{\pi}_i^{(a)}$ of the carrier also as $\pi_i^{(a)}$ from now on. It contains n real positive parameters.

Now let us calculate the row transfer matrix energy per site in the stationary state:

$$h_i^{(a)} = \lim_{L \rightarrow \infty} \frac{1}{L} E_i^{(a)}. \tag{20}$$

See (4) for the definition of $E_i^{(a)}$. On average the carrier $x \in B_i^{(a)}$ and a local state $y \in B_s^{(r)}$ arrive at a vertex with the probability $\pi_i^{(a)}(x)\pi_s^{(r)}(y)$. Their encounter produces the local energy $H(x \otimes y)$. Thus we have

$$h_i^{(a)} = \sum_{x \otimes y \in B_i^{(a)} \otimes B_s^{(r)}} H(x \otimes y) \pi_i^{(a)}(x) \pi_s^{(r)}(y). \tag{21}$$

We call this *stationary local energy* for the carrier from $B_i^{(a)}$ or for the time evolution $T_i^{(a)}$. Note that $h_i^{(a)}$ does depend on the choice of the set of local states $B_s^{(r)}$ although it is suppressed in the notation.

To write (21) more concretely, consider the irreducible decomposition of the tensor product $V(i\varpi_a) \otimes V(s\varpi_r)$. It is multiplicity free as noted in Section 2.1, and results in the identity of the Schur functions as

$$s_{(ia)}(w_1, \dots, w_{n+1}) s_{(s^r)}(w_1, \dots, w_{n+1}) = \sum_{\nu \in \mathcal{P}_{i,s}^{(a,r)}} s_{\nu}(w_1, \dots, w_{n+1}).$$

Here $\mathcal{P}_{i,s}^{(a,r)}$ denotes the set of partitions (Young diagrams) labeling the irreducible components described by the Littlewood–Richardson rule. Concretely one has

$$\mathcal{P}_{i,s}^{(a,r)} = \{ \nu = (\nu_i) : \text{partition} \mid \ell(\nu) \leq \min(n+1, a+r), \nu \supseteq (i^a), \nu \supseteq (s^r), |\nu| = ia + sr \},$$

where $\ell(\nu)$ denotes the length of the partition ν . From the description of the local energy H in Section 2.1, the result (21) is expressed as

$$h_i^{(a)} = \frac{\sum_{\nu \in \mathcal{P}_{i,s}^{(a,r)}} (\sum_{j > \max(a,r)} \nu_j) s_{\nu}(w_1, \dots, w_{n+1})}{s_{(ia)}(w_1, \dots, w_{n+1}) s_{(s^r)}(w_1, \dots, w_{n+1})}. \tag{22}$$

Example 3.3. Consider the case $r = 1$ for the set $B_s^{(r)}$ of local states. Then $\mathcal{P}_{i,s}^{(a,1)} = \{(i + s - k, i^{a-1}, k) \mid k \in [0, \min(i, s)]\}$ for $x \otimes y \in B_i^{(a)} \otimes B_s^{(1)}$. The local energy takes the value $H(x \otimes y) = k$. Thus $h_i^{(a)}$ (22) reads as

$$h_i^{(a)} = \frac{\sum_{k=1}^{\min(i,s)} k s_{(i+s-k, i^{a-1}, k)}(w_1, \dots, w_{n+1})}{s_{(i^a)}(w_1, \dots, w_{n+1})s_{(s)}(w_1, \dots, w_{n+1})}. \tag{23}$$

We will use this formula with $s = 1$ in Section 6.

4. Fermionic form

4.1. Deformed character

Given a tensor product $B = B_{l_1}^{(k_1)} \otimes \dots \otimes B_{l_L}^{(k_L)}$, we set

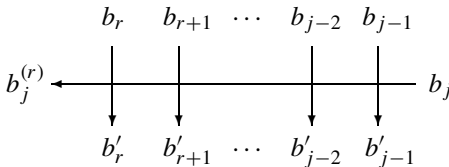
$$\chi_w(B) = \sum_{b_1 \otimes \dots \otimes b_L \in B} w^{D(b)} e^{\text{wt}(b)} \in \mathbb{Z}_{\geq 0}[z_1^{\pm 1}, \dots, z_n^{\pm 1}, w], \tag{24}$$

$$D(b_1 \otimes \dots \otimes b_L) = \sum_{1 \leq i < j \leq L} H(b_i \otimes b_j^{(i+1)}) \in \mathbb{Z}_{\geq 0}. \tag{25}$$

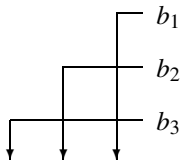
Here w is a parameter having nothing to do with w_1, \dots, w_{n+1} in (15). The element $b_j^{(r)} \in B_{l_j}^{(k_j)}$ ($j \geq r$) is the one occurring at the position r by sending $b_j \in B_{l_j}^{(k_j)}$ to the left by successively applying the combinatorial R as

$$\begin{aligned} b_r \otimes b_{r+1} \otimes \dots \otimes b_{j-1} \otimes b_j &\simeq b_r \otimes b_{r+1} \otimes \dots \otimes b_j^{(j-1)} \otimes b'_{j-1} \simeq \dots \\ &\simeq b_r \otimes b_j^{(r+1)} \otimes \dots \otimes b'_{j-2} \otimes b'_{j-1} \\ &\simeq b_j^{(r)} \otimes b'_r \otimes \dots \otimes b'_{j-2} \otimes b'_{j-1}. \end{aligned} \tag{26}$$

In particular we set $b_r^{(r)} = b_r$. The procedure is depicted as



In contrast to the energy associated with the row transfer matrices (4), the quantity D in (25) corresponds to the energy of a *corner transfer matrix*,⁶ which goes back to [1, chap.13]. In fact, using the Yang–Baxter equation for the combinatorial R it can be identified with the sum of the local energy associated to all the $L(L - 1)/2$ vertices in the following diagram ($L = 3$ example).



This quadrant structure is essentially a combinatorial counterpart of [1, Fig.13.1(b)]. By the definition we have

⁶ For types other than A_n , one needs to add boundary energy. See (101).

$$\chi_w(B)|_{w=1} = \prod_{i=1}^L \sum_{c \in B_i^{(k_i)}} e^{\text{wt}(c)} = \prod_{i=1}^L Q_{l_i}^{(k_i)} \tag{27}$$

due to (14). In this sense $\chi_w(B)$ is a w -deformation of the character $\text{ch}(\otimes_{i=1}^L V(l_i \varpi_{k_i}))$. See [31] for a representation theoretical study.

Example 4.1. From the description of the local energy in Section 2.1 and the Littlewood–Richardson rule (see Section 3.3) we have

$$\chi_w(B_i^{(a)} \otimes B_s^{(1)}) = \sum_{k=0}^{\min(i,s)} w^k s_{(i+s-k, i^{a-1}, k)}(w_1, \dots, w_{n+1}), \tag{28}$$

$$\chi_w(B_i^{(a)} \otimes B_1^{(r)}) = \sum_{k=\max(0, r-a)}^{\min(r, n+1-a)} w^{a+k-\max(a,r)} s_{((i+1)^{r-k}, i^{a-r+k}, 1^k)}(w_1, \dots, w_{n+1}), \tag{29}$$

where $s_\lambda(w_1, \dots, w_{n+1})$ is the Schur polynomial (13).

4.2. Fermionic formula

Given a tensor product $B = B_{l_1}^{(k_1)} \otimes \dots \otimes B_{l_L}^{(k_L)}$ and $\lambda \in P$, we define the Fermionic form⁷ $M(B, \lambda, w) \in \mathbb{Z}_{\geq 0}[w]$ by

$$M(B, \lambda, w) = \sum_m w^{c(m)} \prod_{1 \leq a \leq n, i \geq 1} \begin{bmatrix} p_i^{(a)} + m_i^{(a)} \\ m_i^{(a)} \end{bmatrix}_w \tag{30}$$

$$p_i^{(a)} = \sum_{j=1}^L \delta_{a, k_j} \min(i, l_j) - \sum_{b=1}^n C_{ab} e_i^{(b)}, \tag{31}$$

$$e_i^{(a)} = \sum_{j \geq 1} \min(i, j) m_j^{(a)}, \tag{32}$$

$$c(m) = \frac{1}{2} \sum_{1 \leq a, b \leq n} C_{ab} \sum_{i, j \geq 1} \min(i, j) m_i^{(a)} m_j^{(b)}, \tag{33}$$

$$\begin{bmatrix} m \\ k \end{bmatrix}_w = \frac{(w)_m}{(w)_k (w)_{m-k}}, \quad (w)_m = \prod_{i=1}^m (1 - w^i). \tag{34}$$

The quantity $p_i^{(a)}$ called vacancy and $e_i^{(a)}$ are the same with (5) and (6). The sum \sum_m in (30) is taken over the array of nonnegative integers $m = (m_i^{(a)})_{(a,i) \in [1, n] \times \mathbb{Z}_{\geq 1}}$ satisfying

- (i) nonnegativity of vacancy⁸: $p_i^{(a)} \geq 0$ for all $(a, i) \in [1, n] \times \mathbb{Z}_{\geq 1}$ such that $m_i^{(a)} \geq 1$,

⁷ The $M(B, \lambda, w)$ here corresponds to [10, eq.(4.3)] with q replaced by w^{-1} .

⁸ This finitely many conditions are known to guarantee $p_i^{(a)} \geq 0$ for all $(a, i) \in [1, n] \times \mathbb{Z}_{\geq 1}$.

(ii) weight condition:

$$\sum_{a=1}^n e^{(a)} \alpha_a = \sum_{i=1}^L l_i \varpi_{k_i} - \lambda. \tag{35}$$

By the definition the summand corresponding to $m = (m_i^{(a)})_{(a,i) \in [1,n] \times \mathbb{Z}_{\geq 1}}$ in (30) is zero unless

$$\lambda \in \left(\sum_{i=1}^L l_i \varpi_{k_i} - Q_+ \right) \cap P_+. \tag{36}$$

The necessity for $\lambda \in P_+$ is seen by noting that (35) and (31) imply $\lambda = \sum_{a=1}^n p_\infty^{(a)} \varpi_a$. Given the data $(k_1, l_1), \dots, (k_L, l_L) \in [1, n] \times \mathbb{Z}_{\geq 1}$, there are finitely many choices of $m = (m_i^{(a)})_{(a,i) \in [1,n] \times \mathbb{Z}_{\geq 1}}$ such that the above condition (i) and (ii) are satisfied for some λ obeying (36). Those m are called *configurations*. A configuration is equivalent to an n -tuple of Young diagrams via (44). They obey nontrivial constraints originating from the above (i) and (ii). To determine their asymptotic shape in the large L limit is a main theme of this paper.

Theorem 4.2 ([22]). *For any $B = B_{l_1}^{(k_1)} \otimes \dots \otimes B_{l_L}^{(k_L)}$, the following equality is valid:*

$$\chi_w(B) = \sum_{\lambda} M(B, \lambda, w) \text{ch} V(\lambda), \tag{37}$$

where the sum extends over those λ satisfying (36).

Theorem 4.2 tells that the Fermionic form $M(B, \lambda, w)$ is a w -analogue of the branching coefficient $[\otimes_{i=1}^L V(l_i \varpi_{k_i}) : V(\lambda)]$. The simplest case $L = 1$ of Theorem 4.2 gives (14). Namely one has

$$Q_l^{(k)} = \chi_w(B_l^{(k)}), \tag{38}$$

which is actually independent of w .⁹ Fermionic forms for general affine Lie algebra were introduced for non-twisted [10] and twisted [11] cases inspired by those for the Kostka–Foulkes polynomials [30] which correspond to $A_n^{(1)}$ [19,20].

4.3. Properties of deformed character

Proposition 4.3 (Th.6.1 in [10]). *Let $B = B_{l_1}^{(k_1)} \otimes \dots \otimes B_{l_L}^{(k_L)}$ be arbitrary. For any $(a, i) \in [1, n] \times \mathbb{Z}_{\geq 1}$ the following equality holds¹⁰:*

$$\chi_w(B_i^{(a)} \otimes B_i^{(a)} \otimes B) = \chi_w(B_{i-1}^{(a)} \otimes B_{i+1}^{(a)} \otimes B) + w^\phi \chi_w(\bigotimes_{b \sim a} B_i^{(b)} \otimes B), \tag{39}$$

where $\phi = i + \sum_{j=1}^L \delta_{a, k_j} \min(i, l_j)$.

⁹ For types other than A_n , $\chi_w(B_l^{(k)})$ depends on w in general. Many such examples are available in [10, app.A].

¹⁰ When $i = 1$, the factor $B_0^{(a)}$ should just be dropped.

Actually (39) was shown in [10] by substituting (37) to the three terms and using a decomposition of the Fermionic form. As a corollary of Proposition 4.3 and (27) with empty B we see that the classical character $Q_i^{(k)}$ (14) of the KR module $W_i^{(k)}$ satisfies the Q-system:

$$(Q_i^{(a)})^2 = Q_{i-1}^{(a)} Q_{i+1}^{(a)} + \prod_{b \sim a} Q_i^{(b)}. \tag{40}$$

To validate this at $i = 0$ with $Q_0^{(a)} = 1$, we employ the convention $Q_{-1}^{(a)} = 0$.

For simplicity we use the abbreviation

$$B_i = B_{l_i}^{(k_i)}, \quad Q_i = Q_{l_i}^{(k_i)} \tag{41}$$

in the remainder of this section and (73). The following result resembles the Wick theorem.

Lemma 4.4.

$$\frac{\partial}{\partial w} \log \chi_w(B_1 \otimes \cdots \otimes B_L) |_{w=1} = \sum_{1 \leq i < j \leq L} \frac{1}{Q_i Q_j} \sum_{b \otimes c \in B_i \otimes B_j} H(b \otimes c) e^{\text{wt}(b \otimes c)}. \tag{42}$$

The equality is *invalid* without specialization to $w = 1$.

Proof. From (24) and (25) we have

$$\begin{aligned} \frac{\partial \chi_w(B)}{\partial w} \Big|_{w=1} &= \sum_{b_1 \otimes \cdots \otimes b_L} D(b_1 \otimes \cdots \otimes b_L) e^{\text{wt}(b_1 \otimes \cdots \otimes b_L)} \\ &= \sum_{1 \leq i < j \leq L} \sum_d H(b_i \otimes b_j^{(i+1)}) e^{\text{wt}(d)}, \end{aligned}$$

where the sum is taken over $d = b_1 \otimes \cdots \otimes b_i \otimes b_j^{(i+1)} \otimes b'_{i+1} \otimes \cdots \otimes b'_{j-1} \otimes b_{j+1} \otimes \cdots \otimes b_L \in B_1 \otimes \cdots \otimes B_i \otimes B_j \otimes B_{i+1} \otimes \cdots \otimes B_{j-1} \otimes B_{j+1} \otimes \cdots \otimes B_L$ in the notation of (26). By changing the summation variables, the summand corresponding to the pair $i < j$ is expressed as

$$\begin{aligned} &\left(\sum_{b_1 \otimes \cdots \otimes b_{i-1}} e^{\text{wt}(b_1 \otimes \cdots \otimes b_{i-1})} \right) \left(\sum_{b \otimes c \in B_i \otimes B_j} H(b \otimes c) e^{\text{wt}(b \otimes c)} \right) \\ &\quad \times \left(\sum_{b_{i+1} \otimes \cdots \otimes b_j \otimes \cdots \otimes b_L} e^{\text{wt}(b_{i+1} \otimes \cdots \otimes b_j \otimes \cdots \otimes b_L)} \right), \end{aligned}$$

where $\overset{\vee}{b}_j$ means the absence of the factor. This equals $(\prod_{k=1, k \neq i, j}^L Q_k) \sum_{b \otimes c \in B_i \otimes B_j} H(b \otimes c) e^{\text{wt}(b \otimes c)}$. Therefore (42) follows from (27). \square

5. TBA analysis

5.1. Ensemble of n-tuple of Young diagrams associated with Fermionic form

Having prepared the Fermionic form, we are ready to perform an asymptotic analysis of the randomized BBS on $B = (B_s^{(r)})^{\otimes L}$ in Section 3. According to Proposition 3.2, each local state

obeys the probability distribution $\pi_s^{(r)}$ (16) on $B_s^{(r)}$. In other words a local state $u \in B_s^{(r)}$ occurs with the probability proportional to $e^{\text{wt}(u)} = \prod_{a=1}^n z_a^{u_a}$ for $\text{wt}(u) = u_1 \varpi_1 + \dots + u_n \varpi_n \in P$. We shall concentrate on the regime of the parameters z_1, \dots, z_n such that $z_a > 0$ and $\prod_{b=1}^n z_b^{C_{ab}} > 1$ for all $a \in [1, n]$. In view of $\prod_{b=1}^n z_b^{C_{ab}} = e^{\sum_{b=1}^n C_{ab} \varpi_b} = e^{\alpha_a}$, it means $\alpha_a > 0$ for all the simple roots α_a of A_n . Thus the local states closer to the highest weight element $u_s^{(r)}$ are realized with strictly larger probability. For instance, in case of $B_s^{(r)} = B_1^{(1)}$, this is equivalent to $\pi_1^{(1)}(1) > \pi_1^{(1)}(2) > \dots > \pi_1^{(1)}(n+1)$.

Let $\mathcal{E}_L(r, s)$ be the ensemble of states from $(B_s^{(r)})^{\otimes L}$ generated by i.i.d. probability distribution $(\pi_s^{(r)})^{\otimes L}$. Let further $\mathcal{E}_L^+(r, s)$ be the ensemble of highest states $b_1 \otimes \dots \otimes b_L \in (B_s^{(r)})^{\otimes L}$ whose probability distribution is proportional to $e^{\text{wt}(b_1 \otimes \dots \otimes b_L)}$. The randomized BBS in Section 3 corresponds to $\mathcal{E}_L(r, s)$. It slightly differs from $\mathcal{E}_L^+(r, s)$ in which the local states are not i.i.d. due to the nonlocal constraint of being highest. Both of them induce a probability distribution on the set of n -tuple of Young diagrams μ_1, \dots, μ_n by taking the conserved quantities. In the regime $\alpha_1, \dots, \alpha_n > 0$ under consideration, the highest condition on $b_1 \otimes \dots \otimes b_j \otimes b_{j+1} \otimes \dots \otimes b_L \in (B_s^{(r)})^{\otimes L}$ becomes void almost surely for the right part $b_{j+1} \otimes \dots \otimes b_L$ in the limit $L \gg j \rightarrow \infty$. Since the large L asymptotics μ_1, \dots, μ_n does not depend on the left finite tail of $b_1 \otimes \dots \otimes b_L$, we claim that the two laws on the set of n tuples of Young diagrams induced from $\mathcal{E}_L(r, s)$ and $\mathcal{E}_L^+(r, s)$ coincide. (This ‘‘asymptotic equivalence’’ of $\mathcal{E}_L(r, s)$ and $\mathcal{E}_L^+(r, s)$ is discussed in more detail for $B_s^{(r)} = B_1^{(1)}$ in [23].)

For $\mathcal{E}_L^+(r, s)$, the conserved quantities μ_1, \dots, μ_n are the Young diagrams in the rigged configurations obtained by the KSS bijection. Therefore their (joint) probability distribution is explicitly given by

$$\text{Prob}(\mu_1, \dots, \mu_n) = \frac{1}{Z_L} e^{-\sum_{a=1}^n \beta_a \sum_{i \geq 1} i m_i^{(a)}} \prod_{1 \leq a \leq n, i \geq 1} \binom{p_i^{(a)} + m_i^{(a)}}{m_i^{(a)}}, \tag{43}$$

$$m_i^{(a)} = \# \text{ of length } i \text{ rows in } \mu_a. \tag{44}$$

In what follows we will identify the n -tuple of Young diagrams $\mu = (\mu_1, \dots, \mu_n)$ with the data $m = (m_i^{(a)})_{(a,i) \in [1,n] \times \mathbb{Z}_{\geq 1}}$ by (44). Then $|\mu_a| = \sum_{i \geq 1} i m_i^{(a)} = e_\infty^{(a)}$ holds from (32).

The product of the binomial coefficients in (43) is the one in the Fermionic form (30) at $w = 1$. It accounts for the multiplicity of $\mu = (\mu_1, \dots, \mu_n)$ in the image of the KSS bijection. See the explanation around (7). The vacancy $p_i^{(a)}$ is (31) with the choice $\forall (k_i, l_i) = (r, s)$:

$$p_i^{(a)} = L \delta_{a,r} \min(i, s) - \sum_{b=1}^n C_{ab} e_i^{(b)}. \tag{45}$$

This will serve as the source of (L, r, s) -dependence of $\text{Prob}(\mu_1, \dots, \mu_n)$. The parameters β_1, \dots, β_n are chemical potentials or inverse temperatures in the context of the generalized Gibbs ensemble. As we will see in (57), they are actually the simple roots $\alpha_1, \dots, \alpha_n$. Therefore the factor $e^{-\sum_{a=1}^n \beta_a \sum_{i \geq 1} i m_i^{(a)}}$ in (43) is just $e^{-Ls\varpi_r + \lambda}$ due to (35). Besides the irrelevant constant $e^{-Ls\varpi_r}$, the factor e^λ here indeed incorporates the relative probability $e^{\text{wt}(b_1 \otimes \dots \otimes b_L)}$ adopted in $\mathcal{E}_L^+(r, s)$. Note that $\text{Prob}(\mu_1, \dots, \mu_n) = 0$ unless $m = (m_i^{(a)})_{(a,i) \in [1,n] \times \mathbb{Z}_{\geq 1}}$ is a configuration in the sense explained after (36). Finally Z_L is given by (46).

Our aim is to determine the ‘‘equilibrium’’, i.e., most probable configuration under the probability distribution (43) when L tends to infinity. It will be done by the method of grand canonical ensemble with the partition function, namely the generating function of (43):

$$Z_L = \sum_m e^{-\sum_{a=1}^n e_\infty^{(a)} \beta_a} \prod_{1 \leq a \leq n, i \geq 1} \binom{p_i^{(a)} + m_i^{(a)}}{m_i^{(a)}} \tag{46}$$

$$= e^{-Ls\varpi_r} \sum_\lambda M((W_s^{(r)})^{\otimes L}, \lambda, 1) e^\lambda. \tag{47}$$

The latter expression tells that Z_L is a generating series of the branching coefficients $[V(s\varpi_r)^{\otimes L} : V(\lambda)]$. Numerous combinatorial objects labeling the irreducible components $V(\lambda)$ and their counting formulas are known in combinatorial representation theory and algebraic combinatorics.

In the original work by Bethe himself [2], a considerable effort was devoted to the completeness issue of his own *string hypothesis*. The succeeding development [20,21] assembled the Bethe strings and visualized them as rigged configurations. These works produced the Fermionic counting formula (46) for the representation theoretical quantity (47). A further insight, soliton/string correspondence (see Section 2.3) gained after entering this century, elucidated that the Bethe strings are nothing but the BBS *solitons* for which one can formulate an integrable dynamics based on KR crystals. It endowed the *individual* term in the sum (46) with a natural interpretation as the *partition function* of the BBS with a prescribed soliton content m [25,26]. In short the BBS provided the Fermionic formula with a *refinement* via a *quasi-particle picture*. Physically speaking the BBS solitons are bound states of magnons over a ferromagnetic ground state of an integrable $A_n^{(1)}$ -symmetric spin chain deformed by $U_{q=0}(A_n^{(1)})$.

5.2. TBA equation and Y-system

We are going to apply the idea of TBA [42] to the system governed by the grand canonical partition function (46). Similar problems have been studied in the context of ideal gas of Haldane exclusion statistics. See for example the original works [37,41,13] and a review from the viewpoint of a generalized Q-system [24, sec.13]. In fact our treatment here is a constant (spectral parameter free) version of the TBA analysis in [24, sec.14, 15]. In Theorem 5.1 it will be shown that the results coincide quite nontrivially with those obtained from the crystal theory consideration.

In the large L limit, the dominant contribution in (46) come from those $m = (m_i^{(a)})$ exhibiting the L -linear asymptotic behavior

$$m_i^{(a)} \simeq L\rho_i^{(a)}, \quad p_i^{(a)} \simeq L\sigma_i^{(a)}, \quad e_i^{(a)} \simeq L\varepsilon_i^{(a)}, \quad |\mu_a| = e_\infty^{(a)} \simeq L\varepsilon_\infty^{(a)}, \tag{48}$$

where $\rho_i^{(a)}, \sigma_i^{(a)}, \varepsilon_i^{(a)}$ are of $O(L^0)$. This fact will be justified by invoking the large deviation principle in [23]. From (45) and (32) the scaled variables are related as

$$\sigma_i^{(a)} = \delta_{a,r} \min(i, s) - \sum_{b=1}^n C_{ab} \varepsilon_i^{(b)}, \quad \varepsilon_i^{(a)} = \sum_{j \geq 1} \min(i, j) \rho_j^{(a)}. \tag{49}$$

This is a constant version of the Bethe equation in terms of string density $\rho_i^{(a)}$ and the hole density $\sigma_i^{(a)}$ in [24, eq.(15.6)] $_{|l \rightarrow \infty}$. The equilibrium configuration corresponds to the $\rho = (\rho_i^{(a)})$ that minimizes the “free energy per site”

$$F[\rho] = \sum_{a=1}^n \beta_a \sum_{i=1}^l i \rho_i^{(a)}$$

$$-\sum_{a=1}^n \sum_{i=1}^l \left((\rho_i^{(a)} + \sigma_i^{(a)}) \log(\rho_i^{(a)} + \sigma_i^{(a)}) - \rho_i^{(a)} \log \rho_i^{(a)} - \sigma_i^{(a)} \log \sigma_i^{(a)} \right). \quad (50)$$

This is $(-1/L)$ times logarithm of the summand in (46) to which the Stirling formula has been applied. Note that (48) is consistent with the extensive property of the free energy, which enabled us to remove the system size L as a common overall factor. We have introduced a cut-off l for the index i , which will be sent to infinity later. Accordingly the latter relation in (49) should be understood as $\varepsilon_i^{(a)} = \sum_{j=1}^l \min(i, j) \rho_j^{(a)}$. From $\frac{\partial \sigma_j^{(b)}}{\partial \rho_i^{(a)}} = -C_{ab} \min(i, j)$, one finds that the equilibrium condition $\frac{\partial F[\rho]}{\partial \rho_i^{(a)}} = 0$ is expressed as a TBA equation

$$-i\beta_a + \log(1 + Y_i^{(a)}) = \sum_{b=1}^n C_{ab} \sum_{j=1}^l \min(i, j) \log(1 + (Y_j^{(b)})^{-1}) \quad (51)$$

for $1 \leq i \leq l$ in terms of the ratio

$$Y_i^{(a)} = \frac{\sigma_i^{(a)}}{\rho_i^{(a)}}. \quad (52)$$

The TBA equation is equivalent to the Y-system

$$\frac{(1 + Y_i^{(a)})^2}{(1 + Y_{i-1}^{(a)})(1 + Y_{i+1}^{(a)})} = \prod_{b=1}^n (1 + (Y_i^{(b)})^{-1})^{C_{ab}} \quad (53)$$

for $1 \leq i \leq l$ with the boundary condition

$$Y_0^{(a)} = 0, \quad 1 + Y_{l+1}^{(a)} = e^{\beta_a} (1 + Y_l^{(a)}). \quad (54)$$

The Y-system is known to follow from the Q-system (40) by the substitution (cf. [24, Prop. 14.1])

$$Y_i^{(a)} = \frac{Q_{i-1}^{(a)} Q_{i+1}^{(a)}}{\prod_{b \sim a} Q_i^{(b)}}, \quad 1 + Y_i^{(a)} = \prod_{b=1}^n (Q_i^{(b)})^{C_{ab}}, \quad 1 + (Y_i^{(a)})^{-1} = \frac{(Q_i^{(a)})^2}{Q_{i-1}^{(a)} Q_{i+1}^{(a)}}, \quad (55)$$

where $Q_i^{(a)} \in \mathbb{Z}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ is defined in (14). Now we take the boundary condition (54) into account. The left one $Y_0^{(a)} = 0$ is automatically satisfied due to $Q_{-1}^{(a)} = 0$. See the remark after (40). On the other hand the right condition in (54) is expressed as

$$e^{\beta_a} = \prod_{b=1}^n \left(\frac{Q_{l+1}^{(b)}}{Q_l^{(b)}} \right)^{C_{ab}}. \quad (56)$$

The result [10, Th. 7.1 (C)] tells that $\lim_{l \rightarrow \infty} (Q_{l+1}^{(a)} / Q_l^{(a)}) = e^{\alpha_a}$ in the regime $\prod_{b=1}^n z_b^{C_{ab}} > 1$ under consideration. Thus the large l limit of (56) can be taken, giving

$$e^{\beta_a} = \prod_{b=1}^n z_b^{C_{ab}} = e^{\alpha_a}. \quad (57)$$

In this way the chemical potentials β_a are naturally identified with the simple roots α_a . We shall keep using the both symbols although.

To summarize so far, we have determined the equilibrium configuration ρ_{eq} of $\rho = (\rho_i^{(a)})$ implicitly by (49), (52), (55) and (57) in terms of the chemical potentials β_1, \dots, β_n . The next task is to relate them to the canonically conjugate densities which are “physically more controllable”. It amounts to formulating the equation of state. This we do in the next subsection.

5.3. Equation of state for randomized BBS

From now on we will only treat the equilibrium values and frequently omit mentioning it. Let us calculate the equilibrium value of the free energy per site (50). First we use (52) to rewrite (50) as

$$F[\rho_{\text{eq}}] = \sum_{a=1}^n \beta_a \sum_{i=1}^l i \rho_i^{(a)} - \sum_{a=1}^n \sum_{i=1}^l \left(\rho_i^{(a)} \log(1 + Y_i^{(a)}) + \sigma_i^{(a)} \log(1 + (Y_i^{(a)})^{-1}) \right). \quad (58)$$

On the other hand taking the linear combination of the TBA equation as $\sum_{a=1}^n \sum_{i=1}^l (51) \times \rho_i^{(a)}$ we get

$$\begin{aligned} \sum_{a=1}^n \beta_a \sum_{i=1}^l i \rho_i^{(a)} &= \sum_{a=1}^n \sum_{i=1}^l \rho_i^{(a)} \log(1 + Y_i^{(a)}) \\ &\quad - \sum_{a,b=1}^n C_{ab} \sum_{i,j=1}^l \min(i, j) \rho_i^{(a)} \log(1 + (Y_j^{(b)})^{-1}). \end{aligned} \quad (59)$$

Substituting this into the first term on the RHS of (58) and using $\sigma_i^{(a)}$ from (49) we find

$$\begin{aligned} F[\rho_{\text{eq}}] &= - \sum_{i=1}^l \min(i, s) \log(1 + (Y_i^{(r)})^{-1}) \\ &= - \log \left(Q_s^{(r)} \left(\frac{Q_l^{(r)}}{Q_{l+1}^{(r)}} \right)^s \right) \xrightarrow{l \rightarrow \infty} - \log \left(z_r^{-s} Q_s^{(r)} \right), \end{aligned} \quad (60)$$

where (55) is used and $l \gg s$ is assumed in the second equality.

Now we resort to the general relation

$$F[\rho_{\text{eq}}] = - \lim_{L \rightarrow \infty} \frac{1}{L} \log Z_L. \quad (61)$$

From (48), we see that the equilibrium (most probable) value of the $1/L$ -scaled number of boxes $L^{-1}|\mu_1|, \dots, L^{-1}|\mu_n|$ in the Young diagrams μ_1, \dots, μ_n are that for $\varepsilon_\infty^{(1)}, \dots, \varepsilon_\infty^{(n)}$. Denote them by ν_1, \dots, ν_n . These are parameters representing the *densities* of the boxes in the Young diagrams μ_1, \dots, μ_n . From (46), (48) and (61) one has the relation $\nu_a = \frac{\partial F[\rho_{\text{eq}}]}{\partial \beta_a}$ for $1 \leq a \leq n$. In view of (57) it is convenient to take the linear combination of this as follows:

$$\sum_{b=1}^n C_{ab} \nu_b = \left(\sum_{b=1}^n C_{ab} \frac{\partial}{\partial \beta_b} \right) F[\rho_{\text{eq}}] = z_a \frac{\partial F[\rho_{\text{eq}}]}{\partial z_a}. \quad (62)$$

Substituting (60) we arrive at the equation of state of the system:

$$z_a \frac{\partial}{\partial z_a} \log Q_s^{(r)} = s\delta_{a,r} - \sum_{b=1}^n C_{ab} v_b \quad (1 \leq a \leq n). \tag{63}$$

The LHS is an explicit rational function of z_1, \dots, z_n that can be calculated from (13) and (14). The variables z_1, \dots, z_n are simply related to the chemical potentials β_1, \dots, β_n or equivalently to the fugacity $e^{-\beta_1}, \dots, e^{-\beta_n}$ by (57) as

$$z_a = e^{\sum_{b=1}^n (C^{-1})_{ab} \beta_b} = e^{\sum_{b=1}^n (\min(a,b) - \frac{ab}{n+1}) \beta_b}. \tag{64}$$

Thus (63) relates the densities v_1, \dots, v_n with the fugacity $e^{-\beta_1}, \dots, e^{-\beta_n}$, thereby enabling us to control either one by the other. Set

$$y_a = e^{-\alpha_a}, \quad \overline{Q}_s^{(r)} = z_r^{-s} Q_s^{(r)} \in \mathbb{Z}[y_1, \dots, y_n], \tag{65}$$

where y_1, \dots, y_n are the fugacity mentioned just above since $\alpha_a = \beta_a$ according to (57). Then the equation of state (63) also admits a somewhat simpler presentation as

$$v_a = y_a \frac{\partial}{\partial y_a} \log \overline{Q}_s^{(r)}. \tag{66}$$

To see this, note from $y_a = \prod_{b=1}^n z_b^{-C_{ab}}$ and (65) that (63) is rewritten as

$$\sum_{b=1}^n C_{ab} v_b = s\delta_{a,r} - z_a \frac{\partial}{\partial z_a} \log(z_r^s \overline{Q}_s^{(r)}) = -z_a \frac{\partial}{\partial z_a} \log \overline{Q}_s^{(r)} = \sum_{b=1}^n C_{ab} y_b \frac{\partial}{\partial y_b} \log \overline{Q}_s^{(r)}. \tag{67}$$

In the language of the BBS, the relation $\varepsilon_\infty^{(a)} = v_a$ implies that the equilibrium weight of a site variable $b \in B_s^{(r)}$ is

$$\text{wt}(b) = s\varpi_r - \sum_{a=1}^n v_a \alpha_a \leftrightarrow \#_a(b) = s\theta(a \leq r) + v_{a-1} - v_a \quad (v_0 = v_{n+1} = 0), \tag{68}$$

where $\#_a(b)$ for $a \in [1, n + 1]$ denotes the number of the letter a in $b \in B_s^{(r)}$ regarded as a semistandard tableau of shape (s^r) . The empty space corresponds to the letter 1. Note that the weight $\text{wt}(b)$ specifies an element $b \in B_s^{(r)}$ uniquely if and only if $\min(r, n - r, s) = 1$.

5.4. Difference equation characterizing the equilibrium shape

One should recognize (63) as the $i = \infty$ case of the left relation in (49). It concerns the total number (density) of boxes in the Young diagrams only. However, the relations (49) and (52) hold for any finite i and provide information on the equilibrium shape of these Young diagrams. In fact they can be combined to give the difference equation for the variables $\varepsilon_i^{(a)}$ as

$$\delta_{a,r} \min(i, s) - \sum_{b=1}^n C_{ab} \varepsilon_i^{(b)} = Y_i^{(a)} (-\varepsilon_{i-1}^{(a)} + 2\varepsilon_i^{(a)} - \varepsilon_{i+1}^{(a)}). \tag{69}$$

The quantity in the parenthesis in the RHS is $\rho_i^{(a)}$ (49), which is the $1/L$ -scaled number $m_i^{(a)}$ of length i rows in the Young diagram μ_a . See (44) and (48).

In this way we have characterized the vertically $1/L$ -scaled equilibrium shape of the Young diagrams μ_1, \dots, μ_n under the prescribed densities $(v_1, \dots, v_n) = \lim_{L \rightarrow \infty} L^{-1} (|\mu_1|, \dots, |\mu_n|)$ in terms of the variables $\varepsilon_i^{(a)}$. The procedure consists of the following steps:

- (i) Given the densities ν_1, \dots, ν_n , determine z_1, \dots, z_n by the equation of state (63) or (66).
- (ii) Compute $Y_i^{(a)}$ by substituting those z_1, \dots, z_n into (55) and (14).
- (iii) Find the solution to (69) with the boundary condition $\varepsilon_0^{(a)} = 0, \varepsilon_\infty^{(a)} = \nu_a$.

Once $\varepsilon_i^{(a)}$ is known, the equilibrium Young diagrams are deduced from

$$\varepsilon_i^{(a)} = \lim_{L \rightarrow \infty} \frac{1}{L} (\# \text{ of boxes in the left } i \text{ columns of } \mu_a). \tag{70}$$

See (44), (48) and (49). In the next subsection we present an explicit solution to the step (iii) in terms of the stationary local energy (21).

5.5. Solution to the difference equation by stationary local energy

From (16) the stationary local energy (21) (for the i.i.d. ensemble $\mathcal{E}_L(r, s)$ with $L \rightarrow \infty$) is expressed as

$$h_i^{(a)} = \frac{\sum_{x \otimes y \in B_i^{(a)} \otimes B_s^{(r)}} H(x \otimes y) e^{\text{wt}(x \otimes y)}}{Q_i^{(a)} Q_s^{(r)}}. \tag{71}$$

According to (20) this is equal to the $\lim_{L \rightarrow \infty} \frac{1}{L} E_i^{(a)}$. On the other hand (for the highest state ensemble $\mathcal{E}_L^+(r, s)$ with $L \rightarrow \infty$), the soliton/string correspondence (8) and the definition (48) indicate that the same quantity should also show up in the TBA analysis exactly as $\varepsilon_i^{(a)}$. Thus the asymptotic equivalence of the two ensembles indicates that they coincide. The next theorem, which is our main result in the paper, identifies them rigorously.

Theorem 5.1. *The solution to the difference equation (69) satisfying the boundary condition $\varepsilon_0^{(a)} = 0, \varepsilon_\infty^{(a)} = \nu_a$ is provided by the stationary local energy $\varepsilon_i^{(a)} = h_i^{(a)}$ in (71).*

Theorem 5.1 shows a nontrivial coincidence of the equilibrium configuration, i.e., the n -tuple of Young diagrams, determined by two different approaches:

- stationary local energy for the Markov process of carriers in the randomized BBS,
- difference equation arising from the TBA analysis of the Fermionic formula.

The result may be regarded as *randomized* version of the soliton/string correspondence (8). Being able to give an explicit formula for $\varepsilon_i^{(a)}$ is a very rare event in the actual TBA analyses involving the spectral parameter.

For simplicity we temporarily write the w -deformed character (24) as

$$Q_{l_1}^{(k_1)} * \dots * Q_{l_L}^{(k_L)} = \chi_w(B_{l_1}^{(k_1)} \otimes \dots \otimes B_{l_L}^{(k_L)}). \tag{72}$$

In this notation, Lemma 4.4 reads as

$$\frac{\partial \log(Q_1 * \dots * Q_L)}{\partial w} \Big|_{w=1} = \sum_{1 \leq i < j \leq L} \frac{\partial \log(Q_i * Q_j)}{\partial w} \Big|_{w=1}. \tag{73}$$

Then Theorem 5.1 is summarized in the following formula for the $1/L$ -scaled Young diagrams:

$$\varepsilon_i^{(a)} = \frac{\partial \log(Q_i^{(a)} * Q_s^{(r)})}{\partial w} \Big|_{w=1}. \tag{74}$$

From this and (70) the quantity $\eta_i^{(a)} := \varepsilon_i^{(a)} - \varepsilon_{i-1}^{(a)}$ has the meaning and the explicit formula as

$$\eta_i^{(a)} = \lim_{L \rightarrow \infty} \frac{1}{L} (\# \text{ of boxes in the } i\text{-th column of } \mu_a) = \frac{\partial}{\partial w} \left(\log \frac{Q_i^{(a)} * Q_s^{(r)}}{Q_{i-1}^{(a)} * Q_s^{(r)}} \right) \Big|_{w=1}. \tag{75}$$

5.6. Proof of Theorem 5.1

First we prove

Proposition 5.2. $\varepsilon_i^{(a)} = h_i^{(a)}$ (71) provides a solution to the difference equation (69).

Proof. By substitution of (71) and the formula (55) for $Y_i^{(a)}$, the equation (69) becomes

$$\begin{aligned} \delta_{a,r} \min(i, s) - \sum_{b=1}^n C_{ab} \frac{\sum H(x_i^{(b)} \otimes y) e^{\text{wt}(x_i^{(b)} \otimes y)}}{Q_i^{(b)} Q_s^{(r)}} &= \frac{Q_{i-1}^{(a)} Q_{i+1}^{(a)}}{Q_s^{(r)} \prod_{b \sim a} Q_i^{(b)}} \\ \times \left(\frac{\sum H(x_{i-1}^{(a)} \otimes y) e^{\text{wt}(x_{i-1}^{(a)} \otimes y)}}{Q_{i-1}^{(a)}} + 2 \frac{\sum H(x_i^{(a)} \otimes y) e^{\text{wt}(x_i^{(a)} \otimes y)}}{Q_i^{(a)}} \right. \\ \left. - \frac{\sum H(x_{i+1}^{(a)} \otimes y) e^{\text{wt}(x_{i+1}^{(a)} \otimes y)}}{Q_{i+1}^{(a)}} \right). \end{aligned} \tag{76}$$

Here and in what follows $x_j^{(c)}$ and y should always be summed over $B_j^{(c)}$ and $B_s^{(r)}$, respectively. By removing the denominators, this is cast into

$$\begin{aligned} \delta_{a,r} \min(i, s) Q_s^{(r)} \prod_{b \sim a} Q_i^{(b)} &= 2 Q_i^{(a)} \sum H(x_i^{(a)} \otimes y) e^{\text{wt}(x_i^{(a)} \otimes y)} \\ &\quad - \sum \left(\prod_{b \sim a, c \sim a, c \neq b} Q_i^{(c)} \right) \sum H(x_i^{(b)} \otimes y) e^{\text{wt}(x_i^{(b)} \otimes y)} \\ &\quad - Q_{i+1}^{(a)} \sum H(x_{i-1}^{(a)} \otimes y) e^{\text{wt}(x_{i-1}^{(a)} \otimes y)} - Q_{i-1}^{(a)} \sum H(x_{i+1}^{(a)} \otimes y) e^{\text{wt}(x_{i+1}^{(a)} \otimes y)}. \end{aligned} \tag{77}$$

In the derivation, we have used the Q-system (40) to cancel a factor $Q_i^{(a)}$ in the first term of the RHS. In order to verify (77) we consider the two special cases of (39):

$$Q_i^{(a)} * Q_i^{(a)} * Q_s^{(r)} - Q_{i-1}^{(a)} * Q_{i+1}^{(a)} * Q_s^{(r)} - w^\kappa \prod_{b \sim a} Q_i^{(b)} * Q_s^{(r)} = 0, \tag{78}$$

$$Q_i^{(a)} * Q_i^{(a)} - Q_{i-1}^{(a)} * Q_{i+1}^{(a)} - w^i \prod_{b \sim a} Q_i^{(b)} = 0, \tag{79}$$

where $\kappa = i + \delta_{a,r} \min(i, s)$ and the product over b means the one by $*$. Take the w -derivative of (78) at $w = 1$. By means of Lemma 4.4 or equivalently (73), it leads to

$$\begin{aligned}
 0 = & 2Q_i^{(a)} \sum H(x_i^{(a)} \otimes y) e^{\text{wt}(x_i^{(a)} \otimes y)} - Q_{i+1}^{(a)} \sum H(x_{i+1}^{(a)} \otimes y) e^{\text{wt}(x_{i+1}^{(a)} \otimes y)} \\
 & - Q_{i-1}^{(a)} \sum H(x_{i+1}^{(a)} \otimes y) e^{\text{wt}(x_{i+1}^{(a)} \otimes y)} - \kappa \left(\prod_{b \sim a} Q_i^{(b)} \right) Q_s^{(r)} - \sum_{b \sim a} \left(\prod_{c \sim a, c \neq b} Q_i^{(c)} \right) H(x_i^{(b)} \otimes y) e^{\text{wt}(x_i^{(b)} \otimes y)} \\
 & + \left(\frac{\partial(Q_i^{(a)} * Q_i^{(a)})}{\partial w} - \frac{\partial(Q_{i-1}^{(a)} * Q_{i+1}^{(a)})}{\partial w} - \sum_{b, c \sim a} \left(\prod_{d \sim a, d \neq b, c} Q_i^{(d)} \right) \frac{\partial(Q_i^{(b)} * Q_i^{(c)})}{\partial w} \right) \Big|_{w=1} Q_s^{(r)}.
 \end{aligned} \tag{80}$$

The same calculation for (79) tells that the quantity in the big parenthesis of the last line of (80) is equal to $i \prod_{b \sim a} Q_i^{(b)}$ at $w = 1$. Therefore this term cancels the κ term on the second line of (80) partially. The resulting relation is nothing but (77). \square

Next we verify the boundary condition $h_0^{(a)} = 0, h_\infty^{(a)} = \nu_a$. As the former is obvious, we concentrate on the latter. From (66) and (71) the boundary condition $h_\infty^{(a)} = \nu_a$ in question is stated as

Proposition 5.3.

$$\lim_{i \rightarrow \infty} \frac{\sum_{x \otimes y \in B_i^{(a)} \otimes B_s^{(r)}} H(x \otimes y) e^{\text{wt}(x \otimes y)}}{Q_i^{(a)} Q_s^{(r)}} = y_a \frac{\partial}{\partial y_a} \log \overline{Q}_s^{(r)}. \tag{81}$$

Proposition 5.3 turns out to be reducible to some simple cases. To demonstrate it we utilize the (r, s) -dependence of $h_i^{(a)}$ (71), hence exhibit it as

$$h_{i,s}^{(a,r)} = h_{s,i}^{(r,a)} = \frac{\sum_{x \otimes y \in B_i^{(a)} \otimes B_s^{(r)}} H(x \otimes y) e^{\text{wt}(x \otimes y)}}{Q_i^{(a)} Q_s^{(r)}}. \tag{82}$$

Here the symmetry under the exchange of the indices is due to the invariance of weights and the local energy H by the combinatorial R .

Lemma 5.4.

$$s\delta_{a,r} = 2y_r^{-s} \prod_{t=1}^n (\overline{Q}_s^{(t)})^{C_{rt}} h_{\infty,s}^{(a,r)} - y_r^{-s} \frac{\overline{Q}_{s+1}^{(r)} \overline{Q}_{s-1}^{(r)}}{\prod_{t \sim r} \overline{Q}_s^{(t)}} (h_{\infty,s-1}^{(a,r)} + h_{\infty,s+1}^{(a,r)}) - \sum_{t \sim r} h_{\infty,s}^{(a,t)}. \tag{83}$$

Proof. In terms of $h_{i,s}^{(a,r)}$ in (82), the already established relation (76) or equivalently (77) reads

$$\delta_{a,r} \min(i, s) = 2 \prod_{b=1}^n (Q_i^{(b)})^{C_{ab}} h_{i,s}^{(a,r)} - \frac{Q_{i+1}^{(a)} Q_{i-1}^{(a)}}{\prod_{b \sim a} Q_i^{(b)}} (h_{i-1,s}^{(a,r)} + h_{i+1,s}^{(a,r)}) - \sum_{b \sim a} h_{i,s}^{(b,r)}.$$

Exchange the indices $(a, i) \leftrightarrow (r, s)$ here and apply the symmetry $h_{s,i}^{(r,a)} = h_{i,s}^{(a,r)}$. Then (83) follows from it by taking the limit $i \rightarrow \infty$ and substituting $Q_s^{(r)} = z_r^s \overline{Q}_s^{(r)}$. \square

Lemma 5.5. Let $\zeta_s^{(a,r)} = y_a \frac{\partial}{\partial y_a} \log \overline{Q}_s^{(r)}$ be the RHS of (81). It satisfies

$$s\delta_{a,r} = 2y_r^{-s} \prod_{t=1}^n (\overline{Q}_s^{(t)})^{C_{rt}} \zeta_s^{(a,r)} - y_r^{-s} \frac{\overline{Q}_{s+1}^{(r)} \overline{Q}_{s-1}^{(r)}}{\prod_{t \sim r} \overline{Q}_s^{(t)}} (\zeta_{s-1}^{(a,r)} + \zeta_{s+1}^{(a,r)}) - \sum_{t \sim r} \zeta_s^{(a,t)}. \tag{84}$$

Proof. The Q-system (40) becomes $(\overline{Q}_s^{(r)})^2 = \overline{Q}_{s-1}^{(r)} \overline{Q}_{s+1}^{(r)} + y_r^s \prod_{t \sim r} \overline{Q}_s^{(t)}$ in terms of the variables in (65). This is an identity in $\mathbb{Z}[y_1, \dots, y_n]$. The assertion follows from it by taking the derivative $y_a \frac{\partial}{\partial y_a}$. \square

Proof of Proposition 5.3. From Lemma 5.4 and Lemma 5.5, the quantities $h_{\infty,s}^{(a,r)}$ and $\zeta_s^{(a,r)}$ obey the same difference relation with respect to r which is at most of second order since $t \sim r$ means $t \in \{r \pm 1\} \cap [1, n]$. Moreover they are both 0 at $r = 0$. Therefore $h_{\infty,s}^{(a,r)} = \zeta_s^{(a,r)}$ (81) follows from the $r = 1$ case. It can be slightly rewritten by (65) as

$$\lim_{i \rightarrow \infty} \frac{\sum_{x \otimes y \in B_i^{(a)} \otimes B_s^{(1)}} H(x \otimes y) e^{\text{wt}(x \otimes y)}}{Q_i^{(a)} z_1^s} = y_a \frac{\partial \overline{Q}_s^{(1)}}{\partial y_a}. \tag{85}$$

In the sequel we prove (85). From the w -derivative of (28) at $w = 1$, the LHS of (85) with fixed $i (\geq s)$ is expressed as

$$\frac{\sum_{x \otimes y \in B_i^{(a)} \otimes B_s^{(1)}} H(x \otimes y) e^{\text{wt}(x \otimes y)}}{Q_i^{(a)} z_1^s} = \sum_{k=1}^s k \frac{s_{(i+s-k, i^{a-1}, k)}(w_1, \dots, w_{n+1})}{s^{(i^a)}(w_1, \dots, w_{n+1}) z_1^s}. \tag{86}$$

In the regime $\prod_{b=1}^n z_b^{C_{ab}} > 1$ under consideration, the variables w_a in (15) satisfy $w_1 > w_2 > \dots > w_{n+1}$. Then the large i limit of each summand in RHS of (86) is easily extracted from the determinantal formula in (13). It is decomposed into a product of Schur polynomials, which leads to

$$\begin{aligned} \text{LHS of (85)} &= z_1^{-s} \sum_{k=1}^s k s_{(s-k)}(w_1, \dots, w_a) s_{(k)}(w_{a+1}, \dots, w_n) \\ &= \sum_{k=0}^s k s_{(s-k)}(\overline{w}_1, \dots, \overline{w}_a) s_{(k)}(\overline{w}_{a+1}, \dots, \overline{w}_n). \end{aligned} \tag{87}$$

We have set $\overline{w}_a = w_a/z_1 = y_1 y_2 \dots y_{a-1}$. See (15) and (65). As for RHS of (85), we invoke the formula for $\overline{Q}_s^{(1)}$ as the sum over semistandard tableaux on the Young diagram with length s single row shape. The entry $b \in [1, n+1]$ of the tableaux corresponds to \overline{w}_b . Therefore for any $a \in [1, n]$ we have

$$\begin{aligned} \overline{Q}_s^{(1)} &= \sum_{k=0}^s \sum_{1 \leq b_1 \leq \dots \leq b_{s-k} \leq a} \overline{w}_{b_1} \dots \overline{w}_{b_{s-k}} \sum_{a < c_1 \leq \dots \leq c_k \leq n+1} \overline{w}_{c_1} \dots \overline{w}_{c_k} \\ &= \sum_{k=0}^s s_{(s-k)}(\overline{w}_1, \dots, \overline{w}_a) s_{(k)}(\overline{w}_{a+1}, \dots, \overline{w}_n). \end{aligned} \tag{88}$$

Since the first factor is free from y_a whereas the latter contains it as the overall multiplier y_a^k , the derivative $y_a \frac{\partial \overline{Q}_s^{(1)}}{\partial y_a}$ coincides with (87). This completes the proof of (85) hence that of Proposition 5.3. We have finished the proof of Theorem 5.1. \square

6. Example

In this section we focus on the simplest choice $B_s^{(r)} = B_1^{(1)}$ for the set of local states.

6.1. Explicit formula of the limit shape by Schur functions

We set $p_a = \pi_1^{(1)}(a)$ for $a \in [1, n + 1]$, where a in the RHS signifies the element of $B_1^{(1)}$ corresponding to the semistandard tableau containing a in the single box Young diagram. So p_1 is the density of empty sites and p_a with $a \in [2, n + 1]$ is the density of balls with color a . According to (16), one has $\pi_1^{(1)}(a) = e^{\alpha_1 - \alpha_1 - \dots - \alpha_{a-1}} / Q_1^{(1)}$. Therefore in the regime $\alpha_1, \dots, \alpha_n > 0$ under consideration, $1 > p_1 > p_2 > \dots > p_{n+1} > 0$ holds. Of course $p_1 + \dots + p_{n+1} = 1$ should also be satisfied. For $n = 2$ this notation agrees with Example 3.1. According to (16) we set

$$p_a = \frac{w_a}{w_1 + \dots + w_{n+1}} \quad (1 \leq a \leq n + 1) \tag{89}$$

in terms of $w_j = z_{j-1}^{-1} z_j$ given in (15). The denominator is $Q_1^{(1)}$ (14). Thus we find (cf. [23])

$$z_a = u^{-\frac{a}{n+1}} p_1 p_2 \dots p_a, \quad u = p_1 p_2 \dots p_{n+1} \quad (0 \leq a \leq n + 1). \tag{90}$$

From (68), the ball densities p_1, \dots, p_{n+1} are connected to the Young diagram densities v_1, \dots, v_n as

$$v_a = p_{a+1} + p_{a+2} + \dots + p_{n+1} \quad (1 \leq a \leq n). \tag{91}$$

The equation of state (63) reads

$$z_a \frac{\partial}{\partial z_a} \log \left(\sum_{j=1}^{n+1} \frac{z_j}{z_{j-1}} \right) = \delta_{a,1} - \sum_{b=1}^n C_{ab} v_b \quad (1 \leq a \leq n), \tag{92}$$

where $z_0 = z_{n+1} = 1$ as in (15). One can easily check that (92) is satisfied by z_a and v_a in (90) and (91) provided that $p_1 + \dots + p_{n+1} = 1$ is valid. This essentially achieves the step (i) in Section 5.4. For the remaining steps (ii) and (iii), we have already given the general solution in Theorem 5.1. In the present case the solution $\varepsilon_i^{(a)} = h_i^{(a)}$ can be written down concretely by setting $s = 1$ in Example 3.3:

$$\varepsilon_i^{(a)} = \frac{s_{(i^a, 1)}(w_1, \dots, w_{n+1})}{s_{(i^a)}(w_1, \dots, w_{n+1}) s_{(1)}(w_1, \dots, w_{n+1})} \quad (w_a = u^{-\frac{1}{n+1}} p_a). \tag{93}$$

For simplicity denote the Schur polynomial $s_\lambda(w_1, \dots, w_{n+1})$ by s_λ . Then the quantity (75) is given neatly as

$$\eta_i^{(a)} = \varepsilon_i^{(a)} - \varepsilon_{i-1}^{(a)} = \frac{s_{(i^a, 1)}}{s_{(i^a)} s_{(1)}} - \frac{s_{((i-1)^a, 1)}}{s_{((i-1)^a)} s_{(1)}} = \frac{s_{((i-1)^{a-1})} s_{(i^{a+1})}}{s_{(i^a)} s_{((i-1)^a)} s_{(1)}}, \tag{94}$$

where we have used a bilinear identity among the Schur polynomials.

In the simplest case $n = 1$, the equation of state (92) becomes $v_1 = (1 + z_1^2)^{-1}$. From $s_{(i)}(w_1, w_2) = \frac{z_1^{i+1} - z_1^{-i-1}}{z_1 - z_1^{-1}}$ and $s_{(i,i)}(w_1, w_2) = 1$, the result (94) reduces to

$$\eta_i^{(1)} = \frac{1}{s_{(i)} s_{(i-1)} s_{(1)}} = \frac{\zeta^i (\zeta - 1)^2}{(\zeta + 1) (\zeta^i - 1) (\zeta^{i+1} - 1)}, \quad \zeta = z_1^{-2}. \tag{95}$$

This agrees with a corresponding result in [23].

6.2. Scaling behavior of the width of the Young diagrams

Note that $\eta_i^{(a)}$ (75) is the $1/L$ scaled length of the i -th column of the Young diagram μ_a . Moreover we have used the scaling behavior (48). Thus the above result (94) should be understood to be effective in the range $1 \leq i \leq I_a$ where

$$L\eta_i^{(a)} \simeq 1 \quad (L \gg 1). \tag{96}$$

This yields a crude estimate of the scaling behavior of the width I_a of the Young diagram μ_a as L grows large.

Let us investigate the consequence of (96) closely for the regime $1 > p_1 > p_2 > \dots > p_{n+1} > 0$. From (13) and $w_1 > w_2 > \dots > w_{n+1} > 0$, we see that $s_{(i^a)} = s_{(i^a)}(w_1, \dots, w_{n+1})$ behaves as

$$s_{(i^a)} = \frac{\Delta(w_1, \dots, w_a)\Delta(w_{a+1}, \dots, w_{n+1})}{\Delta(w_1, \dots, w_{n+1})} (w_1 \dots w_a)^{i+n-a+1} (1 + O(e^{-ci})) \quad (i \rightarrow \infty)$$

for some constant $c > 0$, where $\Delta(x_1, \dots, x_m) = \prod_{1 \leq j < k \leq m} (x_j - x_k)$ is the Vandermonde determinant. Applying this to (94) and using (90) we find that $\eta_i^{(a)}$ tends to 0 as $i \rightarrow \infty$ as

$$\eta_i^{(a)} \simeq \frac{\prod_{j=1}^a (p_j - p_{a+1}) \prod_{j=a+1}^{n+1} (p_a - p_j)}{\prod_{j=1}^{a-1} (p_j - p_a) \prod_{j=a+2}^{n+1} (p_{a+1} - p_j)} \frac{1}{p_a} \left(\frac{p_{a+1}}{p_a}\right)^{i+n-a} \quad (i \rightarrow \infty) \tag{97}$$

up to exponentially small corrections. Thus the estimate (96) implies the logarithmic scaling

$$I_a \simeq \frac{\log L}{\log \frac{p_a}{p_{a+1}}} \quad (L \rightarrow \infty) \tag{98}$$

in the leading order. For $n = a = 1$ and $1 - p_1 = p_2 = p$, (97) and (96) lead to $I_1 \simeq \log\left(\frac{(1-2p)^2 L}{1-p}\right) / \log \frac{1-p}{p}$. Incidentally this reproduces μ_L in [29, Th.2(i)] including the coefficient.

The result (98) indicates yet another scaling behavior at $p_a = p_{a+1}$. For simplicity let us consider the most degenerate case of such situations $p_a = \frac{1}{n+1}$ for all $a \in [1, n+1]$. It corresponds to the completely random distribution of the balls and empty sites. Then we have $w_a = 1$ and (94) simplifies to

$$\eta_i^{(a)} = \frac{a(n+1-a)}{(n+1)(i+a-1)(i+a)}. \tag{99}$$

Therefore the estimate (96) gives

$$I_a \simeq \sqrt{\frac{a(n+1-a)L}{n+1}}. \tag{100}$$

This square root scaling behavior is a signal of criticality as observed in [29].

6.3. Numerical check

Here we deal with the $n = 2$ case, i.e., 2-color BBS. The relevant KR crystals are $B_l^{(1)}$ and $B_l^{(2)}$. We parametrize the both by the set $\{x = (x_1, x_2, x_3) \in (\mathbb{Z}_{\geq 0})^3 \mid x_1 + x_2 + x_3 = l\}$. For $B_l^{(1)}$, x_i is the number of letter i contained in the semistandard tableau on the Young diagram of single row shape with length l . For $B_l^{(2)}$, x_1, x_2, x_3 are the number of the columns $\begin{smallmatrix} 2 \\ 3, 3, 2 \end{smallmatrix}$ in the semistandard tableau on the Young diagram of double row shape with length l . The combinatorial

Table 1

Notations and equation numbers except in the first column are those in [27].

	Combinatorial R	local energy $H(x \otimes y)$
$E_l^{(1)}$	$R _{m=1}$ (2.1)	$Q_0(x, y)$
$E_l^{(2)}$	$\vee R _{m=1}$ (2.3)	$P_{-1}(x, y)$

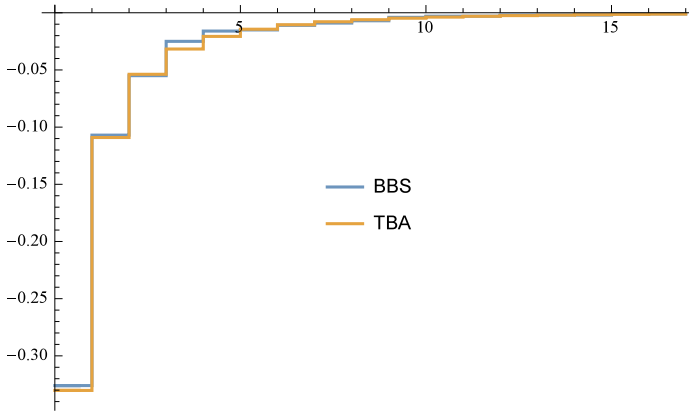


Fig. 1. Vertically $1/L$ scaled Young diagram μ_1 . $L = 1000$, $(p_1, p_2, p_3) = (\frac{7}{18}, \frac{6}{18}, \frac{5}{18})$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

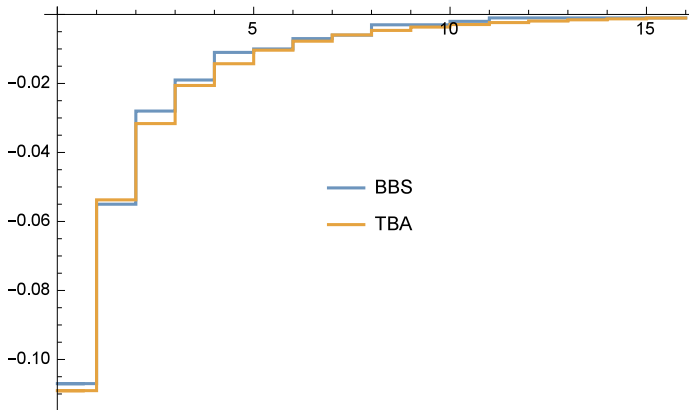


Fig. 2. Vertically $1/L$ scaled Young diagram μ_2 . $L = 1000$, $(p_1, p_2, p_3) = (\frac{7}{18}, \frac{6}{18}, \frac{5}{18})$.

$R : x \otimes y \mapsto \tilde{y} \otimes \tilde{x}$ and the local energy $H(x \otimes y)$ necessary to compute $E_l^{(1)}$ and $E_l^{(2)}$ are those acting on $B_l^{(1)} \otimes B_1^{(1)}$ and $B_l^{(2)} \otimes B_1^{(1)}$, respectively. They are given explicitly by piecewise linear formulas in [27, sec.2.2] $_{|n=3}$ with a slight conventional adjustment. We summarize them in Table 1.

We have generated a BBS state in $\{1, 2, 3\}^L$ with a prescribed ball densities $1 > p_1 > p_2 > p_3 > 0$ and length $L = 1000$ by computer. Calculating the energy $E_l^{(1)}$, $E_l^{(2)}$ by (4) we extract the Young diagrams μ_1 and μ_2 . After scaling by $1/L$ vertically we plot them (called “BBS”) together with the prediction (94) (called “TBA”) in Figs. 1 and 2.

According to (96) we have truncated the scaled μ_1 and μ_2 at the width $I_1 = 17$ and $I_2 = 16$. The agreement of the numerical data from BBS and the TBA prediction is more or less satisfactory.

7. Discussion

7.1. Summary

We have elucidated a new interplay among the randomized BBS, Markov processes of carriers, KR modules/crystals, combinatorial R , local energy, deformed characters, Fermionic formulas, rigged configurations, Q and Y-systems, TBA equations and so forth. Our main result is Theorem 5.1 which identifies the stationary local energy of the KR crystal (71) as the explicit solution to the difference equation (69) originating from TBA. It determines the equilibrium shape of the Young diagrams μ_1, \dots, μ_n in the scaling limit as in (70), (74) and (75). These random Young diagrams arise as the conserved quantities (generalized soliton contents) of the randomized BBS and obey the probability distribution given by the Fermionic form (43).

7.2. Generalization to simply-laced case

Although the above results are concerned with the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ with $\hat{\mathfrak{g}} = A_n^{(1)}$, all the essential ingredients are known or at least conjecturally/conceptually ready for general quantum affine algebras. In particular formulas for the simply-laced cases $\hat{\mathfrak{g}} = A_n^{(1)}, D_n^{(1)}, E_{6,7,8}^{(1)}$ possess a quite similar and simple structure. Let us explain them briefly and conjecturally describe the parallel results on the randomized BBS of type ADE uniformly. We set $\mathfrak{g} = A_n, D_n, E_{6,7,8}$ according to $\hat{\mathfrak{g}} = A_n^{(1)}, D_n^{(1)}, E_{6,7,8}^{(1)}$ and $n = 6, 7, 8$ for $\hat{\mathfrak{g}} = E_{6,7,8}^{(1)}$. The matrix $(C_{ab})_{1 \leq a, b \leq n}$ is to be understood as the Cartan matrix of \mathfrak{g} and the relation $a \sim b$ is defined by $C_{ab} = -1$.

The KR modules $\{W_s^{(r)} \mid (r, s) \in [1, n] \times \mathbb{Z}_{\geq 0}\}$ over $U_q(\hat{\mathfrak{g}})$ are specified in terms of the Drinfeld polynomials. See for example [24, sec.4.2]. The corresponding KR crystals $\{B_s^{(r)} \mid (r, s) \in [1, n] \times \mathbb{Z}_{\geq 0}\}$ have been constructed for $\hat{\mathfrak{g}} = D_n^{(1)}$ in [33], whereas their existence is yet conjectural in general for $\hat{\mathfrak{g}} = E_{6,7,8}^{(1)}$. Here we assume them and denote by $u_s^{(r)} \in B_s^{(r)}$ the unique element of weight $s\varpi_r$. The local energy H should be so taken as $H \in \mathbb{Z}_{\geq 0}$ and $H(u_i^{(a)} \otimes u_s^{(r)}) = 0$ on any $B_i^{(a)} \otimes B_s^{(r)}$. Set $Q_i^{(a)} = \sum_{b \in B_i^{(a)}} e^{\text{wt}(b)}$. Although it is no longer a character of an irreducible \mathfrak{g} module in general, it satisfies the Q-system (40). See for example [10] and [24, sec.13] and references therein.

The $U_q(\hat{\mathfrak{g}})$ BBS is formulated in the same manner as Section 2. Take the set of local states to be $B_s^{(r)}$. We consider the randomized $U_q(\hat{\mathfrak{g}})$ BBS where the local states and the stationary measure of the carrier for the time evolution $T_i^{(a)}$ are given as Proposition 3.2. Then the stationary local energy $h_i^{(a)}$ takes the same form as (21) or equivalently (71).

Concerning the deformed character (24), the corner transfer matrix energy D in (25) needs to be replaced by

$$D(b_1 \otimes \dots \otimes b_L) = \sum_{j=1}^L H(b_j^{\natural} \otimes b_j^{(1)}) + \sum_{1 \leq i < j \leq L} H(b_i \otimes b_j^{(i+1)}), \tag{101}$$

where $b_j^\natural \in B_{l_j}^{(k_j)}$ is the unique element such that $\varphi(b_j^\natural) = l_j \Lambda_0$. See [32, sec.5.1] for a detailed account of this. The first sum on the RHS is referred to as the boundary energy. It is 0 for $A_n^{(1)}$ but is nontrivial for the other types.

The Fermionic form $M(B, \lambda, w)$ is defined by the same formulas as (30)–(35). Theorem 4.2¹¹ is valid for $D_n^{(1)}$ [34,31] and conjecturally valid for $E_{6,7,8}^{(1)}$. Proposition 4.3 has been shown in [10]. Lemma 4.4 is influenced by the boundary energy and should be replaced by

$$\begin{aligned} & \frac{\partial}{\partial w} \log \chi_w(B_1 \otimes \cdots \otimes B_L)|_{w=1} \\ &= \sum_{1 \leq i < j \leq L} \frac{1}{Q_i Q_j} \sum_{b \otimes c \in B_i \otimes B_j} H(b \otimes c) e^{\text{wt}(b \otimes c)} + \sum_{1 \leq i \leq L} \frac{1}{Q_i} \sum_{c \in B_i} H(b_i^\natural \otimes c) e^{\text{wt}(c)}. \end{aligned} \tag{102}$$

As for the TBA analysis, all the relations from (43) until (72) remain unchanged.¹² In particular, the property $\lim_{l \rightarrow \infty} (Q_{l+1}^{(a)} / Q_l^{(a)}) = e^{\tau a}$ used to simplify (56) is valid not only for non-exceptional cases [10, Th. 7.1 (C)] but for all types [12, Prop.5.9].

We conjecture that Theorem 5.1 is also valid for type D_n and $E_{6,7,8}$. In fact admitting Theorem 4.2, it can be shown that $\varepsilon_i^{(a)} = h_i^{(a)}$ provides a solution to the difference equation (69).¹³ The proof uses (102). In particular with the notation (72), its $L = 2$ case captures the stationary local energy $h_i^{(a)}$ as

$$\begin{aligned} \frac{\partial \log(Q_i^{(a)} * Q_s^{(r)})}{\partial w} \Big|_{w=1} &= h_i^{(a)} + \frac{1}{Q_i^{(a)}} \sum_{c \in B_i^{(a)}} H(b_1^\natural \otimes c) e^{\text{wt}(c)} \\ &+ \frac{1}{Q_s^{(r)}} \sum_{c \in B_s^{(r)}} H(b_2^\natural \otimes c) e^{\text{wt}(c)}. \end{aligned} \tag{103}$$

The point is that the effect of extra “boundary terms” containing $b_1^\natural \in B_i^{(a)}, b_2^\natural \in B_s^{(r)}$ is canceled by those in (102), leaving the difference equation unchanged from (69). As for the boundary condition for the difference equation, we conjecture that (81) or equivalently (85) holds universally for type D_n and $E_{6,7,8}$. It is an intriguing relation involving the local energy whose proof will shed new light into the KR crystals and the Q-system.

7.3. Further outlook

We expect the generalization to the non simply-laced cases and twisted affine Lie algebras is also feasible albeit with a slight technical complexity. Another obvious direction of a future research is periodic systems. The generalized BBS for $A_n^{(1)}$ with the quantum space $(B_1^{(1)})^{\otimes L}$ has been studied under the periodic boundary condition [28]. It also has an n -tuple of Young diagrams as a label of iso-level sets for which a Fermionic formula [28, eq.(57), Th.3] for the multiplicity has been obtained under a technical assumption. It will be interesting to analyze it by TBA similarly to this paper. In the simplest case $n = 1$, the Fermionic formula has been fully justified and reduces to

¹¹ For general affine Lie algebra it is often called $X = M$ conjecture [10,11].
¹² The unique exception is the last expression in (64) which is specific to type A_n .
¹³ This assertion is the analogue of Proposition 5.2, which was the “first half” of Theorem 5.1.

$$\frac{L}{L-2M} \prod_{i \geq 1} \binom{p_i^{(1)} + m_i^{(1)} - 1}{m_i^{(1)}} \quad (104)$$

for system size L and M -ball sector with $M < \frac{L}{2}$ in the same notation as (31).¹⁴ So at least in this simplest situation, the scaled limit shape of the Young diagram remains the same as (95).

There are a number of further challenging problems to be investigated. We list a few of them as closing remarks.

- (i) Study the limit shape problem when the BBS states are inhomogeneous as $B_{s_1}^{(r_1)} \otimes \cdots \otimes B_{s_L}^{(r_L)}$ with a given statistical distribution of (r_i, s_i) .
- (ii) Can one architect a BBS like dynamical system whose Markov process of carriers has the stationary measure described by q -characters [6]?
- (iii) Can one extend the TBA analysis so as to include w -binomials in (43) with $w \neq 1$? What is the counterpart of the BBS corresponding to such a generalization?
- (iv) Our TBA analysis in this paper was spectral parameter free. See the remark after (49). Is there any Yang-Baxterization of Theorem 5.1?

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References

- [1] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, 1982.
- [2] H.A. Bethe, Zur Theorie der Metalle, I. Eigenwerte und Eigenfunktionen der linearen Atomkette, *Z. Phys.* 71 (1931) 205–231.
- [3] D.A. Croydon, T. Kato, M. Sasada, S. Tsujimoto, Dynamics of the box–ball system with random initial conditions via Pitman’s transformation, arXiv:1806.02147.
- [4] V.G. Drinfeld, Quantum groups, in: *Proceedings of the International Congress of Mathematicians*, vols. 1, 2, Berkeley, Calif., 1986, Amer. Math. Soc., Providence, RI, 1987, pp. 798–820.
- [5] P.A. Ferrari, C. Nguyen, L. Rolla, M. Wang, Soliton decomposition of the box–ball system, arXiv:1806.02798.
- [6] E. Frenkel, N. Reshetikhin, The q -characters of representations of quantum affine algebras and deformations of W -algebras, *Contemp. Math.* 248 (1999) 163–205.
- [7] K. Fukuda, M. Okado, Y. Yamada, Energy functions in box ball systems, *Int. J. Mod. Phys. A* 15 (2000) 1379–1392.
- [8] W. Fulton, *Young Tableaux: With Applications to Representation Theory*, Cambridge Univ. Press, 1997.
- [9] G. Hatayama, K. Hikami, R. Inoue, A. Kuniba, T. Takagi, T. Tokihiro, The $A_M^{(1)}$ automata related to crystals of symmetric tensors, *J. Math. Phys.* 42 (2001) 274–308.
- [10] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, Y. Yamada, Remarks on Fermionic formula, *Contemp. Math.* 248 (1999) 243–291.
- [11] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, Z. Tsuboi, Paths, crystals and Fermionic formulae, *Math-Phys odyssey 2001*, *Prog. Math. Phys.* 23 (2002) 205–272.
- [12] D. Hernandez, The Kirillov–Reshetikhin conjecture and solutions of T-systems, *J. Reine Angew. Math.* 596 (2006) 63–87.
- [13] K. Iguchi, K. Aomoto, Integral representation for the grand partition function in quantum statistical mechanics of exclusion statistics, *Int. J. Mod. Phys. B* 14 (2000) 485–506.
- [14] R. Inoue, A. Kuniba, T. Takagi, Integrable structure of box–ball systems: crystal, Bethe ansatz, ultradiscretization and tropical geometry, *J. Phys. A, Math. Theor.* 45 (2012) 073001, 64pp.

¹⁴ $p_i^{(1)} = L - 2 \sum_{j \geq 1} \min(i, j) m_j^{(1)}$, $M = \sum_{j \geq 1} j m_j^{(1)}$. One can show (104) $\in \mathbb{Z}_{\geq 0}$.

- [15] M. Jimbo, A q -difference analogue of $U(\mathfrak{g})$ and the Yang–Baxter equation, *Lett. Math. Phys.* 10 (1985) 63–69.
- [16] V.G. Kac, *Infinite Dimensional Lie Algebras*, third ed., Cambridge University Press, 1990.
- [17] S.-J. Kang, M. Kashiwara, K.C. Misra, T. Miwa, T. Nakashima, A. Nakayashiki, Perfect crystals of quantum affine Lie algebras, *Duke Math. J.* 68 (1992) 499–607.
- [18] M. Kashiwara, On crystal bases of the q -analogue of universal enveloping algebras, *Duke Math. J.* 63 (1991) 465–516.
- [19] S.V. Kerov, A.N. Kirillov, N.Yu. Reshetikhin, Combinatorics, the Bethe ansatz and representations of the symmetric group, *J. Sov. Math.* 41 (1988) 916–924.
- [20] A.N. Kirillov, N.Yu. Reshetikhin, The Bethe ansatz and the combinatorics of Young tableaux, *J. Sov. Math.* 41 (1988) 925–955.
- [21] A.N. Kirillov, N.Yu. Reshetikhin, Representations of Yangians and multiplicity of occurrence of the irreducible components of the tensor product of representations of simple Lie algebras, *J. Sov. Math.* 52 (1990) 3156–3164.
- [22] A.N. Kirillov, A. Schilling, M. Shimozono, A bijection between Littlewood–Richardson tableaux and rigged configurations, *Sel. Math. Sov.* 8 (2002) 67–135.
- [23] A. Kuniba, H. Lyu, One-sided scaling limit of random multicolor box-ball system, arXiv:1808.08074.
- [24] A. Kuniba, T. Nakanishi, J. Suzuki, T-systems and Y-systems in integrable systems, *J. Phys. A, Math. Theor.* 44 (2011) 103001 (146pp).
- [25] A. Kuniba, M. Okado, T. Takagi, Y. Yamada, Vertex operators and canonical partition function of the box–ball systems, *RIMS Kôkyûroku* 1302 (2003) 91–107 (in Japanese).
- [26] A. Kuniba, M. Okado, R. Sakamoto, T. Takagi, Y. Yamada, Crystal interpretation of Kerov–Kirillov–Reshetikhin bijection, *Nucl. Phys. B* 740 (2006) 299–327.
- [27] A. Kuniba, M. Okado, Y. Yamada, Box–ball system with reflecting end, *J. Nonlinear Math. Phys.* 12 (2005) 475–507.
- [28] A. Kuniba, T. Takagi, Bethe ansatz, inverse scattering transform and tropical Riemann theta function in a periodic soliton cellular automaton for $A_n^{(1)}$, *SIGMA* 6 (2010) 013 (52pp).
- [29] L. Levine, H. Lyu, J. Pike, Double jump phase transition in a soliton cellular automaton, arXiv:1706.05621.
- [30] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, second ed., Oxford Univ. Press, 1995.
- [31] K. Naoi, Demazure crystals and tensor products of perfect Kirillov–Reshetikhin crystals with various levels, *J. Algebra* 374 (2013) 1–26.
- [32] M. Okado, $X = M$ conjecture, in: *Combinatorial Aspect of Integrable Systems*, MSJ Memoirs 17 (2007) 43–73.
- [33] M. Okado, A. Schilling, Existence of Kirillov–Reshetikhin crystals for nonexceptional types, *Represent. Theory* 12 (2008) 186–207.
- [34] M. Okado, R. Sakamoto, A. Schilling, T. Scrimshaw, Type $D_n^{(1)}$ rigged configuration bijection, *J. Algebraic Comb.* 46 (2017) 341–401.
- [35] R. Sakamoto, Kirillov–Schilling–Shimozono bijection as energy functions of crystals, *Int. Math. Res. Not.* 2009 (2009) 579–614.
- [36] M. Shimozono, Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties, *J. Algebraic Comb.* 15 (2002) 151–187.
- [37] B. Sutherland, Quantum many-body problem in one dimension: thermodynamics, *J. Math. Phys.* 12 (1971) 251–256.
- [38] D. Takahashi, On some soliton systems defined by using boxes and balls, in: *Proceedings of the International Symposium on Nonlinear Theory and Its Applications (NOLTA '93)*, 1993, pp. 555–558.
- [39] D. Takahashi, J. Matsukidaira, Box and ball system with a carrier and ultradiscrete modified KdV equation, *J. Phys. A, Math. Gen.* 30 (1997) L733–L739.
- [40] D. Takahashi, J. Satsuma, A soliton cellular automaton, *J. Phys. Soc. Jpn.* 59 (1990) 3514–3519.
- [41] Y.-S. Wu, Statistical distribution for generalized ideal gas of fractional statistical particles, *Phys. Rev. Lett.* 73 (1994) 922–925.
- [42] C.N. Yang, C.P. Yang, Thermodynamics of a one-dimensional system of bosons with repulsive delta-function interaction, *J. Math. Phys.* 10 (1969) 1115–1122.