Magnetic Monopoles and Gravity

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Dedicated to Prof. Jan Łopuszański
on occasion of his 75th birthday

Abstract. Subject of this talk is an overview of results on self-gravitating non-abelian magnetic monopoles. The coupling to the gravitational field leads to new features absent in flat space: gravitating monopoles have unstable "radial excitations" and they exist only up to a maximal mass (related to a kind of "gravitational confinement" at the Planck scale). In addition to the globally regular monopoles there are "coloured" black holes, i.e., magnetically charged black holes carrying a non-trivial YM field outside their event horizon. The latter give rise to a violation of the "No Hair" Conjecture.

1 Introduction

This talk is an overview of results on self-gravitating magnetic monopoles. It is mainly based on analytical and numerical results obtained in collaboration with P. Breitenlohner and P. Forgács (Breitenlohner et al. 1992, 1995). Many other people, who have contributed in establishing our present understanding of this subject will be mentioned in due course.

As a genuine non-linear structure magnetic monopoles play an important role in the non-perturbative aspects of the Yang-Mills-Higgs (YMH) theory. Originally they were found as solutions of the YMH field equations in flat space, but in a very interesting early paper van Niewenhuizen et.al. (van Nieuwenhuizen et al. 1976) considered also the gravitational self-interaction of monopoles. However these authors made no attempt to actually construct solutions with analytical or numerical methods. Only twenty years later, triggered by the discovery of globally regular solutions of the Einstein-Yang-Mills theory by Bartnik and McKinnon (Bartnik and McKinnon 1988) new interest in the subject arose.

A systematic numerical study of the effects of gravity on magnetic monopoles (Ortiz 1992, Lee et al. 1992, Breitenlohner et al. 1992, 1995) revealed a number of interesting phenomena. In contrast to monopoles in flat space gravitating monopoles allow for "radial excitations", which have some close connection with the solutions discovered by Bartnik and McKinnon. As to

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be expected gravitating monopoles develop a gravitational instability for sufficiently strong gravitational self-force, manifesting itself in some kind of a "Gravitational Confinement". Systematically increasing the strength of the gravitational self-coupling (resp. letting the monopole mass approach the Planck scale) one reaches a limiting solution, which in its exterior part has the geometry of an extremal black hole.

Obviously one may question the physical relevance of monopoles with a mass close to the Planck mass, since on the one hand even in GUTs the monopole mass would be considerably lower and on the other hand at the Planck scale one would expect quantum gravity effects to come into play.

In addition to the regular monopoles there are also black hole solutions carrying a non-trivial exterior YM field ("Coloured Black Holes"). Taking into account their radial excitations, one finds a rich spectrum of such static black hole solutions. This is to be contrasted with Einstein's theory in vacuum resp. with the Einstein-Maxwell theory, where according to a theorem of Israel (Israel 1967 1968) the Schwarzschild resp. Reissner-Nordstrøm solution are the only static black holes. The co-existence of all these black hole solutions with the same magnetic charge gives rise to an interesting violation of the "No-Hair" Conjecture (Chruściel 1994; Bizon 1993).

2 Magnetic Monopoles and Sphalerons in Flat Space

Let me start with a short reminder on the static, spherically symmetric solutions of the YMH system in flat space. For simplicity I restrict myself to the gauge group *SU(2)* from now on.

The so-called 't Hooft-ansatz for the static, spherically symmetric YM field in polar coordinates reads

$$W_0^a = 0 \quad W_i^a = \epsilon_{iak}\frac{x^k}{r^2}(W(r) - 1)\ . \tag{1}$$

Inserting it into the standard YM action

$$S_{YM} = -\frac{1}{4\pi}\int d^4\, x\Big[\frac{1}{4g^2}\mathrm{Tr}F^2\Big] \tag{2}$$

yields the reduced action

$$S_{YM,\mathrm{red}} = -\int dr\Big[\frac{1}{g^2}(W'^2 + \frac{(1-W^2)^2}{2r^2})\Big]\ . \tag{3}$$

A rescaling of the radial coordinate $r \to r/\lambda$ leads to a rescaling of the action $S_{YM,\mathrm{red}} \to \lambda S_{YM,\mathrm{red}}$. This property (related to the scale invariance of the 4-dimensional theory) prevents the existence of any non-trivial stationary point of $S_{YM,\mathrm{red}}$ with finite non-zero action (energy), manifestating the general statement, that the flat YM theory has no solitons (Coleman 1975; Deser 1976).

The situation changes with the inclusion of the Higgs field. Through its "vacuum expectation value" v two scales are introduced, the mass $M_W = gv$ of the YM field and the Higgs mass $M_H = \sqrt{\lambda}v$. There are two different cases

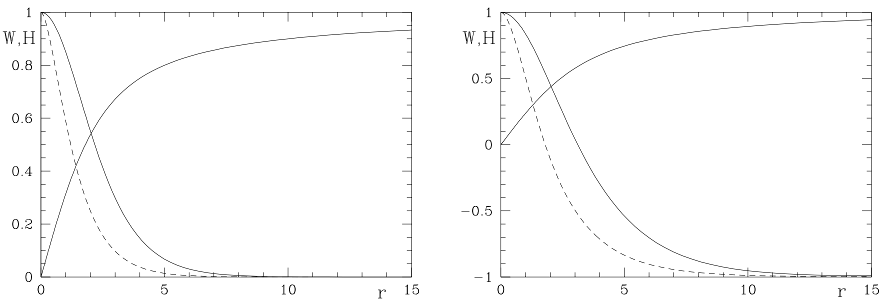

Fig. 1. a) PS-monopole, b) DHN-sphaleron, both for vanishing (solid) and infinite (dashed) Higgs mass

to be considered, leading to rather different types of solutions. The Higgs field can be either in a triplet or in a doublet representation. In either case the action is

$$S_H = \frac{1}{4\pi} \int d^4\, x \Big[\frac{1}{2} |D\phi|^2 - \frac{\lambda}{8} (|\phi|^2 - v^2) \Big] \; . \tag{4}$$

The finite energy solutions in the case of a Higgs triplet are the 't Hooft-Polyakov magnetic monopoles ('t Hooft 1974; Polyakov 1974). They are obtained with the ansatz

$$\phi^a = \frac{x^a}{r} H(r) \; . \tag{5}$$

Inserting this ansatz in the action (4) one gets

$$S_{H,\mathrm{red}} = - \int dr \Big[\frac{r^2}{2} H'^2 + \frac{\lambda r^2}{8} (H^2 - v^2)^2 + W^2 H^2 \Big] \; . \tag{6}$$

In order to obtain finite total energy the Higgs field has to tend to its vacuum value v for $r \to \infty$, forcing in turn $W \to 0$. Outside a "core" of size $1/M_W$ the solution is essentially equal to the embedded abelian Dirac monopole $W \equiv 0$ avoiding the singular center at $r = 0$ of the latter (compare Fig. 1a).

For large values of M_H and hence of β the function $H(r)$ rises quickly to its asymptotic value v. In the limit $\beta \to \infty$ the Higgs field may be replaced by v for all $r > 0$ and its only role is to give a mass to the YM field. The total energy of the solution stays finite in this limit. In fact, it only varies by a factor ≈ 1.8 as β varies from 0 to ∞. Of particular interest is the exact BPS monopole solution for $\beta = 0$ with the simple exact form (using $h = rH$ for convenience)

$$W(r) = \frac{vr}{\sinh(vr)} \qquad h(r) = vr\coth(vr) - 1 \,, \tag{7}$$

satisfying a system of first order equations (Bogomolny equations)

$$rW' = Wh \,, \tag{8a}$$
$$rh' = h + 1 - W^2 \tag{8b}$$

implying the second order field equations. Considered as a solution of a suitable supersymmetric extension (N=2 SUSY-YM theory) it has the special property of being "half supersymmetric", i.e., it is annihilated by one half of the infinitesimal supersymmetry generators. This implies the relation $E = vP$ between the energy of the solution and its magnetic charge P (equalling the central charge of the $N = 2$ SUSY algebra). If "quantizing" the solution does not destroy supersymmetry, i.e., the above relation is preserved, any quantum corrections to its mass have to vanish (Witten and Olive 1978).

Due to the topological character of the magnetic charge, related to the asymptotic vacuum structure of configurations with finite energy, the monopole is a stable solution.

The second possibility is a Higgs field in the doublet representation. The relevant ansatz of the Higgs field is $\Phi^\alpha = H(r)\xi^\alpha$ with some constant spinor ξ. Although this ansatz is not itself spherically symmetric it leads to a consistent reduction. The corresponding reduced action is

$$S_{H,\mathrm{red}} = -\int dr \Big[\frac{r^2}{2}H'^2 + \frac{\lambda r^2}{8}(H^2 - v^2)^2 + \frac{1}{4}(W+1)^2 H^2\Big] \,. \tag{9}$$

The only essential difference of this action to the one for the triplet is the form of the mass term. It destroys the symmetry $W \to -W$ and enforces W to turn to $W = -1$ for $r \to \infty$ in order to have finite total energy (compare Fig. 1b). This asymptotic behaviour implies that the solution has no magnetic charge in contrast to the previous case with $W \to 0$.

Unlike the stable monopole the sphaleron, i.e., the solution minimizing the energy $E = -S$, is unstable. In order to understand this instability it is important to consider the most general spherically symmetric ansatz for the YM field.

$$W_t^a = (0, 0, A_0) \,, \qquad W_\theta^a = (W_1, W_2, 0) \,, \tag{10a}$$
$$W_r^a = (0, 0, A_1) \,, \qquad W_\varphi^a = (-W_2 \sin\theta, W_1 \sin\theta, \cos\theta) \,. \tag{10b}$$

The ansatz used above for the monopole and the sphaleron corresponds to a consistent truncation, putting $A_0 = A_1 = W_2 = 0$ and $W_1 = W$. The sphaleron turns out to be stable under variations staying within the minimal ansatz, but not if $\delta W_2 \neq 0$ and $\delta A_1 \neq 0$. As was discussed by Manton (Manton 1983) this instability may be attributed to the non-trivial topology of the configuration space of the spherically symmetric YM potential, again related to the asymptotic vacuum structure of configurations with finite energy.

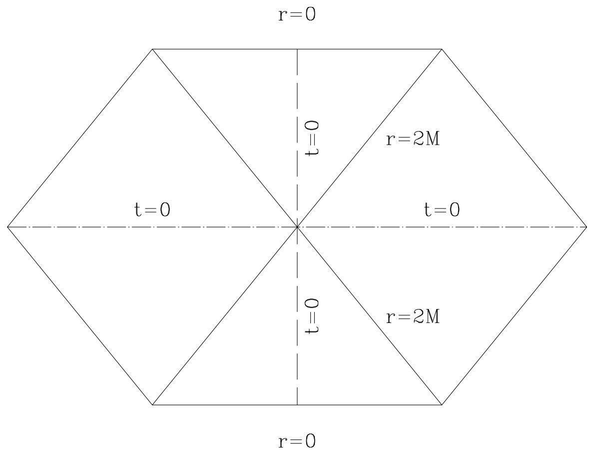

Fig. 2. Conformal diagram for the Schwarzschild solution

3 The Spherically Symmetric Gravitational Field

Spherically symmetric space-times M_4 have the structure of an orthogonal product $M_4 = M_2 \times S^2$ of a 2-dimensional space M_2 with a 2-sphere with a corresponding decomposition of the metric $ds_4^2 = ds_2^2 + r^2 d\Omega^2$, where $d\Omega^2$ is the invariant metric on S^2 and its inverse curvature r is a function on M_2. A convenient parametrization of ds_2^2 is

$$ds_2^2 = A^2 B dt^2 - \frac{dR^2}{B} \ , \tag{11}$$

with arbitrary time resp. radial coordinates t and R. The standard choice for the latter is the "Schwarzschild" coordinate $R \equiv r$, which is possible as long as $dr/dR \neq 0$. We are only interested in static solutions, with A and B independent of the canonical time coordinate ("Killing-time") t. Insertion of the ansatz into the standard Einstein action then yields

$$S_{G,\mathrm{red}} = \frac{1}{2\sqrt{G}} \int dr \Big[A(B + rB' - 1) \Big] \ . \tag{12}$$

The dimensionality of G introduces a mass scale $M_{\mathrm{Pl}} = 1/\sqrt{G}$, the Planck mass. Variation with respect to A and B yields (with suitable boundary condition at infinity) the Schwarzschild solution $A = 1$, $B = 1 - 2M/r$. [1]

[1] Although $S_{G,\mathrm{red}}$ is also just rescaled under a scaling $r \to \lambda r$ similar to $S_{YM,\mathrm{red}}$, there is now a non-trivial stationary point with vanishing action, because $S_{G,\mathrm{red}}$ is indefinite.

As is well known it describes a static black hole of total mass M and event horizon located at the Schwarzschild radius $r_S = 2M$. The fact that its total mass is finite, although the solution has a real singularity at the origin, illustrates a general difference to flat space, where the finiteness of the total energy of a field configuration in general implies some regularity properties. This remark is not made without hindsight, as it explains the unsuitability of the energy (mass) functional for existence proofs of solutions once the gravitational self-interaction is included.

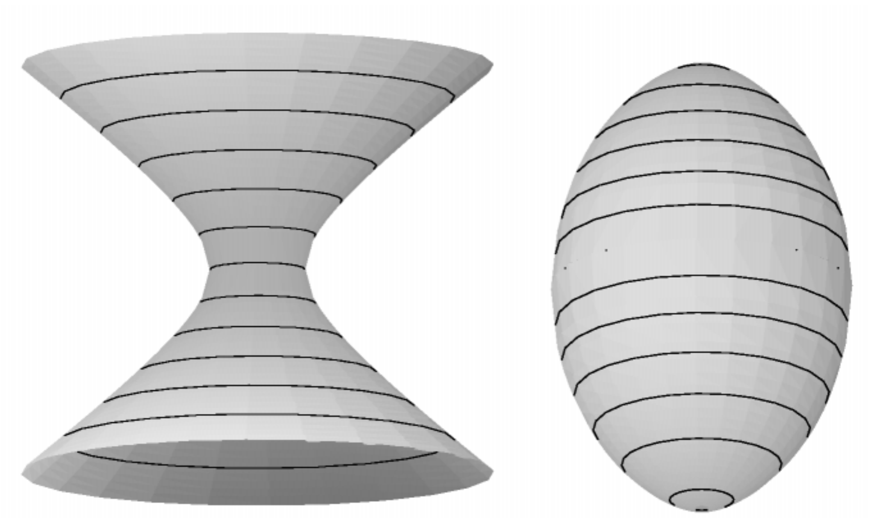

Fig. 3. a) Schwarzschild b.h., $t = 0$ hyperplane for $r > 2M$ (each circle is actually a 2-sphere), b) same for $r < 2M$

The geometry of black hole space-times is best illustrated with their conformal diagram Fig. 2 (Hawking and Ellis 1973). Since we are considering the solutions for fixed t, the hyper-surfaces $t =$ const. are of particular interest. They meet the horizon at the so-called "bifurcation surface" of the horizon. In order to study their geometry it is useful to switch to another radial coordial coordinate, avoiding the apparent singularity of the metric at $r = 2M$. A convenient choice is

$$\rho = \int_{2M}^{r} \frac{dr'}{\sqrt{B}} \; . \tag{13}$$

Due to the square-root ρ is a double-valued function of r, the continuation through the branch point at $r = 2M$ leading to another copy of the original surface and thus giving the surfaces $t =$ const. their famous "wormhole" structure (Fig. 3). A similar construction for the corresponding (time-like)

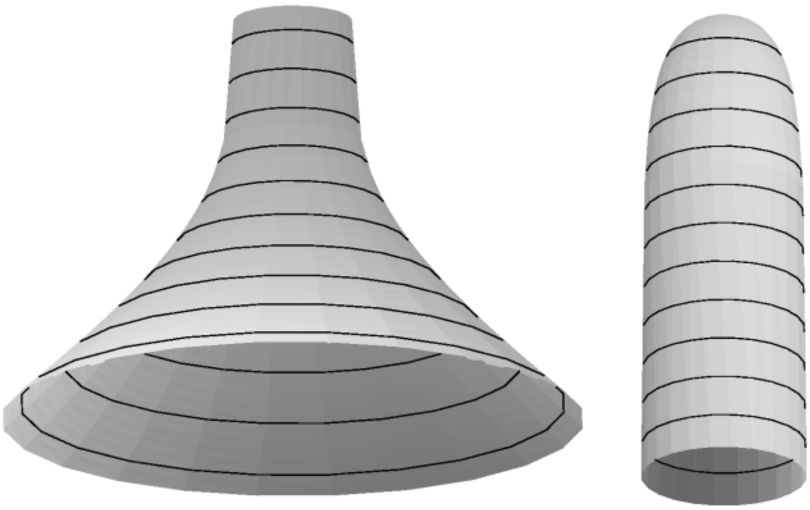

Fig. 4. a) Extremal Reissner-Nordstrøm b.h., part of the t=0 hyperplane for $r > M$, b) same for $r < M$

surfaces inside the horizon (replacing B by $-B$ in (13)) leads to a compact surface with a second singular center ($r = 0$).

Next we consider the static black holes of the Einstein-Maxwell theory, described by the Reissner-Nordstrøm (RN) solution. There are two possibilities for a static, spherically symmetric Maxwell field, the electric monopole with the potential $A_0 = q/r$ or a magnetic monopole with $A_\varphi = q\cos\theta$ with the Dirac string singularity, leading to the same metric given by $A = 1$, $B = 1 - 2M/r + q^2/r^2$. For $M > |q|$ the function B has two zeros, leading to an outer and an inner horizon. Outside the outer horizon the structure of the $t =$ const. surfaces is as before. However, as $|q|$ tends to M the worm-hole develops a long "throat" with $r \approx M$. The limiting case $M = |q|$ represents the "extremal" RN black hole, whose horizon is degenerate, due to the double zero of B at $r = M$. There is no more wormhole, but an infinitely long throat. Also the $t =$ const. (space-like!) surfaces inside the horizon show this infinite throat as $r \to M$ (Fig. 4).

4 Gravitating Monopoles – BPS-Type Solutions

As already mentioned the flat BPS monopole plays a very special role in connection with supersymmetry. Amazingly it is possible to embed the flat solution into certain supergravity theories, satisfying the coupled field equations. The first such embedding is due to Gauntlett et.al. (Gauntlett et al.

1993). The relevant SUGRA theory is the $N = 4$ extended SUGRA coupled to $N = 4$ SUSY-YM, derived from the corresponding $N = 1$ theory in ten dimensions, which itself is a field theory limit of heterotic string theory. Besides the gravitational field the theory contains a dilaton φ and an axion $H_{\lambda\mu\nu}$. For the solution considered, the members of the SUGRA multiplet can be expressed through the YM potential W and the Higgs field h of the flat BPS solution as

$$e^{2\varphi} = \tfrac{2}{r^2}(1 - W^2 + 2h) \qquad g_{\mu\nu} = e^{2\varphi}\eta_{\mu\nu} \tag{14a}$$

$$H_{ij4} = 2\epsilon_{ijk}\tfrac{x^k}{r^4}h(1 - W^2) \qquad H_{ijk} = 0\ . \tag{14b}$$

As in flat space the solution solves first order Bogomolny equations and preserves one half of the supersymmetries.

Another embedding discovered more recently by Chamseddine and Volkov (Chamseddine and Volkov 1977) is even more surprising, since the model contains no Higgs field. The corresponding SUGRA is the $N = 4$ gauged supergravity (Friedman and Schwarz 1978), which may be obtained as a non-trivial Kaluza-Klein reduction from the $N = 1$ SUGRA in ten dimensions (related to type II strings). The YM field results from the non-trivial structure of the internal space $S^3 \times S^3$, on which the compactification is performed. The non-vanishing curvature of the internal space leads to a cosmological constant in four dimensions. After a suitable truncation the 4-dimensional (bosonic) action considered in (Chamseddine and Volkov 1977) is

$$S = -\int d^4x\sqrt{-g}\Big[\frac{R}{2} - \frac{1}{2}(\partial\varphi)^2 + \frac{1}{4g^2}e^{2\varphi}\mathrm{Tr}F^2 - \frac{1}{4}e^{-2\varphi}\Big]\ . \tag{15}$$

The gravitational the gravitational field and the dilaton can be expressed through the flat BPS solution

$$R^2 = 2h - W^2 + 1 \qquad \text{with} \quad R = \frac{r}{\sqrt{2}}e^{-\varphi} \tag{16a}$$

$$B = 1 + \frac{(R^2 + W^2 - 1)^2}{4R^2} \tag{16b}$$

$$A = \frac{r}{\rho} \tag{16c}$$

$$2e^{2\varphi} = A^2B\ , \tag{16d}$$

where the coordinate ρ is chosen such that $ds_2^2 = A^2B(dt^2 - d\rho^2)$. Again the solution satisfies a first order system of Bogomolny equations

$$\rho\frac{d}{d\rho}W = -\frac{W}{2\sqrt{B}}(R^2 + W^2 - 1)\ , \tag{17a}$$

$$\rho\frac{d}{d\rho}\varphi = \frac{1}{4\sqrt{B}}\Big(R^2 - \frac{(W^2 - 1)^2}{R^2}\Big)\ . \tag{17b}$$

The solution is not asymptotically flat due to the cosmological term in the action (not even asymptotically anti-deSitter) and it preserves 1/4 of the supersymmetries.

5 Gravitating Monopoles without SUSY

Let us now proceed to the case of self-gravitating magnetic monopoles without SUSY. Through gravity another mass scale $M_{\rm Pl}$ has entered and we can form two dimensionless ratios $\alpha = \sqrt{G}v = M_W/gM_{\rm Pl}$ and $\beta = M_W/M_H$. As long as $M_W << M_{\rm Pl}$, i.e., $\alpha << 1$ the influence of gravity is small and we expect to find a weakly self-gravitating version of the flat monopole (van Nieuwenhuizen et al. 1976). However, for values $\alpha \approx 1$ the situation changes. The size of the monopole is determined by $R_m = 1/M_W = \sqrt{G}/g\alpha$, whereas its Schwarzschild radius is given by $2GM_{mon} \approx GM_W/g^2 = \alpha\sqrt{G}/g$. For $\alpha \to 1$ both radii approach each other and we expect the monopole to become gravitationally unstable, i.e., we expect regular monopoles to exist only up to some maximal value of α of order one. Since no exact solutions are known (besides the ones involving a dilaton discussed in the previous section) , we have to take recourse to numerical methods for their study.

Combining the flat space ansatz for the YM resp. Higgs field with the one for the static, spherically symmetric gravitational field (11) the reduced Einstein-YM-Higgs (EYMH) action can be expressed as (using Schwarzschild coordinates for simplicity)

$$S = \int dr A \Big[\frac{1}{2}(rB' + B - 1) - r^2 BV_1 - V_2\Big] \,, \tag{18}$$

with

$$V_1 = \frac{(W')^2}{r^2} + \frac{1}{2}(H')^2 \,, \tag{19}$$

and

$$V_2 = \frac{(1-W^2)^2}{2r^2} + W^2H^2 + \frac{\beta^2 r^2}{8}(H^2 - \alpha^2)^2 \,. \tag{20}$$

Through a suitable rescaling we have achieved that the action depends only on the dimensionless parameters α and β.

Upon variation we obtain the corresponding field equations

$$(ABW)' = AW\Big(\frac{W^2-1}{r^2} + H^2\Big) \tag{21a}$$

$$(ABr^2H)' = AH\Big(\frac{\beta^2}{2}r^2(H^2-\alpha^2) + 2W^2\Big) \tag{21b}$$

$$rB' = 1 - B - 2(r^2BV_1 + V_2) \tag{21c}$$

$$rA' = 2r^2V_1A \,. \tag{21d}$$

This system of ODE's has singular points at $r = 0, \infty$ and for $B = 0$. Gravitating monopoles are globally regular solutions of the this singular system. Although it is not too difficult to prove local existence of suitable families of regular solutions the question of global existence is a very difficult problem, still beyond reach (except in some simple cases, e.g., for $\beta = \infty$). Recall that

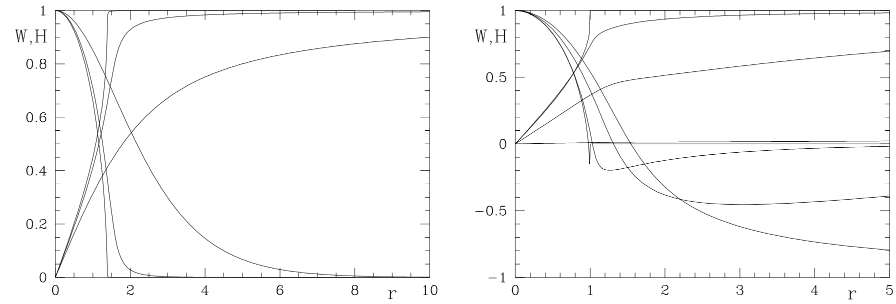

Fig. 5. W and H for ($\beta = 0$) a) the gravitating monopole solutions for $\alpha = 0.05, \alpha_{\max} = 1.403$ and $\alpha_c = 1.386$; b) first radial excitation for $\alpha = 0.01, 0.2, 0.5$ and 0.86.

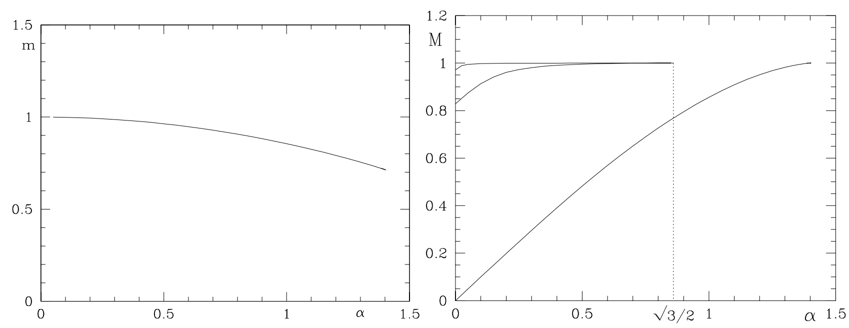

Fig. 6. a) Masses (in units of M_W) of fundamental monopole solutions versus α (for $\beta = 0$); b) Masses (now in units of $M_{\mathrm{Pl}}/\mathrm{g}$) of fundamental monopole solutions and first and second radial excitations versus α (for $\beta = 0$);

the energy functional has no good functional analytic properties in this case.

Thus our knowledge about the solutions is based to a large extent on numerical computations. There are several methods available for that purpose, one sided ("Shooting and aiming") or two sided methods ("matching") (Breitenlohner et al. 1992, 1995) and for stable solutions also relaxation methods may be applied.

The numerical analysis (Ortiz 1992, Lee et al. 1992, Breitenlohner et al. 1992, 1995) revealed that there are self-gravitating versions of the flat-space non-abelian monopoles for values of α ranging from zero up to some maximal value $\alpha_{\max}(\beta)$, fully in accordance with our expectations (Fig. 5a). As α increases the solutions develop a typical limiting behaviour, which may be characterized as "gravitational confinement" of the monopole. As the function B develops a double zero at the finite value $r_l = 1$ (measured in units of $1/M_{\mathrm{Pl}}$), the spatial hyper-surface $t =$ const. develops an infinite throat separating an interior region with a smooth origin and non-trivial YM field from an exterior solution with $W \equiv 0$, which is nothing but an extremal RN black hole. Thus there is a "confined" interior essentially non-abelian region and an outer abelian monopole. All this is much like the $t =$ const. surfaces of the extremal RN solution, with the only difference, that the interior part of the throat is in no sense the continuation of the exterior one. In fact the metric function A blows up along the throat coming from the interior, whereas $A \equiv 1$ for the RN solution. Hence the interior solution has no extremal horizon at $r = r_l$, instead it represents a geodesically complete, asymptotically AdS "cosmological" kind of solution.

Actually, what was just said is only true for not too large values of the parameter β (roughly $\beta < 10$), i.e., for not too large Higgs mass. For larger values of β it is still true that $B(r)$ develops a double zero at some finite value $r_l < 1$, thus there is the infinite throat again, but now $W(r)$ does not tend to zero there and $A(r)$ remains bounded. This means that the limiting solution, obtained for some maximal value $\alpha_{\max}(\beta)$, represents an extremal black hole with "non-abelian hair" (E. Weinberg, private communication). Due to the large difference between the mass scales for W and H it seems that this solution can be obtained numerically only with the use of a suitable relaxation method. Amazingly this extremal black hole is completely regular inside its horizon with a regular origin. At the horizon $W(r)$ and $H(r)$ are continuous, but not C^∞, due to some power behaviour of the type $(r - r_h)^{p_i}$ with some real exponents $p_i > 1$.

The observation, that the onset of a gravitational instability as the parameter α becomes too large, manifests itself within the family of static solutions in the formation of an extremal black hole (as far as the outer part of the solution is concerned) seems to be rather general. A similar phenomenon was observed for rigidly rotating dust discs by Neugebauer and Meinel (Meinel 1997). In their case the exterior part of the solution tends to the extremal Kerr solution, whereas the interior part is again some kind of a "cosmological" solution. In contrast to the flat space monopoles gravitating monopoles

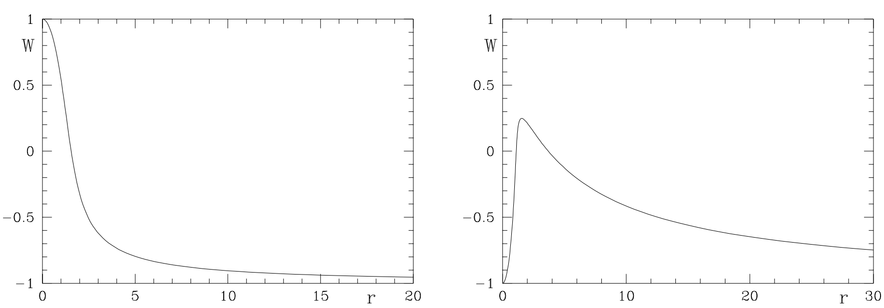

Fig. 7. W for the first two Bartnik-McKinnon solutions

also allow for radial excitations (compare Fig. 5b). As seen from Fig. 6b their mass stays finite (in units of M_{Pl}) as α tends to zero. At this point it is important to observe, that the limit $\alpha \to 0$ can be achieved in two different ways:

i) $G \to 0$, M_W fixed, in which the gravitational field decouples (flat space);
ii) $v = M_W/g \to 0$, G fixed, in which the Higgs field becomes trivial and can be ignored.

Whereas the fundamental monopole tends to its flat version as $\alpha \to 0$, the excited ones have no flat limit, instead tend to a class of solutions without a Higgs field, discovered by Bartnik and McKinnon (Bartnik and McKinnon 1988). There is a countably infinite number of these BM solutions distinguished by the number of zeros of the YM potential W. Their mass is of the order of M_{Pl}, the only scale of the EYM theory. As $r \to \infty$ the function $W(r)$ tends to ± 1 (compare Fig. 7), thus they carry no magnetic charge. In fact, they may be understood as some kind of gravitationally bound sphalerons (Volkov and Gal'tsov 1991; Sudarsky and Wald 1992), in particular as they turn out to be unstable (Straumann and Zhou 1990; Boschung et al. 1994; Volkov 1995). In addition to the "topological" instability of the flat YMH sphalerons however the gravitational BM sphalerons show additional "gravitational" instabilities within the minimal ansatz (Lavrelashvili and Maison 1995).

Turning back to the radially excited monopoles, it appears quite natural (at least for small values of α) to consider them as a Planck scale BM sphaleron sitting inside a $1/M_W$ size monopole. All these radial excitations disappear at the same value of $\alpha = \sqrt{3}/2$ merging in the by now well-known manner with the extremal RN black hole.

A few remarks should be made here about the stability properties of the gravitating monopoles. I shall discuss here only stability against infinitesimal, spherically symmetric perturbations. In view of the time-independence of the solutions this amounts to analyzing the spectrum of perturbations with harmonic time-dependence obeying suitable boundary conditions. Imaginary frequencies correspond to unstable modes of the solution. As to be expected the branch of gravitating monopoles connected to the flat space solution is stable up to α_{max}. All the excited regular monopoles turn out to be unstable (Hollmann 1994).

6 Coloured Black Holes

Apart from the solutions with Minkowskian space-time topology there are non-abelian, "coloured" black holes, parametrized by their radius r_h (the value of r at the event horizon) in addition to α and β (Breitenlohner et al. 1992, 1995, Lee et al. 1992). For $r_h << 1/M_W$ these non-abelian black holes may be interpreted as a tiny Schwarzschild black hole sitting inside a monopole (Kastor and Traschen 1992). On the other hand, when r_h becomes

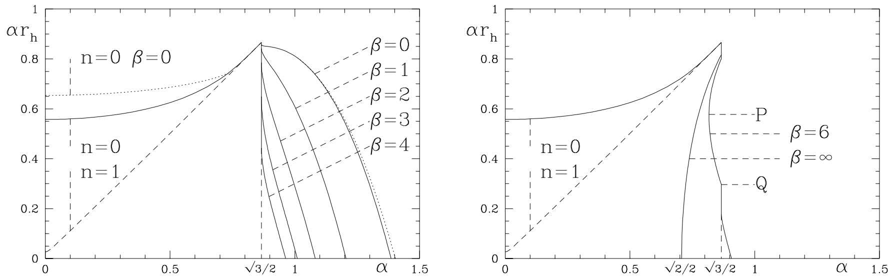

Fig. 8. Domains of existence for non-abelian black holes: a) for $\beta = 0$, 1, 2, 3, and 4; b) for $\beta = 6$ and ∞

bigger than $\approx 1/M_W$ this type of solution disappears and only the abelian RN black holes exist. For $r_h \to 0$ the matter fields tend uniformly to those of the globally regular solutions, whereas for the metrical functions this limit is clearly more delicate.

Detailed numerical analysis reveals that non-abelian black holes exist only in a limited domain of the α-r_h-plane, whose shape undergoes some characteristic changes as β varies from 0 to ∞. Fig. 8 shows some of these domains. Observe that we use αr_h instead of r_h as the abscissa - equivalent to expressing r_h in units of $1/M_W$ - in order to obtain domains remaining bounded for $\alpha \to 0$.

In the following I shall discuss in some more detail the structure of these "Phase Diagrams" and the phenomena happening at their boundaries. Let me start with the case $\beta = 0$. It is appropriate to subdivide the relevant sector $\alpha \geq 0$, $r_h \geq 0$ into the four subregions I-IV (compare Fig. 9).

In regions I and II we find coloured black holes. Above the diagonal, i.e., in regions II and III we have the abelian RN black holes, the extremal RN black holes sitting on the diagonal. Below the diagonal the RN solution has a naked singularity and does not represent a black hole. No black holes, neither abelian nor non-abelian, could be found in region IV. Region I may be subdivided in region Ia, where only the black hole version of the fundamental monopole resides and region Ib, where in addition their radial excitations are found. Thus region Ia contains only one black hole solution [2] for given values of α and r_h, whereas in region Ib countably many solutions exist for any given α and r_h.

In region Ib fundamental and radially excited solutions coexist, whereas in region II even abelian and non-abelian black holes coexist. This establishes an obvious violation of the so-called "No-Hair" Conjecture. According to the latter static black holes of a given mass (or size, i.e., given value of r_h) should be uniquely determined through their "gauge charges" - their magnetic charge

[2] This is not strictly true for small values of β, where two solutions exist in a small interval $\alpha_c(r_h) < \alpha < \alpha_{\max}(r_h)$.

in the present case. However, all these black holes carry the same magnetic charge. Although black holes of the same size differ in general in their mass, degeneracy in mass also occurs. In some cases abelian and non-abelian black holes are even degenerate in mass and size.

As β increases from 0 to $\beta = 4$ the structure of the "Phase Diagram" remains essentially the same, the right boundary curve moving in to the left. However, for $\beta > 4$ this boundary curve develops a second, concave branch (compare Fig. 8b) determined by another mechanism – the formation of a degenerate inner (above P) resp. outer (between P and Q) horizon, leading to extremal black holes with non-abelian hair.

The boundary curve above the diagonal is essentially characterized by the bifurcation of the non-abelian with the abelian RN solution. For a given value of α this happens at some value $r_{\mathrm{c,n}}(\alpha)$ depending on the number n of zeros of W (Fig. 8). Approaching this value from below the value W_h of W at the horizon tends to zero, thus abelian and non-abelian black holes merge.

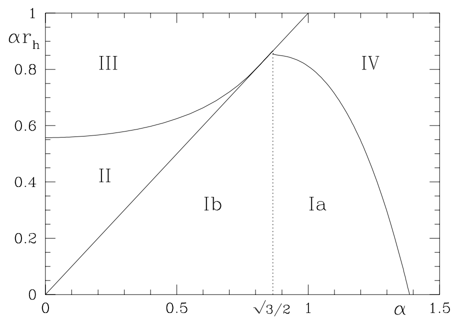

Fig. 9. Domains of existence for abelian and non-abelian black holes, $\beta = 0$

Finally again a remark concerning the stability of the solutions. Similar to the situation with regular monopoles the fundamental coloured black hole solutions are stable, likewise the radially excited ones are unstable. It is, however, interesting to observe that the abelian RN black hole is unstable in the framework of the non-abelian theory for α smaller than some value $\alpha(r_h) < \sqrt{3}/2$ (Bizon and Wald 1991; Lee et al. 1992a, Breitenlohner et al. 1992, 1995). In particular, the extremal RN solution is unstable for $\alpha \leq \sqrt{3}/2$ and stable above this value. At the limiting value $\alpha = \sqrt{3}/2$ the extremal RN solution bifurcates with infinitely many non-abelian solutions and in fact develops infinitely many unstable modes.

7 Additional Remarks

The interpretation of the BM sphalerons as gravitationally bound sphalerons is supported by the fact that similar solutions have been found for a theory where the gravitational field is replaced by a dilaton, serving the same purpose (Lavrelashvili and Maison 1992, 1993; Bizon 1993; Donets and Galtsov 1993). There is also an investigation of magnetic monopoles coupled to gravity and a dilaton (without SUSY) (Forgacs and Gyürüsi 1996). Gravitating monopoles resp.sphalerons for higher gauge groups (*SU(3)* etc.) were studied in (Kleihaus et al. 1995, 1998) with similar results.

Furthermore axially symmetric, static generalizations of the BM solutions were constructed numerically (Kleihaus and Kunz 1997). Similar solutions generalizing multiply charged axially symmetric flat monopoles are expected also with gravity. However more interesting is the question, if there are stationary rotating solutions. It seems that only the neutral BM solutions can rotate (Volkov and Straumann 1997; Brodbeck et al. 1997), whereas rotating magnetic monopoles are excluded (even in flat space) (Brodbeck and Heusler 1997; Heusler et al. 1998).

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