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Abstract

We introduce an $i\epsilon$ prescription for the Sachdev–Ye–Kitaev model both at finite and at zero temperature. This prescription regularizes all the naive ultraviolet divergences of the model. As expected the prescription breaks the conformal invariance, but the latter is restored in the $\epsilon \rightarrow 0$ limit. We prove rigorously that the Schwinger Dyson equation of the resummed two point function at large N and low momentum is recovered in this limit. Based on this $i\epsilon$ prescription we introduce an effective field theory Lagrangian for the infrared SYK model.

1. Introduction and discussion

The Sachdev–Ye–Kitaev (SYK) model [1–7] has been extensively studied recently in the context of the AdS/CFT duality. In its most common form, the SYK model is the one-dimensional field theory for a vector Majorana fermion χ_a with N components with action:

$$\frac{1}{2} \int_{-\beta/2}^{\beta/2} \mathrm{d}\tau \sum_{a} \chi_{a}(\tau) \partial_{\tau} \chi_{a}(\tau) + J \sum_{a^{1},\dots,a^{q}} T_{a^{1}\dots,a^{q}} \int_{-\beta/2}^{\beta/2} \mathrm{d}\tau \ \chi_{a^{1}}(\tau) \dots \chi_{a^{q}}(\tau), \tag{1}$$

where T are time independent quenched random couplings with Gaussian distribution:

$$\mathrm{d}\nu(T) = \left(\prod_{a^1,\dots,a^q} \sqrt{\frac{N^{D-1}}{2\pi}} \,\mathrm{d}T_{a^1,\dots,a^q}\right) \mathrm{e}^{-\frac{1}{2}N^{q-1}\sum_{a^1,\dots,a^q} T_{a^1\dots,a^q}T_{a^1\dots,a^q}}.$$

This model has a large N limit dominated by melonic graphs [3, 4, 8]. The melonic large N limit is universal in random tensors [9], and the quenching can be eliminated if one considers a tensor version of the SYK model [10-15] (see also [16] for a detailed discussion of the leading and next to leading orders in 1/N in various models).

Leaving aside the details of the model, the melonic large *N* limit leads to an 'almost conformal' one dimensional filed theory. This theory (the CFT side of the AdS/CFT) has been studied [3, 4, 17] with various degrees of rigor.

This paper aims to give a rigorous meaning to some of the results obtained so far in this research program.

The trouble with the two point function. Let us briefly review some standard results on the SYK model. Having a *q* fermion interaction and a free propagator:

$$C(\tau, \tau') = \frac{1}{2}\operatorname{sgn}(\tau - \tau'),$$

with antiperiodic boundary conditions at finite temperature, the model defined by equation (1) is power counting super renormalizable: there are no ultraviolet (UV) divergences, and infrared (IR) divergences might exist only at zero temperature. One can then resum the two point function at leading order in N. This resummed two point function, $\mathcal{G}_{\beta}(\tau, \tau')$, is recovered from the Schwinger Dyson equation (SDE):

$$1 = \mathcal{G}_{\beta}C^{-1} - \mathcal{G}_{\beta}\Sigma_{\beta},$$

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taking into account that in the melonic large N limit the self energy factors in terms of two point functions $\Sigma_{\beta}(\tau, \tau') = J^2[\mathcal{G}_{\beta}(\tau, \tau')]^{q-1}$:

$$\delta(\tau_1 - \tau_2) = \partial_{\tau_1} \mathcal{G}_{\beta}(\tau_1 - \tau_2) - J^2 \int_{-\beta/2}^{\beta/2} du \ \mathcal{G}_{\beta}(\tau_1 - u) [\mathcal{G}_{\beta}(u - \tau_2)]^{q-1},$$

where we used the fact that \mathcal{G}_{β} is antisymmetric and translation invariant.

While the SDE can not be solved analytically at arbitrary momentum (except for the degenerate q = 2 case [3]), a solution can be found in the conformal (low momentum, infrared) limit. Indeed, in this limit the first term (free term) can be neglected and the SDE becomes:

$$\delta(\tau) = J^2 \int_{-\beta/2}^{\beta/2} du \ G_{\beta}(u - \tau) [G_{\beta}(u)]^{q-1},$$
(2)

where G_{β} denotes the infrared two point function. Let us, for now, consider the zero temperature case, $\beta \to \infty$ (we will reinstate the finite temperature later on). In order to solve for the infrared resummed two point function one proposes the ansatz:

$$G_{\infty}(\tau) = b \frac{\operatorname{sgn}(\tau)}{|\tau|^{2\Delta}},$$

with $\Delta > 0$. Substituting this in equation (2) one gets [3, 4, 12] the equation:

$$\begin{split} \delta(\tau) &= J^2 b^q \int_{-\infty}^{\infty} \, \mathrm{d} u \, \frac{\mathrm{sgn}(u-\tau)}{|u-\tau|^{2\Delta}} \frac{\mathrm{sgn}(u)}{|u|^{2\Delta(q-1)}} = J^2 b^q \frac{1}{|\tau|^{2\Delta q-1}} \\ &\times [\beta(1-2\Delta, 2\Delta q-1) + \beta(1-2\Delta(q-1), 2\Delta q-1) - \beta(1-2\Delta, 1-2\Delta(q-1))], \end{split}$$

with $\beta(a, b)$ the Euler beta function. This equation is *formally* solved by $\Delta = \frac{1}{q}$ and *b* respecting:

$$1 = J^2 b^q \frac{\pi}{\frac{1}{2} - \frac{1}{q}} \frac{\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}},$$

however it is quite obvious that:

- the left-hand side of the equation, $\frac{1}{|\tau|^{2\Delta q-1}}$, is not a $\delta(\tau)$ function, even for $\Delta = \frac{1}{q}$.
- the integral does not converge absolutely in the $u \sim 0$ region for $\Delta = \frac{1}{q}$, as $2\Delta(q 1) = 2 \frac{2}{q} > 1$. This translates on the right-hand side in the fact that the Euler beta functions are evaluated at negative arguments. While, of course, the beta function can be defined by analytic continuation at negative arguments, its naive integral representation diverges for such values.

The situation only gets worse when one tries to compute the leading order four point function, the spectrum of the four point kernel (which generates the ladder diagrams) [3, 4] or the leading order six point functions [17]: all the integrals one encounters exhibit UV divergences. This should come as no surprise: in the conformal limit the theory is power counting marginal (as one would expect from a conformal field theory).

Of course these divergences have already been noted and discussed in the literature [3, 4]. Physically, they are regulated by the fact that at large momentum one can not use the conformal ansatz G_{β} and one must go back to the full two point function \mathcal{G}_{β} . Using the full two point function regulates all the divergences of the model: after all, we already know that the model is UV finite. However, as the SDE can not be solved analytically at arbitrary momentum, one does not have an explicit formula for \mathcal{G}_{β} . In the absence of such a formula, the procedure applied so far [3, 4] consists in the following:

- In most cases. In most cases one can try to make sense of these integrals by analytic continuation. One can hope that, due to the antisymmetry of the two point function, all the UV divergences are regulated if one defines the integrals by, for instance, a Cauchy principal value. In practice one computes the integrals for values of the parameters (like for instance Δ) for which they converge and then substitutes the relevant values (like $\Delta = \frac{1}{q}$) only at the end. Typically this leads to some Euler $\Gamma(a)$ functions evaluated at arguments *a* with negative real part which are well defined by analytic continuation. However this approach has several drawbacks:
 - sometimes one needs to formally evaluate integrals which are divergent for *any* values of the parameters
 [4], therefore not even the starting point of the analytic continuation is well defined.
 - the classical integral representation for the $\Gamma(a)$ function at $\Re(a) < 0$ requires [18, 19] counterterm subtractions:

$$-p - 1 < \Re(a) < -p,$$
 $\Gamma(a) = \int_0^\infty \mathrm{d}t \ t^{a-1} \left(\mathrm{e}^{-t} - \sum_{q=0}^p \frac{(-t)^p}{p!} \right).$

It is not clear where the counterterms might come from.

- the fact that the two point function is antisymmetric does not eliminate the UV divergences. Indeed, if two vertices of a graph are connected by and even number of edges larger or equal to q/2, the corresponding integral is divergent and symmetric hence the graph is UV divergent¹.
- finally, and most importantly, in the absence of an explicit regularization procedure, there is *a priori* no reason to consider the Cauchy principal value in the first place. In fact it turns out that the $t\epsilon$ regularization we introduce in this paper justifies the use of the Cauchy principal value in some of the cases encountered in [3, 4].

In some cases. In some cases the above procedure fails. This is notably the case (using the notation of [3]) of the h = 2 mode of the four point kernel which leads to a breaking of conformal invariance in the resummed leading order four point function. In this case the UV divergences are crucial and one needs to deal with them carefully. The procedure applied so far [3] (also discussed to a lesser extent in [4]) is to account for the effect of the free term in the SDE using first order perturbation theory in quantum mechanics. This has several drawbacks:

- while first order perturbation theory in quantum mechanics eliminates the divergence, it is difficult to see in what sense such a regularization can be rendered rigorous (the perturbation theory in quantum mechanics usually diverges).
- it is not a priori obvious that this procedure will regulate all the divergences.
- perturbation theory in quantum mechanics is model dependent. In order to study the departure from conformality in the SYK model in a systematic manner, a more appropriate starting point would be a universal regularization procedure.

In this paper we propose an $\iota\epsilon$ prescription for the SYK model which regulates all the UV divergences. The limit $\epsilon \to 0$ can be taken rigorously. Our prescription is a particular kind of cutoff in the frequency space and comes to replacing the low momentum resummed two point function G_{β} by a regulated version G_{β}^{ϵ} . Like the full two point function \mathcal{G}_{β} of the SYK model, the regulated two point function G_{β}^{ϵ} breaks the conformal invariance. Contrary to \mathcal{G}_{β} however, G_{β}^{ϵ} does this in an universal manner.

The interpretation of this prescription is best understood if one takes a quantum field theoretical point of view on the SYK model. The momentum scale at which one feels the breaking of conformal invariance due to the first term of the SDE, where one should start using \mathcal{G}_{β} instead of \mathcal{G}_{β} , plays the role of an ultimate 'physical cutoff scale'. In the case of quantum electrodynamics (QED) for instance this should be taken as the scale at which quantum chromodynamics (QCD) effects come into play; for the standard model as a whole this could be a grand unification scale, or the Plank scale. Its precise value, and the precise way in which it alters the UV behavior of the model should play no role in understanding the departure from conformality in the SYK model (to pursue our comparison, understanding that QED flows to the Gaussian fixed point in the infrared and computing the β function close to the Gaussian fixed point does not depend on the number of quark generations). In order to understand the infrared behavior of the model one needs to introduce a new scale (call it the 'mathematical cutoff scale') and a regularization procedure (for instance a multiplicative momentum cutoff or a Schwinger parametric cutoff). This is an arbitrary UV scale, which can be considered lower that the physical cutoff scale (in QED this would be a cutoff scale in the neighborhood of the Gaussian fixed point). Introducing this new scale and a regularization procedure at this scale allows one to ignore the true UV completion of the theory (in QED, once one introduces a UV cutoff scale, one ignores the rest of the standard model), and study its infrared behavior in a self contained manner.

The *t* ϵ prescription we present here yields the 'mathematical cutoff' of the SYK model. It allows one to study the departures from conformality without needing to resort to the precise UV completion \mathcal{G}_{β} of the model. The ϵ scale is a mathematical artifact which identifies the overall power counting of an effect, but there is no meaning attached to the specific value of ϵ .

An upshot of our $\iota \epsilon$ prescription is that we are able, at finite β , to write down an explicit effective field theory Lagrangian whose large N resummed two point function is G_{β}^{ϵ} . The similarity between the Lagrangian we

¹ One can still attempt to deal with this by resuming families of graphs. This is a formal manipulation, as each individual graph in the family is divergent. Moreover, except is very simple cases, one can not identify appropriate families of graphs to (formally) cancel all the divergences.

R Gurau

propose here and the 'conformal SYK' Lagrangian recently discussed in [20] is only superficial: the two models differ drastically in the infrared. To be precise, in the conformal SYK model of [20], G_{β}^{ϵ} is the *bare* covariance while in our case it is the *effective* two point function. Ergo the infrared behavior of our effective Lagrangian reproduces (a cutoffed version of) the infrared of the genuine SYK model, while the infrared behavior of the conformal SYK model of [20] does not.

A feature of the effective Lagrangian we introduce in this paper is that it *requires* the presence of the regulator ϵ : in the limit $\epsilon \to 0$ the bare covariance diverges². The effective field theory fails in this limit. Below we prove that the effective Lagrangian we propose leads to a sensible theory for ϵ large enough. We conjecture that this is in fact the case for any $\epsilon > 0$.

2. The *i* ϵ regularization

We consider *q*, the number of fermions, to be even and $q \ge 4$ and we denote $\Delta = 1/q$. We posit the *i* ϵ regularization of the two point function in the SYK model:

$$G_{\beta}^{\epsilon}(\tau) = \frac{b}{2\iota\sin(\pi\Delta)} \left[\frac{1}{\left(\frac{\beta}{\pi}\sinh\frac{\pi(\epsilon-\iota\tau)}{\beta}\right)^{2\Delta}} - \frac{1}{\left(\frac{\beta}{\pi}\sinh\frac{\pi(\epsilon+\iota\tau)}{\beta}\right)^{2\Delta}} \right]$$
$$= \frac{b}{\sin(\pi\Delta)} \left(\frac{\beta}{\pi}\right)^{-2\Delta} \frac{\sin\left(2\Delta\arctan\frac{\tan\frac{\pi\tau}{\beta}}{t_{\epsilon}}\right)}{\left[\left(\sinh\frac{\pi\epsilon}{\beta}\right)^{2}\left(\cos\frac{\pi\tau}{\beta}\right)^{2} + \left(\cosh\frac{\pi\epsilon}{\beta}\right)^{2}\left(\sin\frac{\pi\tau}{\beta}\right)^{2}\right]^{\Delta}}$$

Observe that $G_{\beta}^{\epsilon}(\tau) = -G_{\beta}^{\epsilon}(-\tau)$, $G_{\beta}^{\epsilon}(\tau) \ge 0$ for $0 \le \tau \le \beta/2$ and that in the $\epsilon \to 0$ limit one recovers pointwise the conformal two point function at finite temperature [3]:

$$\lim_{\epsilon \to 0} G_{\beta}^{\epsilon}(\tau) = G_{\beta}(\tau) = b \frac{\operatorname{sgn}(\tau)}{\left| \frac{\beta}{\pi} \sin \frac{\pi\tau}{\beta} \right|^{2\Delta}}$$

The zero temperature version is obtained by taking $\beta \rightarrow \infty$:

$$G_{\infty}^{\epsilon}(\tau) = \frac{b}{2\iota\sin(\pi\Delta)} \left[\frac{1}{(\epsilon - \iota\tau)^{2\Delta}} - \frac{1}{(\epsilon + \iota\tau)^{2\Delta}} \right].$$

One can easily write down the momentum space representation at zero temperature:

$$G_{\infty}^{\epsilon}(\tau) = \frac{b}{2\iota\sin(\pi\Delta)\Gamma(2\Delta)} \int_{0}^{\infty} d\omega \ \omega^{2\Delta-1} \ e^{-\epsilon\omega}(e^{\iota\omega\tau} - e^{-\iota\omega\tau}),$$

while the momentum space representation at finite temperature requires a bit more effort (see appendix A):

$$G_{\beta}^{\epsilon}(\tau) = \frac{b}{2\iota\sin(\pi\Delta)\Gamma(2\Delta)} \left(\frac{2\pi}{\beta}\right)^{2\Delta} \sum_{n>0} \frac{\Gamma\left(\frac{\beta}{2\pi}\omega_n + \Delta\right)}{\Gamma\left(\frac{\beta}{2\pi}\omega_n + 1 - \Delta\right)} e^{-\epsilon\omega_n} [e^{\iota\omega_n\tau} - e^{-\iota\omega_n\tau}],$$

where $\omega_n = \frac{2\pi}{\beta} \left(n + \frac{1}{2} \right)$ denotes the fermionic Matsubara frequencies. In particular G_{β}^{ϵ} is a positive operator (as it is diagonal in momentum space and its eigenvalues are positive). Observing that the Matsubara frequencies vary in increments of $\frac{2\pi}{\beta}$, one recovers directly the momentum space representation at zero temperature in the $\beta \to \infty$ limit.

As mentioned in the introduction, it is clear that this $\iota\epsilon$ prescription is an $e^{-\epsilon|\omega|}$ frequency cutoff. The Feynman graphs of the effective theory (each such graph represents the resummation of graphs of the bare model with arbitrary melonic insertions on the edges) have q valent vertices and effective propagators G_{β}^{ϵ} . As $|\sinh(\epsilon \pm \iota\tau)| \ge \sinh(\epsilon)$, at finite temperature the amplitude of a graph with E edges and V internal vertices is bounded up to constants by:

$$\beta^V \frac{1}{\sinh(\epsilon)^E}.$$

Of course this bound can be significantly improved (in particular the marginal power counting of any graph can be recovered easily). At zero temperature the amplitudes are UV finite, but one might encounter IR divergences.

 $^{^2}$ This is again in contrast with the conformal SYK model of [20] whose bare version does not require a regulator ϵ .

The right-hand side of the SDE equation (2) becomes with our regularization:

$$A^{\epsilon}_{\beta}(\tau) = J^2 \int_{-\beta/2}^{\beta/2} \mathrm{d}u \ G^{\epsilon}_{\beta}(u-\tau) [G^{\epsilon}_{\beta}(u)]^{q-1}.$$
(3)

Our first results is presented in the following theorem:

Theorem 1. $A^{\epsilon}_{\beta}(\tau)$ is a well defined distribution for any ϵ and:

$$\lim_{\epsilon \to 0} A^{\epsilon}_{\beta}(\tau) = \delta(\tau), \tag{4}$$

in the sense of distributions.

Proof. See section 3

Observe that A^{ϵ}_{β} can also be viewed as a linear operator on the Hilbert space $L^2[(-\beta/2, \beta/2)]$:

$$(A^{\epsilon}_{\beta}f)(\tau) = \int_{-\beta/2}^{\beta/2} \mathrm{d}\tau' A^{\epsilon}_{\beta}(\tau - \tau')f(\tau'),$$

which commutes with the inverse covariance $(G_{\beta}^{\epsilon})^{-1}$:

$$((G_{\beta}^{\epsilon})^{-1}A_{\beta}^{\epsilon})(\tau_{1}, \tau_{2}) = -J^{2}[G_{\beta}^{\epsilon}(\tau_{1} - \tau_{2})]^{q-1} = (A_{\beta}^{\epsilon}(G_{\beta}^{\epsilon})^{-1})(\tau_{1}, \tau_{2})$$

2.1. Effective field theory

One of the most interesting facts about this $i\epsilon$ regularization is that it allows one to introduce an effective field theory reproducing the IR behavior of the SYK model at all orders in 1/N.

Our aim is to write a field theory whose effective resummed leading order two point function is the IR propagator of the SYK model $G_{\beta}^{\epsilon}(\tau_1, \tau_2)$ and whose interaction that of equation (1). If we take the bare propagator of the effective field theory to be $G_{\beta}^{\epsilon}(\tau_1, \tau_2)$, that is if we consider the conformal SYK model of [20] with momentum cutoff, the effective two point function at leading order in 1/N will be $G_{\beta}^{\epsilon}(\tau_1, \tau_2)$ dressed by melonic radiative corrections. We add to the bare theory a bi local counterterm:

$$\frac{1}{2}\int_{-\beta/2}^{\beta/2} \mathrm{d}\tau_{\mathrm{I}} \mathrm{d}\tau_{\mathrm{2}} \sum_{a} \chi_{a}(\tau_{\mathrm{I}}) \mathbf{A}_{\beta}^{\epsilon}(\tau_{\mathrm{I}}, \tau_{2}) \chi_{a}(\tau_{2}),$$

so as to precisely cancel these radiative corrections at leading order in 1/N and lead to an effective two point function exactly equal to G_{β}^{ϵ} . In order to determine the appropriate counterterm, let us take for the moment some arbitrary $\mathbf{A}_{\beta}^{\epsilon}$ and denote the resummed two point function in the melonic sector with this choice of counterterm $G_{\mathbf{A}_{\beta}^{\epsilon};\beta}^{\epsilon}$. The SDE of this model at leading order in 1/N writes:

$$1 = G^{\epsilon}_{\mathbf{A}^{\epsilon}_{\beta\beta}\beta}[G^{\epsilon}_{\beta}]^{-1} + G^{\epsilon}_{\mathbf{A}^{\epsilon}_{\beta\beta}\beta}\mathbf{A}^{\epsilon}_{\beta} - G^{\epsilon}_{\mathbf{A}^{\epsilon}_{\beta\beta}\beta}\boldsymbol{\Sigma}^{\epsilon}_{\mathbf{A}^{\epsilon}_{\beta\beta}\beta}, \qquad \boldsymbol{\Sigma}^{\epsilon}_{\mathbf{A}^{\epsilon}_{\beta\beta}\beta} = J^{2}[G^{\epsilon}_{\mathbf{A}^{\epsilon}_{\beta\beta}\beta}]^{q},$$

where $\Sigma_{\mathbf{A}_{\beta}^{\epsilon};\beta}^{\epsilon}$ is the self energy at melonic order in the model with counterterm. We now require that $G_{\mathbf{A}_{\beta}^{\epsilon};\beta}^{\epsilon} = G_{\beta}^{\epsilon}$ is a solution of this equation which imposes:

$$\mathbf{A}^{\epsilon}_{\beta}(\tau_{1},\,\tau_{2})=J^{2}[G^{\epsilon}_{\beta}(\tau_{1},\,\tau_{2})]^{q-1}=-([G^{\epsilon}_{\beta}]^{-1}A^{\epsilon}_{\beta})(\tau_{1},\,\tau_{2}),$$

hence the effective field theory action we propose is:

$$S^{\text{eff}} = \frac{1}{2} \int_{-\beta/2}^{\beta/2} d\tau_1 d\tau_2 \sum_a \chi_a(\tau_1) ([G_{\beta}^{\epsilon}]^{-1} (1 - A_{\beta}^{\epsilon}))(\tau_1, \tau_2) \chi_a(\tau_2) + J \sum_{a^1, \dots, a^q} T_{a^1, \dots, a^q} \int_{-\beta/2}^{\beta/2} d\tau \ \chi_{a^1}(\tau) \dots \chi_{a^q}(\tau),$$
(5)

and the random couplings are of course still quenched and distributed on a Gaussian. The bare covariance of the effective SYK field theory is:

$$G_{\beta}^{\epsilon} \frac{1}{1 - A_{\beta}^{\epsilon}},$$

and is a well defined positive operator for ϵ large enough due to the following result.

Theorem 2. For any finite inverse temperature β and for ϵ large enough such that:

$$\frac{1-t_{\epsilon}^2}{1+t_{\epsilon}^2} \left[1 + \frac{2\Delta+1}{\sqrt{\pi}} t_{\epsilon} \right] \leqslant [\tan(\pi\Delta)]^{q-2},$$

R Gurau

the operator A^{ϵ}_{β} is bounded in norm by 1:

$$||A_{\beta}^{\epsilon}||_{\mathrm{op}} = \sup_{f, \, ||f||_2 \leqslant 1} ||A_{\beta}^{\epsilon}f||_2 \leqslant 1,$$

where $\|\cdot\|_2$ denotes the L^2 norm on $L^2[(-\beta/2, \beta/2)]$.

Proof. See section 3.1

The effective field theory will break down at some momentum scale. Indeed, theorem 2 ensures that the effective theory is well defined only for low enough momentum cutoff ϵ^{-1} . From theorem 1 we see that the bare covariance of the model diverges in the $\epsilon \to 0$ limit, hence the effective field theory certainly breaks down in the limit. We conjecture that the effective field theory breaks down only in the $\epsilon \to 0$ limit, that is we conjecture that theorem 2 can be extended to any $\epsilon > 0$.

3. The SDE

In this section we prove theorem 1.

Let us denote:

$$\begin{split} s_{\epsilon-\iota\tau} &= \sinh \frac{\pi(\epsilon-\iota\tau)}{\beta}, \ c_{\epsilon-\iota\tau} = \cosh \frac{\pi(\epsilon-\iota\tau)}{\beta}, \ t_{\epsilon-\iota\tau} = \tanh \frac{\pi(\epsilon-\iota\tau)}{\beta}, \\ s_{\epsilon+\iota\tau} &= \sinh \frac{\pi(\epsilon+\iota\tau)}{\beta}, \ c_{\epsilon+\iota\tau} = \cosh \frac{\pi(\epsilon+\iota\tau)}{\beta}, \ t_{\epsilon+\iota\tau} = \tanh \frac{\pi(\epsilon+\iota\tau)}{\beta}, \\ s_{\epsilon} &= \sinh \frac{\pi\epsilon}{\beta}, \ c_{\epsilon} = \cosh \frac{\pi\epsilon}{\beta}, \ t_{\epsilon} = \tanh \frac{\pi\epsilon}{\beta}, \\ s_{\tau} &= \sin \frac{\pi\tau}{\beta}, \ c_{\tau} = \cos \frac{\pi\tau}{\beta}, \ t_{\tau} = \tan \frac{\pi\tau}{\beta}. \end{split}$$

We start by rewriting $A^{\epsilon}(\tau)$ as a convergent integral more suitable to discuss the $\epsilon \to 0$ limit.

Proposition 1. We have the following integral representation:

$$A_{\beta}^{\epsilon}(\tau) = 2\pi J^{2} \left(\frac{b}{2t\sin(\pi\Delta)}\right)^{q} \left(\frac{\pi}{\beta}\right) \frac{1}{\Gamma(2\Delta)} \\ \times \left\{\frac{-1}{\Gamma(2-2\Delta)} \frac{1}{c_{\epsilon}^{2\Delta(q-1)}} \left[\frac{1}{c_{\epsilon+t\tau}^{2\Delta}[t_{\epsilon+t\tau} + t_{\epsilon}]} + \frac{1}{c_{\epsilon-t\tau}^{2\Delta}[t_{\epsilon-t\tau} + t_{\epsilon}]}\right] + \sum_{r=1}^{q/2-1} {q-1 \choose r} (-1)^{r} \\ \times \int_{1}^{\infty} \frac{dy}{2^{2\Delta(q-1)-1}} \frac{(y-1)^{2\Delta(q-1-r)-1}(y+1)^{2\Delta r-1} - (y+1)^{2\Delta(q-1-r)-1}(y-1)^{2\Delta r-1}}{\Gamma[2\Delta(q-1-r)]\Gamma(2\Delta r)} \\ \times \frac{1}{c_{\epsilon}^{2\Delta(q-1)}} \left[\frac{1}{c_{\epsilon+t\tau}^{2\Delta}[t_{\epsilon+t\tau} + t_{\epsilon}y]} + \frac{1}{c_{\epsilon-t\tau}^{2\Delta}[t_{\epsilon-t\tau} + t_{\epsilon}y]}\right] \right\}.$$
(6)

Proof. See appendix **B**.

From equation (6), one can show that $A^{\epsilon}(\tau)$ is a well defined distribution for any $\epsilon > 0$ and that in the sense of distributions it converges to $\delta(\tau)$. Indeed, let us consider a term in equation (6). When applied on a test function $f(\tau)$ it has the generic form:

$$\left(\frac{\pi}{\beta}\right)\int_{-\beta/2}^{\beta/2} \mathrm{d}\tau \int_{1}^{\infty} \mathrm{d}y \, H(y) \frac{1}{c_{\epsilon}^{2\Delta(q-1)}} \left[\frac{1}{c_{\epsilon+\iota\tau}^{2\Delta}[t_{\epsilon+\iota\tau}+t_{\epsilon}y]} + \frac{1}{c_{\epsilon-\iota\tau}^{2\Delta}[t_{\epsilon-\iota\tau}+t_{\epsilon}y]}\right] f(\tau),\tag{7}$$

where:

$$H(y) = \frac{1}{2^{2\Delta(q-1)-1}} \frac{(y-1)^{2\Delta(q-1-r)-1}(y+1)^{2\Delta r-1} - (y+1)^{2\Delta(q-1-r)-1}(y-1)^{2\Delta r-1}}{\Gamma[2\Delta(q-1-r)]\Gamma(2\Delta r)},$$

is a function such that:

- H(y) is integrable in $y \sim 1$,
- $H(y) \sim y^{2\Delta(q-1)-3}$ for $y \sim \infty$ hence H(y) is integrable for $y \sim \infty$,
- $H(y) \leq 0$ for $y \in [1, \infty]$.

•

We now express $c_{\epsilon \pm \iota \tau}^{-2\Delta}$ and $t_{\epsilon \pm \iota \tau}$ in terms of t_{τ} by the formulae:

$$\frac{1}{c_{\epsilon\pm t\tau}^{2\Delta}} = (1+t_{\tau}^2)^{\Delta} \mathrm{e}^{-2\Delta \ln(c_{\epsilon}\pm is_{\epsilon}t_{\tau})} = \frac{1}{c_{\epsilon}^{2\Delta}} \left(\frac{1+t_{\tau}^2}{1+t_{\epsilon}^2 t_{\tau}^2}\right)^{\Delta} \mathrm{e}^{\mp 2i\Delta \arctan(t_{\epsilon}t_{\tau})},$$
$$t_{\epsilon\pm i\tau} = \frac{s_{\epsilon\pm i\tau}}{c_{\epsilon\pm i\tau}} = \frac{s_{\epsilon}\pm ic_{\epsilon}t_{\tau}}{c_{\epsilon}\pm is_{\epsilon}t_{\tau}} = \frac{t_{\epsilon}\pm it_{\tau}}{1\pm it_{\epsilon}t_{\tau}},$$

and changing variables to $v = \frac{t_{\tau}}{(1+y)t_{\epsilon}}$, equation (7) becomes:

$$\int_{1}^{\infty} dy \int_{-\infty}^{\infty} dv \, R^{\epsilon}(v, y) \, H(y) \, f\left(\frac{\beta}{\pi} \arctan[(1+y)t_{\epsilon}v]\right),$$

$$R_{\beta}^{\epsilon}(v, y) = \frac{2(1-t_{\epsilon}^{2})}{[1+t_{\epsilon}^{2}(1+y)^{2}v^{2}]^{1-\Delta}[1+t_{\epsilon}^{4}(1+y)^{2}v^{2}]^{\Delta}[1+v^{2}(1+yt_{\epsilon}^{2})^{2}]} \times \{\cos[2\Delta\arctan(t_{\epsilon}^{2}(1+y)v)](1+t_{\epsilon}^{2}v^{2}(1+y)(1+yt_{\epsilon}^{2})) + \sin[2\Delta\arctan(t_{\epsilon}^{2}(1+y)v)]v(1-t_{\epsilon}^{2})\}.$$
(8)

Using proposition 2 in the appendix C, and denoting $||f||_{\infty}$ the L^{∞} norm of f (which is a constant, if f is a test function), the integral in equation (8) is bounded by:

$$2\pi K_{\epsilon}||f||_{\infty}\int_{1}^{\infty} \mathrm{d}y |H(y)| \leq K,$$

for some constant *K* independent of ϵ (as $K_{\epsilon} < 3$). By the Lebesgue dominated convergence theorem we can then commute the $\epsilon \rightarrow 0$ limit and the integral and we have:

$$\begin{split} &\lim_{\epsilon \to 0} \left(\frac{\pi}{\beta}\right) \int_{-\beta/2}^{\beta/2} d\tau \int_{1}^{\infty} dy \, H(y) \frac{1}{c_{\epsilon}^{2\Delta(q-1)}} \left[\frac{1}{c_{\epsilon+\pi}^{2\Delta}[t_{\epsilon+\pi} + t_{\epsilon}y]} + \frac{1}{c_{\epsilon-\pi}^{2\Delta}[t_{\epsilon-\pi} + t_{\epsilon}y]} \right] f(\tau) \\ &= 2\pi f(0) \int_{1}^{\infty} dy \, H(y), \end{split}$$

as $\lim_{\epsilon \to 0} R^{\epsilon}_{\beta}(v, y) = \frac{2}{1+v^2}$. We therefore obtain: $\lim_{\epsilon \to 0} \int_{-\infty}^{\beta/2} d\tau \, A^{\epsilon}(\tau) f(\tau)$

$$\lim_{\epsilon \to 0} \int_{-\beta/2} d\tau A_{\beta}(\tau) f(\tau) = f(0)(2\pi J)^2 \left(\frac{b}{2\iota \sin(\pi\Delta)}\right)^q \frac{1}{\Gamma(2\Delta)} \left\{\frac{-1}{\Gamma(2-2\Delta)} + \sum_{r=1}^{q/2-1} \binom{q-1}{r} (-1)^r + \sum_{r=1}^{\infty} \frac{dy}{2^{2\Delta(q-1)-1}} \frac{(y-1)^{2\Delta(q-1-r)-1}(y+1)^{2\Delta r-1} - (y+1)^{2\Delta(q-1-r)-1}(y-1)^{2\Delta r-1}}{\Gamma[2\Delta(q-1-r)]\Gamma(2\Delta r)} \right\}.$$
(9)

The integrals over *y* are evaluated in appendix D and we get:

$$\begin{split} &\lim_{\epsilon \to 0} \int_{-\beta/2}^{\beta/2} \mathrm{d}\tau \, A_{\beta}^{\epsilon}(\tau) f(\tau) = f(0) (2\pi J)^2 \left(\frac{b}{2\iota \sin(\pi\Delta)}\right)^q \frac{1}{\Gamma(2\Delta)} \\ &\times \left\{ \frac{-1}{\Gamma(2-2\Delta)} + \sum_{r=1}^{q/2-1} \binom{q-1}{r} (-1)^r \frac{1}{\Gamma[2\Delta(q-1-r)]\Gamma(2\Delta r)} \right. \\ &\times \left[\left(\frac{1-2\Delta r}{-1+2\Delta}\right) \frac{\Gamma(2\Delta)\Gamma[2-2\Delta-2\Delta r]}{\Gamma(2-2\Delta r)} - \left(\frac{1-2\Delta(q-1-r)}{-1+2\Delta}\right) \frac{\Gamma(2\Delta)\Gamma(2\Delta r)}{\Gamma(2\Delta+2\Delta r)} \right] \right\}. \end{split}$$

Observe that all the terms in the last two lines can be combined in a unique sum over r:

$$f(0)(2\pi J)^2 \left(\frac{b}{2\iota\sin(\pi\Delta)}\right)^q \frac{1}{\Gamma(2\Delta)} \sum_{r=1}^{q-1} {q-1 \choose r} (-1)^r \left(\frac{1-2\Delta r}{-1+2\Delta}\right) \frac{\Gamma(2\Delta)}{\Gamma(2\Delta r)\Gamma(2-2\Delta r)}$$

and using:

$$\sum_{r=1}^{q-1} \binom{q-1}{r} (-1)^r \frac{(1-2\Delta r)}{\Gamma(2\Delta r)\Gamma(2-2\Delta r)} = \frac{1}{\pi} \sum_{r=1}^{q-1} \binom{q-1}{r} (-1)^r \sin(2\pi\Delta r)$$
$$= \frac{(2i)^q}{2\pi} [\sin(\pi\Delta)]^{q-1} \cos(\pi\Delta(q-1)),$$

we finally obtain:

$$\lim_{\epsilon \to 0} \int_{-\beta/2}^{\beta/2} \mathrm{d}\tau \, A_{\beta}^{\epsilon}(\tau) f(\tau) = f(0) \, J^2 b^q \frac{\pi}{\frac{1}{2} - \frac{1}{q}} \frac{\cos \frac{\pi}{q}}{\sin \frac{\pi}{q}},$$

which completes the proof of theorem 1.

3.1. The bare covariance

We now prove theorem 2.

Observe that for any function in $L^2[(-\beta/2, \beta/2)]$, the L^2 norm is bounded by the L^{∞} norm $||f||_2 \leq \beta^{1/2} ||f||_{\infty}$, hence:

$$||A_{\beta}^{\epsilon}||_{\mathrm{op}} = \sup_{f, \, ||f||_2 \leqslant 1} ||A_{\beta}^{\epsilon}f||_2 \leqslant \sup_{f, \, ||f||_{\infty} \leqslant \beta^{-1/2}} ||A_{\beta}^{\epsilon}f||_2.$$

On the other hand:

$$\begin{aligned} (||A_{\beta}^{\epsilon}f||_{2})^{2} &= \int_{-\beta/2}^{\beta/2} \mathrm{d}\tau_{1} \int_{-\beta/2}^{\beta/2} \mathrm{d}\tau \; \bar{A}_{\beta}^{\epsilon}(\tau) \bar{f}(\tau_{1}+\tau) \int_{-\beta/2}^{\beta/2} \mathrm{d}\tau' A_{\beta}^{\epsilon}(\tau') f(\tau_{1}+\tau') \\ &\leqslant \beta ||f||_{\infty}^{2} \left(\int_{-\beta/2}^{\beta/2} \mathrm{d}\tau \; |A_{\beta}^{\epsilon}(\tau)| \right)^{2} \Rightarrow ||A^{\epsilon}||_{\mathrm{op}} \leqslant \int_{-\beta/2}^{\beta/2} \mathrm{d}\tau \; |A_{\beta}^{\epsilon}(\tau)|, \end{aligned}$$

therefore, in the notation of section 3, the operator norm of A_{β}^{A} is bounded by a sum of terms of the form:

$$\int_1^\infty dy \int_{-\infty}^\infty dv \ R^\epsilon(v, y) \ |H(y)|,$$

therefore we obtain a bound:

$$\begin{split} ||A_{\beta}^{\epsilon}||_{\text{op}} &\leqslant K_{\epsilon} (2\pi J)^{2} \left(\frac{b}{2\sin(\pi\Delta)}\right)^{q} \frac{1}{\Gamma(2\Delta)} \sum_{r=1}^{q-1} {q-1 \choose r} \left| \left(\frac{1-2\Delta r}{1-2\Delta}\right) \frac{\Gamma(2\Delta)}{\Gamma(2\Delta r)\Gamma(2-2\Delta r)} \right| \\ &= K_{\epsilon} \frac{1}{\cos(\pi\Delta) [2\sin(\pi\Delta)]^{q-1}} \sum_{r=1}^{q-1} {q-1 \choose r} \sin(2\pi\Delta r) = \frac{K_{\epsilon}}{[\tan(\pi\Delta)]^{q-2}}. \end{split}$$

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Appendix A. The momentum space representation

At finite temperature we use the Fourier transform conventions:

$$\tilde{f}(\omega) = \int_{-\beta/2}^{\beta/2} \mathrm{d}\tau \ \mathrm{e}^{\imath\omega\tau} f(\tau), \qquad f(\tau) = \frac{1}{\beta} \sum_{n \in \mathbb{Z}} \mathrm{e}^{-\imath\omega_n\tau} \tilde{f}(\omega_n),$$

where $\omega_n = \frac{2\pi}{\beta} \left(n + \frac{1}{2} \right)$ are the fermionic Matsubara frequencies. Our aim in this section is to compute the Fourier transform:

$$\tilde{G}^{\epsilon}_{\beta}(\omega) = \frac{b}{2\iota\sin(\pi\Delta)} \left(\frac{\pi}{\beta}\right)^{2\Delta} \int_{-\beta/2}^{\beta/2} d\tau \ e^{\iota\omega\tau} \left[\frac{1}{\left(\sinh\frac{\pi(\epsilon-\iota\tau)}{\beta}\right)^{2\Delta}} - \frac{1}{\left(\sinh\frac{\pi(\epsilon+\iota\tau)}{\beta}\right)^{2\Delta}}\right],$$

where ω is one of the fermionic Matsubara frequencies ω_n .

We denote
$$s_{\epsilon} = \sinh \frac{\pi\epsilon}{\beta}$$
, $c_{\epsilon} = \cosh \frac{\pi\epsilon}{\beta}$, $t_{\epsilon} = \tanh \frac{\pi\epsilon}{\beta}$ and change variables to $t = \tan \frac{\pi\tau}{\beta}$ to get:

$$\frac{b}{2\iota \sin(\pi\Delta)} \left(\frac{\pi}{\beta}\right)^{2\Delta-1} \int_{-\infty}^{\infty} \frac{dt}{1+t^2} \left(\frac{1+\iota t}{1-\iota t}\right)^{\frac{\beta}{2\pi}\omega} (1+t^2)^{\Delta} \left[\frac{1}{(s_{\epsilon}-\iota c_{\epsilon}t)^{2\Delta}} - \frac{1}{(s_{\epsilon}+\iota c_{\epsilon}t)^{2\Delta}}\right]$$

$$= \frac{b}{2\iota \sin(\pi\Delta)} \left(\frac{\pi}{\beta}\right)^{2\Delta-1} (-\iota) \int_{-\iota \infty}^{\iota \infty} dz \ \frac{(1+z)^{\frac{\beta}{2\pi}\omega-1+\Delta}}{(1-z)^{\frac{\beta}{2\pi}\omega+1-\Delta}} \left[\left(\frac{1}{s_{\epsilon}-c_{\epsilon}z}\right)^{2\Delta} - \left(\frac{1}{s_{\epsilon}+c_{\epsilon}z}\right)^{2\Delta}\right].$$
(10)

At $z \sim \infty$ the integrand behaves like z^{-2} hence we can turn the contour of integration on z to run around the positive or negative real axis.

Let us consider $\omega > 0$ (the case $\omega < 0$ is similar). The first term in equation (10) writes:

$$\frac{b}{2\iota\sin(\pi\Delta)} \left(\frac{\pi}{\beta}\right)^{2\Delta-1} (-\iota) \int_{-\iota\infty}^{\iota\infty} dz \ \frac{(1+z)^{\frac{\beta}{2\pi}\omega-1+\Delta}}{(1-z)^{\frac{\beta}{2\pi}\omega+1-\Delta}} \left(\frac{1}{s_{\epsilon}-c_{\epsilon}z}\right)^{2\Delta},$$

having singularities at $z = \pm 1$, t_{ϵ} . We turn the contour to run along the negative real axis. The only factor which has a discontinuity is $(1 + z)^{\frac{\beta}{2\pi}\omega - 1 + \Delta}$ and we obtain:

$$\lim_{\delta \to 0} \int_{-1}^{-\infty} dy \ \frac{1}{(1-y)^{\frac{\beta}{2\pi}\omega+1-\Delta}} \left(\frac{1}{s_{\epsilon} - c_{\epsilon}y} \right)^{2\Delta} \{ (1+y+\imath\delta)^{\frac{\beta}{2\pi}\omega-1+\Delta} - (1+y-\imath\delta)^{\frac{\beta}{2\pi}\omega-1+\Delta} \} \\ = -\int_{1}^{\infty} dx \ \frac{1}{(1+x)^{\frac{\beta}{2\pi}\omega+1-\Delta}} \left(\frac{1}{s_{\epsilon} + c_{\epsilon}x} \right)^{2\Delta} (x-1)^{\frac{\beta}{2\pi}\omega-1+\Delta} [e^{\left(\frac{\beta}{2\pi}\omega-1+\Delta\right)(\imath\pi)} - e^{\left(\frac{\beta}{2\pi}\omega-1+\Delta\right)(\imath\pi)}]$$

Recalling that $\frac{\beta}{2\pi}\omega = n + 1/2$, we have $e^{i(n-1/2+\Delta)\pi} - e^{-i(n-1/2+\Delta)\pi} = 2i(-1)^{n+1}\cos(\Delta\pi)$ and finally the first term in equation (10) becomes:

$$\frac{b}{2\iota\sin(\pi\Delta)} \left(\frac{\pi}{\beta}\right)^{2\Delta-1} 2(-1)^n \cos(\Delta\pi) \int_1^\infty dx \ \frac{1}{(1+x)^{\frac{\beta}{2\pi}\omega+1-\Delta}} \left(\frac{1}{s_{\epsilon}+c_{\epsilon}x}\right)^{2\Delta} (x-1)^{\frac{\beta}{2\pi}\omega-1+\Delta}.$$
 (11)

Observe that this integral is convergent both for $x \sim \infty$ and for $x \sim 1$. We now consider the second term in equation (10):

$$\frac{b}{2\iota\sin(\pi\Delta)} \left(\frac{\pi}{\beta}\right)^{2\Delta-1} \iota \int_{-\iota\infty}^{\iota\infty} dz \ \frac{(1+z)^{\frac{\beta}{2\pi}\omega-1+\Delta}}{(1-z)^{\frac{\beta}{2\pi}\omega+1-\Delta}} \left(\frac{1}{s_{\epsilon}+c_{\epsilon}z}\right)^{2\Delta},$$

having singularities at $z = \pm 1$, $-t_{\epsilon}$. We close again the contour around the negative real axis to obtain:

$$\begin{split} \lim_{\delta \to 0} \int_{-t_{\epsilon}}^{-\infty} \mathrm{d}y \ \frac{1}{(1-y)^{\frac{\beta}{2\pi}\omega+1-\Delta}} \Biggl\{ \frac{(1+y+\imath\delta)^{\frac{\beta}{2\pi}\omega-1+\Delta}}{(s_{\epsilon}+c_{\epsilon}y+\imath\epsilon_{\epsilon}\delta)^{2\Delta}} - \frac{(1+y-\imath\delta)^{\frac{\beta}{2\pi}\omega-1+\Delta}}{(s_{\epsilon}+c_{\epsilon}y-\imath\epsilon_{\epsilon}\delta)^{2\Delta}} \Biggr\} \\ &= -\lim_{\delta \to 0} \int_{t_{\epsilon}}^{\infty} \mathrm{d}x \ \frac{1}{(1+x)^{\frac{\beta}{2\pi}\omega+1-\Delta}} \Biggl\{ \frac{(1-x+\imath\delta)^{\frac{\beta}{2\pi}\omega-1+\Delta}}{(s_{\epsilon}-c_{\epsilon}x+\imath\epsilon_{\epsilon}\delta)^{2\Delta}} - \frac{(1-x-\imath\delta)^{\frac{\beta}{2\pi}\omega-1+\Delta}}{(s_{\epsilon}-c_{\epsilon}x-\imath\epsilon_{\epsilon}\delta)^{2\Delta}} \Biggr\}. \end{split}$$

The integral splits into an integral over the interval $(t_{\epsilon}, 1)$ and a second integral over the interval $(1, \infty)$ (as $t_{\epsilon} < 1$). Taking the limit $\delta \to 0$ the first integral contributes:

$$\int_{t_{\epsilon}}^{1} \mathrm{d}x \ \frac{1}{(1+x)^{\frac{\beta}{2\pi}\omega+1-\Delta}} (1-x)^{\frac{\beta}{2\pi}\omega-1+\Delta} \frac{1}{(c_{\epsilon}x-s_{\epsilon})^{2\Delta}} [\mathrm{e}^{-2\Delta(t\pi)}-\mathrm{e}^{-2\Delta(-t\pi)}],$$

while the second one is:

$$\int_{1}^{\infty} dx \ \frac{1}{(1+x)^{\frac{\beta}{2\pi}\omega+1-\Delta}} \frac{(x-1)^{\frac{\beta}{2\pi}\omega-1+\Delta}}{(c_{\epsilon}x-s_{\epsilon})^{2\Delta}} [e^{\left(\frac{\beta}{2\pi}\omega-1+\Delta\right)(t\pi)} e^{-2\Delta(t\pi)} - e^{\left(\frac{\beta}{2\pi}\omega-1+\Delta\right)(-t\pi)} e^{-2\Delta(-t\pi)}].$$

Recalling again that $\omega = \frac{2\pi}{\beta}(n+1/2)$, we have $e^{i(n-1/2-\Delta)\pi} - e^{-i(n-1/2-\Delta)\pi} = 2i(-1)^{n+1}\cos(\pi\Delta)$, hence finally the second term in equation (10) is:

$$\frac{b}{2\iota\sin(\pi\Delta)} \left(\frac{\pi}{\beta}\right)^{2\Delta-1} (-2)\sin(2\pi\Delta) \int_{t_{\epsilon}}^{1} dx \ \frac{1}{(1+x)^{\frac{\beta}{2\pi}\omega+1-\Delta}} \frac{(1-x)^{\frac{\beta}{2\pi}\omega-1+\Delta}}{(c_{\epsilon}x-s_{\epsilon})^{2\Delta}} \\ + \frac{b}{2\iota\sin(\pi\Delta)} \left(\frac{\pi}{\beta}\right)^{2\Delta-1} 2(-1)^{n+1}\cos(\pi\Delta) \int_{1}^{\infty} dx \ \frac{1}{(1+x)^{\frac{\beta}{2\pi}\omega+1-\Delta}} \frac{(x-1)^{\frac{\beta}{2\pi}\omega-1+\Delta}}{(c_{\epsilon}x-s_{\epsilon})^{2\Delta}}.$$
 (12)

Adding up equations (11) and (12) the integrals from 1 to ∞ cancel and we obtain:

$$\tilde{G}^{\epsilon}_{\beta}(\omega) = \frac{b}{2\iota\sin(\pi\Delta)} \left(\frac{\pi}{\beta}\right)^{2\Delta-1} (-2)\sin(2\pi\Delta) \int_{t_{\epsilon}}^{1} \mathrm{d}x \ \frac{1}{(1+x)^{\frac{\beta}{2\pi}\omega+1-\Delta}} \frac{(1-x)^{\frac{\beta}{2\pi}\omega-1+\Delta}}{(c_{\epsilon}x-s_{\epsilon})^{2\Delta}}$$

which is an absolutely convergent integral. We now change variable to $x = \tanh(s + \frac{\pi}{\beta}\epsilon)$ and obtain:

$$\tilde{G}^{\epsilon}_{\beta}(\omega) = \frac{b}{2\iota\sin(\pi\Delta)} \left(\frac{\pi}{\beta}\right)^{2\Delta-1} (-2)\sin(2\pi\Delta) \ \mathrm{e}^{-\omega\epsilon} \int_{0}^{\infty} \mathrm{d}s \ \mathrm{e}^{-\frac{\beta}{\pi}\omega s} \frac{1}{[\sinh(s)]^{2\Delta}},$$

and changing again variables to $y = e^{-2s}$, the integral can be explicitly evaluated in terms of an Euler beta function with positive arguments:

$$\tilde{G}^{\epsilon}_{\beta}(\omega) = \frac{b}{2\iota\sin(\pi\Delta)} \left(\frac{2\pi}{\beta}\right)^{2\Delta-1} (-2)\sin(2\pi\Delta) \frac{\Gamma\left(\frac{\beta}{2\pi}\omega + \Delta\right)\Gamma(1-2\Delta)}{\Gamma\left(\frac{\beta}{2\pi}\omega + 1 - \Delta\right)} e^{-\omega\epsilon}.$$

Appendix B. Proof of the proposition 1

Substituting G_{β}^{ϵ} in equation (3) we get:

$$\begin{aligned} A^{\epsilon}(\tau) &= J^{2} \left(\frac{b}{2\iota \sin(\pi\Delta)}\right)^{q} \left(\frac{\pi}{\beta}\right)^{2\Delta q} \int_{-\beta/2}^{\beta/2} \mathrm{d}u \\ &\times \left[\frac{1}{\left(\sinh\frac{\pi[\epsilon - \iota(u - \tau)]}{\beta}\right)^{2\Delta}} - \frac{1}{\left(\sinh\frac{\pi[\epsilon + \iota(u - \tau)]}{\beta}\right)^{2\Delta}}\right] \left[\frac{1}{\left(\sinh\frac{\pi(\epsilon - \iota u)}{\beta}\right)^{2\Delta}} - \frac{1}{\left(\sinh\frac{\pi(\epsilon + \iota u)}{\beta}\right)^{2\Delta}}\right]^{q-1}. \end{aligned}$$

Recalling that $\sinh(z \pm ix) = \sinh(z)\cos(x) \pm i\cosh(z)\sin(x)$, $\Delta q = 1$, changing variable to $t = \tan \frac{\pi u}{\beta}$ and expanding the binomial, $A^{\epsilon}(\tau)$ becomes:

$$J^{2}\left(\frac{b}{2\iota\sin(\pi\Delta)}\right)^{q}\left(\frac{\pi}{\beta}\right)\sum_{r=0}^{q/2-1} {\binom{q-1}{r}} (-1)^{r} \int_{-\infty}^{\infty} dt \left[\frac{1}{[s_{\epsilon+\iota\tau}-\iotac_{\epsilon+\iota\tau}t]^{2\Delta}}-\frac{1}{[s_{\epsilon-\iota\tau}+\iotac_{\epsilon-\iota\tau}t]^{2\Delta}}\right] \\ \times \left[\frac{1}{(s_{\epsilon}-\iotac_{\epsilon}t)^{2\Delta(q-1-r)}}\frac{1}{(s_{\epsilon}+\iotac_{\epsilon}t)^{2\Delta r}}-\frac{1}{(s_{\epsilon}-\iotac_{\epsilon}t)^{2\Delta r}}\frac{1}{(s_{\epsilon}+\iotac_{\epsilon}t)^{2\Delta(q-1-r)}}\right].$$

Taking into account that:

$$\Re(t_{\epsilon\pm\iota\tau}) = \frac{s_{\epsilon}c_{\epsilon}}{\cos\left(\frac{\pi\tau}{\beta}\right)^2 + s_{\epsilon}^2} > 0, \qquad t_{\epsilon} > 0,$$

one can use (absolutely convergent) Schwinger parametric representations to rewrite $A^{\epsilon}(\tau)$ as:

$$J^{2}\left(\frac{b}{2t\sin(\pi\Delta)}\right)^{q}\left(\frac{\pi}{\beta}\right)\sum_{r=0}^{q/2-1} {q-1 \choose r} (-1)^{r} \int_{-\infty}^{\infty} dt \int_{0}^{\infty} d\alpha \ d\alpha_{1} d\alpha_{2} \ \frac{\alpha^{2\Delta-1}\alpha_{1}^{2\Delta(q-1-r)-1}\alpha_{2}^{2\Delta r-1}}{\Gamma(2\Delta)\Gamma[2\Delta(q-1-r)]\Gamma(2\Delta r)} \\ \times \left[\frac{1}{c_{\epsilon+t\tau}^{2\Delta}} e^{-\alpha t_{\epsilon+\tau}+t\alpha t} - \frac{1}{c_{\epsilon-t\tau}^{2\Delta}} e^{-\alpha t_{\epsilon-\tau}-t\alpha t}\right] \frac{1}{c_{\epsilon}^{2\Delta(q-1)}} e^{-t_{\epsilon}(\alpha_{1}+\alpha_{2})} (e^{tt(\alpha_{1}-\alpha_{2})} - e^{-tt(\alpha_{1}-\alpha_{2})}),$$

where the integral over α_2 is absent for r = 0. The integral over t can now be computed and we get:

$$2\pi J^{2}\left(\frac{b}{2\iota\sin(\pi\Delta)}\right)^{q}\left(\frac{\pi}{\beta}\right)\sum_{r=0}^{q/2-1} {q-1 \choose r} (-1)^{r} \int_{0}^{\infty} d\alpha \, d\alpha_{1} d\alpha_{2} \, \frac{\alpha^{2\Delta-1}\alpha_{1}^{2\Delta(q-1-r)-1}\alpha_{2}^{2\Delta r-1}}{\Gamma(2\Delta)\Gamma[2\Delta(q-1-r)]\Gamma(2\Delta r)} \\ \times \frac{1}{c_{\epsilon}^{2\Delta(q-1)}} e^{-t_{\epsilon}(\alpha_{1}+\alpha_{2})} \left[\frac{1}{c_{\epsilon+\iota\tau}^{2\Delta}}e^{-\alpha t_{\epsilon+\iota\tau}} + \frac{1}{c_{\epsilon-\iota\tau}^{2\Delta}}e^{-\alpha t_{\epsilon-\iota\tau}}\right] [\delta(\alpha+\alpha_{1}-\alpha_{2}) - \delta(\alpha-\alpha_{1}+\alpha_{2})].$$

Changing variables to $\alpha_1 = \alpha U$ and $\alpha_2 = \alpha V$ and integrating over α we get:

$$2\pi J^{2} \left(\frac{b}{2t\sin(\pi\Delta)}\right)^{q} \left(\frac{\pi}{\beta}\right) \sum_{r=0}^{q/2-1} {q-1 \choose r} (-1)^{r} \frac{1}{\Gamma(2\Delta)} \int_{0}^{\infty} dU dV \frac{U^{2\Delta(q-1-r)-1}V^{2\Delta r-1}}{\Gamma[2\Delta(q-1-r)]\Gamma(2\Delta r)}$$
$$\times \frac{1}{c_{\epsilon}^{2\Delta(q-1)}} \left[\frac{1}{c_{\epsilon+t\tau}^{2\Delta}[t_{\epsilon+t\tau}+t_{\epsilon}(U+V)]} + \frac{1}{c_{\epsilon-t\tau}^{2\Delta}[t_{\epsilon-t\tau}+t_{\epsilon}(U+V)]}\right] [\delta(1+U-V) - \delta(1-U+V)],$$

where we recall that for r = 0 the integral over V is absent. Integrating once using the δ functions we obtain:

$$2\pi J^{2} \left(\frac{b}{2\iota\sin(\pi\Delta)}\right)^{q} \left(\frac{\pi}{\beta}\right) \frac{1}{\Gamma(2\Delta)} \left\{\frac{-1}{\Gamma(2-2\Delta)} \frac{1}{c_{\epsilon}^{2\Delta(q-1)}} \left[\frac{1}{c_{\epsilon+\iota\tau}^{2\Delta(q-1)}} + \frac{1}{c_{\epsilon-\iota\tau}^{2\Delta}[t_{\epsilon-\iota\tau} + t_{\epsilon}]} + \frac{1}{c_{\epsilon-\iota\tau}^{2\Delta}[t_{\epsilon-\iota\tau} + t_{\epsilon}]}\right] \right\}$$
$$+ \sum_{r=1}^{q/2-1} \left(\frac{q-1}{r}\right) (-1)^{r} \int_{1}^{\infty} dV \frac{(V-1)^{2\Delta(q-1-r)-1}V^{2\Delta r-1} - V^{2\Delta(q-1-r)-1}(V-1)^{2\Delta r-1}}{\Gamma[2\Delta(q-1-r)]\Gamma(2\Delta r)} \\\times \frac{1}{c_{\epsilon}^{2\Delta(q-1)}} \left[\frac{1}{c_{\epsilon+\iota\tau}^{2\Delta}[t_{\epsilon+\iota\tau} + t_{\epsilon}(2V-1)]} + \frac{1}{c_{\epsilon-\iota\tau}^{2\Delta}[t_{\epsilon-\iota\tau} + t_{\epsilon}(2V-1)]}\right] \right\},$$

and finally, changing variables to y = 2V - 1 proves proposition 1

Appendix C. Bound on the integral in equation (8)

Proposition 2. For any $y \ge 1$ and $\Delta \le \frac{1}{2}$ we have:

$$\int_{-\infty}^{\infty} \mathrm{d} v \ R^{\epsilon}(v, y) \leqslant 2\pi K_{\epsilon}, \qquad K_{\epsilon} = \frac{1 - t_{\epsilon}^2}{1 + t_{\epsilon}^2} \bigg[1 + \frac{2\Delta + 1}{\sqrt{\pi}} t_{\epsilon} \bigg].$$

Proof. Let us first find a bound for the integral:

$$I_{\Delta} = \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{(z^2+1)^{1-\Delta}}.$$

Observe that $\lim_{\Delta\to 0} I_{\Delta} = \pi$ and the integral is convergent for $0 \le \Delta \le \frac{1}{2}$. The integrand has two cuts, $(\iota, \iota\infty)$ and $(-\iota, -\iota\infty)$. We deform the contour of integration to run around the cut $(\iota, \iota\infty)$ and the integral becomes:

$$\int_{1}^{\infty} (\iota d\rho) \lim_{\epsilon \to 0} [e^{-(1-\Delta)\ln[1+(\iota\rho+\epsilon)^{2}]} - e^{-(1-\Delta)\ln[1+(\iota\rho-\epsilon)^{2}]}]$$

= $2\sin[(1-\Delta)\pi] \int_{1}^{\infty} \frac{d\rho}{(\rho^{2}-1)^{1-\Delta}} = \sum_{\rho=y^{-1/2}} \sin[(1-\Delta)\pi] \frac{\Gamma(\frac{1}{2}-\Delta)\Gamma(\Delta)}{\Gamma(\frac{1}{2})}$

Since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$, we have:

$$I_{\Delta} \leqslant \sqrt{\pi} \frac{\Gamma\left(\frac{1}{2} - \Delta\right)}{\Gamma(1 - \Delta)} \leqslant \sqrt{\pi}$$

as $\Delta \leq \frac{1}{2}$ and Γ is strictly increasing for positive real arguments.

Now, going back to $R^{\epsilon}_{\beta}(v, y)$, we use:

 $\cos[2\Delta \arctan(t_{\epsilon}^{2}(1+y)\nu)] \leq 1, \qquad \sin[2\Delta \arctan(t_{\epsilon}^{2}(1+y)\nu)] \leq 2\Delta t_{\epsilon}^{2}(1+y)\nu,$

to obtain a bound (observe that
$$R^{\epsilon}_{\beta}(v, y) \ge 0$$
):

$$R^{\epsilon}(v, y) \leqslant \frac{2(1 - t_{\epsilon}^{2})\{1 + v^{2}t_{\epsilon}^{2}(1 + y)(1 + yt_{\epsilon}^{2}) + 2\Delta t_{\epsilon}^{2}(1 + y)v^{2}(1 - t_{\epsilon}^{2})\}}{[1 + t_{\epsilon}^{2}(1 + y)^{2}v^{2}]^{1 - \Delta}[1 + t_{\epsilon}^{4}(1 + y)^{2}v^{2}]^{\Delta}[1 + v^{2}(1 + yt_{\epsilon}^{2})^{2}]} \\ \leqslant \frac{2(1 - t_{\epsilon}^{2})}{1 + yt_{\epsilon}^{2}} \Bigg[\frac{1 + yt_{\epsilon}^{2}}{1 + v^{2}(1 + yt_{\epsilon}^{2})^{2}} + (2\Delta + 1)t_{\epsilon} \frac{t_{\epsilon}(1 + y)}{[1 + t_{\epsilon}^{2}(1 + y)^{2}v^{2}]^{1 - \Delta}} \Bigg],$$

therefore:

$$\int_{-\infty}^{\infty} \mathrm{d}v \, R^{\epsilon}(v, y) \leqslant \frac{2(1-t_{\epsilon}^2)}{1+yt_{\epsilon}^2} [\pi + t_{\epsilon}(2\Delta+1)\sqrt{\pi}] \leqslant 2\pi \frac{1-t_{\epsilon}^2}{1+t_{\epsilon}^2} \bigg[1 + \frac{2\Delta+1}{\sqrt{\pi}} t_{\epsilon} \bigg].$$

Appendix D. The integrals in equation (9)

Proposition 3. For $\Re(a) > 0$, $\Re(b) > 0$ and $\Re(a + b) < 2$, $a + b \neq 1$ we have:

$$F(a, b) = \int_{1}^{\infty} \frac{dy}{2^{a+b-1}} \left[(y-1)^{a-1}(y+1)^{b-1} - (y+1)^{a-1}(y-1)^{b-1} \right]$$
$$= \left(\frac{1-b}{1-a-b} \right) \frac{\Gamma(2-a-b)\Gamma(a)}{\Gamma(2-b)} - \left(\frac{1-a}{1-a-b} \right) \frac{\Gamma(2-a-b)\Gamma(b)}{\Gamma(2-a)}$$

Proof. The integral is clearly convergent in 0. At infinity, due to the subtraction, the integrand behaves like y^{a+b-3} , hence the integral converges for $\Re(a + b) < 2$. Changing variables to $x = \frac{2}{1+y}$, the integral becomes

$$\frac{1}{2^{a+b-1}} \int_{1}^{0} \left(-2\frac{\mathrm{d}x}{x^{2}}\right) \left[\left(\frac{2}{x}-2\right)^{a-1} \left(\frac{2}{x}\right)^{b-1} - \left(\frac{2}{x}\right)^{a-1} \left(\frac{2}{x}-2\right)^{b-1}\right] \\ = \int_{0}^{1} \mathrm{d}x \ x^{-a-b} \left[(1-x)^{a-1} - (1-x)^{b-1}\right].$$

Observe that the two terms can not be integrated separately, as each integral would diverge in $x \sim 0$. However, the difference is convergent in $x \sim 0$ as the behavior is tamed by the explicit subtraction. We observe that

$$x^{-a-b} = \frac{1}{1-a-b} [x^{1-a-b}]'(1-x) + x^{1-a-b},$$

hence we get:

$$F(a, b) = \left[\frac{x^{1-a-b}}{1-a-b}[(1-x)^a - (1-x)^b]\right]_0^1 + \frac{1}{1-a-b}\int_0^1 dx \, x^{1-a-b}[a(1-x)^{a-1} - b(1-x)^{b-1}] \\ + \int_0^1 dx \, x^{1-a-b}[(1-x)^{a-1} - (1-x)^{b-1}].$$

As $\Re(a + b) < 2$ the boundary terms cancel and all the integrals are convergent and can be expressed in terms of Euler Γ functions.

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References

- [1] Sachdev S and Ye J 1993 Gapless spin fluid ground state in a random, quantum Heisenberg magnet Phys. Rev. Lett. 70 3339
- [2] Kitaev A 2015 A simple model of quantum holography KITP Strings Seminar and Entanglement 2015 Program (12 February, 7 April and 27 May 2015
- [3] Maldacena J and Stanford D 2016 Remarks on the Sachdev-Ye-Kitaev model Phys. Rev. D 94 106002
- [4] Polchinski J and Rosenhaus V 2016 The spectrum in the Sachdev-Ye-kitaev model J. High Energy Phys. JHEP04(2016)001
- [5] Fu W, Gaiotto D, Maldacena J and Sachdev S 2017 Supersymmetric Sachdev–Ye–Kitaev models Phys. Rev. D 95 026009
- [6] Gross D J and Rosenhaus V 2017 A generalization of Sachdev–Ye–Kitaev J. High Energy Phys. 2 JHEP02(2017)093
- [7] Das S R, Jevicki A and Suzuki K 2017 Three-dimensional view of the SYK/AdS duality
- [8] Jevicki A, Suzuki K and Yoon J 2016 Bi-local holography in the SYK model J. High Energy Phys. JHEP07(2016)007
- [9] Gurau R 2016 Random Tensors (Oxford: Oxford University Press)
- [10] Witten E 2016 An SYK-like model without disorder
- [11] Gurau R 2017 The complete 1/N expansion of a SYK-like tensor model Nucl. Phys. B 916 386-401
- [12] Klebanov I R and Tarnopolsky G 2017 Uncolored random tensors, melon diagrams, and the SYK models Phys. Rev. D 95 046004
- [13] Peng C, Spradlin M and Volovich A 2017 A supersymmetric SYK-like tensor model J. High Energy Phys. 5 JHEP05(2017)062
- [14] Krishnan C, Sanyal S and Subramanian P N B 2017 Quantum chaos and holographic tensor models *J. High Energy Phys.* **3** JHEP03 (2017)056
- [15] Peng C 2017 Vector models and generalized SYK models J. High Energy Phys. 5 JHEP05(2017)129
- [16] Bonzom V, Lionni L and Tanasa A 2017 Diagrammatics of a colored SYK model and of an SYK-like tensor model, leading and next-toleading orders J. Math. Phys. 58 052301
- [17] Gross D J and Rosenhaus V 2017 The bulk dual of SYK: cubic couplings J. High Energy Phys. 5 JHEP05(2017)092
- [18] Bergere M C and David F 1979 Integral representation for the dimensionally regularized massive Feynman amplitude J. Math. Phys. 20 1244
- [19] Bergere M C and David F 1981 Integral representation for the dimensionally renormalized Feynman amplitude Commun. Math. Phys. 811
- [20] Gross D J and Rosenhaus V 2017 A line of CFTs: from generalized free fields to SYK J. High Energy Phys. JHEP07(2017)086