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ON THE FINITENESS OF SCALING SUM RULES*

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ABSTRACT

The convergence of sum rules relating the matrix elements of local operators to integrals over deep-inelastic structure functions is studied critically. It is found that the matrix elements may always be written as the $q^2 \to -\infty$ limit of finite expressions, regardless of the (Regge) asymptotic behavior of the structure functions or the possible occurrence of J=0 fixed singularities. The correct form of the sum rule for the operator Schwinger term is taken as a paradigm case. It is derived from the Bjorken-Johnson-Low theorem and agrees with the results of parton model and light-cone analyses. It readily encompases the results of second order ϕ^3 theory (where the Schwinger term diverges logarithmically) and second order vector gluon theory (where it vanishes). Sufficient conditions for the finiteness of the operator Schwinger term are the scaling of the longitudinal structure function and the absence of J=0 fixed singularities with nonpolynomial residues. The treatment is readily applicable to other scaling and fixed q^2 sum rules needing regulation. A compendium of these is given.

I. INTRODUCTION

The study of the deep-inelastic limit of lepton-nucleon scattering has produced a profusion of sum rules similar to those of old-fashioned equal-time current algebra. These sum rules generally relate integrals over deep-inelastic structure functions to the single-nucleon matrix elements of local operators. Some, like the Bjorken [1] limit of the Adler sum rule [2] for neutrino scattering, relate convergent integrals over the data (assuming the Pomeranchuk theorem to apply to highly virtual processes) to known static properties of the nucleon. Others however, such as the sum rule of Jackiw, van Royen and West [3] for the operator Schwinger term, appear to diverge if the structure functions possess the Regge behavior expected of them. It is very important to understand whether such divergences are actual, indicating for example that the operator Schwinger term is truly infinite in the presence of Regge behavior, or artifactual, indicating a breakdown in the method of derivation — since it is not known *a priori* whether or not objects such as the operator Schwinger term are finite.

Our object is to present a straightforward and systematic analysis of these apparent divergences employing several of the techniques for writing finite sum rules which have been developed over the past few years [4,5,6], and using the Schwinger term sum rule as an example [7]. These techniques and our conclusions are quite general and apply to a wide class of sum rules, some of which we shall enumerate. While many of the considerations in this paper already appear in the literature, we feel that the consistency, generality and utility of these techniques are not widely recognized [8] and warrant a unified presentation.

For the case of the Schwinger term, we find that if the sum rule of Ref. [3] is divergent due to asymptotic Regge behavior, the derivation is invalid [9]. Even if Regge asymptotics are such that the integral converges, the sum rule of Ref. [3] obtains only if $\alpha=0$ fixed singularities in the real part of the virtual Compton amplitudes have residues which are polynomials in q^2 (the square of the photon's mass). If they do not, the sum rule of Ref. [3] may be modified by an additive, unknown constant. In the presence of leading Regge behavior further modifications occur: We obtain a sum rule which is a generalization of that proposed originally in Ref. [4]. It represents an analytic continuation in the Regge intercept α, of the sum rule derived if all trajectories have $\alpha<0$, to values of α in the range $0<\alpha\leq 1$. (A term with $\alpha=0$ in the imaginary part is an exception. The sum rule remains well defined but is not the analytic continuation of the result for $\alpha\neq 0$.) The modified version of the sum rule leads us to conclude that the existence of the Bjorken limit of the longitudinal structure function and the absence of a certain class of non-polynomial fixed singularities are sufficient to guarantee the finiteness of the Schwinger term.

The paper is organized as follows.

In Section II we give a straightforward derivation of the sum rule of Ref. [3] based on the Bjorken-Johnson-Low (BJL) theorem [10, 11]. In the presence of Regge trajectories with intercepts $\alpha\geq 0$, or of $\alpha=0$ fixed singularities, the derivation is invalid, and the Schwinger term is related to a subtraction constant rather than an integral over a structure function.

In Section III we parameterize the expected Regge asymptotic behavior and use dispersion relations to relate the subtraction constant to a finite integral over the structure function (with its leading Regge behavior subtracted off) and the residue of a possible α=0 fixed singularity. The versions of the sum rule found in the literature may be obtained from this general result with appropriate additional assumptions.

In Section IV we argue that this general result is not an artifact of our particular derivation (via the BJL theorem) but may equally well be obtained from light-cone analyses or in the parton model. The possible α=0 fixed singularity which entered the BJL derivation is found to correspond, in the light cone analysis, to a δ-function of the structure function in the Bjorken limit and to be absent in the parton model.

In Section V we study second order perturbation theory in ϕ^3 theory and the vector gluon model and obtain results which exemplify the conclusion of Sections III and IV. In particular the rather subtle workings of the sum rule in the vector gluon model, pointed out by Corrigan and by Zee [7], fit simply into our scheme.

Section VI contains a summary of our conclusions and an enumeration of other sum rules to which they apply. We tabulate the sum rules as they occur in the literature and indicate the modifications discovered by our analysis.

In the Appendix we demonstrate how to deal with an α=0 term in the imaginary part and with Regge cuts.

II. DERIVATION OF THE SCHWINGER TERM SUM RULE FROM THE BJL THEOREM

The proton matrix element of the equal-time commutator of the electromagnetic charge and current densities defines the Schwinger term [12]:

$$<P|\left[J_0(\vec{y},0),\ J_i(0)\right]|P> = i\,\partial_i\,\delta^3(\vec{y})\ S \quad . \tag{1}$$

The BJL theorem [10, 11] relates S to a limit of the forward Compton amplitude

$$T_{\mu\nu}(P,q) = i\int d^4y\ e^{iq\cdot y}\ \theta(y_0) <P|\left[J_\mu(y),\ J_\nu(0)\right]|P> + (\text{polynomials in q and P}) \tag{2}$$

whose absorptive part $\left(W_{\mu\nu} = \frac{1}{2\pi}\,\mathrm{Im}\,T_{\mu\nu}\right)$ describes inelastic electroproduction.

To isolate the Schwinger term consider T_{0i} and take the limit $q_0 \to i\infty$ with P and $\vec{q}$ fixed ($\lim_{BJL}$). Then

$$\lim_{BJL}\ T_{0i} = -\frac{1}{q_0}\int d^4y\ e^{iq\cdot y}\,\delta(y_0) <P|\left[J_0(y),\ J_i(0)\right]|P> + 0\left(\frac{1}{q_0^2}\right) + \text{polynomial}$$

$$= -\frac{q_i}{q_0}\ S + 0\left(\frac{1}{q_0^2}\right) + \text{polynomial} \tag{3}$$

where the right hand side has been obtained by partial integration of Eq. (2).

To study T_{0i} we write the covariant decomposition

$$T_{\mu\nu}(P,q) = \left(q_\mu q_\nu - q^2 g_{\mu\nu}\right) t_L(q^2,\nu) + \frac{1}{M^2}\left(\nu\,(P_\mu q_\nu + P_\nu q_\mu) - q^2 P_\mu P_\nu - \nu^2 g_{\mu\nu}\right) t_2(q^2,\nu) \tag{4}$$

where $\nu = P\cdot q$ and t_L and t_2 are free of kinematic singularities and related to the conventional amplitudes [13] by

$$t_L = \frac{1}{q^2}\left(\frac{\nu^2}{M^2 q^2}\ T_2 + T_1\right) \ , \qquad t_2 = -\frac{1}{q^2}\ T_2 \quad .$$

For convenience we choose the coordinate system in which $\vec{P}=0$ so that Eq. (3) becomes:

$$\lim_{BJL} q_0 q_i \left(t_L(q^2,\nu) + t_2(q^2,\nu)\right) = -\frac{q_i}{q_0} S + 0\left(\frac{1}{q_0^2}\right) + q_i P(q^2,\nu) \tag{5}$$

where $P(q^2,\nu)$ is some polynomial in q^2 and ν, which is to be identified with terms on the right-hand side which do not vanish as $q_0 \to i\infty$. Identifying the coefficient of $1/q_0$, we obtain

$$S = -\lim_{BJL} q^2 \left(\bar{t}_L(q^2,\nu) + \bar{t}_2(q^2,\nu)\right) \quad , \tag{6}$$

where $\bar{t}_L$ and $\bar{t}_2$ are defined to have all those terms subtracted off which in the limit vanish like $1/\nu$ or more slowly.

Fixed q^2 dispersion relations may now be used to relate the Schwinger term to electroproduction data. Here the question of Regge-asymptotic behavior becomes crucial. We consider two cases.

Case I. <u>Both t_L and t_2 obey unsubtracted dispersion relations</u> in ν for for fixed, spacelike q^2. Let $w_i = \frac{1}{2\pi} \mathrm{Im}\, t_i$ then

$$t_i(q^2,\nu) = 4\int_{-q^2/2}^{\infty} \frac{\nu' d\nu' \, w_i(q^2,\nu')}{\nu'^2-\nu^2} \quad , \qquad i=L,2 \quad . \tag{7}$$

A Regge analysis indicates that the integral over w_L does not converge,

$$\lim_{\nu\to\infty} w_L(q^2,\nu) \sim \nu^{\alpha} \quad , \qquad \alpha \leq 1 \quad , \tag{8a}$$

while the w_2 integral is convergent,

$$\lim_{\nu\to\infty} w_2(q^2,\nu) \sim \nu^{\alpha-2} \quad , \qquad \alpha \leq 1 \quad . \tag{8b}$$

Of course it is not certain that Regge considerations apply in the limit $q^2 \to -\infty$ where we shall study Eq. (7). In this spirit we assume for the moment that both dispersion integrals of Eq. (7) are convergent.

The relation between w_L and w_2 and the conventional structure functions is

$$-q^2 w_L(q^2,\nu) = \frac{\omega}{2} F_2(q^2,\omega) - F_1(q^2,\omega) = F_L(q^2,\omega)$$

$$-\frac{q^2\nu}{M^2} w_2(q^2,\nu) = F_2(q^2,\omega) \tag{9}$$

where $\omega = -2\nu/q^2$ and

$$F_2(q^2,\omega) = \frac{\nu}{M^2} W_2(q^2,\nu)$$

$$F_1(q^2,\omega) = W_1(q^2,\nu) \ .$$

The Bjorken limit ($\lim_{Bj}$) is $\nu, -q^2 \to \infty$, ω fixed and finite and by scaling we mean that the functions $F_{1,2}(\omega) = \lim_{Bj} F_{1,2}(q^2,\omega)$ exist.

In terms of F_L and F_2, Eqs. (7) read:

$$t_L(q^2,\nu) = -\frac{4}{q^2}\int_1^\infty \frac{\omega' d\omega'}{\omega'^2 - \omega^2} F_L(q^2,\omega') \ , \tag{10a}$$

$$t_2(q^2,\nu) = \frac{8M^2}{q^4}\int_1^\infty \frac{d\omega'}{\omega'^2 - \omega^2} F_2(q^2,\omega') \ . \tag{10b}$$

In the BJL limit $q^2 \to -\infty$ and $\omega^2 \to 4M^2/q^2$. Combining Eqs. (6) and (10) and assuming scaling we obtain the sum rule of Jackiw, van Royen and West [3]

$$S = 4\int_1^\infty \frac{d\omega}{\omega} F_L(\omega) \ . \tag{11}$$

Case II. The dispersion relation for t_L requires one subtraction as dictated by Regge theory.

Then Eq. (10a) must be replaced by

$$t_L(q^2,\nu) = t_L(q^2,0) - \frac{4\omega^2}{q^2}\int_1^\infty \frac{d\omega'}{\omega'(\omega'^2-\omega^2)} F_L(q^2,\omega') \tag{12}$$

and Eqs. (6) and (12) yield

$$S = - \lim_{q^2 \to -\infty} q^2 \bar{t}_L(q^2,0) \quad . \tag{13}$$

That is, the Schwinger term is given by the subtraction constant. It is already apparent that any divergence of S must arise from the $q^2 \to -\infty$ limit.

Even if the dispersion relation for t_L does not need a subtraction one cannot rule out the possibility of an additive constant (one more generally a polynomial in ν^2) with arbitrary q^2 dependence. The effect of such a constant is to reduce Case I to Case II. (We exclude higher order terms in ν^2 by requiring the full amplitude to behave as ν^α with $\alpha \leq 1$.)

To summarize we have found that if

a) the structure functions $F_{L,2}(q^2,\omega)$ scale in the Bjorken limit,

and b) Im $t_L(q^2,\nu)$ is bounded by $\beta(q^2)\nu^\alpha$, $\alpha<0$, as $\nu \to \infty$, so that an unsubtracted dispersion relation is permitted,

and c) no additive constant enters this fixed q^2 dispersion relation,

then the sum rule of Jackiw, van Royen and West, Eq. (11), may be obtained.

If a) holds and b) fails or if a) and b) hold and c) fails, then Eq. (13) must be used instead.

III. A FINITE SCHWINGER TERM SUM RULE

As shown in the last section, in a wide class of circumstances the attempt to derive a Schwinger term sum rule results merely in identifying S with the $q^2 \to -\infty$ limit of a subtraction constant. Here we show how to relate the subtraction constant to finite, regulated integrals over the structure function $F_L(q^2,\omega)$.

Suppose $t_L(q^2,\nu)$ obeys the subtracted dispersion relation, Eq. (12), allowing for Regge asymptotics. Let $F_L^R(q^2,\omega)$ represent the leading ($\alpha > 0$) Regge behavior of $F_L(q^2,\omega)$; for example:

$$F_L^R(q^2,\omega) = \sum_{\alpha>0} \gamma(\alpha, q^2) \, |\omega|^{\alpha} \, \epsilon(\omega) \quad . \tag{14}$$

Other possible parameterizations are discussed below. The leading Regge behavior of $t_L(q^2,\nu)$ may be constructed uniquely from $F_L^R(q^2,\omega)$ by means of a subtracted dispersion relation

$$\begin{aligned} t_L^R(q^2,\nu) &= -\frac{4\omega^2}{q^2} \int_0^\infty \frac{d\omega' \, F_L^R(q^2,\omega')}{\omega'(\omega'^2-\omega^2)} \\ &= \frac{2\pi}{q^2} \sum_{\alpha>0} \gamma(\alpha,q^2) \left[\frac{(\omega)^\alpha + (-\omega)^\alpha}{\sin \pi\alpha}\right] \quad . \end{aligned} \tag{15}$$

We have defined $t_L^R(q^2,\nu)$ with no subtraction constant since it parameterizes those terms which grow as $\nu \to \infty$.

Subtracting Eq. (15) from Eq. (12) and letting $\nu \to \infty$ we obtain

$$\lim_{\nu\to\infty} \left[t_L(q^2,\nu) - t_L^R(q^2,\nu)\right] = t_L(q^2,0) + \frac{4}{q^2}\int_0^\infty \frac{d\omega}{\omega} \tilde{F}_L(q^2,\omega) \tag{16}$$

where

$$\tilde{F}_L(q^2,\omega) = F_L(q^2,\omega) - F_L^R(q^2,\omega)$$

and the integral converges provided there is no $\alpha=0$ piece in $F_L(q^2, \omega)$. We consider later the interesting question of such asymptotically constant parts in the imaginary part. Here we merely note that they are not allowed in a simple Regge pole picture, because of the signature factor which is real at $\alpha=0$.

Equation (16) defines the residue of an $\alpha=0$ singularity in the real part of $t_L(q^2, \nu)$. There is good reason to believe that such singularities are fixed [5,14,15,16] (i.e., they occur at $J=\alpha(t)$ where $\alpha(t)=0$ for all t) and have residues which are polynomial in q^2 for amplitudes free of kinematic singularities [5,15]. We shall refer to such singularities as fixed poles (although Kronecker delta [16] is perhaps more correct for t_L). Defining the residue

$$t_L^{F.P.}(q^2) = \lim_{\nu \to \infty} \left[t_L(q^2, \nu) - t_L^R(q^2, \nu) \right] \tag{17}$$

we conclude from Eqs. (16) and (17) that

$$\lim_{q^2 \to -\infty} q^2 \bar{t}_L(q^2, 0) = -4 \lim_{q^2 \to -\infty} \int_0^\infty \frac{d\omega}{\omega} \widetilde{F}_L(q^2, \omega)$$

if $t_L^{F.P.}(q^2)$ is a polynomial.

More generally

$$S = -\lim_{q^2 \to -\infty} q^2 \bar{t}_L(q^2, 0) = \lim_{q^2 \to -\infty} \left[4 \int_0^\infty \frac{d\omega}{\omega} \widetilde{F}_L(q^2, \omega) - C(q^2) \right] \tag{18}$$

where

$$C(q^2) = q^2 \bar{t}_L^{F.P.}(q^2)$$

and $\bar{t}_L^{F.P.}(q^2)$ is the nonpolynomial fixed pole residue. The polynomial part (together with any term in $\widetilde{F}_L(q^2, \omega)$ which grows like a power of q^2 as $q^2 \to -\infty$) does not contribute to $\bar{t}(q^2, 0)$ and is identified with the polynomial occurring in the BJL limit, Eq. (5). It is tempting to take the $q^2 \to -\infty$ limit under the integral in Eq. (18) and replace $\widetilde{F}_L(q^2, \omega)$ by $\widetilde{F}_L(\omega)$, assuming scaling. However in Section V we find that the order of limit and integral is essential in reconciling the results of Corrigan and Zee [7]. Finally we perform the integration from $\omega=0$ to $\omega=1$ $\left(\text{where } \widetilde{F}_L(q^2, \omega) = -\widetilde{F}^R_L(q^2, \omega)\right)$ to obtain our Schwinger term sum rule

$$S = \lim_{q^2 \to -\infty} -q^2 \bar{t}_L(q^2, 0) = \lim_{q^2 \to -\infty} \left[4 \int_1^\infty \frac{d\omega}{\omega} \widetilde{F}_L(\omega, q^2) - 4 \sum_{\alpha>0} \frac{\gamma(\alpha, q^2)}{\alpha} - C(q^2) \right] . \tag{19}$$

We now consider the analyticity of the Schwinger term sum rule in α. Assume scaling and take $F_L(\omega)$ to be the sum of an asymptotically vanishing piece $f(\omega)$ and a single trajectory contributing $\gamma(\alpha)\omega^\alpha$. Equation (19) gives

$$S = 4 \int_1^\infty \frac{d\omega}{\omega} f(\omega) - 4 \frac{\gamma(\alpha)}{\alpha} - C$$

for $\alpha>0$ <u>and</u> for $\alpha<0$, indicating that it is the analytic continuation of

$$S = 4 \int_1^\infty \frac{d\omega}{\omega} F_L(\omega) - C$$

to $\alpha>0$ (provided of course $\gamma(\alpha)$ is analytic in α).

It remains to consider the possibility of an asymptotically constant ($\alpha=0$) piece in $F_L(q^2, \omega)$, which would correspond to Re $t_L(q^2, \omega)$ growing like log ν as $\nu \to \infty$, q^2 fixed. Such a term must be subtracted in defining the fixed pole residue through Eq. (17). Unfortunately the standard Regge parameterization, Eq. (14), is not suitable since Eq. (15) makes no sense for $\alpha=0$. This is an artifact of the location of the Regge threshold at $\omega=0$. To avoid this complication we choose

$$F^R_L(q^2, \omega) = \gamma(0, q^2) \left[\theta(\omega-1) - \theta(-\omega-1)\right] \tag{20}$$

corresponding to

$$t_L^R(q^2,\nu) = \frac{2\pi}{q^2}\,\gamma(0,q^2)\,\log\,(1-\omega^2) \quad . \tag{21}$$

Proceeding as before we obtain

$$S = \lim_{q^2\to-\infty}\left\{4\int_1^\infty \frac{d\omega}{\omega}\left[F_L(q^2,\omega) - \gamma(0,q^2)\right] - C(q^2)\right\} \tag{22}$$

which is finite (unless the $q^2 \to -\infty$ limit diverges). Clearly Eq. (22) is not the analytic continuation of Eq. (19) to α=0; indeed the latter appears to be singular as $\alpha \to 0$. This is because the $\nu \to \infty$ limit which defines the fixed pole is not uniform in α. The combination $\left[t_L^R(q^2,\nu) - \frac{4}{q^2}\frac{\gamma(\alpha,q^2)}{\alpha}\right]$, where $t_L^R(q^2,\nu)$ is the general Regge form of Eq. (15), agrees with the special Regge form of Eq. (21), in the limit $\alpha \to 0$ followed by $\nu \to \infty$. It is this that ensures the absence of a counter term at α=0 in Eq. (19).

The nonanalyticity in α is not an artifact of choosing a Regge form for $\alpha > 0$, Eq. (15), which cannot be continued to α=0. In the Appendix we use the parameterization

$$t_L^R(q^2,\omega,\alpha) = \frac{2\pi}{q^2}\,\gamma(\alpha,q^2)\left[\frac{(1+\omega)^\alpha + (1-\omega)^\alpha - 2}{\sin\pi\alpha}\right]$$

which is analytic in α and derive both Eq. (19) for α>0 and Eq. (22) for α=0. In the Appendix we also give a regularization scheme for Regge cuts with $\alpha_c > 0$. These present no problems in principle in obtaining a finite Schwinger term, given scaling.

In its final form (Regge poles only) the sum rule is

$$S = \lim_{q^2\to-\infty}\left[4\int_1^\infty \frac{d\omega}{\omega}\,\widetilde{F}_L(q^2,\omega) - 4\sum_{\alpha>0}\frac{\gamma(\alpha,q^2)}{\alpha} - C(q^2)\right] \tag{23}$$

where

$$\widetilde{F}_L(q^2,\omega) = F_L(q^2,\omega) - \sum_{\alpha \geq 0} \gamma(\alpha, q^2)\, \omega^{\alpha}$$

$$C(q^2) = q^2\, \bar{t}_L^{\,F.P.}(q^2) \ .$$

Clearly S is finite if $F_L(q^2,\omega)$ scales and $C = \lim_{q^2 \to -\infty} C(q^2)$ exists. Moreover, if for example

$$\lim_{Bj} F_L(q^2,\omega) = \sum_{i=0}^{n} (q^2)^i\, F_L^i(\omega)$$

then the Schwinger term is given by $F_L^0(\omega)$ and the higher order terms are identified with polynomials in the BJL limit. If on the other hand $F_L(q^2,\omega)$ grows logarithmically with q^2 (or with a fractional power of q^2) in the Bjorken limit we conclude that S is actually infinite.

The sum rule of Ref. [4] (see Eq. (5.14) therein) may be obtained from our general result by further assuming that

a) any α=0 fixed pole in Re $t_L(q^2,\nu)$ has a residue which is polynomial in q^2 (so that C=0),

b) there is no α=0 term in $F_L(\omega)$,

c) there is no operator Schwinger term.

On the other hand the result of Ref. [3], Eq. (11), assumes that both real and imaginary parts of $t_L(q^2,\nu)$ have $\alpha<0$ (vanishing asymptotically).

Attempts to calculate the Schwinger term from the sum rule will depend heavily upon assumptions beyond the existence of the Bjorken limit. However the existence of the Bjorken limit and the absence of nonpolynomial fixed poles are sufficient to guarantee the finiteness of S. If $F_L(\omega) = 0$ (implying that the ratio of longitudinal to transverse photoabsorption cross sections vanishes in the Bjorken limit) and there are no nonpolynomial fixed poles then S=0. This is the case in parton models with spin 1/2 charged constituents. However light cone analyses of the quark-vector gluon model only give $F_L(\omega) = 0$ and make no commitment on fixed poles. It is interesting that a fixed pole in ρ electroproduction could give

$$\bar{t}_L^{F.P.}(q^2) = \frac{C}{q^2 - m_\rho^2}$$

being a nonpolynomial fixed pole in Compton scattering.

IV. OTHER METHODS OF REGULATING SUM RULES

The regulation prescription developed from the BJL expansion and dispersion relations may also be obtained from light-cone analyses or the parton model. Here we present brief descriptions of the origin of the regulating terms from these starting points. Analogous prescriptions can presumably be developed in other popular theories of inelastic processes.

A. Light Cone

A coordinate space derivation [16] of the Schwinger term sum rule begins with the decomposition:

$$\langle P| \left[J_\mu(y),\ J_\nu(0)\right] |P\rangle = (g_{\mu\nu}\,\Box - \partial_\mu\partial_\nu)\, V_L(y^2, y\cdot P)$$
$$+ \left[P_\mu P_\nu\,\Box + g_{\mu\nu}(P\cdot\partial)^2 - (P_\mu\partial_\nu + P_\nu\partial_\mu)\,P\cdot\partial\right] V_2(y^2, y\cdot P) \quad . \tag{24}$$

For simplicity we assume Bjorken scaling for $F_{2,L}(q^2,\omega)$ — the more general treatment is more complex but no more enlightening — which requires the following singularity in $V_L(y^2, y\cdot P)$ [3] :

$$V_L(y^2, y\cdot P) = \frac{1}{2\pi i}\ \epsilon(y\cdot P)\left[\delta(y^2)\ f_L(y\cdot P) + \ \theta(y^2)\ \bar{f}_L(y^2, y\cdot P)\right] \tag{25}$$

where $\bar{f}_L(y^2, y\cdot P)$ does not contribute to $F_L(\omega)$. V_2 does not enter the sum rule and we ignore it henceforth. After Fourier transformation we find

$$F_L(\omega) = \frac{1}{4\pi\omega}\int_{-\infty}^{\infty} d\lambda\ e^{-i\lambda/\omega}\ f_L(\lambda) \quad . \tag{26}$$

Inversion of Eq. (26) is delicate. If $\lim\limits_{\omega\to\infty} F_L(\omega) = 0$ we obtain

$$f_L(\lambda) = 4\int_1^{\infty} \frac{d\omega}{\omega}\ \cos\lambda/\omega\ F_L(\omega) \tag{27}$$

and from Eqs. (1), (24) and (25)

$$S = f_L(0)$$

so that Eq. (27) at $\lambda=0$ gives the naive sum rule of Ref. [3], Eq. (11). However in the presence of leading Regge behavior

$$F_L(\omega) = \tilde{F}_L(\omega) + \sum_{\alpha>0} \gamma(\alpha) \, |\omega|^{\alpha} \, \epsilon(\alpha) \tag{28}$$

we obtain

$$f_L(\lambda) = 4\int_0^\infty \frac{d\omega}{\omega} \cos \lambda/\omega \;\; \tilde{F}_L(\omega) + 2 \sum_{\alpha>0} \gamma(\alpha) \int_{-\infty}^{\infty} dx \, e^{-i\lambda x} \, \left|\frac{1}{x}\right|^{\alpha+1}$$

$$= 4\int_0^\infty \frac{d\omega}{\omega} \cos \lambda/\omega \;\; \tilde{F}_L(\omega) - 2 \sum_{\alpha>0} \gamma(\alpha) \frac{\Gamma(\alpha+1)}{\sin \pi\alpha/2} \, |\lambda|^{\alpha} \quad ,$$

where the second integral has been interpreted as a distribution.

At $\lambda=0$ we obtain

$$S = f_L(0) = 4\int_0^\infty \frac{d\omega}{\omega} \tilde{F}_L(\omega) = 4\left[\int_1^\infty \frac{d\omega}{\omega} \tilde{F}_L(\omega) - \sum_{\alpha>0} \frac{\gamma(\alpha)}{\alpha}\right] \tag{29}$$

which is the scaling version of the general result of Eq. (23), except for the term $C = \lim_{q^2 \to -\infty} C(q^2)$ and the possibility of an $\alpha=0$ term in $F_L(\omega)$. Both these considerations are included in a light-cone BJL treatment presented below. Regarding C note that a term in $F_L(q^2, \omega)$ with the singular behavior

$$\lim_{Bj} \omega F_L(q^2, \omega) = -\frac{C}{2} \delta\left(\frac{1}{\omega}\right) \tag{30}$$

cannot be ruled out a priori. Such a term will not be encountered in the integral of Eq. (23) but contributes to Eq. (29). This is the formal expression of a non-polynomial fixed pole in this coordinate space derivation.

We now give the light-cone BJL derivation which, correctly treated, relates the Schwinger term to a subtraction constant. We take the limit $q_- \to \infty$

$\left(q_\pm = \frac{1}{\sqrt{2}}(q_0 \pm q_3)\right)$ with the remaining components of q and P fixed ($\lim_{LC}$) so that $\omega = 1/x = -P_+/q_+$. This limit isolates light-cone commutators [17], and in particular [18]:

$$\lim_{LC} T^{++}(P,q) = \text{polynomial}$$

$$- \frac{q_+ q_+}{q_-} \int d^4y\, e^{iq\cdot y}\, \delta(y_+)\, V_L(y^2, y\cdot P) + 0\left(\frac{1}{q_-^2}\right) .$$

Equivalently

$$\lim_{Bj} \left(-q^2\, \bar{t}_L(q^2,\nu)\right) = \frac{ix}{2} \int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda x}\, \epsilon(\lambda)\, f_L(\lambda) \quad . \tag{31}$$

The left-hand side of Eq. (31) is given by a once subtracted dispersion relation and the right-hand side may be so written using the representation

$$\int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda x}\, \epsilon(\lambda)\, f_L(\lambda) = \frac{-i}{\pi} \int_{-\infty}^{\infty} \frac{dx'}{x-x'} \int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda x'}\, f_L(\lambda)$$

and subtracting at $x=\infty$. Thus we obtain

$$\lim_{q^2\to-\infty} \; -q^2\, \bar{t}_L(q^2,0) + 2\int_{-1}^{1} \frac{dx'\, F_L\left(\frac{1}{x'}\right)}{x-x'}$$

$$= f_L(0) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{x'dx'}{x-x'} \int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda x'}\, f_L(\lambda) \quad . \tag{32}$$

Equating subtraction constants we obtain

$$S = -\lim_{q^2\to-\infty} q^2\, \bar{t}_L(q^2,0) \tag{33}$$

which was obtained before from the BJL limit, so that the sum rule follows from the analysis of Section III. Equating absorptive parts we obtain Eq. (26).

A term with $\alpha=0$ in $F_L(\omega)$ may now be accommodated by proceeding from Eq. (33) as in Section III and we see that the constant C of Eq. (30) must be the light-cone expression of a nonpolynomial fixed pole (if any).

Finally we remark on the difficulty of inverting Eq. (26) when $F_L(\omega)$ contains an $\alpha=0$ piece. If we proceed as in Eq. (20) and write

$$F_L(\omega) = \widetilde{F}_L(\omega) + \gamma(0)\left[\theta(\omega-1) - \theta(-\omega-1)\right]$$

then

$$f_L(\lambda) = 4\int_1^\infty \frac{d\omega}{\omega} \cos \lambda/\omega \, \widetilde{F}_L(\omega) + f_L^R(\lambda)$$

where

$$f_L^R(\lambda) = 4\int_0^1 \frac{dx}{x} \cos \lambda x$$

and for consistency with the momentum space calculation of Section III we must require $f_L^R(0) = 0$. This is assumed but not explained in Ref. [16].

In conclusion the coordinate space derivation gives the regulated sum rule of Eq. (29). The light cone BJL limit gives the same result as the $q_0 \to i\infty$ BJL limit, Eq. (33), indicating that there is no problem with an $\alpha=0$ piece in $F_L(\omega)$ and that a nonpolynomial fixed pole corresponds to the behavior of Eq. (30).

B. Parton Model

The parton model supposes the nucleon to behave in current scattering processes as though composed of pointlike particles. An operator Schwinger term arises, by gauge invariance, from the local "seagull" coupling to spin-zero partons. The seagull operator is $-2g_{\mu\nu} :\phi^*(y)\, \lambda^2\, \phi(y):$ where λ is the parton's charge and $\phi(y)$ its field and the Schwinger term is given by

$$S = 2 \langle P | \phi^*(0)\, \lambda^2\, \phi(0) | P \rangle$$

where the matrix element simply counts spin-zero charged partons. The correspondence with the previous section can be seen by calculating $f_L(y\cdot P)$ from Eqs. (24) and (25) using

$$J_\mu(y) = i\phi^*(y)\overleftrightarrow{\partial}_\mu \, \lambda\phi(y)$$

and using the leading light-cone singularity of the free field theory propagator to obtain (for $y^2=0$)

$$f_L(y\cdot P) = \langle P|\phi^*(y)\,\lambda^2\,\phi(0)|P\rangle + (y \to -y) \quad .$$

As is well known the charged-squared-weighted probability function for spin-zero partons carrying a fraction $x=1/\omega$ of the longitudinal momentum in the infinite momentum frame is $\frac{2}{x} F_L\left(\frac{1}{x}\right)$ (note that $F_L=0$ for spin-half partons) which implies

$$\begin{aligned} S &= 2 \langle P|\phi^*(0)\,\lambda^2\,\phi(0)|P\rangle \\ &= 2\int_0^1 dx\, \frac{2}{x} F_L\left(\frac{1}{x}\right) = 4\int_1^\infty \frac{d\omega}{\omega} F_L(\omega) \quad . \end{aligned} \tag{34}$$

This is of course the sum rule of Jackiw, van Royen and West, Eq. (11), and is correct in the absence of leading Regge behavior and nonpolynomial fixed pole residues.

As emphasized by Landshoff, Polkinghorne and Short [19], $F_L(\omega)$ is related to the imaginary part of the off-mass-shell, spin-averaged, forward parton-proton scattering amplitude $A(s,\mu^2)$, where $s = (P+p)^2$, $\mu^2=p^2$ for proton and parton momenta P and p respectively. The leading behavior of Im $A(s,\mu^2)$ as $s\to\infty$ determines the leading behavior of $F_L(\omega)$ as $\omega\to\infty$. If

$$\mathrm{Im}\, A(s,\mu^2) \sim \beta_\alpha(\mu^2)\, s^\alpha$$

then

$$F_L(\omega) \sim \gamma_\alpha \omega^\alpha$$

where [19]

$$\gamma_\alpha = \frac{1}{8\pi^2} \int_0^\infty dz\, z^\alpha \int_{-\infty}^{-z} d\mu^2 \beta_\alpha(\mu^2) \quad .$$

The Schwinger term, S, is given by the diagram of Fig. 1a, in terms of the full amplitude $A(s,\mu^2)$. If $A(s,\mu^2)$ satisfies an unsubtracted dispersion relation

$$A(s,\mu^2) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{ds' \, \mathrm{Im}\, A(s',\mu^2)}{s' - s} \tag{35}$$

we obtain the result of Eq. (34). If however $\mathrm{Im}\, A(s,\mu^2)$ has leading Regge behavior ($\alpha \geq 0$) or if Eq. (35) is modified by an additive constant the Schwinger term sum rule is modified.

Consider the subtracted dispersion relation

$$A(s,\mu^2) = A(0,\mu^2) + \frac{s}{\pi} \int_{-\infty}^\infty \frac{ds' \, \mathrm{Im}\, A(s',\mu^2)}{s'(s'-s)} \quad , \tag{36}$$

then the subtraction constant $A(0,\mu^2)$ may be thought of as contributing to S via the diagram of Fig. 1b. Note that only the absorptive part of $A(s,\mu^2)$ contributes to $F_L(\omega)$. We may however express S in terms of $F_L(\omega)$ by relating $A(0,\mu^2)$ to an integral over $\left[\mathrm{Im}\, A(s,\mu^2) - \mathrm{Im}\, A^R(s,\mu^2)\right]$ in analogy with the treatment of $t_L(q^2,\nu)$ in Section III. Since hadronic amplitudes (e.g., parton-proton amplitudes) should not be afflicted by α=0 fixed poles [15] we obtain

$$A(0,\mu^2) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{ds'}{s'} \left[\mathrm{Im}\, A(s',\mu^2) - \mathrm{Im}\, A^R(s',\mu^2)\right] \tag{37}$$

where we have assumed a simple Regge parameterization for $\mathrm{Im}\, A^R(s,\mu^2)$ with $\alpha > 0$. (A term with α=0 requires the more careful treatment of Section III.)

Combining Eqs. (36) and (37) we obtain

$$A(s,\mu^2) = \frac{1}{\pi}\int_{-\infty}^{\infty} ds' \left[\frac{\mathrm{Im}\, A(s',\mu^2)}{s'-s} - \frac{\mathrm{Im}\, A^R(s',\mu^2)}{s'}\right] . \tag{38}$$

The Schwinger term is then given by

$$S = 4\int_0^{\infty} \frac{d\omega}{\omega}\left[F_L(\omega) - \sum_{\alpha>0} \gamma_\alpha\, \omega^\alpha\right] \tag{39}$$

where the two terms of Eq. (39) correspond to the two terms of Eq. (38). Evaluating the integral from ω=0 to 1 we obtain

$$S = 4\left[\int_1^{\infty} \frac{d\omega}{\omega}\, \widetilde{F}_L(\omega) - \sum_{\alpha>0} \frac{\gamma_\alpha}{\alpha}\right]$$

$$\widetilde{F}_L(\omega) = F_L(\omega) - \sum_{\alpha>0} \gamma_\alpha \omega^\alpha \tag{40}$$

which is the scaling version of Eq. (23), supposing no α=0 term in $F_L(\omega)$ and no α=0 fixed pole.

The absence of any additional term C in Eq. (40) is a consequence of the assumption that the hadronic parton-proton amplitude has no fixed pole at α=0. The restriction on α=0 terms in the imaginary part may be relaxed by taking a more careful Regge parameterization of $A(s,\mu^2)$.

V. VALIDITY OF THE SCHWINGER TERM SUM RULE IN PERTURBATION THEORY

A. ϕ^3 Theory

A ϕ^3 model in second order perturbation theory provides the simplest verification of the sum rule. Consider first the box diagrams of Figs. 2a and 2b, from which we obtain

$$F_L(\omega) = \frac{g^2}{32\pi^2}\left(\frac{\omega-1}{\omega\mu^2 + (\omega-1)^2 M^2}\right) , \quad \omega > 1 \tag{41}$$

where M and μ are the masses of the charged and neutral scalar particles. The Schwinger term may be calculated directly from the contact term of Fig. 2c which gives

$$S = \frac{g^2}{8\pi^2}\int_0^1 dx \left(\frac{1-x}{x\mu^2 + (1-x)^2 M^2}\right) \tag{42}$$

where x is a Feynman parameter.

Comparing Eqs. (41) and (42) we see that the diagrams of Fig. 2 satisfy

$$S = 4\int_1^{\infty} \frac{d\omega}{\omega} F_L(\omega)$$

and none of the complexities of Section III occur.

However it is well known that S is logarithmically divergent to second order. The infinity arises from the diagrams of Fig. 3, which do not contribute to $F_L(\omega)$. The contact term of Fig. 3c gives a logarithmic divergence

$$S = \frac{g^2}{8\pi^2\mu^2} \log (\Lambda^2/M^2) \tag{43}$$

where Λ is some cutoff. However the sum of the diagrams of Fig. 3 gives a finite contribution to t_L:

$$q^2 t_L(q^2, \nu) = \frac{-g^2}{8\pi^2\mu^2}\left[\beta \log\left(\frac{\beta+1}{\beta-1}\right) - 2\right] = C(q^2) \quad , \tag{44}$$

where

$$\beta = (1-4M^2/q^2)^{1/2} \quad ,$$

corresponding to a nonpolynomial fixed pole.

The sum rule of Eq. (23) is then satisfied, in the form

$$S = - \lim_{q^2 \to -\infty} C(q^2)$$

with matching logarithmic divergences.

Note that that in parton model terminology the diagrams of Fig. 3 correspond to a fixed pole in the hadronic parton-proton amplitude.

B. Vector Gluon Model

The vector gluon model is more challenging [20]. The box diagrams (Figs. 2a and 2b) give [7]

$$F_L(q^2, \omega) = \frac{g^2}{8\pi^2\omega}\,\theta(\omega^2-1) + 0\left(\frac{1}{q^2}\log(-q^2)\right) \tag{45}$$

and there is no contact term. Moreover Corrigan [21] has shown that

$$\lim_{\nu\to\infty} q^2 t_L(q^2, \nu) = q^2\frac{g^2}{\pi^2}\int_0^1 \frac{dx\, x^2(1-x)}{q^2 x(1-x) - M^2} = C(q^2) \tag{46}$$

corresponding to a nonpolynomial fixed pole.

Combining Eqs. (45) and (46) we see that the sum rule is satisfied, in the form

$$S = \lim_{q^2 \to -\infty} \left[4 \int_1^\infty \frac{d\omega}{\omega} F_L(q^2, \omega) - C(q^2) \right] = 0 \quad . \tag{47}$$

As expected the Schwinger term vanishes.

Zee [7] takes a superficially different approach. He studies the case of zero fermion mass (the attendant infrared problems do not appear in the Bjorken limit). He uses the DGS representation to calculate $F_L(q^2, \omega)$ in the Bjorken limit and concludes that

$$\lim_{Bj} \frac{1}{x} F_L\left(q^2, \frac{1}{x}\right) = \frac{g^2}{8\pi^2} \left[\theta(1-x^2) - 2\delta(x)\right] \tag{48}$$

where the δ-function at x=0 results from a careful consideration of the limiting procedure. His version of the Schwinger term sum rule reads

$$S = 2 \int_{-1}^{1} dx \lim_{q^2 \to -\infty} \left[\frac{1}{x} F_L\left(q^2, \frac{1}{x}\right) \right] = 0 \quad . \tag{49}$$

It is important to realize that Eqs. (47) and (49) are completely equivalent. In the first case the integral is to be evaluated before letting $q^2 \to -\infty$, so that any δ-function singularity in the Bjorken limit will not be encountered. The nonpolynomial fixed pole then cancels the integral. If on the other hand one insists on formulating the sum rule as

$$S = 2 \int_{-\infty}^{\infty} \frac{dx}{x} \widetilde{F}_L\left(\frac{1}{x}\right)$$

it is necessary to treat $\frac{1}{x} F_L\left(\frac{1}{x}\right)$ (and its regulated version) as a distribution with singularities at x=0 which reflect the existence of nonpolynomial fixed poles (as indicated in Eq. (30)). We prefer the first approach since it circumvents the delicate manipulations necessary in Zee's work and permits an interpretation of the sum rule even when S is divergent (as in second order ϕ^3 theory).

It is perhaps appropriate to note here how the sum rule is expected to be fulfilled in the canonical vector gluon model. Although S and $F_L(\omega)$ both vanish according to canonical manipulations it is not obvious that C=0. The vanishing of C is ensured by requiring that real and imaginary parts of $t_L(q^2, \nu)$ scale with the same power of q^2 in the Bjorken limit. This additional assumption is usually incorporated into light-cone analyses [18].

VI. SUMMARY AND COMPENDIUM OF REGULATED SUM RULES

We have shown that the sum rules of inelastic lepton scattering are finite unless the $q^2 \to -\infty$ limit diverges. The generic form of sum rules involving amplitudes satisfying once subtracted dispersion relations is illustrated by the Schwinger term sum rule

$$S = \lim_{q^2 \to -\infty} \left[4 \int_1^\infty \frac{d\omega}{\omega} \tilde{F}_L(q^2, \omega) - 4 \sum_{\alpha > 0} \frac{\gamma(\alpha, q^2)}{\alpha} - C(q^2) \right] \tag{50}$$

where

$$\tilde{F}_L(q^2, \omega) = F_L(q^2, \omega) - \sum_{\alpha \geq 0} \gamma(\alpha, q^2)\, \omega^\alpha \qquad \text{for } \omega > 0 \ .$$

It should be noted that:

1) All Regge contributions with effective intercept greater than <u>or equal to</u> zero have been removed from $F_L(q^2, \omega)$. The integral of Eq. (50) is therefore always convergent (see comment 4 apropos of Regge cuts).

2) There is no counter term for an α=0 piece in $F_L(q^2, \omega)$. Therefore the sum rule need not be analytic in α at α=0, though $t_L(q^2, \nu)$ may be.

3) $C(q^2) = q^2 \bar{t}_L^{\,F.P.}(q^2)$ where $\bar{t}_L^{\,F.P.}(q^2)$ is the nonpolynomial residue of an α=0 fixed pole (if any). Depending upon the kinematics, other sum rules of the form of Eq. (50) may have contributions from polynomial fixed poles.

4) Regge cuts with branch points α_c>0 present no fundamental problems — regulation schemes similar to Eq. (50) will guarantee convergent integrals. (See the appendix for a possible parameterization.)

5) This form of the sum rule encompasses the several perturbation theory calculations of S which exist in the literature.

The procedure we have presented may be applied to sum rules satisfying the following criteria [6]:

1) They must be valid in free field theory — BJL, parton or light-cone techniques should not yield results incompatible with free field theory.

2) In their unregulated form they must not diverge worse than linearly in the presence of α=1 (Pomeron) leading Regge behavior — we have only discussed subtraction of terms down to α=0, additional subtractions require further discussion.

We have searched the literature for such sum rules and we present here a list of those involving the structure functions for inelastic electron scattering (including polarized targets) and inelastic neutrino scattering on unpolarized targets. We define the amplitudes for forward virtual Compton scattering by the decomposition

$$\begin{aligned} T^{e}_{\mu\nu} &= -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) T^{e}_{1}(q^2,\nu) \\ &+ \frac{1}{M^2}\left(P_\mu - \frac{\nu}{q^2} q_\mu\right)\left(P_\nu - \frac{\nu}{q^2} q_\nu\right) T^{e}_{2}(q^2,\nu) \\ &+ \frac{i}{M^2}\, \epsilon_{\mu\nu\alpha\beta}\, q^\alpha s^\beta A^{e}_{1}(q^2,\nu) \\ &+ \frac{i}{M^4}\, \epsilon_{\mu\nu\alpha\beta}\, q^\alpha \left(s^\beta P\cdot q - P^\beta s\cdot q\right) A^{e}_{2}(q^2,\nu) \quad , \end{aligned} \tag{51a}$$

where s is the polarization vector (s^2=$-M^2$, s·P=0).

For the weak currents (average over the nucleon spin) we write

$$T^{\nu,\bar{\nu}}_{\mu\nu} = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) T^{\nu,\bar{\nu}}_1(q^2,\nu)$$

$$+ \frac{1}{M^2}\left(P_\mu - \frac{\nu}{q^2} q_\mu\right)\left(P_\nu - \frac{\nu}{q^2} q_\nu\right) T^{\nu,\bar{\nu}}_2(q^2,\nu)$$

$$- \frac{i}{2M^2} \epsilon_{\mu\nu\alpha\beta} P^\alpha q^\beta T^{\nu,\bar{\nu}}_3(q^2,\nu) + \frac{1}{M^2} q_\mu q_\nu T^{\nu,\bar{\nu}}_4(q^2,\nu)$$

$$+ \frac{1}{2M^2} (P_\mu q_\nu + P_\nu q_\mu) T^{\nu,\bar{\nu}}_5(q^2,\nu) \quad . \tag{51b}$$

The structure functions for inclusive electron scattering are defined by

$$W_i = \frac{1}{2\pi} \operatorname{Im} T^e_i \qquad i=1,2 \quad ,$$

$$G_i = \frac{1}{2\pi} \operatorname{Im} A^e_i \qquad i=1,2 \quad ,$$

and for inclusive neutrino scattering by

$$W^\pm_i = \frac{1}{2\pi} \operatorname{Im}\left(T^\nu_i \pm T^{\bar{\nu}}_i\right) \qquad i=1,5 \quad .$$

In Table I we list the scaling behavior predicted by the quark-vector gluon model and in Table II give regulated and unregulated forms of the sum rules with appropriate references. All these sum rules can be modified by non-polynomial fixed poles. The fixed q^2 sum rules may be invalid and the scaling sum rules may be modified by an additional constant if there are nonpolynomial fixed poles. Note that the scalar densities which occur in Table II for the quark-vector gluon model scaling sum rules are

$$\langle P|\bar{\psi}(0)\, m\{\lambda_+,\lambda_-\}\psi(0)|P\rangle = 2M\sigma_\pi$$

$$\langle P|\bar{\psi}(0)\, m\, \lambda_Q^2\, \psi(0)|P\rangle = 2M\sigma_Q \tag{52}$$

where

$$\lambda_{\pm} = \frac{1}{2}(\lambda_1 \pm i\lambda_2) \quad ,$$

$$\lambda_Q = \frac{1}{2}\left(\lambda_3 + \frac{1}{\sqrt{3}}\lambda_8\right) \quad ,$$

$$\text{Trace } \lambda_i^2 = 2 \quad ,$$

$\psi(y)$ is the quark field and m the quark mass matrix.

Finally we remark that the most accessible test of the assumption of polynomial fixed pole residues is the Corrigan-Cornwall-Norton sum rule [4] whose fate will be determined by accurate data for $F_2^{ep}(q^2, \omega)$ for large $-q^2$ and large ω.

APPENDIX

Here we discuss the Schwinger term sum rule allowing for an $\alpha=0$ term in the imaginary part of $t_L(q^2,\nu)$. Such a term is somewhat bizarre since $t_L(q^2,\nu)$ is even in ν and a subtle combination between Regge poles is necessary to give a finite real part and nonvanishing imaginary part at $\alpha=0$. We show that even with such a term present the Schwinger term is finite, given scaling. We also indicate how to treat Regge cuts.

We parameterize the Regge terms with $\alpha\geq 0$ by

$$t_L^R(q^2,\nu) = \sum_{1\geq\alpha\geq 0} t_L^R(q^2,\omega,\alpha)$$

where

$$t_L^R(q^2,\omega,\alpha) = \frac{2\pi\gamma(\alpha,q^2)}{q^2}\left[\frac{(1+\omega)^\alpha + (1-\omega)^\alpha - 2}{\sin\pi\alpha}\right] \qquad \text{(A. 1)}$$

with the following properties

1) $t_L^R(q^2,\omega,\alpha)$ is analytic in α for $1\geq\alpha\geq 0$

2) $t_L^R(q^2,0,\alpha) = 0 \qquad \text{for } 1\geq\alpha\geq 0$

3) $t_L^R(q^2,\omega,0) = \dfrac{2\gamma(0,q^2)}{q^2}\log(1-\omega^2)$

4) $t_L^R(q^2,\omega,1) = \dfrac{2\gamma(1,q^2)}{q^2}\left[\omega\log\left(\dfrac{1-\omega}{1+\omega}\right) - \log(1-\omega^2)\right]$

5) $F_L^R(q^2,\omega) = \sum_{1\geq\alpha\geq 0}\gamma(\alpha,q^2)\,(\omega-1)^\alpha\,\theta(\omega-1) \qquad \text{for } \omega>0$

From Eq. (A. 1) and properties 3 and 4 it can be seen that $t_L^R(q^2,\nu)$ contains a constant piece

$$\frac{-4}{q^2}\left[\sum_{1>\alpha>0}\left(\frac{\pi\gamma(\alpha,q^2)}{\sin\pi\alpha}\right) + \gamma(1,q^2)\right]$$

as $\nu\rightarrow\infty$.

Thus the expression for the fixed pole residue now reads

$$t_L^{F.P.}(q^2) = \lim_{\nu \to \infty} \left[t_L(q^2,\nu) - t_L^R(q^2,\nu) \right]$$

$$- \frac{4}{q^2} \left[\sum_{1>\alpha>0} \left(\frac{\pi\gamma(\alpha,q^2)}{\sin \pi\alpha} \right) + \gamma(1,q^2) \right] \quad .$$

Using property 2 we may write a dispersion relation for $t_L^R(q^2,\nu)$, subtracting at $\nu=0$, and repeat the analysis of Section III to obtain

$$S = \lim_{q^2 \to -\infty} \left\{ 4 \int_1^\infty \frac{d\omega}{\omega} \hat{F}_L(q^2,\omega) - 4 \sum_{1>\alpha>0} \left(\frac{\pi\gamma(\alpha,q^2)}{\sin \pi\alpha} \right) - 4\gamma(1,q^2) - C(q^2) \right\} \tag{A. 2}$$

where

$$\hat{F}_L(q^2,\omega) = F_L(q^2,\omega) - F_L^R(q^2,\omega) \quad . \tag{A. 3}$$

We now express Eq. (A. 2) in terms of the more conventional

$$\widetilde{F}_L(q^2,\omega) = F_L(q^2,\omega) - \left[\sum_{1\geq\alpha>0} \gamma(\alpha,q^2)\, \omega^\alpha + \widetilde{\gamma}(0,q^2) \right]$$

where, by property 5,

$$\widetilde{\gamma}(0,q^2) = \gamma(0,q^2) - \gamma(1,q^2)$$

and

$$\widetilde{F}_L(q^2,\omega) - \hat{F}_L(q^2,\omega) = \sum_{1>\alpha>0} \gamma(\alpha,q^2) \left[(\omega-1)^\alpha - \omega^\alpha \right]$$

for $\omega \geq 1$.

Hence

$$\int_1^\infty \frac{d\omega}{\omega} \left[\widetilde{F}_L(q^2,\omega) - \hat{F}_L(q^2,\omega) \right] = \sum_{1>\alpha>0} \gamma(\alpha,q^2) \left[\frac{1}{\alpha} - \frac{\pi}{\sin \pi\alpha} \right] \tag{A. 4}$$

and we recover the familiar result from Eqs. (A. 2) and (A. 4)

$$S = \lim_{q^2 \to -\infty} \left\{ 4 \int_1^\infty \frac{d\omega}{\omega} \widetilde{F}_L(q^2, \omega) - 4 \sum_{1 \geq \alpha > 0} \frac{\gamma(\alpha, q^2)}{\alpha} - C(q^2) \right\}$$

and the integral is finite even if $F_L(\omega, q^2)$ contains an asymptotically constant piece $\widetilde{\gamma}(0, q^2)$.

Regge cuts may be incorporated into our analysis by using the following parameterization, corresponding to a branch point $\alpha_c > 0$

$$t_L^{RC}(q^2, \omega, \alpha_c) = \int_0^{\alpha_c} d\alpha \, t_L^R(q^2, \omega, \alpha)$$

using Eq. (A. 1).

Then

$$F_L^{RC}(q^2, \omega) = \int_0^{\alpha_c} d\alpha \, \gamma(\alpha, q^2) \, (\omega-1)^\alpha \, \theta(\omega-1) \qquad \text{for } \omega>0$$

with leading Regge behavior $\omega^{\alpha_c}/(\log \omega)^n$ as $\omega \to \infty$, where $n \geq 1$ and n depends upon the behavior of $\gamma(\alpha, q^2)$ at $\alpha=\alpha_c$. Then in analogy with Eq. (A. 2)

$$S = \lim_{q^2 \to -\infty} \left\{ 4 \int_1^\infty \frac{d\omega}{\omega} \hat{F}_L(q^2, \omega) + G(\alpha_c, q^2) - C(q^2) \right\}$$

where $G(\alpha_c, q^2)$ is the term in $q^2 t_L^{RC}(q^2, \omega, \alpha_c)$ which is asymptotically constant as $\omega \to \infty$, and is necessarily finite. Moreover $G(\alpha_c, q^2)$ has a finite limit as $q^2 \to -\infty$, given scaling of $F_L(q^2, \omega)$.

From the analyses we conclude

1) The sum rule need not be analytic in α although $t_L(q^2, \nu)$ may be.

2) S is finite given scaling and the existence of $C = \lim_{q^2 \to -\infty} C(q^2)$.

3) Different Regge parameterizations give the same sum rule for S, provided they parameterize the leading ($\alpha \geq 0$) behavior. The regulated integrals (from $\omega = 1$ to ∞) may differ, but the difference is reflected in different counter terms.

4) Regge cuts present no problems in principle.

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$$(P_3^-)^p = - (P_3^-)^n = - 12 \, (P_2^p - P_2^n)$$

where p and n denote proton and neutron targets. This result depends upon the quark charges and is in principle testable at $q^2=0$. It is a dramatic example of the consequence of assuming polynomial fixed pole residues.

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TABLE I

Scaling behavior of structure functions in the limit $q^2 \to -\infty$, $x = -q^2/2\nu$ fixed. (* Indicates scaling behavior specific to quark model, where $F_L(x) = 0$.)

Structure Function	Scaling Function
$W_1(q^2, \nu)$	$F_1(x)$
$\left(\frac{\nu}{M^2}\right) W_2(q^2, \nu)$	$F_2(x)$
$\left(\frac{\nu}{M^2}\right) W_3(q^2, \nu)$	$F_3(x)$
$\left(\frac{\nu}{M^2}\right)^2 W_4(q^2, \nu)$	$F_4(x)$ *
$\left(\frac{\nu}{M^2}\right)^2 W_5(q^2, \nu)$	$F_5(x)$ *
$W_L(q^2, \nu) - W_2(q^2, \nu)$	$F_L(x)$
$\left(\frac{\nu}{M^2}\right) W_L(q^2, \nu)$	$F_G(x)$ *
$\left(\frac{\nu}{M^2}\right) G_1(q^2, \nu)$	$S_1(x)$
$\left(\frac{\nu}{M^2}\right)^2 G_2(q^2, \nu)$	$S_2(x)$

where $W_L(q^2, \nu) = \left(1 - \frac{\nu^2}{M^2 q^2}\right) W_2(q^2, \nu) - W_1(q^2, \nu)$

TABLE II

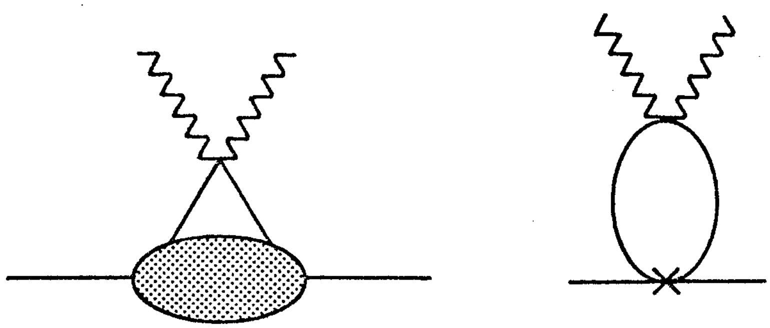
Compendium of Sum Rules

(S, σ_π and σ_Q are defined in Eqs. (1) and (52). Scaling functions are defined in Table I.)

Unregulated	Ref.	Regulated	Ref.
Scaling Sum Rules			
$S = 4\int_0^1 \frac{dx}{x} F_L(x)$	3	$= 4\int_0^\infty \frac{dx}{x} \tilde{F}_L(x)$	4, 5
$\sigma_\pi = 2M\int_0^1 dx\, x F_4^+(x)$	22	$= 2M\int_0^\infty dx \left(x\tilde{F}_4^+(x) - \frac{1}{4}\tilde{F}_5^+(x)\right)$	6
$= \frac{M}{2}\int_0^1 dx \left(2F_G^+(x) + 4x\, F_4^+(x) - F_2^+(x)\right)$	23, 24	$= \frac{M}{2}\left\{2\int_0^\infty dx \left(\tilde{F}_G^+(x) + 2x\,\tilde{F}_4^+(x)\right) - \int_0^1 dx\, F_2^+(x)\right\}$	23
$\sigma_Q = M\int_0^1 dx \left(2F_G(x) - F_2(x)\right)$	25	$= M\left\{2\int_0^\infty dx\, \tilde{F}_G(x) - \int_0^1 dx\, F_2(x)\right\}$	23
Fixed Mass Sum Rules			
$\int_{-q^2/2}^\infty d\nu\, W_3^-(q^2,\nu) = 0$	26	$\int_{-q^2/2}^\infty d\nu\, \tilde{W}_3^-(q^2,\nu) = P_3^-$	6
$\int_{-q^2/2}^\infty d\nu\, G_2(q^2,\nu) = 0$	27	$\int_{-q^2/2}^\infty d\nu\, \tilde{G}_2(q^2,\nu) = 0$	6
		$\int_{-q^2/2}^\infty d\nu\, \nu\tilde{W}_2(q^2,\nu) = q^2 P_2$	4

FIGURE CAPTIONS

1. Schwinger term in the parton model.
2. Second order ϕ^3 theory contributions giving finite Schwinger term.
3. Second order ϕ^3 theory contributions giving logarithmically divergent Schwinger term.



(a) (b)

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Fig. 1

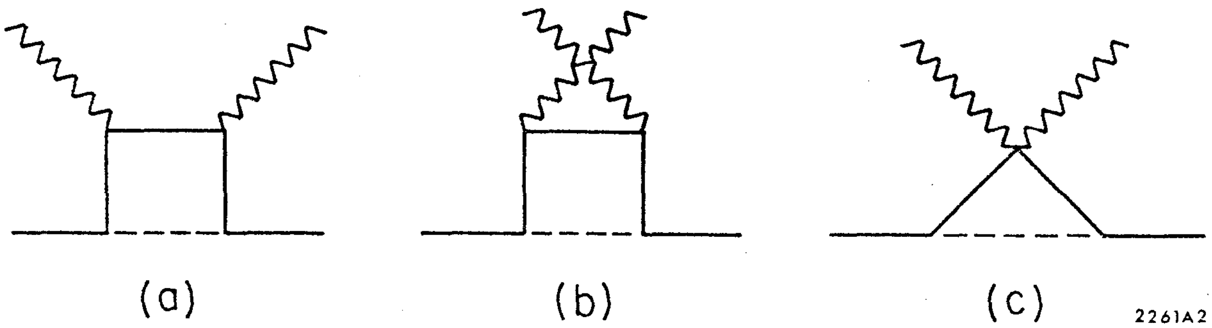

Fig. 2

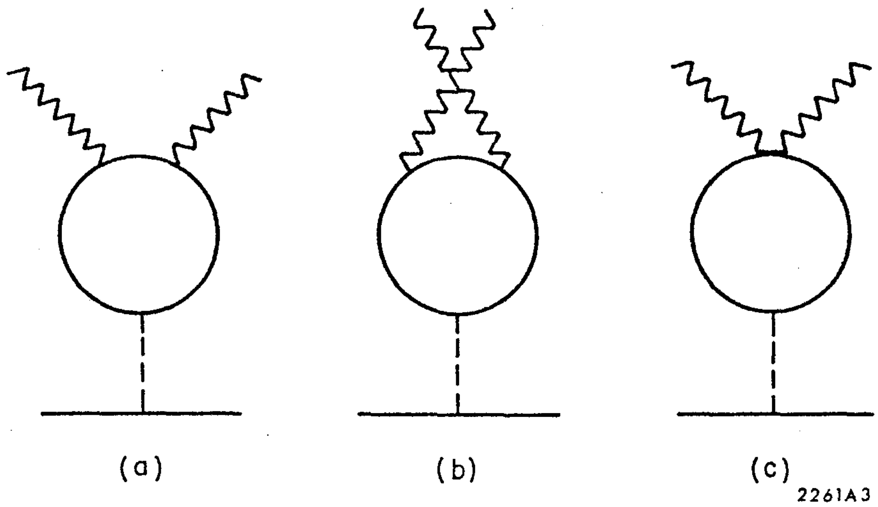

Fig. 3