Longitudinal Space Charge Effect in Proton Linacs

T. Nishikawa and S. Okumura

Department of Physics and Institute for Nuclear Study,
University of Tokyo,
Tokyo, Japan

1. Introduction

A number of studies have been made to evaluate the current limited by space charge effect on the longitudinal motion in proton linacs[1,2,3,4,5]. Most of them assumed a stationary distribution of space charge so that dynamical effects caused by acceleration were neglected. Consequently it has been stated that the space-charge-limited current would be determined at the low energy or injection end of the accelerator. Using an ellipsoidal bunch model, however, Gluckstern pointed out that the space charge effect may become important rather at higher energies as the beam is accelerated and damped longitudinally[5]. In addition, even if we neglect the dynamical effects, there are different conclusions on the space charge limit depending upon methods and assumptions used. For example, Kapchinsky and Kronrod[2] calculated the self-consistent stable phase limits using an integral equation for the potential kernel with the assumption of uniform-phase-space charge distribution. They proved that the space charge effect on the stable phase limits is considerably weaker than the result given by a uniformly charged ellipsoidal model[1], leading to the maximum current of the order of one hundred milliampares at the space charge parameter μ being one. A similar analysis has been given by Morton including corrections due to the image charges and the adjacent bunches[3]. At the last Conference, however, Bondarev and Vlasov[4], commented these analyses that irrespective of the assumption of the charge distributions, the maximum current with space charge effect would be given at a region of $\mu = 0.3 \sim 0.4$ and would consequently be lower than that obtained by Kapchinsky and Kronrod. We shall re-examine these space charge problems including some further effects dicarded in the previous papers. Details of our analytic method has been given in an unpublished report[6].

2. Space-charge-limited Current by Simple Ellipsoidal Bunch Model

Hamiltonian for the phase oscillations under a potential function V(X) is written as

$$\frac{H}{m_0 c^2 \omega} = \frac{1}{2\gamma_s^3 \beta_s^2} (\gamma - \gamma_s)^2 + \left(\frac{\lambda k_1}{2\pi}\right)^2 \gamma_s^3 \beta_s^4 V(X)$$

and if we neglect shift of the phase stable point due to the space charge field, then the equation of motion is given, in a good approximation, by

$$(\gamma_s \beta_s)^{-3} \frac{d}{dz} (\gamma_s^3 \beta_s^3 \frac{dX}{dz}) + k_1^2 \left[(1-\mu) X - \frac{X^2}{2|\varphi_s|} \right] = 0 \qquad (1)$$

where

$$X = \varphi - \varphi_s \ (\varphi_s : \text{synchronous phase angle}),$$

$$k_1^2 = \left(\frac{2\pi e\, E_0 T\, |\sin \varphi_s|}{mc^2 \lambda \gamma_s^3 \beta_s^3} \right), \qquad (2)$$

$$\mu = 1 - \left(\frac{\partial^2 V(X)}{\partial X^2} \right)_{X=0}$$

For a beam bunch of a uniformly charged ellipsoid, Lapostolle calculates the space charge field[7] and Gluckstern gives the formula[5],

$$\mu = \frac{90\, I \gamma_s^{-3} \beta_s \lambda^2 f}{2\pi a a' b\, E_0 T\, |\sin \varphi_s|} \qquad (3)$$

where b, a, and a' are the semi-longitudinal and transverse axes of the ellipsoid, I the beam current averaged over the bunches, and $f(\sqrt{aa'}/b)$ the function depending on the beam shape. f is approximated by

$$f \approx \frac{1}{3} \frac{\sqrt{aa'}}{b} = \frac{2\pi \sqrt{aa'}}{3\beta_s X_b \lambda} \qquad (4)$$

in the region of $\sqrt{aa'}/b$ from 0.2 to 1.2, and almost all cases actually considered will satisfy this condition for $\sqrt{aa'}/b$. Hence, equ. (3) written in terms of the total phase spread of $2X_b$ becomes

$$\mu = \zeta \frac{1}{\beta_s X_b^2} \qquad (5)$$

$$\zeta = \frac{\pi}{3} \frac{180\, I\, \eta}{E_0 T\, |\sin \varphi_s| \sqrt{aa'}} \qquad (5')$$

$$X_b = \frac{2\pi b}{\beta_s \lambda}, \qquad (5'')$$

where η is a function of $b/\sqrt{aa'}$ and γ_s, which has approximately a constant value of one.

Now the space-charge-limited current previously given[1,4,5] by the ellipsoidal bunch model is derived in our terms as follows:
Space charge effect is given by the simple replacements in the equation without space charge,

$$k_1 \rightarrow k_1 \sqrt{1-\mu}$$

$$|\varphi_s| \rightarrow |\varphi_s| (1-\mu).$$

Correspondingly, the stability limits would become

$$X_1 = 2 |\varphi_s| (1-\mu)$$

$$X_2 = -|\varphi_s| (1-\mu). \qquad (6)$$

X_1 would correspond to the potential maximum where the restorting force becomes zero, and X_2 the limit at the other side where the potential energy is equal to that of at X_1. It is noted here that equ. (1) and (6) are applicable only when the beam bunch is spread over the region from X_2 to X_1 having the center at $X = 0$. For the present, however, we shall assume the self-consistent value of the beam spread as

$$2X_b = X_1 - X_2 = 3|\varphi_s| (1-\mu). \qquad (7)$$

Inserting equ. (7) into (5), we get a relation

$$\mu(1-\mu)^2 = \frac{4\zeta}{9\beta_s |\varphi_s|^2}. \qquad (8)$$

Neglecting effect of acceleration, we get the maximum value of the left hand side at $\mu = \frac{1}{3}$. From equ. (5'), the maximum current being able to be trapped into the stable orbits is

$$I_{max} = \frac{E_0 T \beta_s |\varphi_s|^3 \sqrt{aa'}}{180\pi}, \qquad (9)$$

where we let $\sin|\varphi_s| \approx |\varphi_s|$ and $\eta = 1$. The space-charge-limited current thus evaluated is about 30 mA for the typical parameters of the present *Brookhaven AGS Linac and yields too small values for almost all cases of* existing linacs.

This analysis, however, has some problems because we have used only the interior potential of the ellipsoid with the center at the synchronous phase angle. Thus we shall re-examine the self-consistent stable phase limits including the exterior potential and the shift of center of the ellipsoid from the synchronous angle. Also, we shall give brief discussions on effects of the charge distribution and the acceleration.

3. Effect of Bunch Center and Shift of Synchronous Phase Angle

As pointed out by Kapchinsky and Kronrod, the space charge, in general, displaces the synchronous phase angle. This is explained in terms of the uniformly charged ellipsoid as the effect due to the shift of the bunch center from the synchronous angle. Consider the center being at φ_0 and define the parameter

$$\delta = \varphi_0 - \varphi_s \; .$$

The equation of motion becomes

$$\gamma_s^{-3}\beta_s^{-3}\frac{d}{dz}\left(\gamma_s^{3}\beta_s^{3}\frac{dX}{dz}\right) + k_1^2 \left[(1-\mu)X - \frac{X^2}{2|\varphi_s|} + \mu\delta\right] = 0 \qquad (10)$$

For the fullbucket case, we may assume that the center of the bunch is at the midpoint of the stable region, or

$$\delta = \frac{X_1 + X_2}{2} \; . \qquad (11)$$

The solution which would correspond to the maximum of the potential is now given by

$$X_1 = \left(1 - \frac{3}{2}\mu + \sqrt{1 - \frac{3}{4}\mu^2}\right) |\varphi_s| \; . \qquad (12)$$

The other limit of the phase stable region, which satisfies the condition $V(X_2) = V(X_1)$, is

$$X_2 = \left(1 - 2\sqrt{1 - \frac{3}{4}\mu^2}\right) |\varphi_s| \; . \qquad (12')$$

Thus the total spread of the phase stable region will be

$$X_1 - X_2 = 3|\varphi_s|\left[\sqrt{1 - \frac{3}{4}\mu^2} - \frac{\mu}{2}\right] \approx 3|\varphi_s| \quad (1-\mu) \quad \left(1 + \frac{\mu}{2}\right) .$$

Letting $(X_1 - X_2) = 2X_b$, we can rewrite equ. (8) thus

$$\mu(1-\mu)^2\left(1 + \frac{\mu}{2}\right)^2 = \frac{4\zeta}{9\beta_s|\varphi_s|^2} \; . \qquad (13)$$

The maximum value of the left-hand side of equ. (13) occurs at $\mu = 0.4$, and the limiting current so evaluated increases 55 percent, including the 20 percent increase in the phase stable region.

The shift of the synchronous phase angle is given by

$$X_s = \left(1 - \frac{\mu}{2} - \sqrt{1 - \frac{3}{4}\mu^2}\right) \varphi_s \; . \qquad (14)$$

X_1, X_2 and X_s are shown in Fig. 1 as function of μ. The straight-lines in this figure give the limits determined by the simple equ. (6). The calculated shift of the synchronous phase angle is the order of -0.1 rad. at most and agrees well with the result obtained by Kapchinsky and Kronrod. They assumed a uniform charge density in phase space resulting a wider stable region.

4. Exterior Potential of Beam Bunch and Self-consistent Limiting Current without Acceleration

As noted earlier, equ. (1) is only applicable to the interior of the bunch and the phase stable limits given by equs. (6) or (12) are valid when the beam spreads over the region between X_1 and X_2. If we assume the beam spread as $2X_b = X_1 - X_2$, then it becomes questionable whether the limit given by X_1 in (6) or (12) does really correspond to the potential maximum or not.

For simplicity, first we shall consider a uniformly charged spherical bunch having its center at the synchronous angle. Then the interior potential function which results in the equation of motion (1) is

$$V_{int} = \frac{1}{2}\left((1-\mu)X^2 - \frac{X^3}{3|\varphi_s|} + \text{const.} \right) , \qquad (15)$$

where μ is given by (3) letting a = a' = b or f = 1/3. On the other hand, the exterior potential of a spherical bunch becomes

$$V_{ext} = \frac{1}{2}\left(X^2 - \frac{X^3}{3|\varphi_s|} + 2\mu\frac{X_b^{\,3}}{|X|}\right) . \qquad (15')$$

From the continuity condition at $|X| = X_b$, the potential constant in equ. (15) is $3\mu X_b^2$.

Consider the region of X>0, and if we assume that $X_{b1} = X_1 = 2|\varphi_s|(1-\mu)$, then the first derivative at X_1 is certainly zero both for the interior and the exterior potentials. The second derivative at this point is negative for the interior potential so far as $\mu \leq 1$, but for the exterior potential

$$\left(\frac{\partial^2 V_{ext}}{\partial X^2}\right)_{X=X_1} = -1 + 4\mu \begin{array}{l} < 0, \text{ if } \mu < 0.25, \\ > 0, \text{ if } \mu > 0.25. \end{array}$$

Therefore, for a smaller μ (< 0.25), X_1 corresponds to the potential maximum, however for a larger μ (μ > 0.25) this becomes an inflection point and the maximum moves outside the bunch (Fig. 2 (a)).

For μ > 0.25, therefore, we shall consider a beam which has a wider spread, $X_{b1} > X_1$. In Fig. 2 (b), shapes of the potential are sketched for such cases. If the beam is slightly wider than that is given by equ. (7), then there exist three points where the first derivative of the potential becomes zero; one is the maximum at $X_1 = 2|\varphi_s|(1-\mu)$ being inside the bunch, the second is the minimum just outside the bunch, and the last is the maximum near $X = 2|\varphi_s|$ except the case of lower μ ($\leq$ 0.5). As the beam spread increases, the minimum approaches to the maximum outside the bunch, and they are reduced to an inflection point at

$$X_{b1\,max} = \frac{3}{2}|\varphi_s| (4\mu)^{-\frac{1}{3}} \; . \qquad (16)$$

The slef-consistent beam spread for μ >0.25 will be approximated by this value of X_{b1}. This corresponds to the case of an unstable bunch, however, if the bunch becomes only a little narrower, then the maximum and minimum are readily separated giving a stable bunch. Including the bunch center effect discussed above, we get the other limit from the relation $V(X_{b2}) = V(X_{b1})$.

So far we have assumed a spherical bunch, however it is not difficult to generalize our method to an ellipsoidal bunch[5]. Using Lapostolle's formula for the exterior potential of a uniformly charged ellipsoid[7], we obtain the phase stable limits by numerical calculations as shown in Fig. 3. In this figure, effect of the shape of the ellipsoid is shown by the parameter f ; it is obvious that the prolate ellipsoid ($f < \frac{1}{3}$) gives a wider stable region than the case of the spherical bunch ($f = \frac{1}{3}$).

Now the current associated with the self-consistent beam spread can be determined as follows:

1) Assuming a and b (or X_b), we can immediately determine f.
2) We get corresponding μ values from Fig. 3 using the relation between μ, X_b, and f.
3) Using the assumed values for E_0, T, φ_s and β_s we get the current I from equs. (5) and (5').

By these means, it turns out that the space-charge-limited current has a maximum at $\mu = 1$. Here, the potential becomes almost flat over the beam spread so that the area of the phase stable region disminishes as $\Delta\gamma \sim 0$. This corresponds to the nearly stationary charge distribution since the forces acting on the particles become very small. The maximum accelerating current thus calculated is about 65 mA for the AGS Linac, which is more than 2 times as large as the value given by equ. (9). Thus, the analysis based on the ellipsoidal model agrees with the results obtained by Kapchinsky and Kronrod and also by Morton using the integral equation with the uniform-phase-space charge distribution. Using Morton's computational program, Benton and Chasman obtained the limiting current of about 85 mA for the above case[8].

5. Effect of Charge Distribution

Since the charge distribution in the ellipsoidal model is more concentrated into the inner part of the bunch than in the case of uniform-phase-space distribution, lower limiting current may result from the dense charge distribution. Therefore, we shall consider the another extreme of the charge distribution given by the uniformly charged cylindrical bunch. We again neglect the shift of the center from the synchronous angle. Then, for the oscillations with smaller amplitudes the space charge parameter μ is approximately

$$\mu = \frac{90\,\Gamma_s^3 \beta_s \lambda^2 I}{3\pi a^2 b\, E_0 T \sin|\varphi_s|} \left[1 - \frac{b}{\sqrt{a^2 + b^2}} \right]. \tag{17}$$

Comparing this to equ. (3) for the ellipsoidal bunch, we see that the function f is now replaced by

$$f' = \frac{2}{3} \left[1 - \frac{b}{\sqrt{a^2 + b^2}} \right]. \tag{17'}$$

Thus the cylindrical bunch having the same μ and the same a (= a') and b as the corresponding ellipsoidal ones results in 70 percent larger current. Bondarev and Vlasov have also pointed out a similar relation between the uniform-phase-space distribution and uniform ellipsoidal distribution[4]. It should be noted, however, that, since the charges spread out much more than the ellipsoidal bunch, the self-consistent beam spread for the cylindrical bunch can not be given by the simple replacement of f in Fig. 3. For a higher μ the cylindrical bunch will give a narrower stable region than the corresponding ellipsoidal bunch. Therefore, the limiting current would not increase by as much as 70 percent.

6. Effect of Acceleration

As pointed out by Gluckstern, if the phase stable limits are given by equ. (6), then the beam spread may cross over these limits at an intermediate energy, since the space charge parameter μ increases and approaches one during the acceleration. By taking the exterior potential and the shift of the bunch center, however, we showed that the phase stability expands into a wider region, especially when μ becomes large. Hence, it will be assured that if the particles are trapped once into the stable region at low energy, then they will not be lost thereafter.

It is not easy to evaluated the space-charge-limited current including these dynamic effects due to acceleration. For example, if we inject more particles than the numbers corresponding to the limiting current at injection ($\mu_0 = 1$), then the motions in the initial stage will be chaotic. Even for the case of $\mu_0 > 1$, there remains a phase stable region due to the infinitely high potential wall in the negative X side. Thus at least the particles injected into such a region will be accepted into stable orbits. During the acceleration, unstable particles will be lost completely from the stable region and the remaining ones will tend to a stationary distribution. Hence, the space-charge-limited current will be determined from the self-consistent stable beam spread at a higher energy where the stationary distribution is nearly established.

A computational study has been made dividing a cylindrical beam bunch into thin discs, which move back and forth depending on the potential with space charge effect.* Neglecting effect of transverse motions, we assume that the beam radius is constant over the accelerator length (0.5 or 0.3 cm). The phase spread of bunches at the injection is also fixed to be a constant from -79° to 31°. The values of other parameters are taken from the tentative choises for the injector of a 40 GeV PS project in Japan.

After being injected into the linac, the number of particles in a bunch will decrease on account of phase motion. This is shown by cutting off the particles outside the region of injection phase spread. The result is shown in Fig. 4 where the beam current along the accelerator length are plotted for several values of the injection current. It should be noted that the nearly stationary distribution is achieved at a position of $\beta_s = 0.07 \sim 0.08$ for the range of currents considered here. This is because the phase stable region decreases for higher currents while the velocity of phase motions slows down due to space charge effect.

If we assume a stationary distribution, it is easy to show from the analytic method that the space-charge-limited current, I_{max}, will be in proportion to β_s^2. Thus effect of acceleration will be considered in the first order as the change of β_s from the value at the injection to a higher one where the stationary distribution is nearly established. Taking the latter as $\beta_s = 0.07 \sim 0.08$, we get an estimate of the space-charge-limited current three or four times as large as the value evaluated at the injection ($\beta_{s0} = 0.04$).**

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* Similar computational study is also given by Benton and Chasman[8].

** Gluckstern investigates similar effect of acceleration for the case without space charge ; the paper will be submitted at the conference (private communication).

Fig. 5 shows the computational results of the ourput current after accelerations taking the injection current as the abscissa. Almost the same curve can be reproduced by the analytic method described above, and we can assure a limiting current of more than 100 mA from the longitudinal phase motion in proton linacs. For a more complete analysis, it will be necessary to include transverse space charge effect which couples to the longitudinal motion.

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L.C. Teng (NAL): In the usual longitudinal space charge calculations for circular machines, the bunch is usually quite long compared to the transverse dimension of the chamber, so the vacuum chamber shielding is very important on the field. Admittedly in this case the bunch is much shorter and the bores large compared to the bunch, but I wonder whether the shielding effect of the drift tube bore is important.

Gluckstern: It can be important. The expressions that Lapostolle uses in the computations do include these image effects due to the bore, including the region between drift tubes assuming that the bunch is between two parallel planes. I believe they are part of the Brookhaven but not of the Los Alamos computation. If the size of the beam gets to be 80% or 90% of the bore radius, these image effects can be seen, but for beam sizes of the order of half of the bore radius these are unobservable and lost in the noise of the calculation.

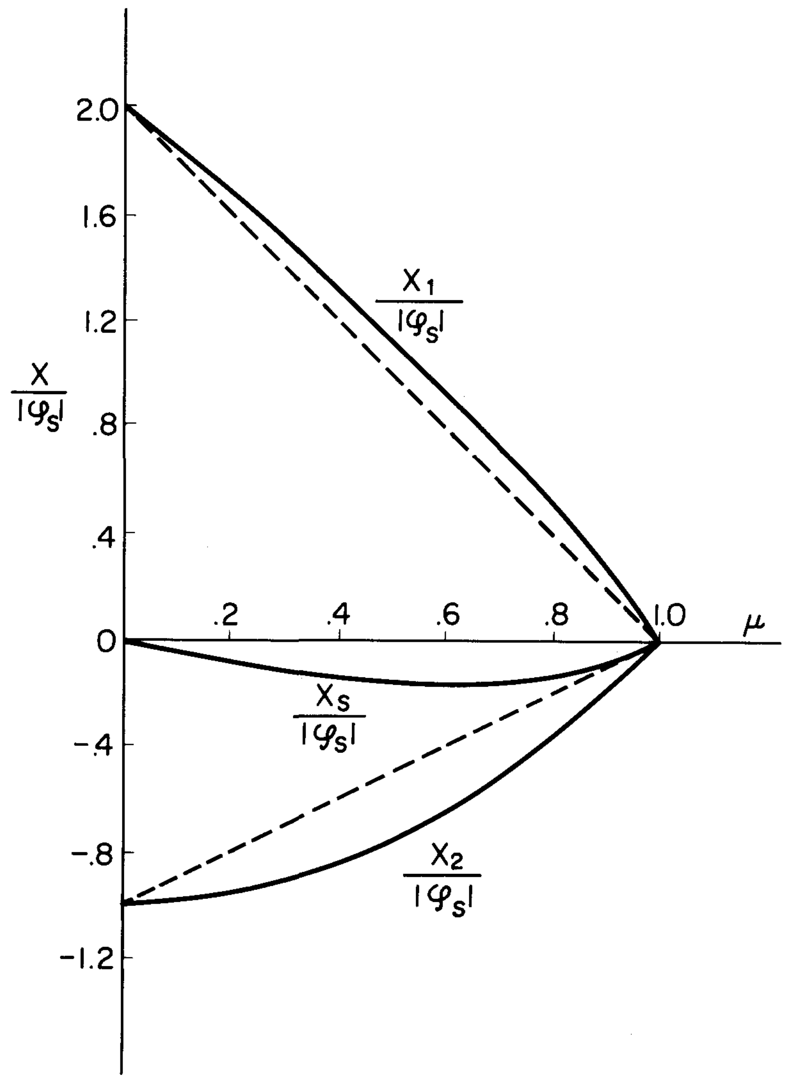

Fig. 1. Effect of Bunch Center and Shift of Synchronous Phase Angle

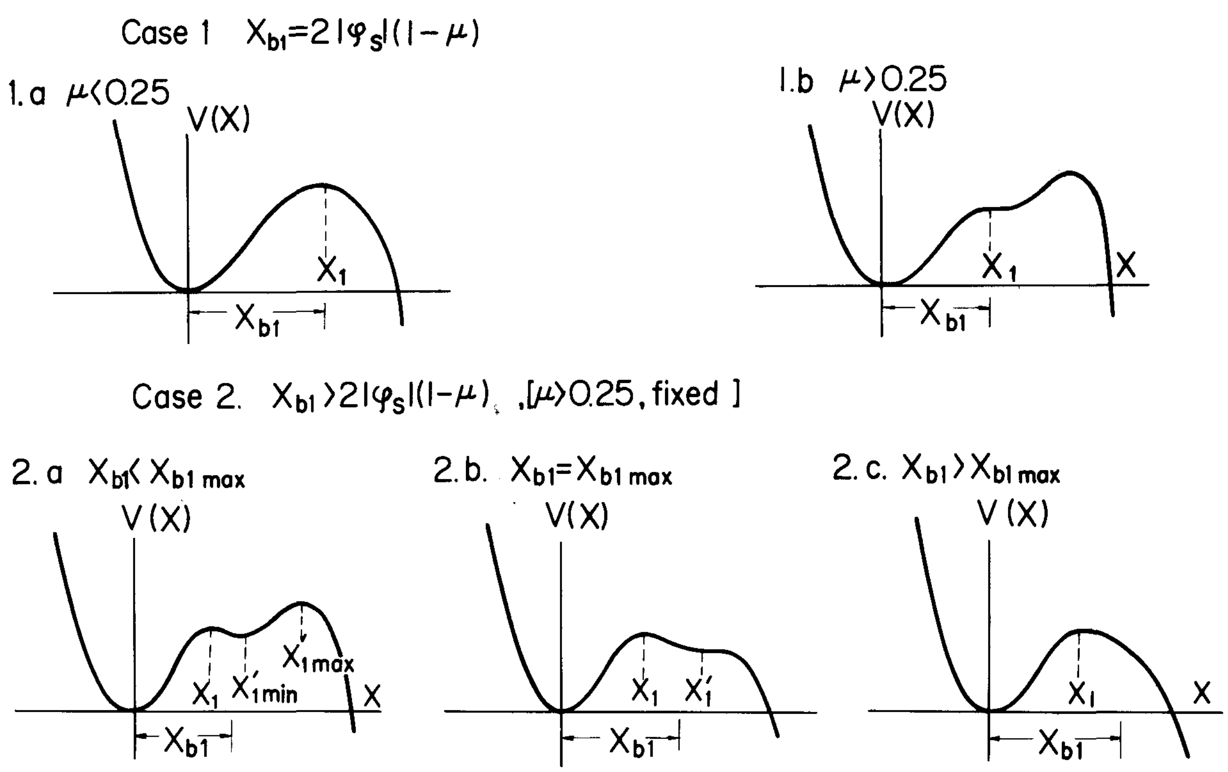

Fig. 2. Schematic Diagram of Longitudinal Potential with Space Charge due to Uniformly-charged Spherical Bunch

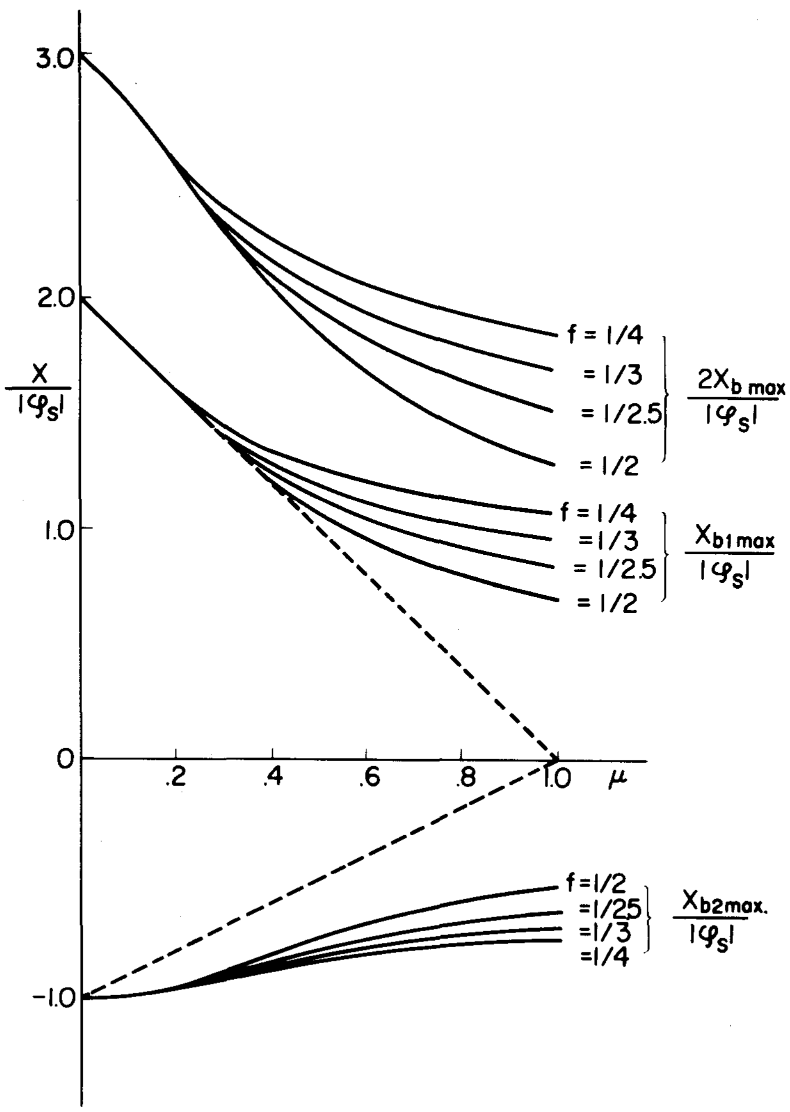

Fig. 3. Self-consistent Phase Stable Limits for Ellipsoidal Bunches (For Spherical Bunch, f = 1/3)

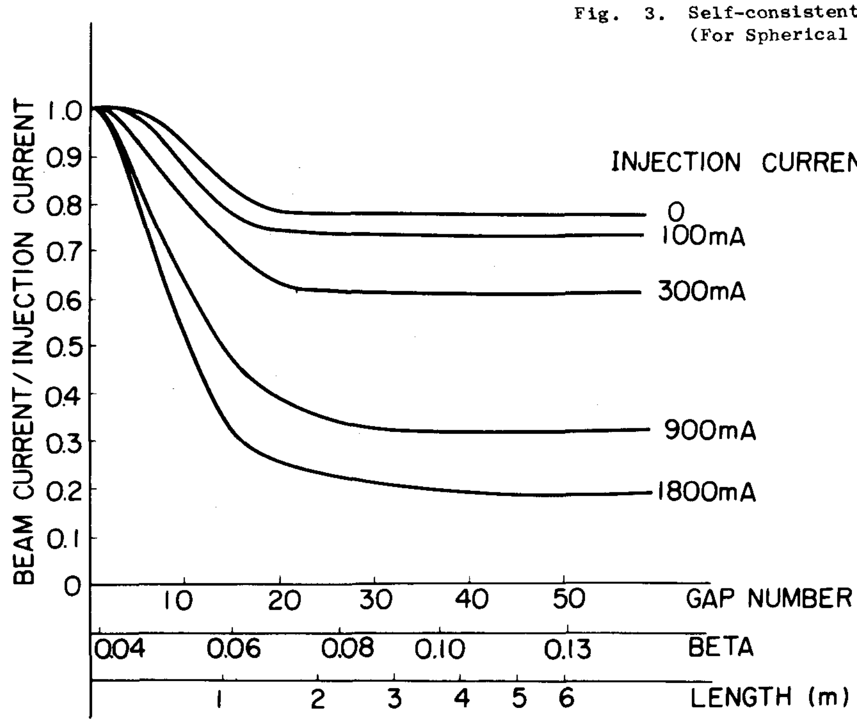

Fig. 4. Variation of Beam Currents during Acceleration with Space Charge (Beam Radius : 0.5 cm, Injection Phase Spread : $-79^{\circ} \sim 31^{\circ}$).

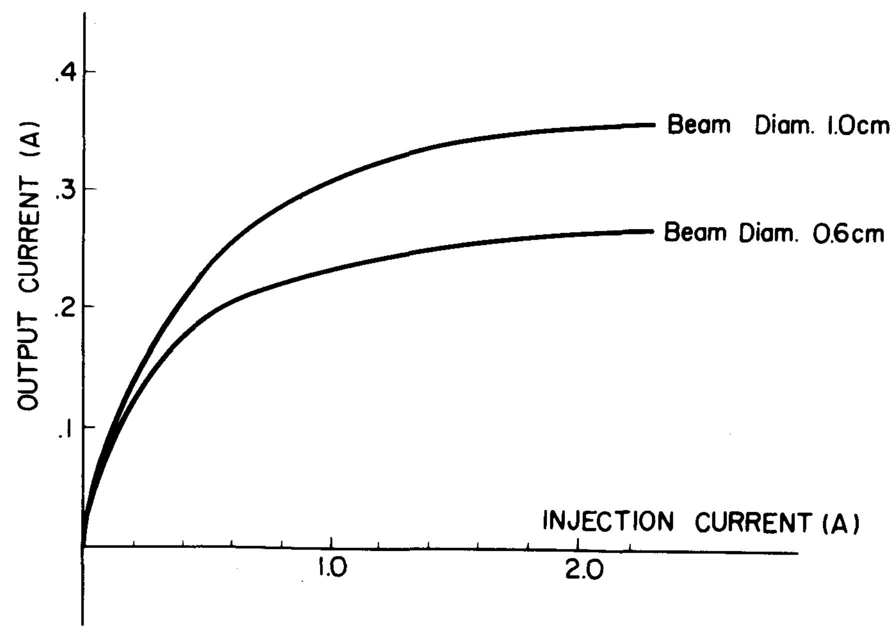

Fig. 5. Computational Relations between Output Current and Injection Current.