THE MASS FUNCTION IN THE ADHESION MODEL

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Abstract

We present an analytical and numerical derivation of the mass function in the adhesion model. The scaling properties of the mass function at small masses are shown to be connected with the existence of a Devil's staircase in the Lagrangian map.

One important characteristic of large-scale structures in the Universe is the number density of masses, here referred to as the 'mass function'. Observations suggest that it follows a power law at small masses followed by a rapid cut-off at larger masses [1]. It is shown here that this property can be understood within a simplified model of the dynamics of the large scale structures, called the adhesion model. It is an extension of the so-called Zeldovich approximation. This approximation transforms the Jean-Vlasov-Poisson equations into a 'free-flight' problem (equivalent to a Burgers equation with no viscosity). Its main conditions of applicability are i) non-crossing of the particles orbits; ii) equivalence of the Vlasov-Poisson and the hydrodynamical Euler description; iii) Lagrangian invariance of the gravity; iv) proportionality of the initial velocity and gravitational acceleration (see [6]). Solutions of the Zeldovich approximation can be obtained by a simple mapping between initial (Lagrangian) position and present (Eulerian) position. However, after some time, this mapping develops singularities, corresponding to situations where particles, thrown with large initial velocities, catch up with slower particles and overtake them. This situation corresponds to the formation of a caustic in the density field.

This situation is not very realistic. In a true Universe with self-gravitating particles, caustic are avoided by the mutual interactions, which modify locally the gravitational acceleration and the velocity of the particles. To take into account this effect (and also to remove the singularity

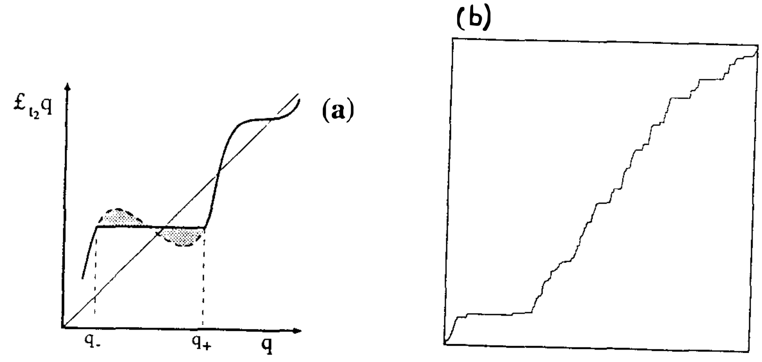

Figure 1: (a) Lagrangian map in the presence of a shock; (b) Lagrangian map for Brownian initial velocities ($h = 1/2$).

in the mapping), Gurbatov and Saichev [2] suggested to introduce an infinitely small viscosity, to mimick gravitational sticking. The caustics are then replaced by shocks, corresponding to formation of "clumps" in the density field (fig 1a), and the dynamics can then be described by a Burgers equation

$$\partial_t v + (v\nabla)v = \nu\nabla^2 v, \tag{1}$$

where $v(x,t)$ is the velocity field and ν is the velocity. In the limit $\nu \rightarrow 0$, the shape of the shocks is independent of the details of the dissipation mechanism, and exact solutions of the Burgers equation can be obtained via Legendre transforms (see [6]). This allows both analytical predictions on the behaviour of the solutions, via probabilistic tools developed by Sinai [5], and high resolution numerical solutions, thanks to an efficient Legendre transform algorithm developed by Noullez and Vergassola [7]. All the simulations presented here were performed on a Sun workstation, using the Fast Legendre Transform algorithm.

In the cosmological context, the initial density field is often taken as scale free, with a power law power spectrum of index n. This corresponds in the adhesion approximation to scaling initial velocities:

$$v_0(x + \lambda l) - v_0(x) = \lambda^h (v_0(x + l) - v_0(x)), \tag{2}$$

where h is the scaling exponent, so that, in a d-dimensional space, $n + d = 2(1 - h)$. With such scaling initial conditions, shocks occur everywhere, and the Lagrangian mapping initial $\rightarrow$ present position takes a very special shape, shown in Fig.1b. This shape is characterized by successive increasing 'plateaus' of different size (corresponding to the shocks), which are organized in a self-similar way: successive zooms inside the structure always lead to the same structure. This self-similarity is the signature of a fractal behaviour. The structure seen in Fig. 1b is called a Devil's staircase. This is a non-decreasing function which varies only over a fractal set of zero measure. The most famous Devil's staircase is obtained by computing the mass of a Cantor set as a function of the position. The corresponding shock structure in the

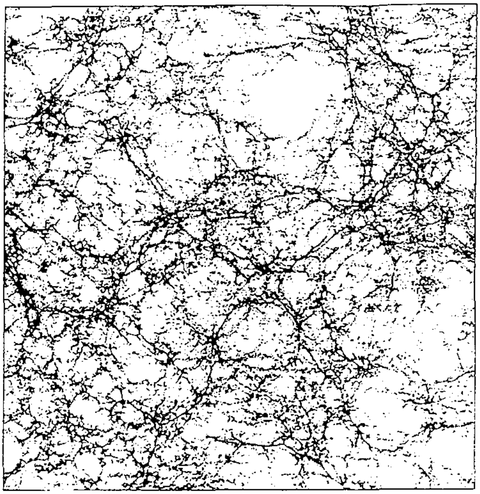

Figure 2: Mass density field in 2D for Brownian initial conditions ($h = 1/2$).

Eulerian space (at a given time), is depicted in Fig. 2, for a 2D simulation. A complex pattern of thin shock lines intersecting at nodes can be observed, reminiscent of what is observed in complete N-Body simulations.

Using these simulations, the mass function of the shocks $n(m)$ was computed for various initial conditions (various h) (see fig. 3 for the 1D case). It was found to obey a simple scaling behaviour, depending on the dimension d and of the scaling exponent of the initial velocity field, h:

• at small masses, $n(m)$ follows a power-law:

$$n(m) \sim m^{-h-1}; \tag{3}$$

Note that the power-law is independent of the dimension. It might be due to the fact that the smallest masses belong to $d-1$ dimensional structures (nodes in 1D, shock lines or walls in 2D or 3D), so that the dynamics transverse to the shock is 1D in all cases. This is however a mere speculation.

• at large masses, $n(m)$ is a streched exponential:

$$\log(n(m)) \sim -m^{-2(1-h)/d}. \tag{4}$$

In that case, the result depends on the dimension. This is because strong shocks are made of nodes in any dimension, which correspond to d-dimensional polyhedra of mass m and length $L \sim m^{1/d}$ in the Lagrangian space.

The scaling properties of the mass function in the adhesion approximation, reminiscent of what is observed in the real Universe, is actually the signature of the Devil's staircase appearing in the Lagrangian map. The (analytical) proof of this result requires complex probability tools and can be found in [6]. A heuristic insight of the result can however be given, if one accepts that initial conditions with scaling exponent h give rise to a Devil's staircase of dimension h in the Lagrangian map. The underlying Cantor set, obtained by removing recursively smaller and smaller intervals, is then of dimension $d_F = h$. Since the Lagrangian map increases only

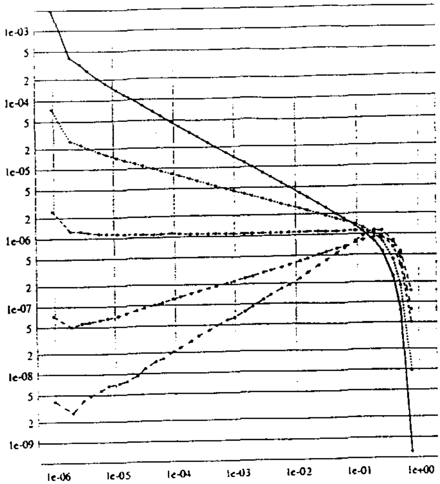

Figure 3: Cumulated mass function for 1D solutions of the adhesion models. The scaling exponent range from $h = 1/2$ to $h = -1/2$ from top to bottom, in steps of 0.25. Observe the wide range of scales, which has the exponent $-h$.

at regular points and remains constant at the location of shock points, the regular points correspond to the points in the Cantor set, while shocks correspond to the removed intervals. A well known result in fractal theory is that the distribution of the removed intervals (number of removed intervals of size l) follows a power-law $N(l) \sim l^{-d_F}$, where d_F is the dimension of the corresponding Cantor set. Since the mass was initially uniformly distributed over the space, the length of the removed interval (shock) is directly proportional to its mass. This means that the cumulated mass distribution obeys $N(m) \sim m^{-d_F}$, or, since $d_F = h$ and $n(m) = dN/dm$, $n(m) \sim m^{-h-1}$.

The present study provides a complete understanding of the mass function in the adhesion model. It is interesting to compare our results with the outcome of a heuristic theory of structure formation by Press and Schechter [4], which predicts a mass function:

$$n(m) \sim m^{-2+(1-h)/d} \exp\Big(-(\frac{m}{m_*})^{2(1-h)/d}\Big). \qquad (5)$$

We find therefore that the Press and Schechter and the adhesion model predictions agree in their functional form at large masses for any space dimension, but agree only in one dimension at small masses. The reasons for this agreement/discrepancy are not firmly established yet. We may note that Press and Schechter theory is based on isotropic collapse, which may be adequate to describe node (large mass) formation, but not structures with small masses, which are formed by a compression in one direction. Also, Press and Schechter theory is based on local collapse arguments, leading to the so-called "cloud in cloud problem": one cannot discreminate between real strutures, or structures which appear locally bounded but which actually belong to a larger structure. In the adhesion model, the theory is based on conditions of non-collapse of an extended halo, which avoids this problem.

It is yet not clear why Press and Schechter theory has been so succesfull in explaining mass functions derived from real N-Body experiments. Maybe the structure formation in N-Body simulations is more isotropic and more local than in the adhesion model, and so closer to the Press and Schechter assumptions. Maybe it is just an artefact of the N-Body mass function definition (based on friend of friend algorithm) which does not coincide with the adhesion definition, based on the Lagrangian map.

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