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The Physics & Mathematics of Microstates in String Theory

and a Monstrous Farey Tail

UvA Dissertation by
Paul de Lange

Paul de Lange   The Physics and Mathematics of Microstates in String Theory

Microstates Moonshine Monster

Paul de Lange

A dissertation that delves into physical and mathematical aspects of string theory. In the first part ot this book, microscopic porperties of string theoretic black holes are investigated. The second part is concerned with the moonshine phenomenon. The theory of generalized umbral moonshine is developed. Also, concepts in monstrous moonshine are used to study the entropy of certain two dimensional conformal field theories.

Paul de Lange (1988) is born in ‘s-Gravenhagen. He obtained his BSc in mathematics and physics and his MSc in theoretical physics at the Universiteit van Amsterdam. In 2016 he defended this PhD thesis in Amsterdam.

The Physics and Mathematics of Microstates in String Theory

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The Physics and Mathematics of Microstates in String Theory

and a Monstrous Farey Tail

ACADEMISCH PROEFSCHRIFT

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op gezag van de Rector Magnificus
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ingestelde commissie,
in het openbaar te verdedigen in de Aula der Universiteit
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door

Paul de Lange

geboren te 's-Gravenhage

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Publications

This thesis is based on the following publications:

Structure of Six-Dimensional Microstate Geometries
Paul de Lange, Daniel Mayerson, Bert Vercnocke
JHEP **09** 075 (2015) [dLMV15]

Symmetric Products and AdS_3/CFT_2 in the Grand Canonical Ensemble
Paul de lange, Alex Maloney, Erik Verlinde
to appear (2016) [dLMV16]

Generalized Umbral Moonshine
Miranda Cheng, Paul de Lange, Daniel Whalen
to appear (2016) [CdLW16]

Quantum Quivers and Melting Molecules
Dionysios Anninos, Tarek Anous, Paul de Lange,
George Konstantinidis
JHEP **03** 066 (2015) [AAdLK15]

Voor mijn ouders

What's puzzling you is the nature of my game

The Rolling Stones
Sympathy for the Devil

Contents

How to read. And a word on notation.

This dissertation is divided into four chapters. Each chapter builds on a publication. However, the text in the dissertation is not restricted to these publications. As a whole, this dissertation can be read as four distinct but not perpendicular angles on the study of microscopic degrees of freedom in string theory.

There are two ways to read the cohesion of this work. The first two and the final chapter can be considered as the physics, as referred to in title of the dissertation. The third chapter covers mathematical topics. Only the final chapter relies explicitly on results obtained in chapter three.

Another way to interpret the structure of this work is to view chapters one and two as the first part, encompassing the study of black hole microstates. The final two chapters can then be seen as a second part, spanning different manifestations of the moonshine phenomenon in string theory.

Each chapter is preceded by a brief and general introduction that is directed to the novice in the subject matter. Also, the preface is directed to the most general audience and the reader, whatever his background may be, is warmly welcomed te read this invitation to this dissertation. The main body of the chapters is of a technical nature and knowledge of string theory is presumed in formulating the content.

Finally, a word on notation is in order. Natural units are used for the speed of light, Planck's constant and Boltzmann's constant: $c = 1$, $\hbar = 1$, $k_B = 1$, but not for Newton's constant. This choice was made because many different dimensions are considered in this work, and we want to keep track of this by writing the D-dimensional Newton's constant as G_D.

The author aimed at using a consistent notation throughout this dissertation. As it covers both topics in physics and mathematics, this may occasionally result in irregular notation. Most notably J.P. Serre's convention is adapted to denote fields in boldface instead of blackboard script. Hence, the fields $\mathbb{F}_q$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ will be denoted by $\mathbf{F}_q$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$. We also use boldface for rings, e.g. $\mathbf{Z}$ for the ring of integers and $\mathbf{Z}_p$ for the ring $(\mathbb{Z}/p\mathbb{Z})^*$. The finite group $\mathbb{Z}/n\mathbb{Z}$ is also denoted by $\mathbf{Z}_n$, and sometimes even abbreviated to $\mathbf{Z}_n = n$. Technically this choice is made to denote the monster group in the conventional blackboard script, $\mathbb{M}$, without confusing it for a field. But also because the author agrees with J.P. Serre in that blackboard script is a boldface script for the blackboard, and not strictly necessary à propos to printed text.

Preface

Suppose — suppose — the end of the world is near, and whatever your fate may be — a colossal comet, nuclear annihilation, a virulent virus — you have reconciled with the depressing fact that the eradication of mankind is inevitable. And imagine that in this horrible scenario you get the chance to, in our Final Day, write down one idea, one piece of information in one sentence, and store it in a vault for the next intelligent generation to be found. What then would you pass on to these archaeologists of the distant future? What single idea is so simple and slim that it can be expressed in a single sentence, while being deep enough to expose the complexity and intelligence that characterizes contemporary mankind?
When confronted with this happily hypothetic conundrum, the physicist Richard P. Feynman opted for the idea that

"... all things are made of atoms — little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another."

He commented that

"In that one sentence, you will see, there is an enormous amount of information about the world, if just a little imagination and thinking are applied."

Feynman's imagination was rather exceptional and his thinking notoriously deep, but it is true that the "atomic hypothesis" is remarkably fruitful and abundant in conceptual consequences. It is the fundamental premise of chemistry, and it teaches us why and how substances react with each other to form new types of matter. It clarifies and classifies the diversity of matter we encounter in Nature. It is through atoms and molecules that we understand why water freezes and evaporates, why substances in general change phases at certain temperatures, and why matter expands and compresses as the temperature is changed. Even the notion of temperature itself radically changes when we accept that matter is made of atoms.

Despite drifting around in philosophy for thousands of years, it was only until the early 19^{th} century that the atom got promoted to a scientific concept, when Dalton used it to explain the "law of multiple proportions" (the observation that elements react in ratios of small integers), and the first periodic table was published in 1869. That this took a while is not that surprising: the length scale that is associated with a typical atom is 10^{-9} meter, and on scales well above that, say that of the centimeter and up, for a lot of phenomena in nature we do not need the atoms to describe them accurately. Kepler's laws of planetary motion, Newton's laws of mechanics and Bernouilli's equation of fluid dynamics are all independent of atomic knowledge. They describe physics at a vastly larger scale than that of the atom and it is only at the small scale that these laws start being inaccurate. Vice versa, however, we can understand these theories that apply at the larger scales as being laws that originate from the microscopics, and it is by averaging over the small scale physics that we recover the large scale physics. Unfortunately, averaging is an irreversible procedure. Sole knowledge of the average, or "effective" laws gives us little, if any, insight in the microscopics. We do realize that we can only be glad that such effective or average theories do exist. To describe, say, the flow of water, we certainly do not want to keep track of al the atoms that constitute this liquid. A liter of water is made up of more than 10^{25} H_2O molecules.

Occasionally even at large scales, such "effective" descriptions of the phenomena can harbor signs of their own underlying microscopic description. Phase transitions of fluids are not described by the theory of Bernouilli and can only be understood through knowledge of the microscopic nature of the substance, and the spectrum of blackbody radiation requires a fundamental, quantized energy-scale. Another example can be found in the anomalous behavior of Mercury around its perihelion, that was in its time already a hint that the theories of Newton and Kepler were not complete. It was Einstein's theory of General Relativity that cured this discrepancy.

Where by now we understand that matter is made of atoms, and atoms are made of quarks and electrons, gravity has always been very proficient in hiding its true microscopic nature. From a conceptual and fundamental point of view there has to be such a micro- or quantum-gravity, as the absence of a quantum picture of gravity would violate some deep principles within quantum mechanics. But also, akin of the theory of fluids, occasionally, gravity gives us subtle hints regarding its own microscopic nature.

Most famously, Einstein's theory of gravity predicts the existence of black holes: massive, dense objects with a gravitational pull so extreme not even light can escape them. Today, black holes surpassed the status of theoretic constructs and there are plentiful up in the heavens. After depleting their fuel, a massive star will collapse into a black hole, and we have many reasons to believe that Sagitarrius A*, sitting in center of our very own Milky Way, is in fact a massive black hole.

Black holes are large scale structures. They are surrounded by a large "horizon" — a surface beyond which there is no return — that is of the order of its own mass, and for a ballpark figure: the estimated horizon radius of Sagitarrius A* is about 20 times the radius of the sun. That black holes are surrounded by this obscuring horizon makes them inherently hard — if not impossible — to observe directly. Still, today no one dares to claim what exactly goes on behind the horizon.

A first step towards the "atomic" description of a black hole was made by Stephen Hawking. In a famous computation Hawking showed that black holes are in fact not entirely black, but rather a very dark purple. At the very edge of the horizon they radiate weakly, and due to this radiation we can associate a specific temperature with a black hole. It is as if — albeit very dim and weakly — black holes leak some of the information they hide behind their horizon away to their distant observers. Admittedly, the radiation is so weak and the associated "Hawking temperature" so low, that Hawking's observation is of a rather unpractical, unobservable nature, but its conceptual consequences can not be overestimated.

Explicitly, Hawking's result states that the temperature of a black hole is inversely proportional to its mass: if we increase the mass of a black hole by a factor of two, the temperature decreases by a factor of one half. Smaller black holes are hotter. Hawking's result is very universal, and the derivation does not rely on too many details of the black hole in question. This means that whatever microscopic description of black holes and gravity one cooks up, it has to reproduce Hawking's result to be viable.

We can compare this situation with the more mundane study of liquids and gasses, where the concept of temperature is intimately related with a microscopic description of the substance. Intuitively we may think of temperature as a measure of how rocky the molecules of a gas are wiggling in space. Much like in the case of these gasses, the Hawking temperature provides us a window into the black hole microscopics. In fact, contemporaries of Hawking were quick to show that from the temperature we can derive how many of these black holes "atoms" or microstates there should be. This led to a surprising conclusion. The intuition of the physicist would lead him to predict that the number of microstates should depend on the *volume* of a system. However, it was shown that the amount of microstates of a black hole depends only on the *area* of the horizon. So the challenge set by these results is that for a theory of quantum gravity — a microscopic theory of gravity — to be feasibly, it has to reproduce this peculiar result that the number of microstates of a black hole is a function of its area only.

This is a dissertation in string theory. And string theory is a quantum theory of gravity. Among its fundamental constituents are strings — tiny objects that have a length and a tension. Much like the string of a violin, these strings can be excited, and in string theory it is one of these excitations that represents the graviton: the microscopic, quantum particle of gravity.
Being a microscopic theory of gravity, it should pass the black hole "test": it has

to reproduce the peculiar fact that the number of microstates inside a black hole depends of its area only. And it is one of the milestones of string theory that indeed, for specific black holes that occur in the theory, string theory manages to explain what these microstates are, and counting these states one finds agreement with the formulae of Hawking and others. On the day this dissertation is printed string theory is the only theory of quantum of gravity that can claim this success.

String theory does not answer all the questions about black holes and their microscopic description. For example, string theory does not yet very concretely explains the microscopic nature of realistic black holes, that is black holes in four dimensions such as Sagittarius A*. But even for the black holes where string theory has been successful in describing its microscopic nature, there are still many open questions, both of a technical and a conceptual nature.

1 Black holes and microstates

String theory is a theory of gravity. In its low energy limit it reduces to known theories of supergravity, most notably the ten dimensional supergravities of type IIA and type IIB. As such, the theory has black holes as solutions to its equations. But string theory is much more than a theory of gravity: beside the string and its graviton excitation, there are many elementary objects that are fundamental to understand the full theory. There are D-branes, orientifolds, exotic branes, and monopoles, to name but a few of these object. And it is in these objects that we, in some examples, can understand what the microscopic degrees of freedom are.

In a famous example, that is worked out explicitly in chapter I of this dissertation, a four dimensional supersymmetric black hole is constructed in string theory. Its microscopic degrees of freedom are understood to be due to a fivebrane and the degeneracy is computed as an index of this fivebrane as it warps itself around the compact, small dimensions. A general strategy to compute the entropy of this brane-system is to consider the two-dimensional conformal field theory living on the world-volume. In this two-dimensional field theory we can consider the partition function $Z(T)$ as a function of the temperature T. This function captures all the information about the number of states of the system and we can extract the behavior of the number of states at high temperature using Cardy's formula.

1.1 The Cardy formula

We want to know the entropy of the conformal field theory living on the world-volume of the brane so we can compare it with the black hole entropy. To this end we consider the partition function $Z(\tau)$ of the conformal field theory:

$$Z(\tau) = \sum_n d(n) e^{2\pi i E_n \tau}. \tag{1}$$

In this notation, $d(n)$ denotes the number of ways the system can arrange itself so that it has total energy E_n. It is convenient to introduce the notation $q = e^{2\pi i \tau}$ so that the partition function is $Z = Z(q) = \sum_n d(n) q^n$, and the degeneracy $d(n)$ can be read off as $d(n) = \frac{1}{2\pi i} \oint_\gamma Z(q) dq$.

A partition function is a tool we encounter in mathematics an physics alike. In physics, we use partition functions to describe statistical and thermal properties of a system with a large number of microscopic particles. The partition function $Z(\tau)$ packs information about the degeneracies at different energy scales and from it thermal properties such as internal heat, entropy and specific heat can be extracted, typically by taking suitable (logarithmic) derivatives of $Z(\tau)$.

In mathematics, we use the partition function in similar vain as a tool to package the degeneracies of some mathematical quantity at some index level. Let me specify this with the canonical example that will occur later in this dissertation.

Consider the following mathematical problem: Let $n \in \mathbb{N}$ be an integer. We can 'split' up n additively by writing it as the sum of smaller integers and we can typically do so in many ways:

$$4 = 1 + 1 + 1 + 1 = 1 + 3 = 2 + 1 + 1 = 2 + 2.$$

In this case there are four different ways of splitting up the number 4. We say there are four partitions of 4. Actually, we like to call 4 itself its own partition, the 'trivial' partition, so there are five in total. Now consider the more general question: how many partitions does a general integer n have. Let's denote the number of partitions of n by the symbol $p(n)$. We introduce a 'formal' symbol $x$and construct a function $F(x)$ — the partition function for the number of partitions — by taking the n^{th} power of x and multiplying this with $p(n)$, in the end summing over everything:

$$\begin{aligned} F(x) = \sum_n p(n) x^n &= p(0) + p(1)x + p(2)x^2 + p(3)x^3 + p(4)x^4 + \dots \\ &= 1 + 1 \cdot x + 2 \cdot x^2 + 3 \cdot x^3 + 5 \cdot x^4 + \dots \end{aligned}$$

It is convenient to introduce such a function $F(x)$ because in mathematics it is somewhat easier to manipulate functions than it is to manipulate lists of numbers. For example, using function analysis, we can now quite easily deduce how many partitions very large numbers have. The answer turns out to be:

$$p(n) \sim \exp\left(\pi \sqrt{\frac{2n}{3}}\right) \quad \text{as } n \to \infty \tag{2}$$

For the details of this derivation, see chapter III.

In physics, we can write a partition function as $Z = \sum_n d(n) q^n$ where $d(n)$ denotes the number of microscopic configurations corresponding to an energy E_n. Here $d(n)$ is a quantity of interest as it gives the entropy of the system: $S = \log d(n)$. Just like in the case of the partition of integers, we would like to use methods in functional analysis to estimate the entropy — the logarithm of the number of microstates — from this partition function.

For conformal field theories — the theories relevant for the microscopic counting of black holes in string theory — Cardy used just such techniques to compute this entropy, a famous result known as the Cardy entropy or the Cardy formula. Because it plays a rather central role in this thesis lets consider the derivation of the entropy of a conformal field theory and focus on the regime of its validity.

Consider a conformal field theory with a central charge c and a set of primary operators $\{\Phi_i\}$ of conformal dimension $D\Phi_i = \Delta_i \Phi_i$. We will be interested in the degeneracy $d(\Delta)$ of primary operators Φ having a fixed conformal dimension Δ. To that end we introduce the partition sum

$$Z(\tau) = \sum_\Delta d(\Delta) e^{2\pi i \tau(\Delta - c/24)} \tag{3}$$

The parameter τ is related to the temperature T as $\tau = i\beta/2\pi + \mu$ with $\beta = 1/T$. We will focus on the purely imaginary case, $\mu = 0$. The conformal invariance allows us to define the theory on a torus $\mathbb{T} = \mathbf{C}/(\mathbf{Z} \oplus \tau\mathbf{Z})$ and reparametrization invariance on the torus implies the modular invariance of $Z(\tau)$:

$$Z(\tau) = Z(-1/\tau) \tag{4}$$

The low temperature behavior of $Z(\tau)$ is determined by the ground state, that is the state with $\Delta = 0$, so that, exploiting the modular invariance in equation (4), the high temperature limit reads

$$\lim_{\tau \to 0} Z(\tau) = \exp\left(\frac{2\pi i}{\tau}\frac{c}{24}\right)(1 + \ldots)\,. \tag{5}$$

Taking a Laplace transform of the partition sum (3) we can compute the degeneracies

$$d(\Delta) = \int_{-i\infty}^{i\infty} d\tau e^{-2\pi i \tau(\Delta - c/24)} Z(\tau). \tag{6}$$

For high temperature we may plug in expression (5) to obtain

$$d(\Delta) = \int_{-i\infty}^{i\infty} d\tau \exp\left(-2\pi i \tau(\Delta - c/24) + \frac{2\pi i}{\tau}\frac{c}{24}\right)(1 + \ldots)\,. \tag{7}$$

For large Δ we may evaluate the formula (7) at the saddle point

$$\tau_*^2 = -\frac{c/24}{\Delta - c/24} \tag{8}$$

to obtain Cardy's formula

$$d(\Delta) = \exp 4\pi \sqrt{\frac{c}{24}\left(\Delta - \frac{c}{24}\right)}. \tag{9}$$

We understand that the derivation holds only for high temperature $\tau \to 0$ and small saddle point values for τ_*, that is for

$$\Delta \gg \frac{c}{12} \tag{10}$$

This will be important especially in chapter IV of this thesis.

In the example worked out in the first chapter — the four dimensional black hole in M-theory — the Cardy formula of the conformal field theory on the brane world-volume matches the macroscopic area formula computed on the gravitational side. This result, first obtained by [SV96] (the example we work it is based on [MSW97]), is one of the benchmark successes of string theory.
So string theory manages, for a specific genre of black holes (black holes with supersymmetry), to account for the entropy by computing the Cardy entropy of the conformal field theory living on the brane world-volume. Albeit being a very important, pretty and satisfactory answers, it leaves a lot of questions about black holes — even the string theoretic, supersymmetric ones — unaddressed. This dissertation delves into some of the technical and conceptual conundrums that the string theoretic account of black holes entropy gives us. We lay out four different lines of flight in this dissertation, each chapter devoted to such a line. The questions that are addressed in the chapters are laid out in the following section.

2 Four questions, four chapters

This thesis consists of four chapters, each aiming at addressing a question that directly or indirectly relates to (black hole) microscopics in string theory.

2.1 Gravitational microstates and fuzzballs

String theory reduces to familiar theories of supergravity in its low energy limit. In this low energy reduction, we find black hole solutions. To suppress quantum gravitational effects, we need to take the string coupling, g_s, large: $g_s \gg 1$. It is in this regime that we can really trust the black hole solution and its corresponding area entropy formula.

However, to get into the regime where we can compute the microscopic entropy — the regime where we can reasonably describe the relevant physics in terms of a conformal field theory and use Cardy's formula — we should consider string theory at very small string coupling, $g_s \ll 1$. At this regime in the moduli space, the string length is larger than the Schwarzchild radius and the black hole physics will be subject to large, stringy corrections. The fact that the microscopic entropy still matches the macroscopic area formula can be understood by the fact that in the field theory we are computing a supersymmetric index that is topologically protected and is constant on the moduli space.

Still, this begs the following question: consider, at very small string coupling, one of these microscopic degrees of freedom in the field theory. Slowly crank up the coupling constant g_s to adiabatically end up in the gravitational regime, where everything is described in terms of supergravity. What then, in this regime, should be the representation or interpretation of this microstate. Where does it fit inside the supergraviational theory?

Sameer Mathur proposed that in the gravitational regime, the microscopic degree of freedom is a *fuzzball*: a solution to Einstein's equation that asymptotically looks like the black hole solution it is the microstate of, without having a horizon or a singularity. Mathur proposed that around the horizon there are as many of these fuzzballs as the macroscopic area formula for black holes computes.

In the first chapter we show how such microstate geometries can support the mass of the black hole they constitute. We show how a no-go theorem, preventing such smooth horizonless solutions to support a non-zero mass, can be circumvented in higher dimensions by taking into account non-trivial topology on the higher dimensional space-time manifold. We elucidate this principle by going over a wide variety of microstate geometries in six dimensions — supersymmetric and non-supersymmetric — and show how the non-trivial topology can support the non-zero mass for these black holes.

2.2 Conformal quantum mechanics

The Cardy formula — the formula that is ultimately used to count the mircroscopic degrees of freedom of string theoretic black holes at weak string coupling — is a formula that counts the number of primary operators of large conformal dimension in a two-dimensional conformal field theory. In the spirit of AdS_{D+1}/CFT_D dualities [Mal99, GKP98, Wit98a], in the successful cases where the Cardy formula does count black hole microstates, such black holes have a three-dimensional AdS_3 factor in their near-horizon geometry. Even in the cases where the near horizon geometry has an AdS_2 factor, this factor can typically be seen as the base of some Hopf-fibration that lifts to the same AdS_3 factor, and we can still apply the techniques of two-dimensional conformal field theory.

However, there are examples in string theory of black holes with an AdS_2 factor in their near-horizon geometry, where it is not at all clear how to uplift such solutions to a larger AdS_3 structure. When we compactify M-theory on a generic Calabi-Yau three-fold with no $U(1)$ isometry, it is not clear at all if this is possible. In these cases accounting for the black hole microscopics in terms of a field theory has faced problems indeed.

Rather that to fit the AdS_2 structure inside an AdS_3 space, we consider applying the AdS_{D+1}/CFT_D philosophy directly to the AdS_2 space. The field theory then, following the dictionary of AdS/CFT, should be one-dimensional, hence a (super-symmetric) quantum mechanics. Inspired by brane constructions and the works [DM96, Den02] we consider quiver quantum mechanics. In the second chapter we show for a class of quiver quantum mechanical systems that arise from D-brane set-ups that they have a conformal symmetry in a rather non-trivial fashion. We also comment on the stability of the vacua of the solution space against thermal perturbations, thus testing if these ground states could account for non-supersymmetric microstates.

2.3 Patterns in the tail of Cardy's formula

The Cardy formula is an expression for the high energy degeneracies of conformal primaries in a two-dimensional conformal field theory. One formulates the partition sum $Z = \sum d(n)q^n$ of the conformal field theory in question, and Cardy's formula estimates $d(n)$ for large n. In its form of euqation (**??**), it is an approximation, with corrections of order $n^{-1/2}$, and one may wonder if there are better approximations that take into account next-to leading order contributions in n as well.

Conformal field theories have a huge number of symmetries. In fact, being defined on a torus, conformal field theoretic partition functions enjoy modular invariance. The modular group is an infinite dimensional group, and this infinite number of symmetries can be exploited to not just improve the accuracy of Cardy's result, but in fact render it into an exact expression. In the mathematics literature this was already known and worked out by Rademacher in e.g. [Rad38]. With Rademacher's result we can, from rather minimal information about a conformal field theory, exactly compute all the degeneracies $d(n)$.

The partition function harbors thermal and entropic information. But it can also be interpreted from a Hamiltonian point of view as a trace over the Hilbert space $\mathcal{H}$ of the system:

$$Z = \mathrm{Tr}_{\mathcal{H}} q^H. \tag{11}$$

From this point of view, the coefficients $d(n)$ enclose information of the Hilbert space. For example, if some group G acts on the Hilbert space, than the partition function organizes itself in the irreducible representations of G, and the coefficients $d(n)$ contain information about the dimensions of these irreducible representations.

When compactifying string theory on the K3 manifold — the only non-trivial Calabi-Yau two-fold — the physics at small string coupling is captured by a modified form of the partition function $\mathcal{Z}(\tau, z)$, a form that takes into account a conserved charge in the supersymmetric algebra. Although technically a bit more intricate, this modified partition function, called the elliptic genus, is in many ways like a partition function. Notably, it enjoys a form of modular invariance. And just like in the case of modular functions, there is an analogue of

Rademacher's result in this case. We expand the function $\mathcal{Z}(\tau, z)$:

$$\mathcal{Z}(\tau, z) = \left(\sum D(n)q^n\right)\mathcal{F}(\tau, z) + \mathcal{G}(\tau, z) \tag{12}$$

where we used some techniques in the theory of such elliptic genera (we refer to chapter III for the technical details, where for example the functions $\mathcal{F}$ and $\mathcal{G}$ are given explicitly). Upon inspection of the coefficients $D(n)$, for example with Rademacher's technique, we recognize the dimensions of irreducible representations of a large but finite group one would not at all a priori expect to play a role in the physics. This coincidence — the relation between a modular object and a large finite group — is very surprising and so historically referred to as 'moonshine'.

In the mathematical community there was, already in 70's, an example known of such 'moonshine'. In this famous monstrous moonshine phenomenon, the modular object was a bit simpler, and the finite group way larger and more complicated. There, the observation led to a conjecture — the monstrous monshine conjecture — that was ultimately proven by Borcherds, an effort awarded with the Fields medal.

In the third chapter of this thesis we review monstrous moonshine, give a brief summary of how Borcherds came to his proof, and discuss some more examples of occurrences of such moonshinesque behavior in functions we encounter in string theory. We consider a generalized moonshine behavior and give evidence for generalized moonshine for a list of groups called the umbral groups. We also briefly mention the physical relevance by highlighting a connection with the Farey tail — a gravitational interpretation of the Rademacher expansions of such elliptic genera.

2.4 The Cardy regime for the BTZ black hole

The black holes for which string theory has been successful in counting the microscopic degrees of freedom are notably black hole solutions in four ([MSW97]) and five ([SV96]) dimensions.

In three dimensions, Einstein's theory of gravity is trivial; there is no propagating degree of freedom. Still, surprisingly, this theory has a black hole solution: the BTZ black hole [BTZ92]. Being of a rather topological nature, it still obeys the famous result that its entropy is proportional to its area, and for these black holes we would like to know what the microscopic degrees of freedom really are.

A big hint towards the fact that the BTZ macroscopic entropy really is counting something is that, once again, the entropy can be naturally cast in the form of a Cardy formula. This hints towards the existence of some conformal field theory that is the effective field theory of some brane set-up.

That we find a Cardy like formula is rather surprising for the following reason: the BTZ black hole entropy can be cast in the form of a Cardy formula only at temperatures of the order of one, $T \sim 1$, wheres the Cardy formula only applies to very high energies, hence high temperatures $T \gg 1$ (known as the Cardy regime).

In the final chapter we investigate this discrepancy in limits. Taking up ideas of Witten from his revisitation of three-dimensional gravity [Wit07], we consider a rather putative conformal field theory that has a lot of the desired properties one needs for the field theory to be a candidate dual of three dimensional gravity. This conformal field theory is the symmetric product orbifold theory of the monster conformal field theory. The monster theory we already encounter in chapter III in the context of monstrous moonshine, and in the final chapter we use some techniques developed in the third chapter to show how in the context of three dimensional gravity, the Cardy regime should really not be the high energy expansions (or high temperature limit) but rather the limit of large central charge — the limit where quantum gravitational effects on the horizon of the black hole can be neglected. This is also the limit we would take in a bona fide AdS/CFT scenario.

In arguing for the large central charge regime, we develop a framework where we introduce a conjugate variable for the central charge in the theory space of symmetric orbifolds. We interpret this chemical potential rather physically and consider the grand canonical ensemble of these theories. We also comment on the phase transitions we encounter as we vary both the temperature and this new chemical potential.

I

The Fuzz about Black Holes

The laws we use to describe Nature depend on the scale at which we do experiments. If we want to write down the laws governing planetary motion in our solar system, Schrödinger's equation — the equation that describes the evolution of (sub)atomic particles — will not appear on the blackboard. And vice versa, Hubble's law — the law that describes the expansion of the universe — will not be taught in chemistry class.

Another example is the study of water, where a scale is set by the temperature. At low temperature where water is in its solid state (ice), we use the physics of lattices and crystals. At room temperature we use the hydrodynamic differential equations to describe the streaming of water and at high temperatures we use the statistical and thermal physics of gasses.

At very high temperature we describe water in terms of a microscopic degree of freedom: the H_2O molecule. The liquid phase and its hydrodynamics is an effective description of a large number of these molecules. The molecules are still there, but their bulk behavior is described by laws and equations that differ from those at high temperature: different scales, different physics.

The same principle hold in the study of the universe. We use Newton's and Kepler's laws to study the planetary motions, but to understand the expansion of the universe, or motion in strong gravitational fields, we need the laws of Einstein. Newton's laws are approximations to Einstein's equations, and apply only at different scales namely in weak gravitational fields.

The Einstein equation,

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 8\pi G_D T_{\mu\nu} \tag{1}$$

is a differential equation for the metric $g_{\mu\nu}$. Schwarzschild famously found the spherical solutions in vacuum that would lead to the notion of black holes: massive and dense objects that allow for a surface outside its matter distribution, such that anything crossing this surface — the horizon — would never come back out again. We still lack a complete picture of what a black hole precisely is, and what is going on behind this obscure horizon. What prevents us from really understanding black holes is that we have no definite microscopic, fundamental description of a strong gravitational field. Its like we're studying water, without having knowledge of the H_2O molecule.

As was told in the preface of this dissertation, the first leap towards such a microscopic understanding of strong gravitational fields and black holes was made by Stephen Hawking in [Haw75] . With a computation in quantum field theory in curved space-time he showed that black holes are actually not totally black, but radiate particles at the horizon, inducing a temperature — the Hawking temperature — that is inversely proportional to M, the black hole mass:

$$T_H = \frac{1}{8\pi M G_D}. \tag{2}$$

In thermal physics, temperature is intimately related with the amount of microscopic information there is availably in a closed system. In fact, in a closed system of fixed volume, the first law of thermodynamics states that the change in internal energy U is proportional to the change in entropy S, and that the constant of proportionality is the temperature:

$$dU = TdS. \tag{3}$$

Integrating this relation with respect to the energy gives us an expression for the entropy in terms of the temperature. There is another interpretation of entropy due to Boltzmann, and that is in terms of the number of ways the microscopic degrees of freedom can occupy some closed system. Or rather, the logarithm thereof, as: $S = k_b \log N$. The number N can really be thought of as the number of degrees of freedom a system has. Hence knowing the entropy grants us direct information about the microscopics of the system.

Integrating the first law of thermodynamics in the context of black holes, teaches us that the entropy of a black hole is proportional to A, its area, as was noted by Bekenstein in [Bek73]. Hawking's result allows us to fix the constant of proportionality, to obtain the celebrated formula

$$S = \frac{A}{4G_D}, \tag{4}$$

now known as the Bekenstein-Hawking entropy, $S_{B.H.}$.

This is a very big hint towards finding a microscopic description of strong gravitational fields. Whatever microscopic degree of freedom we suggest, its entropy should be proportional to the area.

String theory offers such a microscopic picture of this black hole entropy. It is one of the mile stone success of string theory that it not only proposes a microscopic degree of freedom that accounts for the entropy, but it also matches the universal relation by Bekenstein and Hawking.

Being a huge success, it does not solve all the questions about the microscopics. Going back to the analogy with water, understanding the molecular structure of water is one thing, but still we would like to understand how the hydrodynamic equations of water come about and what the role of the molecules is at is at room temperature.

We start this chapter with showing a beautiful example of a microscopic account in string theory, of actually M-theory. Later on we address the following issue: starting with the string theoretic microstates of a black hole at weak coupling, how do we understand these microstates as we turn up the coupling, eventually ending up in the supergravity regime.

Taking a lead of the fuzzball program (see [Mat05] for a review) — the program that proposes the interpretation of these microstates in terms of smooth horizonless solutions to Einstein's equations — we set out to show how in the context of six-dimensional supergravity. This is a very natural arena for the important class of three-charge black holes. We show how these solutions can support the mass of the black hole in question. But first we briefly go through the famous derivation of black hole microstates in string theory and introduce the concept of microstate geometries.

1 Black hole microstates in M-theory

In the low energy limit, string and M-theory are effectively described by supergravity ([GSW88a, GSW88b, Pol07a, Pol07b] is the canonical literature). For string theory, the description is in terms of a ten dimensional supergravity of type-IIA or type-IIB. For M-theory the low energy effective theory is in terms of eleven dimensional supergravity. We can make contact with lower dimensional physics by compactifying these theories on manifolds, leaving us with lower dimensional theories of supergravity, coupled to matter content, depending on the manifold of compactification. As most theories of gravity, these lower dimensional theories admit black hole (or black brane) solutions that have a corresponding Bekenstein-Hawking entropy S_{BH}. At this point, the manifold of compactification looks as if its only role is to get rid of superfluous dimensions. This is not the case: if we want to identify the S_{BH} with a microscopic entropy, we must search within this manifold. Typically, a system of branes nests itself inside the compactification manifold. On these branes we can study the world-volume, that typically has an effective description in terms of a conformal field theory in two dimensions. We then can use Cardy's formula to identify a statistical entropy with this CFT. We will show an example where this statistical entropy matches — in the appropriate regimes — the Bekenstein-Hawking entropy S_{BH}.

1.1 Black holes in M-theory

We will show an example of microstate counting in string theory. We treat the case of extremal black hole solutions in $\mathcal{N} = 2$ supergravity in four dimensions [MSW97]. This theory finds a realization in compactified M-theory, where we can compute the number of microstates by counting the index of wrapped M5-branes. But first we compute the Bekenstein-Hawking entropy of the extremal black hole solutions.

Macroscopics

In the low energy limit, M-theory is effectively described by eleven dimensional supergravity. We make contact with four dimensional physics by compactifying this theory on $M \times S^1$. Picking M to be a Calabi-Yau three-fold $M = \mathrm{CY}_3$, we are left with $\mathcal{N} = 2$, $d = 4$ supergravity on the noncompact $\mathbf{R}^4$ [GSW88b]. In this theory we will be considering two types of multiplets, the bosonic content of which is:

- ◇ A gravity multiplet $\left(g_{\mu\nu}, X^0_\mu\right)$
- ◇ Vector multiplets $\left(X^\alpha_\mu, z^\alpha\right)$, $\alpha = 1 \dots n_v$.

The number n_v depens on the specific type of Calabi-Yau. In fact, $n_v = h^{1,1}(X)$, where $h^{1,1}(X)$ is the Hodge-number of $(1,1)$-forms on X.
Writing $F^I_{\mu\nu}$ for the field-strengths ($I = 0, \dots, n_v$), we can write down the four dimensional effective action

$$I = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \Bigg(R + 2g_{\alpha\beta} \partial^\mu z^a \partial_\mu z^b + \frac{1}{4} \left(\mathcal{N}_{AB} F^{+A} \wedge F^{+B} - \bar{\mathcal{N}}_{AB} F^{-A} \wedge F^{-B} \right) \Bigg) \tag{5}$$

where we have split F^I into F^{+I} and F^{-I} ($I = 0 \dots n_v$), its self and anti-self dual parts, respectively. The moduli space parameterized by $z = \{z^\alpha\}$ is equipped with the metric $g_{\alpha\beta}(z, \bar{z})$ and has the structure of a special Kähler manifold [Gre96].
The theory admits extremal charged black hole solutions that have a global $\mathrm{AdS}_2 \times S^2$ topology, and a metric:

$$ds^2 = -\frac{r^2}{|Z|^2} dt^2 + |Z|^2 \left(\frac{dr^2}{r^2} + r^2 d\Omega^2 \right) \tag{6}$$

The function $|Z|$ is the central charge and equals for extremal black holes the mass of the black hole: $|Z| = M$. It also relates to the area as $A = 4\pi|Z|^2$ so that

$$S_{BH} = \pi |Z|^2 \tag{7}$$

Much like in the case of the Reissner-Nördstrom black hole, where in the extremal case $M^2 = q^2 + p^2$, in the $\mathcal{N} = 2$ caes, Z is a function of all the magnetic and electric carges (p^A, q_A), where

$$p^A = \frac{1}{4\pi} \int_{\Sigma_2} F^A \tag{8}$$

$$q_A = \frac{1}{4\pi} \int_{\Sigma_2} G_A \tag{9}$$

where $G_A = \text{Im} \star F_I + \text{Re} F_I$, $F_I = \mathcal{N}_{IJ} F^J$. The Dirac-Schwinger-Zwanziger quantization condition dictates that $p \cdot q \in \mathbf{Z}$.
In principal the central charge could've depended on the scalar moduli z as well. However, the scalars are fixed at the horizon in terms of the charges by the attractor mechanism [FKS95]. We will express the central charge as follows. First we will express $|Z|$ as a function of both the electric and magnetic charges and the scalar fields. Then we solve for the scalars in terms of the charges and so find an expression for $|Z| = |Z(p^A, q_A)|$. We first need to introduce some tools in special geometry [Str90].
The $\mathcal{N} = 2$ d=4 supergravity is special because the scalar fields define a special Kähler manifold (see e.g. [FVP12]). It is convenient to parameterize the scalar z^α in terms of X^A, $A = 0 \dots n_v$, with the constraint

$$N_{AB} X^A \bar{X}^B = -1/G_4 = -e^{-K(X,\bar{X})} \tag{10}$$

The function $K = K(X, \bar{X})$ is tha Kähler potential that induces that ω, the Kähler form: $\omega = \partial\bar{\partial} K$, that in turn defines the metric on the moduli space, $i\omega_{\alpha\beta} = g_{\alpha\beta}$. A central role in special geometry is played by the prepotential $F(X)$:

$$F(X) = -\frac{d_{abc} X^a X^b X^c}{6X^0} \tag{11}$$

where $d_{abc} = \int_{\text{CY}_3} \alpha^a \wedge \alpha^b \wedge \alpha^c$ is the triple intersection form on CY_3, $\{\alpha^a\}$ an integral basis of $H^2(\text{CY}_3, \mathbf{Z})$ (or $H^{1,1}_{\bar{\partial}}(\text{CY}_3)$). This prepotential allows a formulation of the vector matter action in superspace, $I = I_g + I_v$, $I_v = \text{Im} \int d^4x d^4\theta F(X)$. We also introduce the derived quantities $F_A = \partial_A F(X)$, $F_{AB} = \partial_A \partial_B F(X)$. Now the doublet (X^A, F_A) is a symplectic section and the equations of motions are invariant under $\text{Sp}(2n_v + 2)$ rotations of this doublet. With these notions we can conveniently write the central charge as

$$|Z|^2 = Y^a q_a - F_A(Y) p^a, \tag{12}$$

where $Y^A = \bar{Z} e^{K/2} X^A$. The attractor equations that follow from supersymmetry are

$$p^A = 2\text{Re}(iY^A), \tag{13}$$

$$q_A = 2\text{Re}(i\bar{Z} e^{K/2} F_A). \tag{14}$$

Solving these equations for the scalar moduli and plugging them into the expression for the central charge leads to the compact expression for the entropy

$$S_{BH} = 2\pi \sqrt{|D| \hat{q}_0} \tag{15}$$

where we used the abbreviations $|D| = d_{abc}p^a p^b p^c$, $\hat{q}_0 = q_0 - \frac{1}{2}q^2$, $d_{abc}p^c = -\frac{q_a q_b}{6q^2}$. Demanding weak curvature of the solution requires

$$|D| \gg |M|, \tag{16}$$

$|M|$ the volume of the CY_3.

Microscopics

We computed the Bekenstein-Hawking entropy of extremal $\mathcal{N}=2$ black holes in four dimensions. Here we will show how to account for this entropy microscopically in M-theory.

In M-theory, (black hole) configurations with electric and magnetic charge vector (p^0, p^A, q_0, q_A) are obtained by wrapping an M5-brane on a five-cycle in the manifold of compactification $S^1 \times M$ with M Calabi-Yau, with total momentum q_0 on S^1 and KK-monopole charge p_0 on S^1. For simplicity we take $p_0 = 0$ throughout.

We take a four-cycle $P \in H_4(M, \mathbf{Z})$ and wrap the M5-brane on $P \times S^1$. The four-cycle P has a Poincaré dual two-form $[P] \in H^2(M, \mathbf{Z})$. Let ω_A be a basis of two-forms, $\langle \omega_A \rangle_{\mathbf{Z}} = H_2(M, \mathbf{Z})$. so that $[P]$ decomposes as $P = \sum_A p^A \omega_A$. The charges q_A are obtained by turning on self-dual anti-symmetric tensors on the M5-brane world-volume.

In the limit where the volume of the Calabi-Yau is much smaller than the radius R of S^1, $|M| \ll R^6$, the low-energy dynamics of the M5-brane world-volume is described by an $\mathcal{N} = (0,4)$ sigma model on $\mathbf{R}_t \times S^1$. This is SCFT has central charge c_L and left-moving momentum $\bar{q}_0$. The Cardy-formula gives us, for $q_0 \gg c_L$, the entropy

$$S_{micro} = 2\pi \sqrt{\frac{c_L \bar{q}_0}{6}}. \tag{17}$$

Note that for large q_0 we can approximate $q_0 = \bar{q}_0$. We first compute the central charge. The central charge c_L is the sum of all moduli: the moduli d_P of the M5-brane on P, the number b_2^- of left-moving scalars on the S^1 and three additional degrees of freedom for three spatial translations degrees of freedom:

$$c_L = d_P + b_2^- + 3. \tag{18}$$

Using Riemann-Roch, we can compute d_P as

$$d_P = 2w - 2, \quad w = \int_M e^P \mathrm{Td}(M) = \int_M \left(\frac{P^3}{6} + \frac{1}{12} P c_2(M) \right). \tag{19}$$

Also b_2^- can be computed with indices as $b_2^- = \frac{1}{2}(\chi - \sigma)$ with $\chi = \int_P c_2(P)$, the Euler characteristic of P, and $\sigma = -\frac{2}{3}\chi + \frac{1}{3}\int_P c_1(P) \wedge c_1(P)$ the signature of

P. We can compute these Chern-classes of P in terms of Chern-classes of M. Taking this together gives

$$c_L = \int_M P^3 + c_2 \cdot P = |D| + c_2 \cdot P \tag{20}$$

so that:

$$S_{micro} = 2\pi\sqrt{|D|q_0} + \mathcal{O}(1/|D|). \tag{21}$$

It can be shown that turning on self-dual anti-symmetric two-forms on the M5 world-volume induces exactly the shift in $q_0 \mapsto \hat{q}_0 = q_0 - \frac{1}{2}q^2$. So indeed:

$$S_{BH} = S_{micro} \tag{22}$$

to leading order in D^{-1}. In fact the one-loop corrections (the D^{-1} terms) also match on both sides, see [MSW97].

Note that this computation is executed in the Coulomb branch, where the five-branes are far apart and graviton scattering between them can be ignored. This condition is met if

$$|D| \ll |M| \tag{23}$$

which is the exact opposite regime of the regime (16). The volume of the Calabi-Yau — a hypermultiplet scalar — acts as a coupling constant. The microscopic entropy S_{micro} counts BPS states and - assuming no wall-crossing phenomena - we expect that the quantity $S_{micro}(p^A, q_A; |M|)$ is constant on the hypermultiplet moduli-space, or at least constant under continues changes in $|M|$.

2 Microstate geometries

In the previous section we saw an example of a general principle at play for black holes in string theory. We find a configuration of D- or M-branes that produces a black hole geometry in the decoupling regime. This geometry has an associated $S_{BH}(g)$, where g is a coupling constant that needs to be small in order for quantum gravity effects to suppressed at the horizon. Then, going back to the D-brane picture, we go to a limit where the brane ensemble is described in terms of a (S)CFT, where we can compute a microscopic index S_{micro} with Cardy's formula. To stay in the Coulomb branch, the coupling constant g should be large compared to the charges. We then compare $S_{BH}(g)$ and $S_{micro}(g)$ and indeed, in a lot of realizations in string theory, we find to leading order in a large-charge expansions that $S_{BH}|_{g\ll 1} = S_{micro}|_{g\gg 1}$. The fact that these indices agree despite being computed in different regimes of g is understood as invariance of the BPS or Witten index under continuous deformations: The BPS is a topological and hence protected quantity on the moduli space.

This does not however address the following question: Start, at weak coupling $g \ll 1$ with one of the microstates in the D- or M-brane configuration. Now continuously crank up the coupling to $g \gg 1$, ending up in the regime where we can trustfully describe physics in terms of a gravitational theory. What is then - in terms of this gravitational theory - the physical picture of this microstate? Or, in terms of Einstein's equations: what type of solution does it correspond to?

Mathur proposed an answer to this question in an effort that is now collectively called the *fuzzball* program. His resolution to the problem is that in a theory of gravity, for a black hole with Bekenstein-Hawking entropy S_{BH}, there are in that theory $e^{S_{BH}}$ smooth and horizonless solutions that are to be identified with the black hole microstates. These solutions he called *fuzzballs*, and should asymptote to the black hole geometry.

The first question to ask is: do fuzzballs exist? Are there smooth horizonless solutions to Einstein's equations?

2.1 A no-go theorem and its refute

Of course there are smooth horizonless solutions to Einstein's equations in D dimensions. The maximally symmetric solutions flat space and AdS in D dimensions are smooth and horizonless. But for a fuzzball to be the microstate of a generic black hole, we want it to support the mass of the black hole. So we are really looking for a smooth horizonless solution to Einstein's equations with non-vanishing ADM-mass. But here we encounter the no-go theorem:
For a solution of Einstein's equation to support non-vanishing ADM mass, it must have a horizon
or, more catchy: *There are no solitons without horizons.* This seems to kill Mathur's fuzzball conjecture right away. But a no-go theorem is as strong as the assumptions that go into it. And indeed, it is in weakening those assumptions that we learn how to circumvent the no-go and find existence of non-trivial fuzzballs. Let's make this more rigorous.

Consider Einstein gravity in four dimensions with vector matter turned on. The action reads

$$S = \frac{1}{16\pi G_4} \int d^4x \left(R - \tfrac{1}{4}F^2\right) \tag{24}$$

with the Einstein quation

$$R_{\mu\nu} = 8\pi G_4 \left(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T\right), \tag{25}$$

where $T^{\mu\nu} = -F^{\mu\lambda}F^{\nu}_{\lambda} + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$ and $dF = 0$. If a solution $g_{\mu\nu}$ to equation (25) has a timelike Killing vector $K = (K_\mu)$, that leaves the the fields invariant, $\mathcal{L}_K F = 0$, we can define a globally conserved quantity, the conserved mass:

$$M = \int_{S^2} \star dk = -2\int_\Sigma K^\mu R_{\mu\nu} dx^\nu = 16\pi G_4 \int_\Sigma K^\mu F_{\mu\lambda} F^{\lambda}_{\nu} dx^\nu \tag{26}$$

where the S^2 in the integral is a closed space-like surface at infinity.
Now Cartan's "master formula" states that $\mathcal{L}_K F = d(i_K F) + i_K(dF)$. so that $\mathcal{L}_K F = 0$ implies $d(i_k F) = 0$. In four dimensions, $d(i_k F) = 0$ implies $i_k F = 0$, unless there are non-trivial one-cycles, so that under the assumption that space-time is simply-connected, we can conclude that $M = 0$. The upshot of the circumvention of this no-go theorem in higher dimensions is that, in higher dimensions, the formula $d(i_K F) = 0$ may still allow for the presence of non-trival forms, say $i_K F = \omega + d\eta$, $d\omega = 0$. The form ω may then support non-vanishing mass. We will illustrate this idea for a wide variety of solutions in six dimsional supergravity, commenting along the way on how to reduce from six to five dimensions.

3 Microstate geometries in six dimensions

Six dimensions is natural arena to study three-charged black holes. Also, if we want to understand non-extremal and non-supersymmetric black holes the JMaRT solution of [JMRT05] is a good starting point, and this solution has is naturally described in six dimensions.

Motivated by the no-go theorem of the previous section, we here set out to explore how microstate geometries in six dimensions can support mass. As we saw, the resolution will lie in understanding the global properties and topology of the space-time of these solutions. But first we set up the techniques to compute mass in six dimensional gravity: Komar integrals and the Smarr formula.

3.1 Smarr formula in six dimensions

We discuss Komar integrals, the relation to the energy and tension of a solution, a Smarr formula for smooth horizonless solutions using topology and their application to six-dimensional supergravity with tensor multiplets.

3.1.1 Komar integrals

Any Killing vector K of a metric on a D-dimensional Lorentzian spacetime defines a conserved quantity through a Komar integral:

$$\mathcal{Q}_K = \frac{1}{8\pi G_D}\int_{\partial V_\infty} \star dK = \frac{1}{8\pi G_D}\int_{\partial V_\infty} (\partial_\mu K_\nu - \partial_\mu K_\nu) d\Sigma^{\mu\nu}\,, \tag{27}$$

where we integrate over a closed spatial surface at infinity. Killing vectors enjoy the property $\nabla^2 K_\mu = -R_{\mu\nu}K^\nu$. With the help of Stokes' theorem, we can then rewrite this as a bulk integral over a volume V on a spatial hypersurface with boundary $\partial V_\infty \cup \partial V_{\rm int}$:

$$\mathcal{Q}_K = -\frac{1}{4\pi G_D}\int_V \star(K^\mu R_{\mu\nu}dx^\nu) - \frac{1}{8\pi G_D}\int_{\partial V_{\rm int}} dS^{\mu\nu}(\partial_\mu K_\nu - \partial_\mu K_\nu)\,. \quad (28)$$

For a spacetime with a timelike Killing vector K, one usually relates the Komar integral to the ADM mass. However, this is only valid for an energy-momentum tensor that asymptotically approaches that of a weak static dust source, with $T_{00} \gg T_{0i}, T_{ij}$ and $\partial_0 g_{ij} = 0$ asymptotically. For a string-like object spanning the y direction, we expect that T_{00} and T_{yy} will be of the same order, so we need to slightly modify the story.

3.1.2 ADM integrals

We now review the relevant results of [TZ01]. To relate the Komar integral to physical quantities such as the ADM energy, we consider an energy-momentum tensor that has asymptotically $p{+}1$ dominating diagonal components $T_{00}, T_{aa}, a = 1\dots p$ and $p < D-3$. We assume all other components of the energy-momentum tensor are subleading compared to these. We take the p coordinates to be compact and consider the linearization around a flat metric, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with Minkowski reference metric

$$ds_D^2 = -dt^2 + \sum_{a=1}^p dy^a dy^a + \sum_{i=1}^n dx^i dx^i\,, \qquad n = D - p - 1\,. \quad (29)$$

We write the Einstein equation as

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G_D T_{\mu\nu}\,. \quad (30)$$

The energy density $\mathcal{E}$, average tension $\mathcal{T}$ and angular momentum density $\mathcal{J}$ are

$$\mathcal{E} = \int d^n x \langle T_{00}\rangle\,, \quad (31)$$

$$\mathcal{T} = -\frac{1}{p}\sum_{a=1}^p \int d^n x \langle T_{aa}\rangle\,, \quad (32)$$

$$\mathcal{J}_{ij} = \int d^n x (x_i \langle T_{j0}\rangle - x_j \langle T_{i0}\rangle)\,. \quad (33)$$

with the average over the compact space $\langle X\rangle = 1/V_p \int d^p y X$. From the linearized Einstein equation, one can then deduce the relations to the linearized metric components $h_{\mu\nu}$ [TZ01]:

$$\mathcal{E} = -\frac{1}{16\pi G_D (n-2)}\int_{\partial V_\infty} dS_i \partial_i((n-1)h_{00} - h_{aa})\,, \quad (34)$$

$$\mathcal{T} = -\frac{1}{p}\frac{1}{16\pi G_D (n-2)}\int_{\partial V_\infty} dS_i \partial_i(p\, h_{00} - (n+p-2)h_{aa})\,. \quad (35)$$

These are the formulae that relate the asymptotic expansion of an extended object (where T_{aa} is not negligible compared to T_{00}) to its mass and tension. After dimensional reduction over the p internal directions, the ADM mass in $D-p$ dimensions is given by $\mathcal{E}$. The angular momentum density can still be read off from the off-diagonal metric components:

$$g_{0i} = \frac{16\pi G_D}{\Omega_{D-2}} \frac{x^j J^{ji}}{\rho^n} \ + \ \dots\,, \tag{36}$$

where Ω_{D-2} is the volume of the unit $(D-2)$-sphere and ρ the radius in the four spatial dimensions.

3.1.3 Normalization of the Komar integrals

We now discuss the relation of the Komar integral to the energy density and tension.

Timelike Killing vector.

One readily shows that for a timelike Killing vector K that asymptotes to $K_\infty = \partial_t$, we have the normalization

$$\mathcal{E} - \frac{p}{(D-3)}\mathcal{T} = -\frac{1}{16\pi G_D}\frac{(D-2)}{(D-3)}\int_{\partial V_\infty} dS_{\mu\nu}(\partial^\mu K^\nu - \partial^\nu K^\mu)\,. \tag{37}$$

For $p=0$, we retrieve the usual relations between the ADM mass $M=\mathcal{E}$ and the asymptotic form of the metric components [Pee00, GW14]

$$g_{00} \ = \ -1 + \frac{16\pi G_D}{(D-2)\,\Omega_{D-2}}\frac{M}{\rho^{D-3}} + \dots\,, \tag{38}$$

$$g_{ij} \ = \ \left(1 + \frac{16\pi G_D}{(D-2)\,(D-3)\,\Omega_{D-2}}\frac{M}{\rho^{D-3}}\right)\delta_{ij} + \dots\,. \tag{39}$$

Null Killing vector.

Most of this chapter is concerned with supersymmetric solutions in six dimensions. For these, it is useful to discuss $p=1$ and consider null coordinates:

$$u = \frac{t-y}{\sqrt{2}}\,, \qquad v = \frac{t+y}{\sqrt{2}}\,. \tag{40}$$

For a null Killing vector K that asymptotically becomes $K_\infty = \partial_u$, one finds:

$$\mathcal{E} + \mathcal{T} = -\,\frac{1}{8\pi G_D}\frac{(n+p-1)}{(n-2)}\int_{\partial V_\infty} dS_{\mu\nu}(\partial^\mu K^\nu - \partial^\nu K^\mu)\,. \tag{41}$$

Note that these results, as in [TZ01], are in principle only valid for time-independent metric perturbations. Metrics with a null Killing vector ∂_u do not in general have to be time-independent. However, the time-dependence of the metric is heavily constrained. Since we average (integrate) over the internal, compact direction y, the resulting averaged metric must be time-independent and the results for the Komar integrals remain valid.

The normalization of the Komar integral (27), which we use in a 6D supergravity context for strings ($p = 1$), implies that:

$$\mathcal{Q}_K = -\frac{1}{2}\left(\mathcal{E} + \mathcal{T}\right). \tag{42}$$

3.1.4 Six-dimensional supergravity

Here we discuss the six-dimensional setup relevant for the three-charge black hole. First we consider an arbitrary number n_T of tensor multiplets; for superstrata in six dimensions, $n_T = 2$. We also explicitly give the formulas for $n_T = 1$, which is relevant for all of the examples we discuss except the superstrata of section 4.1.4.

Minimal supergravity with n_T tensor multiplets

The six-dimensional supergravity theories of relevance to this chapter have an $SO(n, m)$ global symmetry, with n the number of tensors in the gravity multiplet. In the D1-D5-P frame, the relevant six-dimensional theories are obtained by a compactification on T^4 or K3, which respectively give $\mathcal{N} = (2, 2)$-supergravity with $SO(5, 5)$ global symmetry and $\mathcal{N} = (2, 0)$-supergravity with an $SO(5, 21)$ symmetry group.

Luckily, we do not need the full details of these extended supergravity theories. Rather, we can consider a consistent truncation to 'minimal' six-dimensional supergravity with only $\mathcal{N} = (1, 0)$ supersymmetry. This theory has $SO(1, n_T)$ global symmetry where n_T is the number of tensor multiplets and is in principle arbitrary as it is unfixed by supersymmetry. For our purposes, n_T will be either 1 or 2, see appendix B for more details on the reduction from 10D. Even though we focus on the theory with $SO(1, n_T)$ global symmetry, our results and in particular the Komar integrals (50) and (51) below are straightforwardly extended to the bosonic sector of six-dimensional supergravity theories with more supersymmetry, by formally replacing the $SO(1, n_T)$ metric η_{rs} with the metric of the appropriate global symmetry group.

When $n_T > 1$, the equations of motion of the tensor fields do not follow from an action. We can still consider the 'pseudo-action' [FRS98, Ric01] for the bosonic fields[1]

$$\mathcal{L} = \frac{1}{4}R - \frac{1}{2}\partial_\mu v_r \partial^\mu v^s - \frac{1}{12}\mathcal{M}_{rs} G^r_{\mu\nu\rho} G^{s\,\mu\nu\rho}, \tag{43}$$

that captures the equations of motion of the scalar fields and the metric. The scalars parametrize the coset $SO(1,n_T)/SO(n_T)$. They can be organized in the $SO(1,n_T)$-matrix $V = \begin{pmatrix} v_r \\ x_r^M \end{pmatrix}$ with $M = 1\dots n_T$ and $r = 0\dots n_T$.[2] They enter the tensor dynamics through the scalar metric $\mathcal{M} = \eta V^T V \eta$, with η the $SO(1,n_T)$-metric, or in index notation

$$\mathcal{M}_{rs} = v_r v_s + x_r^M x_s^M\,. \tag{44}$$

The dynamics of the $n_T + 1$ tensor fields G^r are captured by the self-duality conditions and Bianchi identities

$$\mathcal{M}_{rs} G^s = \eta_{rs} \star G^s\,, \qquad dG^r = 0\,, \tag{45}$$

where $\star$ is the six-dimensional Hodge star operator. Finally, the Einstein equation reads:

$$R_{\mu\nu} = 2\partial_\mu v^r \partial_\nu v_r + \frac{1}{2}\mathcal{M}_{rs} G^r_{\mu\alpha\beta} G_\nu^{s\,\alpha\beta}\,. \tag{46}$$

Smarr formula

We are concerned with field configurations that respect the symmetry of a Killing vector K. This means the Lie derivative of the fields with respect to K vanishes:

$$\mathcal{L}_K g_{\mu\nu} = 0\,, \quad \mathcal{L}_K v^r = 0\,, \quad \mathcal{L}_K G^r = 0\,. \tag{47}$$

Since dG^r and $\mathcal{L}_K = d\, i_K + i_K\, d$, we can write the three-form and its dual as

$$i_K G^r = d\Lambda^r + H^r\,, \tag{48}$$

for some globally defined one-forms Λ^r and closed but not exact two-forms H^r. The Einstein equation (46) becomes

$$K^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\rho \left(\mathcal{M}_{rs}\Lambda^r_\sigma G_\nu^{s\,\rho\sigma}\right) + \frac{1}{2}\left(\mathcal{M}_{rs} H_{\rho\sigma} G_\nu^{s\,\rho\sigma}\right)\,. \tag{49}$$

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[1] To avoid confusion with standard notation H for harmonic forms, we do not follow the notation of [FRS98, Ric01] for the three-forms and the kinetic matrix. To convert, use $G^r = H^r_{theirs}$ and $\mathcal{M}_{rs} = (G_{rs})_{theirs}$.

[2] It is customary to write the $SO(1,n_T)$ conditions $V\eta V^T = V^T\eta V = \eta$ in component notation as $v_r v^r = 1, v^r x_r^M = 0, v_r v_s - x_r^M x_s^M = \eta_{rs}$.

Then the Komar integral (28) is:

$$\mathcal{Q}_K = -\frac{1}{8\pi G_6}\int_V \mathcal{M}_{rs} H^r_{\rho\sigma} G^s_{\nu}{}^{\rho\sigma} dV^\nu \\ - \frac{1}{8\pi G_6}\int_{\partial V_{int}} \left(\mathcal{M}_{rs}\Lambda^r_\sigma G^s_{\mu\nu}{}^\sigma dS^{\mu\nu} + (\partial_\mu K_\nu - \partial_\mu K_\nu)\right). \quad (50)$$

As in [GW14], we find that we can support matter (non-zero Komar integrals) with horizons or with topology. For trivial topology, $H^r = 0$ and the Smarr formula (50) relates the Komar integral to horizon quantities (area, charges and angular momenta). If also no horizons are present, the right-hand side of (50) is zero and we get a vanishing Komar integral for the Killing vector K.

We are interested in spacetimes without inner boundaries. With (45), we find

$$\mathcal{Q}_K = -\frac{1}{4\pi G_6}\int_V \eta_{rs} H^r \wedge G^s\,, \quad (51)$$

so that only non-trivial topology can allow for non-zero Komar integrals.

One tensor multiplet

For many of the solutions in this chapter we can restrict to $SO(1,1)$ supergravity with $n_T = 1$. Including only one extra tensor multiplet in addition to the minimal supergravity multiplet is convenient as it allows for a Lagrangian description of the theory. The single self-dual three-form G^+ of the gravity multiplet can be combined with the single anti self-dual three-form G^- of the tensor multiplet in one unconstrained three-form $G = \frac{1}{2}(G^+ + G^-)$. The action becomes

$$\mathcal{L} = \frac{1}{4}R - \frac{1}{2}\partial_\mu X \partial^\mu X - \frac{1}{12}e^{2\sqrt{2}X} G_{\mu\nu\rho}G^{\mu\nu\rho}. \quad (52)$$

We introduce the dual three-form (equivalent to (45)):

$$\tilde{G} = e^{2\sqrt{2}X} \star G\,. \quad (53)$$

To compare to the discussion of section 3.1.4, we can choose $G^0 = G, G^1 = \tilde{G}$. The $SO(1,1)$ metric is then $\eta = \sigma_1$, and one can choose the $SO(1,1)$ scalar matrix as $V = \exp(\sqrt{2}X\sigma_3)$, where σ_i are the Pauli matrices.

The Einstein equation can be (re)written as:

$$R_{\mu\nu} = 2\partial_\mu X \partial_\nu X + \frac{1}{2}\left(e^{2\sqrt{2}X} G_{\mu ab} G_\nu{}^{ab} + e^{-2\sqrt{2}X}\tilde{G}_{\mu ab}\tilde{G}_\nu{}^{ab}\right). \quad (54)$$

The Komar integral (51) is then

$$\mathcal{Q}_K = -\frac{1}{4\pi G_6}\int_V \left(H \wedge \tilde{G} + \tilde{H} \wedge G\right), \quad (55)$$

with the harmonic forms $H, \tilde{H}$ defined through

$$i_K G = d\Lambda + H\,, \qquad \tilde{i}_K \tilde{G} = d\tilde{\Lambda} + \tilde{H} \tag{56}$$

for some global one-forms Λ.

Supersymmetry

Let us also mention the fermionic content of the $SO(1,1)$ theory. The gravity multiplet consists of $(e_\mu, \psi^\alpha_\mu, B^+_{\mu\nu})$ with B^+ a self-dual tensor such that $G^+ \equiv dB^+ = \star G^+$. The tensor multiplet consists of $(B^-_{\mu\nu}, \chi^\alpha, X)$ with $G^- \equiv dB^- = -\star G^-$. The supersymmetry transformations of the fermions are:

$$\delta\psi^\alpha_\mu = (\partial_\mu - \frac{1}{4} e^{\sqrt{2}X} G^+_{\mu\nu\sigma}\gamma^{\nu\sigma})\varepsilon^\alpha\,, \tag{57}$$

$$\delta\chi^\alpha = \frac{1}{2i}(\sqrt{2}\gamma^\mu\partial_\mu X + \frac{1}{6} e^{\sqrt{2}X} G^-_{\mu\nu\rho}\gamma^{\mu\nu\rho})\varepsilon^\alpha. \tag{58}$$

Given a Killing spinor ϵ^α we can construct the bilinear vector:

$$K_\mu \varepsilon^{\alpha\beta} = \bar{\epsilon}^\alpha \gamma_\mu \varepsilon^\beta, \tag{59}$$

which is always a *null* Killing vector, $K \cdot K = 0$. The supersymmetry equations imply (using the form notation $K \equiv K^\mu g_{\mu\nu} dx^\nu$):

$$dK = 2e^{\sqrt{2}X} i_K G^+ = i_K(e^{\sqrt{2}X} G + e^{-\sqrt{2}X}\tilde{G})\,, \tag{60}$$

$$i_K dX = 0\,, \tag{61}$$

since the self-dual part of G is given by $G^+ = 1/2(G + e^{-2\sqrt{2}X}\tilde{G})$. Using $i_K \star G = \star(G \wedge K)$, this allows us to write the null charge associated with K as

$$\mathcal{Q}_K = \frac{1}{8\pi G_6}\int_{\partial V_\infty} \star dK = -\frac{1}{8\pi G_6}\int_{\partial V_\infty} \left(\tilde{G} + G\right) \wedge K\ , \tag{62}$$

where we have assumed that $X = 0$ at infinity, which we can always do for asymptotically flat spacetimes. In the microstate geometries of section 4, we find that $\partial V_\infty = S^1 \times S^3$, and the Killing vector K projected on this spacelike surface is (proportional to) the isometry along the compact S^1. In the notation of the metric (64) below, $K = -dv$ at spatial infinity. This means we simply get:

$$\mathcal{Q}_K = -\frac{L_v}{8\pi G_6}\int_{S^3} \left(\tilde{G} + G\right) = -\frac{L_v \pi}{4G_6}(Q_e + Q_m), \tag{63}$$

where $L_v = 2\pi R_v$ is the size of the S^1 direction parametrized by v (at constant time). This relation is thus simply the BPS condition in 6D relating the null charge associated to K to the electric and magnetic charges of the solution.

4 Examples in six dimensions

Having set up the formalism, we are now ready to work out in detail the (null) Komar integrals for known six-dimensional microstate geometries. We first analyze suppersymmetric extremal solutions in section 4.1 and treat non-extremal examples in section 4.2.

4.1 Supersymmetric examples

We now analyze in detail the null Komar integral for known smooth supersymmetric solutions to six-dimensional supergravity. The structure of supersymmetric solutions in 6D minimal supergravity was studied in [GMR03] and including an additional vector multiplet and one tensor multiplet in [CMC04]. Using the Killing spinors of such supersymmetric solutions, one can always construct a null Killing vector which locally is $V = \partial_u$. The metric can then be shown to take the form:

$$ds_6^2 = -2H^{-1}(dv + \beta_i dx^i)[du + \omega_i dx^i + \frac{\mathcal{F}}{2}(dv + \beta_i dx^i)] + H dx_4^2, \tag{64}$$

where dx_4 is the line element on the 4D "base space" $\mathcal{B}$, the one-forms $\beta = \beta_i dx^i, \omega = \omega_i dx^i$ only have legs on $\mathcal{B}$ and the functions $H, \beta_i, \omega_i, \mathcal{F}$ are in general functions of v and all of the 4D base coordinates x^i. The conditions that these functions (and the three-form and scalar) must satisfy for supersymmetric solutions can be found in [CMC04], or [BGSW12] whose conventions and notation we follow. Note that the ansatz (64) only holds for sections 4.1.1-4.1.3, in section 4.1.4 we extend the ansatz for two tensor multiplets.

4.1.1 General expectations

It is instructive to first work out the ADM integrals $\mathcal{E}$ and $\mathcal{T}$ for the three-charge solutions of our interest. Asymptotically, the metric (64) approaches that of the three-charge black string for which $H = (Z_2 Z_3)^{-1/2}, \mathcal{F} = -Z_1, \omega = 0, \beta = 0$ and $Z_i = 1 + Q_i/\rho^2$, with ρ the standard radial coordinate of the 4D base $\mathcal{B} = \mathbf{R}^4$. The asymptotic metric perturbation in the coordinates t, y (40) is

$$h_{00} = \frac{1}{2}\frac{Q_2 + Q_3 + Q_1}{\rho^2} + O(\rho^3)\,, h_{yy} = \frac{1}{2}\frac{-Q_2 - Q_3 + Q_1}{\rho^2} + O(\rho^3)\,. \tag{65}$$

and we find that

$$\mathcal{E} = \frac{\pi L_y}{4\pi G_6}\left(Q_2 + Q_3 + \frac{1}{2}Q_1\right)\,, \qquad \mathcal{T} = \frac{\pi L_y}{4\pi G_6}\left(Q_2 + Q_3 - \frac{1}{2}Q_1\right)\,, \tag{66}$$

with $y \sim y + L_y$. Note that $\mathcal{E}$ is the ADM mass after dimensional reduction over the y-circle.[3] Using (42), we anticipate that the Komar integral will be:

$$\mathcal{Q}_K = -\frac{1}{2}(\mathcal{E} + \mathcal{T}) = -\frac{\pi L_y}{4G_6}(Q_2 + Q_3)\,, \tag{67}$$

and does not involve the momentum charge Q_1.

4.1.2 The uplift of five-dimensional microstate geometries

As a warm-up, we consider the uplift of five-dimensional microstate geometries. Komar integrals and Smarr formulae for those geometries were discussed at length in [GW14], hence we do not go into much detail here. The solutions are completely smooth multi-centered solutions of the 5D STU model with three gauge fields A^I ($I = \{1, 2, 3\}$) and three scalars X^I, constrained by $X^1X^2X^3 = 1$. The 5D Lagrangian is given by (161). The 6D theory of minimal supergravity coupled to one tensor multiplet (52) gives exactly this STU model when dimensionally reduced to 5D. See appendix A for more details.

The 5D solutions that we are interested in are given by the metric [GG05, EEMR05, BW05]:

$$ds_5^2 = -Z^{-2}(dt + k)^2 + Zds_4^2, \qquad Z = (Z_1Z_2Z_3)^{1/3}. \tag{68}$$

where the 4D base space $\mathcal{B}$ is Gibbons-Hawking: it is a $U(1)$ fibration with coordinate ψ over flat $\mathbf{R}^3$. The solutions are then determined by specifying the poles of eight functions V, K^I, L_I, M, which are harmonic functions on $\mathbf{R}^3$. For instance, we have $Z_I = L_I + C_{IJK}K^JK^K/2V$ with $C_{IJK} = |\epsilon_{IJK}|$. These eight harmonic functions must satisfy stringent conditions in order for the full 5D spacetime to be completely regular and asymptotically flat [BW08, GW14].

The gauge potentials in 5D are:

$$A^I = -Z_I^{-1}(dt + k) + B^I, \tag{69}$$

where B^I is a magnetic potential (only well-defined locally). The scalars are given by:

$$X^I = \frac{Z}{Z_I}. \tag{70}$$

For asymptotically flat 5D spacetimes, we have asymptotically:

$$Z_I \sim 1 + \frac{Q_I}{4r} = 1 + \frac{Q_I}{\rho^2}, \tag{71}$$

[3] Note that the dimensional reduction in section 4.1.2 and appendix A.2 is a reduction over the spacelike v-circle, which will give a different resulting 5D ADM mass in terms of Q_1, see eq. (73).

with r the usual radial coordinate on $\mathbf{R}^3$ and $\rho = 4r$ is the radial coordinate on the four-dimensional base. In microstate geometry literature, the charges Q_I are normalized through the asymptotic expansion of the electric field in 5D as $F_{0\rho} \sim 2\frac{Q_I}{\rho^3}$ and not with factors involving the volume of the three sphere that are more common from Gaussian integrals. This means that we have:

$$-\frac{1}{16\pi G_5}\int_{\partial V_\infty} \star_5 F_I = \frac{\pi}{4G_5} Q_I\,. \tag{72}$$

For the six-dimensional metric, scalar and tensor solutions see eqs. (170).

The topology of the base

The poles of V ('centers') indicate where the ψ-fibre degenerates in the 4D base space (although the complete 5D spacetime is always completely smooth). Since the ψ-fibre degenerates at each center, we can construct non-contractible compact two-cycles in the 4D space, which are also compact two-cycles in the full 5D geometry. These two-cycles are constructed by taking the ψ-fibration over an arbitrary path in $\mathbf{R}^3$ between two centers. This completely determines the 5D homology structure of simply connected solutions. For $N = 2p + 1$ centers, the global topology is that of a p-fold connected sum of $(S^2 \times S^2)$ with a point removed, for $N = 2p$ centers the topology is $(\mathbf{R}^2 \times S^2)\#(S^2 \times S^2)\# \ldots \#(S^2 \times S^2)$.[4]

The five-dimensional ADM mass of these solutions can be written as [GW14]

$$M_{ADM,5D} = -\frac{1}{32\pi G_5} C_{IJK}\alpha^I \int_{\Sigma_4} F^J \wedge F^K = \frac{\pi}{4G_5}\alpha^I Q_I = \frac{\pi}{4G_5}(Q_1+Q_2+Q_3), \tag{73}$$

where $\alpha^I = 1$ for asymptotically flat solutions and Σ_4 is a spacelike surface of constant time. The integral of $F^J \wedge F^K$ is computed "entirely with cohomology", by calculating the flux of the F^I over the non-trivial compact two-cycles of the geometry as well as the intersection number of these two-cycles.

The topology of the uplift

The six-dimensional uplift of (68) is a non-trivial fibration of the new coordinate v. From the expression for the three-form:

$$2\,G = (X^3)^{-2} \star_5 F^3 + F^2 \wedge (dv + A^1), \tag{74}$$

we can easily see that we have:

$$2\,i_K G = d\left(\lambda_2(dv + A^1)\right) + d\left(Z_1^{-1} Z_2^{-1}(dt + k)\right) + F^1, \tag{75}$$

[4]We only discuss $V = \sum_i q_i/|x - x_i|$ with $|q_i| = 1$, such that the centers are smooth points in the full space, and $\sum_i q_i = 1$, such that the space is asymptotically flat.

where we have defined $\lambda_I = Z_I^{-1} - 1$. The form given in the first term, $\lambda_2(dv + A^1)$, is well-defined. The second term is $Z_1^{-1} Z_2^{-1}(dt + k)$ and is also a well-defined form (as discussed in [GW14]). This implies the cohomology split:

$$2\,\Lambda = \lambda_2(dv + A^1) + Z_1^{-1} Z_2^{-1}(dt + k), \tag{76}$$

$$2H = F^1. \tag{77}$$

Similarly, we can find $\tilde{\Lambda}, \tilde{H}$ by switching the roles of Z_2 and Z_3 in the above expressions. Note that also $2\tilde{H} = F^1$.

The null charge is then:

$$\mathcal{Q}_K = -\frac{1}{4\pi G_6} \int_V \left(H \wedge \tilde{G} + \tilde{H} \wedge G\right) \tag{78}$$

$$= -\frac{1}{16\pi G_6} \int_V \left(F^1 \wedge (F^3 \wedge dv) + F^1 \wedge (F^2 \wedge dv)\right) \tag{79}$$

$$= \frac{L_v}{16\pi G_6} \int_{\Sigma_4} \left(F^1 \wedge F^3 + F^1 \wedge F^2\right) \tag{80}$$

$$= -\frac{L_v \pi}{4G_6}(Q_2 + Q_3), \tag{81}$$

where we used the cohomological computation of the integral $F^I \wedge F^J$ in 5D over Σ_4 [GW14], and $V = S^1(v) \times \Sigma_4$. We see that the null charge is simply the sum of electric and magnetic (string) charges. Note that in five dimensions, Q_1 is on the same footing as $Q_{2,3}$, but in six dimensions it is a momentum charge and does not appear in the null charge $\mathcal{Q}_K$.

The analysis above shows us that we clearly still have non-trivial compact two-cycles in six dimensions which are given by the trivial uplift of the two-cycles of the five-dimensional solution. These are the cycles supporting the cohomological flux $H, \tilde{H} \sim F^1$. The S^1-fibration of the coordinate v over the compact two-cycles of the five-dimensional geometry also introduces new non-trivial three-cycles. Over these cycles, the cohomology elements $F^{2,3} \wedge dv$ have non-zero flux.

However, this is not quite the end of the story. In 6D, we must also have a non-trivial three-sphere at infinity. Indeed, the (electric string) charge in 6D is defined as:

$$Q_e = \frac{1}{2\pi^2} \int_{S^3(\infty)} e^{2\sqrt{2}X} \star G, \tag{82}$$

where S^3 is the S^3 at infinity perpendicular to the string which is along v. Since the equation of motion for the three-form is simply $d(e^{2\sqrt{2}X} \star G) = 0$, this S^3 at infinity must be non-contractible to be able to support non-zero flux for smooth solutions free of singularities. Note that this non-trivial three-cycle is absent in the original 5D geometry. This can be explained by the fact that this three-cycle must be homologically equivalent to an $S^1(v)$ fibration over a two-cycle in the 4D base (which we mentioned above). These new (compared to 5D) non-trivial three-cycles in constant time-slices of the six-dimensional geometry are an interesting feature of the $S^1(v)$ uplift.

4.1.3 D1-D5 microstate geometries and supertubes

We are now ready to discuss the topology and the Komar integral for more generic solutions of the D1-D5-P system. In this section, we first focus on the D1-D5 supertube solutions of Lunin and Mathur [LM01, LMM02]. As we explain in section 4.1.4, the result (102) for the Komar integral is the same for more generic D1-D5 supertubes and D1-D5-P superstrata, since those describe wiggles of the D1-D5 supertube and are topologically equivalent.

The D1-D5 Lunin-Mathur geometries are solutions to six-dimensional supergravity with only one tensor multiplet:

$$ds^2 = -\frac{2}{\sqrt{Z_1 Z_2}}(dv+\beta)(du+\omega) + \sqrt{Z_1 Z_2}ds_4^2, \tag{83}$$

$$e^{2\sqrt{2}X} = \frac{Z_1}{Z_2}, \tag{84}$$

$$2B = -Z_1^{-1}(du+\omega)\wedge(dv+\beta) + \gamma_2. \tag{85}$$

Here ds_4^2 is the 4D flat metric with coordinates x_i ($i = \{1,\dots,4\}$) and $a_1, \gamma_2, \beta, \omega$ are forms on the 4-manifold. The D1-D5 microstate is completely determined by profile functions $g_i(v), i = 1\dots 4$ with $0 \le v \le L$. Certain important functions are given by (for the complete list of fields, see for example [GMPR13]):

$$Z_2 = 1 + \frac{Q_5}{L}\int_0^L \frac{1}{|x_i - g_i(v')|^2}dv', \quad Z_1 = 1 + \frac{Q_5}{L}\int_0^L \frac{|\dot{g}_i(v')|^2}{|x_i - g_i(v')|^2}dv', \tag{86}$$

$$A = -\frac{Q_5}{L}\int_0^L \frac{\dot{g}_j(v')dx^j}{|x_i - g_i(v')|^2}dv', \qquad dB = -\star_4 dA, \tag{87}$$

$$\beta = \frac{-A+B}{\sqrt{2}}, \qquad \omega = \frac{-A-B}{\sqrt{2}}, \tag{88}$$

$$d\gamma_2 = \star_4 dZ_2. \tag{89}$$

Perhaps the easiest explicit profile is the once-wound circle, given by (with $L = 2\pi R_y$):

$$g_1(v) = a\cos(v/R_y), \qquad g_2(v) = a\sin(v/R_y), \qquad g_3(v) = g_4(v) = 0. \tag{90}$$

Then we can parametrize the (flat) 4D metric as:

$$ds_4^2 = \frac{f}{r^2+a^2}dr^2 + f d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\psi^2, \tag{91}$$

and the above functions become:

$$Z_1 = 1 + \frac{Q_1}{f}, \qquad Z_2 = 1 + \frac{Q_5}{f}, \tag{92}$$

$$A = -a\sqrt{Q_1 Q_5}\frac{\sin^2\theta}{f}d\phi, \qquad B = -a\sqrt{Q_1 Q_5}\frac{\cos^2\theta}{f}d\psi, \tag{93}$$

$$f = r^2 + a^2\cos^2\theta, \tag{94}$$

where $Q_1 = a^2 R_y^2/Q_5$, and the D1-D5 string at $x_i = F_i(v)$ is now at $r = 0, \theta = \pi/2$ $(f = 0)$.

Topology and homology

The topology of the D1-D5 system with once-wound circular profile was discussed in [LMM02]. Any D1-D5 geometry with profile $g_i'(v)$ that can be continuously deformed into a circle will share the same topology of $\mathbf{R}^2 \times S^3$. At infinity we have an $S^3(\theta, \phi, \psi)$ of the 4D base, which deforms continuously to the non-trivial $S^3(\theta, \tilde{\phi}, \tilde{\psi})$ in the interior with $\tilde{\phi} = \phi + t/R, \tilde{\psi} = \psi + y/R$, while $S^1(y)$ (keeping $\tilde{\psi}$ fixed) shrinks to zero size in the interior.

Hence we clearly have exactly one non-trivial three-cycle given by the three-sphere at infinity, and one non-trivial (non-compact) two cycle, given by the volume element of the $\mathbf{R}^2$ factor. The three-cycle is again needed in this singularity-free geometry in order for the geometry to be able to support non-zero three-form flux. The intersection number between the two-cycle and the three-cycle is simply $+1$ (with suitable orientations of the cycles).

Cohomology and null charge

For a general D1-D5 geometry, we have:

$$2\, i_k G = d(Z_1^{-1}(dv + \beta)) \tag{95}$$

$$= \frac{1}{\sqrt{2}} d\left(Z_1^{-1}(dy + B) + Z_1^{-1}(dt - A)\right). \tag{96}$$

Note that there is no obvious easy split to be made by defining $\lambda_1 = Z_1^{-1} - 1$ and splitting off terms proportional to λ_1. This is because the fibres A, B typically have singularities on the string profile and/or in the origin. So we can leave the well-behaved one-form Λ implicit:

$$2\, H \equiv i_K G - d\Lambda = d(Z_1^{-1}(dv + \beta)) - d\Lambda, \tag{97}$$

since the integrals we will perform are independent of Λ anyway. In the explicit example of the once-wound circular profile, we can easily see that

$$\frac{1}{L_v}\int_{\mathbf{R}^2} H = \frac{1}{L_v}\left(\frac{L_v}{2}\right) = \frac{1}{2}, \tag{98}$$

where we integrate the $\mathbf{R}^2$ cycle from the string profile (at $r = 0, \theta = \pi/2$) to $r = \infty$, and we used that $Z_1^{-1}(f = 0) = 0$ and $Z_1^{-1}(r = \infty) = 1$.

We see that H is the cohomological dual of the non-trivial two-cycle in the geometry, as expected. The harmonic part of the three-form G and its dual $\tilde{G}$ are both proportional to the volume form of the non-trivial three-cycle S^3:

$$\frac{1}{2\pi^2}\int_{S^3(\infty)} G = Q_5, \qquad \frac{1}{2\pi^2}\int_{S^3(\infty)} \tilde{G} = Q_1, \tag{99}$$

as these parts precisely define the D1 and D5 charges of the geometry. Putting this together gives for the null charge:

$$\mathcal{Q}_K = -\frac{1}{4\pi G_6}\int_{\mathbf{R}^2\times S^3}\left(H\wedge\tilde{G} + \tilde{H}\wedge G\right) \tag{100}$$

$$= -\frac{1}{4\pi G_6}\left(\int_{\mathbf{R}^2} H\right)(+1)\left(\int_{S^3}\tilde{G}\right) - \frac{1}{4\pi G_6}\left(\int_{\mathbf{R}^2}\tilde{H}\right)(+1)\left(\int_{S^3} G\right) \tag{101}$$

$$= -\frac{L_v\pi}{4G_6}\left(Q_1 + Q_5\right), \tag{102}$$

where we used the intersection number to split the integral into separate integrals over the non-trivial cycles.

4.1.4 D1-D5-P superstrata

The most general three-charge microstate geometries that fall within six-dimensional supergravity arise from reduction on a rigid T^4 [GMPR13]. These solutions excite all IIB supergravity fields in ten dimensions (metric, Ramond-Ramond fields $C_{(0)}, C_{(2)}, C_{(4)}$, as well as $B_{(2)}$ and the dilaton ϕ_1). The solutions can be interpreted as solutions in minimal supergravity in six dimensions coupled to *two* tensor multiplets, see appendix B.

These solutions require extending the results of section 4.1.3 in two ways: considering an extra tensor multiplet, and adding the momentum charge P. Only then can we cover both generic D1-D5 geometries with a rigid T^4 [KST07] and the D1-D5-P superstrata [BGR+15]. However, these more general solutions are topologically equivalent to the D1-D5 supertubes (83). We will show that the Komar integral is unchanged.

The general superstrata solutions as given in [GR13, BGR+15], in six-dimensional language, fit within the ansatz [GMPR13, BGR$^+$15]:

$$ds^2 = \frac{\mathcal{P}}{Z_1 Z_2}\left(-\frac{2}{\sqrt{\mathcal{P}}}(dv+\beta)\left[du+\omega+\frac{\mathcal{F}}{2}(dv+\beta)\right]+\sqrt{\mathcal{P}}ds_4^2\right), \tag{103}$$

$$e^{2\phi} = \frac{Z_1^2}{\mathcal{P}}, \tag{104}$$

$$\chi = \frac{Z_4}{Z_1}, \tag{105}$$

$$2B = -\frac{Z_2}{\mathcal{P}}(du+\omega)\wedge(dv+\beta) + a_1\wedge(dv+\beta)+\gamma_2, \tag{106}$$

$$B' = -\frac{Z_4}{\mathcal{P}}(du+\omega)\wedge(dv+\beta) + a_4\wedge(dv+\beta)+\delta_2, \tag{107}$$

$$\mathcal{P} = Z_1 Z_2 - Z_4^2, \tag{108}$$

where, similar to the D1-D5 ansatz, ds_4^2 is the 4D flat metric and $\beta, \omega, a_1, a_4, \gamma_2, \delta_2$ are forms on this 4D base. We refer to [GMPR13, BGR$^+$15] for the full set of supersymmetry equations and equations of motion and only quote those that we need:

$$d\gamma_2 = \star_4 dZ_2\,, \qquad d\delta_2 = \star_4 dZ_4\,. \tag{109}$$

The tensor B comes from the dimensional reduction of $C_{(2)}$ while B' descends from $B_{(2)}$ in 10D; the scalar ϕ is simply the 10D dilaton while χ is the 10D axion $C_{(0)}$. For more information on the dimensional reduction from 10D to 6D and the realization of the $SO(1,2)$ symmetry, see appendix B. This ansatz reduces to the D1-D5 ansatz (83) when $Z_4 = a_4 = \delta_2 = 0$; the tensor multiplet parametrized by the fields B', χ is set to zero, truncating the $SO(1,2)$ theory down to $SO(1,1)$.

The tensor multiplet scalars $\tau = \chi + ie^{-\phi}$ parametrize the coset $SO(1,2)/SO(2)$. While B and its field strength $G = dB$ are unconstrained, the tensor B' satisfies a duality relation. Indeed, the field strength:

$$G' = dB' - 2\frac{\chi}{e^{-2\phi}+\chi^2}dB, \tag{110}$$

is anti self-dual in six dimensions:

$$G' = -\star G'. \tag{111}$$

Thus, we find the correct tensor field content for the $SO(1,2)$ theory of minimal supergravity with two tensor multiplets.

The null charge is given by (see also appendix B):

$$\mathcal{Q}_K = -\frac{1}{4\pi G_6}\int_V \left(H\wedge\tilde{G} + \tilde{H}\wedge G\right) + \frac{1}{8\pi G_6}\int_V (H'\wedge G')\,, \tag{112}$$

where $H, \tilde{H}$ are defined as in (56), similarly H' is the harmonic part of $i_K G'$, and the dual form $\tilde{G}$ is now defined by:

$$\tilde{G} = \frac{e^{2\phi}}{1+e^{2\phi}\chi^2}\star G. \tag{113}$$

For the superstrata of [BGR+15], the terms in (112) involving $G, \tilde{G}$ can easily be seen to give the same contribution $\sim (Q_1 + Q_5)$ as for the D1-D5 microstates above. The term involving G' does not contribute. It is easiest to realize this by seeing that dB' and χdB fall off too fast at infinity to have a non-zero integral $\int_{S^3_\infty} G'$; in essence, this is because Z_4 falls off faster at infinity than Z_1 or Z_2 (which give the Q_1, Q_5 contributions to the null charge as in the D1-D5 case above).[5] We conclude that:

$$\mathcal{Q}_K = -\frac{L_v \pi}{4 G_6} \left(Q_1 + Q_5\right) , \tag{117}$$

just as for the D1-D5 supertube.

That the null charge gives the same result for D1-D5-P superstrata as for the D1-D5 supertubes is not so surprising from a topological point of view. The important thing to note is that a generic superstratum solution has the same topology as the D1-D5 round supertube. Superstrata describe fluctuations on top of a topologically non-trivial S^3 (shape modes depending on two variables), just as generic two-charge supertubes describe one-dimensional shape modes on the S^3. This is the same S^3 present for the round supertube discussed in section 4.1.3, and therefore supertubes and superstrata have a similar topological three-cycle.

[5] To see this fall-off explicitly we quote the behaviour for the most general D1-D5 supertube invariant under T^4 rotations. This has five profile components $g_i, i = 1 \ldots 4$ and g_5, and the fields are [GR14]:

$$\begin{aligned}
&Z_2 = 1 + \tfrac{Q_5}{L} \int_0^L \tfrac{1}{|x_i - g_i(v')|^2}\, dv' , && Z_4 = -\tfrac{Q_5}{L} \int_0^L \tfrac{\dot{g}_5(v')}{|x_i - g_i(v')|^2}\, dv' ,\\
&Z_1 = 1 + \tfrac{Q_5}{L} \int_0^L \tfrac{|\dot{g}_i(v')|^2 + |\dot{g}_5(v')|^2}{|x_i - g_i(v')|^2}\, dv' && d\gamma_2 = *_4 dZ_2 \qquad d\delta_2 = *_4 dZ_4 ,\\
&A = -\tfrac{Q_5}{L} \int_0^L \tfrac{\dot{g}_j(v')\, dx^j}{|x_i - g_i(v')|^2}\, dv' && dB = - *_4 dA ,\\
&\beta = \tfrac{-A+B}{\sqrt{2}} && \omega = \tfrac{-A-B}{\sqrt{2}} \quad \mathcal{F} = 0 , \quad a_1 = a_4 = x_3 = 0 ,
\end{aligned} \tag{114}$$

An explicit example is a round profile in the $\mathbf{R}^4$ base and a non-zero g_5 component:

$$g_1(v) = a\cos(v/R_y), \qquad g_2(v) = a\sin(v/R_y), \qquad g_3(v) = g_4(v) = 0, \qquad g_5(v) = -\frac{b}{k}\sin(v/R_y) . \tag{115}$$

The D1-D5 seed solution of [BGR+15] starts from such a profile. Then we have that

$$Z_1 = 1 + \frac{Q_1}{f} + c_1 \frac{\sin^{2k}\theta\cos(2k\phi)}{(r^2+a^2)^k f} , \qquad Z_2 = 1 + \frac{Q_5}{f} , \qquad Z_4 = c_4 \frac{\sin^k\theta\cos(k\phi)}{\sqrt{r^2+a^2} f} , \tag{116}$$

where $c_1 = \frac{Q_1 a^2 b^2}{2a^2+b^2}$ and $c_4 = \sqrt{\frac{Q_1 Q_5}{a+2+b^2/2}}\, b a^k$ are constants. Clearly Z_4 falls off too fast for the $H' \wedge G'$-term to contribute to the Komar integral. For superstrata solutions, we refer to [GR13, BGR+15].

4.2 Non-extremal example

We now discuss the JMaRT solutions of [JMRT05], which have an interpretation as microstate geometries of the five-dimensional overspinning three-charge black hole. In the IIB frame, these are smooth solitons, with a natural interpretation in six-dimensional supergravity after dimensional reduction on the compact T^4.

4.2.1 Metric and gauge fields

The solitons are obtained by demanding the metric ansatz appropriate for describing the non-extremal three-charge black hole to be smooth. Usually, the five-dimensional physical charges are quoted, which in this case are the ADM mass $M_{ADM,5D}$, the electric charges Q_1, Q_5, Q_p, and the two angular momenta J_ψ, J_ϕ:[6]

$$M_{ADM,5D} = \frac{L_y \pi}{4G_6}\frac{m}{2}\sum_I \cosh 2\delta_I\,, \quad J_\psi = -\frac{L_y\pi}{4G_6} m(a_1 c_1 c_2 c_3 - a_2 s_1 s_2 s_3)\,, \tag{118}$$

$$Q_I = \frac{m}{2}\sinh 2\delta_I\,, \qquad J_\phi = -\frac{L_y\pi}{4G_6} m(a_2 c_1 c_2 c_3 - a_1 s_1 s_2 s_3)\,, \tag{119}$$

given in terms of parameters $m, \delta_1, \delta_5, \delta_p, a_1, a_2$ and with the notation $s_i = \sinh\delta_i, c_i = \cosh\delta_i$. The supersymmetric limit is $m, a_1, a_2 \to 0$, $\delta_i \to \infty$ while keeping $Q_I, m/\sqrt{a_i}$ fixed. We note that the 6D ADM mass (for the asymptotically $\mathbf{R}^{4,1}\times S^1$ spacetime) is actually:

$$M_{ADM,6D} = \frac{L_y\pi}{4G_6}\frac{m}{2}\left(\cosh 2\delta_1 + \cosh 2\delta_5 + 2\cosh 2\delta_p\right), \tag{120}$$

so the contribution due to the momentum charge (which is the charge from the graviphoton in reducing from 6D to 5D) is different.

We choose to write the metric and gauge fields in the notation of [CM14]. The metric, scalar and gauge field in 6D are (note that $B = -C_2/2$, with C_2 the

[6] Standard conventions in the literature are to take $G_5 = \pi/4$, which would render the prefactor $L_y\pi/(4G_6) = 1$. As in the rest of the paper, we instead choose to keep the explicit factors of G_6 in all of the relevant formulae. We also choose a normalization for the Q_I that is the same as the rest of the paper, instead of the usual normalization which would include a factor of $L_y\pi/(4G_6)$ in the definition of the Q_I as well.

RR two-form of [JMRT05]):

$$\begin{aligned}
ds_6^2 &= \frac{1}{H_p(H_1H_5)^{1/2}}\Bigg[-H_m\,(dt+k)^2 \\
&\quad + H_p^2\left((dy+B_p^m+\frac{c_p}{s_p}k)+\frac{c_p}{s_p}(H_p^{-1}-1)(dt+k)\right)^2\Bigg] \\
&\quad + (H_1H_5)^{1/2}ds_4^2, \qquad (121)\\
e^{2\sqrt{2}X} &= \frac{H_1}{H_5}, \qquad (122)\\
-2B &= \frac{c_1}{s_1}dt\wedge dy - \frac{c_1}{s_1}H_1^{-1}(dt+k)\wedge dy - B_1\wedge dz - \frac{c_1c_p}{s_1s_p}H_1^{-1}dt\wedge dk \\
&\quad - \frac{s_p}{c_p}dt\wedge B_1 - \frac{c_1}{s_1}H_1^{-1}dt\wedge B_3 \\
&\quad + ms_5c_5\frac{r^2+a_2^2+ms_1^2}{fH_1}\cos^2\theta d\psi\wedge d\phi\,. \qquad (123)
\end{aligned}$$

where the quantities used are defined by:

$$\begin{aligned}
ds_4^2 &= f\left(\frac{r^2}{g}dr^2+d\theta^2+\sin^2\theta d\phi^2+\cos^2\theta d\psi^2\right) \\
&\quad + H_m^{-1}\left(a_1\cos^2\theta d\psi+a_2\sin^2\theta d\phi\right)^2 - \left(a_2\cos^2\theta d\psi+a_1\sin^2\theta d\phi\right)^2, \qquad (124)\\
k &= \frac{m}{f}\Big[-\frac{c_1c_5c_p}{H_m}\left(a_1\cos^2\theta d\psi+a_2\sin^2\theta d\phi\right) \\
&\quad + s_1s_5s_p\left(a_2\cos^2\theta d\psi+a_1\sin^2\theta d\phi\right)\Big], \qquad (125)\\
B^{(i)} &= \frac{m}{fH_m}\frac{c_1c_5c_p}{s_Ic_I}\left(a_1\cos^2\theta d\psi+a_2\sin^2\theta d\phi\right). \qquad (126)
\end{aligned}$$

Everything is built from the following functions:

$$\begin{aligned}
H_i &= 1+\frac{ms_i^2}{f}, \qquad (127)\\
H_m &= 1-\frac{m}{f}, \qquad (128)\\
f &= r^2+a_1\sin^2\theta+a_2^2\cos^2\theta, \qquad (129)\\
g &= (r^2+a_1^2)(r^2+a_2^2)-mr^2 = (r^2-r_+^2)(r^2-r_-^2). \qquad (130)
\end{aligned}$$

The three-form is simply $G = dB$. The dual potential, $\tilde{G} = d\tilde{B}$ is then given by:

$$\tilde{B} = B \text{ with } s_1 \leftrightarrow s_5; c_1 \leftrightarrow c_5; H_1 \leftrightarrow H_5. \qquad (131)$$

4.2.2 Constraints for smooth solutions

Smooth JMaRT solutions are determined for fixed charges Q_1, Q_5, Q_p, by two integers m, n. One can extend these to include $\mathbf{Z}_k$ orbifolds with k an integer. They have the following relations between their parameters:

$$r_+^2 = -\, a_1 a_2 \frac{s_1 s_5 s_p}{c_1 c_5 c_p}\,, \tag{132}$$

$$M = a_1^2 + a_2^2 - a_1 a_2 \left[\frac{c_1^2 c_5^2 c_p^2 + s_1^2 s_5^2 s_p^2}{s_1 c_1 s_5 c_5 s_p c_p} \right]. \tag{133}$$

The constant t slices have the topology of $\mathbf{R}^2 \times S^3/\mathbf{Z}_k$. The non-contractible S^3 is spanned at the origin $r = r_+$ by the coordinates $\theta, \tilde{\psi}, \tilde{\phi}$, with the identifications

$$\tilde{\psi} = \psi - \frac{s_p c_p}{a_2 c_1 c_5 c_p - a_1 s_1 s_5 s_p} y\,, \qquad \tilde{\phi} = \phi - \frac{s_p c_p}{a_1 c_1 c_5 c_p - a_2 s_1 s_5 s_p} y\,, \tag{134}$$

The following quantization conditions ensure that the identification $y \to y + 2\pi R$ is a closed orbit:

$$\frac{s_p c_p}{a_2 c_1 c_5 c_p - a_1 s_1 s_5 s_p} R = m\,, \qquad \frac{s_p c_p}{a_1 c_1 c_5 c_p - a_2 s_1 s_5 s_p} R = n\,, \tag{135}$$

for integers m, n.

The $\mathbf{R}^2$ factor has a smooth origin at $r = r_+$, where the $t = constant$ part of the metric has the form (up to irrelevant constant prefactors)

$$ds^2|_{dt=0} = d\rho^2 + \frac{\rho^2}{R^2} dy^2\,, \tag{136}$$

with the identification $y \sim y + 2\pi R k$ and the radius given by

$$R = \frac{M s_1 c_1}{\sqrt{a_1 a_2}} \frac{\sqrt{s_1 c_1 s_5 c_5 s_p c_p}}{c_1^2 c_5^2 c_p^2 - s_1^2 s_5^2 s_p^2}\,. \tag{137}$$

4.2.3 Komar integral

We want to study the Komar integral, which reduces for this topology to

$$\mathcal{Q}_K = -\frac{1}{4\pi G_6} \int_V \left(H \wedge \tilde{G} + \tilde{H} \wedge G \right) \tag{138}$$

$$= -\frac{1}{4\pi G_6} \left(\int_{\mathbf{R}^2} H \int_{S^3} \tilde{G} + \int_{\mathbf{R}^2} \tilde{H} \int_{S^3} G \right) \tag{139}$$

The non-contractible S^3 is homologically equivalent to the one at infinity appearing in Gauss' law. Hence we can perform the S^3 integral at spatial infinity:

$$\frac{1}{4\pi G_6}\int_{S^3(\infty)} G = -\frac{1}{8\pi G_6}\lim_{r\to\infty}\int d\left[ms_5c_5\frac{r^2+a_2^2+ms_1^2}{fH_1}\cos^2\theta d\psi\wedge d\phi\right] \tag{140}$$

$$= -\frac{\pi}{4G_6}\lim_{r\to\infty} ms_5c_5\frac{r^2+a_2^2+ms_1^2}{fH_1}\cos^2\theta\Big|_{\theta=0}^{\theta=\pi/2} = \frac{\pi}{4G_6}Q_5 \tag{141}$$

To obtain the H-integral, we can in principle split the interior product of the Killing vector with the three-form as

$$i_K G = d\Lambda + H\,. \tag{142}$$

However, for our purposes we do not need to do this explicitly: the integral of $i_K G$ and of H are identical, as the contribution of $d\Lambda$ for Λ a well-defined one-form cancels anyway.

To make contact with the supersymmetric limit later, we consider the Killing vector

$$K = \partial_t + \alpha\partial_y\,. \tag{143}$$

with α a constant. Then we find that locally

$$d\omega \equiv i_K G|_{t=const.}\,, \tag{144}$$

$$\omega = \frac{c_1}{s_1}H_1^{-1}\left(dy + (\frac{c_p}{s_p}-\alpha)k + B^{(p)}\right) - \frac{c_1}{s_1}dy + \left(\frac{s_p}{c_p}-\alpha\right)B^{(1)}\,. \tag{145}$$

The one-form ω is zero at infinity and well-behaved at any finite distance, but note that it is not globally well-defined. The integral $\int_{\mathbf{R}^2} i_K G_3$ only receives a contribution from the origin $r = r_+$. A short computation shows that for constant $\tilde\psi, \tilde\phi$:

$$B^{(i)}|_{r=r_+} = -\frac{s_p c_p}{s_i c_i}dy\,, \qquad k|_{r=r_+} = 0\,. \tag{146}$$

and hence the first bracket in (145) does not contribute in the $\mathbf{R}^2$–integral. The other terms give:

$$\int_{\mathbf{R}^2} H = \int_{\mathbf{R}^2} i_K G = -L_y\omega_y|_{r=r_+} = L_y\left(\frac{c_1}{s_1}+\frac{s_p^2-\alpha s_p c_p}{s_1c_1}\right)$$

$$= L_y\frac{M_1+M_p-\alpha Q_p}{Q_1}\,, \tag{147}$$

using the notation

$$M_i = \frac{m}{2}\cosh(2\delta_i)\,, \tag{148}$$

which gives the contribution to the 5D ADM mass in the i-channel (so that $M_{ADM,5D} = (L_y\pi)/(4G_6)\sum_i M_i$).

In the end, we find that (139) becomes

$$\mathcal{Q}_K = -\frac{L_y\pi}{4G_6}\left(\frac{M_5+M_p-\alpha Q_p}{Q_5}Q_5 + \frac{M_1+M_p-\alpha Q_p}{Q_1}Q_1\right) \tag{149}$$

$$= -\frac{L_y\pi}{4G_6}\left(M_1+M_5+2(M_p-\alpha Q_p)\right)\,. \tag{150}$$

For $\alpha = 0$, we have $K = \partial_t$ and we retrieve the 6D ADM mass (120) for the Komar charge $\mathcal{Q}_K$. Note that each term of the second line contributes to the M_p-channel. Also, in a sense, the non-extremality resides only in the integral over H; the integrals over S^3 of $G_3, \tilde{G}_3$ contribute the charge. For $\alpha = 1$, so that $K = \partial_t + \partial_y$, the Komar charge in the supersymmetric limit becomes the usual null charge $\mathcal{Q}_K = -(L_y\pi)/(4G_6)(Q_1+Q_5)$.

Appendix

A Uplift of five-dimensional multi-center solutions

A.1 General reduction

Reducing 6D minimal supergravity plus a tensor multiplet gives the STU model in 5D. The 6D metric $\hat{g}_{ab}$ decomposes into the 5D metric g_{ab}, a graviphoton A^1_a, and a scalar ϕ_2. The three-form gives two gauge fields: $\hat{G}_{abc} \sim (\star_5 F^2)_{abc}$ and $\hat{G}_{ab6} \sim F^3_{ab}$. Finally, our 6D scalar gives a scalar in 5D $\hat{X} = \phi_1$. We can then reparametrize the 5D scalars ϕ_1, ϕ_2 to get the usual three constrained scalars X^I of the STU model.

We use hats to denote 6D quantities in this section; unhatted quantities, such as indices, are 5D. We start with the 6D Lagrangian:

$$\sqrt{-\hat{g}}\,\mathcal{L}_6 = \sqrt{-\hat{g}}\left[\hat{R} - 2\partial_{\hat{\mu}}X\partial^{\hat{\mu}}X - \frac{1}{3}e^{2\sqrt{2}\hat{X}}\hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{G}^{\hat{\mu}\hat{\nu}\hat{\rho}}\right]. \tag{151}$$

We call the (spacelike) coordinate along which we reduce y. The reduction ansatz for the metric is:

$$d\hat{s}^2 = e^{\phi_2/\sqrt{6}}ds_5^2 + e^{-3\phi_2/\sqrt{6}}(dy + A^1_a dx^a)^2, \tag{152}$$

with inverse:

$$(\partial\hat{s})^2 = e^{-\phi_2/\sqrt{6}}(\partial s_5)^2 - 2e^{-\phi_2/\sqrt{6}}A^{1\mu}\partial_\mu\partial_y + (e^{3\phi_2/\sqrt{6}} + e^{-\phi_2/\sqrt{6}}(A^1)^2)\partial_y^2. \quad (153)$$

The Einstein-Hilbert Lagrangian then reduces to:

$$\frac{1}{G_6}\sqrt{-\hat{g}}\hat{R} = \frac{1}{G_5}\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi_2)^2 - \frac{1}{4}e^{-4\phi_2/\sqrt{6}\phi_2}(F^1)^2\right], \quad (154)$$

where $G_6 = L_y G_5$. Note that $\sqrt{-\hat{g}} = e^{\phi_2/\sqrt{6}}\sqrt{-g_5}$.

The kinetic term for the 6D scalar $\hat{X}$ gives the contribution:

$$\frac{1}{G_6}\sqrt{-\hat{g}}\left[-2\partial_{\hat{\mu}}X\partial^{\hat{\mu}}X\right] = \frac{1}{G_5}\sqrt{-g}\left[-2(\partial\phi_1)^2\right]. \quad (155)$$

Finally, reducing the three-form can be done most easily using form notation. The reduction ansatz is:

$$2\,\hat{G} = e^{-2\sqrt{2}\phi_1 + 2\phi_2/\sqrt{6}} \star_5 F^3 + F^2 \wedge (dy + A^1), \quad (156)$$

which also implies:

$$2\,\hat{\star}\,\hat{G} = e^{2\phi_2/\sqrt{6}} \star_5 F^2 + e^{-2\sqrt{2}\phi_1}F^3 \wedge (dy + A^1). \quad (157)$$

Then the reduction of the kinetic term is:

$$\begin{aligned} 2\,e^{2\sqrt{2}X}\hat{\star}\hat{G} \wedge \hat{G} = dy \wedge \Bigg[&\frac{1}{2}e^{-2\sqrt{2}\phi_1 + 2\phi_2/\sqrt{6}}F_3 \wedge \star_5 F_3 \\ &+ \frac{1}{2}e^{2\sqrt{2}\phi_1 + 2\phi_2/\sqrt{6}} \star_5 F_2 \wedge F_2 + F^3 \wedge F^2 \wedge A^1\Bigg]. \end{aligned} \quad (158)$$

Summarizing, the reduction gives us the 5D Lagrangian:

$$\begin{aligned} \sqrt{-g}\,\mathcal{L}_5 = \sqrt{-g}\Bigg[&R - \frac{1}{2}(\partial\phi_2)^2 - \frac{1}{4}e^{-4\phi_2/\sqrt{6}\phi_2}(F^1)^2 - 2(\partial\phi_1)^2 \\ &- \frac{1}{4}e^{2\sqrt{2}\phi_1 + 2\phi_2/\sqrt{6}}(F^2)^2 - \frac{1}{4}e^{-2\sqrt{2}\phi_1 + 2\phi_2/\sqrt{6}}(F^3)^2\Bigg] \\ &- \frac{1}{4}\epsilon^{\mu\nu\rho\sigma\lambda}A^1_\mu F^2_{\nu\rho}F^3_{\sigma\lambda}. \end{aligned} \quad (159)$$

To bring this to the usual STU form, we can define:

$$X_1 = e^{2\phi_2/\sqrt{6}}, \qquad X_2 = e^{-\phi_2/\sqrt{6} - \sqrt{2}\phi_1}, \qquad X_3 = e^{-\phi_2/\sqrt{6} + \sqrt{2}\phi_1}, \quad (160)$$

so that $X^1X^2X^3 = 1$, and the Lagrangian can be written as:

$$\mathcal{L}_5 = R - \frac{1}{4}\frac{1}{(X^I)^2}(F^I)^2 - \frac{1}{2}\frac{(\partial X^I)^2}{(X^I)^2} - \frac{1}{4}e^{-1}\epsilon^{\mu\nu\rho\sigma\lambda}A^1_\mu F^2_{\nu\rho}F^3_{\sigma\lambda}, \quad (161)$$

with sum over $I = \{1,2,3\}$ implied. This is the usual form of the STU Lagrangian. We can also write this as:

$$\mathcal{L}_5 = R - \frac{1}{2} Q_{IJ} F^I_{\mu\nu} F^{J\,\mu\nu} - Q_{IJ} \partial_\mu X^I \partial^\mu X^J - \frac{1}{24} e^{-1} C_{IJK} \epsilon^{\mu\nu\rho\sigma\lambda} A^I_\mu F^J_{\nu\rho} F^K_{\sigma\lambda}, \quad (162)$$

where we have $C_{IJK} = |\epsilon_{IJK}|$ and:

$$\frac{1}{6} C_{IJK} X^I X^J X^K = 1, \quad (163)$$

$$Q_{IJ} := \frac{9}{2} X_I X_J - \frac{1}{2} C_{IJK} X^K, \quad (164)$$

$$X_I := \frac{1}{6} C_{IJK} X^J X^K. \quad (165)$$

A.2 Uplifting SUSY solutions

The most general 6D supersymmetric metric can be written as [GMR03, CMC04]:

$$ds_6^2 = -2H^{-1}(dv + \beta)[du + \omega + \frac{\mathcal{F}}{2}(dv + \beta)] + H dx_4^2, \quad (166)$$

$$= -H^{-1}\mathcal{F}[dv + \beta + \mathcal{F}^{-1}(du + \omega)]^2 + H^{-1}\mathcal{F}^{-1}(du + \omega)^2 + H dx_4^2. \quad (167)$$

The rewriting of the metric in the second line shows us that we can reduce along v as long as it is a spacelike coordinate, i.e. $\mathcal{F} < 0$ everywhere. The reduction gives us:

$$\begin{aligned} ds_5^2 &= -H^{-4/3}\mathcal{F}^{-2/3}(du + \omega)^2 + H^{2/3}(-\mathcal{F}^{1/3}) dx_4^2, \\ e^{-3\phi_2/\sqrt{6}} &= H^{-1}(-\mathcal{F}), \\ A^1 &= \beta + \mathcal{F}^{-1}(du + \omega). \end{aligned} \quad (168)$$

We see that the 6D null coordinate u becomes a timelike coordinate in 5D [GMR03].

With the metric, gauge fields and scalars in 5D given by (68)-(70), we can then identify the appropriate 6D quantities in terms of the 5D ones as follows:

$$\mathcal{F} = -Z_1, \qquad \omega = k, \qquad \beta = B^1, \qquad H = (Z_2 Z_3)^{1/2}. \quad (169)$$

For reference, the full 6D fields are given by:

$$
\begin{aligned}
ds_6^2 &= -\frac{1}{Z_1 (Z_2 Z_3)^{1/2}} (du + k)^2 + (Z_2 Z_3)^{1/2} ds_4^2 \\
&\quad + \frac{Z_1}{(Z_2 Z_3)^{1/2}} (dv - Z_1^{-1}(du + k) + B^1)^2, \\
e^{\sqrt{2}X} &= e^{\sqrt{2}\phi_1} = X_1^{1/2} X_3 = \frac{Z_2^{1/2}}{Z_3^{1/2}}. \\
2G &= X_3^{-2} \star_5 F^3 + F^2 \wedge (dv + A^1).
\end{aligned} \tag{170}
$$

B Rigid T^4 Reduction of IIB and $SO(1,2)$ Truncation

The reduction of IIB supergravity to six-dimensional $\mathcal{N} = (1,0)$ supergravity with 2 tensor multiplets goes in two steps. In a first step, reduction of the bosonic sector on a rigid T^4 gives a theory with $SO(2,2)$ global symmetry [DLP99]. Then the compatibility with D1-D5-P supersymmetries as in [GMPR13] leads to the bosonic sector of the $SO(1,2)$ invariant supergravity.

First, we reduce IIB supergravity on a T^4, keeping only the components of the fields with indices over the remaining six dimensions. This gives us two dilatons (from the 10D dilaton ϕ and the breathing mode of the T^4); two axions (from the 10D axion $C_{(0)}$ and from the only relevant component of $C_{(4)}$), along with the two reduced three-forms coming from the potentials $C_{(2)}$ and $B_{(2)}$. The reduction ansatz is [LLPT99, DLP99]:

$$
\begin{aligned}
ds^2_{10,str} &= e^{\phi_1/2}\left(e^{\phi_2/2} ds_6^2 + e^{-\phi_2/2} ds^2_{T_4}\right), &\quad C_{(0)} &= \chi_1, \\
\phi &= \phi_1, & C_{(2)} &= C_{(2)}, \\
B_{(2)} &= B_{(2)}, & C_{(4)} &= -\chi_2 \,\mathrm{vol}(T_4) + \cdots ,
\end{aligned} \tag{171}
$$

where $ds^2_{T_4}$ and $\mathrm{vol}(T^4)$ are the flat metric and flat volume element on T^4. The $\cdots$ in $C_{(4)}$ are other terms that follow from the self-duality condition $F_{(5)} = \star F_{(5)}$. Note that we use the IIB supergravity conventions as in [BGR+15]. The resulting 6D Lagrangian is [DLP99]:

$$
\begin{aligned}
\mathcal{L}_{6D,SO(2,2)} = R &- \frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 - \frac{1}{2}e^{2\phi_1}(\partial\chi_1)^2 - \frac{1}{2}e^{2\phi_2}(\partial\chi_2)^2 \\
&- \frac{1}{12}e^{-\phi_1-\phi_2} H^2_{(3)} - e^{\phi_1-\phi_2}\frac{1}{12}F^2_{(3)} + \chi_2 H_{(3)} \wedge dC_{(2)},
\end{aligned} \tag{172}
$$

with $F_{(3)} \equiv dC_{(2)} - C_{(0)}H_{(3)}$. This reduction/truncation has an $SO(2,2) \cong SL(2)_1 \times SL(2)_2$ symmetry where each $\tau_i = \chi_i + ie^{-\phi_i}$ parametrizes an $SL(2)/SO(2)$ coset. The $SO(2,2)$ is not a symmetry of the tensor Lagrangian, but rather of the equations of motion and Bianchi identities. Those can be written as Bianchi identities of an $SO(2,2)$ vector of field strengths G^r with components

$$\begin{pmatrix} G^1 \\ G^2 \end{pmatrix} \equiv \begin{pmatrix} dB_{(2)} \\ dC_{(2)} \end{pmatrix}\,, \qquad \begin{pmatrix} G^3 \\ G^4 \end{pmatrix} \equiv \begin{pmatrix} \frac{d\mathcal{L}_{6D}}{dG^2} \\ -\frac{d\mathcal{L}_{6D}}{dG^1} \end{pmatrix} = -e^{\phi_2}(i\sigma_2)\cdot\mathcal{M}_1\cdot\begin{pmatrix} \star G^1 \\ \star G^2 \end{pmatrix} + \chi_2 \begin{pmatrix} G^1 \\ G^2 \end{pmatrix}\,. \tag{173}$$

Those tensors obey the duality relation (compare (45)):

$$\mathcal{M}_{rs}G^s = \eta_{rs} \star G^s\,, \tag{174}$$

with the off-diagonal $SO(2,2)$ metric $\eta = (i\sigma_2)\otimes(i\sigma_2)$ and scalar matrix

$$\mathcal{M} = \mathcal{M}_2(\tau_2)\otimes\mathcal{M}_1(\tau_1), \quad \text{with} \quad \mathcal{M}_i = V_iV_i^T, \quad V_i = \begin{pmatrix} e^{-\frac{1}{2}\phi_i} & \chi_i e^{\frac{1}{2}\phi_i} \\ 0 & e^{\frac{1}{2}\phi_i} \end{pmatrix}. \tag{175}$$

It is important to realize that this $SO(2,2)$ theory cannot be the bosonic part of any supergravity theory. One can perform a further truncation to obtain a theory that can be the bosonic part of $SO(1,2)\cong SL(2)$ supergravity by setting $\tau_2 = f(\tau_1)$ with f an $SL(2)$-transformation. This identifies a 'diagonal' $SL(2)$ subgroup in $SO(2,2)\cong SL(2)_1\times SL(2)_2$. The four tensors G^r then decompose in a singlet and a triplet under this truncation. Consistency of the truncation requires that we put the singlet to zero.

We are interested in solutions with the supersymmetries of the D1-D5-P system [GMPR13], giving the truncation:

$$\tau_2 = -\frac{1}{\tau_1}. \tag{176}$$

The τ_2 equation of motion then requires that we put the singlet G^1+G^4 to zero. The remaining three field strengths are

$$\hat{G}^1 = \frac{1}{2}(G^3 - G^2)\,, \quad \hat{G}^2 = \frac{1}{2}(G^2 + G^3)\,, \quad \hat{G}^3 = \frac{1}{2}(G^4 - G^1)\,, \tag{177}$$

Dropping the hats again, G^r then obeys the self-duality relation with the $SO(1,2)$ matrix

$$V = \exp(\chi E_+)\exp(\phi H/2)\,, \qquad E_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\,, \qquad H = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{178}$$

and the Komar integral (51) applies.

To make the connection to the theory with one tensor multiplet clear, we write the vanishing singlet as an anti self-duality constraint on a three-form G':

$$G' = -\star G', \qquad G' = dB_{(2)} + \chi_2 dC_{(2)} = \frac{e^{-2\phi_1}H_{(3)} - \chi_1 F_{(3)}}{e^{-2\phi_1}+\chi_1^2}. \tag{179}$$

We can then take $G \equiv (1/2)dC_{(2)}$ to be the (unrestricted) tensor that is the combination of the other self-dual and anti self-dual tensors. In section 4.1.4, we take $\phi = \phi_1, \chi = \chi_1$ and $B = (1/2)C_{(2)}, B' = B_{(2)}$. An obvious further truncation of this $SO(1,2)$ theory is to take $G' = 0, \chi_1 = 0$ which leaves us with the $SO(1,1)$ sector used in large parts of this paper, after the identification $\phi_1 = \sqrt{2}X$.

For the $SO(1,2)$ theory with the unrestricted three-form G and the anti self-dual three-form G' as defined above, the generalization (50) of (55) for the null charge reduces to:

$$\mathcal{Q}_K = -\frac{1}{4\pi G_6}\int_V \left(H \wedge \tilde{G} + \tilde{H} \wedge G\right) + \frac{1}{8\pi G_6}\int_V \left(H' \wedge G'\right) , \tag{180}$$

where $H, \tilde{H}$ are defined as in (56), keeping in mind the $SO(1,2)$-generalized definitions for the dual form:

$$\tilde{G} = \frac{e^{2\phi_1}}{1 + e^{2\phi_1}\chi_1^2} \star G, \tag{181}$$

The harmonic form H' is defined by the split:

$$i_K G' = d\Lambda' + H', \tag{182}$$

where Λ' is a globally defined one-form.

II

Black Chemistry

In the previous chapter we have learned how to describe the microstates of a black hole in terms of primary fields of the effective conformal field theory living on the world-volume of a compactified system of D- or M-branes. We used techniques of two dimensional conformal field theory - notably Cardy's formula - and hence we were tacitly concerned with black holes that have an AdS_3 factor in their near-horizon geometry. Even the four dimensional black-holes we encountered in the introduction of chapter I could be seen as the reduction from five dimensions — where the black hole has an AdS_3 factor — to four dimensions. In the spirit of the AdS/CFT conjecture, we then indeed expect such black holes to have a description in terms of a two dimensional CFT. This in not necessarily a generic set-up.

There is namely a class of extremal black holes that arises in string theory and M-theory that can be obtained by considering a Calabi-Yau compactification, with *generic* $SU(3)$ holonomy, of eleven dimensional M-theory down to five non-compact dimensions. Typically, such a Calabi-Yau will not have $U(1)$ isometries that we can easily T-dualize along. Yet, wrapping branes around the supersymmetric cycles of the Calabi-Yau yields extremal five-dimensional black holes [BMPV97, KKV99], which again have an $SL(2,\mathbf{R})$ isometry in the near horizon. This AdS_2 does not naturally fit inside an AdS_3 with a two-dimensional CFT dual.[1] Remarkably, attempts to count the microstates of such supersymmetric black holes have faced significant difficulties [HKMT09]. In fact, whenever the precise counting of microstates has been successful it has involved a Cardy formula [SV96, Vaf98, MSW97]. Lacking the larger Virasoro structure, one may wonder where the 'isolated' $SL(2,\mathbf{R})$ of these black holes originates and to what extent it is robust. Perhaps an additional motivation for understanding such an 'isolated' AdS_2 geometry is the emergence of an $SL(2,\mathbf{R})$ symmetry in the world-line data of the static patch of de Sitter space [AHH12, Ann12]. It is interesting to note that the static patch of four-dimensional de Sitter space is conformally equivalent to $\text{AdS}_2 \times S^2$, whose 'isolated' $SL(2,\mathbf{R})$ does not seem to reside within a larger structure containing a Virasoro algebra.

One particular way the $SL(2,\mathbf{R})$ isometeries of the black hole manifest themselves is in the worldine dynamics of D-particles propagating in the near horizon region. We might then ask whether there are microscopic models, such as matrix

[1] One might also consider this AdS_2 as a degenerate limit of the warped AdS_3/NHEK near horizon geometry of the rotating black hole, in the limit of vanishing angular momentum. The $SL(2,\mathbf{R})$ might then be a global subgroup of the full symmetries associated to the duals of such geometries [GS09, ALP+09, Ann09, GHSS09, GS11, DHH12].

quantum mechanics models with a large number of ground states, whose effective eigenvalue dynamics describe an $SL(2,\mathbf{R})$ invariant multiparticle theory.[2]

Some insight into these issues can be provided by studying certain quiver quantum mechanics models, which capture the low energy dynamics of strings connecting a collection of wrapped branes [DM96, Den02, DM11, BBdB+12, LWY12, MPS12, MPS13] in a Calabi-Yau compactification of type IIA string theory to four-dimensions. Under certain conditions these quiver theories have an exponential number of (supersymmetric) ground states whose logarithm goes as the charge of the branes squared, which is the same scaling as the entropy of a supersymmetric black hole in $\mathcal{N}=2$ supergravity. It has been argued [DM11, BBdB+12] that the near horizon AdS_2 of these supersymmetric black holes is related to the exponential explosion in the number of ground states in the quiver quantum mechanics. The states in question are referred to as pure-Higgs states since they reside in the Higgs branch of the quantum mechanics, where all the branes sit on top of each other. Interestingly, going to the Coulomb branch after integrating out the massive strings stretched between the wrapped branes, leads a non-trivial potential and velocity dependent forces governing the wrapped brane position degrees of freedom in the non-compact space. Moreover, whenever the Higgs branch has an exponentially large number of ground states, the Coulomb branch exhibits a family of supersymmetric scaling solutions [Den02, DM11, BWW08] continuously connected to its origin. The equations determining the positions of the wrapped branes in such supersymmetric zero energy scaling solutions are reproduced in four-dimensional $\mathcal{N}=2$ supergravity [Den02], believed to be the appropriate description of the system in the limit of a large number of wrapped branes.

In this note we would like to touch upon some of these issues. We do so by discussing two aspects of the Coulomb branch of such quiver theories, particularly those describing three wrapped branes containing scaling solutions.

First we derive the Coulomb branch Lagrangian of a three node quiver model (see figure II.1), and establish the existence of a low energy scaling limit where the theory exhibits the full $SL(2,\mathbf{R})$ symmetry of conformal quantum mechanics [**?**, **?**, **?**, **?**]. These scaling theories have velocity dependent forces, a non-trivial potential as well as a metric on configuration space. It is also worth noting that the full quiver quantum mechanics theory is itself *not* a conformal quantum mechanics (and most certainly not a two-dimensional conformal field theory). The emergence of a full $SL(2,\mathbf{R})$ symmetry rather than only a dilatation symmetry in the scaling limit is not guaranteed, and is reminiscent of the emergence of a

[2]The original $N \times N$ Hermitean matrix models [BIPZ78] in the double scaling limit has eigenvalue dynamics described by free fermions which naturally have an $SL(2,\mathbf{R})$ symmetry. Of course, such models contain only the eigenvalue degrees of freedom due to the $U(N)$ gauge invariance that allows for a diagonalization of the matrix, and hence do not have $\mathcal{O}(N^2)$ degrees of freedom. These models are dual to strings propagating in two dimensions (for some reviews see [Kle91, GM93, Pol94]).

full $SL(2,\mathbf{R})$ in the near horizon geometry of extremal black holes.

Second, we study the behavior of the Coulomb branch upon integrating out the strings in a thermal state, rather than in their ground state. At sufficiently high temperatures, the Coulomb branch melts into the Higgs branch. This is reminiscent of the gravitational analogue where increasing the temperature of a black hole increases its gravitational pull, or a particle falling back into the finite temperature de Sitter horizon.

1 Quiver quantum mechanics

In this section we discuss the quiver quantum mechanics theory and its Coulomb branch. These theories constitute the low energy, non-relativistic and weakly coupled sector of a collection of branes along the supersymmetric cycles of a Calabi-Yau three fold. The wrapped branes look pointlike in the four-dimensional non-compact Minkowski universe.

1.1 Full quiver theory

The $\mathcal{N}=4$ supersymmetric quiver quantum mechanics comprises the following fields: chiral multiplets $\Phi^\alpha_{ij}=\{\phi^\alpha_{ij},\psi^\alpha_{ij},F^\alpha_{ij}\}$ and vector multiplets $\mathbf{X}_i=\{A_i,\mathbf{x}_i,\lambda_i,D_i\}$. The ψ^α_{ij} are the fermionic superpartners of the ϕ^α_{ij}, the λ_i are the fermionic superpartners of the scalars $\mathbf{x}_i$, A_i is a $U(1)$ connection, and F^α_{ij} and D_i are auxiliary scalar fields. The Φ^α_{ij} transform in the $(\bar{\mathbf{1}}_i,\mathbf{1}_j)$ of the $U(1)_i\times U(1)_j$. The index $\alpha=1,2,\ldots,|\kappa_{ij}|$ denotes the specific arrow connecting node i to node j (see figure II.1). The chiral multiplets encode the low energy dynamics of strings stretched between the wrapped D-branes of mass m_i sitting at three-vector positions $\mathbf{x}_i$ in the non-compact four-dimensions. The index $i=1,2,\ldots,N$ denotes the particular wrapped D-brane. The electric-magnetic charge vector, $\Gamma_i=(Q^{(i)}_I,P^{(i)}_I)$, of the wrapped branes depends on the particular cycles that they wrap, and the Zwanziger-Schwinger product of their charges are given by the $\kappa_{ij}=(P^{(i)}_I Q^{(j)}_I-Q^{(i)}_I P^{(j)}_I)$. The κ_{ij} count the number of intersection points in the internal manifold between wrapped branes i and j. In what follows we measure everything in units of the string length l_s which we have set to one.

The Lagrangian $L=L_V+L_C+L_W$ for the three-node quiver quantum mechanics [Den02] describing the low energy non-relativistic dynamics of three wrapped branes which are pointlike in the $(3+1)$ non-compact dimensions is given by:

$$L_V=\sum_{i=1}^{3}\frac{\mu^i}{2}\left(\dot{\mathbf{q}}^i\cdot\dot{\mathbf{q}}^i+D^iD^i+2i\bar{\lambda}^i\dot{\lambda}^i\right)-\theta^iD^i\ , \tag{1}$$

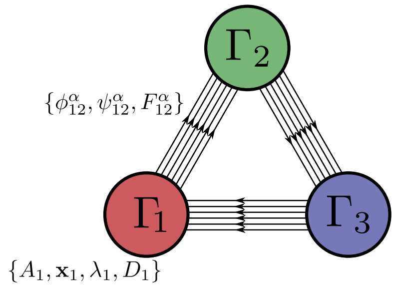

Figure II.1: A 3-node quiver diagram which captures the field content of the Lagrangian $L = L_V + L_C + L_W$, each piece of which is given in (1), (2), and (5). This quiver admits a closed loop if κ^1, $\kappa^2 > 0$ and $\kappa^3 < 0$.

and,

$$L_C = \sum_{i=1}^{3} |\mathcal{D}_t \phi^i_\alpha|^2 - \left(\mathbf{q}^i \cdot \mathbf{q}^i + s_i D^i\right) |\phi^i_\alpha|^2 + |F^i_\alpha|^2 + i\bar{\psi}^i_\alpha \mathcal{D}_t \psi^i_\alpha$$
$$- s_i \bar{\psi}^i_\alpha (\boldsymbol{\sigma} \cdot \mathbf{q}^i)\psi^i_\alpha + i\sqrt{2}\left(s_i \bar{\phi}^i_\alpha \lambda^i \epsilon \psi^i_\alpha - \text{h.c.}\right) \ . \quad (2)$$

In (1-2) and what follows we will mostly work with the relative degrees of freedom (i.e. $\mathbf{x}_{ij} \equiv \mathbf{x}_i - \mathbf{x}_j$, $D_{ij} \equiv D_i - D_j$, etc.) since the center of mass degrees of freedom decouple and do not play a role in our discussion. The notation we use is somewhat non-standard (for example $(\mathbf{q}^1, \mathbf{q}^2, \mathbf{q}^3) \equiv (\mathbf{x}_{12}, \mathbf{x}_{23}, \mathbf{x}_{13})$) and is given in appendix A along with all of our conventions. We note that the relative Lagrangian is only a function of two of the three vector multiplets, since $\mathbf{q}^3 = \mathbf{q}^1 + \mathbf{q}^2$, $D^3 = D^1 + D^2$ and $\lambda^3 = \lambda^1 + \lambda^2$. The s_i encode the orientation of the quiver. For the majority of our discussion we choose $s_1 = s_2 = -s_3 = 1$, corresponding to a closed loop like the one in figure II.1. The reduced masses μ^i (which we denote in superscript notation in (1-2)) are related to the masses $m_i \sim 1/g_s$, where g_s is the string coupling constant, of the wrapped branes sitting at the $\mathbf{x}_i$ by:

$$\mu^1 = \frac{m_1 m_2}{m_1 + m_2 + m_3} \ , \quad \mu^2 = \frac{m_2 m_3}{m_1 + m_2 + m_3} \ , \quad \mu^3 = \frac{m_1 m_3}{m_1 + m_2 + m_3} \ . \quad (3)$$

The superpotential, which is allowed by gauge invariance only when the quiver has a closed loop, is given by:

$$W(\phi) = \sum_{\alpha,\beta,\gamma} \omega_{\alpha\beta\gamma} \phi^1_\alpha \phi^2_\beta \phi^3_\gamma + \text{higher order terms} \ , \quad (4)$$

(where we take coefficients $\omega_{\alpha\beta\gamma}$ to be arbitrary) and contributes the following piece to the Lagrangian:

$$L_W = \sum_{i=1}^{3}\left(\frac{\partial W(\phi)}{\partial \phi^i_\alpha}F^i_\alpha + \text{h.c.}\right) + \sum_{i,j=1}^{3}\left(\frac{\partial^2 W(\phi)}{\partial \phi^i_\alpha \partial \phi^j_\beta}\psi^i_\alpha \epsilon \psi^j_\beta + \text{h.c.}\right) . \quad (5)$$

In this note we only consider cubic superpotentials and ignore the higher order terms. This is consistent so long as the ϕ^i_α are small, which in turn can be assured by taking the $|\theta^i|$ sufficiently small [DM11].

The theory contains a manifest $SO(3)$ global R-symmetry. In the absence of a superpotential, the Lagrangian is diagonal in the arrow (Greek) indices and thus the theory also exhibits a $U(|\kappa^1|)\times U(|\kappa^2|)\times U(|\kappa^3|)$ global symmetry under which the ϕ^i_α transform as $U(|\kappa^i|)$ vectors. The superpotential explicitly breaks this symmetry down to the $U(1)_1 \times U(1)_2 \times U(1)_3$ gauge symmetry.

The theory can be obtained by dimensionally reducing an $\mathcal{N}=1$ gauge theory in four-dimensions to the $(0+1)$-dimensional worldline theory. It can also be viewed as the dimensional reduction of the $\mathcal{N}=2$ two-dimensional σ-models studied extensively, for example, in [Wit93]. In order for the theory to have supersymmetric vacua we also demand that the Fayet-Iliopoulos constants sum to zero: $\theta_1+\theta_2+\theta_3=0$.

In units where $\hbar=1$ is dimensionless, and where we choose dimensions for which $[t]=1$, the dimensions of energy are automatically set to $[E]=-1$. We also find the following dimensional assignments: $[\phi]=1/2$, $[D]=-2$, $[\mathbf{x}]=-1$, $[\mu]=3$, $[\omega_{\alpha\beta\gamma}]=-3/2$, $[\psi]=0$, $[\lambda]=-3/2$, $[F]=-1/2$ and $[\theta]=1$. The mass squared of the ϕ^i_α fields upon integrating out the auxiliary D fields is given by $M^2_{ij}=(|\mathbf{x}_{ij}|^2+\theta_i/m_i-\theta_j/m_j)$. For physics whose energies obey $E^2/M^2 \ll 1$ we can integrate out the massive ϕ^i_α fields and study the effective action on the Coulomb branch. Finally, notice also that the coupling $\omega_{\alpha\beta\gamma}$ has positive units of energy and is thus strong in the infrared limit since the natural dimensionless quantity is $\omega_{\alpha\beta\gamma}/E^{3/2}$. The contribution from ϕ^i_α loops grows as κ^i and thus the effective coupling constant at low energies is given by $g_{\text{eff}} \sim g_s\kappa$. In the large g_{eff} limit, the wrapped branes backreact and the appropriate description of the system is given by four-dimensional $\mathcal{N}=2$ supergravity [Den02].

1.2 Some properties of the ground states

An immediate question about the above model regards the structure of the ground states $\Psi_g\left[\Phi^i_\alpha,\mathbf{Q}^i\right]$ of the theory, satisfying: $\hat{\mathbf{H}}\,\Psi_g\left[\Phi^i_\alpha,\mathbf{Q}^i\right]=0$. Though an explicit expression for the full $\Psi_g\left[\Phi^i_\alpha,\mathbf{Q}^i\right]$ remains unknown, the degeneracy of ground states has been extensively studied [Den02, DM11, BBdB$^+$12, LWY12, MPS12, MPS13]. In particular, the degeneracy of ground states localized near $\mathbf{Q}^i=0$, i.e. the ground states of the Higgs branch of the theory, were shown to grow exponentially in κ^i when the theory contains a superpotential, the quiver admits closed loops (e.g. $\kappa^1,\kappa^2>0$ and $\kappa^3<0$) and the κ^i obey the triangle inequality (i.e. $|\kappa^2|+|\kappa^3|\geq|\kappa^1|$ and cyclic permutations thereof). This growth is related to the exponential explosion in the Euler characteristic of the complete intersection manifold $\mathcal{M}$ [DM11] given by imposing the constraints from the F-term ($\delta_{F^i_\alpha} L|_{F^i_\alpha=0} = 0$) onto the D-term constraints ($\delta_{D^i} L|_{D^i=0} = 0$).[3] Since κ^i goes as the charge squared of the associated $U(1)$ gauge symmetry, the number of ground states scales in the same way as the Bekenstein-Hawking entropy of the associated black hole solutions in the large g_{eff} limit. Though a complete match between the ground states of a single Abelian quiver model and the entropy of a BPS black hole in $\mathcal{N} = 2$ supergravity is not known,[4] the vast number of microstates makes these systems potentially useful candidate toy models to study features of extremal or near extremal black holes.

Other pieces of Ψ_g localized near $\Phi^i_\alpha = 0$, i.e. the (quantum) Coulomb branch of the theory, have also been studied [Den02]. Unlike the Higgs branch quiver with a closed loop, superpotential and an exponential growth in its number of ground states, it was found that the number of Coulomb branch ground states grows only polynomially in the κ^i. Interpreting the $\mathbf{q}^i$ as the relative positions of wrapped branes, these ground states can be viewed as describing various multi-particle configurations, as we will soon proceed to describe in further detail. To each ground state in the quantum Coulomb branch there exists a corresponding ground state in the Higgs branch, but the converse is not true. Another way to view this statement is that whenever a given Ψ_g has non-trivial structure in the $\mathbf{Q}^i$ directions and peaks sharply about $\Phi^i_\alpha = 0$, it will also have a non-trivial structure in the Φ^i_α directions and peak sharply about $\mathbf{Q}^i = 0$ but not vice versa.

2 Coulomb branch and a scaling theory

For large enough $|\mathbf{q}^i|$ we can integrate out the massive Φ^i_α's (in their ground state) from the full quiver theory (1). This can be done exactly given that the Φ^i_α appear quadratically in (1) whenever the superpotential vanishes. One finds the bosonic quantum effective Coulomb branch Lagrangian (up to quadratic order in $\dot{\mathbf{q}}^i$ and D^i):

$$L_{c.b.} = \frac{1}{2}\sum_{i=1}^{2} G_{ij}\ \left(\dot{\mathbf{q}}^i \cdot \dot{\mathbf{q}}^j + D^i\, D^i\right) - \sum_{i=1}^{3} s_i|\kappa^i|\mathbf{A}^d(\mathbf{q}^i)\cdot\dot{\mathbf{q}}^i - \left(\frac{s_i\,|\kappa^i|}{2|\mathbf{q}^i|} + \theta^i\right) D^i\ . \quad (6)$$

[3] As an example, we can take $\theta^1, \theta^2 < 0$. Then the complete intersection manifold is given by setting $\phi^3_\alpha = 0$, imposing the κ^3 F-term constraints: $\omega_{\alpha\beta\gamma}\phi^1_\alpha\phi^2_\beta = 0$, inside a $\mathbf{CP}^{\kappa^1-1} \times \mathbf{CP}^{\kappa^2-1}$ space coming from the D-term constraints: $|\phi^1_\alpha|^2 = -\theta^1$ and $|\phi^2_\alpha|^2 = -(\theta^2 - \theta^1)$. The space is a product of $\mathbf{CP}^k$'s since we have to identify the overall phase of the ϕ^i_α due to the $U(1)$ gauge connection. When $|\kappa^1| + |\kappa^2| - 2 \geq |\kappa_3|$, which for large κ amounts to the κ^i satisfying the triangle inequality, the number of constraints become less or equal to the dimension of $\mathbf{CP}^{\kappa^1-1} \times \mathbf{CP}^{\kappa^2-1}$, allowing for more complicated topologies for $\mathcal{M}$.

[4] Indeed, there are several quiver diagrams with the same net charges and one might suspect that all such quivers are required to obtain the correct entropy of the supersymmetric black hole (see for example [LWY14]).

The terms linear in $\dot{\mathbf{q}}^i$ and D^i follow from a non-renormalization theorem [Den02], whereas the quadratic piece in $\dot{\mathbf{q}}^i$ is derived in appendix D.1. Recall that the system is only a function of $\mathbf{q}^1$ and $\mathbf{q}^2$ since $\mathbf{q}^3 = \mathbf{q}^1 + \mathbf{q}^2$. The three-vector $\mathbf{A}^d$ is the vector potential for a magnetic monopole:

$$\mathbf{A}^d(\mathbf{x}) = \frac{-y}{2r(z \pm r)}\hat{x} + \frac{x}{2r(z \pm r)}\hat{y} \ , \tag{7}$$

and G_{ij} is the two-by-two metric on configuration space:

$$[G_{ij}] = \begin{pmatrix} \mu^1 + \mu^3 + \frac{1}{4}\frac{|\kappa^1|}{|\mathbf{q}^1|^3} + \frac{1}{4}\frac{|\kappa^3|}{|\mathbf{q}^1+\mathbf{q}^2|^3} & \mu^3 + \frac{1}{4}\frac{|\kappa^3|}{|\mathbf{q}^1+\mathbf{q}^2|^3} \\ \mu^3 + \frac{1}{4}\frac{|\kappa^3|}{|\mathbf{q}^1+\mathbf{q}^2|^3} & \mu^2 + \mu^3 + \frac{1}{4}\frac{|\kappa^2|}{|\mathbf{q}^2|^3} + \frac{1}{4}\frac{|\kappa^3|}{|\mathbf{q}^1+\mathbf{q}^2|^3} \end{pmatrix} . \tag{8}$$

Upon integrating out the auxiliary D^i-fields, we obtain a multi-particle quantum mechanics with (bosonic) Lagrangian:

$$L_{c.b.} = \frac{1}{2}\sum_{i=1}^{2} G_{ij}\,\dot{\mathbf{q}}^i \cdot \dot{\mathbf{q}}^j - \sum_{i=1}^{3} s_i|\kappa^i|\mathbf{A}^d(\mathbf{q}^i)\cdot\dot{\mathbf{q}}^i - V(\mathbf{q}^i) \ . \tag{9}$$

That the quantum effective Coulomb branch theory has a non-trivial potential $V(\mathbf{q}^i)$ should be contrasted with other supersymmetric cases such as interacting D0-branes or the D0-D4 system [DKPS97] where the potential vanishes and the non-trivial structure of the Coulomb branch comes from the moduli space metric. The potential $V(\mathbf{q}^i)$ is also somewhat involved and is given in appendix D.2.

2.1 Supersymmetric configurations

The supersymmetric configurations of the Coulomb branch consist of time independent solutions which solve the equations $V(\mathbf{q}^i) = 0$. For (6), this amounts to:

$$\frac{s_i\,|\kappa^i|}{|\mathbf{q}^i|} + \frac{s_3\,|\kappa^3|}{|\mathbf{q}^3|} + 2\theta^i = 0 \ , \quad i = 1, 2 \ . \tag{10}$$

In appendices B and C we review that these supersymmetric configurations are robust against corrections of the Coulomb branch theory from the superpotential and from integrating out higher orders in the auxiliary D fields.

2.1.1 Bound states

There are *bound state solutions* [Den02, Den00] of (10) which are triatomic (or more generally N-atomic if dealing with N wrapped branes) molecular like configurations. Of the original nine degrees of freedom, three can be removed by fixing the center of mass. Then the bound state condition (10) fixes another two-degrees of freedom. Thus, bound state solutions have a four-dimensional classical moduli space. Due to the velocity dependent terms in the Lagrangian, the flat directions in the moduli space are dynamically inaccessible at low energies – the particles resemble electrons in a magnetic field. Several dynamical features of the three particles were studied in [NO08, AAD$^+$13].

2.1.2 Scaling solutions

There are also *scaling solutions* [DM11] of (10) which are continuously connected to the origin $|\mathbf{q^i}| = 0$. They occur whenever the κ^i form a closed loop in the quiver diagram (e.g. $\kappa^1, \kappa^2 > 0$ and $\kappa^3 < 0$) and obey the same triangle inequality ($|\kappa^2| + |\kappa^3| \geq |\kappa^1|$ and cyclic permutations thereof) that the $|\mathbf{q}^i|$ are subjected to. These solutions can be expressed as a series:

$$|\mathbf{q}^i| = |\kappa^i| \sum_{n=1}^{\infty} a_n \lambda^n \ , \quad \lambda > 0 \ . \tag{11}$$

The coefficient $a_1 = 1$, while the remaining a_n can be obtained by systematically solving (10) in a small λ expansion, and will hence depend on θ^i. The moduli space of the scaling solutions is given by the three rotations as well as the scaling direction parameterized by λ. Though the angular directions in the moduli space are dynamically trapped due to velocity dependent forces, the scaling direction is not and constitutes a flat direction even dynamically.

Requiring that the series expansion converges, i.e. $(a_{n+1}\lambda^{n+1})/(a_n\lambda^n) \ll 1$, leads to the condition:

$$\lambda \ll \frac{1}{\theta} \ . \tag{12}$$

Because of this condition on the λ's, one should be cautious when dealing with such scaling solutions. They occur in the near coincident limit of the branes where the bifundamentals that we have integrated out become light. In order for the mass of the bifundamentals, $M_{ij}^2 = \big(|\mathbf{x}_{ij}|^2 + \theta_i/m_i - \theta_j/m_j\big)$, to remain large we require:

$$\left(\frac{\theta}{\mu}\right)^{1/2} \ll |\mathbf{q}^i| \ . \tag{13}$$

Taking $\mu^i = \nu\,\widehat{\mu}^i$ and $|\mathbf{q}^i| = |\widehat{\mathbf{q}}^i|/\nu^\alpha$ the inequalities (12) and (13) can be satisfied in the limit $\nu \to \infty$, with $\widehat{\mu}^i$, $\widehat{\mathbf{q}}^i$, κ^i and θ^i fixed and furthermore $\alpha \in (0, 1/2)$.

Notice that (12) implies that the distances between particles in a scaling regime $\sim \lambda\kappa$ is much less than the typical inter-particle distance of a bound state $\sim \kappa/\theta$.

2.2 Scaling theory

To isolate the physics of the Coulomb branch in the scaling regime we take an infrared limit of the Lagrangian (6), pushing the $\mathbf{q}^i$ near the origin and dilating the clock t. In particular, we would like the $\sim \kappa/|\mathbf{q}^i|^3$ part of the metric in configuration space to dominate over the $\sim \mu$ piece leading to:

$$|\mathbf{q}^i| \ll \left(\frac{\kappa}{\mu}\right)^{1/3} . \tag{14}$$

Additionally we must satisfy the inequalities (12) and (13). Again taking $\mu^i = \nu\,\widehat{\mu}^i$, $\mathbf{q}^i = \widehat{\mathbf{q}}^i/\nu^\alpha$ and in addition $t = \nu^\alpha\,\widehat{t}$ with fixed $\widehat{\mu}^i$, $\widehat{\mathbf{q}}^i$, κ^i and θ^i, we can also satisfy (14) in the limit $\nu \to \infty$, so long as we also ensure $\alpha \in (1/3, 1/2)$.[5] The rescaling of t is required to maintain a finite action in the scaling limit. In type II string compactifications $\mu \sim 1/l_P \sim \sqrt{v}/g_s l_s$, where l_P is the four-dimensional Planck length and v is the volume of the Calabi-Yau in string units [Den02], therefore the $\nu \to \infty$ limit corresponds to a parametrically small string coupling. Furthermore the scaling throat deepens as we increase the mass of the wrapped branes.

The rescaling above amounts simply to setting the θ^i and μ^i to zero in (6) and replacing $\mathbf{q}^i$ and t with $\widehat{\mathbf{q}}^i$ and $\widehat{t}$. We call the remaining Lagrangian with vanishing θ^i and μ^i the *scaling theory*. Notice from equation (75) that the potential $V(\mathbf{q}^i)$ in this limit becomes a homogeneous function of order one, i.e. $V(\nu\,\widehat{\mathbf{q}}^i) = \nu V(\widehat{\mathbf{q}}^i)$ and the linear in velocity term is retained.

2.2.1 Consistency of the small D^i expansion

One can infer from the supersymmetry variations [Den02] that supersymmetric configurations have vanishing D^i. Furthermore, as we have already noted, we have performed an expansion in small D^i fields prior to integrating them out (see appendix C for more details) in order to obtain the Coulomb branch. Thus, in order for a scaling theory to exist and be consistent with a small D^i expansion, it must be the case that zero energy scaling configurations exist.

Had we considered a two-particle theory, where no such scaling solutions can exist, taking a small $\mathbf{q}$ limit would be inconsistent with the small D expansion. That is because the dimensionless small quantity in our perturbation series is actually $D/|\mathbf{q}|^2$ and the supersymmetric configuration $D = 0$ occurs at $|\mathbf{q}| = -\kappa/2\theta$. Expanding the non-linear D equation (C.1) (see appendix C) in powers of $\epsilon \equiv D/|\mathbf{q}|^2$, while imposing $|\mathbf{q}| \gg (\theta/\mu)^{1/2}$, one finds the following consistency condition:

$$\epsilon = \left(\frac{\theta}{\mu}\right)\frac{1}{|\mathbf{q}|^2} + \left(\frac{\kappa}{2\mu}\right)\frac{1}{|\mathbf{q}|^3}\left(1 - \frac{1}{2}\epsilon + \mathcal{O}(\epsilon^2)\right) . \tag{15}$$

[5] Another limit one might imagine is given by: $|\mathbf{q}^i| \gg (\kappa/\mu)^{1/3}$, $|\mathbf{q}^i| \ll \kappa/\theta$. In this case the metric on configuration space remains flat while the potential scales like $\sim 1/|\mathbf{q}|^2$.

Indeed, the first term on the right hand side is negligible by construction, since we took $|\mathbf{q}| \gg (\theta/\mu)^{1/2}$ to keep the strings massive. Smallness of the second term in the equation would require $|\mathbf{q}|^3 \gg \kappa/\mu$, in contradiction with the condition (14) required to isolate the scaling theory.

We now consider the three-node case for s_i that admit a closed loop in the quiver. For small $\epsilon_2 \equiv D^2/|\mathbf{q}^2|^2$ (taking all masses the same and $\theta_1 = \theta_2 = \theta$ and again imposing $|\mathbf{q}^2| \gg (\theta/\mu)^{1/2}$) the equation of motion of the auxiliary D^2-field (in the form of (53)) is given by:

$$\epsilon_2 = \frac{3\theta}{2\mu|\mathbf{q}^2|^2} + \frac{|\kappa^2|}{2\mu|\mathbf{q}^2|^3}\left(\delta - \frac{1}{2}\epsilon_2 + \mathcal{O}(\epsilon_2^2)\right) \tag{16}$$

where $\delta \equiv 1 - \frac{|\mathbf{q}^2|}{2\,|\kappa^2|}\left(\frac{|\kappa^1|}{|\mathbf{q}^1|} + \frac{|\kappa^3|}{|\mathbf{q}^3|}\right)$ measures how close the configuration is to the scaling solution. For sufficiently small $\delta \sim \epsilon \ll 1$ we can consistently satisfy (16) in addition to imposing the scaling inequalities (14).

In this sense the scaling theory is in fact a theory of the deep infrared configurations residing parametrically near the zero energy scaling solutions. This is consistent with the dilation of time required to obtain the scaling theory.

3 Conformal quivers: emergence of $SL(2,\mathbf{R})$

In this section we uncover that the bosonic scaling theory action, i.e. (9) with the θ^i and μ^i set to zero, has an $SL(2,\mathbf{R})$ symmetry. This is the symmetry group of conformal quantum mechanics [**?**]. The group $SL(2,\mathbf{R})$ is generated by a Hamiltonian H, a dilatation operator D and a special conformal transformation K, with the following Lie algebra:

$$[H,D] = -2iH\ , \qquad [H,K] = -iD\ , \qquad [K,D] = 2iK\ . \tag{17}$$

As we have already mentioned, the full $SL(2,\mathbf{R})$ symmetry is not guaranteed by the existence of time translations and dilatations alone [BPMSV00]. This is suggested by the fact that the H and D operators form a closed subalgebra of the full $SL(2,\mathbf{R})$. The presence of a full $SL(2,\mathbf{R})$ is actually quite remarkable, particularly given the specific form of the scaling theory Lagrangian which has velocity dependent forces and a non-trivial potential.

As discussed in the previous section, it is not true that, for any finite κ^i, μ^i and θ^i, the Coulomb branch can be described precisely by the scaling theory action. There will always be small corrections that break its manifest scaling symmetry: $\mathbf{q}^i \to \gamma\ \mathbf{q}^i$, and $t \to t/\gamma$. This is comforting, given that the full quiver theory has a finite number of ground states—yet a conformal quantum mechanics has a diverging number of arbitrarily low-energy states, with a density of states that behaves as dE/E. The corrections serve as a cutoff for the infrared divergence in the number of states, such that the Coulomb branch can fit consistently inside the full quiver theory (related discussions can be found in [BBdB$^+$12, dBESMVdB09]).

3.1 Conditions for an $SL(2,\mathbf{R})$ invariant action

The conditions under which an action will be $SL(2,\mathbf{R})$ invariant (up to possible surface terms) have been studied extensively in [MS00, BPMSV00, Pap00]. Showing that a general theory with bosonic Lagrangian describing N degrees of freedom :

$$L = \frac{1}{2}\,\dot{q}^i\,G_{ij}\,\dot{q}^j - A_i\,\dot{q}^i - V(q)\ , \quad i = 1,2,\ldots,N \tag{18}$$

has an $SL(2,\mathbf{R})$ symmetry is equivalent to finding a solution to the following equations [Pap00]:

$$2\,\nabla_{(i}Z_{j)} = G_{ij}\ , \tag{19}$$
$$-Z^i\partial_i V = V\ , \tag{20}$$
$$2\,Z_i = \partial_i f\ , \tag{21}$$
$$Z^j F_{ji} = 0\ , \quad F_{ij} \equiv \partial_{[i}A_{j]}\ . \tag{22}$$

Equations (19) and (20) ensure the existence of a dilatation symmetry. In particular, equation (19) implies that the metric on configuration space allows for a conformal Killing vector field (also referred to as a homothetic vector field). Equations (21) and (22) ensure the that the action remains invariant under special conformal transformations, where f is an arbitrary function of the q^i. Indices are raised and lowered with the metric G_{ij}.

Interestingly, equation (21) imposes that the conformal Killing form Z_i of the metric be exact, which is generically not the case. Hence the existence of a dilatation symmetry does not necessarily imply the symmetry of the full conformal group.

Once a solution to (19-22) is found, the three conserved quantities are then given by [Pap00]:

$$\mathcal{Q}_n = \frac{1}{2}t^{n+1}\,\dot{q}^i\,G_{ij}\,\dot{q}^j - (n+1)t^n Z^i\,G_{ij}\,\dot{q}^j + t^{n+1}V(q) + F_n\ , \quad n = -1,0,1 \tag{23}$$

where $F_{-1} = F_0 = 0$ and $F_1 = f$. The charge $\mathcal{Q}_{-1}$ is the Hamiltonian, whereas $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are related to dilatations and special conformal transformations, respectively. These three charges generate the $SL(2,\mathbf{R})$ algebra (17) (up to factors of i) under the Poisson bracket. In what follows, we find Z^i and f for the scaling theory described in section 2.2.

3.2 Two particles

As a warm up, we can study a simple model consisting of a two node quiver with an equal number of arrows going to and from each node, with a total number $\kappa > 0$ arrows altogether. This is the theory describing, for example, the low energy dynamics of a wrapped D4-D0 brane system (see [DKPS97]). The bosonic Lagrangian for the relative position $\mathbf{q} = (q^x, q^y, q^z)$ on the Coulomb branch, in the scaling limit, is given by:

$$L = \frac{\kappa}{2} \frac{\dot{\mathbf{q}}^2}{|\mathbf{q}|^3} \ . \tag{24}$$

The above Lagrangian is also the non-relativistic limit of one describing a BPS particle in an $AdS_2 \times S^2$ background. The wordline theory has three degrees of freedom and a diagonal metric on configuration space: $g_{ij} = \kappa\ |\mathbf{q}|^{-3}\ \delta_{ij}$. In addition to the Hamiltonian H, the above theory has a dilatation operator D and special conformal generator K given by:

$$D = i \left(q^i\ \partial_i + |\mathbf{q}|^{-9/2}\ \partial_i\ |\mathbf{q}|^{9/2}\ q^i \right)\ , \quad K = 2\,\kappa\,|\mathbf{q}|^{-1}\ , \tag{25}$$

such that the $SL(2,\mathbf{R})$ algebra is satisfied. It is thus a simple example of a conformal quantum mechanics.

3.3 $SL(2,\mathbf{R})$ symmetry of the full three particle scaling theory

For our particular problem, it is useful to note that the index structure of the relative coordinates, i.e. $i = 1, 2, 3$, is trivially tensored with the spatial index implicit in bold vector symbols (e.g. $\mathbf{q} = (q^x, q^y, q^z)$). Introducing $Q^\alpha \equiv \left(\mathbf{q}^1, \mathbf{q}^2\right)$, with $\alpha = 1, 2, \ldots, 6$, our Lagrangian (9) (with μ^i and θ^i set to zero, and s_i that admit a closed loop) takes the form:

$$L_{c.b.} = \frac{1}{2}\ \dot{Q}^\alpha\ G_{\alpha\beta}\ \dot{Q}^\beta - A^{(Q)}_\alpha\ \dot{Q}^\alpha - V(Q^\alpha)\ , \tag{26}$$

where $G_{\alpha\beta}$ and the six-dimensional vector potential can be extracted from (9). An expression for the vector potential is simple to write down and is given by:

$$\begin{aligned} A^{(Q)}_\alpha = &\left(\ s_1\,|\kappa^1|\mathbf{A}^d(\mathbf{q}^1) + s_3\,|\kappa^3|\mathbf{A}^d(\mathbf{q}^1 + \mathbf{q}^2)\ ,\right. \\ &\left. s_2\,|\kappa^2|\mathbf{A}^d(\mathbf{q}^2) + s_3\,|\kappa^3|\mathbf{A}^d(\mathbf{q}^1 + \mathbf{q}^2)\ \right)\ . \end{aligned} \tag{27}$$

where $\mathbf{A}^d(\mathbf{x})$ is the vector potential for a magnetic monopole and is given in (7). It is straightforward to check, using Mathematica for example, that the conditions (19-22) are indeed satisfied. We find $Z^\alpha = -Q^\alpha$ and $f = 2\ Q^\alpha G_{\alpha\beta} Q^\beta$. The explicit generators of the $SL(2,\mathbf{R})$ are given by:

$$H = -\frac{1}{2}\left(\frac{1}{\sqrt{G}}\partial_\alpha\sqrt{G} - iA^{(Q)}_\alpha\right) G^{\alpha\beta}\left(\partial_\beta - iA^{(Q)}_\beta\right) + V(Q^\alpha)\ , \tag{28}$$

$$D = i\left(Q^\alpha\partial_\alpha + \frac{1}{\sqrt{G}}\partial_\alpha\sqrt{G}Q^\alpha\right)\ , \tag{29}$$

$$K = 2\ Q^\alpha G_{\alpha\beta} Q^\beta\ , \tag{30}$$

where we have used that $Q^\alpha A^{(Q)}_\alpha = 0$. The generator K simplifies to:

$$K = \frac{|\kappa^1|}{2|\mathbf{q}^1|} + \frac{|\kappa^2|}{2|\mathbf{q}^2|} + \frac{|\kappa^3|}{2|\mathbf{q}^3|} \ . \tag{31}$$

We thus conclude that the bosonic scaling theory is an interacting multi-particle $SL(2,\mathbf{R})$ invariant conformal quantum mechanics with velocity dependent forces. The theory admits a further $SO(3)$ symmetry acting on the spatial three-vectors.

3.3.1 N-particle case

Though we do not prove it here, it is natural to conjecture that the N-particle scaling theory will be an N-particle conformal quantum mechanics. Indeed, for any closed loop of N_i-particles we expect that in the corresponding throat there will be a conformal quantum mechanics with D and K as in (28) except Q^α runs over all the $3N_i$ bosonic degrees of freedom. The one loop metric on configuration space in the Coulomb branch, i.e. the coefficient of $\dot{\mathbf{q}}^i \cdot \dot{\mathbf{q}}^j$, will be equivalent to that of $D^i\,D^j$ (which is far easier to compute) upon integrating out the massive strings for general quivers. Interestingly, when there are more than three nodes, there may be several distinct closed loops allowing for their own scaling throat and several distinct scaling throats may coexist, within its residing a decoupled conformal quantum mechanics theory.

3.3.2 Superconformal quantum mechanics

It is worth remarking that the Coulomb branch is in fact a supersymmetric quantum mechanics with four supercharges $\mathcal{Q}^i_S$ and an $SO(3)$ global R-symmetry group. This follows from effective field theory. We are integrating out the heavy chiral Φ^i_α multiplet in its supersymmetric ground state. Thus, the low energy effective theory of the vector multiplet will be endowed with the four supercharges of the parent quiver theory. Of course, there could be anomalies that arise in the process. For instance, the discrete time reversal symmetry of the full quiver theory is violated in the Coulomb branch by the linear in velocity terms. This is an anomaly which does not spoil the supersymmetry of the Coulomb branch, and occurs due to a zero mode in the functional determinant of the ψ^i_α fermions. As shown in [**?**] for the two node case, one can compute the supercharges of the low energy effective theory in a systematic fashion using perturbation theory.

We have shown above that the bosonic Lagrangian of this theory exhibits an $SL(2,\mathbf{R})$ symmetry. In order to establish that this extends to a superconformal quantum mechanics we must establish the existence of four supersymmetric special conformal generators S^i. These will be given by the commutator $S^i = i[\mathcal{Q}^i_S, K]$. As discussed in [MS00], one must also ensure that the commutator $[\mathcal{Q}^i_S, D] = -i\mathcal{Q}^i_S$ is satisfied in order to have a superconformal system. This is guaranteed due to the manifest scale invariance of the supersymmetric action: $\mathbf{q}^i \to \gamma\,\mathbf{q}^i$, $\lambda^i \to \gamma^{3/2}\,\lambda^i$ with $t \to t/\gamma$.[6] Hence the scaling theory is a superconformal multiparticle theory.

3.3.3 Gravity

As discussed in the introduction, the emergence of a full $SL(2,\mathbf{R})$ in the deep scaling regime is reminiscent of the emergence of a full $SL(2,\mathbf{R})$ in the deep AdS_2 throat in gravity, which could end at a horizon or cap off at the locations of entropyless D-particles. The symmetry group manifests itself in the dynamics of wrapped branes moving in a 'geometry' resulting from integrating out the interconnecting strings and extends upon similar observations made for other multiparticle systems in [MS00, BPMSV00, MS99]. However, we must point out that the conformal quantum mechanics obtainde here comes directly from the low energy dynamics of strings interacting with wrapped branes, i.e. the quiver theory, rather than from a gravity calculation of the moduli space of a multi-black hole system. It would in fact be interesting to repeat such gravitational Ferrel-Eardley type calculations [BPMSV00, MS99, FE87, GR86, MSS02] for the low energy velocity expansion of the corresponding one-half BPS scaling solutions [Den00] in $\mathcal{N}=2$ supergravity. This might give an operational meaning to the AdS_2/CQM correspondence [Str99, ANT08, Sen11, MMS99, CJPS11] (particularly in the large $g_s\kappa$ limit and within the AdS_2 scaling region). Recall that in the classical supergravity limit, the gravitational backreaction of a brane becomes parametrically small given that its mass goes like $1/l_P$.[7] Furthermore, as emphasized in the introduction, our system is a quiver quantum mechanics that does not necessarily reside inside a larger two-dimensional conformal field theory, and thus the $SL(2,\mathbf{R})$ may be of the 'isolated' type.

3.4 Wavefunctions in the scaling theory

Given the existence of an $SL(2,\mathbf{R})$ invariant scaling theory, one natural question is whether a quantum state will stay localized within the scaling regime or if its wavefunction will spread away from the scaling regime, or even fall back into the Higgs branch (where $\langle|\mathbf{q}^i|\rangle = 0$). One particular direction in which a wavefunction might easily spread is the scaling direction where the potential is identically zero and where there are (classically) no velocity dependent magnetic

[6] Though we have not presented the piece of the Lagrangian quadratic in λ^i, which will be of the form $\sim \kappa\,\bar{\lambda}\,\dot{\lambda}/|\mathbf{q}|^3$ in the scaling region, the linear piece is fixed by supersymmetry [Den02] to be (50), which is enough to read off the scaling dimension of λ^i.

[7] This approach is reminiscent of attempts to match the Coulomb branch of the BFSS matrix model [BFSS97, BB97] with eleven-dimensional supergravity calculations. A basic difference is the presence of the ability to zoom into a deep AdS_2 throat in the geometry and that the microscopic quiver model is vector like rather than matrix like.

forces.

Studying the quantum mechanics problem of the full three-particle scaling theory is difficult. Instead, we will look at the wavefunctions of the simpler wrapped D4-D0 model discussed in section 3.2. The zero angular momentum piece of the Schrödinger equation in spherical coordinates (i.e. $q^x = q\cos\theta\sin\phi$, $q^y = q\sin\theta\sin\phi$ and $q^z = q\cos\phi$, $q \equiv |\mathbf{q}|$) is:

$$-\frac{1}{2\sqrt{g}}\,\partial_i\,\sqrt{g}\,g^{ij}\,\partial_j\,\psi_E = -\frac{1}{2\kappa}\,q^{5/2}\,\frac{\partial}{\partial q}q^{1/2}\frac{\partial}{\partial q}\,\psi_E = E\,\psi_E\ , \tag{32}$$

which can easily be solved: $\psi_E(q) \sim \sqrt{q}\exp\left(\pm 2i\sqrt{2\kappa E/q}\right)$.[8] These energy eigenstates are non-normalizable at small q with respect to the covariant inner product:

$$\langle\psi_1|\psi_2\rangle = \int d^3q\,\sqrt{g}\,\psi_1^*(\mathbf{q})\,\psi_2(\mathbf{q})\ . \tag{33}$$

Non-normalizability of energy eigenstates is common for scale invariant theories [**?**], the non-normalizability means that energy eigenstates leak into the Higgs branch.

As in [**?**], we can consider instead eigenstates $\psi_\lambda(q)$ of the $H + aK$ operator with eigenvalue $\lambda \equiv \sqrt{a}\left(n+\frac{1}{2}\right) \in \mathbf{R}$, where a is a constant with appropriate dimensions. If we define a variable $x = q/(\kappa\sqrt{a})$, then the zero-angular momentum wavefunctions $\psi_\lambda(x)$ satisfy

$$\frac{1}{2}\left(-x^{5/2}\,\frac{\partial}{\partial x}x^{1/2}\frac{\partial}{\partial x} + \frac{4}{x}\right)\,\psi_\lambda = \left(n+\frac{1}{2}\right)\,\psi_\lambda\ , \tag{34}$$

The normalizable wavefunctions are given by confluent hypergeometric functions:

$$\psi_\lambda(x) = \mathcal{N}e^{-2/x}\,U\left(\frac{1-n}{2},\,\frac{3}{2},\,\frac{4}{x}\right)\ , \tag{35}$$

where $\mathcal{N}$ is a finite normalization factor that depends on κ and a. Whenever $n \in \mathbf{N}$, the expression for $U\left(\frac{1-n}{2},\,\frac{3}{2},\,\frac{4}{x}\right)$ simplifies significantly. We can also obtain asymptotic expressions near $x = 0$ and $x = \infty$:

$$U\left(\frac{1-n}{2},\,\frac{3}{2},\,\frac{4}{x}\right) = x^{-n/2}\left(2^{n-1}\sqrt{x} + \mathcal{O}\left(x^{3/2}\right)\right)\ ,\quad \text{for } x\sim 0\ , \tag{36}$$

$$= \frac{\sqrt{\pi\,x}}{2\,\Gamma\left[(1-n)/2\right]} - \frac{2\sqrt{\pi}}{\Gamma\left[-n/2\right]} + \mathcal{O}\left(x^{-1/2}\right)\ ,\quad \text{for } x^{-1}\sim 0\ . \tag{37}$$

[8] If we include the angular variables we would find $\psi_E^{lm} = Q_l^E(q)Y_l^m(\theta,\phi)$ with Y_l^m spherical harmonics and Q_l^E can be written in terms of Bessel functions. Here we only look at $l = 0$ modes, $\psi_E(q) = Q_0^E(q)$.

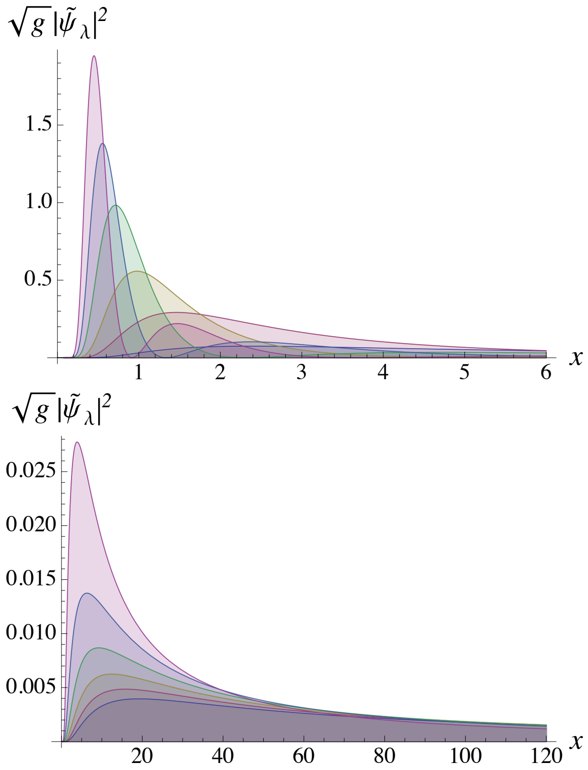

Figure II.2: Examples of $H + a\,K$ eigenfunctions. Left: Plot of $\tilde{\psi}_\lambda(x)$ for $n \in (0.2, 5.2)$ in unit increments. Right: Plot of $\tilde{\psi}_\lambda(x)$ for $n \in (-5.7, -0.7)$ in unit increments.

The dependence on κ and a drops out of all expectation values of operators as a function of x labeled $\mathcal{O}(x)$, hence we can instead normalize the wavefunctions $\tilde{\psi}_\lambda(x) \equiv \psi_\lambda(x)|_{\kappa,a=1}$ and reinstate the dependence on κ and a in expectation values of operators by looking at the dependence of $\mathcal{O}(x)$ on the variable x and how it transforms when we take $q \to x\kappa\sqrt{a}$. Examples of $\tilde{\psi}_\lambda(x)$ are displayed in figure II.2.

To understand whether these states leak back into the Higgs branch, we consider for example $\langle x \rangle_\lambda$. This is only finite if n is an odd positive integer for which $\langle x \rangle_\lambda = 8$ (or equivalently $\langle q \rangle_\lambda = 8\kappa\sqrt{a}$), otherwise $\langle x \rangle_\lambda$ is infinite. This can be seen from the fact that for positive and odd n, $\Gamma[(1-n)/2]$ diverges and the $\mathcal{O}(\sqrt{x})$ term in the large x expansion (37) dissappears. However, for any $\lambda(n)$ there exists a number $s < 1$ such that $\langle x^s \rangle_\lambda$ is finite. Similarly, when n is an odd positive integer, there exists a number $s > 1$ such that $\langle x^s \rangle_\lambda$ is infinite.

Thus a large class of $H + aK$ eigenstates do not leak into the Higgs branch for large κ.

Finding the wavefunctions in the three-node Coulomb branch is clearly more complicated. There are now six-degrees of freedom and a non-trivial potential $V(\mathbf{q}^i)$. However, at the classical level, the scaling direction remains flat and motion along it is uninhibited, neither by the potential nor by any velocity dependent magnetic forces. So if the wavefunction has any chance of spreading, it will do so along the scaling direction. Our previous analysis of the two particle wavefunctions, which deals essentially with the scaling direction, suggests that at least some $H + aK$ eigenstates do not leak into the Higgs branch.[9]

We end this section with an important question. The $SL(2,\mathbf{R})$ symmetry we have been discussing so far resides in the Coulomb branch of the full quiver theory. In particular it resides in the scaling regime of the Coulomb branch which is connected to the Higgs branch at its tip, i.e. where all the $\mathbf{q}^i$ become vanishingly small. How does the $SL(2,\mathbf{R})$ structure act on (if at all) the Higgs branch degrees of freedom? Perhaps a way to answer this question is via the Higgs-Coulomb map of [BV99] (further developed in the context of quiver theories in [BBdB$^+$12]). Indeed, an $SL(2,\mathbf{R})$ symmetry acting on the states residing in the Higgs branch would resonate closely with the appearance of an $SL(2,\mathbf{R})$ isometry of the AdS_2 in the near horizon region of the extremal black hole.

4 Melting molecules

We have seen that the low energy excitations of the scaling regime in the Coulomb branch are described by a multiparticle $SL(2,\mathbf{R})$ invariant quantum mechanics. It was noted that this is reminiscent of the $SL(2,\mathbf{R})$ that appears in the near horizon geometry of an extremal black hole or the AdS_2 geometry outside a collection of extremal black holes that reside within a scaling throat. If we heat up such a collection of black holes (for example by making the centers slightly non-extremal) they will fall onto each other due to gravity's victory over electric repulsion. Naturally, then, we might ask what happens to our scaling theories and more generally the Coulomb branch configurations upon turning on a temperature?[10] This amounts to integrating out the massive strings, i.e. the chiral multiplet, in a thermal state as opposed to their vacuum state. In this section we assess the presence of bound states and scaling solutions as we vary the temperature. Roughly speaking, this is the weak coupling version of

[9] We also note that the quantization of the classical solution space of scaling configurations was considered in [dBESMVdB09], where it was found that the expectation value of the total angular momentum operator was non-vanishing.

[10] Aspects of supersymmetric quantum mechanics at finite temperature are discussed in [Fuc85, DKM86, RR88]. An incomplete list of more recent studies including numerical simulations is [KLL01b, KLL01a, KNT07, CW07, AHNT08, LSWY15, IKRS13, ABT11].

the calculations in [AAB+12, CV12, CMV13, AADP15] where the multicentered solutions were studied at finite temperature.[11] We find that the bound (but non-scaling) configurations persist at finite temperature until a critical temperature after which they either classically roll toward the origin or become metastable. This amounts to the 'melting' of the bound state. Scaling solutions on the other hand do not persist at finite temperature and instead the potential develops a minimum at the origin, even at small temperatures.

As was alluded to, what we find is somewhat analogous to heating up a collection of extremally charged black holes, known as the Majumdar-Papapetrou geometries [Maj47, Pap47], by making the masses of the centers slightly larger than their charges. Doing so will cause the black holes to collapse into a single center configuration upon the slightest deviation from extremality.

4.1 Two nodes - melting bound states

Consider the low energy dynamics of two wrapped branes, which is given by a two node quiver. There can be no gauge-invariant superpotential in this case since the quiver admits no closed loops. The relative Lagrangian is:

$$\begin{aligned} L_V &= \frac{\mu}{2}\left(\dot{\mathbf{q}}^2 + D^2 + 2i\bar{\lambda}\dot{\lambda}\right) - \theta D \ , \\ L_C &= |\mathcal{D}_t\phi_\alpha|^2 - \left(|\mathbf{q}|^2 + D\right)|\phi_\alpha|^2 + i\bar{\psi}_\alpha\mathcal{D}_t\psi_\alpha - \bar{\psi}_\alpha\mathbf{q}\cdot\boldsymbol{\sigma}\psi_\alpha - \\ &\quad i\sqrt{2}\left(\bar{\phi}_\alpha\psi^\alpha\epsilon\lambda - \bar{\lambda}\epsilon\bar{\psi}_\alpha\phi_\alpha\right) \ , \end{aligned}$$

where $\mathcal{D}_t\phi_\alpha \equiv (\partial_t + iA)\,\phi_\alpha$. The matter content is given by a chiral multiplet $\Phi_\alpha = \{\phi_\alpha, \psi_\alpha\}$ and a vector multiplet $\mathbf{Q} = \{A, \mathbf{q}, D, \lambda\}$ (A is a one-dimenional $U(1)$ connection). The index $\alpha = 1, 2, \ldots, \kappa$ (where $\kappa > 0$) is summed over.

If we keep $|\mathbf{q}|$ constant and set to zero the fermionic superpartners λ, we can integrate out the ϕ_α and ψ_α fields to get the effective bosonic action on the position degrees of freedom:

$$\begin{aligned} L_{\text{eff}} = \frac{\mu}{2}D^2 - \theta D - \kappa \log\det\left(-(\partial_t + iA)^2 + |\mathbf{q}|^2 + D\right) \\ + \kappa\log\det\left(-(\partial_t + iA)^2 + |\mathbf{q}|^2\right) \ . \end{aligned} \tag{38}$$

If we wish to study the system at finite temperature $T \equiv \beta^{-1}$, we must Wick rotate to Euclidean time $t \to it_E$ and compactify $t_E \sim t_E + \beta$. Notice we cannot

[11] In fact, [AAB+12, AADP15] explored how the effective potential of a D-particle/black hole bound state changed as the temperature of the black hole was increased. Hence, a closer analogy to [AAB+12] would be to consider a four-node mixed Higgs-Coulomb branch as in figure II.8. One node is 'pulled' away from the remaining three, such that one can integrate out all connecting heavy strings in some thermal state. The remaining three nodes, which we choose to contain a closed loop (and hence have exponentially many ground states), are in the Higgs branch which is also considered to be at some finite temperature. One could then study the Coulomb branch theory for the position degree of freedom of the far away wrapped brane as a function of the temperature.

set the zero mode of A to zero since one can have non-trivial holonomy around the thermal circle. The ϕ_α fields have periodic boundary conditions and the ψ_α fields have anti-periodic boundary conditions around the thermal circle. The operator ∂_{t_E} has as eigenvalues the Matsubara frequencies $\omega_n = 2\pi nT$ for the bosonic case and $\omega_n = 2\pi(n+1/2)T$ for the fermionic case, where $n \in \mathbf{Z}$. Hence, it is our task to evaluate:

$$L_{\text{eff}}^{(T)} = \frac{\mu}{2}D^2 - \theta D - \frac{\kappa}{\beta}\log\det\left(4\pi^2(n+a)^2T^2 + |\mathbf{q}|^2 + D\right) + \frac{\kappa}{\beta}\log\det\left(4\pi^2(n+1/2+a)^2T^2 + |\mathbf{q}|^2\right) , \quad (39)$$

where we have defined $a \equiv A/2\pi T$. Notice that $L_{\text{eff}}^{(T)}$ depends on the connection A which we eventually need to path-integrate over. Upon evaluating the determinant (see appendix E for details), one finds the effective thermal potential is given by:

$$V(T,D,a) = -\frac{\mu}{2}D^2 + \theta D + \kappa T \log\left(\frac{\cosh\left(\frac{\sqrt{|\mathbf{q}|^2+D}}{T}\right) - \cos(2\pi a)}{\cosh\left(\frac{|\mathbf{q}|}{T}\right) + \cos(2\pi a)}\right) , \quad (40)$$

which has the correct low temperature limit $\lim_{T\to 0} V(T,D,a) = -\mu D^2/2 + \theta D - \kappa|\mathbf{q}| + \kappa\sqrt{|\mathbf{q}|^2 + D}$, in accordance with the results of [Den02]. Our task is to now perform the path integral of $e^{-\beta V(T,a)}$ over D and a. In doing so, we must be careful to ensure that $D > -|\mathbf{q}|^2$ such that integrating out the string is justifiable. Differentiating (40) with respect to a we find that $V(T,D,a)$ has a minimum at $\cos(2\pi a) = 1$ for all T and $D > -|\mathbf{q}|^2$. Solving the saddle point equations for D must be done numerically.

The potential V has two scaling symmetries associated to it. Configurations related by:

$$T \to \gamma T\,, \quad |\mathbf{q}| \to \gamma\,|\mathbf{q}|\,, \quad D \to \gamma^2 D\,, \quad \mu \to \gamma^{-2}\delta\,\mu\,, \quad \theta \to \delta\,\theta\,, \quad \kappa \to \gamma\delta\,\kappa\,, \quad (41)$$

have respective potentials related by: $V(\mu,\theta,\kappa,|\mathbf{q}|,D,T) \to \delta\gamma^2 V(\mu,\theta,\kappa,|\mathbf{q}|,D,T)$. The qualitative features of the potential thus only depend on scale invariant quantities:

$$\tilde{\kappa} \equiv \kappa\left(\frac{\mu}{|\theta|^3}\right)^{1/2} , \quad \tilde{T} \equiv T\left(\frac{\mu}{|\theta|}\right)^{1/2} , \quad |\tilde{\mathbf{q}}| \equiv |\mathbf{q}|\left(\frac{\mu}{|\theta|}\right)^{1/2} , \quad (42)$$

and the scaling-invariant potential is given by $\tilde{V} \equiv V\frac{\mu}{|\theta|^2}$. Note that our finite temperature analysis breaks down for $T \sim 1$ in string units, where the effects of massive string modes start to kick in. However, we may still consider large $\tilde{T}$

as long as we restrict ourselves to parameter regions where our approximation remains valid, e.g. $T \ll 1$ and $\mu/|\theta| \gg 1$. We use the scaling symmetries to fix $\mu = |\theta| = 1$ and study the thermal phases of the theory as a function of $\tilde{T}$ and $\tilde{\kappa}$, noting that we can always extract physical quantities by properly rescaling μ and θ.

4.1.1 Thermal phases

Having derived the thermal effective potential we can now describe the different thermal configurations for the two-node quiver. The physically distinguishable phases have potentials $\tilde{V}$ with the following distinct properties:

1. A single stable minimum away from $|\tilde{\mathbf{q}}| = 0$.
2. Two non-degenerate minima away from $|\tilde{\mathbf{q}}| = 0$.
3. Two degenerate minima away from $|\tilde{\mathbf{q}}| = 0$.
4. Two minima with one at $|\tilde{\mathbf{q}}| = 0$ where:
   (a) the minimum at $|\tilde{\mathbf{q}}| = 0$ is the global minimum,
   (b) the minimum at $|\tilde{\mathbf{q}}| > 0$ is the global minimum.
5. A single minimum at $|\tilde{\mathbf{q}}| = 0$.

Examples of each of these thermal configurations are displayed in figures IV and II.4.

Depending on the location in the $(\tilde{\kappa}, \tilde{T})$-plane, the system will be driven (either through thermal activation or quantum tunneling) to the most stable configuration. If the system is in a metastable configuration, such a process can take exponentially long. At high enough temperatures, the system will eventually fall back to $|\tilde{\mathbf{q}}| = 0$ where the Higgs and the Coulomb branch meet, mimicking gravitational collapse or the melting of the molecule.

Note that for bound states in $\mathcal{N} = 2$ supergravity, the effective potential of a small probe around a large hot black hole, as studied in [AAB+12], never exhibited two minima away from $|\tilde{\mathbf{q}}| = 0$ (that is we never noticed potentials of types 2-4b). Since supergravity is a good effective description at large $g_{\text{eff}} \sim g_s\kappa$, it would be interesting to see if this behavior matches qualitatively in the Coulomb branch as $\tilde{\kappa}$ is increased. An example of this is shown in figure II.5.

A way to think about the high temperature melting transition is to integrate out the auxiliary D-fields first, which induces a quartic interaction $\left(|\phi_\alpha|^2\right)^2/\mu$ for the bifundamentals. Since this interaction is relevant, it will play a minor effect at sufficiently large temperatures, namely for $T \gg \mu^{-1/3}$. Thus, at high temperatures we are dealing with a collection of κ non-interacting complex bosonic and fermionic degrees of freedom with square masses $(|\mathbf{q}|^2 + \theta/m)$ and $|\mathbf{q}|^2$ respectively. In addition, due to the $U(1)$ connection we must only consider the gauge

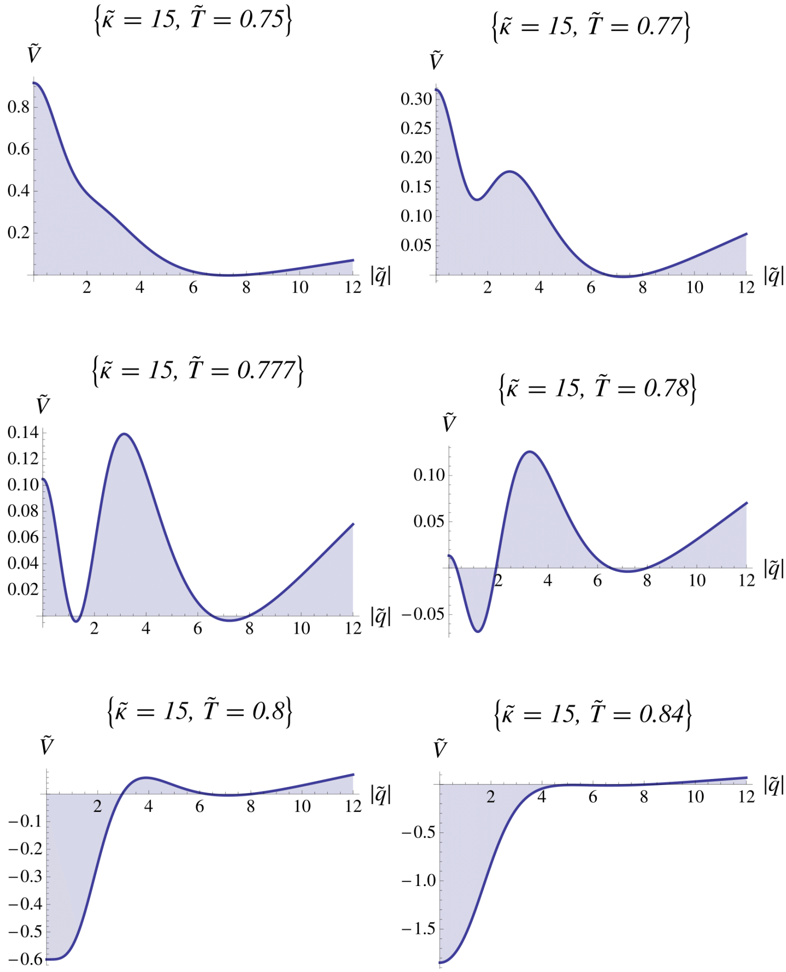

Figure II.3: Thermal effective potentials of a two node quiver ($\theta = -1$ and $\mu = 1$). As the temperature is increased the system explores various thermal configurations of stable and metastable minima. From top left to bottom right the system is of type $1 \to 2 \to 3 \to 2 \to 4a \to 5$.

invariant states: the spectrum is constrained to those states annihilated by the $U(1)$ charge operator. A particularly convenient gauge is the temporal gauge, $A = 0$. At some fixed high temperature, the number of gauge invariant modes that aren't Boltzmann suppressed increases with decreasing mass and thus the dominant contribution to the free energy will come from for the lightest possible mass, i.e. $|\mathbf{q}| = 0$. This can be viewed as an entropic effect [Ver11].

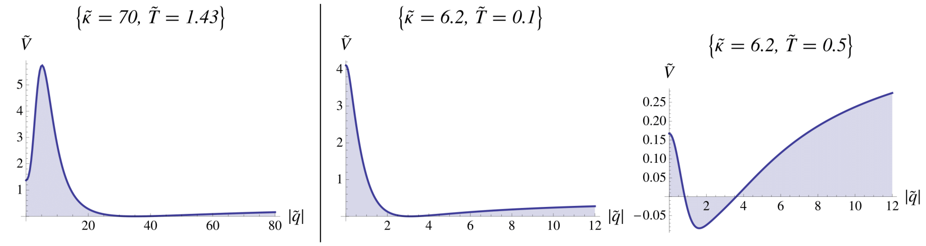

Figure II.4: Thermal effective potentials of a two node quiver (for $\theta = -1$ and $\mu = 1$). *Left*: An example of phase type 4b. *Right*: A case where the potential of the supersymmetric minimum decreases as the temperature is increased. A similar observation was made for supersymmetric bound states in [AAB$^+$12].

4.2 Three nodes - unstable scaling solutions

A similar analysis for three nodes gives rise to the following effective thermal potential:

$$V(T, D^i, a^i) = \sum_{i=1}^{3} -\frac{\mu^i}{2} D^i D^i + \theta^i D^i + |\kappa^i|\, T \log\left(\frac{\cosh\left(\frac{\sqrt{|\mathbf{q}^i|^2 + s_i D^i}}{T}\right) - \cos(2\pi a^i)}{\cosh\left(\frac{|\mathbf{q}^i|}{T}\right) + \cos(2\pi a^i)}\right) . \tag{43}$$

Our main interest to understand the behavior of the scaling potential at finite temperature, and in particular the flat scaling direction. Thermal effects will kick in when $|\mathbf{q}^i| \lesssim T$. Since zero temperature scaling solutions occur for arbitrarily small $|\mathbf{q}^i|$, it is sufficient to perform a small temperature expansion. The dimensionless quantities capturing the thermal transition are $|\mathbf{q}^i|/T$. As in the two-node case, one must integrate out the $U(1)$ connections a^i and the D^i. For the case where the $\kappa^1 = \kappa^2 = -\kappa^3 > 0$ we can identify a critical point along the scaling direction $|\mathbf{q}^i| = \lambda|\kappa^i|$: $a^1 = 1$, $a^2 = 0$, $D^1 = 0$ and $D^2 = 0$. On this saddle the potential becomes:

$$V^*(T) = 2\, T \sum_{i=1}^{3} |\kappa^i| \, \log\left[\tanh\left(\frac{\lambda|\kappa^i|}{2T}\right)\right] , \tag{44}$$

which is negative definite. Of course there could be more dominant saddles that make the potential even lower than the above. So more generally, this procedure must be done numerically, even in the small temperature expansion, as the equations governing the connections are intractable analytically. It amounts

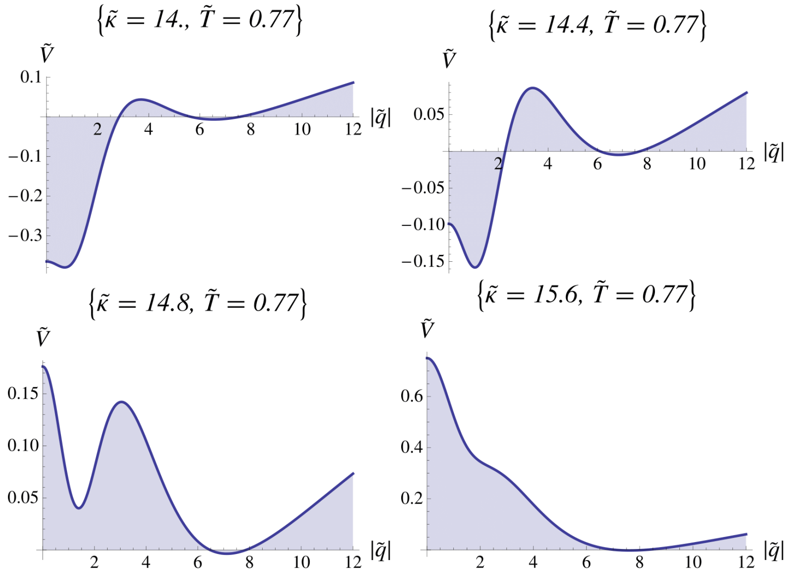

Figure II.5: Thermal effective potentials of a two node quiver ($\theta = -1$ and $\mu = 1$). As $\tilde{\kappa}$ is increased we note that the first minimum disappears.

to numerically minimizing a function of four variables, namely a^1, a^2, D^1 and D^2 (recall that $a^3 = a^1 + a^2$ and $D^3 = D^1 + D^2$).

Since we are interested in the scaling branch we can set $\theta^i = \mu^i = 0$ in our analysis.[12] The resulting potential (with $\theta^i = \mu^i = 0$) then exhibits the following scaling relation:

$$\kappa^i \to \delta\,\kappa^i\ , \quad T \to \gamma\, T\ , \quad |\mathbf{q}^i| \to \gamma\,|\mathbf{q}^i|\ , \quad D^i \to \gamma^2\, D^i\ , \tag{45}$$

such that $V(\kappa^i, |\mathbf{q}^i|, D^i, T) \to \gamma\,\delta\,V(\kappa^i, |\mathbf{q}^i|, D^i, T)$. From the scaling symmetries we observe that the numerical value of the temperature is of little meaning, all that matters for the thermal phase structure is whether or not it vanishes. This is to be expected since we are dealing with a scale invariant system.

In figure II.6 we display the potential at $T = 0$ (left) and $T \neq 0$ (right) in the scaling direction $|\mathbf{q}^i| = \lambda\,|\kappa^i|$ for $\kappa^1 = \kappa^2 = -\kappa^3 = 1$. At $T = 0$, we naturally find that it is vanishing for all λ. At $T \neq 0$ we see that for radial values larger than the temperature, the scaling direction is still effectively flat, but for radii of order the temperature the thermal potential quickly falls. The numerics are in good agreement with the analytic expression (44). Hence, at finite temperature

[12] The reason we set $\theta^i = 0$ is for computational simplicity and numerical clarity. We could have started a Coulomb branch with non-zero θ^i's which has scaling solutions for $|\mathbf{q}^i| = \lambda\,\kappa^i + \mathcal{O}(\lambda^2)$ in the limit $\lambda \to 0$ and found similar results.

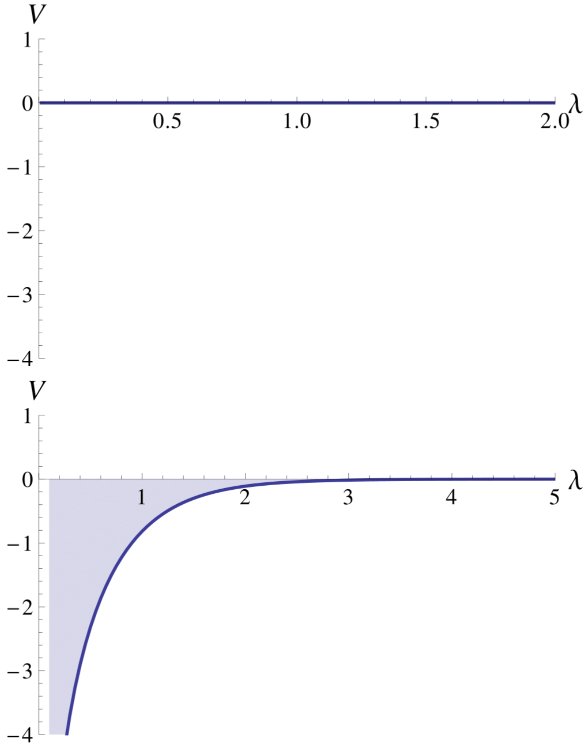

Figure II.6: Thermal potential along the scaling direction $|\mathbf{q}^i| = \lambda\,\kappa^i$ for $\kappa^1 = \kappa^2 = -\kappa^3 = 1$ at $T = 0$ (left) and $T \neq 0$ (right).

the system falls back into its Higgs branch. It would be interesting to explore this phenomenon in more generality for a larger number of nodes.

5 Emergence and Holography

We have observed the emergence of a full $SL(2,\mathbf{R})$ symmetry from a quiver quantum mechanics model which itself is not a conformal quantum mechanics. The $SL(2,\mathbf{R})$ manifested itself in the effective theory of the position degrees of freedom of the D-particles, once the heavy strings were integrated out. Supersymmetry played an important role in the previous discussion given that the form of the quiver Lagrangian (1) is heavily constrained by it. But as was mentioned in the introduction, the $SL(2,\mathbf{R})$ symmetry appears for geometries that need not be supersymmetric. In this final speculative section, we consider the idea that randomness (in the Wignerian sense) is behind the $SL(2, \mathbf{R})$ and observe that dilatations imply special conformal transformations for one dimensional systems whose dilatation symmetry is geometrized into an additional radial direction. Finally, we discuss some possible extensions.[13]

5.1 Random Hamiltonians and emergent $SL(2, \mathbf{R})$?

It would be interesting to understand whether the emergence of such an $SL(2, \mathbf{R})$ from systems with large numbers of degrees of freedom, such as matrix quantum mechanics, could be more general. For instance, if there are a large number of almost degenerate vacua in the putative microscopic dual of AdS_2, one might imagine an approximate scale invariance emerging at large N due to the formation of almost continuous bands of low energy eigenvalues.

More generally, if the Hamiltonian is sufficiently complicated due to the multitude of internal cycles being wrapped by the branes, one might imagine drawing it from a random ensemble $\mathcal{H}_N$ of $N \times N$ Hermitean matrices. We can then formulate the following problem. Draw an $N \times N$ Hermitean matrix $H \in \mathcal{H}_N$ and assess (under some as of yet unspecified measure, perhaps the Frobenius norm is a possibility) how well the matrix equations (17) can be satisfied, given arbitrary Hermitean matrices D and K. In particular, are the equations better satisfied as we increase N.[14] The emergence of an $SL(2, \mathbf{R})$ from a random ensemble of Hamiltonians, if true, would be similar in spirit to the emergence of the Wigner distribution of eigenvalue spacings from random matrices that is almost universal to quantum systems that become chaotic in the classical limit.

Interestingly, an ensemble of random Hamiltonians with a scale invariant distribution of eigenvalues was found in [AOI10]. We hope to return to this question in the future.

5.2 Holographic considerations

If one assumes that there exists a gravitational dual to the theory, and in addition that a scaling symmetry $t \to \lambda^a t$ exists along with a radial transformation $z \to \lambda^2 z$ one obtains the two-dimensional metric (see for example [Nak15]):

$$ds^2 = -\frac{dt^2}{z^a} + \frac{dz^2}{z^2} \ , \tag{46}$$

which is nothing more than AdS_2 (albeit in unusual coordinates). The isometry group of AdS_2 is $SL(2, \mathbf{R})$ and thus, holographically a dilatation symmetry seems to imply the existence a full $SL(2, \mathbf{R})$ symmetry. This is not true in higher dimensions unless one also assumes Lorentz invariance of the boundary metric.

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[13]We would like to acknowledge Frederik Denef, Diego Hofman and Sean Hartnoll for many interesting discussions leading to these ideas.

[14]Of course, given the fact that $SL(2, \mathbf{R})$ has no finite dimensional unitary representations, the equations can only be satisfied exactly for $N = \infty$.

Another feature of conformal quantum mechanics that contrasts with higher dimensional conformal field theory is that there is no normalizable $SL(2,\mathbf{R})$ invariant ground state. On the other hand, d-dimensional conformal field theories have an $SO(d,2)$ invariant ground state wavefunctional. We view this as a hint that whatever the holographic description of AdS_2 is, it is not necessarily a conformal quantum mechanics off the bat. Indeed, the quiver quantum mechanics whose ground state degeneracies count a large fraction of the microstates of a black hole with an AdS_2 near horizon are *not* conformal quantum mechanics in and of themselves. Instead, the $SL(2,\mathbf{R})$ symmetry of AdS_2 might only become exact in some kind of large N limit,[15] as opposed to the $SO(d,2)$ symmetries of AdS_{d+1} which should persist at finite N (assuming the β-function of the dual CFT vanishes at finite N as is the case for $\mathcal{N}=4$ SYM).

5.3 Possible extensions

As a final note, we would like to mention some open questions and extensions to our discussion. We have left question of superpotential corrections to the Coulomb branch of the three-node quiver untouched. Though the superpotential will not affect the ground state energy or existence of scaling solutions, it will correct the higher powers in the velocity expansion and it would be interesting to understand whether the $SL(2,\mathbf{R})$ symmetry is preserved by such corrections. The quartic interaction coming from integrating out the D-terms, $\left(|\phi_\alpha|^2\right)^2/\mu$, is dominated by an expansion in cactus diagrams in the large κ limit. There is no such cactus diagram expansion for the interaction coming from integrating out the F-term, the quartic bosonic interaction being: $\sum_{\alpha,\beta,\gamma,\tilde\beta,\tilde\gamma}\omega_{\alpha\beta\gamma}\omega^*_{\alpha\tilde\beta\tilde\gamma}\phi^i_\beta\bar\phi^i_{\tilde\beta}\phi^j_\gamma\bar\phi^j_{\tilde\gamma}$. Thus we are confronted with how (if at all) does one organize a large κ expansion in this case. By dimensional analysis, and inspection of the two-loop Feynman diagrams (see figure II.7), we expect the velocity squared piece of the Coulomb branch Lagrangian in the scaling regime to become (up to $\mathcal{O}(1)$ coefficients):

$$\delta L_{c.b.} \sim |\kappa|\;\dot{\mathbf{q}}\cdot\dot{\mathbf{q}}\left(\frac{1}{|\mathbf{q}|^3}+\frac{(|\kappa|\,|\omega|)^2}{|\mathbf{q}|^6}+\mathcal{O}\left(\frac{|\kappa|^5|\omega|^4}{|\mathbf{q}|^9}\right)\right)\;, \tag{47}$$

where $|\omega|^2\sim|\kappa|^{-3}\sum_{\alpha,\beta,\gamma}|\omega_{\alpha\beta\gamma}|^2$. In order for the superpotential contribution to be subleading we would further require:

$$|\mathbf{q}| \gg (|\kappa|\omega_{\alpha\beta\gamma})^{2/3}\;, \tag{48}$$

in addition to the condition (14) that forces the system into the scaling regime. Thus, we clearly see that the effect of the superpotential (a relevant deformation) becomes important in the deep infrared region of the scaling regime, where detailed stringy physics begins to manifest itself and potentially destroys the $SL(2,\mathbf{R})$.

[15]This $SL(2,\mathbf{R})$ might persist in a perturbative treatment of a supergravity solution but not at finite N, unless of course it resides in a much larger Virasoro structure.

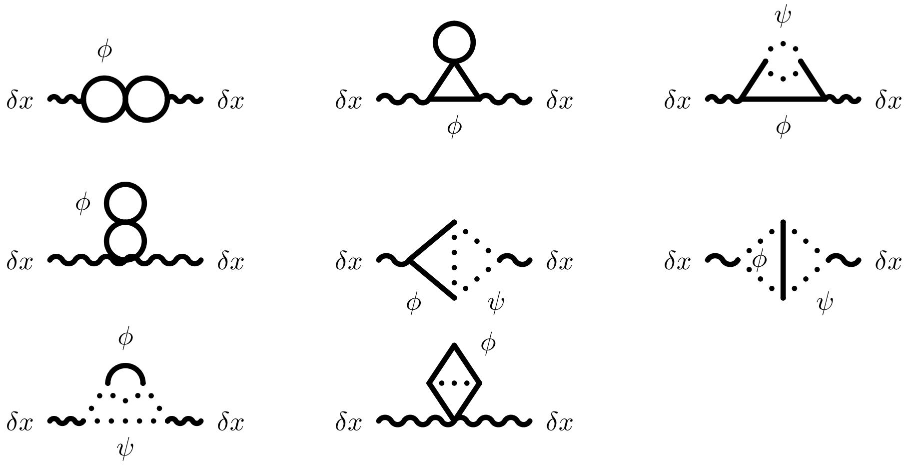


Figure II.7: 2-loop Feynman diagrams contributing to the $\delta x \delta x$ term of the effective Lagrangian. Solid lines represent ϕ, while dotted lines correspond to ψ propagators.

We have also left untouched the issue of larger numbers of nodes. In such a case one could consider mixed Higgs-Coulomb branches. For example, we could consider a four-node quiver where three nodes are in their Higgs branch and the remaining one residing far away in the Coulomb branch (see also [IKLL02, IP08]), as shown in figure II.8. Given the exponentially large number of ground states in the Higgs branch of a three node closed loop quiver, heating such a system up might provide a useful toy model for a D-particle falling into a black hole.

Appendix

A Notation

For convenience, we have used a compact notation whereby latin superscripts denote relative degrees of freedom or arrow directions in the quiver, for example: $(\mathbf{q}^1, \mathbf{q}^2, \mathbf{q}^3) \equiv (\mathbf{x}_{12}, \mathbf{x}_{23}, \mathbf{x}_{13})$ or $(\phi^1_\alpha, \phi^2_\beta, \phi^3_\gamma) \equiv (\phi^\alpha_{12}, \phi^\beta_{23}, \phi^\gamma_{13})$. The supermultiplets in this notation are: $\Phi^i_\alpha = (\phi^i_\alpha, \psi^i_\alpha, F^i_\alpha)$ and the relative vector multiplet is denotes as $\mathbf{Q}^i = (A^i, \mathbf{q}^i, \lambda^i, D^i)$. The only exception to the rule is given by $\theta^1 \equiv \theta_1$, $\theta^2 \equiv -\theta_3$ and $\theta^3 = 0$.

Furthermore, we have: $\mathbf{q}^3 = \mathbf{q}^1 + \mathbf{q}^2$, $D^3 = D^1 + D^2$, $\lambda^3 = \lambda^1 + \lambda^2$. So with

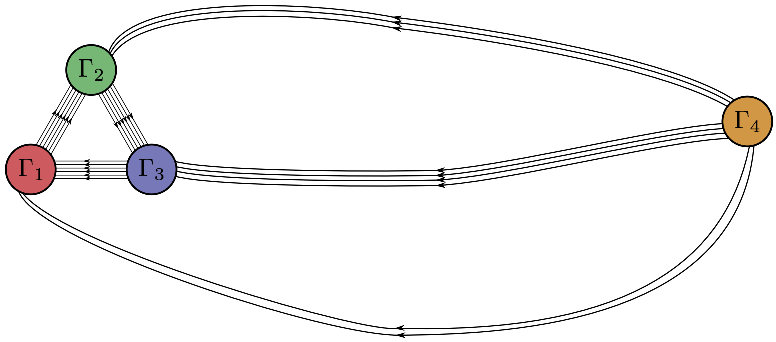


Figure II.8: A schematic representation of a system in a mixed Higgs-Coulomb branch. The long arrows represent very massive strings. Note that there is a closed loop connecting Γ_1, Γ_2 and Γ_3.

respect to the vector multiplet fields, the *relative* Lagrangian is only a function of $\mathbf{q}^1$, $\mathbf{q}^2$, D^1, D^2, λ^1 and λ^2. We also have: $|\phi^i_\alpha|^2 \equiv \sum_\alpha |\phi^i_\alpha|^2$.

The orientation of the quiver is encoded by the s^i. For three nodes we deal with the case of a quiver with a closed loop, and without loss of generality the particular choice $s_1 = 1$, $s_2 = 1$ and $s_3 = -1$, corresponding to $\kappa^1, \kappa^2 > 0$ and $\kappa^3 < 0$. An example of a quiver without closed loops is $s_1 = -1$, $s_2 = 1$, $s_3 = -1$.

The spinors $(\psi^i_\alpha)_a$ and $(\lambda^i)_a$ with $a = 1, 2$ transform in the $\mathbf{2}$ of the $SO(3)$ and the anti-symmetric symbol $\epsilon^{12} = +1$ is such that $\lambda^i \epsilon \psi^i_\alpha \equiv (\lambda^i)_a \epsilon^{ab} (\psi^i_\alpha)_b$. The $\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ are the Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{49}$$

For example $\bar{\psi}^i_\alpha \sigma^x \psi^i_\alpha \equiv (\bar{\psi}^i_\alpha)^a (\sigma^x)_a{}^b (\psi^i_\alpha)_b$. Finally the $U(1)$ covariant derivative $\mathcal{D}_t \phi^i_\alpha \equiv \left(\partial_t + iA^i\right) \phi^i_\alpha$.

We will occasionally revert back to the original $\mathbf{x}_i$ notation, particularly in the appendices.

B Superpotential corrections of the Coulomb branch

It is interesting to compute the effects on the Coulomb branch dynamics due to quantum corrections from the superpotential. This amounts to a two-loop calculation. It is important to note that due to a non-renormalization theorem in [Den02], the linear piece of the $\mathcal{N} = 4$ supersymmetric quantum mechanics

Lagrangian is constrained to be of the form:

$$L^{(1)} = \sum_i \left(-U_i(\mathbf{x})D_i + \mathbf{A}_i(\mathbf{x}) \cdot \dot{\mathbf{x}}_i\right) + \sum_{i,j} \left(C_{ij}(\mathbf{x})\bar{\lambda}_i\lambda_j + \mathbf{C}_{ij}(\mathbf{x}) \cdot \bar{\lambda}_i\boldsymbol{\sigma}\lambda_j\right) \ , \quad (50)$$

with:

$$\mathbf{C}_{ij} = \nabla_i U_j = \nabla_j U_i = \frac{1}{2}\left(\nabla_i \times \mathbf{A}_j + \nabla_j \times \mathbf{A}_i\right) \ , \qquad C_{ij} = 0 \ . \quad (51)$$

This form receives *no* corrections due to quantum effects originating from the superpotential. This means, in particular, that the supersymmetric configurations of the Coulomb branch which solve $U_i = 0$ for all i, both bound and scaling, are unmodified in the presence of a superpotential.

On the other hand, the quadratic piece can receive quantum corrections in the presence of a superpotential. An example of a Feynman diagram contributing to the D_iD_j term is given in figure II.9.

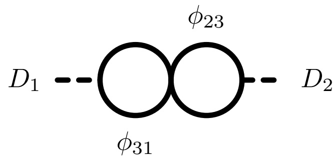


Figure II.9: Example of Feynman diagram contributing to D_iD_j from the superpotential.

C Finite D-terms

When going to the Coulomb branch one integrates out the massive scalars, expands in small D/r^2, solves for the D equations of motion and feeds the solution back into the action. We would like to show here that the supersymmetric configurations are in fact preserved for *finite* D.

In the case of two node quivers, upon integrating out the chiral multiplet degrees of freedom $(\phi_\alpha, F_\alpha, \psi_\alpha)$ one finds the following non-linear equations for D (we take $\theta_1 + \theta_2 = 0$ and $\kappa > 0$):

$$\mu D - \theta = \frac{\kappa}{2\sqrt{|\mathbf{q}|^2 + D}} \ , \quad \theta \equiv \theta_1 - \theta_2 \ . \quad (52)$$

Given that the above equation is a cubic equation in D, one can find analytic solutions, which we are given in appendix C.1. It is straightforward to see that at $|\mathbf{q}| = -\kappa/(2\theta)$, $D = 0$ is a solution to the non-linear equations. Plugging $D = 0$

back into the Lagrangian (which for zero velocity is the potential itself) shows that at $|\mathbf{q}| = -\kappa/(2\theta)$ the system has a zero energy. Away from $|\mathbf{q}| = -\kappa/(2\theta)$ the Lagrangian receives finite D corrections. It is not hard to convince one's self, however, that the potential will never acquire another minimum due to finite D effects. We argue this in appendix C.1.

For three nodes the non-linear equations of D^i become (we take $\theta_1+\theta_2+\theta_3 = 0$ and pick the κ's to form a closed loop in the quiver diagram):

$$\mu^1 D^1 + \mu^3 D^3 - \theta^1 = \frac{|\kappa^1|}{2\sqrt{|\mathbf{q}^1|^2 + D^1}} - \frac{|\kappa^3|}{2\sqrt{|\mathbf{q}^3|^2 - D^3}} \ , \tag{53}$$

and cyclic permutations thereof. These equations can no longer be solved analytically. If we set the θ's to zero we can see that at $|\mathbf{q}^i| = \lambda\,|\kappa^i|$, $D^i = 0$ solves the linear equations and so the scaling solutions persist at finite D for zero θ. For non-zero θ's, setting $|\mathbf{q}^i| = \lambda\,|\kappa^i| + \mathcal{O}(\lambda^2)$ and expanding in small λ, we find that again $D^i = 0$ is a consistent solution order by order in the λ perturbation theory. The existence of new bound states or scaling solutions which are non-supersymmetric due to finite D effects becomes far more intricate in this case. A preliminary numerical scan seems to suggest there are none and we hope to report further on this in the future.

C.1 Non-linear D-term solutions for two-node quivers

For two-nodes, the D-term equation is given by (52). This equation can be solved analytically for D. Supersymmetric bound states are found when $\kappa\,\theta < 0$ at $|\mathbf{q}| = -\kappa/(2\theta)$. In what follows, we choose $\kappa > 0$ and allow θ to have any sign. Equation (52) is equivalent to solving

$$\left(D - \frac{\theta}{\mu}\right)^2 \left(D + |\mathbf{q}|^2\right) - \frac{\kappa^2}{4\mu^2} = 0 \ , \tag{54}$$

which can be turned into a depressed cubic of the form $x^3 + px + s = 0$ using the subsitution $D \to x - \frac{1}{3}\left(|\mathbf{q}|^2 - \frac{2\theta}{\mu}\right)$ and identifying

$$p \equiv -\frac{1}{3}\left(|\mathbf{q}|^2 + \frac{\theta}{\mu}\right)^2 \ , \quad \text{and} \quad s \equiv -\frac{\kappa^2}{4\mu^2} + \frac{2}{27}\left(|\mathbf{q}|^2 + \frac{\theta}{\mu}\right)^3 \ . \tag{55}$$

Since $p < 0$, the three roots of this equation may be written as [wik]:

$$D_k = -\frac{1}{3}\left(|\mathbf{q}|^2 - \frac{2\theta}{\mu}\right) + 2\sqrt{-\frac{p}{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{3s}{2p}\sqrt{-\frac{3}{p}}\right) - k\frac{2\pi}{3}\right) \ , \tag{56}$$

for $k = 0, 1, 2$. For all roots to be real, the arguments of the arccos must be between $[-1, 1]$, which implies $4p^3 + 27s^2 \leq 0$ or $|\mathbf{q}|^2 \geq -\frac{\theta}{\mu} + \frac{3}{2}\left(\frac{\kappa^2}{2\mu^2}\right)^{1/3}$. The $k = 0$ branch is real for all values of $|\mathbf{q}|$.

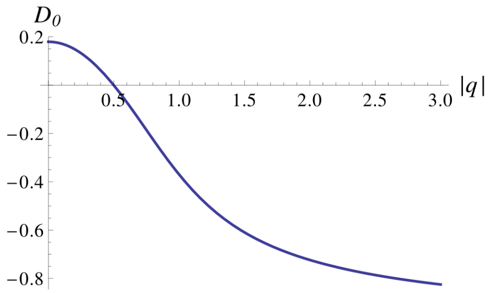


Figure II.10: Plot of D_0 for $\mu = -\theta = \kappa = 1$. Notice that the solution is real for all values of $|\mathbf{q}|$.

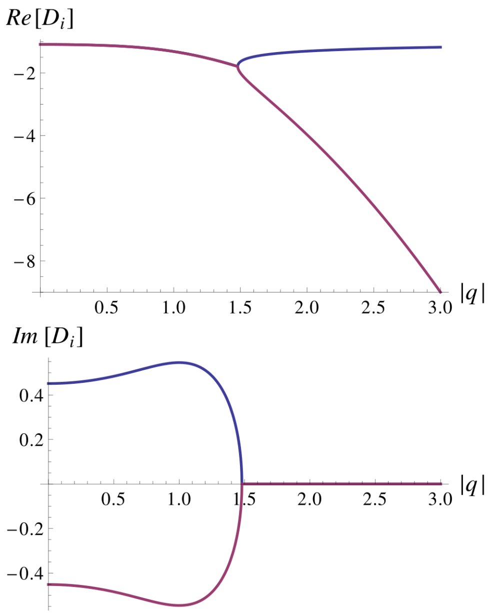


Figure II.11: Left: Plot of $\Re[D_1]$ (blue) and $\Re[D_2]$ (violet). Right: Plot of $\Im[D_1]$ (blue) and $\Im[D_2]$ (violet). Both plots are for $\mu = -\theta = \kappa = 1$. Notice that when complex, the solutions form a conjugate pair.

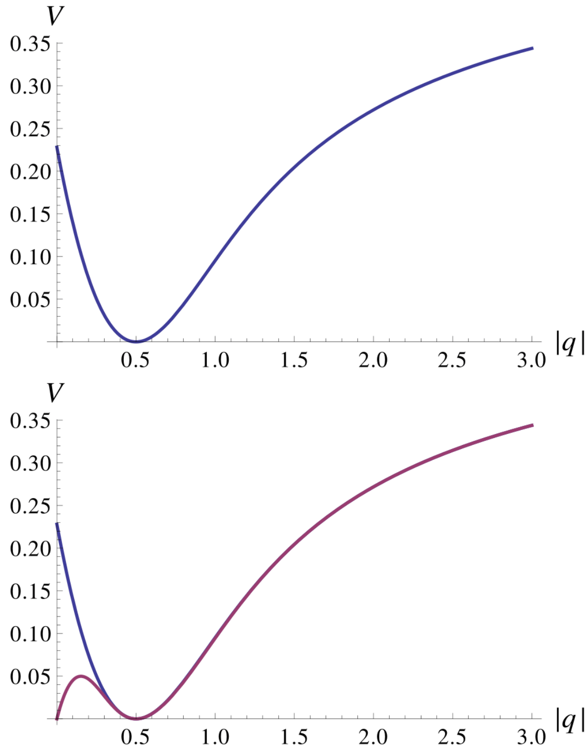


Figure II.12: Left: Plot of V evaluated on D_0 for $\mu = -\theta = \kappa = 1$. Right: Plot of V evaluated on the perturbative D solution in violet, compared with the full non-perturabative D_0 in blue.

In figures II.10 and II.11 we show some plots of the solutions. In figure II.12 we display the effective potential for the D_0 solution and its comparison to the effective potential obtained by expanding D to second order in $D/|\mathbf{q}|^2$ and then evaluating the potential on-shell. We notice that the position of the minimum is unaffected. Evaluating the potential on the D_1 and D_2 results in an unphysical complex potential for small values of $|\mathbf{q}|$. Scanning the parameters for all possible combinations of signs of (κ, θ) results in no other bound states.

D Three node Coulomb branch

In this appendix we present some details leading to the Coulomb branch Lagrangian of a three-node quiver. As was shown in [Den02], the $\mathcal{N} = 4$ supersymmetric quantum mechanics of a $U(1)$ vector multiplet has a Lagrangian whose linear piece in the velocity and D fields is completely fixed by supersymmetry.

Explicit expressions are given in (50) and (51).

In section D.1 below, we show that the coefficient of the $\dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_j$ term of the Lagrangian is the same as that of the $D_i D_j$ term, upon integrating out the chiral matter. In section D.2 we integrate out the auxiliary D fields in the Coulomb theory and obtain an expression for the effective potential on the position degrees of freedom.

D.1 Second order Lagrangian for three-node quiver

Recall the Lagrangian of a n-node quiver theory, after setting the fermionic λ fields to zero, is given by:[16]

$$L = \sum_i \frac{m_i}{2}\left(|\dot{\mathbf{x}}_i|^2 + D_i^2\right) - \theta_i D_i + \sum_{j\to i}\left(|\dot{\phi}_{ij}|^2 + F_{ij}^2 + i\bar{\psi}_{ij}\dot{\psi}_{ij}\right) - \sum_{j\to i}\left[\left(|\mathbf{x}_{ij}|^2 + D_{ij}\right)|\phi_{ij}|^2 + \bar{\psi}_{ij}\,\boldsymbol{\sigma}\cdot\mathbf{x}_{ij}\,\psi_{ij}\right] . \quad (57)$$

For the case $n = 3$ there are three pairs of (i,j) to be considered: $(1,2)$, $(2,3)$ and $(3,1)$. We do not consider contributions from the superpotential in this appendix. We take $\kappa_{12} > 0$, $\kappa_{23} > 0$ and $\kappa_{31} > 0$.

The ϕ_{ij} propagator is given by:

$$D_{\phi_{ij}}(\omega) = \frac{i}{\omega^2 - |\mathbf{x}_{ij}|^2} \quad (58)$$

and the ψ_{ij} propagator is given by:

$$D_{\psi_{ij}}(\omega) = \frac{-i}{\omega^2 - |\mathbf{x}_{ij}|^2}\begin{pmatrix} \omega - z_{ij} & -x_{ij} + iy_{ij} \\ -iy_{ij} - x_{ij} & \omega + z_{ij}\end{pmatrix}, \quad (59)$$

where $\mathbf{x}_{ij} = (x_{ij}, y_{ij}, z_{ij})$ (recall that each lower index denotes its corresponding node). We evaluate our momentum integrals on the imaginary axis, which is guaranteed to give the same result had we evaluated the integral along a real contour with the usual Feynman pole prescription, to lowest order in the the external momenta l. Higher order terms in the external momenta will differ by factors of i.

$D_i\ D_j$ term

First let us consider the diagonal terms and more specifically the contribution to $D_1 D_1$. The two diagrams that contribute are shown in figure II.13. The first

[16] For the purposes of this subsection we use the original notation, i.e. $\mathbf{x}_i$, λ_i, ϕ_{ij}, ψ_{ij}, $\mathbf{x}_{ij} \equiv (\mathbf{x}_i - \mathbf{x}_j)$. and so on. We also supress the Greek indices on the chiral multiplet fields. Also, we do not decouple the center of mass degrees of freedom and switch off the superpotential.

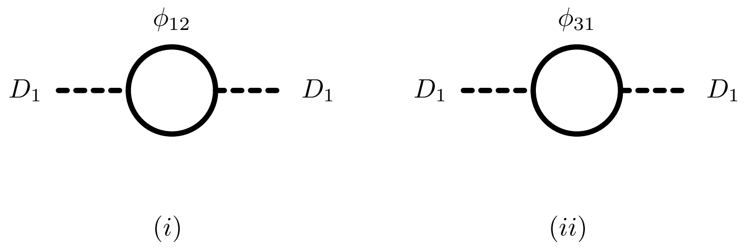


Figure II.13: 1-loop Feynman diagrams contributing to the D_1^2 term of the effective Lagrangian.

diagram, series expanded for small external momentum l, is given by

$$\frac{1}{2}(-i)^2\kappa_{12}\int\frac{d\omega}{2\pi}D_{\phi_{12}}(\omega)D_{\phi_{12}}(\omega+l)=\frac{\kappa_{12}}{8|\mathbf{x}_{12}|^3}-\frac{l^2\kappa_{12}}{32|\mathbf{x}_{12}|^5}\ , \tag{60}$$

while the second is given by

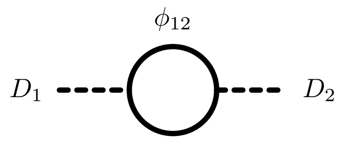
$$\frac{1}{2}(-i)^2\kappa_{31}\int\frac{d\omega}{2\pi}D_{\phi_{31}}(\omega)D_{\phi_{31}}(\omega+l)=\frac{\kappa_{31}}{8|\mathbf{x}_{31}|^3}-\frac{l^2\kappa_{31}}{32|\mathbf{x}_{31}|^5}\ . \tag{61}$$

Therefore, ignoring the l^2 terms, the total contribution to the effective Lagrangian is

$$L_{eff,D_1^2}=\frac{D_1^2}{8}\left(\frac{\kappa_{31}}{|\mathbf{x}_{31}|^3}+\frac{\kappa_{12}}{|\mathbf{x}_{12}|^3}\right)\ . \tag{62}$$

The D_2^2 and D_3^2 terms can be obtained by cyclic permutation.

Now let us consider the off diagonal terms and more specifically D_1D_2. The only diagram that contributes to 1-loop order is:



This diagram differs only by a sign and a factor of 2 compared to (i) of figure II.13. Therefore, the D_1D_2 term of the effective Lagrangian is

$$L_{eff,D_1D_2}=-D_1D_2\left(\frac{\kappa_{12}}{4|\mathbf{x}_{12}|^3}\right)\ . \tag{63}$$

Other off diagonal terms can be obtained by cyclic permutation.

$\delta\mathbf{x}_i \cdot \delta\mathbf{x}_j$ term

In order to obtain the correction to the quadratic velocity terms in the Lagrangian, we consider fluctuations the bosonic degrees of freedom $\mathbf{x}_i + \delta\mathbf{x}_i$. Terms of this form can be divided in terms that are diagonal in the quiver such as $\delta\mathbf{x}_1 \cdot \delta\mathbf{x}_1$ and others that are off diagonal such as $\delta\mathbf{x}_1 \cdot \delta\mathbf{x}_2$. For each of these there is a further subdivision to terms that are diagonal in the vector component such as $\delta x_1 \delta x_1$ and $\delta x_1 \delta x_2$ and terms that are off diagonal in the vector component such as $\delta x_1 \delta y_1$ and $\delta x_1 \delta y_2$.

We begin by considering terms that are off diagonal in the quiver index, and off diagonal in the vector index. For concreteness we will study $\delta x_1 \delta y_2$. The diagrams that contribute are shown in figure II.14. The first diagram, series

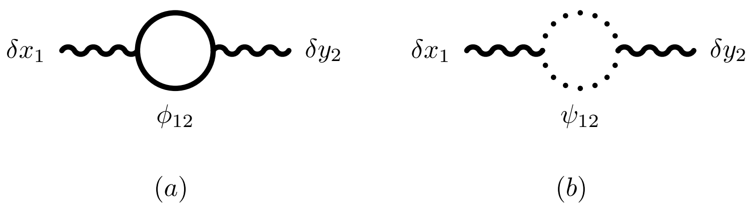


Figure II.14: 1-loop Feynman diagrams contributing to the $\delta x_1 \delta y_2$ term of the effective Lagrangian.

expanded for small external momentum l, is given by:

$$4x_{12}y_{12}(-i)i\kappa_{12}\int \frac{d\omega}{2\pi} D_{\phi_{12}}(\omega)D_{\phi_{12}}(\omega+l) = -\kappa_{12}\left(\frac{x_{12}y_{12}}{|\mathbf{x}_{12}|^3} - \frac{l^2 x_{12}y_{12}}{4|\mathbf{x}_{12}|^5}\right) + \mathcal{O}(l^3)\ . \tag{64}$$

The second diagram, series expanded for small external momentum l, is given by:

$$\kappa_{12}(-i)i\int \frac{d\omega}{2\pi}\mathrm{tr}(\sigma_1 D_{\psi_{12}}(\omega)\sigma_2 D_{\psi_{12}}(\omega+l)) = -\kappa_{12}\left(-\frac{x_{12}y_{12}}{|\mathbf{x}_{12}|^3} + \frac{l z_{12}}{2|\mathbf{x}_{12}|^3} + \frac{l^2 x_{12}y_{12}}{4|\mathbf{x}_{12}|^5}\right) + \mathcal{O}(l^3)\ . \tag{65}$$

Summing the two, the only term that remains uncancelled is the term linear in l.

Next we consider term off diagonal in the quiver index, but diagonal in the vector index, for example $\delta\mathbf{x}_1^1 \delta\mathbf{x}_2^1$. The contributing diagrams are shown in figure II.15. The first diagram is given by (64) with y_{12} replaced by x_{12}. The second

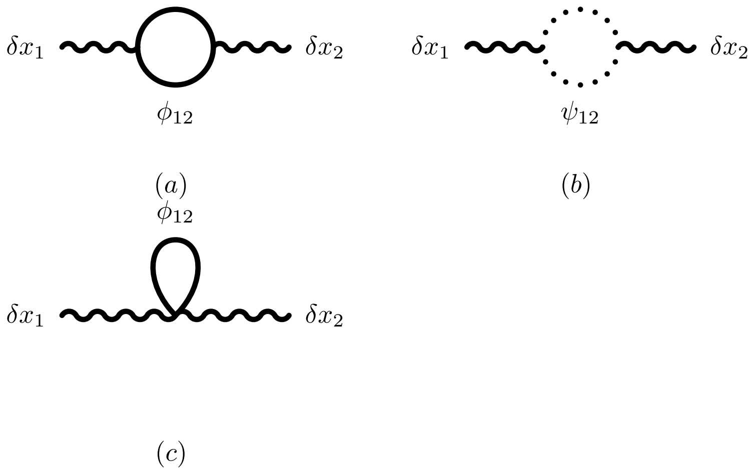


Figure II.15: 1-loop Feynman diagrams contributing to the $\delta x_1 \delta x_2$ term of the effective Lagrangian.

diagram gives:

$$\begin{aligned} &\kappa_{12}(-i)i \int \frac{d\omega}{2\pi} \mathrm{tr}(\sigma_1 D_{\psi_{12}}(\omega)\sigma_1 D_{\psi_{12}}(\omega + l)) \\ &\qquad = -\kappa_{12}\left(|y_{12}|^2 + |z_{12}|^2\right)\left(\frac{1}{|\mathbf{x}_{12}|^3} - \frac{l^2}{4|\mathbf{x}_{12}|^5}\right) + \mathcal{O}(l^3) \ . \end{aligned} \tag{66}$$

The third diagram is given by:

$$2i\kappa_{12} \int \frac{d\omega}{2\pi} D_{\phi_{12}}(\omega) = \frac{\kappa_{12}}{|\mathbf{x}_{12}|} \ . \tag{67}$$

Let us now consider terms that are diagonal in the quiver index. The Feynman diagrams contributing to $\delta \mathbf{x}_1^a \ \delta \mathbf{x}_1^b$ are shown in figure II.16. One finds after similar calculations the following contribution to the effective Lagrangian:

$$\begin{aligned} L_{eff}(\delta\vec{x}_1\delta\vec{x}_1) = \kappa_{12} &\left(\frac{lz_{12}\delta x_1 \delta y_1}{4|\mathbf{x}_{12}|^3} + \frac{lx_{12}\delta y_1 \delta z_1}{4|\mathbf{x}_{12}|^3} + \frac{ly_{12}\delta z_1 \delta x_1}{4|\mathbf{x}_{12}|^3}\right) \\ + \kappa_{31} &\left(\frac{lz_{31}\delta x_1 \delta y_1}{4|\mathbf{x}_{31}|^3} + \frac{lx_{31}\delta y_1 \delta z_1}{4|\mathbf{x}_{31}|^3} + \frac{ly_{31}\delta z_1 \delta x_1}{4|\mathbf{x}_{31}|^3}\right) \end{aligned} \tag{68}$$

$$- \frac{\kappa_{12} l^2 \delta\mathbf{x}_1 \cdot \delta\mathbf{x}_1}{8|\mathbf{x}_{12}|^3} - \frac{\kappa_{31} l^2 \delta\mathbf{x}_1 \cdot \delta\mathbf{x}_1}{8|\mathbf{x}_{31}|^3} \ . \tag{69}$$

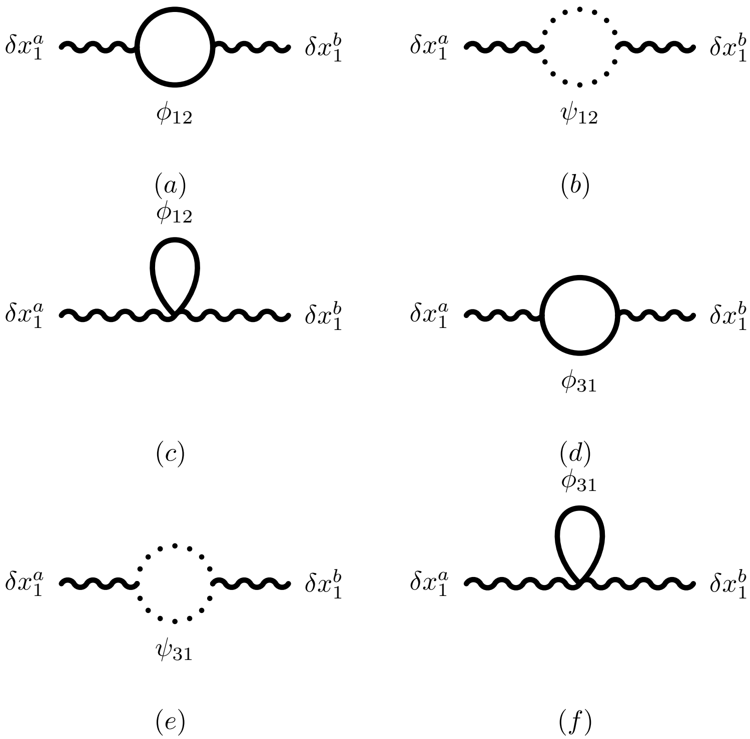


Figure II.16: 1-loop Feynman diagrams contributing to the $\delta\mathbf{x}_1^a\ \delta\mathbf{x}_1^b$ term of the effective Lagrangian.

Quadratic Lagrangian

Having computed all relevant Feynman diagrams that amount to integrating out the ϕ_{ij} and ψ_{ij} fields we can now write the resulting quadratic piece of the effective Lagrangian:

$$L_{eff}^{(2)} = \sum_i \frac{m_i}{2}\left(|\dot{\mathbf{x}}_i|^2 + D_i^2\right) + \frac{1}{8}\sum_{i\to j}\frac{|\kappa_{ij}|}{|\mathbf{x}_{ij}|^3}\ \left(|\dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j|^2 + (D_i - D_j)^2\right)\ . \quad (70)$$

Notice that the coefficient of $\dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_j$ indeed matches that of $D_i\ D_j$. This can be simply generalized to the N-particle case, where the Lagrangian takes the same form as (70). As was noted before, the linear piece was fixed by supersymmetry and given by (50) and (51) [Den02].

D.2 Coulomb branch potential

In this appendix we give the necessary formulae for the three-node Coulomb branch potential. Upon integrating out the chiral fields Φ^α in (1), while keeping the $\mathbf{q}^i$ fixed and time independent, and expanding up to quadratic order in the D^i fields, the D-dependent piece of the Lagrangian is:

$$L_D = \sum_{i=1}^{3} \left(\frac{\mu^i}{2} D^i\, D^i - \theta^i D^i - s_i\, \frac{|\kappa^i|}{2|\mathbf{q}^i|} D^i + \frac{|\kappa^i|}{8|\mathbf{q}^i|^3} D^i\, D^i \right) \, . \tag{71}$$

Noting that $D^3 = D^1 + D^2$, we must integrate out D^1 and D^2 from L_D. The resulting on-shell Lagrangian is simply the potential $-V(\mathbf{q}^i)$. The equations of motion for D^i, using $s_1 = s_2 = -s_3 = 1$, are given by:

$$\begin{aligned} \big(\mu^1 + \mu^3 + \frac{|\kappa^1|}{4|\mathbf{q}^1|^3} + \frac{|\kappa^3|}{4|\mathbf{q}^3|^3}\big) D^1 + \Big(\mu^3 + \frac{|\kappa^3|}{4|\mathbf{q}^3|^3}\Big) D^2 & \\ = \theta^1 + \frac{s_1|\kappa^1|}{2|\mathbf{q}^1|} + \frac{s_3|\kappa^3|}{2|\mathbf{q}^3|} & \end{aligned} \tag{72}$$

$$\begin{aligned} \big(\mu^2 + \mu^3 + \frac{|\kappa^2|}{4|\mathbf{q}^2|^3} + \frac{|\kappa^3|}{4|\mathbf{q}^3|^3}\big) D^2 + \Big(\mu^3 + \frac{|\kappa^3|}{4|\mathbf{q}^3|^3}\Big) D^1 & \\ = \theta^2 + \frac{s_2|\kappa^2|}{2|\mathbf{q}^2|} + \frac{s_3|\kappa^3|}{2|\mathbf{q}^3|} & \end{aligned} \tag{73}$$

Writing the above equations as: $a_{ij} D^i = c_i$, the solution is simply given by:

$$D^1 = \frac{a_{12}c_2 - a_{22}c_1}{a_{12}^2 - a_{11}a_{22}} \, , \qquad D^2 = \frac{a_{12}c_1 - a_{11}c_2}{a_{12}^2 - a_{11}a_{22}} \, . \tag{74}$$

Notice that the supersymmetric solution $D^i = 0$ is indeed given by $c_1 = c_2 = 0$ which is nothing more than equation (10).

Scaling Limit

In the scaling regime, where $\mu^i = 0$, $\theta^i = 0$, and $\kappa^1 > 0$, $\kappa^2 > 0$ and $\kappa^3 < 0$ (or equivalently $s_1 = s_2 = -s_3 = 1$), the potential can be written as:

$$\begin{aligned} V(\mathbf{q}^i) = \frac{1}{\alpha+\beta+\gamma} \Bigg(& \frac{\alpha+\beta}{2\gamma} |\mathbf{q}^3|^4 + \frac{\alpha+\gamma}{2\beta} |\mathbf{q}^2|^4 + \frac{\beta+\gamma}{2\alpha} |\mathbf{q}^1|^4 \\ & - |\mathbf{q}^1|^2 |\mathbf{q}^2|^2 - |\mathbf{q}^1|^2 |\mathbf{q}^3|^2 - |\mathbf{q}^2|^2 |\mathbf{q}^3|^2 \Bigg) \, , \end{aligned} \tag{75}$$

where

$$\alpha = \frac{|\mathbf{q}^1|^3}{\kappa^1} \, , \quad \beta = \frac{|\mathbf{q}^2|^3}{\kappa^2} \, , \quad \text{and } \gamma = -\frac{|\mathbf{q}^3|^3}{\kappa^3} \, . \tag{76}$$

E Thermal determinant

The thermal effective potential for the two-node quiver at finite temperature can be derived from (38):

$$L_{\text{eff}} = \frac{\mu}{2}D^2 - \theta D - \kappa \ln\det\left(-(\partial_t + iA)^2 + |\mathbf{q}|^2 + D\right) + \kappa \ln\det\left(-(\partial_t + iA)^2 + |\mathbf{q}|^2\right)$$

Wick rotating $t \mapsto it_E$ and periodically identifying $t_E \sim t_E + \beta$, introduces the thermal ensemble. The operator ∂_{t_E} has eigenvalues on the Matsubara frequencies: $\omega_n = 2\pi nT$ for bosons and $\omega_n = 2\pi(n+\frac{1}{2})T$ for fermions, with $n \in \mathbf{Z}$. With $a \equiv A/2\pi T$ and denoting $|\mathbf{q}|^2 \equiv s$ this yields:

$$L^{(T)}_{\text{eff}} = \frac{\mu}{2}D^2 - \theta D - \frac{\kappa}{\beta}\sum_{n\in\mathbf{Z}} \ln\left(4\pi^2(n+a)^2T^2 + s + D\right)$$
$$+ \frac{\kappa}{\beta}\sum_{n\in\mathbf{Z}} \ln\left(4\pi^2(n+1/2+a)^2T^2 + s\right) \ ,$$

where we have now included a $1/\beta$ in the coefficient of the sums to cancel out the contribution from the integral over Euclidean time in $S_E = \int_0^\beta dt_E\, L^{(T)}_{\text{eff}}$. Taking a derivative with respect to s:

$$\frac{dL^{(T)}_{\text{eff}}}{ds} = \frac{\kappa}{\beta}\sum_{n\in\mathbf{Z}} f(n) = -\frac{\kappa}{\beta}\sum_{n\in\mathbf{Z}} \frac{1}{4\pi^2(n+a)^2T^2 + s + D}$$
$$+ \frac{\kappa}{\beta}\sum_{n\in\mathbf{Z}} \frac{1}{4\pi^2(n+1/2+a)^2T^2 + s} \ . \quad (77)$$

We will treat the two sums separately using contour integration methods. We write (77) as $f(n) = -f_1(n) + f_2(n)$, denoting the two summands respectively. Consider:

$$\oint_{\mathcal{C}} \pi f_1(z)\cot(\pi z) = \oint_{\mathcal{C}} F(z)dz$$
$$= 2\pi i\left(\text{Res}F(z)|_{z=z^+} + \text{Res}F(z)|_{z=z^-} + \sum_{n\in\mathbf{Z}} \text{Res}F(z)|_{z=n}\right) \ ,$$

with $z^{\pm} = -a \pm \frac{i}{2\pi T}\sqrt{s+D}$ and $\text{Res}F(z)|_{z=z^{\pm}} = \pm\frac{i\cot(\mp\frac{i}{2T}\sqrt{s+D}+\pi a)}{4T\sqrt{s+D}}$ and $\mathcal{C}$ a circle of radius R around the origin with $R \to \infty$. Given that $F(z)$ decays rapidly enough, the integral evaluates to zero along $\mathcal{C}$ and we obtain:

$$-\sum_{n\in\mathbf{Z}} f_1(n) = -\frac{i}{4T\sqrt{s+D}}\left(\cot\left(\frac{i}{2T}\sqrt{s+D}+\pi a\right) - \cot\left(-\frac{i}{2T}\sqrt{s+D}+\pi a\right)\right)$$
$$(78)$$

For the fermionic sum, the procedure is completely analgous, and (using $\cot(x+\pi/2) = -\tan(x)$) we find:

$$\sum_{n\in\mathbf{Z}} f_2(n) = -\frac{i}{4T\sqrt{s}}\left(\tan\left(\frac{i\sqrt{s}}{2T}+\pi a\right) - \tan\left(-\frac{i\sqrt{s}}{2T}+\pi a\right)\right) . \tag{79}$$

We need to integrate these two sums with respect to s to obtain $L_{\text{eff}}^{(T)}$. Using the following identities: $\int \frac{\cot(a\sqrt{x}+b)}{\sqrt{x}}dx = 2\log\sin(a\sqrt{x}+b)/a$ and $\int \frac{\tan(a\sqrt{x}+b)}{\sqrt{x}}dx = -2\log\cos(a\sqrt{x}+b)/b$, we find:

$$L_{\text{eff}}^{(T)} = \frac{\mu}{2}D^2 - \theta D - \kappa T \log\left(\frac{\cosh\left(\frac{\sqrt{|\mathbf{q}|^2+D}}{T}\right) - \cos(2\pi a)}{\cosh\left(\frac{|\mathbf{q}|}{T}\right) + \cos(2\pi a)}\right) , \tag{80}$$

as claimed.

III

Monstrous Farey Tails

In mathematics and in physics alike, equations and relations are significant or — more poetically — *beautiful* or *deep* or *serious*, if the equation or relation connects concepts that were a priori not at all connected and conceptually far apart.

With this notion equalities of the sort $2+2=4$ are not very deep, as they relate natural numbers to natural numbers. But the arithmetic relation $e^{\pi i} = -1$ can be dubbed *deep* as it relates the transcendental, imaginary and integer numbers in an elegant way.

In his 'A Mathematician's Apology' [Har40], G.H. Hardy went further, and wrote that

The 'seriousness' of a mathematical theorem lies not in its practical consequences, which are usually negligible, but in the significance of the mathematical ideas which it connects. We may say, roughly, that a mathematical idea is 'significant' if it can be connected, in a natural and illuminating way, with a large complex of other mathematical ideas. Thus a serious mathematical theorem, a theorem which connects significant ideas, is likely to lead to an important advance in mathematics itself and even in other sciences.

With this notion, the moonshine conjecture[1] can be considered as one of the more serious relations in contemporary mathematics. It connects the branches: finite group theory, Kac-Moody algebra's, representation theory, modular forms and conformal field theory, to name but a few.

In this chapter we first introduce all the objects needed to formulate the monstrous moonshine conjecture. We articulate the actual conjecture by Conway and Norton and give a brief overview of the proof by Borcherds. We then go on to introduce generalizations of the monstrous moonshine theorem: to umbral groups and to Norton's 'generalized (monstrous) moonshine'. In the end we will connect these generalizations to what we may call 'generalized umbral moonhine'. For more introductory reading material we refer to [Bor98, Gan04, Gan06] and references therein.

[1] In the general setting I refrain from using the term 'theorem', as not all aspects of moonshine are proven at this point. In the context of the monster group, a proof is established and we may say monstrous moonshine theorem.

1 Monstrous moonshine

The Princeton Companion to Mathematics [Pri08] opens its lemma on Monstrous Moonshine with the wisecrack:

In 1978 McKay noticed that $196884 \approx 196883$.

Really what McKay noticed was that $1 + 196883 = 196884$. As it stands, this is indeed not deep at all and. But when we learn what these numbers represent we should be more baffled: the first, 1 and 196883, represent the dimensions of irreducible representations of the monster group. The second represents a Fourier coefficient in the j-function. These two objects, the monster group and the j-function, both play prominent roles in two entirely different branches of mathematics (finite groups and modular forms respectively) and that there is such a direct relation between the two is surprising and mysterious (or moonshine) indeed.

Before going on about the monstrous moonshine conjecture, let us introduce these two objects: the monster group and the j-function.

1.1 The monster group

Over the course of the 20^{th} century, the finite simple groups have been classified in a true tour de force (see for example [Gor82] and references therein). The declared completion of the classification in [Gor82] is a little premature and should be supplemented with [ALSS11] . It is the cumulative work of hundreds of mathematicians — with or without help of computers — written down in hundreds of journal publications on thousands of pages.

A bit like in the classification of semi-simple Lie algebras, the finite simple groups are classified as follows. First there are a couple of infinitely big families: the cyclic groups of prime order, the alternating groups of degree bigger than five and the simple groups of Lie type. And then there are 26 groups that do not fit in any of these families. These are called the *sporadic* simple groups. The monster group $\mathbb{M}$ is the largest finite sporadic group. It really is monstrous as it has an order of

$$|\mathbb{M}| = 2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8 \times 10^{53}. \quad (1)$$

The group was predicted to exist in the seventies by Griess and Fischer, and in 1982 Griess constructed $\mathbb{M}$ as the automorphism group of a non-associative algebra, without using a computer, see [Gri82].

The monster group has 194 conjugacy classes and hence 194 irreducible representations. The first nontrivial representation has dimension 196883. A small portion of the 194 by 194 character table is printed in table (III.1).

From this table we can read off the dimensions d_n of the irreducible representations, $d_n = \chi_n(1\text{A})$. These numbers will be important for the observation

Table III.1: A portion of the character table of the Monster group

$[g]$	1A	2A	2B	3A	3B	3C	...
χ_1	1	1	1	1	1	1	
χ_2	4196883	4371	275	782	53	−1	
χ_3	21296876	91884	−2324	7889	−130	248	
χ_4	842609326	1139374	12974	55912	−221	−248	
χ_5	18538750076	8507516	123004	249458	1598	248	
χ_6	19360062527	9362495	−58305	297482	1508	−247	
$\vdots$							

of moonshine later. We will now switch gears and introduce another important object we need to formulate the monstrous moonshine conjecture: the j-function.

1.2 Modular forms, functions and curves

That modular forms arise in the moonshine story is probably not very surprising, as modular forms pop up pretty much always when integers and number theory are mentioned. As Martin Eichler apocryphally stated: "there are five elementary arithmetical operations: addition, subtraction, multiplication, division, and modular forms". Indeed in modern number theory, modular forms play a central role and they play a big role in the story of moonshine. We start with a crash course in the theory of modular forms and functions, highlighting only the most elementary concepts of the theory. We refer to [Ser73, RS11, BGHZ08] for more thorough coverage of the subject matter.

Let $\mathcal{H}$ denote the upper-half plane: $\mathcal{H} = \{z \in \mathbf{C} \,|\, \mathrm{Im}(z) > 0\}$. A modular form f of weight k on the (congruence) subgroup $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ is complex function f on the upper half-plane, $f : \mathcal{H} \to \mathbf{C}$ that satisfies:

1. f is holomorphic on $\mathcal{H}$
2. $f(\gamma \cdot \tau) = (a\tau + b)^k f(\tau)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ \gamma \cdot \tau = \frac{a\tau+b}{c\tau+d}$
3. $f(\tau)$ is holomorphic at $\tau = i\infty$

A modular form on $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ is called a cusp form if it vanishes at the boundary of the upper half-plane, that is if $f(i\infty) = 0$. Also we call f a modular function if it is meromorphic at $\mathcal{H}$ but has $k = 0$, $f(\gamma \cdot \tau) = f(\tau)$. We will later need the notion of modular forms or functions with automorphic factors, where condition 2 is relaxed to

$$f(\gamma \cdot \tau) = \epsilon(\gamma)(a\tau + b)^k f(\tau),\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ |\epsilon(\gamma)| = 1$$

Modular forms of weight k (and cusp forms) on a congruence subgroup Γ form a vector space $M_k(\Gamma)$ (or S_k for cusp forms) over the complex numbers. Taking for now $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ we will briefly list some properties of these vector spaces. We will for now write $M_k(\Gamma) = M_k$ and $S_k(\Gamma) = S_k$. The dimensions of the vector spaces are:

$$\dim_{\mathbf{C}}(M_k) = \begin{cases} \lfloor k/12 \rfloor, & \text{if } k \equiv 2(12) \\ \lfloor k/12 \rfloor + 1, & \text{otherwise} \end{cases} \tag{2}$$

$$\dim_{\mathbf{C}}(S_k) = \begin{cases} \lfloor k/12 \rfloor - 1, & \text{if } k \equiv 2(12) \\ \lfloor k/12 \rfloor, & \text{otherwise} \end{cases} \tag{3}$$

so the first non-trivial cusp-form arises at dimensions 12. It plays a central role in the theory of modular forms and in string theory: S_{12} is spanned by

$$\Delta(\tau) = q \prod_{n=1} (1 - q^n)^{24}, \tag{4}$$

where $q = \exp(2\pi i \tau)$.

As $T = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$, and $|q(\tau)| < 1$ for $\tau \in \mathcal{H}$, we can view a modular form $f(\tau)$ as a funcion $f = f(q(\tau))$ with a well-defined (Fourier) series expansion

$$f(q) = \sum_{n=-\infty}^{\infty} a_n q^n, \tag{5}$$

with $a_n = 0$ for almost all $-\infty < n < 0$ for modular functions, $a_0 = 0$ for cusp forms and $a_n = 0$ for all $-\infty < n < 0$ in the case of modular forms.

It is through these Fourier coefficients a_n that we make contact with number theory. As an example, consider again $\Delta(q)$. This function is related to partitions of integers. A partition $\lambda \vdash n$ of a natural number n is a sum decomposition of n in smaller integers, for example $1 + 3 \vdash 4$. If we denote the number of partitions of an integer by $p(n)$ then the generating function of $p(n)$ can be shown to be

$$F(q) = \sum_{n=0}^{\infty} p(n) q^n = \prod_{n=0}^{\infty} \frac{1}{1 - q^n} \tag{6}$$

and indeed, $qF(q)^{-24} = \Delta(q)$.

Pointwise multiplication is a map $\cdot : M_k \times M_{k'} \to M_{k+k'}$. This map endows the total space of modular forms, $M = \bigoplus_k M_k$ with the structure of a ring. This ring is generated by the two modular function $E_4(\tau)$ and $E_6(\tau)$:

$$M = \langle E_4, E_6 \rangle_{\mathbf{C}} \tag{7}$$

where E_4 and E_6 are known as Eisenstein series of weight 4 and 6 respectively. An Eisenstein series E_{2l} of weight l is the sum

$$E_{2l} = \frac{1}{\zeta(2l)} \sum_{(m,n)\in\mathbf{Z}'} (m + n\tau)^{-2l} \tag{8}$$

where the factor $\zeta(2l)$ ($\zeta(z)$ the Riemann zeta function) is conventional, chosen so that, for example, $E_4^2 = E_8$ and $E_4E_6 = E_{10}$, in accordance with the dimension formulas (2). The notation $\mathbf{Z}' = \mathbf{Z} - \{0\}$ is used in summations.

We can use $\Delta(q)$ and the Eisenstein series to construct a modular invariant function that is holomorphic everywhere except for a simple pole at the cusp $\tau = i\infty$. Defining $g_2 = 60E_2$, we introduce the j-function, or Klein's invariant $j(\tau)$:

$$j(\tau) = 1728\frac{g_2^3}{\Delta}. \tag{9}$$

The j-function defines a bijection between the fundamental domain $\mathcal{H}/\mathrm{SL}_2(\mathbf{Z})$ and $\mathbf{C}^\times$. It has a q-series that starts with

$$j(q) = \sum_{n=-1}^{\infty} a_n q^n = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \dots \tag{10}$$

The j-function is special in that all modular functions are rational functions of $j(q)$. It is called the generator of the function field on the modular curve $X(1) = \mathcal{H}/\mathrm{SL}_2(\mathbf{Z})$. For later reference, we want to make this a little more precise before going further.

Let $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ be a commensurable subgroup. For our purposes we will be interested in the cases $\Gamma \in \{\Gamma(N), \Gamma_0(N), \Gamma_1(N)\}$ where

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a \equiv d \equiv \pm 1(N), b \equiv c \equiv 0(N) \right\} \tag{11}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | c \equiv 0 \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a \equiv d \equiv \pm 1(N), c \equiv 0(N) \right\}$$

Another important subgroup is the Fricke group, defined as follows. Let F_n be the Fricke involution $F_n : \tau \mapsto -1/N\tau$. Then the Fricke group $\Gamma_0(n)_+$ is defined as

$$\Gamma_0(n)^+ = \langle \Gamma_0(n), F_n \rangle \tag{12}$$

These groups act on the upper half-plane $\mathcal{H}$, with coordinate τ, as fractional linear transformations on τ. Upon taking the quotient of $\mathcal{H}$ with a congruence subgroup Γ, we get a variety $X(\Gamma)$, referred to as a modular curve. The most common ones are $X(N) = \mathcal{H}/\Gamma(N)$, $X_0(N) = \mathcal{H}/\Gamma_0(N)$ and $X_1(N) = \mathcal{H}/\Gamma_1(N)$.

For each modular curve $X(\Gamma)$ we define the cusps as the points we need to add to $X(\Gamma)$ to compactify it to $\overline{X(\Gamma)}$. For the case $X(1) = \mathcal{H}/\Gamma(1)$ ($\Gamma(1) = \mathrm{SL}_2(\mathbf{Z})$), we can simply compactify $\mathcal{H}^\times = \mathcal{H} \cup \{\infty\}$ and $\overline{X(1)} = \mathcal{H}^\times/\Gamma(1)$. For other congruence subgroups we may need more points — cusps — to add to compactify the corresponding modular curve. The number of points equals the number of orbits of Γ on $\mathbf{Q} \cup \{\infty\}$.

Now consider on the variety $X(\Gamma)$ the field of functions

$$k(\overline{X(\Gamma)}) = \{f : \overline{X(\Gamma)} \to \mathbf{C}^\times \mid f \text{ meromorphic}\}. \tag{13}$$

The set is a field under pointwise addition and multiplication. For the modular curves, these fields are finitely generated, and when the modular curve has genus zero — the only case we will be considering — the modular function field is generated by a single function. We call this function the Hauptmodul of the modular curve. Sometimes we will write J_Γ for the Hauptmodul corresponding to Γ. The j-function is the Hauptmodul for the modular curve $X(1)$:

$$k(\overline{X(1)}) = \langle\, j \,\rangle_{\mathbf{C}}. \tag{14}$$

This generator is unique upto a multiplicative factor and an additional shift (it is the unique generator of the affine projective function field). We will also use the shifted j-function $J(q)$ a lot:

$$J = j - 744. \tag{15}$$

For another example, the Hauptmodul for $X_0(2)$ is denoted $j_{\Gamma_0(2)}$, and has a q-series

$$j_{\Gamma_0(2)} = q^{-1} + 276q - 2048q^2 + 11202q^3 + \dots \tag{16}$$

Typically, for N low enough, the Hauptmoduls of $\Gamma_0(N)$ can be expressed as quotients of Dedekind η-function. For example,

$$j_{\Gamma_0(2)} = \eta(\tau)^{24}/\eta(2\tau)^{24} + 24 \tag{17}$$

$$j_{\Gamma_0(3)} = \eta(\tau)^{12}/\eta(3\tau)^{12} + 12. \tag{18}$$

The Fricke group $\Gamma_0(2)^+$ is genus-zero as well[2]. The Hauptmodul reads

$$\begin{aligned} j_{\Gamma_0(2)^+} &= \left(\left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^{12} + 2^6\left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{12}\right)^2 \\ &= q^{-1} + 104 + 4372q + 96256q^2 + 1240002q^3 + \dots \end{aligned} \tag{19}$$

We have now introduced all the objects we need to state the original monstrous moonshine conjecture.

[2]The group $\Gamma_0(p)^+$ has genus zero if and only if $p \in \{2, 3, 5, 7, 11, 13, 17, 23, 29, 31, 41, 47, 49, 71\}$. Note that this is the set of prime divisors of the order of $\mathbb{M}$.

1.3 The monstrous moonshine conjecture

As eluded to in the beginning of this section, the mathematician John McKay observed that there is a similarity between the coefficient a_n in the q-series in expression (10), and the dimensions of the irreducible representations of the monster table as listed in table (III.1). In fact, using a_n for the coefficients of the j-functions and $d_n = \chi_n(1\text{A})$, we can note the following identities:

$$\begin{aligned} a_1 &= d_1 + d_2 \\ a_2 &= d_1 + d_2 + d_3 \\ a_3 &= 2d_1 + 2d_2 + d_3 + d_4 \\ a_4 &= 3d_1 + 3d_2 + d_3 + 2d_4 + d_5 \end{aligned} \tag{20}$$

McKay took these observations to suspect the existence of an infinite dimensional (graded) module $V_{\mathbb{M}}$ over the monster group,

$$V_{\mathbb{M}} = \bigoplus_{n=-1}^{\infty} V_n \tag{21}$$

were, comparing with table (III.1), $V_{-1} = \chi_1$, $V_0 = \{0\}$, $V_1 = \chi_1 + \chi_2$, $V_2 = \chi_1 + \chi_2 + \chi_3$, so that indeed,

$$\sum_{n=-1}^{\infty} \dim(V_n) q^n = 1 + 196994q^2 + \ldots = J(q) \tag{22}$$

As it stands the observation is completely trivial and vacuous, as we can at will add trivial representations in the graded module to obtain the j-function. But the existence of a module means we could take characters of other group elements as well.

We now only evaluated the characters at the identity element of the monster group. Should the monster module $V_{\mathbb{M}}$ exist, this reasoning should apply for all elements of the monster group, that is, for every element $g \in \mathbb{M}$ we can write down

$$T_g(\tau) = \sum_{n=-1}^{\infty} \mathrm{ch}_{V_n}(g) q^n. \tag{23}$$

where ch_{V_n} is now the character of V_n. These series were suggested by Thompson in [Tho79] and are known as McKay-Thompson series. The functions $T_g(\tau)$ are class functions and only depend on the conjugacy class of g in $\mathbb{M}$, $T_g(\tau) = T_{[g]}(\tau)$. With this notation, the proposal is that $T_{[1A]} = J(q)$. We are now ready to state the monstrous moonshine conjecture.

Conjecture. *[Conway-Norton] For each $g \in \mathbb{M}$, the McKay-Thompson series $T_g(\tau)$ is the Hauptmodul $J_{\Gamma_g}(\tau)$ for some commensurable genus-zero subgroup Γ_g of $SL_2(\mathbf{Z})$.*

We see now that indeed the conjecture not only is about the j-function, but also applies for example to the Hauptmodul $j_{\Gamma_0(2)}$ as in equation (16). Comparing with the character table (III.1) and write $j_{\Gamma_0(2)} - 24 = \sum_{n=-1}^{\infty} a_n[2B]q^n$:

$$\begin{aligned} a_1[2B] =& \chi_1(2B) + \chi_2(2B) \\ a_2[2B] =& \chi_1(2B) + \chi_2(2B) + \chi_3(2B) \\ & \vdots \end{aligned} \tag{24}$$

and we may write $T_{2B}(\tau) = j_{\Gamma_0(2)} - 24$. The theorem states that for each conjugacy class of $\mathbb{M}$ we can find such a Hauptmodul. For example also: $T_{2A} = j_{\Gamma_0(2)^+} - 104$.
From a technical point of view it seems natural that only genus-zero subgroups arise as there we have a canonical choice, up to an overall factor and an additional constant, for the Hauptmodul. Subgroups Γ of higher genus have function fields defined on them that are generated by more than one generator and it seems like there is no such canonical choice. Geometrically it is less understood why moonshine at this point only works for genus zero.
The theorem (1.3) as it stands is proven by Borcherds, a proof famously awarded with the Fields medal in 1984. His proof can be deconstructed into four steps.

1. Construct a module V that has the additional structure of a "Vertex Operator Algebra" (VOA)
2. From V, construct a Lie algebra $\mathfrak{M}$ togehter with a group action by the monster group $\mathbb{M}$.
3. Use the Weyl-Kac denominator formula to show that the functions $T_g(\tau)$ are "completely replicable".
4. Show that the $T_g(\tau)$ are Hauptmoduls for genus-zero subgroups of $\mathrm{SL}_2(\mathbf{Z})$.

The module in step (1) was constructed by Frenkel, Lepowsky and Meurman [FLM88]. They created V as the vertex algebra of physical states of bosonic strings moving on a target space $(\mathbf{R}^{26}/\Lambda)/\mathbf{Z}_2$, with Λ the Leech lattice, and they showed that the monster group $\mathbb{M}$ acts naturally on this space as a group of global isometries. Its explicit construction is rather tedious and lengthy (500+ pages) and we will not describe it here and luckily we do not need all the details of the construction. The VOA of [FLM88] is a graded module, and has the property that $\sum_{n=-1}^{\infty} \dim(V_n)q^n = J(\tau)$, so this is a big step towards the proof of moonshine indeed. One basically would like to show that this module is in fact the sought for graded module in the conjecture (1.3).

Borcherds needs step (2) for the following reason. If we can promote the VOA of [FLM88] to a Lie algebra $\mathfrak{M}$, and can show that it is a (generalized)

Kac-Moody algebra (gKMA, also sometimes called Borcherds algebra), we can use the Kac-Weyl denominator formula to generate replication formulas for the functions $T_g(\tau)$. We mean by replication formulas for some functions, formulas that generate the relations between the Fourier coefficients of the functions. The technical definitions is as follows: let H_n be the Hecke-like operator acting on a function $f_g(\tau)$ as:

$$H_n f_g(\tau) = \sum_{\substack{ad=n \\ b \bmod d}} f_{g^a}\left(\frac{a\tau + b}{d}\right) \tag{25}$$

for g a group element of some group G (here, of course, $G = \mathbb{M}$). The f_g is called replicable if each $H_n f_g(\tau)$ has a Fourier expansion $H_n f_g(\tau) = \sum_{n=-1}^{\infty} a_i q^i$. Replicability will allow us to construct all the functions $T_g(\tau)$ from a small amount of data. We then only need to show that these functions are Hauptmoduls.

We briefly introduce the concept of a gKMA. First, recall the following properties of a finite dimensional Lie algebra $\mathfrak{g}$:

1. $\mathfrak{g}$ has an invariant symmetric bilinear form $(,)$
2. $\mathfrak{g}$ has a Cartan involution ω, $\omega^2 = \mathrm{id}$
3. $\mathfrak{g}$ is graded, $\mathfrak{g} = \bigoplus_{n\in\mathbf{Z}} \mathfrak{g}_n$, $\mathfrak{g}_n$ finite dimensional and $\omega|_{\mathfrak{g}_0} = -1$, $\mathfrak{g}_0$ the Cartan subalgebra
4. $(g, \omega(g)) > 0$ for all $g \in \mathfrak{g}_n$, $g \neq 0$.

Now if we relax condition (4) to the condition:

4' $(g, \omega(g)) > 0$ for all $g \in \mathfrak{g}_n$, $g \neq 0$ and $n \neq 0$,

we promote $\mathfrak{g}$ to a Generalized Kac-Moody algebra. Many of the properties and concepts of the theory of Lie algebras carry over to gKMA's rather straightforwardly. For the purpose of this review, we will need the Weyl-Kac denominator formula, that is indeed just the analogue of Weyl's denominator formula for the case of Lie algebra's. For a gKMA $\mathfrak{G}$, let W be its Weyl group, ρ the Weyl vector, $\{\alpha\}$ the set of roots. For $w \in W$, denote $\det w = \pm 1$: $\det(w) = +1$ if w is the product of an even number of reflections, $\det(w) = -1$ if its the product of an odd number of reflections. Furthermore let $\varepsilon(\alpha) = (-1)^n$ if α is the sum of n orthogonal (imaginary) simple roots that are all orthogonal to the lowest weight vector. Let $\varepsilon(\alpha) = 0$ if otherwise. With these concepts, the Weyl-Kac denominator formula reads

$$e^\rho \prod_{\alpha>0} (1 - e^\alpha)^{\mathrm{mult}(\alpha)} = \sum_{w\in W} \det(w) w(e^\rho \sum_\alpha \varepsilon(\alpha) e^\alpha). \tag{26}$$

For example, when applied to $\mathfrak{sl}_2[z, z^{-1}]$, this reduces to the Jacobi triple product identity.
In the seminal [Bor92] Borcherds constructs from V a gKMA $\mathfrak{M}$ that is $\mathbf{Z} \oplus \mathbf{Z}$-graded and has the monster group $\mathbb{M}$ acting on it. The simple roots of this gKMA are the vectors $\beta_{1,n} = (1, n)$ for $n = -1$ or $n \in \mathbf{N}$, and these simple roots $\beta_{1,n}$ each have multiplicity c_n, where c_n is still the coefficient of the j-function. It can be shown that the multiplicity of a root $\alpha_{m,n} = (m, n) \neq (0, 0)$ are cnm. If we now write $p = e^{(1,0)} = e^{2\pi i\sigma}, q = e^{(0,1)} = e^{2\pi i\tau}$, we find a denominator identity for $\mathfrak{M}$:

$$p^{-1} \prod_{\substack{m>0 \\ n\geq -1}} (1 - p^m q^n)^{c_{mn}} = j(\sigma) - j(\tau). \tag{27}$$

This truly beautiful formula was noted earlier by Zagier, Norton and Koike, none of whom bothered to publish their proof before Borcherds. This formula already induces relations between the coefficients of $j(\tau)$, for example $2(c_4 - c_3) = c_1^2 - c_1$. In fact, formula (27) induces enough non-trivial polynomial relations for all the coefficients to be computable form only the first five coefficients $c_1, \ldots c_5$. To give an example, one such polynomial relation reads $c_{4n+2} - c_{2n+2} = \sum_{k=1}^{n} c_k c_{2n-k+1}$. In chapter IV of this dissertation we will even give this formula a physical interpretation, and think of it as a grand canonical partition function.

Conway and Norton had already conjectured in [CN79] the existence of relations similar to the expression (27) for the McKay-Thompson series T_g, namely

$$p^{-1} \exp\left(- \sum_{\substack{k,m>0 \\ n\in\mathbf{Z}}} c_{mn}(g) \frac{p^{mk} q^{nk}}{k} \right) = T_g(\sigma) - T_g(\tau), \tag{28}$$

where $T_g(\tau) = \sum_{n\in\mathbf{Z}} c_n(g) q^n$.

Borcherds proved these relations using the gKMA $\mathfrak{M}$. The Hauptmoduls now are completely replicable: there are all sorts of polynomial relations between the coefficients of the McKay-Thompson series, and we only need the first couple of coefficients to determine all the coefficients of the Hauptmoduls (preferably on a computer).

1.4 Generalized monstrous moonshine

A few years after the "discovery" of the monstrous moonshine, Norton proposed a generalization of the conjecture. In [Nor87] he proposed that for every commuting pair in the monster group, $(g, h) \in \mathbb{M} \times \mathbb{M}$, we can attach a function $T_{g,h}(\tau)$ such that

1. $T_{g,h}(\tau) = T_{g^k,h^k}(\tau)$

2. $\xi_\gamma T_{g,h}(\gamma\tau) = T_{g^a h^c, g^b h^d}(\tau)$ for $\Gamma_{g,h} \subset \mathrm{SL}_2(\mathbf{Z})$ and $\xi_\gamma^{24} = 1$
3. $T_{g,h}(\tau)$ is a Hauptmodul for some genus-zero $\Gamma_{g,h} \subset \mathrm{SL}_2(\mathbf{Z})$ with $\Gamma(M) \subset \Gamma_{g,h}$
4. $T_{e,h}(\tau) = T_h(\tau)$ where $T_h(\tau)$ the McKay-Thompson series of monstrous moonshine
5. $T_{g,h}(\tau) = \sum_{n=-1}^{\infty} \hat{\mathrm{ch}}_{C_g^n}(h) q^n$

where $C_g = \bigoplus_{n=-1}^{\infty} C_g^n$ is an infinite dimensional graded module of the centralizer subgroup $C_{\mathbb{M}}(h)$ and $\hat{ch}_H(h)$ denotes the projective character of the projective representation of a group H. The reason behind this proposal can be made more clear and intuitive by thinking about the functions $T_{g,h}(\tau)$ as twisted and twined partition functions of some orbifolded conformal field theory. A lot of the concepts in a vertex operator algebra — the structure introduced to prove the monstrous moonshine conjecture — have an analogy in terms of objects in a (rational, meromorphic) conformal field theory. The vertex algebra is like the chiral (algebra) $\mathcal{H}$ of vertex operators corresponding to a sector of representations of the Virasoro algebra that contains the vacuum, with a group G ($G = \mathbb{M}$ for monstrous moonshine) acting as the group of endomorphisms of $\mathcal{H}$. We will be interested in dividing the algebra by its group of endomorphsims and will be looking at the orbifolded theory $\mathcal{H}/G$. In this theory, some states are projected out, but there will also arise new sectors that pick up a G-valued holonomy, see below. Then, for example, the McKay-Thompson series $T_g(\tau)$ of monstrous moonshine is really the graded trace Z_g:

$$T_g(\tau) = Z_g(\tau) = \mathrm{Tr}_{\mathcal{H}}\left(g q^{L_0 - 1}\right) \tag{29}$$

for $g \in G = \mathrm{End}(\mathcal{H})$. Note that $Z_e = J(\tau)$. For each $h \in G$ there is a unique twisted module $\mathcal{H}_h$. Physically his is the sector of twisted states: states $\phi(z)$ that pick up a monodromy h: $\phi(e^{2\pi i} z) = h \cdot \phi(z)$. We refer to [Don92] for a formal definition of a twisted module.

For an element $g \in C_G(h)$ — the centralizer of h in G — g induces a linear map on $\mathcal{H}_h$ so we can define the twisted orbifold trace function, or twist-h twine-g partition function

$$Z_{g,h}(\tau) = \mathrm{Tr}_{\mathcal{H}_h}\left(g q^{L_0 - 1}\right). \tag{30}$$

Note that $\mathcal{H} = \oplus_{[h]} \mathcal{H}_h$ where $[h]$ denotes the conjugacy class of h. We can now see that $Z_{g,h} = \zeta_\gamma Z_{g^k, h^k}$, $|\zeta_\gamma| = 1$. If we think in terms of operators $\phi(z)$ on the torus $\mathbb{T} = \mathbf{Z} \oplus \tau\mathbf{Z}$, then the partition function $Z_{g,h}$ can be seen as the partition function of states with boundary condition

$$\phi(z+1) = h \cdot \phi(\tau), \;\; \phi(z+\tau) = g \cdot \phi(z) \tag{31}$$

That is why we sometimes use a box notation:

$$Z_{g,h}(\tau) = {}_{g}\underset{h}{\square}(\tau) \qquad (32)$$

where the box is a pictorial representation of the torus $\mathbb{T}$ and g and h denote te boundary condition along its cylces. Let A and B denote the two cycles of the torus: $H_1(\mathbb{T}, \mathbf{Z}) = \langle A, B\rangle_{\mathbf{Z}}$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$ we have an action on the cycles (A, B):

$$\gamma \cdot (A, B) = (cB + dA, aB + bA) \qquad (33)$$

so that the partition function transforms as

$$_{g}\underset{h}{\square}(\tau) = \xi(\gamma, g, h)\, {}_{g^a h^b}\underset{g^c h^d}{\square}(\gamma\tau). \qquad (34)$$

The partition function ${}_{g}\underset{h}{\square}(\tau)$ is invariant under modular transformations up to a phase for some subgroup $\Gamma_{g,h} \subset \mathrm{SL}_2(\mathbf{Z})$.
The phase $\xi(\gamma, g, h)$ is, for a fixed γ, a map $\xi : G \times G \to \mathbf{C}^{\times}$. As explained in the appendix, the pahse ξ is actually a two-cocycle in the group cohomology $H^2(C_G(h), U(1))$ that can be universally computed from the third cohomology [DW90] $\psi_h : H^3(G, U(1)) \to H^2(C_G(h), U(1))$. Unfortunately, for the monster group the group cohomology $H^3(\mathbb{M}, U(1))$ is not computed yet. For some of the centralizers however we can compute the group cohomology and determine the phases $\xi(\gamma, g, h)$.

For low orders of group elements, some of the centralizer subgroups are listed in table (III.2), where we see the Baby Monster $\mathbb{B}$ (the second largest simple finite

Table III.2: Centralizer subgroups of $\mathbb{M}$ for low $p = ord(h)$

p	2	3	5	7	11	13
$C_h(\mathbb{M})$	2.$\mathbb{B}$	3.Fi	$5 \times$ HN	$7 \times$ He	$11 \times \mathrm{M}_{12}$	$5 \times L_3(3)$

sporadic group), the Fisher group Fi, the Harada-Norton group, the Mathieu group M_{12} and the atlas group $L_3(3)$. In this notation, we abbreviated $\mathbf{Z}_n = n$. Also, $G \times H$ denotes the direct product of groups and $A = G.H$ denotes the group A that is not isomorphic to $G \times H$ but has G and H as normal subgroups (that is, A is the nontrivial extension of G by H).

All the listed groups are finite simple groups and can be found in the ATLAS [CCN+85]. It is conjectured that for all these groups, there is again a Hauptmodul with a McKay-Thompson series $T_h(\tau)$ that is conjectured to be a genus-zero Hauptmodul. As will be explained later, but can already be seen from the defintions, the independent orbits of functions $T_{g,h}$ under the action of

$\mathbb{M} \times \mathrm{SL}_2(\mathbb{Z})$ are classified by the rank-two Abelian subgroups of $\mathbb{M}$. Consider to this end the rank-two subgroup $\langle g, h\rangle \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ with g, h, gh all in class $2A$. First of all,

$$T_{e,g}(\tau) = T_{e,h}(\tau) = T_{2A}(\tau) = q^{-1} + 4372q + 96256q^2 + \dots \tag{35}$$

$$T_{g,e} = T_{h,e} = T_{e,g}(\tau/2) = q^{-1/2} + 4372q^{1/2} + 96256q + \dots \tag{36}$$

$$T_{g,g} = T_{h,h} = -iT_{g,e}(\tau+1) = q^{-1/2} + 4372q^{1/2} - 96256q + \dots \tag{37}$$

Here we see that $T_{g,e}(\tau+1) = iT_{g,g}(\tau)$, and there is a non-trivial multiplier system.

Finally the funtion $T_{g,h}$ is a bit harder to find but constructed in [**?**]:

$$T_{g,h}(\tau) = \sqrt{J(\tau) - 984} = q^{-1/2} - 492q^{1/2} - 22590q^{3/2} + \dots \tag{38}$$

For another example, the case of the Harada-Norton group HN, the McKay-Thompson series reads

$$\begin{aligned} T_{5A,e}(\tau) = j_{\Gamma_0(5)^+} - 6 &= \left(\frac{\eta(\tau)}{\eta(5\tau)}\right)^6 + 5^3\left(\frac{\eta(5\tau)}{\eta(\tau)}\right)^6 \\ &= \frac{1}{q} + 134q + 760q^2 + 3345q^3 + 12256q^4 + 39350q^5 + \dots \end{aligned} \tag{39}$$

2 Umbral moonshine

A natural question to ask is: does the moonshine phenomenon only occur for the monster group and the Hauptmoduls? Or are there more groups that have infinite dimensional modules, the McKay-Thompson series of which are Hauptmoduls, or other modular objects? The answer is yes. Well, first of all there are the groups that arise as centralizer subgroups as in section 1.4, and they have their McKay-Thompson series. But there is an interesting generalization of monstrous moonshine to groups that are not per se centralizer subgroups of the monster, and where the modular objects are not Hauptmoduls but more intricate. We will discuss the generalization of moonshine for the monster groups to moonshine for the “umbral groups” in this .

2.1 Mathieu moonshine

In [EOT11] Eguchi, Ooguri and Tachikawa reported on a phenomenon very much like that of moonshine. The function in question is not the j-function, but the elliptic genus of K3. This is a topological quantity encountered in string theory when compactifying heterotic string theory on a K3 manifold. Consider a two dimensional sigma model with target space K3. Let L_0 and $\bar{L}_0$ denote the zero modes of the Virasoro operators and J_0^3 the zero mode of the z-component of the affine $\mathfrak{su}(2)$ algebra that arises as the R-symmetry of the $\mathcal{N}=4$ sigma model on K3. If we furthermore let F_L and F_R denote the left and right moving fermion numbers, the elliptic genus is defined as a modified Witten index, namely

$$Ell[K3](\tau;z) = \mathrm{Tr}_R(-1)^{F_L+F_R} q^{L_0}\bar{q}^{\bar{L}_0} y^{4J_0^3}, \tag{40}$$

where $q=e^{2\pi i\tau}, y=e^{\pi i z}$. For the manifold K3 this quantity was computed in [EOTY89] to be

$$Ell[K3](\tau;z) = 8\left[\left(\frac{\theta_2(\tau;z)}{\theta_2(\tau;0)}\right)+\left(\frac{\theta_3(\tau;z)}{\theta_3(\tau;0)}\right)+\left(\frac{\theta_4(\tau;z)}{\theta_4(\tau;0)}\right)\right], \tag{41}$$

where $\{\theta_i(\tau;z)$ are the Jacobi theta functions.

On special values the elliptic genus reduces to more familiar topological quantities and we can check that indeed $\mathrm{Ell}[K3](\tau;0)=\chi(K3)=24, \mathrm{Ell}[K3](\tau,1/2)=\sigma(K3)=16+\mathcal{O}(q)$. The elliptic genus is known to be a weak Jacobi form [Wit87]. We follow the conventions of [EZ85] and refer to appendix (A)for more details on (weak) Jacobi forms. For now it suffices to state that a Jacobi form $\phi(\tau,z)$ of level 1, weight k and index m is a function $\phi:\mathcal{H}\times\mathbf{C}\to\mathbf{C}$ with the following properties:

1. $\phi(\gamma\cdot\tau,\gamma\cdot z) = (c\tau+d)^k\exp(\frac{2\pi imcz^2}{c\tau+d})\phi(\tau,z)$ where $\gamma\in\Gamma\subset\mathrm{SL}_2(\mathbf{Z})$
2. $\phi(\tau,z+\lambda\tau+\mu) = \exp -2\pi im(\lambda^2\tau+2\lambda z)\phi(\tau,z)$ for $\lambda,\mu\in\mathbf{Z}$
3. $\phi(\tau,z) = \sum_{\substack{n\geq 0\\ r^2\leq 4mn}} c(n,r)q^n y^r$, $q=\exp 2\pi i\tau$, $y=\exp 2\pi iz$.

A Jacobi form is called weak if $c(n,r)=0$ unless $n\geq 0$. The space of Jacobi forms of weight-0 and index-1 is actually one-dimensional so with this knowledge the result (41) is rather trivial.

Expanding the elliptic genus on K3 in terms of characters of the $\mathcal{N}=4$ superconformal algebra, the authors of [EOT11] noticed that

$$Ell[K3](\tau;z) = 24\mathrm{ch}^R_{h=\frac{1}{4},\ell=0}(\tau;z) + H^{(2)}(\tau)\frac{\theta_1(\tau;z)^2}{\eta(\tau)^3} \tag{42}$$

$$H^{(2)}(\tau) = -2q^{-1/8} + q^{-1/8}\sum_{n=1}^{\infty}A_n q^n, \tag{43}$$

where the first few coefficients A_n read $A_1=45, A_2=231, A_3=770, A_4=2277, A_5=5796, A_6=13915,\ldots$. The first five coefficients are also the dimensions of irreducible representation of the Mathieu group M_{24}, another group in the list of sporadic simple groups, see table (III.3) where we printed de dimensions of some of the irreducible representation χ_i of M_{24} (where i is ordered along

Table III.3: Dimensions of the irreducible representations of M_{24}

i	1	2	3	4	5	6	...	10	11	...	20	...
$\mathrm{Tr}(\chi_i)$	1	23	45	$\overline{45}$	231	$\overline{231}$	...	770	$\overline{770}$	...	2277	...

with the size of the dimension of χ_i, and an overline denotes a dimension of a conjugate representation).

The coefficient A_6 we can also write as the linear combination of dimensions of irreducible representations of M_{24}. This of course resonates a lot with the monstrous moonshine conjecture. With the monstrous moonshine in mind, one is now immediately led to conjecture the existence of a graded M_{24}-module[3] $K^{(2)}$:

$$K^{(2)} = \bigoplus_{n=0}^{\infty} K^{(2)}_{n-1/8} \tag{44}$$

with the property that $\sum_{n=0}^{\infty} \dim\left(K^{(2)}_{n-1/8}\right) q^{n-1/8} = H^{(2)}(\tau)$. Again this would trigger the study of the McKay-Thomson series of this module: $H^{(2)}_g(\tau) = \sum_{n=0}^{\infty} \mathrm{ch}_{K_{n-1/8}}(g) q^{n-1/8}$, hoping we can classify these functions like in the case of monstrous moonshine, where the McKay-Thompson series were conjectured to be the genus-zero Hauptmoduls.

The first challenge is to find a precise articulation of a conjecture, analogous to Conway and Norton's conjecture (1.3). What troubles a precise analogue is that the functions $H_g(\tau)$ turn out to be not exactly modular functions. Instead, for every g in M_{24}, they are *mock modular forms*[**?**]f weight 1/2 of $\Gamma_g \subset \mathrm{SL}_2(\mathbf{Z})$ [EH11]. Mock modular forms are peculiar objects that were already studied by Ramanujan in his "Lost Notebooks" [Ram88]. They were rigorously defined only quite recently by Zwegers in [Zwe08] and we give his definition here:

A *(weakly holomorphic) mock modular form* of weight k on the congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ is a holomorphic function $h(\tau)$ on the upper half-plane $\mathcal{H}$ if it obeys the criteria:

1. $h(\tau)$ is holomorphic
2. $|h(\tau)| \leq C|\exp(c\tau)|$ for some constants C, c as $\tau \to \alpha \in \mathbf{Q}$ for all $\alpha \in \mathbf{Q}$
3. There is a holomorphic modular form $f(\tau)$ of weight $2-k$ on Γ such that the *completion* $\hat{h}(\tau) = h(\tau) + (4i)^{k-1} \int_{-\bar{\tau}}^{\infty} (z+\tau)^{-k} \overline{f(-\bar{z})} dz$ is a holomorphic

[3] The superscript (2) on the module $K^{(2)}$ will become clear in a bit

from of weight k on Γ with multiplier system $\chi : \Gamma \to \mathbf{C}^{\times}$. We call $f(z)$ the *shadow* of $h(\tau)$.

In condition (3), $\hat{h}(\tau)$ need not be holomorphic, and what we mean by a multiplier system χ for a modular form $F(\tau)$ of weight k is that $F(\gamma\tau) = \chi(\gamma)(c + d\tau)^{-k}F(\tau)$. It is in this definitions that all $H_g(\tau)$ are mock modular forms of weight 1/2. For example, $H^{(2)}$ as in expression (43) is a mock modular form in the sense that

$$\hat{H}(\tau) = H(\tau) + 24(4i)^{-1/2}\int_{\bar{\tau}}^{\infty}(z+\tau)^{-1/2}\overline{\eta(-\bar{z})^3}dz \tag{45}$$

transforms as a weight-1/2 modular form on $\Gamma(1)$, with shadow $\eta(z)^3$.

The trouble in finding a direct analogue for conjecture (1.3) is that first of all here Γ_g is not per se a genus-zero subgroup. But it may not even matter if it would, as there is no clear way in which we can see any of these $H_g(\tau)$ as the generator of some function field. In [CD12] this was keenly solved with a method inspired by concepts borrowed from physics: the Farey tail and Rademacher series. Before we go on to explain how the genus-zero property of monstrous moonshine is mimicked in the case at hand, the case of the Mathieu group M_{24}, we will introduce Rademacher series and the closely related Farey tail expansion.

2.2 Farey tails and Rademacher series

As was noted in section 1.2, the theory of modular functions and forms makes contact with number theory through the Fourier expansions of a modular function or form $f(\tau) = \sum_{n=-\Delta}^{\infty} a_n q^n$. It is these a_n that harvest interesting relations between natural numbers, so in the theory of modular forms knowledge of these coefficients a_n is fundamental.
Let's make this more concrete through an example. As discussed in section 1.2, there is an intimate relation between partitions of integers and modularity. Recall that the partition function

$$F(\tau) = \sum_{n=0}^{\infty} p(n)q^n, \tag{46}$$

where $p(n)$ is the number of partition of the integer n, is related to the modular (cusp) form $\Delta(\tau)$, and in fact: $qF(\tau)^{-24} = \Delta(\tau)$. What does the modularity of Δ teaches us about the coefficients $p(n)$?
First of all, as was already observed by Hardy and Ramanujan [GH18] using the modularity of $\Delta(\tau)$ we can estimate the $p(n)$ for large n as follows:

$$\log p(n) \simeq \pi\sqrt{\frac{2n}{3}} + \mathcal{O}(n^{-1}). \tag{47}$$

In fact, Hardy and Ramanujan concluded the following in [GH18]. Consider a weight-k modular form $f(\tau)$ on $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ with Fourier expansion $f(\tau) = \sum_{n=-c}^{\infty} a_n q^n$. Then the coefficient a_n is approximated by

$$a_n = \tfrac{1}{2}\sqrt{2}\left|\frac{c}{24}\right|^{1/4-k/2} (n-c/24)^{k/2-3/4}\, e^{2\pi\sqrt{|c/6|(n-c/6)}}\left(1+\mathcal{O}(n^{-1/2})\right), \quad (48)$$

where we assumed $a_{-c} = 1$, but this is not important. The derivation of equation (48) is straightforward. It is basically an approximation of the contour integral $a_n = \oint f(q) q^{-n-1} dq$ on its saddle-point after performing an S-transformation ($S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ for $\Gamma = \mathrm{SL}_2(\mathbf{Z})$). This formula, due to Hardy and Ramanujan, is actually Cardy's formula in a mathematical disguise (or rahter, vice versa). In the derivation of [Car86], Cardy assumes the modularity of a partition function $Z(\tau) = \sum_n a_n q^n$ of a two-dimensional CFT of a torus $\mathbf{C}/(\mathbf{Z} \oplus \tau\mathbf{Z})$. The c in (48) is then just the central charge of the CFT and n plays the role of L_0, the energy eigenvalue, and indeed:

$$S[CFT] = \log a_n \sim 2\pi\sqrt{\tfrac{c}{6}\,(L_0 - c/24)}. \quad (49)$$

Going back to the coefficients of modular forms, the derivation of the formula (48) only uses one element of the group Γ. But Γ is infinite dimensional and we expect to be able to get a better approximation. We can actually exploit the infinity of Γ to get an *exact* answer for the coefficients. The first example of an exact computation of coefficients of a modular object was by Rademacher, and is was on computing the coefficients of the j-functions $j(\tau) = \sum_{n=-1}^{\infty} c_n q^n$. The derivation [Rad38] uses a nice way to draw the contour using Ford circles based on Farey sequences. The result is

$$c_n = \frac{2\pi}{\sqrt{n}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_1\left(\frac{2\pi\sqrt{n}}{k}\right), \quad (50)$$

with $n > 0$. Here $I_k(z) = i^{-k} J_k(ik)$ is the imaginary Bessel function of the first kind, $J_k(z) = \frac{1}{\pi}\int_0^{\pi} \cos(k\tau - z\sin(\tau)))d\tau$ for $k \in \mathbf{Z}$. Slightly more general, we can perform such a "Rademacher series" for any weight-k modular form $f(\tau)$ on Γ [Rad43]. To this end, first split $f(\tau)$ into its polar part $f^-(\tau)$ and its non-polar part, $f = f^- + \hat{f}$, $f = \sum_{n=-c}^{\infty} q^n$, $f^- = \sum_{n<0} a_n q^n$. We can then write a Rademacher series for the coefficients a_n:

$$a_n = 2\pi \sum_{i<0} \left(\frac{n - c/24}{|i - c/24|}\right) a_i \sum_{r=1}^{\infty} \frac{1}{r} \mathrm{Kl}(n - c/24, i - c/24; r) \times$$
$$I_{1-k}\Big(\frac{4\pi}{r}\sqrt{|i - c/24|(n - c/24)}\Big) \quad (51)$$

where we used the Kloosterman sum $\mathrm{Kl}(n,i;r) = \sum_{d(r)} \exp 2\pi i \left(d\frac{n}{r} + d^{-1}\frac{i}{r}\right)$ where $d(r)$ means d mod r. Note that in equation (50), $A_k(n) = \mathrm{Kl}(n,-1,k)$. It is from the asymptotics of the Bessel function $I_\alpha(z) \sim \frac{1}{2\pi z}e^z$ as $\mathrm{Re}(z) \to \infty$ that the expression (48) is an approximation of equation (51).
Expression (51) looks rather cumbersome indeed. We can however simplify the expression as follows. First, map the function $f(q)$ to its Farey Transform $\mathcal{Z}_f = (q\partial_q)^{1-w} f(q)$. Note that $\mathcal{Z}_\bullet : M_k \to M_{2-k}$, that is Farey's transform maps weight-k modular forms to weight-$(2-k)$ modular forms. Now let $\mathcal{Z}_f^- = \mathcal{Z}_{f^-}$, the polar part of the transform. That is, let $\tilde{a}_n$ be the Fourier coefficients of $\mathcal{Z}_f = \sum_{-c}^{\infty} \tilde{a}_n q^n$, then $\mathcal{Z}_f^- = \sum_{n<0} \tilde{a}_n q^n$. Then the Rademacher series (51) can be recast as

$$\mathcal{Z}_f = \sum_{\Gamma\backslash\Gamma_\infty} (c\tau + d)^{w-2} \mathcal{Z}_f^-(q_\gamma) \tag{52}$$

where $\Gamma_\infty = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle \subset \Gamma$ and $q_\gamma = \gamma \cdot q = \exp 2\pi i \left(\frac{a\tau+b}{c\tau+d}\right)$. We see that the Rademacher series of the Farey transform is really just a Poincaré series. In the context of physics this series is also referred to as a Farey tail.

Lets illustrate these relations by applying them to the j-function $j(\tau) = \sum_{n=-1}^{\infty} c_n q^n$. The Rademacher series simplifies to (50) and the Farey tail becomes

$$j'(\tau) = \sum_{\Gamma\backslash\Gamma_\infty} \frac{\exp\left(-2\pi i \frac{a\tau+b}{c\tau+d}\right)}{(c\tau+d)^2} \tag{53}$$

Now recall from the constructions in the proof of the moonshine conjecture that the partition function $j(\tau)$ can be seen as the partition of a conformal field theory at central charge $c = 24$. Witten proposed in [?] to investigate the resemblances between this extremal CFT and pure gravity at Chern-Simons level $k = 24$. For the AdS/CFT conjecture to hold in this picture, we should have the equalities of partition functions $\mathcal{Z}_{\mathrm{AdS}} = \mathcal{Z}_{\mathrm{CFT}}$ where $\mathcal{Z}$ is the partition function. In [MS98] it was shown that in three-dimensional gravity, the solutions come in an $\mathrm{SL}_2(\mathbf{Z})$ family and we can write the partition function of AdS_3 as

$$\mathcal{Z}_{AdS_3} = \sum_{G \in \text{geometries}} e^{S[G]} = \sum_{\Gamma\backslash\Gamma_\infty} M(\tau;\gamma) q_\gamma \tag{54}$$

where $M(\tau;\gamma) = \frac{1}{(c\tau+d)^2}$, the measure factor for the gravity partition sum. So the Farey tail expansions can through AdS/CFT duality be read as a sum over saddle points - geometries - of gravity in AdS_3. In the original proposal [?], the Farey tail was executed in a more rigorous setting, where AdS/CFT stands firm. The modular object in question there is not the j-function, but the elliptic genus $\chi(\tau, z; k)$ of the Hilbert scheme of k points on $K3$. For large k we believe the AdS/CFT duality to hold in this context, so that $\chi(\tau, z; k)$ can be seen as a gravitational partition function, on $AdS_3 \times S^3 \times K3$. The elliptic genus $Ell[K3](\tau, z)$ on $K3$ itself we gave in equation (41) and if we decompose it in its Fourier coefficients $Ell[K3](\tau, z) = \sum_{\substack{n \geq 0 \\ r^2 \leq 4n}} c(n, r) q^n y^r$ we can find the elliptic genus $\chi(\tau, z, k)$ on $Hilb^k(K3)$, the Hilbert scheme of k point on $K3$ using the generating function from DMVV, [DMVV97]:

$$\sum_{k=1}^{\infty} \chi(q, y; k) p^k = \prod_{\substack{n>0, m \geq 0 \\ \ell \in \mathbf{Z}}} (1 - p^n q^m y^\ell)^{-c(nm, \ell)} \tag{55}$$

We can subsequently use a Farey transform that generalizes to Jacobi forms. In this case, take an index-k weight-w Jacobi form $\phi(\tau, z)$. Then its Farey Transform $\mathcal{F}_\phi$ is defined as $\mathcal{F}_\phi = \left| q\partial_q - \frac{1}{4k}(y\partial_y)^2 \right|^{3/2-w} \cdot \phi(\tau, z)$. We can then perform the Rademacher series and identify all terms with quantities in a Chern-Simons theory on $SU(2) \times S^3 \times K3$.
It is a beautiful result that the Rademacher series has an application in physics. Here we will use the Rademacher series in finding the equivalent of the genus zero property (that was so important in formulating the monstrous moonshine conjecture) in the formulation of moonshine for the mock modular forms $H_g^{(2)}(\tau)$.

The trick is to first find a set of conditions that is equivalent to the genus-zero property of the Hauptmoduls for some Γ in monstrous moonshine, without explicitly having to compute the function-fields associated to that Γ. In [DF11] it was shown that such a condition exist indeed in terms of a Rademacher sums as follows, and we will first illustrate the idea for monstrous moonshine.

Consider for $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ the Γ-invariant function $T_g : \mathcal{H} \to \mathbf{C}$, T_g holomorphic with a simple pole in q at the cusp $\tau = i\infty$. Also, for the same Γ, consider the following *Rademacher sum*:

$$R_\Gamma(\tau) = \mathrm{Reg}\left(\sum_{\gamma \in \Gamma \backslash \Gamma_\infty} q_\gamma^{-1} \right) \tag{56}$$

The convergence properties of the sum in (56) are subtle and we need some regularization. For example, for the case $\Gamma = \Gamma(1) = \mathrm{SL}_2(\mathbf{Z})$, we know the sum should be $R_{\Gamma(1)}(\tau) \in k(X(1))$, that is the Rademacher sum should be proportional to the j-function up to a shift. In his work [Rad39], Rademacher already noticed that

$$j(\tau) = -12 + \lim_{K \to \infty} \sum_{\substack{|c|,|d| \geq K \\ (c,d)=1}} \mathrm{e}\left(-\frac{a\tau + b}{c\tau + d} \right) - \mathrm{e}\left(-\frac{a}{c} \right) \tag{57}$$

where in the sum the subtraction by $\mathrm{e}\left(-\frac{a}{c}\right)$ is the unique factor that renders the sum convergent without spoiling the $\Gamma(1)$-invariance. This is the regularization procedure for all congruence subgroups in $\mathrm{SL}_2(\mathbf{Z})$. We also introduced the notation $\mathrm{e}(x) = \exp(2\pi i x)$ for brevity.
Going back to the monstrous moonshine, the following was shown in [DF11]:
Let $T : \mathcal{H} \to \mathbf{C}$ be a holomorphic function, invariant under $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$. Furthermore let $T(\tau)$ have a simple pole in $q = \mathrm{e}(\tau)$ at the cusp at infinity. Then $k(X(\Gamma)) = \langle T \rangle$, that is T generates the function of fields on the modular curve $X(\Gamma)$, iff $T = R_\Gamma$. Following up, the interesting conclusion in [DF11] is that:

Conjecture. *[Conway-Norton-Duncan-Frenkel] For each $g \in \mathbb{M}$, the McKay-Thompson series $T_g(\tau)$ satisfies $T_g(\tau) = R_{\Gamma_g}(\tau)$, with Γ_g the invariance group of $T_g(\tau)$.*

With this conjecture in hand, we can try to formulate a conjecture for the Mathieu group, as we no longer should need the genus-zero property. That does need however the expansion of the Rademacher apparatus to the more general mock modular forms. Recall that the modular objects in the Mathieu moonshine conjecture, $H_g(\tau)$, are mock modular forms of weight 1/2. In [DF11] the techniques of Rademacher summation were suitably generalized for the purpose of Mathieu moonshine. In a slim notation the Rademacher sum for some $\Gamma_g \subset \mathrm{SL}_2(\mathbf{Z})$ is

$$R_{n|h}(\tau) = \mathrm{Reg}\left(\sum_{\gamma \in \Gamma_0(n)\backslash\Gamma_\infty} -2q_{\gamma[n,h]}^{-1/8}\right). \tag{58}$$

The input of the Rademacher sum is now the integers (n, h). They encode information about the cycle-shape of $g \in M_{24}$, written as a permutation on 24 letters. The integer n is the order of g and h the length of the shortest cycle in this representation. As always, further input is the polar order, $-1/8$ in this case, and the congruence subgroup $\Gamma_0(n)$. The action of $\gamma[n, h]$ on q now also depends on the cycle information (n, h). We will write the rather cryptic notation in expression (59) out more explicitly, along the lines of (57).

$$R_{n|h} = \lim_{K\to\infty} \sum_{\gamma \in B_K} \mathrm{e}\left(-\frac{cd}{nh}\right) \mathrm{e}\left(-\frac{\gamma\tau}{8}\right) \frac{\varepsilon^{-3}(\gamma)}{(c\tau + d)^{1/2}} \tag{59}$$

where $\varepsilon(\gamma)$ is the multiplier system of $\eta(\tau)$ and B_K is the "rectangle" $B_K = \{\gamma \bmod \Gamma_\infty | 0 < c < K, -K^2 < d < K^2\}$.
We are now ready to state the Mathieu Moonshine conjecture [CD12]

Conjecture. *[Cheng-Duncan] For each element $g \in M_{24}$ the McKay-Thompson series $H_g^{(2)}$ satisfies $H_g^{(2)} = R_{n|g}$.*

Note that besides having the conceptual advantage of circumventing the genus-zero property of monstrous moonshine, conjecture (2.2) has the computational advantage that with the input of just the polar term $q^{-1/8}$, the cycle-shape of g and the congruence subgroup $\Gamma_0(n)$, we can simply compute $R_{n|h}$ on a computer and compare results with the McKay-Thompson series. The conjecture (2.2) has been verified for a large number of coefficients. What remains to be constructed is the actual module $K^{(2)}$ as in equation (44). At press time, this is still an open problem.

2.3 The umbral groups

The elliptic genus (41) on K3, $Ell[K3](\tau, z)$, is a (weak) Jacobi form of weight 0 and index 1. Upon decomposition into characters of the $\mathcal{N} = 4$ superconformal algebra, the mock modular object $H^{(2)}$ naturally arises, and this object is subject to the Mathieu moonshine conjecture (2.2). Inspired by this conjecture, we may wonder if we can generalize this conjecture to weak Jacobi forms of higher index m. Lets restrict to $J^{ext}_{0,m}$, the space of weak Jacobi forms of weight 0 and index m that have an expansion in terms of characters of the $\mathcal{N} = 4$ superconformal algebra. It was shown in [CDH14] that dim $J^{ext}_{0,m-1} = 1$ if $m - 1|12$ and zero otherwise. So in this generalizations we can hope to find an analogue of the conjecture (2.2) only for the integers $m = 2$ (which corresponds to Mathieu moonshine) and $m = 3, 4, 5, 7, 13$. In analogue with the decomposition of $Ell[K3](\tau, z)$ we may take an extremal Jacobi form $Z^{(\ell)}(\tau) = 2\phi_1^{(\ell)}$ with ϕ_1^{ℓ} a weak extremal Jacobi form of index $\ell - 1$, $\ell \in \Lambda = \{2, 3, 4, 5, 7, 13\}$. We first split of $\psi^{(\ell)} = \Psi_{1,1} Z^{(\ell)}$, where

$$\Psi_{1,1} = -1 \frac{\theta_1(\tau, 2z)\eta(\tau)^3}{\theta_1(\tau, z)^2}. \tag{60}$$

Then we take the "finite part" $\psi^{(\ell),F}$ (where $\psi^{(\ell)} = \psi^{(\ell),F} + \psi^{(\ell),P}$ and $\psi^{(\ell),P}$ the polar part. Now, as is typical for Jacobi forms, we may decompose $\psi^{(\ell),F}$ in a theta-expansion

$$\psi^{(\ell),F}(\tau, z) = \sum_{r=1}^{\ell-1} H_r^{(\ell)}(\tau)\hat{\theta}_r^{(\ell)}(\tau, z) \tag{61}$$

where the index-m theta functions are $\hat{\theta}_r^{(m)} = \theta_{-r}^{(m)} - \theta_r^{(m)}$, with (see appendix (A) for more on theta expansions and the notation used in [EZ85])

$$\theta_r^{(m)}(\tau, z) = \sum_{n \in \mathbf{Z}} q^{(2mn+r)^2/4m} y^{2mn+r}. \tag{62}$$

Note that here $H_1^{(2)}$ is the mock modular form that we already encountered in conjecture (2.2), $H_1^{(2)} = H_e^{(2)}$. This explains the superscript notation. The other objects, $H_r^{(\ell)}$, are *vector valued mock modular forms*. This means that the completion (recall the definition (3) of mock modular forms) $\hat{H}_r^{(\ell)}$ of $H_r^{(\ell)}$ transforms as $\hat{H}_r^{(\ell)}(\tau) = \nu_{rs}(\gamma)\hat{H}(\gamma\tau)_s^{(\ell)}(c\tau+d)^{-k}$ for weight k (here $k = 1/2$), $\nu(\gamma)$ a matrix-valued function on Γ for $\gamma \in \Gamma$. They all have an expansions $H_r^{(\ell)}(\tau) = -2\delta_{r,1}q^{-1/4\ell} + \mathcal{O}(q^{1/4\ell})$ and it actually follows from spectral flow of the $\mathcal{N} = 4$ superconformal algebra that the mock modular forms have an expansion

$$H_r^{(\ell)} = \sum_n c_r(n - r^2/4\ell)q^{n-r^2/4\ell}. \tag{63}$$

Now that we have found the proper generalization for the object $H^{(2)}(\tau)$ of Mathieu moonshine, we can look at its Fourier coefficients to see if we can recognize the dimensions of irreducible representations of other (simple) groups. We take some examples from [CDH14]:

$$H_1^{(2)} = 2q^{-1/8}\Big(-1 + 45q + 231q^2 + 770q^3 + \mathcal{O}(q^4)\Big) \tag{64}$$

$$H_1^{(3)} = 2q^{-1/12}\Big(-1 + 16q + 55q^2 + 144q^3 + \mathcal{O}(q^4)\Big) \tag{65}$$

$$H_2^{(3)} = 2q^{2/3}\Big(10 + 44q + 110q^2 + 280q^3 + \mathcal{O}(q^4)\Big) \tag{66}$$

$$H_1^{(4)} = 2q^{-1/16}\Big(-1 + 7q + 21q^2 + 43q^3 + \mathcal{O}(q^4)\Big) \tag{67}$$

$$H_2^{(4)} = 2q^{3/4}\Big(8 + 24q + 56q^2 + 112q^3 + \mathcal{O}(q^4)\Big) \tag{68}$$

$$H_3^{(4)} = 2q^{7/16}\Big(3 + 14q + 28q^2 + 69q^3 + \mathcal{O}(q^4)\Big) \tag{69}$$

In $H_1^{(2)}$ we of course recognize the dimensions of the irreducible representations of M_{24} and, after pulling out our ATLAS of Finite Groups [CCN+85] we recognize in $H_{1,2}^{(3)}$ the characters of the group $2.M_{12}$, where M_{12} denotes another Mathieu group and $n.G$ means the group with normal subgroup $\mathbf{Z}_n$ such that $(n.G)/\mathbf{Z}_n = G$. In the context of finite groups we will occasionally write $n = \mathbf{Z}_n$. At $\ell = 4$ we recognize in the series (67) and (68) the dimensions of the irreducible representations of the group $2.AGL_3(2)$, the affine linear group in three dimensions over the field $\mathbf{F}_2$. In fact this pattern permeates for all ℓ, that is, for each ℓ we can find a group $G^{(\ell)}$ such that the dimensions of its irreducible representations are recognized in the functions $H_r^{(\ell)}$. As the functions that are attached do these groups are (vector valued) mock modular forms, and as such are characterized by their shadow, the groups that correspond to these functions are called *umbral* groups (the word *umbra* stand for "a shaded area ..." in the Merriam-Webster). The index ℓ is referred to as the *lambency* (where *lambent* is Latin for licking, as the flame of a candle that licks like a tongue in the darkness, or "softly bright or radiant" in the same dictionary — expressing the moonshine of it all). The umbral groups in question are listed in table (III.4).

Table III.4: The umbral groups $G^{(\ell)}$, $(\ell-1)|12$

l	2	3	4	5	6	7	13
G^ℓ	M_{24}	$2.\mathrm{M}_{12}$	$2.\mathrm{AGL}_3(2)$	$\mathrm{GL}_2(5)/2$	$\mathrm{GL}_2(3)$	$\mathrm{SL}_2(3)$	4

Knowing the drill by now, we are led to conjecture the existence of modules

$$K^{(\ell)} = \bigoplus_{0<r<l} K_r^{(\ell)} \tag{70}$$

$$K_r^{(\ell)} = \bigoplus_{k\in\mathbf{Z}} K_{r,k-r^2/4\ell}^{(\ell)} \tag{71}$$

where the grading in the module (71) is in accordance with the expansion of the functions (63). Note that the modules $K^{(\ell)}$ are bi-graded over $\mathbf{Z}\times\mathbf{Q}$. These modules first of all should have the property that

$$H_r^{(\ell)} = \sum_{\substack{k\in\mathbf{Z} \\ r^2-4k\ell<0}} \dim\left(K_{r,k-r^2/4\ell}^{(\ell)}\right) q^{k-r^2/4\ell}. \tag{72}$$

And here we again want to associate functions $H_{g,r}^{(\ell)}$ to all group elements $g \in G^{(\ell)}$ so that these functions should equal graded characters over these bi-graded modules $K^{(\ell)}$:

$$H_{g,r}^{(\ell)} = \sum_{\substack{k\in\mathbf{Z} \\ r^2-4k\ell<0}} \mathrm{ch}_{K_{r,k-r^2/4\ell}^{(\ell)}}(g) q^{k-r^2/4\ell}. \tag{73}$$

Just like in the case of Mathieu moonshine, there is now the nuisance of reformulating a moonshine conjecture like in the case of the monster group, as there is no direct analogue of the genus-zero property. We saw in section 2.2 how to formulate the Mathieu moonshine conjecture using the Rademacher sum. Here we will have to do the same. We only need to extend the Rademacher sum techniques discussed in section 2.2 to vector valued modular forms. The idea is very much like in formula (59), just a little more cumbersome, and we will not spell out all details here but rather refer to [CD12] for technical details. Importantly, the conjecture of umbral moonshine can be extended for all umbral groups:

Conjecture. *[Cheng-Duncan] For the umbral groups $G^{(\ell)}$, with $\ell \in \{2,3,4,5,7,13\}$, we have $H_g^{(\ell)} = R_{n|g}^{(\ell)}$*

where $R_{n|g}^{(\ell)}$ is a generalization of the Rademacher sum to $(\ell-1)$-vector-valued mock modular objects of weight-1/2.

After the first umbral moonshine paper [CDH14], it was noted that the umbral groups $G^{(\ell)}$ in table (III.4) are all automorphism groups of Niemeir lattices. There are 23 such lattices X all with their own group of automorphisms G^X and in [CDH13] the umbral moonshine conjecture was expanded to a conjecture on umbral moonshine for all these groups G^X, each with their own vector valued mock modular form that acts as the McKay-Thompson series of a graded module G^X.
In [DGO15], it was proven that the umbral moonshine modules $K^{(l)}$ exist, although no explicit construction was given. Earlier in [Gan12] already established this for the group M_{24}.

3 Generalized umbral moonshine

We have now formulated Umbral Moonshine, and one might expect a form of generalized moonshine for the umbral groups, that is the existence of mock modular forms $H^{(l)}_{g,h}$ for the umbral group $G^{(\ell)}$ so that the coefficients of $H^{(\ell)}_{g,h}$ contain information about the centralizer subgroups $C_{G^\ell}(g) \subset G^{(\ell)}$.

In [GPRV13, GPV13] this idea was applied to the biggest of the umbral groups, the Mathieu group M_{24}. As stated in the introduction and as we will see below, a crucial ingredient in computing the properties of the twisted and twined partition function $Z^{(\ell)}_{g,h}(\tau)$ under modular transformations is the third cohomology group $H^3(G^{(\ell)}, U(1))$. For the monster group we unfortunately do knot yet know what this group is explicitly, but for the Mathieu group M_{24} we do know the result [DSE09]:

$$H^3\left(\mathrm{M}_{24}, U(1)\right) \cong \mathbf{Z}_{12} \tag{74}$$

This will allow a computation of the modular properties of the twisted twined elliptic genera. Taking up the original observation on the K3 elliptic genus as described in section 2.1 and the concepts of generalized umbral moonshine in section 1.4 the the following construction follows quite naturally:
Consider a K3 sigma model $\mathcal{H}$, a supersymmetric conformal field theory, with some automorphism group $G = End(\mathcal{H})$. Let $g, h \in G$ and $[g,h] = 1$. Let $\mathcal{H}_g$ denote the g-twisted subsector $\mathcal{H}_g \subset \mathcal{H}$. Here we implicitely presume the existence of a vertoex operator algebra that has M_{24} as global group of endomorphisms, although an exlplicit construction is lacking at this point.

The element h induces a linear action on $\mathcal{H}_g$ and we may consider the g-twisted h-twined genus $\phi_{g,h}(\tau, z)$:

$$\phi_{g,h}(\tau, z) = \mathrm{Tr}_{\mathcal{H}_g}\left(h(-1)^{F_L+F_R} q^{L_0} \bar{q}^{\bar{L}_0} y^{4J_0^3}\right) \tag{75}$$

Note that $\phi_{e,e}$ is just the elliptic genus, $\phi_{e,e} = Ell[K3](\tau, z)$ and $\phi_{e,h}$ are the h-twined genera as in [EH11, Che10, GHV10]. Combining the properties of te Mathieu genus and its expansion in terms of characters of the $\mathcal{N}=4$ SCFT algebra, and the properties of the generalized moonshine functions, the g-twisted h-twined genera are in [GPRV13] conjectured to enjoy the following properties for $\phi_{g,h}(\tau,z)$:

1. *Ellipticity and Modularity (weak Jacobi)*:
   $\phi_{g,h}(\tau, z+\kappa\tau+\kappa') = \mathrm{e}(-\kappa^2+2\kappa z)\phi_{g,h}(\tau,z)$
   $\phi_{g,h}(\gamma\tau,\gamma z) = \chi_{g,h}(\gamma)\mathrm{e}\left(\frac{cz^2}{c\tau+d}\right)\phi_{g^a h^c, g^b h^d}(\tau,z)$
2. *Conjugation*:
   $\phi_{g,h}(\tau,z) = \xi_{g,h}(k)\phi_{g^k,h^k}(\tau,z)$
3. *Expansion in Projective Characters*:
   $\phi_{g,h}(\tau,z) = \sum_{0\le r\in\lambda_g+\frac{1}{N}\mathbf{Z}} \mathrm{Tr}_{\mathcal{H}_{g,r}}(\rho_{g,r}(h))\,\mathrm{ch}_{\frac{1}{4}+r,\ell}(\tau,z)$
4. *Consitency*:
   $\phi_{e,e} = Ell[K3](\tau,z)$ and $\phi_{e,g}$ as in [Che10, GHV10]
5. *Group Cohomology*:
   $\chi_{g,h}\in H^2(C_{\mathrm{M}_{24}}(g),U(1)), \xi_{g,h}\in H^2(C_{\mathrm{M}_{24}}(g),U(1))$.
   Moreover there is an $\omega\in H^3(\mathrm{M}_{24},U(1))$ and a surjective map $\psi_g : H^3(\mathrm{M}_{24},U(1)) \to H^2(C_{\mathrm{M}_{24}}(g),U(1))$ so that ω and ψ_g completely determine the two-cycles $\chi_{g,h}$ and $\xi_{g,h}$.

The first condition states that all g-twisted h-twined genera are weak Jacobi forms of weight 0 and index 1 and with a multiplier system $\chi_{g,h}$ under a subgroup $\Gamma_{g,h}\subset \mathrm{SL}_2(\mathbf{Z})$. The second states that the genera are not quite class functions but rather 'projective' class functions, that is class functions up to a phase. In the third we see the generalized moonshine property. In this notation, N denotes the order of the twist, $\mathrm{ord}(g)=N$. The factor $\lambda_g\in\mathbf{Q}$ is the overall fractional power in the q-expansion. For example, $\lambda_e = 1/8$. We also denote with $\mathrm{ch}_{\frac{1}{4}+r,\ell}$ the characters of the $\mathcal{N}=4$ algebra (actually the Ramond sector therof). The $\mathcal{H}_{g,r}$ now iss the graded g-twisted module, $\mathcal{H}_g = \bigoplus_r \mathcal{H}_{g,r}$ and each component carries a projective representation $\rho_{g,r}$ of the centralizer subgroup $C_{\mathrm{M}_{24}}(g)$. We refer to the appendix for more on projective representations and the relation with group cohomology. The fourth statement is simply a statement of consistency that ensures normalizations or on par with those of earlier results. Finally, the fifth condition states that we can actually compute the phases and multipliers from group theory.

3.1 Classification of the M_{24} genera

For each pair of commuting elements $(g,h)\in M_{24}\times M_{24}$ we in principle have an associated g-twisted h-twined genus $\phi_{g,h}(\tau,z)$ obeying the conditions 1-5 in the list above. But actually from the condition we see that many such paris are related by an $\mathrm{SL}_2(\mathbf{Z})$ transformation or an action from the group by conjugation. In fact, we can extend the action of $\mathrm{SL}_2(\mathbf{Z})$ to an action of the full $\mathrm{GL}_2(\mathbf{Z})$ by noting that

$$\phi^*_{g,h}(\tau,z) = \chi_{g,h}\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)\phi_{g,h^{-1}}(\tau,z) \tag{76}$$

where $\phi^*(\tau,z) = \overline{\phi(-\bar{\tau},-\bar{z})}$ and $\bar{z}$ denotes complex conjugation. So we may append the matrix $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ to $\mathrm{SL}_2(\mathbf{Z})$ giving the full general linear group $\mathrm{GL}_2(\mathbf{Z})$.

So the first thing we will want to do is classify the independent genera, that is, take the set $\mathcal{P} = \{(g,h) \in M_{24} \times M_{24} \,|\, [g,h] = 1\}/\,(GL_2(\mathbf{Z}) \times M_{24})$. The action of $GL_2(\mathbf{Z})$ on a pair (g,h) just generates the rank two Abelian subgroup $\langle g,h\rangle \subset M_{24}$. Together with the action of M_{24} onto itself by conjugation this means that the independent g-twisted h-twined genera $\phi_{g,h}$ are in one-to-one correspondence with M_{24}-conjugacy classes of rank-two Abelian subgroups $\langle g,h\rangle \subset M_{24}$. The Abelian subgroups of rank-one correspond to the functions $\phi_{e,g}$. These functions are already computed in [GPRV13, GPV13]. There are indeed 21 such rank-one Abelian subgroups in M_{24}. There are however a total of 34 rank-two Abelian subgroups $\langle g,h\rangle$ hence 34 possible independent g-twisted h-twined genera $\phi_{g,h}$.

But upon closer inspection at the conditions 1-5 in the twisted twined genera, we can conclude that actually some may be obstructed on group theoretic grounds. We can actually find two different such obstructions:

1. Consider g,h,k pairwise commuting elements of M_{24}. Conjugation invariance up to phases inplies that $\phi_{g,h} = \xi_{g,h}(k)\phi_{g,h}$ so that is we may find such a triplet (g,h,k) with $\xi_{g,h}(k) \neq 1$ we may conclude that $\phi_{g,h} = 0$.

2. Consider a pair (g,h) so that $(g^k,h^k) = (g^{-1},h^{-1})$ for some $k \in M_{24}$. Conjugation and modularity imply $\phi_{g,h} = \chi_{g,h}\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)\xi_{g^{-1},h^{-1}}(k)\phi_{g,h}$. So for such a pair (g,h) that has $\chi_{g,h}\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)\xi_{g^{-1},h^{-1}}(k) \neq 1$ we may conclude that $\phi_{g,h} = 0$.

The quest is now to find cases in the 34 rank-two Abelian subgroups that are not obstructed. For analyses like this, and the classification of the 34 mentioned rank-two subgroups and their conjugation calsses, we use GAP [GAP]. So first we have to understand how to compute the phases and multipliers from the group theory. To this end we go back to the interpretation of the twisted and twined genera in terms of partition functions of an orbifolded conformal field theory. In [DVVV89] it was shown that the partition sum $Z(\tau;M/G)$ of a conformal field theory on the orbifold $M/G, G \subset End(M)$ can be expanded as

$$Z(\tau;M/G) = \sum_{[A]} \frac{1}{|C_G([A])|} \sum_{g\in C_G(g_{[A]})} Z_{g_{[A]},h} \tag{77}$$

where still

$$Z_{g,h}(\tau) = \mathrm{Tr}_{\mathcal{H}_g}\left(\rho_g(h)q^{L_0-1}\right) = g\underset{h}{\square}(\tau) \tag{78}$$

As can be seen in the list 1-5, the twisted sectors $\mathcal{H}_g$ form projective representations of $C_G(g)$, where in this context $G = M_{24}$. We refer to appendix C for more on projective representations, but a projective representation ρ of G has

$$\rho(g_1)\rho(g_2) = c(g_1, g_2)\rho(g_1 g_2) \tag{79}$$

where $c(\cdot,\cdot) : G \times G \to U(1)$ is a 2-cocycle, $c \in H^2(G, U(1))$. Hence, again see appendix for details, we can conclude that

$$Z_{g,h} = \frac{c_g(h,k)}{c_g(k,h^k)} Z_{g^k,h^k} \tag{80}$$

where $c_g \in H^2(C_G(g), U(1))$. In [Ban90] the modular properties of twisted twined paritition functions are worked out:

$$Z_{g,h}(\tau + 1) = c_g(g,h) Z_{g,gh}(\tau) \tag{81}$$

$$Z_{g,h}(-1/\tau) = c_h(g, g^{-1}) Z_{h,g^{-1}}(\tau) \tag{82}$$

Together we can now express the multipliers and the phases for the genera in terms of the two-cycles c_g:

$$\chi_{g,h}\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \frac{1}{c_h(g, g^{-1})} \tag{83}$$

$$\chi_{g,h}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = c_g(g,h) \tag{84}$$

$$\chi_{g,h}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = \frac{1}{c_g(h, h^{-1})} \tag{85}$$

$$\xi_{h,g}(k) = \frac{c_g(h,k)}{c_g(k,h^k)} \tag{86}$$

Finally, there is a surjective map $\psi_g : H^3(G, U(1)) \to H^2(C_G(g), U(1))$ that takes a 3-cocycle α and sends it to a 2-coycle c_g. This map is inspired by work on Chern-Simons theory on finite groups, and was first introduced in [DW90]. The map reads, explicitly:

$$\psi_g(\alpha) = c_g(h_1, h_2) = \frac{\alpha(g, h_1, h_2)\alpha(h_1, h_2, g^{h_1 h_2})}{\alpha(h_1, g^{h_1}, h_2)} \tag{87}$$

The trick is then to find the unique 3-cocycle α that reproduces the phases and multipliers for the known and trivial genera $\phi_{e,h}$ so we can subsequently compute, from this α, all the phases and multipliers for $\phi_{g,h}$.
Here we first of all need to be able to do explicit computations on $H^3(G, U(1))$. Fortunately, the group structure of $H^3(M_{24}, U(1))$ is known as
$H^3(M_{24}, U(1)) \cong \mathbf{Z}_{12}$, but we will also need to be able to do explicit computations on the cocycles, that is find actual representations of the cycles. For the group M_{24}, that has $|M_{24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 11 \cdot 23 = 244823040$ this is hard if not

impossible to do by hand. For all the computations in the cocyles, we therefore used GAP [GAP], the system for computational discrete algebra. In GAP we used the package HAP [HAP], a package for homological algebra programming that provides functions for group (co)homology. We refer to the arXiv files of [GPRV13, GPV13] for the transcripts of the GAP files with a complementary text file on how to open and read them.

First, after listing all the 34 rank-two Abelian subgroups of $G^{(2)} = M_{24}$, we have to scan these groups and check if they give rise to an obstruction. After tossing out all the obstructed subgroups, we are left with a smaller set

$$\mathcal{A}^{(\ell)} = \{\langle g,h\rangle \in \mathcal{P} \mid (g,h) \text{ unobstructed}\}. \tag{88}$$

For the group M_{24} we list the unobstructed rank-two Abelian subgroups in table (III.5). We listed the group-structure of the rank-two Abelian subgroup as well as the congruence subgroup under which the corresponding function has the conjectured modular behavior. This congruence subgroup is $\pi(\tilde{\Gamma}_{g,h})$, where

$$\tilde{\Gamma}_{g,h} = \left\{\left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right), k\right) \in \mathrm{SL}_2(\mathbf{Z}) \times \mathrm{M}_{24} \mid (g^a h^c)^k, (g^b h^d)^k = (g,h) \text{ or } (g^{-1}, h^{-1})\right\} \tag{89}$$

and $\pi : \mathrm{SL}_2(\mathbf{Z}) \times \mathrm{M}_{24} \to \mathrm{SL}_2(\mathbf{Z})$ denotes the projection on the special linear matrix. Finally in the table we listed the names of the conjugacy classes of the group elements according to the conventions in GAP which are as close to the ATLAS conventions as possible.

Table III.5: The unobstructed rank-two Abelian subgroups of M_{24}

#	Structure	$\Gamma_{g,h}$	Element names
1	$\mathbf{Z}_2 \times \mathbf{Z}_4$	$\Gamma_0(2)$	$(2A)(2B)^2(4A)^4$
2	$\mathbf{Z}_4 \times \mathbf{Z}_4$	$\Gamma_0(2)$	$(2A)^3(2A)^8(4B)^4$
3	$\mathbf{Z}_4 \times \mathbf{Z}_4$	$\Gamma_0(2)$	$(2A)^3(2A)^8(4B)^4$
4	$\mathbf{Z}_2 \times \mathbf{Z}_8$	$\Gamma_0(4)$	$(2A)(2B)^2(4B)^4(8A)^8$
5	$\mathbf{Z}_3 \times \mathbf{Z}_3$	$\Gamma(1)$	$(3A)^8$
6	$\mathbf{Z}_3 \times \mathbf{Z}_3$	$\Gamma_0(3)$	$(3A)^2(3B)^6$

Knowledge of the congruence subgroups and the multiplier system and phases proves to be sufficient to construct the g-twisted h-twined genera for the M_{24} explicitly.

To see this, first of all we note that the space of weak Jacobi forms $\mathcal{J}_{k,m}$ forms a ring, generated over the modular forms (see appendix (A) for details):

$$\mathcal{J}_{k,m} = \langle \phi_{0,1}, \phi_{-2,1}\rangle_{M^*} \tag{90}$$

that is to say, let $\phi_{k,m}$ be a weight-k index-m weak Jacobi form. Then

$$\phi_{k,m} = \sum_{i=0}^{m} \omega_i(\tau)\phi_{0,1}(\tau,z)^{m-i}\phi_{-2,1}(\tau,z)^i \tag{91}$$

with $\omega_i(\tau) \in M_{k+2i}(\Gamma_{g,h},\chi)$ a modular form on $\Gamma_{g,h}$ with multiplier system χ. We refer to the appendix (A) for details and definitions on $\phi_{0,1}$ and $\phi_{-2,1}$.

In the case of interest — index 1, weight 0 weak Jacobi forms with appropriate multiplier systems — this extends to the observation that

$$\phi(\tau,z) = c_0\phi_{0,1}(\tau,z) + c_1\omega_2(\tau)\phi_{-2,1}(\tau,z) \tag{92}$$

where $c_0 \in \mathbf{C}$ and $\omega_2(\tau)$ a weight-2 modular function on $\Gamma_{g,h}$ with multiplier χ. For weak Jacobi forms with non-trivial multiplier system this readily yields that $c_0 = 0$. For the cases at hand[4] we can write $\omega_2(\tau)$ as an eta-product:

$$\omega_2(\tau) = \frac{\prod \eta^{d_i}(n_i\tau)}{\prod \eta^{d_j}(n_j\tau)} \tag{93}$$

We know the multiplier systems of the eta functions $\eta(n\tau)$ so we just need to find an eta-product that matches the multiplier χ.
As an example, consider the example of group #4 in the list (III.5). The rank-two Abelian subgroup is generated by an element $g \in$ 2B and $h \in$ 8A. The congruence subgroup is $\Gamma_0(4)$ and from the cohomology we can, using GAP and HAP read off the multiplier system. To be completely specific, we take a representation where M_{24} is considered as a subgroup of S_{24}, generated by the three elements $M_{24} = \langle\sigma_1,\sigma_2,\sigma_3\rangle$ with (and we omit cycles of length one):

$$\begin{aligned}
\sigma_1 &= (1,2,\ldots,22,23)\\
\sigma_2 &= (3,17,10,7,9)(4,13,14,19,5)(8,18,11,12,23)(15,20,22,21,16)\\
\sigma_3 &= (1,24)(2,23)(3,12)(4,16)(5,18)(6,10)\\
&\quad(7,20)(8,14)(9,21)(11,17)(13,22)(15,19)
\end{aligned}$$

In this presentation we take for $g \in$ 2B and $h \in$ 8A:

$$\begin{aligned}
g &= (1,10)(2,14)(3,8)(4,5)(6,22)(7,20)(9,18)\\
&\quad(11,23)(12,24)(13,19)(15,16)(17,21)\\
h &= (2,14)(3,9,8,18)(4,6,21,19,15,24,20,11)\\
&\quad(5,22,17,13,16,12,7,23)
\end{aligned}$$

With these cycle presentations we can find elements r and k such that $((gh^4)^k,(h^{-1})^k) = (g,h)$ and $(g^r,(gh)^r) = (g,h)$ with subsequent multiplier

[4] that is, the cases where $M_2(\Gamma_{g,h},\chi)$ does not contain newforms

relations (83)-(86).

$$\phi_{2B,8A}(\tau+1) = c_g(g,h)\phi_{g,gh} = c_g(g,h)\frac{c_g(gh,r)}{c_g(r,h)}\phi_{g,h} = i\phi_{g,h} \tag{94}$$

$$\phi_{2B,8A}(\tfrac{\tau}{-4\tau+1}) = \frac{\prod_{i=1}^{3} c_h(h,gh^i)}{c_h(g,g)c_{gh^4}(h,h^{-1})}\frac{c_{g^4h}(h^{-1},k)}{c_{g^4h}(k,h)}\phi_{g,h}(\tau) = -\phi_{g,h}(\tau) \tag{95}$$

Using the slightly abbreviated notation $[a,b;c,d] = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we summarize this as $T = [1,1;0,1], P = [1,0;-4,1], \chi(T) = i, \chi(P) = -1$ for $T, P \in \Gamma_0(4)$.
The only eta-product that satisfies these properties is $\omega_2(\tau) = \eta(\tau)^2\eta(2\tau)^2$. Hence:

$$\phi_{2B,8A} = c_1\eta(\tau)^2\eta(2\tau)^2\phi_{-2,1}. \tag{96}$$

From the appendix A we read off that $\phi_{-2,1} = -\frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^6}$. We fix the constant c_1 by matching the coefficients with those of the projective character table of $C_{M_{24}}(2B)$ (this is an overall normalization so this does not at all render the result trivial) and we find that with this requirement $c_1 = -4$. Hence we indeed make contact with the function that is found in [GPRV13], where they namey find

$$\phi_{2B,8A} = 2\frac{\eta(2\tau)^2}{\eta(\tau)^4}\vartheta_1(\tau,z)^2 \tag{97}$$

The other five functions can be found in the same way, using the eta-products, or using the techniques of [GPRV13]. We list the results in table (III.6):

Table III.6: The unobstructed g-twisted h-twined genera $\phi_{g,h}$ for M_{24}.

#	$\langle g,h\rangle$	$\phi_{g,h}$
1	$\langle 2B, 4A_2\rangle$	$4\frac{\eta(2\tau)^2}{\eta(\tau)^4}\vartheta_1(\tau,z)^2$
2	$\langle 4B, 4A_3\rangle$	$2\sqrt{2}\frac{\eta(2\tau)^2}{\eta(\tau)^4}\vartheta_1(\tau,z)^2$
3	$\langle 4B, 4A_4\rangle$	$2\sqrt{2}\frac{\eta(2\tau)^2}{\eta(\tau)^4}\vartheta_1(\tau,z)^2$
4	$\langle 2B, 8A_{1,2}\rangle$	$2\frac{\eta(2\tau)^2}{\eta(\tau)^4}\vartheta_1(\tau,z)^2$
5	$\langle 3A, 3A_3\rangle$	0
6	$\langle 3A, 3B_1\rangle$	0

Now we are ready to check the generalized moonshine conjecture. Again with the aid of GAP we print the character tables for the projective representations of the relevant centralizer subgroups $C_{M_{24}}([g])$. For the worked out case $g \in [2B]$ we print the table and the decomposition of the graded module.

3.2 The type-A umbral groups

In this chapter we have introduced the concept of moonshine: the conjecture that for specific groups there exist infinite graded modules such that the characters of these modules equal certain specific (mock) modular objects. Orginally, the conjecture was a statement about the monster group $\mathbb{M}$ and the $j-$function and other Hauptmoduls j_Γ, but we saw how the conjecture extends to the so-called umbral groups $G^{(\ell)}$, with mock modular forms $H_g^{(\ell)}$ attached to the conjugacy classes $[g]$ of $G^{(\ell)}$.
We also introduced a generalization of moonshine for the monstrous group (where computations and proofs are still sparse) where Hauptmoduls were attached to conjugay classes of centralizer subgroups of the monster group. We worked this out more explicitly for the largest umbral group M_{24}.
It is natural to extend this computation to all umbral groups $G^{(\ell)}$. That is it is natural to conjecture the existence of infite graded modules for all the centralizer subgroups $C_{G^{(\ell)}}(g)$ where the characters of these modules are again mock modular, or in terms of physics, where the g-twisted h-twined genus is a weak Jacobi form with certain phases under some subgroup $\Gamma \in \mathrm{SL}_2(\mathbf{Z})$. In this final section we will present evidence that points towards the existence of such modules.
In doing so we follow the rationale of finding the generalized g-twisted h-twined genera for M_{24}. That is, first we compute all rank-two subgroups of the umbral groups $G^{(\ell)}$ and their conjgacy calsses, as they are in one to one correspondence with the twisted twined genera $\phi_{g,h}^{(\ell)}$. The amount $\#\mathcal{A}^{(\ell)}$ of such rank-two Abelian subroups is listed in table (III.7). What we see right away from table (III.7) is

Table III.7: Umbral groups, their third cohomology and the number $\#\mathcal{A}^{(\ell)}$ of rank-two Abelian subroups

ℓ	2	3	4	5	7
$G^{(\ell)}$	M_{24}	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$SL_2(3)$
$H^3\left(G^{(\ell)}, U(1)\right)$	12	$8 \oplus 24$	$3 \oplus 4 \oplus 8$	$3 \oplus 4 \oplus 4$	24
$\#\mathcal{A}^{(\ell)}$	34	18	13	5	0

that lambency $\ell = 7$ is obstructed from generalized moonshine right away, as there are no rank-two Abelian subgroups in $\mathrm{SL}_2(3)$. In the table we also included the GAP/HAP computation of the third cohomology group $H^3(G^{(\ell)}$, as the multiplier systems and the phases are set by an elements $\alpha_{(\ell)} \in H^3(G^{(\ell)})$ through the map (87). Then, for all these rank-two Abelian subgroups we compute which ones are obstructed. The unobstructed subgroups are then listed in tables (III.8),(III.9),(III.10) ending up, for the lambencies $\ell = 3, 4, 5$, with a small list of admissible independent genera $\phi_{g,h}^{(\ell)}$.

In this document we only consider lambencies $\ell \leq 7$ and we refer to [CdLW16] for the higher lambencies.

Table III.8: The non-trivial rank-two Abelian subgroups of $G^{(3)}$

#	Structure	$\Gamma_{g,h}$	Generators	Element names
1	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\Gamma(1)$	$\langle$2B,2a$\rangle$	$(2B)(2C)^2$
2	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\Gamma_0(2)$	$\langle$2B,2b$\rangle$	$(2A)(2B)^2(2C)$
3	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\Gamma(1)$	$\langle$2B,2e$\rangle$	$(2C)^3$
4	$\mathbf{Z}_2 \times \mathbf{Z}_4$	$\Gamma_0(2)$	$\langle$2B,4a$\rangle$	$(2B)(2C)^2(4B)^2(4C)$
5	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\Gamma_0(2)$	$\langle$2C,2b$\rangle$	$(2C)^3$
6	$\mathbf{Z}_3 \times \mathbf{Z}_3$	$\Gamma_0(3)$	$\langle$3A,3a$\rangle$	$(3A)^6(3B)^2$
7	$\mathbf{Z}_3 \times \mathbf{Z}_3$	$\Gamma_0(3)$	$\langle$3B,3a$\rangle$	$(3B)^8$
8	$\mathbf{Z}_3 \times \mathbf{Z}_3$	$\Gamma_0(3)$	$\langle$3B,3b$\rangle$	$(3B)^8$
9	$\mathbf{Z}_3 \times \mathbf{Z}_6$	$\Gamma_0(6)$	$\langle$3B,6g$\rangle$	$(2A)(3B)^8(6B)^8$
10	$\mathbf{Z}_3 \times \mathbf{Z}_6$	$\Gamma_0(6)$	$\langle$3B,6h$\rangle$	$(2A)(3B)^8(6B)^8$
11	$\mathbf{Z}_4 \times \mathbf{Z}_2$	$\Gamma_0(4)$	$\langle$4A,2a$\rangle$	$(2A)(2B)(2C)(4A)^4$
12	$\mathbf{Z}_6 \times \mathbf{Z}_3$	$\Gamma_0(6)$	$\langle$6A,3a$\rangle$	$(2A)(3A)^6(3B)^2(6A)^6(6B)^2$

Table III.9: The unobstructed rank-two Abelian subgroups of $G^{(4)}$

#	Structure	Generators	$\Gamma_{g,h}$	Element names
1	$\mathbf{Z}_2 \times \mathbf{Z}_4$	$\langle$2B,4b$\rangle$	$\Gamma_0(4)$	$(2A)(2B)^2(4A)^4$
2	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\langle$2C,2c$\rangle$	$\Gamma(1)$	$(2C)^3$
3	$\mathbf{Z}_4 \times \mathbf{Z}_4$	$\langle$4A,4c$\rangle$	$\Gamma_0(4)$	$(2A)(2B)^2(4A)^4(4B)^8$

Table III.10: The unobstructed rank-two Abelian subgroups of $G^{(5)}$

#	Structure	Generators	$\Gamma_{g,h}$	Element names
1	$\mathbf{Z}_2 \times \mathbf{Z}_4$	$\langle$4A,4c$\rangle$	$\Gamma^{(5)}_{4A,4c}$	(2A)(2B)(2C)

From the tables (III.8), (III.9) and (III.10) we see that there is, in the independent twisting twining genera, only a rather small number of conjugacy classes of $G^{(\ell)}$ that acts as a twist element for the group. In these tables we identified:

$$\Gamma^{(5)}_{4A,4c} = \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 3 \\ 1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 3 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 2 \\ 2 & 3 \end{smallmatrix}\right) \rangle \tag{98}$$

For example, for lambency $\ell = 5$, we notice that only the twists occur for $g \in \{2A, 4A\}$ and we will see that the $g \in 2A$ twist is trivial in a sense also so that only twist $g \in 4A$ So we will only consider these twists, as all others are in some orbit of these or the trivial cases that stem from cyclic Abelian subgroups (that is are in the orbit of the known umbral twinging genera $\phi^{(\ell)}_{e,h}$. In the appendix we print all the projective character tables for the centralizer subgroup $C_{G^{(\ell)}}([g])$ for g a twist element that occurs in the tables (III.8), (III.9) and (III.10) of independent generalized genera. We give the structural descriptions of the centralizers themselves in tables (III.11), (III.12), (III.13). We also indicate in these tables the degree of the central extension of the centralizer subgroups. If the degree of the central extension is one, then the projective representation is actually just equal to the ordinary representation. We computed the centarlizer subgroups with GAP and used the notation where $\mathbf{Z}_n = n$. Also, $G = L : M$ denotes the semi-direct product, that is, L is normal in G, $G = LM$ and $L \cap M = \{e\}$. The group Q_k is the generalized quaternionic group:

$$Q_k := \langle a, x \,|\, x^2 = a^{2^{k-1}},\ a^{2^k} = 1,\ a^k = a^{-1} \rangle \tag{99}$$

Table III.11: The structure descriptions of the centralizer subgroups for $\ell = 3$

g	$C_{G^{(3)}}(g)$	[Schur(C) : C]
2B	$((((2 \times Q_8) : 2) : 2) : 3) : 2$	4
2C	$((((2 \times Q_8) : 2) : 2) : 3) : 2$	4
3A	$3 \times \mathrm{SL}_2(3)$	3
3B	$2 \times (((3 \times 3) : 3) : 2)$	2
4A	$4 \times A_5$	2

Table III.12: The structure descriptions of the centralizer subgroups for $\ell = 4$

g	$C_{G^{(4)}}(g)$	[Schur(C) : C]
2B	$(((4 \times 4) : 2) : 2) : 3$	4
2C	$(2 \times 2 \times 2 \times 2) : 2$	16
4A	$(4 \times 4) : 2$	2

Table III.13: The structure descriptions of the centralizer subgroups for $\ell = 5$

g	$C_{G^{(5)}}(g)$	$[\mathrm{Schur}(C) : C]$
2B	$(4 \times 4) : 2$	4
4A	$4 \times S_3$	2

3.3 The umbral twisted twined genera

It is natural to extend the conjectures of generalized monstrous moonshine and umbral moonshine to a combined conjecture that we may dub 'generalized umbral moonshine'. The logic is the following: Consider again the (weak) Jacobi form $\phi_h^{(\ell)}(\tau, z)$ we encountered for umbral moonshine. We first introduced an meromorphic Jacobi form of weight 0 and index $\ell - 1$ by splitting of $\psi_h^{(\ell)} = \Psi_{1,1}(\tau, z)\phi_h^{(\ell)}(\tau, z)$ (see equation (60) for a recollection of the definition of $\Psi_{1,1}$). Then we introduce for $\psi_h^{(\ell)}(\tau, z)$ its finite (non-polar) part

$$\psi_h^{F,(\ell)}(\tau, z) = \sum_{r=1}^{\ell-1} H_{r,h}^{(\ell)}(\tau)\hat{\theta}_r^{(\ell)}(\tau, z) \tag{100}$$

where now the $H_{r,h}^{(\ell)}$ are vector valued mock modular forms that are, in the umbral moonshine conjecture, the graded traces of the graded umbral modules $K_r^{(\ell)}$. The theta functions are given in equation (62).

We want to generalized this à la generalized monstrous moonshine. Our approach is as follows. First, in all the functions above, we replace the twining element h by the 'twisted twined' notation $h \mapsto (e, h)$, where $e \in G^{(\ell)}$ is just the identity element or trivial twist. This then gives rise to the Jacobi forms $\phi_{e,h}^{(\ell)}$.

Next we conjecture the existence of Jacobi forms $\phi_{(g,h)}^{(\ell)}$ for $(g, h) \in \{G^{(\ell)} \times G^{(\ell)} \,|\, gh = hg\}$. We conjecture that all such $\phi_{g,h}$ are Jacobi forms of index $\ell - 1$ obeying the conditions of generalized Mathieu moonshine (3)-(5), only with condition 1. and 5. replaced by 1.' and 5'. where in 5'. only the group M_{24} in 5. is replaced by the umbral group $G^{(\ell)}$ and in 1. the transformation property of the Jacobi form of index 1 is replaced by the transformation properties of a Jacobi form of index $\ell - 1$ weight 0:

1. *Ellipticity and Modularity (weak Jacobi)*:
$\phi_{g,h}(\tau, z + \kappa\tau + \kappa') = \mathrm{e}\left((-\kappa^2\tau - 2\kappa z)(\ell - 1)\right) \phi_{g,h}(\tau, z)$
$\phi_{g,h}(\gamma\tau, \gamma z) = \chi_{g,h}(\gamma)\mathrm{e}\left((\ell - 1)\frac{cz^2}{c\tau+d}\right) \phi_{g^a h^c, g^b h^d}(\tau, z)$

We can now just like in the case of umbral moonshine introduce the meromorphic weight 1 index $\ell - 1$ Jacobi form $\psi_{g,h}^{(\ell)}(\tau, z) = \Psi_{1,1}\phi_{g,h}^{(\ell)}(\tau, z)$ and consider the theta-decomposition of the finite part

$$\psi^{F,(\ell)}_{g,h} = \sum_{r=1}^{(\ell-1)} H^{(\ell)}_{r\,(g,h)} \hat{\theta}^{(\ell-1)}_{r} \tag{101}$$

Now the consistincy condition $\phi_{e,h} = \phi_h$ is replaced by the condition for the mock modular forms $H^{(\ell)}_{r\,(g,h)}$ that

$$H^{(\ell)}_{r\,(e,h)} = H^{(\ell)}_{r,h} \tag{102}$$

where the $H^{(\ell)}_{r,h}$ are the mock modular functions of umbral moonshine as in [CDH14, CDH13].
The generalized umbral moonshine conjecture now states the following:
Consider an umbral group $G^{(\ell)}$, and element $g \in G^{(\ell)}$ and the centralizer subgroup $C_{G^{(\ell)}}(g)$. We conjecture the existence of an infinitely graded module $\mathcal{C}^{(\ell)}_g$ of the group $C_{G^{(\ell)}}(g)$:

$$\mathcal{C}^{(\ell)}_g = \bigoplus_{0<r<\ell} \mathcal{C}^{(\ell)}_{g,r} \tag{103}$$

$$\mathcal{C}^{(\ell)}_{g,r} = \bigoplus_{k\in\mathbf{Z}} \mathcal{C}^{(\ell)}_{g;r,k-r^2/4\ell} \tag{104}$$

where the functions $H^{(\ell)}_{r\,(g,h)}$ are

$$H^{(\ell)}_{r\,(g,h)} = \sum_{\substack{k\in\mathbf{Z}\\ r^2-4k\ell<0}} \hat{ch}^{(\ell)}_{\mathcal{C}_{g;r,k-r^2/4\ell}}(h) q^{k-r^2/4\ell} \tag{105}$$

and again $\hat{ch}$ denotes projective character.

Our task in this section is, to find the functions $H^{(\ell)}_{r,\,(g,h)}$ and compute its q-expansions. In doing so we departure from the work on generalized Mathieu moonshine in the following way. In trying to find the functions $H^{(\ell)}_{r\,(g,h)}$, we assume the modules (103) to exist and the functions $H^{(\ell)}_{r\,(g,h)}$ to obey the conjecture (105) indeed.

Next we start with the 'trivial' functions $H^{(\ell)}_{r\,(e,h)}$ that were already computed in [CDH14, CDH13].

Then using the generalized umbral moonshine conjecture, we compute all the functions that are in the orbit of $H^{(\ell)}_{r\,(e,g)}$ for some g, under the group action of $G^{(\ell)} \times \mathrm{GL}_2(\mathbf{Z})$, that is we compute all $H^{(\ell)}_{r,(g'h')}$ where $(g', h') = (kh^c k^{-1}, kh^d k^{-1})$ for some $k \in G^{(\ell)}$. This gives us a lot of functions and we can hope that this already gives rise to a unique decomposition of the module in therms of projective representations of the centralizer subgroups in question by demanding decomposition in terms or projective characters. Generically however only knowledge of the functions $H^{(\ell)}_{r/(g'h')}$ is not enough and there will not be a unique such decompostions in the relevant projective chracters.

Note the following however. For lambency higher than $\ell = 2$, the umbral groups $G^{(\ell)}$ have a non-trivial centre. For the groups we consider here ($\ell = 3, 4, 5$), this centre is

$$Z(G^{(\ell)}) = \mathbf{Z}_2 = \langle z_{(\ell)} \rangle. \tag{106}$$

In all stated cases, the element $z_{(\ell)} \in Z(G^{(\ell)})$ is a member of the conjugacy class $2A$ in the ATLAS definitions.
Note that quite trivially for the central element $z_{(\ell)}$, the centralizer subgroup $C_{G^{(\ell)}}(z_{(\ell)})$ is just the group itself, $C_{G^{(\ell)}}(z_{(\ell)}) = G^{(\ell)}$. Hence it is natural to conjecture that

$$H^{(\ell)}_{r\,(z,h)} = J_{rs} H^{(\ell)}_{s\,(e,h)} \tag{107}$$

where J is a unitary matrix with complex elements J_{rs} that are $U(1)$-valued.

We will illustrate how to obtain functions $H^{\ell}_{r,(g,h)}$ where (g,h) are not in the orbit of (e,h') by taking the example of $\ell = 5$, $g = 4A$. We know that there should be genuine generalized umbral moonshine functions in this case from table **??**. First we compute via the Rademacher summation method the coefficients of all the functions $H^{(\ell)}_{r;(g,h)}$ such that $g \in 4A$ and $(g,h) \sim (e,h')$or (z,h'') where z a central element of $G^{(\ell)}$ and $\sim$ denotes equivalence under the action of $\mathrm{SL}_2(\mathbf{Z}) \times G^{(5)}$. These coefficients are printed in tables (III.14, III.15) with black letters for the conjgacyclasses, as opposed to the red conjgacy classes in the centralizer $C_{G^{(5)}}(4A)$.
Next we demand a decomposition in terms of projective characters of $C_{G^{(5)}}(4A)$. We printed the projective character tables for the twist $g = 4A$ in the appendix, in table III.26. We find under this assumption a unique choice for the Fourier coefficients of the new functions. In this case, the new functions read

$$H^{(5)}_{r,(4A,4c)} \sim H^{(5)}_{r,(4A,4d)} \sim H^{(5)}_{r,(4A,2b)} \sim H^{(5)}_{r,(4A,2c)}. \tag{108}$$

Notice that for the "new" functions — the functions

$$H^{(5)}_{1,(4A,h)}(\tau) = \sum_{n=0}^{\infty} b_n q^{\frac{40n+1}{80}} \tag{109}$$

$$H^{(5)}_{2,(4A,h)}(\tau) = \sum_{n=0}^{\infty} c_n q^{\frac{40n+9}{80}} \tag{110}$$

Table III.14: McKay–Thompson series $H^{(5)}_{4A,h;1}$

$[h]$	1a	2a	4a	4b	4c	4d	2b	2c	3a	6a	12a	12b
$n/80$												
1	1	1	1	1	1	1	1	1	1	1	1	1
41	4	4	-4	4	0	0	0	0	-2	-2	2	-2
81	3	3	3	3	1	1	1	1	0	0	0	0
121	7	7	-7	7	1	1	1	1	1	1	-1	1
161	12	12	12	12	0	0	0	0	0	0	0	0
201	12	12	-12	12	0	0	0	0	0	0	0	0
241	20	20	20	20	0	0	0	0	2	2	2	2
281	28	28	-28	28	0	0	0	0	-2	-2	2	-2
321	36	36	36	36	0	0	0	0	0	0	0	0
361	49	49	-49	49	1	1	1	1	1	1	-1	1

Table III.15: McKay–Thompson series $H^{(5)}_{4A,h;2}$

$[h]$	1a	2a	4a	4b	4c	4d	2b	2c	3a	6a	12a	12b
$n/80$												
9	3	3	3	3	1	1	1	1	0	0	0	0
49	5	5	-5	5	1	1	1	1	1	1	1	1
89	8	8	8	8	0	0	0	0	-2	-2	2	-2
129	12	12	-12	12	0	0	0	0	0	0	0	0
169	17	17	17	17	1	1	1	1	1	1	-1	1
209	24	24	-24	24	0	0	0	0	0	0	0	0
249	36	36	36	36	0	0	0	0	0	0	0	0
289	47	47	-47	47	1	1	1	1	1	1	1	1
329	60	60	60	60	0	0	0	0	0	0	0	0
369	84	84	-84	84	0	0	0	0	0	0	0	0

$$H^{(5)}_{4A,h;3}(\tau) = H^{(5)}_{4A,h;2}(\tau)$$

$$H^{(5)}_{4A,h;4}(\tau) = H^{(5)}_{4A,h;1}(\tau)$$

with h in red — the coefficients obey

$$b_n = \begin{cases} 1 & \text{if } 40n+1 = k^2 \text{ for some } k \in \mathbf{N} \\ 0 & \text{else} \end{cases} \tag{111}$$

(112)

$$c_n = \begin{cases} 1 & \text{if } 40n + 9 = k^2 \text{ for some } k \in \mathbf{N} \\ 0 & \text{else} \end{cases} \tag{113}$$

$$\tag{114}$$

so that we may write these functions as theta functions, namely

$$H^{(5)}_{(4A,4c),1}(\tau) = \sum_{n\in\mathbf{Z}} q^{(10n+1)^2/80} \tag{115}$$

$$H^{(5)}_{(4A,4c),2}(\tau) = \sum_{n\in\mathbf{Z}} q^{(10n+3)^2/80} \tag{116}$$

or

$$H^{(5)}_{(4A,4c),1} = \theta^{(5)}_1(\tau/4, 0) \tag{117}$$

$$H^{(5)}_{(4A,4c),2} = \theta^{(5)}_3(\tau/4, 0) \tag{118}$$

Finally, we may write the finite part of full Jacobi form as

$$\varphi^F_{4A,4c}(\tau, z) = \sum_{r=1}^{4} H^{(5)}_{r,(4A,4c)}(\tau)\hat{\theta}^{(5)}_r(\tau, z) \tag{119}$$

Appendix

A (Weak) Jacobi Forms

In this appendix we collect some relevant facts about (weak) Jacobi forms of weight k and index m. All of the following is taken from [EZ85], also following the notation of said citation.

A Jacobi form, that is a function $\phi(\tau, z)$ with conditions (2.1)-(3) is called weak if $c(n, r) = 0$ unless $n \geq 0$. Condition (2) implies a periodicity of the coefficients $c(n, r)$ of the Jacobi forms, namely

$$c(n, r) = c(n', r') \quad \text{if } r' \equiv r \bmod 2m,\ n \equiv n + \frac{r'^2 - r^2}{4m} \tag{120}$$

This, in turn, implies that we can expand a Jacobi form in terms of theta functions. For a Jacobi form $\phi(\tau, z)$ we have

$$\phi(\tau, z) = \sum_{\mu \bmod 2m} h_\mu(\tau) \theta_r^{(m)}(\tau, z) \tag{121}$$

where the $h_\mu(\tau)$ are a set of $2m$ weight $k - 1/2$ vector valued modular forms

$$h_\mu(\tau) = q^{-\mu^2/4m} \sum_{n \geq 0} c_{n,\mu} q^n \tag{122}$$

and the index-m theta functions are

$$\theta_r^{(m)} = \sum_{n \in \mathbf{Z}} q^{(2mn+r)^2/4m} y^{2mn+r}. \tag{123}$$

The space of weak Jacobi forms $\mathcal{J}_{k,m}$ of weight k and index m is isomorphic to a direct sum of vector spaces of modular forms, namely

$$\mathcal{J}_{k,m} \cong \bigoplus_{i=k}^{k+2m} M_i, \quad k \text{ even} \tag{124}$$

$$\mathcal{J}_{k,m} \cong \bigoplus_{i=k+1}^{k+2m-3} M_i, \quad k \text{ odd} \tag{125}$$

and in fact, for k even, $\mathcal{J}_{k,m}$ is generated by two weak Jacobi forms, $\phi_{0,1}$ and $\phi_{-2,1}$, that is for $\phi_{k,m} \in \mathcal{J}_{k,m}$:

$$\phi_{k,m} = \sum_{i=0}^{m} \omega_i \phi_{-2,1}^i \phi_{0,1}^{m-i} \tag{126}$$

where $\omega_i \in M_{k+2i}$. Conventionally the generators are

$$\phi_{-2,1} = -\frac{\vartheta_1^2(\tau, z)}{\eta^6(\tau)} \tag{127}$$

$$\phi_{0,1} = -\frac{c}{\pi^2}\mathfrak{p}\phi_{-2,1} \tag{128}$$

where ϑ_1 denotes the first Jacobi theta function and $\mathfrak{p}$ the Weierstrass function.

B Group Cohomology

In this appendix we give a brief hands-on introduction to cohomology of discrete groups, without assuming any knowledge about homological algebra, that is we introduce the subject explicitely thourgh te definitions of co-chains. As an application we will want to take G finite, but the statements below apply to generic discrete groups.
Let A be a G-module and let i be an integer, $i \geq 0$. The set of i-cochains is the set $C^i(G, A)$ of functions from G^i to A

$$C^i(G, A) = \left\{ f : G^i \to A \right\}. \tag{129}$$

We can define a differential map $d^i : C^i(G, A) \to C^{i+1}(G, A)$ as

$$\begin{aligned} d^i(f)(g_0, g_1, g_2, \dots g_n) &= g_0 \cdot f(g_1, g_2, \dots g_n) \\ &+ \sum_{k=0}^{n} (-1)^k f(g_0, \dots, g_{k-1}g_k, g_{k+1}, \dots, g_n) + (-1)^{n+1} f(g_0, \dots, g_{n-1}). \end{aligned} \tag{130}$$

This map enjoys the property that $d^{i+1} \circ d^i = 0$. Now we set $B^0(G, A) = 0$ and for $i \geq 1$:

$$Z^i(G, A) = \ker d^i, \quad B^i(G, A) = \operatorname{im} d^{i-1} \tag{131}$$

$$H^i(G, A) = Z^i(G, A)/B^i(G, A). \tag{132}$$

We refer to $Z^i(G, A)$ as the set of i-cocycles and $B^i(G, A)$ as the set of i-boundaries of G with coefficients in A. The group $H^i(G, A)$ is the i^{th} cohomology group of G with coefficients in A. It is a measure of how non-exact cochains are. Note that the zeroth group cohomology is the set of elements in A fixed by G: $H^0(G, A) = A^G = \{a \in A \,|\, ga = a \text{ for all } g \in G\}$. For our purpouses we will be intereseted in trivial G-modules only. For such a trivial G-module A, note that $H^1(G, A) \cong \operatorname{Hom}(G, A)$, the group of homomorphisms form G onto A.
Also for trivial modules, the second cohomlogy group classifies the central extensions of a group G. A central extension E of G by A is a short exact sequence

$$1 \to A \to E \to G \to 1 \tag{133}$$

with im $A \lhd E$ a normal subroup and $G/E \cong A$, such that $A \subset Z(A)$, where $Z(E)$ is the center of E. Two (central) extensions E_1 and E_2 are called isomorphic if there is an isorphism $\phi : E_1 \to E_2$ such that the follow diagram commutes:

$$\begin{array}{ccccc} A & \longrightarrow & E_1 & \longrightarrow & G \\ \downarrow & & \downarrow\phi & & \downarrow \\ A & \longrightarrow & E_2 & \longrightarrow & G \end{array} \tag{134}$$

The set of isomorphism classes of such central extensions is then isomorphic to $H^2(G, A)$. This will be important to us in the context of projective representations, as explained in appendix (C).

The notion of cohomology for finite groups makes contact with the topological notion of cohomology as follows. Let X be a topological space that has $\pi_n X = 0$ for all $n \geq 2$, where π_n denotes the n^{th} homotopy group. Such a space X is sometimes called aspherical. Let $\pi_1 X = G$ be a discrete group. With these presumptions we call X the classifying space BG of G, $X = BG$. Topologically, X depends only on $\pi_1 X = K$. In a broader context, BG is also called an Eilernberg-Maclane space. For example, the unit circle S^1 is the classifying space of $\mathbf{Z}$, $S^1 \cong B\mathbf{Z}$. Small finite groups can give rise to rather complicated calssifying spaces, as the example $B\mathbf{Z}_2 \cong \mathbf{RP}^\infty$ examplifies.

The n^{th} cohomology of BG with coeficfcients in the G-module A is isomorphic to the n^{th} group cohomlogy:

$$H^n(BG, A) \cong H^n(G, A) \tag{135}$$

This theorem may sometimes help us in computing the cohomlogy of discrete groups. Computing classifying space, however, is notoriously hard and the isomorphism (135) is primarily used vice versa.

Rather, we used GAP with the HAP package [GAP, HAP] to do the computation in group cohomlogy. To get some idea what type of group may arise, consider the permutation group on three elements, $G = S_3$ and its trivial $\mathbf{Z}$-module. The cohomology groups are then computed with GAP:

$$\begin{aligned} &H^1(S_3, \mathbf{Z}) \cong 0 \\ &H^2(S_3, \mathbf{Z}) \cong \mathbf{Z}_2 \\ &H^3(S_3, \mathbf{Z}) \cong \mathbf{Z}_6 \\ &H^4(S_3, \mathbf{Z}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_{12} \\ &H^5(S_3, \mathbf{Z}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_{12} \\ &H^6(S_3, \mathbf{Z}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_{12} \end{aligned}$$

and the cohomlogy $H^n(S_3, \mathbf{Z})$ stabalizes for $n \geq 6$: $H^n(S_3, \mathbf{Z}) \cong H^6(S_3, \mathbf{Z})$ for $n \geq 6$.

C Projective Representations

Projective representations play a central role in the theory of generalized moonshine. A projective representation ρ of G is a group homomorphism

$$\rho : G \to \mathrm{PGL}(V, \mathbf{F}) \tag{136}$$

from G onto the projective linear group $\mathrm{PGL}(V, \mathbf{F}) = \mathrm{GL}(V, \mathbf{F})/\mathbf{F}^*$ where $\mathrm{GL}(V, \mathbf{F})$ is the group of general linear transformations on a vector space V over the field $\mathbf{F}$. We will be interested in the case $\mathbf{F} = \mathbf{C}$ only, but the theory applies to any field $\mathbf{F}$.

One can formulate a projective representation in terms of a regular representation as follows. First of all note that $\mathrm{PGL}(V, \mathbf{F})$ is a central extension of $\mathrm{GL}(V, \mathbf{F})$. In fact for $\pi : \mathrm{GL}(V, \mathbf{F}) \to \mathrm{PGL}(V, \mathbf{F})$, where π maps an element of $\mathrm{SL}_2(\mathbf{Z})$ to its corresponding equivalence class in $\mathrm{PSL}_2(\mathbf{Z})$, we have $\ker(\pi) = Z\left(\mathrm{GL}(V, \mathbf{F})\right)$ where $Z\left(\mathrm{GL}(V, \mathbf{F})\right) = \{A \in \mathrm{GL}(V, \mathbf{F}) \,|\, A = \mathrm{diag}(a_i, \ldots a_i)\}$ for some $a_i \in \mathbf{F}$. This allows us to pull back ρ to a regular linear representation

$$\sigma : C \to \mathrm{GL}(V, \mathbf{F}) \tag{137}$$

where C is a central extenstion of G. We will often refer to C as the Schur cover of G, and will sometimes write $C = \mathrm{Schur}(G)$.

Projective representations of a group G are classified by the second group cohomology of G, $H^2(G, \mathbf{F}^*)$. We can see this as follows. A projective representation ρ satisfies

$$\rho(gh) = c(g, h)\rho(g)\rho(h) \tag{138}$$

for $g, h \in G$ and $c(g, h) : G \times G \to \mathbf{F}^*$ a 2-coycle of G. Associativity of G implies the condition on the cocyle

$$c(h, k)c(g, hk) - c(g, h)c(gh, k) = 0. \tag{139}$$

The cocyle c depends on the choice of projective representation. We can chooce another projective representation $\rho'(g) \cong \varphi(g)\rho(g) \in \mathrm{PGL}_2(\mathbf{Z})$ where $\varphi : G \to \mathbf{F}^*$ is a group homomorphism. This yields the equivalence

$$c'(g, h) = \varphi(gh)\varphi^{-1}(g)\varphi^{-1}(h)c(g, h) \cong c(g, h) \tag{140}$$

That is to say that a projective representation defines a unique cocycle in $H^2(G, \mathbf{F}^*)$ and vice versa. the cocycle $c(g, h)$ is sometimes referred to as the Schur multiplier. Note finally that the characters of projective representations fail to be class functions. Let $\chi = \mathrm{Tr}_\rho$ denote the character of a projective representation ρ, then clearly $\chi(g)\chi(h) = c(g, h)\chi(gh)$.

In printing the projective character tables for the centralizer subgroups that we encounter in moonshine we also extensively used the following consitency condition. Let g, h be commuting elements in G, $gh = hg$ and let $\tilde{g} \in C$ be a lift of an element $g \in G$ to the central extension, so that $\lambda : C \to G$, $\lambda(\tilde{g}) = g$. The regular characters $\tilde{\chi} = \mathrm{Tr}_\sigma$ then should satisfy

$$\tilde{\chi}\left(\tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}\right) = \frac{c(h,g)}{c(g,h)} \tag{141}$$

D Projective Character Tables

We print out the (projective) character tables. The names of the conjugacyclasses are as in GAP.

Table III.16: Projective Charactertable of $C_{G^{(2)}}(2B)$

$[g]$																														
χ_1	4	0	0	4	0	−4	2	0	−2	0	−1	1	−1	−1	1	0	2	0	0	−2	0	0	−1	1	1	−1	0	$2e_1^-$	0	$2e_1^+$
χ_2	4	0	0	−4	0	−4	2	0	2	0	−1	1	1	−1	−1	0	2	0	0	2	0	0	−1	−1	1	1	0	$-2e_1^+$	0	$-2e_1^-$
χ_3	4	0	0	4	0	−4	−2	0	2	0	−1	1	−1	1	−1	0	2	0	0	−2	0	0	−1	1	1	−1	0	$-2e_1^-$	0	$-2e_1^+$
χ_4	4	0	0	−4	0	−4	−2	0	−2	0	−1	1	1	1	1	0	2	0	0	2	0	0	−1	−1	1	1	0	$2e_1^+$	0	$2e_1^-$
χ_5	16	0	0	16	0	−16	4	0	−4	0	−1	1	−1	1	−1	0	0	0	0	0	0	0	1	−1	−1	1	0	0	0	0
χ_6	16	0	0	−16	0	−16	4	0	4	0	−1	1	1	1	1	0	0	0	0	0	0	0	1	1	−1	−1	0	0	0	0
χ_7	16	0	0	16	0	−16	−4	0	4	0	−1	1	−1	−1	1	0	0	0	0	0	0	0	1	−1	−1	1	0	0	0	0
χ_8	16	0	0	−16	0	−16	−4	0	−4	0	−1	1	1	−1	−1	0	0	0	0	0	0	0	1	1	−1	−1	0	0	0	0
χ_9	16	0	0	0	0	16	0	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	1	$\sqrt{5}$	1	$\sqrt{5}$	0	0	0	0
χ_{10}	16	0	0	0	0	16	0	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	1	$-\sqrt{5}$	1	$-\sqrt{5}$	0	0	0	0
χ_{11}	20	0	0	20	0	−20	−2	0	2	0	1	−1	1	1	−1	0	2	0	0	−2	0	0	0	0	0	0	0	$2e_1^-$	0	$2e_1^+$
χ_{12}	20	0	0	−20	0	−20	−2	0	−2	0	1	−1	−1	1	1	0	2	0	0	2	0	0	0	0	0	0	0	$-2e_1^+$	0	$-2e_1^-$
χ_{13}	20	0	0	20	0	−20	2	0	−2	0	1	−1	1	−1	1	0	2	0	0	−2	0	0	0	0	0	0	0	$-2e_1^-$	0	$-2e_1^+$
χ_{14}	20	0	0	−20	0	−20	2	0	2	0	1	−1	−1	−1	−1	0	2	0	0	2	0	0	0	0	0	0	0	$2e_1^+$	0	$2e_1^-$
χ_{15}	24	0	0	24	0	−24	0	0	0	0	0	0	0	0	0	0	−4	0	0	4	0	0	−1	1	1	−1	0	0	0	0
χ_{16}	24	0	0	−24	0	−24	0	0	0	0	0	0	0	0	0	0	−4	0	0	−4	0	0	−1	−1	1	1	0	0	0	0
χ_{17}	32	0	0	0	0	32	0	0	0	0	−2	−2	0	0	0	0	0	0	0	0	0	0	2	0	2	0	0	0	0	0
χ_{18}	48	0	0	0	0	48	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−2	0	−2	0	0	0	0	0

Table III.17: Projective Charactertable of $C_{G^{(3)}}(2B)$

$[g]$	$1a$	$2a$	$3a$	$4a$	$2b$	$8a$	$8b$	$4b$	$6a$	$2c$	$6b$	$6c$	$8c$	$8d$	$4c$	$4d$	$4e$	$2d$	$2e$	$2f$	$2g$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	−1	1	−1	1	−1	−1	1	1	−1	1	1	−1	−1	1	−1	1	1	1	1	1
χ_3	2	0	−1	0	2	0	0	2	−1	0	−1	−1	0	0	2	0	2	2	2	2	2
χ_4	2	0	−1	0	2	ϕ	$-\phi$	0	1	0	1	−1	$-\phi$	ϕ	0	0	0	2	−2	−2	−2
χ_5	2	0	−1	0	2	$-\phi$	ϕ	0	1	0	1	−1	ϕ	$-\phi$	0	0	0	2	−2	−2	−2
χ_6	3	−1	0	1	−1	−1	−1	3	0	−1	0	0	1	1	−1	1	−1	3	−1	3	3
χ_7	3	1	0	−1	−1	1	1	3	0	1	0	0	−1	−1	−1	−1	−1	3	−1	3	3
χ_8	3	−1	0	−1	3	1	1	−1	0	−1	0	0	1	1	−1	−1	−1	3	3	3	3
χ_9	3	1	0	1	3	−1	−1	−1	0	1	0	0	−1	−1	−1	1	−1	3	3	3	3
χ_{10}	3	−1	0	1	−1	1	1	−1	0	−1	0	0	−1	−1	3	1	−1	3	−1	3	3
χ_{11}	3	1	0	−1	−1	−1	−1	−1	0	1	0	0	1	1	3	−1	−1	3	−1	3	3
χ_{12}	4	0	1	0	4	0	0	0	−1	0	−1	1	0	0	0	0	0	4	−4	−4	−4
χ_{13}	4	0	1	−2	0	0	0	0	−1	0	1	−1	0	0	0	2	0	−4	0	4	−4
χ_{14}	4	0	1	2	0	0	0	0	−1	0	1	−1	0	0	0	−2	0	−4	0	4	−4
χ_{15}	4	−2	1	0	0	0	0	0	1	2	−1	−1	0	0	0	0	0	−4	0	−4	4
χ_{16}	4	2	1	0	0	0	0	0	1	−2	−1	−1	0	0	0	0	0	−4	0	−4	4
χ_{17}	6	0	0	0	−2	0	0	−2	0	0	0	0	0	0	−2	0	2	6	−2	6	6
χ_{18}	6	0	0	0	−2	ϕ	$-\phi$	0	0	0	0	0	ϕ	$-\phi$	0	0	0	6	2	−6	−6
χ_{19}	6	0	0	0	−2	$-\phi$	ϕ	0	0	0	0	0	$-\phi$	ϕ	0	0	0	6	2	−6	−6
χ_{20}	8	0	−1	0	0	0	0	0	−1	0	1	1	0	0	0	0	0	−8	0	−8	8
χ_{21}	8	0	−1	0	0	0	0	0	1	0	−1	1	0	0	0	0	0	−8	0	8	−8

Table III.18: Projective Charactertable of $C_{G^{(3)}}(2C)$

$[g]$	$1a$	$2a$	$2b$	$4a$	$6a$	$3a$	$8a$	$8b$	$2c$	$2d$	$2e$	$4b$	$4c$	$4d$	$6b$	$6c$	$8c$	$8d$	$4e$	$2f$	$2g$
χ_1	2	−2	0	0	1	−1	$-\varphi$	φ	0	0	0	2	$-\varphi$	φ	−1	1	0	0	0	2	−2
χ_2	2	−2	0	0	1	−1	φ	$-\varphi$	0	0	0	2	φ	$-\varphi$	−1	1	0	0	0	2	−2
χ_3	2	2	0	2	−1	−1	0	0	0	0	0	0	$-\varphi$	$-\varphi$	−1	−1	$-\varphi$	$-\varphi$	0	2	2
χ_4	2	2	0	2	−1	−1	0	0	0	0	0	0	φ	φ	−1	−1	φ	φ	0	2	2
χ_5	4	4	0	4	1	1	0	0	0	0	0	0	0	0	1	1	0	0	0	4	4
χ_6	4	−4	0	0	−1	1	0	0	0	0	0	4	0	0	1	−1	0	0	0	4	−4
χ_7	4	−4	0	0	−1	1	0	0	0	0	0	0	0	0	−1	1	−2	2	0	−4	4
χ_8	4	−4	0	0	−1	1	0	0	0	0	0	0	0	0	−1	1	2	−2	0	−4	4
χ_9	4	4	0	0	1	1	−2	−2	0	0	0	0	0	0	−1	−1	0	0	0	−4	−4
χ_{10}	4	4	0	0	1	1	2	2	0	0	0	0	0	0	−1	−1	0	0	0	−4	−4
χ_{11}	6	−6	0	0	0	0	$-\varphi$	φ	0	0	0	−2	φ	$-\varphi$	0	0	0	0	0	6	−6
χ_{12}	6	−6	0	0	0	0	φ	$-\varphi$	0	0	0	−2	$-\varphi$	φ	0	0	0	0	0	6	−6
χ_{13}	6	6	0	−2	0	0	0	0	0	0	0	0	φ	φ	0	0	$-\varphi$	$-\varphi$	0	6	6
χ_{14}	6	6	0	−2	0	0	0	0	0	0	0	0	$-\varphi$	$-\varphi$	0	0	φ	φ	0	6	6
χ_{15}	8	8	0	0	−1	−1	0	0	0	0	0	0	0	0	1	1	0	0	0	−8	−8
χ_{16}	8	−8	0	0	1	−1	0	0	0	0	0	0	0	0	1	−1	0	0	0	−8	8

Table III.19: Projective Charactertable of $C_{G^{(3)}}(3A)$

$[g]$	$1a$	$3a$	$3b$	$3c$	$3d$	$3e$	$3f$	$6a$	$12a$	$3g$	$3h$	$6b$	$6c$	$12b$	$6d$	$4a$	$6e$	$6f$	$2a$	$6g$	$6h$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	ζ_3^2	ζ_3	ζ_3	ζ_3^2	1	ζ_3^2	ζ_3	1	1	ζ_3	ζ_3^2	ζ_3	1	ζ_3	1	ζ_3^2	ζ_3^2	1	1	1
χ_3	1	ζ_3	ζ_3^2	ζ_3^2	ζ_3	1	ζ_3	ζ_3^2	1	1	ζ_3^2	ζ_3	ζ_3^2	1	ζ_3^2	1	ζ_3	ζ_3	1	1	1
χ_4	1	1	1	ζ_3^2	ζ_3^2	ζ_3^2	ζ_3	ζ_3	ζ_3	ζ_3	ζ_3	ζ_3	ζ_3^2	ζ_3^2	1	1	ζ_3^2	1	1	ζ_3^2	ζ_3
χ_5	1	1	1	ζ_3	ζ_3	ζ_3	ζ_3^2	ζ_3^2	ζ_3^2	ζ_3^2	ζ_3^2	ζ_3^2	ζ_3	ζ_3	1	1	ζ_3	1	1	ζ_3	ζ_3^2
χ_6	1	ζ_3^2	ζ_3	1	ζ_3	ζ_3^2	1	ζ_3^2	ζ_3	ζ_3	ζ_3^2	1	1	ζ_3^2	ζ_3	1	ζ_3	ζ_3^2	1	ζ_3^2	ζ_3
χ_7	1	ζ_3	ζ_3^2	1	ζ_3^2	ζ_3	1	ζ_3	ζ_3^2	ζ_3^2	ζ_3	1	1	ζ_3	ζ_3^2	1	ζ_3^2	ζ_3	1	ζ_3	ζ_3^2
χ_8	1	ζ_3	ζ_3^2	ζ_3	1	ζ_3^2	ζ_3^2	1	ζ_3	ζ_3	1	ζ_3^2	ζ_3	ζ_3^2	ζ_3^2	1	1	ζ_3	1	ζ_3^2	ζ_3
χ_9	1	ζ_3^2	ζ_3	ζ_3^2	1	ζ_3	ζ_3	1	ζ_3^2	ζ_3^2	1	ζ_3	ζ_3^2	ζ_3	ζ_3	1	1	ζ_3^2	1	ζ_3	ζ_3^2
χ_{10}	2	-1	-1	-1	-1	2	-1	1	0	2	-1	1	1	0	1	0	1	1	-2	-2	-2
χ_{11}	2	$-\zeta_3^2$	$-\zeta_3$	$-\zeta_3^2$	-1	$2\zeta_3$	$-\zeta_3$	1	0	$2\zeta_3^2$	-1	ζ_3	ζ_3^2	0	ζ_3	0	1	ζ_3^2	-2	$-2\zeta_3$	$-2\zeta_3^2$
χ_{12}	2	$-\zeta_3$	$-\zeta_3^2$	$-\zeta_3$	-1	$2\zeta_3^2$	$-\zeta_3^2$	1	0	$2\zeta_3$	-1	ζ_3^2	ζ_3	0	ζ_3^2	0	1	ζ_3	-2	$-2\zeta_3^2$	$-2\zeta_3$
χ_{13}	2	$-\zeta_3$	$-\zeta_3^2$	-1	$-\zeta_3^2$	$2\zeta_3$	-1	ζ_3	0	$2\zeta_3^2$	$-\zeta_3$	1	1	0	ζ_3^2	0	ζ_3^2	ζ_3	-2	$-2\zeta_3$	$-2\zeta_3^2$
χ_{14}	2	$-\zeta_3^2$	$-\zeta_3$	-1	$-\zeta_3$	$2\zeta_3^2$	-1	ζ_3^2	0	$2\zeta_3$	$-\zeta_3^2$	1	1	0	ζ_3	0	ζ_3	ζ_3^2	-2	$-2\zeta_3^2$	$-2\zeta_3$
χ_{15}	2	-1	-1	$-\zeta_3$	$-\zeta_3$	$2\zeta_3$	$-\zeta_3^2$	ζ_3^2	0	$2\zeta_3^2$	$-\zeta_3^2$	ζ_3^2	ζ_3	0	1	0	ζ_3	1	-2	$-2\zeta_3$	$-2\zeta_3^2$
χ_{16}	2	-1	-1	$-\zeta_3^2$	$-\zeta_3^2$	$2\zeta_3^2$	$-\zeta_3$	ζ_3	0	$2\zeta_3$	$-\zeta_3$	ζ_3	ζ_3^2	0	1	0	ζ_3^2	1	-2	$-2\zeta_3^2$	$-2\zeta_3$
χ_{17}	2	$-\zeta_3^2$	$-\zeta_3$	$-\zeta_3$	$-\zeta_3^2$	2	$-\zeta_3^2$	ζ_3	0	2	$-\zeta_3$	ζ_3^2	ζ_3	0	ζ_3	0	ζ_3^2	ζ_3^2	-2	-2	-2
χ_{18}	2	$-\zeta_3$	$-\zeta_3^2$	$-\zeta_3^2$	$-\zeta_3$	2	$-\zeta_3$	ζ_3^2	0	2	$-\zeta_3^2$	ζ_3	ζ_3^2	0	ζ_3^2	0	ζ_3	ζ_3	-2	-2	-2
χ_{19}	3	0	0	0	0	3	0	0	-1	3	0	0	0	-1	0	-1	0	0	3	3	3
χ_{20}	3	0	0	0	0	$3\zeta_3$	0	0	$-\zeta_3^2$	$3\zeta_3^2$	0	0	0	$-\zeta_3$	0	-1	0	0	3	$3\zeta_3$	$3\zeta_3^2$
χ_{21}	3	0	0	0	0	$3\zeta_3^2$	0	0	$-\zeta_3$	$3\zeta_3$	0	0	0	$-\zeta_3^2$	0	-1	0	0	3	$3\zeta_3^2$	$3\zeta_3$

Table III.20: Projective Charactertable of $C_{G^{(3)}}(3B)$

$[g]$	$1a$	$2a$	$6a$	$3a$	$3b$	$6b$	$3c$	$3d$	$3e$	$3f$	$6c$	$6d$	$6e$	$2b$	$2c$	$6f$	$6g$	$6h$	$6i$	$6j$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	-1	1	1	-1	1	1	1	1	1	-1	1	1	-1	-1	1	1	1	1
χ_3	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
χ_4	1	-1	-1	1	1	-1	1	1	1	1	-1	1	-1	-1	1	1	-1	-1	-1	-1
χ_5	2	0	0	-1	2	0	2	-1	2	-1	1	0	-2	-2	0	0	-2	-2	1	1
χ_6	2	0	0	-1	2	0	2	-1	2	-1	-1	0	2	2	0	0	2	2	-1	-1
χ_7	2	0	0	2	2	0	2	-1	-1	-1	-2	0	-2	-2	0	0	-2	1	1	1
χ_8	2	0	0	2	2	0	2	-1	-1	-1	2	0	2	2	0	0	2	-1	-1	-1
χ_9	2	0	0	-1	2	0	2	2	-1	-1	1	0	-2	-2	0	0	-2	1	-2	1
χ_{10}	2	0	0	-1	2	0	2	-1	-1	2	1	0	-2	-2	0	0	-2	1	1	-2
χ_{11}	2	0	0	-1	2	0	2	2	-1	-1	-1	0	2	2	0	0	2	-1	2	-1
χ_{12}	2	0	0	-1	2	0	2	-1	-1	2	-1	0	2	2	0	0	2	-1	-1	2
χ_{13}	3	-1	$-\zeta_3^2$	0	$3\zeta_3^2$	$-\zeta_3$	$3\zeta_3$	0	0	0	0	$-\zeta_3$	$3\zeta_3$	3	-1	$-\zeta_3^2$	$3\zeta_3^2$	0	0	0
χ_{14}	3	-1	$-\zeta_3$	0	$3\zeta_3$	$-\zeta_3^2$	$3\zeta_3^2$	0	0	0	0	$-\zeta_3^2$	$3\zeta_3^2$	3	-1	$-\zeta_3$	$3\zeta_3$	0	0	0
χ_{15}	3	1	ζ_3^2	0	$3\zeta_3^2$	ζ_3	$3\zeta_3$	0	0	0	0	$-\zeta_3$	$-3\zeta_3$	-3	-1	$-\zeta_3^2$	$-3\zeta_3^2$	0	0	0
χ_{16}	3	1	ζ_3	0	$3\zeta_3$	ζ_3^2	$3\zeta_3^2$	0	0	0	0	$-\zeta_3^2$	$-3\zeta_3^2$	-3	-1	$-\zeta_3$	$-3\zeta_3$	0	0	0
χ_{17}	3	-1	$-\zeta_3^2$	0	$3\zeta_3^2$	$-\zeta_3$	$3\zeta_3$	0	0	0	0	ζ_3	$-3\zeta_3$	-3	1	ζ_3^2	$-3\zeta_3^2$	0	0	0
χ_{18}	3	-1	$-\zeta_3$	0	$3\zeta_3$	$-\zeta_3^2$	$3\zeta_3^2$	0	0	0	0	ζ_3^2	$-3\zeta_3^2$	-3	1	ζ_3	$-3\zeta_3$	0	0	0
χ_{19}	3	1	ζ_3^2	0	$3\zeta_3^2$	ζ_3	$3\zeta_3$	0	0	0	0	ζ_3	$3\zeta_3$	3	1	ζ_3^2	$3\zeta_3^2$	0	0	0
χ_{20}	3	1	ζ_3	0	$3\zeta_3$	ζ_3^2	$3\zeta_3^2$	0	0	0	0	ζ_3^2	$3\zeta_3^2$	3	1	ζ_3	$3\zeta_3$	0	0	0

Table III.21: Projective Charactertable of $C_{G^{(4)}}(2B)$

$[g]$	$1a$	$2a$	$3a$	$3b$	$2b$	$6a$	$6b$	$6c$	$4a$	$6d$	$4b$	$6e$	$6f$	$4c$	$2c$	$2d$	$2e$
χ_1	2	0	-1	1	0	1	-1	1	0	1	0	-1	1	-2	-2	2	2
χ_2	2	0	-1	1	0	-1	-1	1	0	-1	2	1	-1	0	-2	-2	-2
χ_3	2	0	-1	1	0	1	1	-1	-2	-1	0	1	1	0	2	2	-2
χ_4	2	0	$-\zeta_3$	ζ_3^2	0	ζ_3^2	$-\zeta_3$	ζ_3^2	0	ζ_3	0	$-\zeta_3^2$	ζ_3	-2	-2	2	2
χ_5	2	0	$-\zeta_3^2$	ζ_3	0	ζ_3	$-\zeta_3^2$	ζ_3	0	ζ_3^2	0	$-\zeta_3$	ζ_3^2	-2	-2	2	2
χ_6	2	0	$-\zeta_3$	ζ_3^2	0	$-\zeta_3^2$	$-\zeta_3$	ζ_3^2	0	$-\zeta_3$	2	ζ_3^2	$-\zeta_3$	0	-2	-2	-2
χ_7	2	0	$-\zeta_3^2$	ζ_3	0	$-\zeta_3$	$-\zeta_3^2$	ζ_3	0	$-\zeta_3^2$	2	ζ_3	$-\zeta_3^2$	0	-2	-2	-2
χ_8	2	0	$-\zeta_3$	ζ_3^2	0	ζ_3^2	ζ_3	$-\zeta_3^2$	-2	$-\zeta_3$	0	ζ_3^2	ζ_3	0	2	2	-2
χ_9	2	0	$-\zeta_3^2$	ζ_3	0	ζ_3	ζ_3^2	$-\zeta_3$	-2	$-\zeta_3^2$	0	ζ_3	ζ_3^2	0	2	2	-2
χ_{10}	4	0	1	-1	0	1	-1	1	0	-1	0	1	1	0	4	-4	4
χ_{11}	4	0	ζ_3	$-\zeta_3^2$	0	ζ_3^2	$-\zeta_3$	ζ_3^2	0	$-\zeta_3$	0	ζ_3^2	ζ_3	0	4	-4	4
χ_{12}	4	0	ζ_3^2	$-\zeta_3$	0	ζ_3	$-\zeta_3^2$	ζ_3	0	$-\zeta_3^2$	0	ζ_3	ζ_3^2	0	4	-4	4
χ_{13}	6	0	0	0	0	0	0	0	0	0	0	0	0	2	-6	6	6
χ_{14}	6	0	0	0	0	0	0	0	0	0	-2	0	0	0	-6	-6	-6
χ_{15}	6	0	0	0	0	0	0	0	2	0	0	0	0	0	6	6	-6

Table III.22: Projective Charactertable of $C_{G^{(4)}}(2C)$

$[g]$	$1a$	$2a$	$2b$	$2c$	$4a$	$2d$	$2e$	$2f$	$2g$	$4b$	$4c$	$2h$	$2i$	$2j$
χ_1	2	0	2	0	0	0	0	0	0	0	2	−2	0	−2
χ_2	2	0	2	0	0	0	0	0	0	0	−2	−2	0	−2
χ_3	2	0	−2	−2	0	0	0	0	0	0	0	2	0	−2
χ_4	2	0	2	0	0	0	0	0	0	−2	0	2	0	2
χ_5	2	0	2	0	0	0	0	0	0	2	0	2	0	2
χ_6	2	0	−2	2	0	0	0	0	0	0	0	2	0	−2
χ_7	2	0	−2	0	$-2i$	0	0	0	0	0	0	−2	0	2
χ_8	2	0	−2	0	$2i$	0	0	0	0	0	0	−2	0	2

Table III.23: Projective Charactertable of $C_{G^{(4)}}(4A)$

$[g]$	$1a$	$2a$	$2b$	$4a$	$4b$	$4c$	$8a$	$4d$	$8b$	$4e$	$4f$	$2c$	$4g$	$4h$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	1
χ_3	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1
χ_4	1	-1	1	-1	1	1	-1	1	-1	1	1	1	1	1
χ_5	1	-1	-1	1	1	$-i$	i	i	$-i$	$-i$	-1	1	i	-1
χ_6	1	-1	-1	1	1	i	$-i$	$-i$	i	i	-1	1	$-i$	-1
χ_7	1	1	-1	-1	1	$-i$	$-i$	i	i	$-i$	-1	1	i	-1
χ_8	1	1	-1	-1	1	i	i	$-i$	$-i$	i	-1	1	$-i$	-1
χ_9	2	0	2	0	-2	0	0	0	0	0	-2	2	0	-2
χ_{10}	2	0	-2	0	-2	0	0	0	0	0	2	2	0	2
χ_{11}	2	0	0	0	0	$-1-i$	0	$-1+i$	0	$1+i$	$2i$	-2	$1-i$	$-2i$
χ_{12}	2	0	0	0	0	$-1+i$	0	$-1-i$	0	$1-i$	$-2i$	-2	$1+i$	$2i$
χ_{13}	2	0	0	0	0	$1-i$	0	$1+i$	0	$-1+i$	$-2i$	-2	$-1-i$	$2i$
χ_{14}	2	0	0	0	0	$1+i$	0	$1-i$	0	$-1-i$	$2i$	-2	$-1+i$	$-2i$

Table III.24: Projective Charactertable of $C_{G^{(5)}}(2A)$

$[g]$	$1a$	$2a$	$4a$	$4b$	$3a$	$6a$	$2b$	$2c$	$4c$	$4d$	$12a$	$12b$	$5a$	$10a$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1
χ_3	1	-1	$-i$	i	1	-1	1	-1	$-i$	i	$-i$	i	1	-1
χ_4	1	-1	i	$-i$	1	-1	1	-1	i	$-i$	i	$-i$	1	-1
χ_5	4	4	-2	-2	1	1	0	0	0	0	1	1	-1	-1
χ_6	4	4	2	2	1	1	0	0	0	0	-1	-1	-1	-1
χ_7	4	-4	$-2i$	$2i$	1	-1	0	0	0	0	i	$-i$	-1	1
χ_8	4	-4	$2i$	$-2i$	1	-1	0	0	0	0	$-i$	i	-1	1
χ_9	5	5	-1	-1	-1	-1	1	1	1	1	-1	-1	0	0
χ_{10}	5	5	1	1	-1	-1	1	1	-1	-1	1	1	0	0
χ_{11}	5	-5	$-i$	i	-1	1	1	-1	i	$-i$	$-i$	i	0	0
χ_{12}	5	-5	i	$-i$	-1	1	1	-1	$-i$	i	i	$-i$	0	0
χ_{13}	6	6	0	0	0	0	-2	-2	0	0	0	0	1	1
χ_{14}	6	-6	0	0	0	0	-2	2	0	0	0	0	1	-1

Table III.25: Projective Charactertable of $C_{G^{(5)}}(2B)$

$[g]$	$1a$	$2a$	$4a$	$4b$	$2b$	$2c$	$2d$	$2e$	$4c$	$4d$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	-1	1	1	-1	-1	1	1
χ_3	1	1	-1	-1	1	1	1	1	-1	-1
χ_4	1	1	1	1	1	1	-1	-1	-1	-1
χ_5	1	-1	i	$-i$	1	-1	-1	1	$-i$	i
χ_6	1	-1	$-i$	i	1	-1	-1	1	i	$-i$
χ_7	1	-1	i	$-i$	1	-1	1	-1	i	$-i$
χ_8	1	-1	$-i$	i	1	-1	1	-1	$-i$	i
χ_9	2	2	0	0	-2	-2	0	0	0	0
χ_{10}	2	-2	0	0	-2	2	0	0	0	0

Table III.26: Projective Charactertable of $C_{G^{(5)}}(4A)$

$[g]$	1a	2a	4a	4b	4c	4d	2b	2c	3a	6a	12a	12b
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1
χ_3	1	1	1	1	−1	−1	−1	−1	1	1	1	1
χ_4	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1
χ_5	1	−1	$-i$	i	i	$-i$	−1	1	1	−1	$-i$	i
χ_6	1	−1	i	$-i$	$-i$	i	−1	1	1	−1	i	$-i$
χ_7	1	−1	$-i$	i	$-i$	i	1	−1	1	−1	$-i$	i
χ_8	1	−1	i	$-i$	i	$-i$	1	−1	1	−1	i	$-i$
χ_9	2	2	−2	−2	0	0	0	0	−1	−1	1	1
χ_{10}	2	2	2	2	0	0	0	0	−1	−1	−1	−1
χ_{11}	2	−2	$-2i$	$2i$	0	0	0	0	−1	1	i	$-i$
χ_{12}	2	−2	$2i$	$-2i$	0	0	0	0	−1	1	$-i$	i

IV

The Dark Side of the Moon

1 Introduction

The account of the microscopic degrees of freedom of a black hole is one of the milestone successes of string theory. Since the first report [SV96], the microstates of many types of black holes have been understood in terms of degrees of freedom of string theory. In [SV96], the Bekenstein-Hawking entropy of a five dimensional supersymmetric black hole solution is computed. This black hole solution is realized as a near horizon geometry of a stack of D-branes, the theory of which has an effective description in terms of a 2-dimensional Conformal Field Theory. One can associate a microscopic entropy to this effective theory with the Cardy formula [Car86], and it is shown that this microscopic entropy correctly reproduces the Bekenstein Hawking entropy of the black hole geometry, at least to leading order in the charges.

1.1 The BTZ black hole and the Hawking-Page phase transition

In three dimensions, it has been noted that the Bekenstein-Hawking entropy of the BTZ black hole has the form of a Cardy formula as well [**?**]. The BTZ black hole is a solution to three-dimensional Einstein(-Maxwell) gravity with positive cosmological constant $\Lambda = -1/\ell^2$ with ℓ the AdS radius. The solution reads

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2} dt^2 + \frac{r^2 \ell^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2 \left(d\phi + \frac{r_+ r_-}{r^2} \right) \quad (1)$$

where the mass and the angular momentum read

$$M = \frac{r_+^2 + r_-^2}{\ell^2}, \quad J = \frac{2 r_+ r_-}{\ell}. \quad (2)$$

To this solution we can associate a Bekenstein-Hawking entropy

$$S_{BTZ} = \frac{A}{4G_{(3)}} = \pi \sqrt{\frac{\ell(\ell M + J)}{2G_{(3)}}} + \pi \sqrt{\frac{\ell(\ell M - J)}{2G_{(3)}}}. \quad (3)$$

This expression remarkably is of the form of the Cardy formula if we make the association of the central charge in formula (**??**) and further identify

$$M = \frac{1}{\ell}(L_0 + \bar{L}_0) - \frac{c}{12} \tag{4}$$

$$J = L_0 - \bar{L}_0 \tag{5}$$

Then, indeed, if we for now put $J = 0$ and denote by Δ the eigenvalue of L_0 (note that $L_0 = \bar{L}_0$), expression (3) reads

$$S_{BTZ} = 4\pi\sqrt{\frac{c}{24}\left(\Delta - \frac{c}{24}\right)} \tag{6}$$

in agreement with the Cardy formula (9) for $\Delta \gg 1$.
In the canonical ensemble, pure gravity in AdS_3 has two phases — the BTZ and the thermal, global AdS phase — as competing saddle points. In a chiral theory, the competing free energies read

$$\log Z_{BTZ} = \frac{\pi^2 c}{6\beta} \tag{7}$$

$$\log Z_{EAdS} = \frac{c}{24}\beta \tag{8}$$

where $EAdS$ is an abbreviation for Euclidean AdS space, to remind us we're in a thermal ensemble. The theory picks the leading saddle point so to leading order the gravitational free energy read

$$\log Z = \max\{\log Z_{BTZ}, \log Z_{EAdS}\}. \tag{9}$$

and there's a Hawking-Page when $\log Z_{BTZ} = \log Z_{EAdS}$, that is at

$$\beta = 2\pi. \tag{10}$$

Note that the two free energies of BTZ and EAdS are related by the S-transformation

$$S : \beta \to \frac{4\pi^2}{\beta}. \tag{11}$$

This S transformation is part of a larger $SL_2(\mathbf{Z})$ structure we find in three dimensional AdS gravity [MS98, DMMV00]. In fact if we introduce the modular variable $\tau = \frac{i\beta}{2\pi}$, there is, for every $\gamma \in SL_2(\mathbf{Z})$ a solution to pure three dimensional gravity in AdS, and we can compute the free energy as

$$\log Z_\gamma = -\frac{i\pi c}{12}\gamma(\tau), \quad \gamma(\tau) = \frac{a\tau + b}{c\tau + d}. \tag{12}$$

Including all saddles of this sort gives a rich phase-space structure [MW10]. However, we will be interested in purely imaginary potentials τ, and in that case only the BTZ and $EAdS$ contribute to the partition function.

The fact that the BTZ black hole has a Cardy formula is surprising for the following reason. The derivation of the Cardy formula, as worked out in the preface (), holds for a fixed central charge c and $\Delta \to \infty$ (that is, for very high temperature $\beta \to 0$). However, the Bekenstein-Hawking entropy of the BTZ black hole we trust only in the typical semiclassical limit

$$c \to \infty, \quad \Delta \gtrapprox c \tag{13}$$

We would like to understand why and under what circumstances the entropy of a (holographic) CFT obeys the Cardy formula in this different, semicalssical regime. In [HKS14], this issue was cleared up in a rather general context of CFTs that have holographic properties. We find it instructive to derive the validity of Cardy's formula in the holographic context in a different way. We find that it is instructive to perform the calculation of [Car86] in the grand canonical ensemble, introducing a chemical potential conjugate to the central charge.

Working in the grand canonical ensemble, we investigate the phase space structure, not only as a function of the temperature, but now also as a function of the chemical potential. We find that indeed turning on the chemical potential conjugate to the central charge enriches the phase space, and we find a phase transition a la Hawking and Page, now not just at a critical temperature, but also at a critical chemical potential.

In what follows we will focus on CFTs in two dimensions that have the following characteristics of a holographic field theory: a large N limit and a low lying sparse spectrum. In particular, we will look at symmetric product orbifolds of some seed CFT. For the sake of computation we will mainly be interested in the extremal monster CFT for the seed theory, but some conclusions extend straightforwardly to general symmetric product CFTs, and apply for example to the D1-D5 CFT at the orbifold point. Although it has the named characteristics of a holographic field theory, it is at this point not known if there is a gravitational dual to the symmetric product of the monster theory, nor is there an explicit embedding in string theory known at this point. We will however present evidence that hints towards a higher spin dual.

1.2 The monster CFT and its symmetric products

The monster CFT is a conformal field theory of central charge $c = 24$, first introduced by Frenkel et al., see [FLM88]. The CFT has, per construction, as a partition function the j-invariant

$$Z(q) = j(q) - 744 = J(q) = \sum_{n \geq -1} c(n) q^n = q^{-1} + 196884q + \mathcal{O}(q^2), \tag{14}$$

where we use shifted $J(q) = j(q) - 744$, compared with the more conventional definition of the j-invariant $j(q)$. The j-invariant is a very rich mathematical objects, encountered in the study of modular forms and it is the very j-invariant that has the monstrous moonshine property: its Fourier coefficients are also the dimensions of a particular module of the monster group (the so called monster module, $V^{\#}$). It is because of this that the CFT is called the monster CFT, and that the CFT was constructed in the first place. In [FLM88], the theory was realized as free bosons on $\mathbf{R}^{24}/\mathbf{Z}_2$ modulo the Leech lattice. More details are presented in chapter (III) of this thesis.

The monster theory itself is probably not a complete holographic field theory. There is no dimensionless quantity N that we can take to be very large (as $c = 24$ can hardly be considered large). What we can do however is take the $c = 24$ theory, and take the symmetric product orbifold CFT. Now, the N^{th} symmetric power of the CFT, $(\mathrm{CFT})^N/S_N$, will have central charge $c = 24N$, and we can take $N \gg 1$. This procedure to obtain a putative holographic theory in the context of AdS_3/CFT_2 is inspired by the holographic framework in the context of the D1/D5 CFT (see e.g. [Mal99, AGM$^+$00]). The symmetric product orbifold of a seed CFT with primary operator content $|\Phi_i\rangle$ takes the N-fold tensor product of that CFT but projects out the states that are related by a permutation element $\sigma \in S_N$. This projects out a lot of states (of the order of $N!$) but it includes a new sector of twisted states — states that pick up a monodromy in S_N. See [DHVW85, DHVW86, DVVV89] for more details on the construction.

Due to an identity by Dijkgraaf et al. [DMVV97, Dij98] and Keller [Kel11], there is an expression for the generating function of the partition function (16) of the N^{th} symmetric orbifold:

$$\mathcal{Z}(p,q) = p \prod_{\substack{n\geq 1 \\ m\geq -1}} (1-p^n q^m)^{-c(nm)}. \tag{15}$$

For later use, the notation is such that $q = \exp(2\pi i\tau)$, $p = \exp(2\pi i\rho)$ where $\tau = i\beta+\theta$ and $\rho = i\mu+\phi$. We will mostly be interested in the case $\tau = i\beta, \rho = i\mu$. Also, note that we shifted N by one, to make the formulas more symmetric.

2 The Cardy formula in the grand canonical ensemble

The spectrum of field theories that, in some limit, admit a holographic dual, can typically be divided into two parts: the perturbative states (sometimes called "light states") on the one hand, and the black hole states on the other. The sense in which the perturbative states are called "light" is that normally for a holographic theory we need to take some dimensionless quantity, say N, very large in order for quantum gravitational effects to be suppressed on the bulk side of the duality. The energy of the (primary) states that describe the black holes scales proportionally with N, whereas the energy of the "light" states does not. The energy of the black hole primaries grows parametrically faster than the energy of the light states.

We can single out these primary operators and consider the partition function $Z_{pert.}(q)$ and $Z_{bh}(q;N)$. The function $Z_{pert}(q)$ is the partition function of perturbative states. The partition sum $Z_{bh}(q;N)$ denotes the black hole partition sum and encompasses the primary operators that have a conformal dimension that scales with N.

It is instructive but not always easy to compute the function $Z_{pert}(q)$. We can hope to address questions about the ground state through $Z_{pert}(q)$. But also, investigating its behavior at large energies could tell us something about the nature of the perturbative states: are they higher spin states (with Hagedorn spectrum), is it a free field theory, etc.?

We will be considering such holographic theories that are labelled by an integer N. We will introduce a chemical potential μ, conjugate to N. The reason for this is two-fold. Firstly, we will show that the introduction of this potential elucidates the validity of Cardy's formula outside of the conventional Cardy regime in the realm of holographic field theories. Also, we will be interested in taking this potential more serious and investigate the phase-state structure as a function of μ of the theory in the grand canonical ensemble.

Although we have reason to believe our results are more general we will focus on the extremal monster CFT sitting at $c = 24$. For this theory we will introduce the grand partition function

$$\mathcal{Z}(p,q) = \sum_N p^{N+1} Z_N(q) = \sum_{N,M} D(N,M) p^N q^M. \tag{16}$$

The aim of the first part of this chapter will be to show the following properties of the degeneracies $D(N,M)$:

$$D(N,M) = \begin{cases} \exp 4\pi\sqrt{NM} & \text{if } M \geq N \\ \exp 2\pi(M+N) & \text{if } M < N \end{cases} \tag{17}$$

in the limit where $N \to \infty$. Further still we will show that for polar states $M < 0$, the degeneracy of states is universal in that is does not explicitly depends on the central charge but only on $\Delta = N + M$:

$$D(N,M) = D_\infty(\Delta) \qquad\qquad \text{if } M < 0 \tag{18}$$

That for M above some scale, the spectrum behaves like a Cardy ensemble, and for low M is independent of the central charge is in agreement with the expectation of a putative gravitational dual description. In the intermediate regime the perturbative states grow according to a Hagedorn spectrum, and this is not per se generic. We do however encounter such behavior in the context of $\mathcal{N} = 4$ SYM and holographic higher spin theories such as the $\mathcal{W}_N$ and hs[λ] theories [KP02, HR10, GG11, GG14]. We will actually be able to extract the perturbative part of the spectrum exactly and will show that

$$Z_{pert.}(q) = \frac{1}{q^2 J'(q)} = \prod_\delta (1-q^\delta)^{-d(\delta)}, \quad d(\delta) = \sum_{n+m=\delta} c(nm) \tag{19}$$

In the final part of the chapter we will interpret the chemical potential μ that is dual to the central charge N more physically and investigate the phase diagram in the extended (β, μ) plane.

2.1 Symmetries and poles

The function $\mathcal{Z}(p,q)$ is a function of τ and ρ and is invariant under shifts $\tau \mapsto \tau+1$ and $\rho \mapsto \rho+1$. Also, in the regime $|p|<1, |q|<1$ we may expand $\mathcal{Z}(p,q)$ in a Fourier series

$$\mathcal{Z} = \sum_{\substack{N\geq 0\\ M\geq -N}} D(N,M) p^{N+1} q^{M}. \tag{20}$$

We may recover the Fourier coefficients $D(N,M)$ by means of an inverse Laplace transform

$$D(N,M) = \oint_C dp\, dq\; p^{-N-2} q^{-M-1} \mathcal{Z}(p,q) \tag{21}$$

The integrand is meromorphic so the coefficients $D(N,M)$ can be computed by Cauchy's theorem as the residues at the various poles. Note that the value $D(N,M)$ not only depends on the integrand but also on the contours C in the ρ and τ plane.

To compute the value $D(N,M)$ we need to understand to pole structure of $\mathcal{Z}(p,q)$. To this end reconsider the product identity in (15). From the product identify we understand that there is a pole in $\mathcal{Z}(p,q)$ for all values such that $p^n q^m = 1$, for integers n, m. Recall that the Fourier-expansion (20) is valid in the $|p|<1\,|q|<1$ regime. We hence need $p=q$ and $nm=-1$ so that only $n=1, m=-1$ factor in the product formula gives a simple pole at $p=q$. But the $\mathrm{SL}_2(\mathbf{Z})$ that sends $q \mapsto q_\gamma$ where

$$q_\gamma = \exp\left(2\pi i \gamma(\tau)\right), \qquad \gamma(\tau) = \frac{a\tau+b}{c\tau+d} \tag{22}$$

maps the pole at $p=q$ to any modular image: $p=q_\gamma$. So if we restrict both τ and ρ to their fundamental domains, there is only one simple pole at $p=q$. Also $\mathcal{Z}(p,q)$ is zero only if $\tau \to i\infty$ and $\rho \to i\infty$. From this we can read extract an algebraic closed form of $\mathcal{Z}(p,q)$. Namely, $J(q)$ — the Klein invariant — has one simple pole at $q=0$ and a zero at the cusp $\tau = i\infty$. This together with the pole structure dictates that the partition function $\mathcal{Z}(p,q)$ should upto normalization be

$$\mathcal{Z}(p,q) = \frac{1}{J(p)-J(q)}. \tag{23}$$

This reflects the anti-symmetry, $\mathcal{Z}(p,q) = -\mathcal{Z}(q,p)$. At high energy, the function $\mathcal{Z}(p,q)_>$ reads

$$\mathcal{Z}_>(p,q) = \frac{p^{-1}-q^{-1}}{J(p)-J(q)} \tag{24}$$

and this function has modular invariance in both p and q as well as an exact $\mathbf{Z}_2$ symmetry exchanging p and q. The full automorphism group hence reads

$$G = \mathrm{SL}_2(\mathbf{Z}) \times \mathrm{SL}_2(\mathbf{Z}) \rtimes \mathbf{Z}_2 \simeq \mathrm{O}(2,2;\mathbf{Z}). \tag{25}$$

2.2 The spectrum at high energy: excited states

In the limit of very high energy, $M \geq N$ and large N, we expect the ground state to play a subleading role in computing the degeneracies of the primary operators of the CFT. To compute the high energy degeneracy of states of the theory, we will first split off the ground states of the theory. Observe that for each N, the ground state contribution is q^{-N}, so we can write down the geometric series that is the generating series of all the ground state contributions:

$$\mathcal{Z}_0(p,q) = \sum_{N=1}^{\infty} p^{N+1} q^{-N} = \frac{1}{p^{-1} - q^{-1}} \tag{26}$$

A comment on the radius of convergence is in order. Note that this expression only strictly holds in the regime where $|p/q| < 1$ hence, $\beta < \mu$. If we cross the line $\beta = \mu$ and end up in the regime $\beta > \mu$ we could do an expansion in p^{-N}. This is however an expansion in theories with negative central charge, and this regime is note stable. See section (3) for more about this and a relation with the wall-crossing phenomenon. We single out this ground state contribution by introducing

$$\mathcal{Z}_>(p,q) = \frac{\mathcal{Z}(p,q)}{\mathcal{Z}_0(p,q)} \tag{27}$$

We will start with computing the growth of the degeneracies of $\mathcal{Z}_>(p,q)$, the sum with the ground states factored out. We write

$$\mathcal{Z}_>(p,q) = \sum_{N,M} \mathcal{D}(N,M) p^N q^M = \prod_{n,m\geq 1} (1 - p^n q^m)^{-c(nm)}. \tag{28}$$

Note that this expression is symmetric in p and q and this symmetry implies

$$\mathcal{D}(N,M) = \mathcal{D}(M,N) \tag{29}$$

so that:

$$\mathcal{D}(N,M) = \oint_C d\tau d\rho\, e^{-2\pi i(\rho N + \tau M)} \mathcal{Z}_>(\rho,\tau) \tag{30}$$

where C is a contour around the poles of $\mathcal{Z}_>(\rho,\tau)$. The poles of $\mathcal{Z}_>(\rho,\tau)$ are at all images $\rho = \gamma(\tau)$, $\gamma \in \mathrm{SL}_2(\mathbf{Z})$, but $\gamma \neq \mathrm{id}$. So indeed,

$$\mathcal{D}(N,M) = \oint_C d\tau \sum_{\gamma\neq\mathrm{id}} \mathrm{Res}\big|_{\rho=\gamma(\tau)} \left(e^{-2\pi i(\rho N+\tau M)} \mathcal{Z}_>(\rho,\tau)\right). \tag{31}$$

These poles are at $(\rho - a/c)(\tau + d/c) = -1/c^2$, a condition that is extremized at $\rho\tau = -1$ or, now taking $\tau = i\beta$, $\rho = i\mu$, at $\beta = 1/\mu$. We approximate the sum over the residues by only evaluating the sum at this pole, and to the saddle point approximation. The saddle is at $N/M = \beta^2$. The degeneracies are thus, after performing the integral over μ:

$$\mathcal{D}(N,M) = \int d\beta e^{2\pi(N/\beta + M\beta)} \mathcal{Z}_>(1/\beta, \beta). \tag{32}$$

Now the final point is that, in the Cardy limit, we can evaluate the integral at the saddle and we get

$$\mathcal{D}(N,M) = \exp 4\pi\sqrt{NM} + \dots \tag{33}$$

where the Cardy limit requires $M \geq N$ and N large.

2.3 Including the ground states

For moderate energies $M < N$ we expect the ground states to start playing a role. We will now analyze the full theory, with the ground states included. The full partition function, including the ground states, can be written as $\mathcal{Z} = \mathcal{Z}_0 \times \mathcal{Z}_>$ and we expand as in equation (20). We again compute the degeneracies $D(N,M)$ with the contour integral

$$D(N,M) = \oint_C dpdq\, p^{-N-2} q^{-M-1} \mathcal{Z}_0(p,q) \mathcal{Z}_>(p,q). \tag{34}$$

Plugging in the expansion $\mathcal{Z}_0$ from equation (35), it is not hard to write down the degeneracies $D(N,M)$ in terms of the coefficients $\mathcal{D}(N,M)$ as

$$D(N,M) = \sum_{k=0}^{N} \mathcal{D}(N-k, M+k). \tag{35}$$

From the analysis in the previous section we know that in the large-N limit we may for large M write the $\mathcal{D}(N,M) \simeq \exp\left(4\pi\sqrt{NM}\right)$. In this same limit we may estimate te sum as an integral to obtain

$$D(N,M) \simeq \int_{k=0}^{N} dk\, e^{2\pi\sqrt{(N-k)(M+k)}}. \tag{36}$$

and here still in the large N limit we may, as long as $M < N$, approximate this integral at its saddle point that is at $k = \frac{N-M}{2}$, obtaining

$$\mathcal{D}(N,M) \simeq e^{2\pi(N+M)} = e^{2\pi\Delta}. \tag{37}$$

So indeed, in the regime $M < N$, the degeneracies grow like a Hagedorn spectrum. Note that $N + M = \Delta$ is the conformal dimension of a primary operator

at central charge N. Also, in (37), we see that for $M \geq N$ the leading order contribution comes now from the $k = 0$ saddle and the Cardy regime is found, that is, $D(N, M) \simeq \exp(4\pi NM)$.
In principle, theories that develop a Hagedorn spectrum look problematic. The idea is that for a general theory with a partition function

$$Z(\beta) = \sum_{\Delta} \rho(\Delta) e^{-\beta\Delta}, \tag{38}$$

with a density of states $\rho(\Delta)$ that is exponential, e.g. $\rho(\Delta) = e^{2\pi\Delta}$, the function $Z(\beta)$ may develop, as β approaches 2π, a pole as then

$$Z(\beta) \underset{\Delta\to\infty}{\longrightarrow} \frac{1}{2\pi - \beta}, \tag{39}$$

and this leads to a singularity as $\beta \to 2\pi$. This temperature is called the Hagedorn temperature, $\beta_H = 2\pi$. Theories with resonance modes such as QCD and string theory have such a Hagedorn temperature. In string theory, this singularity is resolved by the creating of black holes. It is at the Hagedorn temperature that in the context of string theory on $AdS_5 \times S^5$, the Hagedorn temperature describes the Hawking-Page phase transition. It is argued that this transition can holographically, in the dual $\mathcal{N} = 4$ theory, can be interpreted as deconfinement [Wit98b, AMM+04].

In the theory we consider, there is a similar principle at play. It is namely below the Hagedorn temperature β_H, that the spectrum is dominated by the perturbative states that have a density of states that grows like $\log \rho \sim N + M$, whereas at and above β_H these particles disappear and are replaced by the states that grow like Cardy's formula, $\log \rho \sim \sqrt{NM}$. Tentatively — in the absence of a rigid holographic dual — calling the former the light states and the latter the BTZ states, we can also here identify the Hagedorn phase transition with a Hawking-Page phase transition.

2.4 Perturbative states and the Hagedorn temperature

So far we focused on the working in the microcanonical ensemble: we analyzed the microscopic degeneracies $D(N, M)$ and investigated their behavior in different regimes. Here we will work in the canonical ensemble, and will choose to work with variables (β, N) and (μ, M) to have more to say about the phases of the theory in different regimes of the parameters.

The amazing formula (15) allows us to compute the full partition function of the N^{th} orbifold, as indeed now: $Z_N(q) = \oint_C \mathcal{Z}(p, q) p^{-N-1} dp$, where C is a contour in the $p-$plane enclosing the poles of $\mathcal{Z}(p, q^*)$ at a fixed $q = q^*$. Knowing where the poles are, we can compute the residues at these poles and using Cauchy's theorem, we can compute $Z_N(q)$.

For a fixed q, there is of course a pole at $p = q$. But as $J(q) = J(q_\gamma)$, where $q_\gamma = \gamma \cdot q$, $\gamma \in \mathrm{SL}_2(\mathbf{Z})$, there is a pole at every $p = q_\gamma$, so we have to sum over all the residues at q_γ to obtain Z_N:

$$Z_N(q) = \sum_{\gamma \in \mathrm{SL}_2/\Gamma_\infty} \mathrm{Res}|_{p=q_\gamma} \mathcal{Z}(p,q) = \sum_{\gamma \in \mathrm{SL}_2/\Gamma_\infty} \frac{q_\gamma^{-N-1}}{J'(q_\gamma)} \tag{40}$$

where $q_\gamma = \gamma \cdot q = e^{2\pi i \frac{a\tau+b}{c\tau+d}}$.

Note that the modular properties of $J(q)$ and $J'(q)$ allow $Z_N(q)$ to be written as

$$Z_N(q) = \frac{1}{J'(q)} \sum_{\gamma \in \mathrm{SL}_2/\Gamma_\infty} \frac{q_\gamma^{-N-1}}{(c\tau+d)^2}. \tag{41}$$

The function

$$1/J'(q) = \sum_{n \geq -2}^{\infty} a_n q^n = \sum_{m \geq -1} m c(m) q^{m-1} = -q^{-2} + \dots \tag{42}$$

plays a special role, as it sits there for all values of N and does not itself depend on N. Recall from the first section, that the spectrum of a holographic theory can typically be divided into (at least) two parts: the light, perturbative states and the black hole states, and that we may introduce separate partition functions $Z_{pert.}(q)$ and $Z_{BH}(q;N)$ that encompass these states.

We want to identify $1/J'(q)$ with the partition function of perturbative states or, in the notation of [BKM15], $1/J'(q) = \rho_\infty(q)$. On the other hand, the factor

$$Z(q;N) = \sum_{\gamma \in \mathrm{SL}_2/\Gamma_\infty} \frac{q_\gamma^{-N-1}}{(c\tau+d)^2} \tag{43}$$

we want to identify with the partition function for BTZ black hole primary states, $Z(g;N) = Z_{BH}(q;N)$.

It is the partition sum $1/J'(q)$ that may tell us something about the nature of the field theory, as the partition function describes the degeneracy of the perturbative spectrum. We will begin by analyzing the behavior of the perturbative degeneracies at large energies.

Note first of all that we can express $1/J'(q)$ in terms of a product expression as follows. First, note that, again using Cauchy's theorem

$$1/J'(q) = \mathrm{Res}|_{p=q} \left(\mathcal{Z}(p,q)\right) \tag{44}$$

so that now we can use the product expression of $\mathcal{Z}(p,q)$ to evaluate this residue:

$$\frac{1}{q^2 J'(q)} = \prod_{n,m} (1-q^{n+m})^{-c(nm)} \tag{45}$$

This product may appear to be one over two indices, n, m, but in fact we can write it as a product over one index, say δ, in the following way. Let $n+m$, then define $d(\delta) = \sum_{k=1}^{\delta} c((\delta-k)k)$. Now:

$$1/J'(q) = q^2 \prod_{\delta=1}^{\infty} (1-q^{\delta})^{-d(\delta)}. \tag{46}$$

This product expression shows that the perturbative spectrum is built out of a gas of non-interacting particles, labelled by δ, each having a partition sum $Z_{\delta} = (1-q^{\delta})^{-d(\delta)}$, and $Z_{pert.}(q) = \prod_{\delta} Z_{\delta}$.

On the other hand, consider again the expression (35), and consider the case where $M < 0$. The coefficients, in the regime $M < 0$, read

$$D(N, M) = \sum_{k=0}^{N+M} \mathcal{D}(N+M-k, k) \equiv D_{\infty}(N+M), \tag{47}$$

so we find that the perturbative part of the spectrum comes from the polar, $M < 0$ states and we may write

$$\sum_{\Delta} D_{\infty}(\Delta) q^{\Delta} = \frac{1}{q^2 J'(q)}. \tag{48}$$

First of all, note that we again may estimate the coefficients D_{∞} by approximating the sum (47) as an integral:

$$D_{\infty}(\Delta) \simeq \int_0^{\Delta} dk e^{2\pi\sqrt{k(\Delta-k)}} \simeq e^{2\pi\Delta}, \tag{49}$$

where we evaluated teh integral, in the limit of large Δ — that is large N — at the saddle $k = \Delta/2$. As was noted in the previous section (35), it is not as problematic as it seems that the perturbative states grow like a Hagedorn density of states, as for $M > N$, that is at the Hagedorn temperature, these states are not in the spectrum and the density of states of the leading contribution starts growing a la Cardy.

3 Phase transitions in the grand canonical ensemble

In this section we will show that the perturbative states of the symmetric product have a Hagedorn growth. We will also investigate the phase space structure, by analyzing phase transitions in the grand canonical ensemble, that is look for Hawking Page phase transitions as a function of both μ and β.

3.1 Extended phase diagram

In the previous section we have seen that the in the symmetric product of the extremal monster CFT we can single out the partition sums $Z_{pert.}(q)$ and $Z_{BH}(N,q)$ and that $Z_{pert.}$ has a Hagedorn spectrum at large energies. Moreover, for finite values of N we have seen how the perturbative spectrum develops a singularity at $\tau = i$, a singularity that is resolved by the addition of black hole primaries in the large N limit. In this large N limit, a sharp phase transition occurs exactly at this Hagedorn temperature [Kel11]. This is the Hawking-Page phase transition. In the previous section we made use of the generating function $\mathcal{Z}(p,q)$, where we treated $p = \exp(2\pi i\rho)$ as a formal generating variable.

In this section we want to give a more physical interpretation to p and identify ρ with the chemical potential - the cost in energy to add or subtract a copy to the symmetric product. Introducing ρ as a chemical potential extends the phase diagram to the (ρ,τ) plane. We have seen that there is a phase transition point at $\tau = i$. We expect that this picture extends to a richer phase diagrams when ρ is turned on and we will show that this is the case indeed. Really, we will be working with purely imaginary potentials only, and set $i\beta = \tau$ and $i\mu = \rho$. Indeed, μ is the actual chemical potential — the conjugate variable to N — and β is the inverse temperature, $\beta = 1/T$. We will mainly be interested in the phase diagram (β,μ).

From the grand canonical partition function we see that a singularity develops at $\beta = \mu$ and at the line $\beta\mu = 1$. We will plot the following contour lines to get a more detailed picture

$$\langle N\rangle = -\partial_\mu \log \mathcal{Z}(\mu,\beta) = \text{constant}. \tag{50}$$

From the expression for the grand canonical partition function we get

$$\langle N\rangle = -1 - \frac{J'(p)}{J(p) - J(q)} \tag{51}$$

and in figure (IV.1) we draw the contour plot with lines of equal charge, as a function of μ and β.

Indeed we recognize the singular lines at $\mu = \beta$ and $\mu\beta = 1$.

4 Automorphic forms and $\mathcal{N} = 2$ BPS states

Naturally we may wonder how specific or general the discussion has been so far. We have been focussing on a theory that has a lot of symmetry and from this point of view it is not clear if the analyses can be straightforwardly extended to a more general set-up.

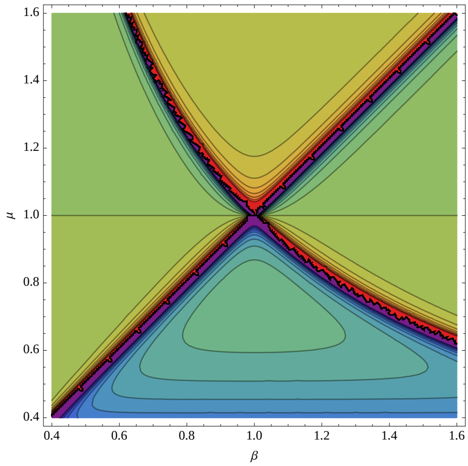


Figure IV.1: Extended phase space contour plot of the extremal monster CFT

Hoewever, at no point in the discussion we exploited the monster symmetry $\mathbb{M}$ — the global symmetry of the target-space. The symmetry we used in deriving the results on the degenracy of states and the phase-space structure was always a symmetry of the world-sheet, that is the $O(2,2;\mathbf{Z})$ symmetry acting on the world-sheet parameters (τ,ρ). This $O(2,2;\mathbf{Z})$ symmetry is more commong in string theory, and automorphism groups of that sort occur, for example, quite generally in the context of heterotic string compactifications. It is in the same context that product formulas like the one in equation (15) are encountered when analyzing the spectrum op BSP states.

In the work on stringy BPS states by Harvey and Moore [HM96], it was in fact shown that when computing certaing integrals in the context of threshold corrections in $\mathcal{N}=2$ heterotic string compactifications, the monstrous product formula (15) can be seen as a specific example in a more general set-up. In a one-loop computation on stringy BPS states, in [HM96] the following integrals are evaluated:

$$\mathcal{I}_{s+2,2} = \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{\mathrm{Im}(\tau)} \left[\left(\sum_{p\in\Gamma^{s+2,2}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} \right) f(q) - c(0) \right]. \tag{52}$$

In this notation, $f(q)$ is a modular function of weight $-s/2$ with expansion

$$f(q) = \sum_{n=-1}^{\infty} c(n) q^n \tag{53}$$

and $\mathcal{F}$ is the fundamental domain of $\mathrm{SL}_2(\mathbf{Z})$. These integrals can be seen as the counterparts of the one-loop computation on the pre-potential of $\mathcal{N}=2$ Yang-

Mills theory [SW94] for $\mathcal{N} = 2$ heterotic string compactifications with vetor matter.

For the case $s = 0$, we may take $f(q) = J(q)$, although in the reference the normalization $f(q) = q^{-1} - 240 + \mathcal{O}(q)$ is chosen. In the normalization on par with the one in this chapter, that is $f(q) = J(q)$, the integral $\mathcal{I}_{2,2}$ can be worked out to give

$$\mathcal{I}_{2,2}(\tau, \rho) = -4 \log \left[e^{-2\pi i \tau} \prod_{\substack{n>0 \\ m\geq -1}} \left(1 - e^{2\pi i (n\tau + m\rho)}\right)^{c(nm)} \right] \tag{54}$$

so that,

$$1/\mathcal{I}_{2,2} = \log\left(\mathcal{Z}(p,q)\right). \tag{55}$$

The formula for $\mathcal{I}_{s+2,2}$ generally have a product expansions, as was shown in Borcherd's work [Bor95] on automorphic forms on the group $O(s + 2, 2; \mathbf{R})$.

It would be interesting to generalize the work carried out in this chapter to the more general epressions for $\mathcal{I}_{s+2,2}$, for $s \neq 0$. In [HM96], explicit computations are done for the case $s = 8$ ($s = 0$ mod 8 is easier for technical reasons), where contact is made with automorphic forms on $O(10, 2; \mathbf{R})$. What prevents us from directly generalizing our results to the case $s \neq 0$ is the presence of a Weyl vector in front of the product expression for these situations [HM96, Bor95]. It would be interesting to understand how to compute the behavior of the degeneracies of (BPS) states and the Cardy regime in this more general set-up.

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Constributions to the Publications

An overview of the author's contributions to the publications upon which this dissertation is built.

Structure of Six-Dimensional Microstate Geometries
Paul de Lange, Daniel Mayerson, Bert Vercnocke
JHEP **09** 075 (2015) [dLMV15]:
The six-dimensional supersymmetric context, the six-dimensional (co-)homology and it's relevant (Komar) mass formulae. Conceptual discussions.

Symmetric Products and AdS_3/CFT_2 in the Grand Canonical Ensemble
Paul de lange, Alex Maloney, Erik Verlinde
to appear (2016) [dLMV16]:
The grand-canconical ensemble and the Cardy formula. Generalizations. Conceptual discussions.

Generalized Umbral Moonshine
Miranda Cheng, Paul de Lange, Daniel Whalen
to appear (2016) [CdLW16] :
The generalized umbral functions. Anlysis of the umbral groups and centralizers. Projective representations of the centralizers. Computer algebraic computations. Conceptual discussions.

Quantum Quivers and Melting Molecules
Dionysios Anninos, Tarek Anous, Paul de Lange, George Konstantinidis
JHEP **03** 066 (2015) [AAdLK15] :
Computation of thermal determinant. Numerical analysis in thermal state. Exact computations in the two-node quiver. Conceptual discussions.

Samenvatting

Het Einde is nabij. Stel. De wereld vergaat in rap tempo door om het even wat — een alles vernietigend virus, Krijt-Paleo komeet of atoomapocalyps — en, stel, je krijgt in dit hopelijk hypotetische scenario de kans om één boodschap achter te laten voor een toekomstige intelligente beschaving. Een zin, een idee waaruit de post-apocalyptische archeologen moeten leren hoe intelligent, complex en beschaafd wij wel niet waren. Welke zin, welk idee is rijk maar bondig genoeg om precies dit uit te drukken? Geconfronteerd met deze goddank denkbeeldige vraag opteerde de natuurkundinge Richard P. Feynman voor het idee dat

“ ... alles bestaat uit atomen — kleine deeltjes die constant rondvliegen, elkaar aantrekkend wanneer ze bij elkaar in de buurt komen, maar elkaar afstotend als ze op elkaar botsen.”

Hij merkte op dat

“ ... je zal merken dat die ene zin zo vol zit met informatie over onze wereld, wanneer we er slechts een beetje verbeeldingskracht en denkwerk op loslaten.”

De verbeeldingskracht van Feynman was op zijn minst opmerkelijk en zijn denkwerk bijzonder diep, maar het is waar dat de “atoomhypothese” bijzonder vruchtbaar is en de conceptuele consequenties rijk van tal. Het formuleert de fundamentele premisse van de scheikunde. Het leert ons waarom en hoe materie met elkaar reageert en nieuwe materie maakt. Het verklaart en klassificeert de verscheidenheid aan materie die we waarnemen in de Natuur. Het is door de atomen en moleculen dat we begrijpen waarom water bevriest en verdampt en waarom stoffen in het algemenen verschillende fases aannemen, en waarom gassen uitzetten en inkrimpen wanneer we de temperatuur veranderen. Zelfs het begrip temperatuur krijgt een veel concretere betekenis wanneer we de accepteren dat materie uit atomen bestaat.

Hoewel het al duizenden jaren rondzwerft in de filosofie, wordt het atoom pas in het begin van de negentiende eeuw opgewaardeerd naar de wetenschappelijke status wanneer Dalton het gebruikt om de “wet van meervoudige verhoudingen” (de observatie dat elementen typisch reageren in verhoudingen van kleine gehele getallen) uit te leggen, en de eerste periodieke tabel der elementen wordt in 1869 gepubliceerd. Dat dit zo laat in de ontwikkeling van de moderne wetenschap plaats vindt hoeft niet verassend te zijn: de lengteschaal van een generiek atoom is 10^{-9} meter, en op veel grotere lengteschalen — die van de centimeter en hoger — hebben we het atomaire concept niet vaak nodig om natuurlijke fenomenen nauwkeurig te beschrijven. De planetaire bewegingswetten van Kepler, de wetten van Newton en Bernouilli's vergelijkingen voor vloeistofdynamica staan los van de kennis van het atoom. Deze wetten beschrijven de natuur op lengteschalen veel groter dan die van het atoom, en het is pas wanneer we observaties doen op de kleinste schaal dat deze wetten onnauwkeurig worden. Andersom echter, kunnen we deze "grote" wetten begrijpen als komende van een onderliggende, microscopische beschrijving, en het is door gemiddelden te nemen over de wetten die het kleinste beschrijven, dat we de wetten van de lange lengteschaal terugvinden. Helaas is uitmiddelen een onomdraaibaar proces en geeft enkel kennis over deze gemiddelde — "effectieve" — wetten weinig tot geen inzicht in de microscopische achtergrond. We moeten ons wel beseffen dat we enkel blij mogen zijn dat zulke effectieve, gemiddelde wetten überhaubt bestaan. Om het stromen van water te beschrijven zouden we zeker niet alle watermoleculen in de gaten willen blijven houden. Een liter water bestaat uit meer dan 10^{25} H_2O moleculen.

Af en toe tonen deze "effectieve" wetten dat ze niet compleet zijn en dat er een fundamentelere wet achter moet zitten. Faseovergangen van vloeistoffen worden niet beschreven door de theorie van Bernouilli en kunnen alleen begrepen worden met een microscopische kijk. En het afwijkende gedrag van Mercurius rond het perihelion toont aan dat de wetten van Newton en Kepler niet compleet zijn: Einstein's theorie van de algemene zwaartekracht was nodig om dit te begrijpen.

We begrijpen nu dat materie uit atomen bestaat, en atomen op hun beurt uit quarks en electronen, maar de zwaartekracht is altijd erg getalenteerd geweest in het verbergen van haar eigen microscopische beschrijving. We weten echter dat er zo'n "atomaire" beschrijving van zwaartekracht moet bestaan, omdat er anders fundamentele wetten van de quantummechanica worden geschonden. Maar ook hier, net als bij vloeistofdynamica, geeft bij tijd en wijle zwaartekracht een hint naar haar eigen microscopische natuur.

Een bekende consequentie van Einstein's theorie voor de zwaartekracht zijn zwarte gaten: massieve, compacte objecten met een aantrekkingskracht die zo intens is dat zelfs het licht er niet van aan kan ontsnappen. Zwarte gaten zijn recentelijk de status van theoretisch concept gepasseerd en we hebben veel aanwijzingen dan Sagitarrius A*, in het midden van ons Melkwegstelsel, een zwart gat is.

Zwarte gaten zijn objecten van de grote lengteschaal. Ze zijn omringd door een grote "horizon" — een oppervlak waarachter er geen omkeren meer mogelijk is — die van de orde is van hun eigen massa. Dit beteket dat de straal van de horizon van Sagitarrius A* zo'n 20 maal de zonneradius bedraagt. Deze horizon maakt het nagenoeg onmogelijk om een zwart gat direct waar te nemen, en tot vandaag de dag weet niemand precies wat er achter deze obscure horizon precies plaatsvindt.

Een eerste stap richting de "atomaire" beschrijving van zwarte gaten werd gedaan door Stephen Hawking. Met een befaamde berekening liet hij zien dat zwarte gaten eigenlijk niet helemaal gitzwart zijn, maar een licht paarse gloed uitstralen aan de rand van hun horizon. Door deze straling kunnen we aan een zwart gat een temperatuur associeren. Het is alsof zwarte gaten door deze straling een heel klein beetje van hun informatie prijs geven aan waarnemers die er ver weg van staan. Toegegeven, deze straling is zo zwak en de "Hawking temperatuur" zo laag, dat deze vinding van een onpraktische, onwaarneembare aard is. Maar de conceptuele consequenties kunnen niet overschat worden.

In concreto is Hawking's conclusie dat de temperatuur van een zwart gat omgekeerd evenredig afhangt van haar massa: als we de massa vergroten met een factor twee dan verlaagt de temperatuur met een factor een half. Kleine zwarte gaten zijn warmer. Dit resultaat is erg universeel en hangt niet af van specifieke details van het zwarte gat. Dit betekent dat wanneer we een atomaire beschrijving opschrijven voor zwarte gaten en zwaartekracht, dat deze dan dit resultaat zal moeten reproduceren.

We kunnen deze situatie vergelijken met de meer mundane studie van vloeistoffen en gassen, waar het concept van een temepratuur imtiem is gelieerd aan een microscopische beschrijving van de stof. We kunnen intuitief nadenken over temperatuur als een maat voor hoe hard de moleculen op en neer bewegen in de ruimte. We kunnen dan, vergelijkbaar met de theorie van gassen, de Hawking temperatuur beschouwen als een eerste hint richting de atomen van een zwart gat. Tijdgenoten van Hawking konden al snel concluderen dat we met behulp van de temperatuur kunnen berekenen *hoeveel* van deze atomen er moeten zijn. De conclusie was opvallend. De intuitie van een natuurkundige zou hem doen vermoeden dat het aantal deeltjes dat nodig is om een systeem, in dit geval een zwart gat, te beschrijven af zou moeten hangen van het *volume* van dit systeem. Er werd echter opgemerkt dat het aantal deeltjes in het geval van een zwart gat enkel af hangt van het *oppervlak* van haar horizon. De uitdaging is dus om een microscopische, "quantum" theorie van de zwaartekracht te vinden die dit opmerkelijke resultaat reproduceert.

Dit is een dissertatie over snaartheorie. En snaartheorie is een quantumtheorie van zwaartekracht. Een van haar fundementele bouwstenen is de snaar — een klein object met een lengte en een spanning. Net als bij de snaar van een viool kunnen de snaren aangeslagen worden, en in snaartheorie is het een van deze aangeslagen toestanden die het gravition beschrijft: het miscroscopische deeltje van de zwaartekracht. Daar het een quantum theorie van de zwaartekracht is moet het Hawking's zwarte gaten "test" doorstaan: het moet het opmerkelijke resultaat reproduceren dat het aantal microscopische toestanden van een zwart gat enkel afhangt van de oppervlakte van haar horizon. Het is een van de mijlpalen van de theorie dat inderdaad voor specifieke zwarte gaten die we tegenkomen binnen de snaartheorie de voorspelling van Hakwing precies uitkomt. Op de dag dat deze thesis wordt afgedrukt is snaartheorie de enige quantumtheorie van zwaartekracht die dit resultaat kan claimen.

Snaartheorie beantwoord (nog) lang niet alle vragen over zwarte gaten en hun microscopische beschrijving. Snaartheorie is bijvoorbeeld nog niet erg succesvol geweest in het begrijpen van realistische zwarte in vier dimensies zoals Sagitarrius A*. Maar zelfs in de succesvolle scenario's blijven er nog veel vragen — van technische en conceptuele aard — onbeantwoord. Deze thesis probeert bij te dragen aan ons begrip van de microscopische aard van zwarte gaten vanuit een snaartheoretisch perspectief door enkele van deze technische en conceptele vragen op te lossen.

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