ON THE RIGOROUS DEFINITION OF SUPERFLUIDITY AND SUPERCONDUCTIVITY

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It is the purpose of this report to give a rigorous definition of superfluidity and to show that neither the off-diagonal long range order (ODLRO) nor the Bose-Einstein condensation can be a sufficient condition for superfluidity. The sufficient condition for superfluidity in a system possessing the ODLRO is also presented.

We start with the Hamiltonian

$$H = \sum_j \vec{p}_j^{\,2}/2m + V(\vec{r}_1, \cdots, \vec{r}_N)$$

together with the cyclic condition

$$V(\vec{r}_1, \cdots, \vec{r}_{j-1}, \vec{r}_j + \vec{L}, \vec{r}_{j+1}, \cdots, \vec{r}_N) = V(\vec{r}_1, \cdots, \vec{r}_j, \cdots, \vec{r}_N)$$

where $\vec{L} = (L, 0, 0)$ or $(0, L, 0)$ or $(0, 0, L)$. The impurity potential as well as the potential for interparticle forces are included in V. Suppose a uniform external field be switched on[1]. It exerts the force $E(t) = E\exp(\varepsilon t)$ $(\varepsilon > 0)$ on a particle along the positive direction of the x-axis. The field is described by the vector potential $\vec{F}(t) = (F(t), 0, 0)$ with $\partial_t F(t) = E(t)$. Then the Hamiltonian is $\mathcal{H} = \sum_j \frac{1}{2m}(\vec{p}_j + \vec{F}(t))^2 + V$. In the infinite past the system is supposed to be in the equilibrium state described by the canonical or the grand canonical density matrix $\rho \propto \exp\{-\beta(H-\mu N)\}$. The above situation represents the system contained in a slowly rotating torus. We follow the logic of Kubo's linear response theory[2]. The linear response current against the external field $F(t)$ is

$$\overline{J(t)} = \left\{ \frac{N}{m} - 2 \left\langle\!\!\left\langle J \frac{H^\times}{(H^\times)^2 + \varepsilon^2} J \right\rangle\!\!\right\rangle \right\} F(t)$$

where $J \equiv \sum_j p_j^x/m$, $H^\times \cdots = [H, \cdots]$, $\langle \cdots \rangle = \mathrm{Tr}\{\cdots\}$, and $\langle\!\langle \;\; \rangle\!\rangle$ denotes the average over impurity distributions. The presence of impurities is crucial in a consideration on superfluidity. The response current is expressed as $j(t) \equiv N^{-1}\overline{J(t)} \equiv j_1(t) + j_2(t)$, where $j_1(t) = \Lambda F(t)$ and $j_2(t) = \sigma_\varepsilon E(t)$ with

$$\Lambda \equiv \frac{1}{m} - \frac{2}{N}\left\langle\!\left\langle J \; \frac{Q}{H^{\times}} J \right\rangle\!\right\rangle \quad \text{and} \quad \sigma_{\varepsilon} \equiv \frac{2}{N}\left\langle\!\left\langle J\left(\frac{Q}{H^{\times}} \frac{\varepsilon}{(H^{\times})^2 + \varepsilon^2}\right)J\right\rangle\!\right\rangle .$$

In the above the projection operator Q is defined by

$$(QJ)_{nm} = J_{nm} \quad \text{for } E_n \neq E_m \quad \text{and} \quad 0 \quad \text{for } E_n = E_m .$$

It is only $j_2(t)$ that corresponds to the ordinary Ohmic current. In fact, if we take the thermodynamic limit (denoted by Lim) and then put $\varepsilon \to 0$, we obtain

$$\sigma = \mathrm{Lim}\,\frac{2}{N}\left\langle\!\left\langle J \; \frac{\delta(H^{\times})\; Q}{H^{\times}} \; J \right\rangle\!\right\rangle$$

which is a generalized version of the Greenwood formula[3] for the Ohmic conductivity. The other component $j_1(t)$ represents the persistent current, which is non-dissipative: If Λ is non-vanishing, then a finite current exists even for an infinitesimal $E(t)$. We have the following;

$\mathrm{Lim}\ \Lambda > 0$	:	Superfluidity.
$\mathrm{Lim}\ \Lambda = 0$	:	Normal fluidity.
$\mathrm{Lim}\ \Lambda < 0$	:	The assumed (ground) state is unstable.

Note that the perfect conductor $\sigma = \infty$ and the superconductor $\mathrm{Lim}\ \Lambda > 0$ are entirely different.

To see that the ODLRO does not necessarily lead to $\mathrm{Lim}\ \Lambda > 0$, we consider here the ground state of ideal Bosons in a large one-dimensional ring with a single δ-function potential at the origin. The single particle Hamiltonian is $H_1 = -(1/2m)(d/dx)^2 + g\,\delta(x)$. The even parity solution of $H_1 \psi_k = E_k \psi_k$ is $\psi_k \propto \cos[k((L/2) - x)]$ for $0 < x < L$, where k is determined by $\tan(kL/2) = mg/k$ and $E_k = k^2/2m$. The odd parity solutions of $H_1 \phi_n = E_n \phi_n$ is $\phi_n \propto \sin[(2\pi n/L)((L/2) - x)]$ with $n = 1, 2, \cdots$ and $E_n = (2\pi n/L)^2(1/2m)$. The ground state of H_1 is described by ψ_{k_0} where k_0 is the minimum value of k. For large L, $k_0 = (\pi/L) - (2\pi/L^2 mg) + \cdots$. Thus

$$\frac{2}{N}\left\langle \Psi_0 \left| J \; \frac{1}{H^{\times}} J \right| \Psi_0 \right\rangle = \frac{4}{m} \sum_n \frac{|\langle \phi_n | p^x | \psi_{k_0} \rangle|^2}{(2\pi n/L)^2 - k_0^2}$$

$$\xrightarrow{\mathrm{Lim}} \frac{64}{\pi^2 m} \sum_{n=1}^{\infty} \frac{4n^2}{(4n^2 - 1)^3} = \frac{1}{m} .$$

The Thomas-Kuhn sum rule $\Lambda = 0$, which holds in any non-cyclic system, has been restored by a single scatterer in the cyclic system. Our system exhibits the condensation $\langle\!\langle \Psi_0 | a_0^{\dagger} a_0 | \Psi_0 \rangle\!\rangle = 8N/\pi^2$ and the ODLRO $\langle\!\langle \Psi_0 | \psi^{\dagger}(x')\,\psi(x'') | \Psi_0 \rangle\!\rangle = 2\,(N/L) \neq 0$, though there is no superfluidity.

Finally a system of interacting Bosons containing low concentration of impurities is considered. Suppose that the system possesses the ODLRO $\lim_{|\vec{r}' - \vec{r}''| \to \infty} \mathrm{Lim} \langle\langle \psi(\vec{r}') \; \psi^\dagger(\vec{r}'') \rangle\rangle \equiv \langle \Phi^2 \rangle \neq 0$. We start with the formula[1]

$$\Lambda = -2 N^{-1} L^2 \langle\langle J(x')(Q/H^\times) J(x'') \rangle\rangle$$

where $J(x') \equiv (1/2m) \Sigma_j \{ p_j^x \, \delta(x_j - x') + \delta(x_j - x') \, p_j^x \}$

and $x' \neq x''$. We consider the case $|x' - x''| \to \infty$. It is proved that

$$\langle\langle J(x') \frac{Q}{H^\times} J(x'') \rangle\rangle = \int dy'dz' \int dy''dz'' \lim_{\varepsilon \to +0} i \left(\int_0^\infty - \int_{-\infty}^0\right) dt \, e^{-\varepsilon |t|} \Gamma(\vec{r}'t|\vec{r}''0)$$

where

$$\Gamma(\vec{r}' \; t \mid \vec{r}'' \; 0) = (1/2m^2) \{ \langle\langle (\partial' \psi^\dagger(\vec{r}'t)) \psi(\vec{r}'t) \psi^\dagger(\vec{r}''0) (\partial'' \psi(\vec{r}''0)) \rangle\rangle$$
$$+ \langle\langle \psi^\dagger(\vec{r}'t) (\partial' \psi(\vec{r}'t)) (\partial'' \psi^\dagger(\vec{r}''0)) \psi(\vec{r}''0) \rangle\rangle \} - (1/4m^2) \partial' \partial'' \langle\langle \rho(\vec{r}'t) \rho(\vec{r}''0) \rangle\rangle$$

with $\partial = \partial/\partial x$ and $\rho(x)$ = the density operator. The last term in Γ does not contribute when $|x' - x''| \to \infty$. We adopt the decoupling approximation, which is believed to be valid in this limit.

$$\Gamma(\vec{r}' \; t \mid \vec{r}'' \; 0) = (\langle \Phi^2 \rangle / 2m^2) \partial' \partial'' \{ \langle\langle \psi^\dagger(\vec{r}'t) \psi(\vec{r}''0) \rangle\rangle + \langle\langle \psi(\vec{r}'t) \psi^\dagger(\vec{r}''0) \rangle\rangle \}$$

Then $$\langle\langle J(x') \frac{Q}{H^\times} J(x'') \rangle\rangle = \frac{\langle \Phi^2 \rangle}{2m^2} \frac{1}{\Omega} \sum_{k \neq 0} k^2 e^{ik(x'-x'')} \int_{-\infty}^{\infty} \frac{\langle A(k,\omega) \rangle}{\omega} d\omega$$

where $k = 2\pi n/L$, Ω = volume , and

$$A(k,\omega) = \langle a_k \delta(H^\times - \mu - \omega) a_k^\dagger \rangle - \langle a_k^\dagger \delta(H^\times + \mu + \omega) a_k \rangle$$

with $a_k \equiv \Omega^{-1/2} \int \psi(\vec{r}) \, e^{-ikx} \, d\vec{r}$.

The quantity $\int d\omega (\langle A(k,\omega) \rangle / \omega)$ is obtained by a general argument.

The result is $\int_{-\infty}^{\infty} d\omega \frac{\langle A(k,\omega) \rangle}{\omega} = \frac{m^2 \langle \Phi^2 \rangle}{k^2 f(k)}$, where $f(k)$ may be a function of k in the impure system. In the case of pure systems considered by the Josephson-Baym theory[4,5], $f(k) = \rho_s$.

The condition for superfluidity is found to be $f(k)$ = constant $\equiv \tilde{\rho}_s$ (for small k including the smallest one, namely, $k = 2\pi/L$).

$$\because \; \Lambda = - \frac{m \langle \Phi^2 \rangle^2}{\rho \; \tilde{\rho}_s} \sum_{k \neq 0} \exp[\, ik(x'-x'')] = \frac{m \langle \Phi^2 \rangle^2}{\rho \; \tilde{\rho}_s} > 0$$

where $\rho = mN/\Omega$. If ρ_s is defined by the London equation, then $\tilde{\rho}_s = \langle \Phi^2 \rangle^2 / \rho_s$. The condition $f(k) \to \tilde{\rho}_s$ ($k \to 0$) is not trivial. In the system of ideal (non-interacting) Bosons containing macroscopic number of impurities, $f(k)$ diverges as k^{-2} , which is rigorously proved. In this case $\Lambda = 0$.

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