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Page 9 - Replace "$d(r_o \phi)/dz$" by "$d(r_o \phi_1)/dz$" in Eq. (5.5b).

Page 10 - Delete "(6.4b)" on the second line from the bottom of page. Insert the equation number "(6.4a)" on the third line from the bottom of page.

Page 19 - Replace "$\xi \ell^3$" by "$\xi \ell^2$" on the 19th line from top of page.

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A CRUDE ESTIMATE OF THE STARTING CURRENT FOR LINEAR-ACCELERATOR BEAM BLOWUP IN THE PRESENCE OF AN AXIAL MAGNETIC FIELD

ABSTRACT

Cheng's simple treatment of Wilson's work on beam blowup in linear accelerators is extended to the case where an axial magnetic field is used for focusing. The starting current for beam blowup increases steadily with the magnetic field strength, and linearly if the magnetic field is sufficiently strong. Simple expressions for the growth rate of oscillations and the blowup time are also derived. The focusing action of an axial magnetic field is rather inefficient, especially for short accelerator sections and for electron beams of higher energies. Beam blowup in linear accelerators may be suppressed more effectively by using shorter sections of nonuniformly loaded waveguides. Nevertheless, the axial magnetic field remains an ultimate means for increasing the beam current in linear accelerators. The limiting value of beam current imposed by the problem of beam blowup may be increased by about two orders of magnitude by using superconducting coils supplying fields of several tens of kilogauss.

I. INTRODUCTION

The phenomenon of pulse shortening, also known as beam blowup, was first reported in 1958 by Boag and Miller.[1] This undesirable effect exhibited by high current linear accelerators was, for a number of years, a matter of considerable concern. Much experimental and theoretical work has been done and published.[2-11] The starting current for this effect to occur varies with the design of the accelerator waveguide and with the length of beam current pulse. The reported values of the starting current range from about 80 ma to 650 ma and from about 40 to 90 per cent of the designed value for obtaining maximum conversion efficiency.[12] The cause of beam blowup is the excitation of a field in the second frequency passband. This field is a mixture of the TM and TE waves; it can effectively blow the electron beam away from the axis, contrary to the case of a pure TM or TE field. In the latter case, either TM or TE, the radial electric and magnetic forces on an electron approximately cancel each other, and their resultant is proportional to the factor $(1 - v_p^2/c^2)$, which tends to zero as the phase velocity of the field approaches the velocity of light.

All the accelerator structures reported to suffer from beam blowup are uniformly loaded waveguides. Tests on two linear accelerators of constant-gradient (nonuniform loading) design were reported by Haimson.[12] Neither of them has shown this undesirable effect. In one case the injected beam current was 740 ma; in the other the current was 2.70 amps. If these constant-gradient accelerators would still suffer from beam blowup at higher currents, the defect is unimportant because the testing currents are already greater than their respective values for maximum conversion efficiency.

Quite recently, Jarvis, Saxon, and Crowley-Milling[13] have reported their experimental work on pulse shortening experienced by a 5-MeV electron beam passing through a short section of uniformly loaded accelerator waveguide. They measured the starting current, the frequency and power of the generated back-wave wave, and several other characteristics such as the rate of build-up of oscillations and the total phase slippage between the generated field and the electtron beam. They also estimated the noise level, from which the field builds up, by injecting small amounts of rf power at about the oscillation frequency.

By injecting a pulsed electron beam of 5 MeV with a 4.3-μsec pulse length through one S-band accelerator section 90 cm long, the starting current for backward wave oscillation was found to be about 310 ma and the total phase slippage about 1.17 π radians. The corresponding values calculated according to Cheng's results[10] are 144 ma (for an infinitely long pulse) and 0.80 π , respectively. The agreement between theory and experiment is reasonably good, indicating that Cheng's simple treatment of Wilson's work[8] is useful despite its many simplifying approximations.

Jarvis' group also experimented with the use of an axial magnetic field of about 280 gauss. They observed that the pulse shortening effect was less severe during the first few minutes after switching on, and more severe subsequently in the steady state. The transitory effect could not be readily understood; the steady effect of greater beam loss accompanied by less amplitude modulation of the generated field is attributed by the authors to better current transmission.

The purpose of this note is to extend Cheng's treatment to calculate the starting current for beam blowup as a function of the axial magnetic field. We are presently more interested in the comparison of the two cases, with and without the magnetic focusing field, than in obtaining accurate values for either case. From this calculation we find that a magnetic field of 280 gauss is too weak to provide effective focusing action. The starting current is increased by only about three per cent. In view of possible changes in some experimental conditions, it may be appreciated that Jarvis' group did not obtain conclusive results regarding the focusing effects of an axial magnetic field.

II. EQUATIONS OF ELECTRON MOTION

In circular cylindrical coordinates (r, ϕ, z), the three components of the vector equation of electron motion,

$$\frac{d}{dt}(m\vec{v}) = e\vec{E} + \frac{e}{c}\vec{v} \times \vec{H} \quad , \tag{2.1}$$

are as follows:

$$\frac{d}{dt}(m\dot{r}) = eE_r + \frac{e}{c}(r\dot{\phi}H_z - \dot{z}H_\phi) + mr\dot{\phi}^2 \tag{2.2a}$$

$$\frac{d}{dt}(mr^2\dot{\phi}) = erE_\phi + \frac{e}{c}\, r\,(\dot{z}H_r - \dot{r}H_z) \;, \tag{2.2b}$$

$$\frac{d}{dt}(m\dot{z}) = eE_z + \frac{e}{c}(\dot{r}H_\phi - r\dot{\phi}H_r) \;. \tag{2.2c}$$

$$(\dot{q} = dq/dt)$$

The electric field $\vec{E}$ has no dc part, but the magnetic field $\vec{H}$ has. Let $\vec{H}_o$ denote the dc magnetic focusing field. Let $\vec{E}'$ and $\vec{H}'$ denote the rf part of $\vec{E}$ and $\vec{H}$, respectively.

$$\vec{E} = \vec{E}' \quad , \tag{2.3a}$$

$$\vec{H} = \vec{H}_o + \vec{H}' \;. \tag{2.3b}$$

The dc magnetic field $\vec{H}_o$ is predominantly along the z-direction, and is approximately independent of z. Thus, we obtain from Eqs. (2.2a) and (2.2b)

$$\frac{d}{dt}(m\dot{r}) = eE_r' + \frac{e}{c}(r\dot{\phi}H_z' - \dot{z}H_\phi') + (\frac{e}{c}\, r\dot{\phi}H_{oz} + mr\dot{\phi}^2) \quad , \tag{2.4a}$$

$$\frac{d}{dt}(mr^2\dot{\phi} + \frac{e}{2c} r^2 H_{oz}) = erE'_\phi + \frac{e}{c} r(\dot{z}H'_r - \dot{r}H'_z) . \qquad (2.4b)$$

The rf field $(\vec{E}', \vec{H}')$ may contain both the axially symmetric field in the fundamental passband and the ϕ-dependent field of a higher frequency in the second passband. Both frequencies may be induced by the beam or supplied externally. All other fields in higher frequency passbands are assumed negligible. The fundamental field is a pure TM wave; it plays only a minor role in determining the transverse components of electron motion. In fact, the only transverse force arising from the fundamental field is in the r-direction, which approaches zero as $(1 - \dot{z}/c)$ does, when $\dot{z}$ approaches c. If the effect of variation of electron energy or mass on beam blowup is only secondary, we may disregard the presence of the fundamental field altogether. Thus, the fields $\vec{E}'$ and $\vec{H}'$ in Eqs. (2.4a) and (2.4b) will, henceforth, be considered to consist of only the ϕ-dependent field in the second passband. The particle mass m in these equations will be considered a constant parameter, so Eq. (2.2c) will not be used subsequently.

III. THE RF FIELD IN THE SECOND PASSBAND

The rf field responsible for beam blowup has been discussed by several authors.[14-17] Let the field $(\vec{E}', \vec{H}')$ be given by

$$\vec{E}' = A'\vec{\Psi}(r, \phi, z; \omega t) + B'\vec{\Psi}(r, \phi + \frac{\pi}{2}, z; \omega t + \theta) , \qquad (3.1a)$$

$$\vec{H}' = A'\vec{\Omega}(r, \phi, z; \omega t) + B'\vec{\Omega}(r, \phi + \frac{\pi}{2}, z; \omega t + \theta) . \qquad (3.1b)$$

Here, A' and B' are two unknown amplitude factors; θ is an unknown phase angle. Near the axis ($r = 0$) of the disk-loaded waveguide, assumed lossless, the components of the vector functions $\vec{\Psi}$ and $\vec{\Omega}$ are, in their simple forms, the following:

$$\Psi_z(r, \phi, z; \omega t) = \frac{r}{a^*} \cos\phi \cos(\beta z - \omega t) . \qquad (3.2a)$$

$$\Psi_r(r, \phi, z; \omega t) = \frac{ka^*}{4}(1 + \frac{r^2}{a^{*2}}) \cos\phi \sin(\beta z - \omega t) . \qquad (3.2b)$$

$$\Psi_\phi(r, \phi, z; \omega t) = -\frac{ka^*}{4}(1 - \frac{r^2}{a^{*2}}) \sin\phi \sin(\beta z - \omega t) . \qquad (3.2c)$$

$$\Omega_z(r, \phi, z; \omega t) = -\frac{r}{a^*} \sin \phi \cos (\beta z - \omega t) . \tag{3.3a}$$

$$\Omega_r(r, \phi, z; \omega t) = \frac{ka^*}{4} \left(1 - \frac{r^2}{a^{*2}} - \frac{4}{k^2 a^{*2}}\right) \sin \phi \sin (\beta z - \omega t) . \tag{3.3b}$$

$$\Omega_\phi(r, \phi, z; \omega t) = \frac{ka^*}{4} \left(1 + \frac{r^2}{a^{*2}} - \frac{4}{k^2 a^{*2}}\right) \cos \phi \sin (\beta z - \omega t) . \tag{3.3c}$$

These expressions satisfy Maxwell's equations exactly when $\beta = k \equiv \omega / c$, and only approximately otherwise. The constant a^* is the radius at which E'_ϕ is supposed to vanish. The two fields of amplitudes A' and B' are linearly independent of each other because of different ϕ-dependences. Notwithstanding the rather crude boundary condition $E'_\phi (r = a^*) = 0$, experiments have shown that the approximation is sufficiently good, qualitatively. The value of a^* as measured by Jarvis' group[13] is not far different from the disk hole radius (usually denoted by a).

IV. THE DC OR ZEROTH-ORDER SOLUTION

Before the rf oscillation starts to build up, we have from Eqs. (2.4a) and (2.4b)

$$\frac{d}{dt}(m\dot{r}_o) = \frac{e}{c} r_o \dot{\phi}_o H_{oz} + m r_o \dot{\phi}_o^2 , \tag{4.1a}$$

$$\frac{d}{dt} (m r_o^2 \dot{\phi}_o + \frac{e}{2c} r_o^2 H_{oz}) = 0 , \tag{4.1b}$$

by putting $A' = B' = 0$, $r = r_o$, and $\phi = \phi_o$.

From Eq. (4.1b) we obtain immediately

$$m r_o^2 \dot{\phi}_o + \frac{e}{2c} r_o^2 H_{oz} = p_{o\phi} = \text{const.} \tag{4.2}$$

This equation states that under the dc condition the generalized angular momentum of the particle, $p_{o\phi}$, is conserved. Thus, $p_{o\phi}$ should always have the same value as when the particle was emitted from the cathode.

The optimum focusing condition calls for the cathode to be completely shielded from the dc magnetic field.[18] Under this condition, the magnitude of $p_{o\phi}$ is small compared to the magnitude of either $m r_o^2 \dot{\phi}_o$ or $(e/2c) r_o^2 H_{oz}$ at any axial distance reasonably far from the cathode. Hence, Eq. (4.2) may be approximated by

$$m r_o^2 \dot{\phi}_o + \frac{e}{2c} r_o^2 H_{oz} = 0 \ ,$$

i.e.,

$$\dot{\phi}_o = -eH_{oz}/2mc = -\omega_L/\gamma \ , \tag{4.3}$$

where $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}} = m/m_o$ and $\omega_L = eH_{oz}/2m_o c$, ω_L being the so-called Larmor frequency.

Now let

$$\beta_L = \omega_L/\gamma\dot{z} \cong \omega_L/\gamma c \ . \tag{4.4}$$

Then, from Eq. (4.3),

$$\phi_o = \phi_i - \beta_L z \ . \tag{4.5}$$

Here ϕ_i is the initial value of ϕ at $z = 0$. Using Eqs. (4.3) and (4.4) and noting that m and $\dot{z}$ are constants, we may transform Eq. (4.1a) to the following simple form :

$$\left(\frac{d^2}{dz^2} + \beta_L^2\right) r_o = 0 \ . \tag{4.6}$$

Hence,

$$r_o = r_i \cos \beta_L z + (r_i'/\beta_L) \sin \beta_L z \ , \tag{4.7}$$

r_i and r_i' being the initial value of r and of dr/dz, respectively.

When the rf field ($\vec{E}'$, $\vec{H}'$) has just started to build up, a spiralling electron entering the accelerator section at $t = t_o$ will experience this field whose components are given by Eqs. (3.1), (3.2), and (3.3), in which r and ϕ are, to the 0th-order approximation, given by Eq. (4.7) and Eq. (4.5), respectively, and

$$\beta z - \omega t = (\beta - \frac{\omega}{\dot{z}}) z - \omega t_o \equiv \beta_\delta z - \omega t_o \ . \tag{4.8}$$

This electron will do work on the rf field in order to build it up further. The major part of the work done is through interaction with the axial electric field E'_z. In passing through the whole accelerator section of length ℓ, the work done by one electron is

$$w = -\int_0^{\ell} eE'_z dz \quad , \tag{4.9}$$

and, to the 0th-order approximation,

$$w = -\int_0^{\ell} (er_o/a^*)\left\{A' \cos\phi_o \cos(\beta_\delta z - \omega t_o) - B' \sin\phi_o \cos(\beta_\delta z - \omega t_o - \theta)\right\} dz \tag{4.9a}$$

Let $\bar{w}$ denote the average value of w, averaged over different initial phase angles ϕ_i and over different entrance time angles ωt_o. In other words,

$$\bar{w} = \text{Ave. of } w = \frac{1}{(2\pi)^2}\int_0^{2\pi} d(\omega t_o)\int_0^{2\pi} (d\phi_i)\, w \quad . \tag{4.10}$$

If w is given by Eq. (4.9a), then obviously $\bar{w} = 0$. Similarly, the contribution to the average work done through interaction with the transverse electric field components $E'_r(r_o, \phi_o, z; \omega t)$ and $E'_\phi(r_o, \phi_o, z; \omega t)$ vanishes also. In order to obtain non-zero $\bar{w}$ so that the electron beam will do positive work on the field in the second passband, we must carry out the analysis at least to the first order of rf quantities.

V. THE FIRST-ORDER RF SOLUTION

Let

$$r = r_o + r_1 + \ldots \quad , \tag{5.1a}$$

and

$$\phi = \phi_o + \phi_1 + \ldots \quad . \tag{5.1b}$$

Here, r_1 and ϕ_1 are the first-order variables to be evaluated. Substituting the 0th-order Eqs. (4.1a) and (4.1b) into Eqs. (2.4a) and (2.4b), respectively, and using Eq. (4.3), we obtain

$$\left(\frac{d^2}{dt^2} + \frac{\omega_L^2}{\gamma^2}\right) r_1 = \frac{e}{m}\left\{E'_r + \frac{1}{c}(r_o\dot{\phi}_o H'_z - \dot{z}H'_\phi)\right\} \tag{5.2a}$$

and

$$\frac{d}{dt}\left(r_o^2 \dot{\phi}_1\right) = \frac{e r_o}{m}\left\{ E'_\phi + \frac{1}{c}\left(\dot{z} H'_r - \dot{r}_o H'_z\right)\right\} ,$$

i.e.,

$$\left(\frac{d^2}{dt^2} + \frac{\omega_L^2}{\gamma^2}\right)\left(r_o \phi_1\right) = \frac{e}{m}\left\{ E'_\phi + \frac{1}{c}\left(\dot{z} H'_r - \dot{r}_o H'_z\right)\right\} . \qquad (5.2b)$$

These equations may further be transformed by changing the independent variable from t to z and substituting β_L for $\omega_L/\gamma\dot{z}$. The resulting equations are as follows:

$$\left(\frac{d^2}{dz^2} + \beta_L^2\right) r_1 = \frac{e}{m\dot{z}^2}\left\{ E'_r + \frac{1}{c}\left(r_o \dot{\phi}_o H'_z - \dot{z} H'_\phi\right)\right\} \equiv - Q'_r \qquad (5.3a)$$

$$\left(\frac{d^2}{dz^2} + \beta_L^2\right)\left(r_o \phi_1\right) = \frac{e}{m\dot{z}^2}\left\{ E'_\phi + \frac{1}{c}\left(\dot{z} H'_r - \dot{r}_o H'_z\right)\right\} \equiv - Q'_\phi . \qquad (5.3b)$$

Evidently,

$$- Q'_r \simeq \frac{e}{mc^2}\left(E'_r - H'_\phi\right)$$

$$= \frac{e}{mc^2} \cdot \frac{1}{ka^*}\left\{ A' \cos \phi_o \sin\left(\beta_\delta z - \omega t_o\right) - B' \sin \phi_o \sin\left(\beta_\delta z - \omega t_o - \theta\right)\right\} ; \qquad (5.4a)$$

$$- Q'_\phi \simeq \frac{e}{mc^2}\left(E'_\phi - H'_r\right)$$

$$= \frac{e}{mc^2} \cdot \frac{1}{ka^*}\left\{ A' \sin \phi_o \sin\left(\beta_\delta z - \omega t_o\right) + B' \cos \phi_o \sin\left(\beta_\delta z - \omega t_o - \theta\right)\right\} . \qquad (5.4b)$$

Noting that

$$r_1 = dr_1/dz = 0 \qquad \text{at} \quad z = 0 \tag{5.5a}$$

and

$$r_o\phi_1 = d(r_o\phi_1)/dz = 0 \qquad \text{at} \quad z = 0\,, \tag{5.5b}$$

we obtain without further ado

$$r_1 = \frac{1}{\beta_L}\int_0^z Q_r'(z')\sin(\beta_L z' - \beta_L z)\,dz' \tag{5.6a}$$

and

$$r_o\phi_1 = \frac{1}{\beta_L}\int_0^z Q_\phi'(z')\sin(\beta_L z' - \beta_L z)\,dz'\,. \tag{5.6b}$$

Thus, we are led to consider the following integrals:

$$G_c(z, \omega t_o) = \frac{1}{\beta_L}\int_0^z \cos(\phi_i - \beta_L z')\sin(\beta_\delta z' - \omega t_o)\sin(\beta_L z' - \beta_L z)\,dz'\,. \tag{5.7a}$$

$$G_s(z, \omega t_o) = \frac{1}{\beta_L}\int_0^z \sin(\phi_i - \beta_L z')\sin(\beta_\delta z' - \omega t_o)\sin(\beta_L z' - \beta_L z)\,dz'\,. \tag{5.7b}$$

In terms of these, r_1 and $r_o\phi_1$ finally become:

$$r_1 = -\frac{e}{mc^2}\cdot\frac{1}{ka^*}\left\{A'G_c(z, \omega t_o) - B'G_s(z, \omega t_o + \theta)\right\}\,. \tag{5.8a}$$

$$r_o\phi_1 = \frac{e}{mc^2}\cdot\frac{1}{ka^*}\left\{A'G_s(z, \omega t_o) + B'G_c(z, \omega t_o + \theta)\right\}\,. \tag{5.8b}$$

VI. POWER GERERATED

Having obtained r_1 and $r_o\phi_1$ we may evaluate the axial electric field E_z' as seen by the moving electron. According to Eqs. (3.1a) and (3.2a),

$$E_z' = \frac{A'}{a^*}(r_1\cos\phi_o - r_o\phi_1\sin\phi_o)\cos(\beta_\delta z - \omega t_o)$$

$$-\frac{B'}{a^*}(r_1\sin\phi_o + r_o\phi_1\cos\phi_o)\cos(\beta_\delta z - \omega t_o - \theta)\,. \tag{6.1}$$

Substituting Eqs. (5.8a) and (5.8b) into Eq. (6.1), we obtain

$$-eE'_z = k\kappa \cos(\beta_\delta z - \omega t_o)\left[A'^2\left\{G_c(z, \omega t_o)\cos\phi_o + G_s(z, \omega t_o)\sin\phi_o\right\}\right.$$
$$\left. - A'B'\left\{G_s(z, \omega t_o + \theta)\cos\phi_o - G_c(z, \omega t_o + \theta)\sin\phi_o\right\}\right]$$
$$+ k\kappa \cos(\beta_\delta z - \omega t_o - \theta)\left[B'A'\left\{G_s(z, \omega t_o)\cos\phi_o - G_c(z, \omega t_o)\sin\phi_o\right\}\right.$$
$$\left. + B'^2\left\{G_c(z, \omega t_o + \theta)\cos\phi_o + G_s(z, \omega t_o + \theta)\sin\phi_o\right\}\right], \quad (6.2)$$

where

$$\kappa = \frac{e^2}{mc^2} \cdot \frac{1}{(ka^*)^2} \quad . \quad (6.2a)$$

There are four different terms making up E'_z, identifiable by the factors A'^2, $A'B'$, $B'A'$ and B'^2. There will also be four different terms contributing to the average work $\overline{w}$, done by an electron and to be evaluated according to Eqs. (4.9) and (4.10). Thus,

$$\overline{w} = \overline{w}_{AA} + \overline{w}_{BB} + (\overline{w}_{AB} + \overline{w}_{BA}) , \quad (6.3)$$

where

$$\overline{w}_{AA} = k\kappa A'^2\left[\text{Ave. of} \int_0^\ell \left\{G_c(z, \omega t_o)\cos\phi_o + G_s(z, \omega t_o)\sin\phi_o\right\}\cos(\beta_\delta z - \omega t_o)\, dz\right] , \quad (6.4a)$$

$$\overline{w}_{AB} = -k\kappa A'B'\left[\text{Ave. of} \int_0^\ell \left\{G_s(z, \omega t_o + \theta)\cos\phi_o - G_c(z, \omega t_o + \theta)\sin\phi_o\right\}\cos(\beta_\delta z - \omega t_o)\, dz\right] , \quad (6.4b)$$

$$\bar{w}_{BA} = k\kappa B'A'\left[\text{Ave. of}\int_0^{\ell}\left\{G_s(z,\omega t_o)\cos\phi_o - G_c(z,\omega t_o)\sin\phi_o\right\}\cos(\beta_\delta z - \omega t_o - \theta)\,dz\right], \tag{6.4c}$$

$$\bar{w}_{BB} = k\kappa B'^2\left[\text{Ave. of}\int_0^{\ell}\left\{G_c(z,\omega t_o + \theta)\cos\phi_o + G_s(z,\omega t_o + \theta)\sin\phi_o\right\}\cos(\beta_\delta z - \omega t_o - \theta)\,dz\right]. \tag{6.4d}$$

From these equations we may note that

$$\bar{w}_{AA}/A'^2 = \bar{w}_{BB}/B'^2 \tag{6.5a}$$

and

$$\bar{w}_{AB} + \bar{w}_{BA} = 2k\kappa A'B'\sin\theta\left[\text{Ave. of}\int_0^{\ell}\left\{G_s(z,\omega t_o)\cos\phi_o - G_c(z,\omega t_o)\sin\phi_o\right\}\sin(\beta_\delta z - \omega t_o)\,dz\right] \tag{6.5b}$$

Furthermore,

$$\text{Ave. of}\int_0^{\ell} G_c(z,\omega t_o)\cos\phi_o\cos(\beta_\delta z - \omega t_o)\,dz = \text{Ave. of}\int_0^{\ell} G_s(z,\omega t_o)\sin\phi_o\cos(\beta_\delta z - \omega t_o)\,dz\ ; \tag{6.6a}$$

$$\text{Ave. of}\int_0^{\ell} G_s(z,\omega t_o)\cos\phi_o\sin(\beta_\delta z - \omega t_o)\,dz = -\text{Ave. of}\int_0^{\ell} G_c(z,\omega t_o)\sin\phi_o\sin(\beta_\delta z - \omega t_o)\,dz\ . \tag{6.6b}$$

Hence,

$$\bar{w} = 2k\kappa(A'^2 + B'^2)\,\text{Ave. of}\left[\int_0^{\ell} G_c(z,\omega t_o)\cos\phi_o\cos(\beta_\delta z - \omega t_o)\,dz + \frac{2A'B'\sin\theta}{A'^2 + B'^2}\int_0^{\ell} G_s(z,\omega t_o)\cos\phi_o\sin(\beta_\delta z - \omega t_o)\,dz\right], \tag{6.7}$$

and, when evaluated, this becomes:

$$\bar{w} = \left\{\frac{k\kappa}{16}(A'^2 + B'^2)\ell^3\right\} \frac{1}{\beta_L \ell} \left[\left\{ \frac{\sin^2(\beta_L - \frac{1}{2}\beta_\delta)\ell}{(\beta_L - \frac{1}{2}\beta_\delta)^2 \ell^2} - \frac{\sin^2(\beta_L + \frac{1}{2}\beta_\delta)\ell}{(\beta_L + \frac{1}{2}\beta_\delta)^2 \ell^2} \right\} \right.$$

$$\left. + \frac{2A'B' \sin\theta}{A'^2 + B'^2} \left\{ \frac{\sin^2(\beta_L - \frac{1}{2}\beta_\delta)\ell}{(\beta_L - \frac{1}{2}\beta_\delta)^2 \ell^2} + \frac{\sin^2(\beta_L + \frac{1}{2}\beta_\delta)\ell}{(\beta_L + \frac{1}{2}\beta_\delta)^2 \ell^2} - 2\frac{\sin^2 \frac{1}{2}\beta_\delta \ell}{(\frac{1}{2}\beta_\delta \ell)^2} \right\} \right]. \quad (6.8)$$

For the sake of brevity we introduce the following notations:

$$u_\delta = \frac{1}{2}\beta_\delta \ell \quad . \qquad (6.9a)$$

$$u_- = \frac{1}{2}\beta_\delta \ell - \beta_L \ell \quad . \qquad (6.9b)$$

$$u_+ = \frac{1}{2}\beta_\delta \ell + \beta_L \ell \quad . \qquad (6.9c)$$

$$f_1 = \frac{\sin^2 u_-}{u_-^2} - \frac{\sin^2 u_\delta}{u_\delta^2} \quad . \qquad (6.9d)$$

$$f_2 = \frac{\sin^2 u_\delta}{u_\delta^2} - \frac{\sin^2 u_+}{u_+^2} \quad . \qquad (6.9e)$$

Then, Eq. (6.8) may be written as

$$\bar{w} = \left\{\frac{k\kappa}{16}(A'^2 + B'^2)\ell^3\right\} F \,, \qquad (6.10a)$$

where

$$F = \frac{1}{\beta_L \ell}\left\{(f_1 + f_2) + \frac{2A'B' \sin\theta}{A'^2 + B'^2}(f_1 - f_2)\right\} \quad . \qquad (6.10b)$$

Having obtained $\bar{w}$, the average work done on the field ($\vec{E}'$, $\vec{H}'$) by one electron, we may calculate the total amount of beam power (P) thus consumed by simply multiplying $\bar{w}$ by the total number of electrons entering the accelerator section per unit time. This number is I/e where I is the beam current during the on-time of a pulse. Hence, the rf power generated is

$$P = \bar{w}\,(l/c) \ . \tag{6.11}$$

VII. THE STARTING CURRENT

In order to build up the field $(\vec{E}', \vec{H}')$, the power P generated by the beam must be greater than the Poynting power P', the average flow rate of rf energy,

$$P' = \frac{1}{2\pi}\int_0^{2\pi} d(\omega t)\ \frac{c}{4\pi}\int_0^{a} r\,dr\int_0^{2\pi} d\phi\ \left| E_r' H_\phi' - E_\phi' H_r' \right| . \tag{7.1}$$

Using the field expressions given by Eqs. (3.1), (3.2), and (3.3) and integrating, we obtain

$$P' = (A'^2 + B'^2)\left(\frac{ka^*}{4}\right)^2 \frac{c}{8}\,a^2 \left| 1 - \frac{4}{k^2 a^{*2}} + \frac{1}{3}\,\frac{a^4}{a^{*4}} \right| . \tag{7.2}$$

i.e.,

$$P' = (A'^2 + B'^2)(ka^*/4)^2\ (1/\xi) \ , \tag{7.3}$$

ξ being the so-called series impedance (impedance per unit area). Here, we may note that no cross product term $(A'B')$ arises in P', because the two fields of amplitudes A' and B' have independent ϕ-dependences.

The starting current is obtained, evidently, by considering the relation $P = P'$, i.e.,

$$I = \left\{ \frac{mc^2/e}{\xi} \cdot \frac{(ka^*)^4}{k\ell^3} \right\} \cdot \frac{1}{F} \ . \tag{7.4}$$

For a given electron energy, a given waveguide structure, and for frequencies sufficiently removed from the cutoff frequencies, the quantities inside the curly brackets of Eq. (7.4) are approximately constants and I varies mainly as the factor $(1/F)$, which is a function of β_L, β_δ, B'/A' and θ. The starting current I_s is the minimum value of I; I_s may be obtained approximately by maximizing F.

As given by Eq. (6.10b), F has two terms. The first term is $(f_1 + f_2)/\beta_L \ell$; this is an even function of $\beta_L \ell$ and an odd function of $\beta_\delta \ell$. This is appropriate because the focusing effect should remain the same, when the dc magnetic field H_{oz} is reversed in direction, and because the average work done $\bar{w}$ should change sign when $\beta_\delta = \beta - \omega/\dot{z}$ changes sign. The second term has the factor $(f_1 - f_2)/\beta_L \ell$, which is odd in $\beta_L \ell$ and even in $\beta_\delta \ell$. To bring the second term to the correct symmetry, we postulate that $\sin\theta$ will change sign as (β_δ/β_L) changes sign. In other words, Eq. (6.10b) may be written as

$$F = (1/\beta_L \ell)\left[(f_1 + f_2) + \left\{\chi \ \mathrm{sgn}\,(\beta_\delta/\beta_L)\right\}(f_1 - f_2)\right] \quad , \tag{7.5}$$

where χ is a numeric parameter,

$$|\chi| = \left|2A'B' \sin\theta \Big/ (A'^2 + B'^2)\right| \leqq 1 \quad , \tag{7.6a}$$

and

$$\mathrm{sgn}\,(\beta_\delta/\beta_L) \left\{\begin{array}{l} 1 \\ 0 \\ -1 \end{array}\right. , \quad \text{when} \quad \beta_\delta/\beta_L \left\{\begin{array}{l} > 0 \\ = 0 \\ < 0 \end{array}\right. . \tag{7.6b}$$

From Eq. (7.5) or its previous version, Eq. (6.10b), the following assertions may clearly be made :

If $\quad (f_1 - f_2)/\beta_L \ell \geqq 0, \qquad F \leqq 2\, f_1/\beta_L \ell$.

If $\quad (f_1 - f_2)/\beta_L \ell \leqq 0, \qquad F \leqq 2\, f_2/\beta_L \ell$.

Hence,

$$F \leqq \mathrm{Sup}(2\, f_1/\beta_L \ell \ , \ 2\, f_2/\beta_L \ell) \quad , \tag{7.7}$$

where "Sup" means "the greater of." Noting that

$$f_1(\beta_\delta \, , \, \beta_L) = -\, f_2(\beta_\delta \, , \, -\beta_L)$$

and

$$f_2(\beta_\delta, \, \beta_L) = -\, f_1(\beta_\delta, \, -\beta_L) \quad ,$$

we may rewrite the inequality (7.7) as

$$F \leq \frac{2}{|\beta_L \ell|} \operatorname{Sup} \left\{ f_1(\beta_\delta, |\beta_L|), \ f_2(\beta_\delta, |\beta_L|) \right\} \tag{7.8}$$

Thus, for given β_L, the maximum value of F is

$$F_{max} = \frac{2}{|\beta_L \ell|} \operatorname{Sup} \left[\left\{ f_1(\beta_\delta, |\beta_L|) \right\}_{max}, \ \left\{ f_2(\beta_\delta, |\beta_L \ell|) \right\}_{max} \right], \tag{7.9}$$

and the starting current (for an infinitely long pulse) is

$$I_s = \left\{ \frac{mc^2/e}{\xi} \cdot \frac{(ka^*)^4}{k\ell^3} \right\} \cdot \frac{|\beta_L \ell|}{2} \ \operatorname{Inf.} \left[1/\left\{ f_1(\beta_\delta, |\beta_L|) \right\}_{max}, \ 1/\left\{ f_2(\beta_\delta, |\beta_L|) \right\}_{max} \right]. \tag{7.10}$$

By "Inf" is meant "the smaller of."

For comparison of the starting currents with and without the focusing field, we now calculate F by letting $\beta_L \ell$ approach zero. Since

$$f_1 - f_2 \xrightarrow[\beta_L \ell \to 0]{} \text{const.} \times (\beta_L \ell)^2 ,$$

$$F \xrightarrow[\beta_L \ell \to 0]{} \frac{1}{\beta_L \ell} (f_1 + f_2) \xrightarrow[\beta_L \ell \to 0]{} \frac{4}{u_\delta} \left(\frac{\sin^2 u_\delta}{u_\delta^2} - \frac{\sin 2u_\delta}{2u_\delta} \right) ,$$

i.e.,

$$F \xrightarrow[\beta_L \ell \to 0]{} \frac{16}{(\beta_\delta \ell)^3} \cdot (1 - \cos \beta_\delta \ell - \tfrac{1}{2} \beta_\delta \ell \sin \beta_\delta \ell) \equiv F(0) . \tag{7.11}$$

Let $I_s(0)$ denote the starting current for the case of no focusing field.

$$I_s(0) = \left\{ \frac{mc^2/e}{\xi} \cdot \frac{(ka^*)^4}{k\ell^3} \right\} \cdot \frac{1}{F(0)_{max}} ; \tag{7.12a}$$

$$F(0)_{max} \cong 1.08 , \quad F(0)_{max} \text{ occurs at } \beta_\delta \ell \cong 0.83\pi . \tag{7.12b}$$

Using the experimental data of Jarvis' group,[13] $mc^2/e \cong 5.5 \times 10^6$ volts, $ka^* \simeq 1.22$, $k \simeq 0.928$ cm^{-1}, $\xi \simeq 117$ ohms/cm^2, and $\ell = 90$ cm, we obtain $I_s(0) \simeq 142$ ma.

Equations (7.11) and (7.12) are the same as obtained by Cheng,[10] although the foregoing derivation is different from his. He used only one rf field ($B' = 0$ in our notation), and did not have to average over the initial values of the azimuthal angle ϕ_i. These differences cause no net effect on the magnitude of the starting current.

VIII. THE GROWTH RATE AND BLOWUP TIME

When the electron beam current I is greater than the starting current I_s, the power P spent by the beam will be greater than the time-averaged rf energy flow P'. The rf field $(\vec{E}', \vec{H}')$ must then grow.

Let

$$C'^2 = A'^2 + B'^2$$

and

$$\eta = (1/16)(k\kappa/e)\ell^3 F .$$

We have, according to Eqs. (6.10a) and (6.11),

$$P = \eta I C'^2 \tag{8.1}$$

and, since $P' = P$ when $I = I_s$,

$$P' = \eta I_s C'^2 \tag{8.2}$$

When $P > P'$, the rf energy W' stored in the waveguide section of length ℓ will increase. Clearly, if the waveguide is assumed lossless,

$$\frac{dW'}{dt} = P - P' \tag{8.3}$$

Noting that

$$W' = (P'/|v_g|)\ell = P' t_F , \tag{8.4}$$

v_g being the group velocity and t_F being the filling time, we obtain the growth rate:

$$\frac{1}{C'}\frac{dC'}{dt} = (1/t_F)\,(I-I_s)\Big/2I_s \qquad \text{nepers per unit time} \tag{8.5a}$$

$$\frac{1}{P'}\frac{dP'}{dt} = 8.69\,(1/t_F)\,(I-I_s)\Big/2I_s \qquad \text{decibels per unit time} \tag{8.5b}$$

To illustrate the use of this estimate, we again use the experimental values of Jarvis' group:

$$|v_g| \sim 0.038\ c, \qquad t_F \sim 0.079\ \mu\text{sec}\ ,$$

$$I \sim 350\ \text{ma}\ , \qquad (1/P')(dP'/dt) \simeq 40\ \text{db per}\ \mu\text{sec}\ .$$

Thus, from Eq. (8.5b), $I_s(0) \simeq 202$ ma, as compared to 142 ma calculated previously. The latter value seems to be much too conservative. This is, however, not surprising in view of the crudeness of our analysis.

The blowup time may be defined to be the time required for the rf field $(\vec{E}', \vec{H}')$ to grow from the noise level to a level which is sufficiently high to cause the electrons, assumed to enter the accelerator section with $r_i = r_i' = 0$, to be deflected away from the axis at the exit end $(z = \ell)$ by a distance r_1 equal to the disk-hole radius. Let P_n' and P_B' be the rf noise power and the rf power for blowup, respectively. Let t_B denote the blowup time, and M denote the power amplification in decibels at $t = t_B$. Then, according to Eq. (8.5b),

$$M = 10\log_{10}\frac{P_B'}{P_n'} = 8.69\left(\frac{I}{I_s} - 1\right)\frac{t_B}{2t_F}\ ,$$

i.e.,

$$t_B = 0.230\ M\,t_F\Big/\left(\frac{I}{I_s} - 1\right)\ . \tag{8.6}$$

Substituting into this equation the experimental values of Jarvis' group, $t_F = 0.079\ \mu$sec, $t_B = 4.30\ \mu$sec, and $I = 312$ ma, and the derived value $I_s = 202$ ma obtained above, we obtain their power amplification for beam blowup $M \sim 129$ db.

IX. DISCUSSION

The usefulness of the dc axial magnetic field for suppressing beam blowup is shown by the following equation:

$$I_s/I_s(0) \quad F(0)_{max}/F_{max}$$

$$\sim 0.54 \left|\beta_L \ell\right| \operatorname{Inf}\left[1/\left\{f_1(\beta_\delta, \left|\beta_L \ell\right|)\right\}_{max},\ 1/\left\{f_2(\beta_\delta, \left|\beta_L \ell\right|)\right\}_{max}\right]. \quad (9.1)$$

Since $(f_1)_{max} \leq 1$ and $(f_2)_{max} \leq 1$, we may state a crude, yet conservative estimate. This is

$$I_s/I_s(0) \sim 0.54 \left|\beta_L \ell\right|, \qquad \left|\beta_L \ell\right| >> 1. \quad (9.2)$$

While the estimates of I_s and $I_s(0)$, as stated by Eqs. (7.10) and (7.12a), are not accurate, it is hoped that the estimate of their ratio may be less inaccurate.

From the last equation we find that, in order to increase the starting current by one order of magnitude, it is necessary to have $\left|\beta_L \ell\right| \cong 10/0.54 \cong 5.9\pi$, corresponding to about 3 periods. In other words, the electrons should execute about 3 revolutions in traveling through the accelerator section of length ℓ.

Let us take the data from Jarvis' group for an example. Thus, $\ell = 90$ cm and $\gamma \sim 11$. To increase the starting current by a factor of ten, we should have

$$H_{oz} \sim 1.14 \times 10^{-7} \times (\gamma c/\ell) \times 5.9\pi \cong 7{,}730 \text{ gauss}$$

and, by a factor of two, $H_{oz} \sim 1{,}550$ gauss.

In Jarvis' experiment H_{oz} is about 280 gauss. This gives $\left|\beta_L \ell\right| \cong 0.670$. For this $\beta_L \ell$, f_1 attains its maximum at $\beta_\delta \ell \cong 1.06\pi$, f_2 attains its maximum at $\beta_\delta \ell \cong 1.06\pi - 2\left|\beta_L \ell\right| \cong 0.63\pi$, and $(f_1)_{max} = (f_2)_{max} \cong 0.353$. Thus, from Eq. (9.1),

$$I_s/I_s(0) \cong 0.54 \times 0.67/0.353 \cong 1.03\ .$$

This small increase in the starting current is probably difficult to measure. A focusing field of $\left|\beta_L \ell\right|$ less than unity is too small to exercise much influence on pulse shortening effects.

For fixed $\beta_L \ell$, the required magnetic field varies as (γ/ℓ). An accelerator section twice as long would require a focusing field half as strong in order to maintain the same ratio $I_s/I_s(0)$, if γ may be assumed to be the same. However, the longer accelerator section has much smaller $I_s(0)$, because $I_s(0) \propto 1/\ell^3$ according to Eq. (7.12a).

For fixed magnetic field, $\beta_L \ell \propto \ell/\gamma$. The focusing effect of an axial magnetic field of given strength becomes less effective when the electron energy becomes greater. The ratio $I_S/I_S(0)$ varies as $(1/\gamma)$, if γ is sufficiently small so that $|B_L \ell| >> 1$. This ratio will become practically independent of γ, if γ is sufficiently large so that $|\beta_L \ell| < 1$; $I_S/I_S(0) \to 1$ as $\gamma \to \infty$. Since $I_S(0)$ varies directly as γ, I_S is independent of γ if γ is sufficiently small and varies directly as γ if γ is sufficiently large.

When $|\beta_L \ell| >> 1$, the starting current I_S is directly proportional to the strength of the axial magnetic field. Except for small γ, this field is not an effective means for focusing. As far as the problem of beam blowup is concerned, the starting current may be increased more effectively by at least two different methods now in actual use. One method takes the advantage of the relation $I_S \sim 1/\ell^3$ by using short waveguide sections at the gun end of the accelerator. The other method is to use the variable-impedance or constant-gradient waveguide sections as used in the Stanford two-mile accelerator and in the linacs reported by Haimson. For these structures, the above-described crude theory can hardly be applied. The lack of pulse-shortening effects in these structures may, nevertheless, be explained by simply saying that the effective impedance of the relevant wave in the second passband, corresponding to $(\xi\ell^2)$ in Eq. (7.4), is drastically reduced by adopting the constant-gradient instead of the constant-impedance design.

It is probable that there exist other effective schemes for preventing linac beam blowup. In fact, any scheme which effectively reduces the power amplification of the undesired field should be useful. Even if all these effective methods were employed, the axial magnetic field, inefficient as it is, remains an ultimate means for increasing the linac beam current, especially worthy if it is feasible to use superconducting coils supplying a magnetic field of several tens of kilogauss. From the sole consideration of beam blowup, it would seem possible to increase the electron beam current in a linac from its present-day value of 1 amp or so by about two orders of magnitude.

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