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The geometric algebra of supersymmetric backgrounds

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Abstract: Supersymmetry-preserving backgrounds in supergravity and string theory can be studied using a powerful framework based on a natural realization of Clifford bundles. We explain the geometric origin of this framework and show how it can be used to formulate a theory of 'constrained generalized Killing forms', which gives a useful geometric translation of supersymmetry conditions in the presence of fluxes.

Contents

1 The geometric algebra approach to (s)pinors

Let (M, g) be a paracompact, smooth, connected and oriented pseudo-Riemannian manifold of dimension $d = p+q$, where p and q denote the numbers of positive and negative eigenvalues of the metric. Inhomogeneous differential forms on M form a $\mathbb{Z}$-graded $\mathcal{C}^\infty(M, \mathbb{R})$-module $\Omega(M) \stackrel{\text{def.}}{=} \Gamma(M, \wedge T^*M)$, whose fixed rank components we denote by $\Omega^k(M) = \Gamma(M, \wedge^k T^*M)$. The usual approach to the spin geometry [1] of (M, g) starts with the Clifford bundle $\mathrm{Cl}(T^*M)$ of the pseudo-Euclidean vector bundle T^*M, where the latter is endowed with the pairing $\hat{g}$ induced by g. Then a spinor bundle S over (M, g) is [1] a bundle of modules over the even sub-bundle $\mathrm{Cl}^{\mathrm{ev}}(T^*M)$ while a pinor bundle is a bundle of modules over $\mathrm{Cl}(T^*M)$. Sections of these bundles are called *spinors* and *pinors*, respectively and correspond in physics to particles of arbitrary spin; of course, a pinor is a spinor in a natural fashion. The case of spin 1/2 arises when S is a bundle of *simple* modules over $\mathrm{Cl}^{\mathrm{ev}}(T^*M)$ or $\mathrm{Cl}(T^*M)$, in which case we say that S is a spin or pin bundle, respectively.

The Kähler-Atiyah bundle and Kähler-Atiyah algebra. Since the fibers of $\mathrm{Cl}(T^*M)$ are given by a universal construction, the Clifford bundle is only determined up to isomorphism and hence its association to (M, g) fails to be strictly functorial. This makes the construction less rigid than it can be, thereby inducing subleties in the development of the theory. The problem can be cured by using a particular realization of the Clifford bundle (which in some ways goes back to Chevalley [3] and Riesz [4]) known as the *Kähler-Atiyah bundle* of (M, g). The latter is functorially determined by (M, g), thereby 'rigidifying' part of the theory of (s)pinors. The Chevalley-Riesz construction identifies the underlying vector bundle of $\mathrm{Cl}(T^*M)$ with the exterior bundle $\wedge T^*M$ of M while transporting the Clifford product of the former to a unital, associative but non-commutative fiberwise binary operation on the latter which we denote by $\diamond$ and call the *geometric product* of (M, g). By definition, the Kähler-Atiyah bundle of (M, g) is the bundle of unital associative algebras $(\wedge T^*M, \diamond)$, while the Kähler-Atiyah algebra is its $\mathcal{C}^\infty(M, \mathbb{R})$-algebra of global sections $(\Omega(M), \diamond)$, where we denote the operation induced on global sections

[1] See [2] for the relation with the approach through vector bundles associated with covers of the bundle of pseudo-orthonormal frames.

through the same symbol. The geometric product is uniquely determined by the differential and metric structure of (M, g). Though it is inhomogeneous with respect to the natural $\mathbb{Z}$-grading of the exterior bundle, it does preserve rank parity and hence it is even with respect to the induced $\mathbb{Z}_2$-grading.

Generalized products. The Chevalley-Riesz construction implies that the geometric product admits an expansion into a finite sum of binary fiberwise-bilinear operations $\triangle_k : \wedge T^*M \times_M \wedge T^*M \to \wedge T^*M$ $(k = 0 \ldots d)$ which are homogeneous of degree $-2k$ with respect to the rank grading, being known as the *generalized products* determined by the metric g. This expansion takes the form:

$$\diamond = \sum_{k=0}^{[\frac{d}{2}]} (-1)^k \triangle_{2k} + \sum_{k=0}^{[\frac{d-1}{2}]} (-1)^{k+1} \triangle_{2k+1} \circ (\pi \otimes \mathrm{id}_{\wedge T^*M}) \quad , \tag{1.1}$$

where π is a certain unital automorphism of the Kähler-Atiyah bundle (known as the *parity automorphism*), being defined through:

$$\pi \stackrel{\text{def.}}{=} \oplus_{k=0}^{d} (-1)^k \mathrm{id}_{\wedge^k T^*M} \quad .$$

Connection to 'partial quantization'. Expansion (1.1) can be viewed as the semiclassical expansion of the geometric product when the latter is identified with the star product arising in a certain 'vertical' partial quantization procedure in which the role of the Planck constant is played by the inverse of the overall scale of the metric g. In the classical limit $g \to \infty$ (i.e. when M is of 'infinite size' when measured by g), the geometric product reduces to $\triangle_0$, which coincides with the wedge product $\wedge$. The corrections to this limit occur as powers in the inverse of the size of M, thereby providing a natural realization of 'compactification/decompactification' limit arguments which are sometimes used in supergravity and string theory.

Recursive construction. The higher generalized products $\triangle_k$ $(k = 1 \ldots d)$ depend on g, their action on inhomogeneous differential forms (=sections of $\wedge T^*M$) being determined recursively through:

$$\omega \triangle_{k+1} \eta = \frac{1}{k+1} g^{ab} (e_a \lrcorner \omega) \triangle_k (e_b \lrcorner \eta) = \frac{1}{k+1} g_{ab} (\iota_{e^a} \omega) \triangle_k (\iota_{e^b} \eta) \quad ,$$

where ι denotes the so-called *interior product*, which is defined as the adjoint of the wedge product with respect to the pairing induced by g on $\wedge T^*M$ (see [5] for details). In the formulas above, $(e_a)_{a=1 \ldots d}$ denotes a local frame of TM and $(e^a)_{a=1 \ldots d}$ its dual local coframe (which satisfies $e^a(e_b) = \delta^a_b$ and $g(e^a, e^b) = g^{ab}$, where (g^{ab}) is the inverse of the matrix (g_{ab})). For latter reference, recall that the corresponding contragradient frame $(e^a)^\#$ and coframe $(e_a)_\#$ satisfy $(e^a)^\# = g^{ab} e_b$ and $(e_a)_\# = g_{ab} e^b$, where the $\#$ subscript and superscript denote the (mutually-inverse) musical isomorphisms between TM and T^*M, given respectively by lowering and raising of indices with the metric g. For latter reference the *main antiautomorphism* (also known as *reversion*) of the Kähler-Atiyah bundle is the involutive antiautomorphism which acts on global sections through:

$$\tau(\omega) = (-1)^{\frac{k(k-1)}{2}}\omega \quad , \quad \forall \omega \in \Omega^k(M) \quad .$$

The generalized products satisfy various identities which are consequence of associativity and unitality of the geometric product and of the fact that the volume form $\nu = \mathrm{vol}_M \in \Omega^d(M)$ of (M, g) satisfies:

$$\begin{aligned} \nu \diamond \nu &= (-1)^{q+[\frac{d}{2}]} 1_M = \begin{cases} (-1)^{\frac{p-q}{2}} 1_M \ , & \text{if } d = \text{even} \\ (-1)^{\frac{p-q-1}{2}} 1_M \ , & \text{if } d = \text{odd} \end{cases} \ , \\ \nu \diamond \omega &= \pi^{d-1}(\omega) \diamond \nu \quad , \quad \forall \omega \in \Omega(M) \ . \end{aligned}$$

Hence ν is central in the Kähler-Atiyah algebra $(\Omega(M), \diamond)$ when d is odd and twisted central (i.e., $\nu \diamond \omega = \pi(\omega) \diamond \nu$) when d is even. In Table 1, we indicate the values of $p - q$ (mod 8) for which the volume form ν has the corresponding properties. Various aspects of the geometric algebra formalism are discussed in detail in [5–7].

	$\nu \diamond \nu = +1$	$\nu \diamond \nu = -1$
ν is central	$\mathbf{1}(\mathbb{R}), \mathbf{5}(\mathbb{H})$	$\mathbf{3}(\mathbb{C}), \mathbf{7}(\mathbb{C})$
ν is twisted central	$\mathbf{0}(\mathbb{R}), \mathbf{4}(\mathbb{H})$	$\mathbf{2}(\mathbb{R}), \mathbf{6}(\mathbb{H})$

Table 1: Properties of the volume form.

Reconsidering (s)pinor bundles. A pinor bundle S can now be viewed as a bundle of modules over the Kähler-Atiyah bundle of (M, g), the module structure being defined by a morphism of bundles of algebras which we denote by $\gamma : (\wedge T^*M, \diamond) \to (\mathrm{End}(S), \circ)$. Since we are interested in pinors of spin 1/2, we assume that γ is fiberwise irreducible. Since $\mathrm{Cl}^{\mathrm{ev}}(T^*M)$ identifies with the sub-bundle of algebras $(\wedge^{\mathrm{ev}} T^*M, \diamond)$, a spinor bundle is a bundle of modules over the latter, being called a spin bundle when its fibers are simple modules. A particularly important role in the study of (s)pin bundles is played by the endomorphism $\gamma(\nu) \in \Gamma(M, \mathrm{End}(S))$, which is central or twisted central in the algebra $(\Gamma(M, \mathrm{End}(S)), \circ)$ depending on the value of $p - q$ (mod 8).

Local expressions. Given a local pseudo-orthonormal frame (e_a) of (M, g) with dual local coframe (e^a), a general inhomogeneous form $\omega \in \Omega(M)$ expands as:

$$\omega = \sum_{k=0}^{d} \omega^{(k)} =_U \sum_{k=0}^{d} \frac{1}{k!} \omega^{(k)}_{a_1 \ldots a_k} e^{a_1 \ldots a_k} \quad \text{with} \quad \omega^{(k)} \in \Omega^k(M) \quad , \tag{1.2}$$

where $e^{a_1 \ldots a_k} \stackrel{\mathrm{def.}}{=} e^{a_1} \wedge \ldots \wedge e^{a_k}$ and the symbol $=_U$ means that equality holds only after restriction of ω to U . We let $\gamma^a \stackrel{\mathrm{def.}}{=} \gamma(e^a) \in \Gamma(U, \mathrm{End}(S))$ and $\gamma_a \stackrel{\mathrm{def.}}{=} g_{ab}\gamma^b \in \Gamma(U, \mathrm{End}(S))$ be the contravariant and covariant 'gamma matrices' associated with the given local orthonormal (co)frame and $\gamma_{a_1 \ldots a_k}$ denote the complete antisymmetrization of the composition $\gamma_{a_1} \circ \ldots \circ \gamma_{a_k}$.

Spin projectors and spin bundles. Giving a direct sum bundle decomposition $S = S_+ \oplus S_-$ amounts to giving a *product structure* on S, i.e. a bundle endomorphism $\mathcal{R} \in \Gamma(M, \mathrm{End}(S)) \setminus \{-\mathrm{id}_S, \mathrm{id}_S\}$ satisfying:

$$\mathcal{R}^2 = \mathrm{id}_S \quad .$$

A product structure shall be called a *spin endomorphism* if it also satisfies:

$$[\mathcal{R}, \gamma(\omega)]_{-,\circ} = 0 \quad , \quad \forall \omega \in \Omega^{\mathrm{ev}}(M) \quad .$$

A spin endomorphism exists only when $p - q \equiv_8 0, 4, 6, 7$. When S is a pin bundle, the restriction $\gamma_{\mathrm{ev}} \stackrel{\mathrm{def.}}{=} \gamma|_{\wedge^{\mathrm{ev}} T^*M} : (\wedge^{\mathrm{ev}} T^*M, \diamond) \to (\mathrm{End}(S), \circ)$ is fiberwise reducible iff. S admits a spin endomorphism, in which case we define the *spin projectors* determined by $\mathcal{R}$ to be the globally-defined endomorphisms $\mathcal{P}^{\mathcal{R}}_{\pm} \stackrel{\mathrm{def.}}{=} \frac{1}{2}(\mathrm{id}_S \pm \mathcal{R})$, which are complementary idempotents in $\Gamma(M, \mathrm{End}(S))$. The eigen-subbundles $S^{\pm} \stackrel{\mathrm{def.}}{=} \mathcal{P}^{\mathcal{R}}_{\pm}(S)$ corresponding to the eigenvalues ± 1 of $\mathcal{R}$ are complementary in S and $\mathcal{R}$ determines a nontrivial direct sum decomposition $\gamma_{\mathrm{ev}} = \gamma^+ \oplus \gamma^-$.

The effective domain of definition of γ. We let $\wedge^{\pm} T^*M$ denote the bundle of twisted (anti-)selfdual forms [5]. Its space $\Omega^{\pm}(M) \stackrel{\mathrm{def.}}{=} \Gamma(M, \wedge^{\pm} T^*M)$ of smooth global sections is the $\mathcal{C}^{\infty}(M, \mathbb{R})$-module consisting of those forms $\omega \in \Omega(M)$ which satisfy the condition $\omega \diamond \nu = \pm \omega$. Defining:

$$\wedge^{\gamma} T^*M \stackrel{\mathrm{def.}}{=} \begin{cases} \wedge T^*M \ , & \text{if } \gamma \text{ is fiberwise injective (simple case)} \ , \\ \wedge^{\epsilon_\gamma} T^*M \ , & \text{if } \gamma \text{ is not fiberwise injective (non} - \text{simple case)} \ , \end{cases}$$

$$\wedge^{-\gamma} T^*M \stackrel{\mathrm{def.}}{=} \begin{cases} 0 \ , & \text{if } \gamma \text{ is fiberwise injective (simple case)} \ , \\ \wedge^{-\epsilon_\gamma} T^*M \ , & \text{if } \gamma \text{ is not fiberwise injective (non} - \text{simple case)} \ , \end{cases}$$

one finds that γ restricts to zero on $\wedge^{-\gamma} T^*M$ and to a monomorphism of vector bundles on $\wedge^{\gamma} T^*M$. Due to this fact, we say that $\wedge^{\gamma} T^*M$ is the *effective domain of definition* of γ.

Schur algebras and representation types. Let S be a pin bundle of (M, g) and x be any point of M. The *Schur algebra* of γ_x is the unital subalgebra $\Sigma_{\gamma,x}$ of $(\mathrm{End}(S_x), \circ)$ defined through:

$$\Sigma_{\gamma,x} \stackrel{\mathrm{def.}}{=} \{T_x \in \mathrm{End}(S_x) \mid [T_x, \gamma_x(\omega_x)]_{-,\circ} = 0 \ , \ \forall \omega_x \in \wedge T^*_x M\} \quad .$$

The subset $\Sigma_\gamma = \{(x, T_x) \mid x \in M \ , \ T_x \in \Sigma_{\gamma,x}\} = \sqcup_{x \in M} \Sigma_{\gamma,x}$ is a sub-bundle of unital algebras of the bundle of algebras $(\mathrm{End}(S), \circ)$ called the *Schur bundle* of γ. The isomorphism type of the fiber $(\Sigma_{\gamma,x}, \circ_x)$ is denoted by $\mathbb{S}$, being called the *Schur algebra* of γ. Real pin bundles S fall into three classes: *normal*, *almost complex* or *quaternionic*, depending on whether their Schur algebra is isomorphic with $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Some of the properties of these types are summarized in the tables below. When γ is fiberwise irreducible (i.e. in the case of pin bundles), the Schur algebra depends only on $p - q$ (mod 8), being indicated in parantheses in Tables 1 and 3. The

$\mathbb{S}$	$p-q$ mod 8	$\wedge T_x^*M \approx \mathrm{Cl}(p,q)$	Δ	N	Number of choices for γ	$\gamma_x(\wedge T_x^*M)$	Fiberwise injectivity of γ
$\mathbb{R}$	$\mathbf{0}, \mathbf{2}$	$\mathrm{Mat}(\Delta, \mathbb{R})$	$2^{[\frac{d}{2}]} = 2^{\frac{d}{2}}$	$2^{[\frac{d}{2}]}$	1	$\mathrm{Mat}(\Delta, \mathbb{R})$	injective
$\mathbb{H}$	$\mathbf{4}, \mathbf{6}$	$\mathrm{Mat}(\Delta, \mathbb{H})$	$2^{[\frac{d}{2}]-1} = 2^{\frac{d}{2}-1}$	$2^{[\frac{d}{2}]+1}$	1	$\mathrm{Mat}(\Delta, \mathbb{H})$	injective
$\mathbb{C}$	$\mathbf{3}, \mathbf{7}$	$\mathrm{Mat}(\Delta, \mathbb{C})$	$2^{[\frac{d}{2}]} = 2^{\frac{d-1}{2}}$	$2^{[\frac{d}{2}]+1}$	1	$\mathrm{Mat}(\Delta, \mathbb{C})$	injective
$\mathbb{H}$	$\mathbf{5}$	$\mathrm{Mat}(\Delta, \mathbb{H})^{\oplus 2}$	$2^{[\frac{d}{2}]-1} = 2^{\frac{d-3}{2}}$	$2^{[\frac{d}{2}]+1}$	2 ($\epsilon_\gamma = \pm 1$)	$\mathrm{Mat}(\Delta, \mathbb{H})$	non-injective
$\mathbb{R}$	$\mathbf{1}$	$\mathrm{Mat}(\Delta, \mathbb{R})^{\oplus 2}$	$2^{[\frac{d}{2}]} = 2^{\frac{d-1}{2}}$	$2^{[\frac{d}{2}]}$	2 ($\epsilon_\gamma = \pm 1$)	$\mathrm{Mat}(\Delta, \mathbb{R})$	non-injective

Table 2: Summary of pin bundle types. $N \overset{\text{def.}}{=} \mathrm{rk}_{\mathbb{R}} S$ is the real rank of S while $\Delta \overset{\text{def.}}{=} \mathrm{rk}_{\Sigma_\gamma} S$ is the Schur rank of S. The non-simple cases are indicated through the blue shading of the corresponding table cells. The red color indicates those cases for which a spin endomorphism can be defined.

real Clifford algebra $\mathrm{Cl}(p,q)$ (which, up to isomorphism, coincides with any fiber of the Kähler-Atiyah bundle) is non-simple iff. $p-q \equiv_8 1, 5$ (this is indicated in the tables through the blue shading).

	injective	non-injective
surjective	$\mathbf{0}(\mathbb{R}), \mathbf{2}(\mathbb{R})$	$\mathbf{1}(\mathbb{R})$
non-surjective	$\mathbf{3}(\mathbb{C}), \mathbf{7}(\mathbb{C}), \mathbf{4}(\mathbb{H}), \mathbf{6}(\mathbb{H})$	$\mathbf{5}(\mathbb{H})$

Table 3: Fiberwise character of real pin representations γ.

Fiberwise injectivity and surjectivity of γ. Basic facts from the representation theory of Clifford algebras imply:

1. γ is *fiberwise injective* iff. $\mathrm{Cl}(p,q)$ is simple as an associative $\mathbb{R}$-algebra, i.e. iff. $p-q \not\equiv_8 1, 5$ (the so-called *simple case*).

2. When γ is fiberwise *non-injective* (i.e. when $p-q \equiv_8 1, 5$, the so-called *non-simple case*), we have $\gamma(\nu) = \epsilon_\gamma \mathrm{id}_S$, where the sign factor $\epsilon_\gamma \in \{-1, 1\}$ is called the *signature* of γ. The two choices for ϵ_γ lead to two inequivalent choices for γ. The fiberwise injectivity and surjectivity of γ are summarized in Table 3.

Using the approach outlined above, one can re-formulate numerous constructions which are common in spin geometry and its applications to gravitational physics (in particular, to supergravity and string theory). For example, one can give [7] a unified, systematic and computationally efficient approach to a certain class of Fierz identities which are central in the study of supergravity backgrounds. One can also use this approach to develop [5, 6] certain aspects of spin geometry in a manner which allows progress in the analysis and classification of flux backgrounds as well as in the analysis of effective actions in string theory. Since a full treatment of each of these directions is quite technical and involved, we shall merely illustrate this with an example.

2 Application to general $\mathcal{N}=2$ flux compactifications of eleven-dimensional supergravity on eight-manifolds.

Consider eleven-dimensional supergravity on a connected, oriented eleven-manifold $\tilde{M}$ admitting a spin structure. The physical fields are the metric $\tilde{g}$ (taken to be of mostly plus Lorentzian signature), the three-form potential $\tilde{C}$ with non-trivial four-form field strength $\tilde{G}$ and the gravitino $\tilde{\Psi}_M$. The pin bundle $\tilde{S}$ of $\tilde{M}$ can be viewed as a bundle of simple modules over the Clifford bundle of $T^*\tilde{M}$. The supersymmetry generator is a section of $\tilde{S}$. Vanishing of the supersymmetry variation of the gravitino requires:

$$\delta_{\tilde{\eta}}\tilde{\Psi}_M \stackrel{\text{def.}}{=} \tilde{\mathcal{D}}_M\tilde{\eta} = 0 \quad . \tag{2.1}$$

The 'supercovariant derivative' $\tilde{\mathcal{D}}_M$ takes the form: $T\tilde{M}$,

$$\tilde{\mathcal{D}}_M \stackrel{\text{def.}}{=} \tilde{\nabla}^{\text{spin}}_M - \frac{1}{288}\left(\tilde{G}_{NPQR}\tilde{\Gamma}^{NPQR}{}_M - 8\tilde{G}_{MNPQ}\tilde{\Gamma}^{NPQ}\right) \tag{2.2}$$

in a local orthonormal frame $(\tilde{e}_M)_{M=0\ldots10}$, where $\tilde{\nabla}^{\text{spin}}_M = \partial_M + \frac{1}{4}\tilde{\omega}_{MNP}\tilde{\Gamma}^{NP}$ is the connection induced on $\tilde{S}$ by the Levi-Civita connection of $\tilde{M}$, $\tilde{\omega}_{MNP}$ are the totally covariant spin connection coefficients and $\tilde{\Gamma}_M$ are the gamma 'matrices' of $\mathrm{Cl}(10,1)$ in the irreducible representation characterized by $\hat{\Gamma}_{11} = +\mathrm{id}_{\tilde{S}}$. Setting $\tilde{M} = M_3\times M$ with usual warped product ansatz for $\tilde{g}$ (see [5, 8–10]), condition (2.1) reduces to the following two conditions (known as the *constrained generalized Killing (CGK) pinor equations* [5]) for the internal part $\xi\in\Gamma(M,S)$ of $\tilde{\eta}$, which is a section of the pin bundle S of the internal manifold M:

$$D_m\xi = 0 \quad \text{with} \quad D_m \stackrel{\text{def.}}{=} \nabla^{\text{spin}}_m + A_m \quad , \quad A_m = -\frac{1}{4}f_n\gamma^n{}_m\gamma_9 + \frac{1}{24}F_{mpqr}\gamma^{pqr} + \kappa\gamma_m\gamma_9 \ , \tag{2.3}$$

$$Q\xi = 0 \quad \text{with} \quad Q = \frac{1}{2}\gamma^m\partial_m\Delta - \frac{1}{288}F_{mnpq}\gamma^{mnpq} - \frac{1}{6}f_m\gamma^m\gamma_9 - \kappa\gamma_9 \ , \quad \forall m,n,p,q = 1\ldots 8 \ . \tag{2.4}$$

We refer to conditions (2.3) and (2.4) above as the differential and algebraic constraints, respectively; they have two independent global solutions (ξ_1,ξ_2) when the background preserves $\mathcal{N}=2$ supersymmetry (see [5, 6]). In (2.3) and (2.4), the gamma 'matrices' γ^m transform in the representations of the real Clifford algebra $\mathrm{Cl}(8,0)$, while κ is a positive real number which is proportional to the square root of the cosmological constant of the external AdS_3 space.

The Fierz isomorphism. When (M,g) has Euclidean signature with $d\equiv_8 0,1$, the morphism $\gamma : (\wedge T^*M,\diamond)\to(\mathrm{End}(S),\circ)$ is surjective and has a partial inverse [5] which allows one to translate the CGK pinor conditions into conditions on differential forms on M constructed as bilinear combinations of sections of S. The process is encoded by the so-called *Fierz isomorphism* [5], which identifies the bundle of bispinors $S\otimes S$ with a sub-bundle $\wedge^\gamma T^*M$ of the exterior bundle playing the role of 'effective domain of definition' [2] of γ. As explained in detail in [5, 7, 11, 12], one can construct *admissible bilinear pairings* $\mathcal{B}$ on the pin bundle satisfying certain properties[3], using which one defines a $\mathcal{B}$-dependent bundle isomorphism:

$$E : S\otimes S \to \mathrm{End}(S) \;\; \text{where} \;\; E_{\xi,\xi'}(\xi'') \overset{\text{def.}}{=} \mathcal{B}(\xi'',\xi')\xi \;\; \text{such that} \quad E_{\xi_1,\xi_2}\circ E_{\xi_3,\xi_4} = \mathcal{B}(\xi_3,\xi_2)E_{\xi_1,\xi_4} \;\; ,$$

which induces the Fierz isomorphism $\quad \check{E} = \gamma^{-1}\circ E : S\otimes S \to \wedge^{\gamma}T^*M \quad$ mentioned above.

Lift to a nine-manifold. To investigate general supersymmetric compactifications on eight-manifolds, one must analyze the CGK pinor equations (2.3), (2.4). The requirement of $\mathcal{N}=2$ supersymmetry means that these equations must have two linearly independent global solutions $\xi_1,\xi_2\in\Gamma(M,S)$ — whose values $\xi_1(x),\xi_2(x)\in S_x$ then are [6] also linearly independent at any point x of M. To simplify the analysis, one can lift [6] the problem to the nine-dimensional metric cone $\hat{M}$ over M. The solutions (ξ_1,ξ_2) determine a point on the second Stiefel manifold $V_2(S_x)$ of each fiber S_x of S. Thus solutions can be classified according to the orbit of the representation of $\mathrm{Spin}(8)$ on S_x, which induces a corresponding action on $V_2(S_x)$. Since the latter action fails to be transitive, a generic basis (ξ_1,ξ_2) of solutions of the CGK pinor equations does not determine a global reduction of the structure group of M. Using the natural embedding $\mathrm{Spin}(8)\subset\mathrm{Spin}(9)\subset\mathrm{Cl}(9,0)$, the action of $\mathrm{Spin}(8)$ on $V_2(S_x)$ extends to an action of $\mathrm{Spin}(9)$, which turns out to be transitive — thereby suggesting that an interpretation in terms of reduction of structure group could be given upon passing to some Riemannian nine-manifold naturally associated with M. The action of $\mathrm{Spin}(9)$ can indeed be geometrized [6] by passing to the metric cone $\hat{M}$ over M, whose Clifford bundle is of non-simple type [5–7, 13]. Using the construction of [6], this allows one to provide a global description of 8-manifold compactifications preserving $\mathcal{N}=2$ supersymmetry through reduction of the structure group of $\hat{M}$ (rather than of M itself). We therefore consider the (punctured) metric cone $\hat{M}=(0,+\infty)\times M$ (with squared line element given by $\mathrm{d}s^2_{\mathrm{cone}}=\mathrm{d}r^2+r^2\mathrm{d}s^2$), where the canonical normalized one-form $\theta=\mathrm{d}r$ is a special Killing-Yano form with respect to the metric g_{cone} of $\hat{M}$. We can pull back the Levi-Civita connection of (M,g) along the natural projection $\Pi:\hat{M}\to M$ and also lift the connection (2.3) from S to the pin bundle $\hat{S}$ of $\hat{M}$ – using the fact [6] that $\hat{S}$ can be identified with the Π-pullback of S. We define $\Pi^*(h)=h\circ\Pi$ for $h\in\mathcal{C}^\infty(M,\mathbb{R})$. It turns out that the morphism $\gamma_{\mathrm{cone}}:\wedge T^*\hat{M}\to\mathrm{End}(\hat{S})$ is determined by the morphism γ of the manifold M. The bundle morphism γ_{cone} is fiberwise surjective but not fiberwise injective, so there are two inequivalent choices distinguished by the property $\gamma_{\mathrm{cone}}(\nu_{\mathrm{cone}})=\pm\mathrm{id}_{\hat{S}}$, where ν_{cone} is the volume form of $\hat{M}$. We choose to work with the representation of signature $+1$, which satisfies $\gamma_{\mathrm{cone}}(\nu_{\mathrm{cone}})=+\mathrm{id}_{\hat{S}}$. Then the Kähler-Atiyah algebra $(\Omega(\hat{M}),\diamond^{\mathrm{cone}})$ of the cone can be realized through the truncated model $(\Omega^<(\hat{M}),\blacklozenge^{\mathrm{cone}})$ of [6], where $\Omega^<(\hat{M})=\oplus_{k=0}^4\Omega^k(\hat{M})$ and $\blacklozenge^{\mathrm{cone}}$ is the reduced geometric product of the cone. Rescaling the metric through $g\to 2\kappa^2 g$, the connection induced on $\hat{S}$ by

[2] When $d\equiv_8 0$ in Euclidean signature, the bundle morphism γ is fiberwise injective and we have $\wedge^{\gamma}T^*M=\wedge T^*M$. When $d\equiv_8 1$, the effective domain of γ is a proper sub-bundle of the exterior bundle since γ fails to be injective in that case.

[3] These properties are symmetry, type and — when applicable — isotropy

the Levi-Civita connection of $\hat{M}$ takes the form:

$$\nabla^{\hat{S}}_{e_m} = \nabla^{S}_{e_m} + \kappa\gamma_{9m} \quad , \quad \nabla^{\hat{S}}_{\partial_r} = \partial_r \quad , \quad \text{where} \quad m = 1 \dots 8 \quad .$$

Defining the connection $\hat{D} \stackrel{\text{def.}}{=} D^* = \nabla^{\hat{S},\text{cone}} + A^{\text{cone}}$ as the pullback of (2.3) to $\hat{M}$, the expression for A^{cone} becomes [6]:

$$A^{\text{cone}} = \Pi^*(A) - \frac{1}{2r} e^m_{\text{cone}} \otimes \gamma_{\text{cone}}((e^{\text{cone}}_m)_{\sharp_{\text{cone}}} \wedge \theta) \quad ,$$

where $\Pi^*(A)$ denotes the pull-back connection and $(e^{\text{cone}}_m)_{\sharp_{\text{cone}}}$ is the one-form dual to the base vector e^{cone}_m with respect to the metric on $\hat{M}$. Lifting to the cone and 'dequantizing' as in [5, 6], the connection A_m of (2.3) and the endomorphism Q from (2.4) induce the following inhomogeneous differential forms on $\hat{M}$:

$$\check{A}_m = \frac{1}{4}\iota_{e^{\text{cone}}_m} F + \frac{1}{4}(e^{\text{cone}}_m)_\sharp \wedge f \wedge \theta \quad , \quad \check{Q} = \frac{1}{2} r(\text{d}\Delta) - \frac{1}{6} f \wedge \theta - \frac{1}{12} F - \kappa\theta \; . \tag{2.5}$$

Analysis of CGK pinor equations. We are interested in form-valued pinor bilinears written locally as follows (where the subscript '+' denotes the twisted self-dual part [5, 6]) :

$$\check{E}^{(k)}_{\hat{\xi}_i,\hat{\xi}_j} \equiv \check{E}^{(k)}_{ij} = \frac{1}{k!}\mathcal{B}(\hat{\xi}_i, \gamma^{\text{cone}}_{a_1\dots a_k}\hat{\xi}_j) e^{a_1\dots a_k}_+ \quad , \quad \text{where} \quad a_1 \dots a_k \in \{1 \dots 9\} \quad ,$$

with $i, j \in \{1, 2\}$ and in the weighted sums:

$$\check{E}_{ij} = \frac{N}{2^d} \sum_{k=0}^{d} \check{E}^{(k)}_{ij} \quad ,$$

which are inhomogeneous differential forms generating the algebra $(\Omega^+(\hat{M}), \diamond)$. Here, $N = 2^{\left[\frac{d}{2}\right]}$ is the real rank of the pin bundle S, $d = 9$ is the dimension of $\hat{M}$ and we normalized the two pinors through $\mathcal{B}(\xi_i, \xi_j) = \delta_{ij}$. Using the properties of the admissible bilinear paring on $\hat{S}$, we find:

$$\mathcal{B}(\hat{\xi}_i, \gamma^{a_1\dots a_k}_{\text{cone}}\hat{\xi}_j) = (-1)^{\frac{k(k-1)}{2}} \mathcal{B}(\hat{\xi}_j, \gamma^{a_1\dots a_k}_{\text{cone}}\hat{\xi}_i) \quad , \quad \forall i, j = 1, 2 \quad ,$$

which implies that the non-trivial form-valued bilinears of rank ≤ 4 are three 1-forms V_k dual to the vector fields with local coefficients given (after raising indices to avoid notational clutter) by:

$$V^a_1 = \mathcal{B}(\hat{\xi}_1, \gamma^a_{\text{cone}}\hat{\xi}_1) \quad , \quad V^a_2 = \mathcal{B}(\hat{\xi}_2, \gamma^a_{\text{cone}}\hat{\xi}_2) \quad , \quad V^a_3 = \mathcal{B}(\hat{\xi}_1, \gamma^a_{\text{cone}}\hat{\xi}_2) \tag{2.6}$$

together with one 2-form K, one 3-form Ψ and three 4-forms ϕ_k with strict coefficients given similarly by:

$$K^{ab} = \mathcal{B}(\hat{\xi}_1, \gamma^{ab}_{\text{cone}}\hat{\xi}_2) \; , \; \psi^{abc} = \mathcal{B}(\hat{\xi}_1, \gamma^{abc}_{\text{cone}}\hat{\xi}_2) \; , \; \phi^{abce}_1 = \mathcal{B}(\hat{\xi}_1, \gamma^{abce}_{\text{cone}}\hat{\xi}_1) \; , \; \phi^{abce}_2 = \mathcal{B}(\hat{\xi}_2, \gamma^{abce}_{\text{cone}}\hat{\xi}_2) \; , \; \phi^{abce}_3 = \mathcal{B}(\hat{\xi}_1, \gamma^{abce}_{\text{cone}}\hat{\xi}_2).$$

The forms above are the definite rank components of the truncated inhomogeneous forms:

$$\check{E}^<_{11} = \frac{1}{32}(1 + V_1 + \phi_1) \; , \; \check{E}^<_{12} = \frac{1}{32}(V_3 + K + \psi + \phi_3) \; , \; \check{E}^<_{21} = \frac{1}{32}(V_3 - K - \psi + \phi_3) \; , \; \check{E}^<_{22} = \frac{1}{32}(1 + V_2 + \phi_2)$$

which are the basis elements of the truncated Fierz algebra of [5, 6] and satisfy the truncated Fierz identities[4]:

$$\check{E}^{<}_{ij}\blacklozenge\check{E}^{<}_{kl} = \frac{1}{2}\delta_{jk}\check{E}^{<}_{il} \quad , \quad \forall i,j,k,l = 1,2 \quad . \tag{2.7}$$

The full analysis of these equations is quite involved [14]. We list only some of the relations implied by the truncated Fierz identities:

$$\iota_{V_1}V_3 = 0 \quad , \quad \iota_{V_1}\phi_3 - \psi + V_1\wedge K = 0 \quad , \quad \iota_{V_3}\phi_1 + \psi - V_1\wedge K = 0 \quad . \tag{2.8}$$

Let us also discuss the constraints on these forms implied by the CGK pinor equations. As explained in [5, 6], the differential constraints (2.3) imply:

$$\mathrm{d}\check{E}^{<}_{ij} = e^a\wedge\nabla_a\check{E}^{<}_{ij} \quad \text{where} \quad \nabla_a\check{E}^{<}_{ij} = -[\check{A}_a, \check{E}^{<}_{ij}]_{-,\blacklozenge} \quad , \quad \forall i,j\in\{1,2\} \ , \tag{2.9}$$

whereas the algebraic constraints (2.4) reduce to:

$$\check{Q}\blacklozenge\check{E}^{<}_{ij} \mp \check{E}^{<}_{ij}\blacklozenge\hat{\tau}(\check{Q}) = 0 \quad , \quad \forall i,j\in\{1,2\} \quad . \tag{2.10}$$

The complete set of conditions can be found in [14], only part of which will be reproduced here. The first equation in (2.10) (the one with the *minus* sign) for $\check{E}^{<}_{12}$ leads to the following constraints when separating into rank components:

$$\iota_{f\wedge\theta}K = 0 \quad , \tag{2.11}$$

$$r\iota_{\mathrm{d}\Delta}K + \frac{1}{3}\iota_{f\wedge\theta}\psi - \frac{1}{6}\iota_\psi F - 2\kappa\iota_\theta K = 0 \quad , \tag{2.12}$$

$$\frac{1}{3}\iota_{f\wedge\theta}\phi_3 - \frac{1}{6}F\bigtriangleup_3\phi_3 + r(\mathrm{d}\Delta)\wedge V_3 + 2\kappa V_3\wedge\theta = 0 \quad , \tag{2.13}$$

while using the truncated inhomogeneous form $\check{E}^{<}_{11}$ amounts to:

$$-\frac{1}{3}f\wedge\theta + \frac{1}{3}\iota_{f\wedge\theta}\phi_1 - \frac{1}{6}F\bigtriangleup_3\phi_1 + r(\mathrm{d}\Delta)\wedge V_1 + 2\kappa V_1\wedge\theta = 0 \quad . \tag{2.14}$$

Expanding (2.9) for $\check{E}^{<}_{11}$ and $\check{E}^{<}_{12}$ gives the following expressions for the covariant derivatives of V_1 and V_3:

$$\nabla_m V_{1n} = \frac{1}{2}f_s\theta_p{\phi_1}^{sp}{}_{mn} - \frac{1}{12}F_{spqm}{\phi_1}^{spq}{}_n \quad \text{and} \quad \nabla_m V_{3n} = \frac{1}{2}f_s\theta_p{\phi_3}^{sp}{}_{mn} - \frac{1}{12}F_{spqm}{\phi_3}^{spq}{}_n \quad . \tag{2.15}$$

Upon antisymmetrization, these give the differential constraints:

$$\mathrm{d}V_1 = \iota_{f\wedge\theta}\phi_1 - \frac{1}{2}F\bigtriangleup_3\phi_1 \quad \text{and} \quad \mathrm{d}V_3 = \iota_{f\wedge\theta}\phi_3 - \frac{1}{2}F\bigtriangleup_3\phi_3 \ .$$

Using the differential and algebraic constraints for the form-valued pinor bilinears, one can investigate [14] the geometric implications of the $\mathcal{N}=2$ supersymmetry condition. This analysis, as well as the a discussion of the physics implications, is rather involved and we shall not attempt to summarize it here.

[4] Note that we will omit writing the 'cone' superscript on $\blacklozenge^{\mathrm{cone}}$ to simplify notation.

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