Aspects of T-branes

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Obras que sólo habían sido vistas, hasta entonces, en las arquitecturas imaginarias...

Alejo Carpentier, *El reino de este mundo*

Abstract

This thesis explores the BPS-stability of T-brane configurations of 7-branes including α-corrections as well as stability on compact 4-cycles in the absence of defects and including them.

First, we study α'-corrections in multiple D7-brane configurations with non-commuting profiles for their transverse position fields. We focus on T-brane systems, crucial in F-theory GUT model building. There α'-corrections modify the D-term piece of the BPS equations which, already at leading order, require a non-primitive Abelian worldvolume flux background. We find that α'-corrections may either *i)* leave this flux background invariant, *ii)* modify the Abelian non-primitive flux profile, or *iii)* deform it to a non-Abelian profile. The last case typically occurs when primitive fluxes, a necessary ingredient to build 4d chiral models, are added to the system. We illustrate these three cases by solving the α'-corrected D-term equations in explicit examples, and describe their appearance in more general T-brane backgrounds. Finally, we discuss implications of our findings for F-theory GUT local models.

Secondly, we analyse global aspects of 7-brane backgrounds with a non-commuting profile for their worldvolume scalars, also known as T-branes. In particular, we consider configurations with no poles and globally well-defined over a compact Kähler surface. We find that such T-branes cannot be constructed on surfaces of positive or vanishing Ricci curvature. For the existing T-branes, we discuss their stability as we move in Kähler moduli space at large volume and provide examples of T-branes splitting into non-mutually-supersymmetric constituents as they cross a stability wall.

Lastly, we consider the effects of defects on the stability of T-brane systems. Such defects are induced by the presence of 7-branes on additional four-cycles intersecting the locus of the T-brane system. Coupling of the fields on both stacks modifies the BPS-equations and we find that it allows for T-branes on four-cycles that do not allow for stable T-branes in absence of defects due to the topological obstructions mentioned before. One class of these solutions feature poles in the Higgs-field profile. By performing a Kaluza-Klein expansion we show that in four dimensions the presence of these poles translates to defect-

zero-modes giving a vev to KK-modes. Finally, by taking a suitable limit, we show that in the case of a self-intersecting four-cycle, the defect picture can be linked to an eight-dimensional Higgs-field valued in a larger gauge algebra.

Resumen

Esta tesis explora la estabilidad BPS de configuraciones de T-branas de 7-branas incluyendo tanto correcciones α' como estabilidad en 4-ciclos compactos.

Primero estudiamos correcciones α' en configuraciones de varias D7-branas con un perfil no-conmutante de los campos de posición transversa. Nos enfocamos en sistemas de T-branas, los cuales son esenciales en el contexto de la construcción de modelos GUT en teoría F. En estos sistemas las correcciones en α' modifican los términos D de las ecuaciones BPS, requiriendo un flujo Abeliano no-primitivo ya a primer orden. Encontramos que las correcciones α' pueden por un lado i) dejar este flujo invariante, ii) modificar el perfil del flujo Abeliano no-primitivo, o $i$ii) deformarlo a un perfil no-Abeliano. Este último caso ocurre típicamente cuando se añaden flujos primitivos, un ingrediente necesario para construir modelos quirales en 4d, al sistema. Ilustramos estos tres casos resolviendo las ecuaciones D, incluyendo las correcciones α', en ejemplos explícitos y describiendo su aparición en casos más generales. Por último discutimos las implicaciones de nuestros resultados en modelos locales de GUTs en teoría F.

En segundo lugar analizamos aspectos globales de fondos de 7-branas con perfiles no-conmutantes de los escalares de la teoría en la superficie de la brana, conocidos como T-branas. En particular, consideramos configuraciones sin polos que son globalmente bien definidas sobre superficies de Kähler compactas. Encontramos que no se pueden construir dichas T-branas en superficies con curvatura de Ricci positiva o cero. Discutimos la estabilidad de las T-branas existentes, en función de la posición en el espacio de Kähler moduli en el límite de gran volumen. Además añadimos ejemplos de descomposición de T-branas en sus componentes no-supersimetricos cruzando un muro de estabilidad.

Finalmente consideramos defectos en sistemas de T-branas y sus consecuencias para la estabilidad. Dichos defectos están inducidos por la presencia de 7-branas en cuatro-ciclos adicionales que cortan el locus del sistema de T-branas. El acoplamiento de los campos en ambos conjuntos de branas modifica las ecuaciones BPS. En consecuencia, vemos que permite T-branas en cuatro-ciclos que, en ausencia de defectos topológicos, no dan lugar a configuraciones estables de

T-branas. Una clase de estas soluciones muestra polos en el perfil del campo de Higgs. Haciendo una expansión Kaluza-Klein, demostramos que en cuatro dimensiones se puede entender la presencia de estos polos como modos-cero de defectos dando vev a KK-modes. Por último, tomamos un limite adecuado para demostrar que en cuatro-ciclos con auto-intersecciones, se puede relacionar la perspectiva de defectos con un campo Higgs en una algebra gauge más amplia en ocho dimensiones.

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Chapter 1

Introduction

During the last century our understanding of physics has been deeply altered by insights that eventually led to the formulation of what today constitutes the two pillars of modern physics: The realisation that time is not an absolute quantity, but instead depends on the observer's velocity, led to the formulation of special relativity at the beginning of last century. A decade later, the insight that spacetime itself is dynamically shaped by the matter and energy it contains, rather than being a static stage upon which history unfolds, gave us general relativity and ultimately what lies at the core of present day cosmology and the laws governing the largest structures in our universe. At the same time, looking ever closer at the smallest structures in the matter surrounding us, forced us to first abandon our deterministic world view in lieu of the probabilistics of quantum mechanics and then, in the 1920's, to depart from the idea that, what we see is comprised of inseparable, elementary particles: The central premise of quantum field theory is instead, that fields and their interactions with one another are the fundamental entity; what is conserved is energy, momentum and local charges rather than the number or type of particles. Quantum field theory in its incarnation in particle physics — the standard model — is what forms our current understanding of physics at small scales since its formulation during the 60's and 70's.

Both theories, general relativity in its description of cosmology, as well as the standard model in explaining subatomic interactions are incredibly successful

— perhaps more so than its inventors anticipated. Crucially, both theories contain a characteristic scale at which their effects become relevant compared to classical Newtonian physics. Most fields of science lie entirely within the realm of either one of the two theories, while effects of the other are insignificant. There are however phenomena that require to take into account effects of both theories, such as early universe physics and certain properties of black holes. Moreover, a theory incorporating effects of both realms could potentially also lead to insights in beyond the standard model physics such as for instance dark matter or dark energy. Unfortunately, the task of uniting the two theories into a common framework has proven to be hard, so far. On a technical level, this is because general relativity is a non-renormalisable field theory. Of course it is extremely desirable to find such a unified description. Perhaps the most convincing attempt to do so has been provided in the mid 1980's by string theory, with its study still continuing at present day. A theory parting from the assertion that both general relativity as well as the standard model are effective theories, valid only at low energies, because we are neglecting microscopic degrees of freedom that become important at higher scales. String theory suggests, that matter is not comprised of particles but instead strings, whose vibrations provide these aforementioned additional degrees of freedom.

While string theory does indeed provide a unified quantum description of gravity and the standard model, it is not a is well-understood one. Fifty years after its initial results we are far from a working description of our universe in terms of string theory. More so, there are no falsifiable predictions made by string theory, which led to the famous criticism that anything can be explained using string theory. Of course the lack of such predictions is due to our current inability to construct experiments at the relevant energy scales, rather than a shortcoming of the theory itself. The key difficulty in making predictions with string theory can be understood from the last paragraphs: Most of historic progress in physics was driven by an interchange of experiment and theory; new discoveries required explanations which in turn predicted new discoveries. Today however, the situation is different: Predictions of general relativity as well as the standard model have proven exact to a staggering degree and while there are many unexplained observations, we most often miss a smoking gun in the form of a new particle not contained in the standard model, for instance. More importantly, the energy scale at which new physics becomes visible may be very high: Combining the characteristic scales of gravity and quantum physics determines an energy at which effects of both theories become relevant and thereby provides an upper bound to where we should expect physics beyond the standard model. This upper bound lies sixteen orders of magnitude above current accelerator ranges. While it is of course true that — depending on your personal beliefs — you might expect new physics to be visible at lower scales, such as supersymmetry-breaking scale, GUT-scale, Kaluza-Klein scale or string-scale, there is little justification for those scales to be close to the energies we observe at current colliders. In fact in some cases there exist bounds that place these scales very high compared to energies probed by current experiments. The only exception perhaps being supersymmetry. So one key challenge of string theory is, that it provides a consistent and convincing quantum-theory of gravity, but its characteristic energy scale may lie well out of range of everything we are going to observe today and in the near future. For that reason, string theorists try to understand what predictions can be made at the energies of accelerators and cosmological observations. So far this has proven to be a challenging task. In addition to the lack of experimental data, it is fair to say that today's understanding of string theory is still rudimentary and many of the phenomenology-inspired local models of the last decades may prove to lack a global embedding at closer inspection. Of course these obstructions are no shortcoming of string theory itself or make it any less probable as a candidate for a theory of quantum gravity. Indeed, the simplicity of its assumptions as well as its deep connection with geometry make it very convincing to anyone studying it.

While string theory has been formulated as a perturbative theory of one-dimensional dynamical objects – *strings* – one of the most important insights has been the realisation that at the non-perturbative level, there is more to the story: There are five distinct (but supposedly dual) formulations of superstring theory, all of which contain closed strings, that is to say loops. Three of these theories however also contain open strings, that may end on subloci of the ten-dimensional spacetime. While these subloci were first seen as simple boundary conditions, it was soon realised that they encode non-perturbative objects that are themselves dynamical and should be seen on the same footing as fundamental strings, albeit their excitations lie at higher energies. The dynamics of these so called *D-branes* may be described via the theory of open strings ending on them. These theories are of great phenomenological interest as they naturally give rise to non-Abelian gauge theories. By considering more complicated constructions featuring intersecting stacks of branes, semi-realistic models for particle physics have been designed in the past.

While it is correct to think of individual branes as dynamical objects wrapping certain subloci of the ten-dimensional spacetime, the physics of multiple of such objects is more complicated. We refer the reader to standard text books, such as [1], for a general introduction on these topics. Since there may be strings stretching between these distinct objects, such brane configurations may also form bound states. These bound states were first investigated in [2–5], the latter of which introduced the term *T-brane* for this class of states. On a classical level, one may think of this bound state as being supported by a standing wave on the string stretching between two branes. These states, T-branes, amount to inherently non-Abelian phenomena of the worldvolume gauge theories. In the past these bound states have not only been studied for their phenomenological properties, especially in relation to realistic Yukawa-couplings [3–8], but also because of their role in string dualities [9–13]. Moreover, considering the central role, in particular of 7-branes, it is important to study not only the set of simple intersecting brane configurations, but instead all 7-brane configurations. In the following thesis we will study aspects of stability for this class of string theory vacua in the case of 7-branes.

One of the key tools in the study of T-branes is the worldvolume theory living on a stack of branes [14], which allows to study stability properties as well as dynamics of such a bound states from a local perspective. That is to say it is a theory on the 4-cycle wrapped by the 7-brane stack and does not take into account the embedding into its ambient space. This perspective allows us to study a T-brane state without specifying data about the global threefold geometry. More so, both D7-branes in perturbative IIB as well as (p,q)-7-branes in F-theory [15–17] share the same worldvolume theory, such that this local analysis holds in both contexts. We will review the relevant aspects of this worldvolume theory in chapter 2.

A different perspective on the study of T-brane vacua in perturbative string theory has been provided by Sen's tachyon condensation [18]. Not only does this provide a global point of view on D-brane bound states, but moreover gives insights on how a single bound state may decay into several distinct brane configurations when moving in Kähler moduli space [10]. We will therefore study this complementary perspective to the worldvolume theory in chapter 3.

A particularly interesting aspect in the study of T-branes is its interpretation in terms of string dualities: While these bound states arise completely natural in perturbative string theory as well as in the local worldvolume theory of 7-branes, their interpretation in a global F-theory compactification is less understood. This is due to the fact, that *T-brane data* is not translated to the geometry of the elliptic fibration. Instead to fully specify a global F-theory vacuum, one needs to give extra data. Recently, there have been two independent proposals on how to encode this additional information [9, 11], which we will discuss in chapter 4 along with other recent advances in the study of T-branes.

After these introductory chapters we pass to the focus of this thesis, which is the study of aspects related to T-brane stability. We begin, in chapter 5 by investigating the role of α'-corrections and discuss in particular how they are distinct in the case of intersecting brane configurations in comparison with T-brane states based on our publication [19].

Most analysis of T-branes that take into account D-term stability have been performed either in an ultralocal picture on a patch of flat space or for specific global configurations that allow for simplification of the equations. In chapter 6 we will therefore present our results from [20] on general T-brane vacua on compact 4-cycles. In particular we present a no-go theorem that T-branes with Abelian gauge bundles cannot be stable on 4-cycles with positive or vanishing Ricci-curvature.

In chapter 7 we generalise this analysis to T-brane systems intersecting additional four-cycles. From the point of view of the eight-dimensional field theory, this corresponds to the introduction of defects along the intersection curve. We show that giving a vev to these defect fields allows for T-branes in set-ups that pose topological obstructions to their stability in absence of such defects and in doing so we generalise our previous no-go theorem. Since some of the solutions we find induce poles in the Higgs-field, we adopt a four-dimensional perspective by performing a Kaluza-Klein expansion to show that these poles can be understood as defect-zero-modes giving vevs to higher order KK-modes in the Higgs field. Lastly, in the case of self-intersecting four-cycles, we link the defect picture to a Higgs field valued in a larger gauge-algebra.

Finally, we conclude in chapter 8 and relegate some technical aspects to the appendices.

Chapter 2

The World-volume Theory of 7-Branes

2.1 The maximally symmetric 8d SYM

This thesis is concerned with bound states of 7-branes, such that we should start by reviewing the necessary concepts to describe the physics of these systems. One important tool in doing so is the worldvolume theory living on a stack of 7-branes, which captures the dynamics of open strings ending on it but is blind to the compactification the branes are embedded in. Under this local perspective we need not be concerned whether we are dealing with perturbative D7-branes in a IIB-compactification or if work instead with more general 7-branes coming from F-theory, such that the gauge group living on the brane stack may be of ADE-type. In the following paragraphs we will work out the field content of this worldvolume theory, largely paraphrasing the discussion presented in [14]. The general idea in this reference is, to constrain the possible theories by requiring that they preserve four-dimensional $\mathcal{N} = 1$ supersymmetry.

From the expansion of the DBI-action it is clear that the worldvolume theory of perturbative D7-branes is given by a supersymmetric eight dimensional Yang-Mills and furthermore, by adiabatic arguments, this should continue to hold also for 7-branes of general type. Such theories can be obtained via dimensional reduction from the maximally supersymmetric Yang-Mills in ten dimensions whose field content is given by the ten-dimensional SYM-multiplet consisting of the gauge field A_{10d} and an adjoint-valued fermion Ψ_{10d} transforming in the positive-chirality spinor representation $\mathbf{16}_+$. This theory preserves 16 supersymmetries, the generator ϵ_{10d} of which also transforms under $\mathbf{16}_+$. In the following we will start with the simple case of 7-branes on eight dimensional flat space and then successively generalise this to arrive at 7-branes wrapping an arbitrary 4-cycle S. So in a first step we consider the 10d maximally SYM on ten dimensional Minkowski space and reduce it to eight-dimensions, such that the structure group decomposes as $SO(9,1) \to SO(7,1) \times U(1)_R$. Under this reduction, the ten-dimensional gauge field decomposes into its eight-dimensional cousin A_{8d} and two real scalar fields Φ_8, Φ_9 corresponding to the two compactified dimensions. As usual we may combine them into a complex scalar $\Phi \equiv \Phi_8 + i\Phi_9$ and its conjugate $\bar{\Phi}$, the two of which are charged under the $U(1)_R$-symmetry

$$\Phi : (\mathbf{1}, -1)\,, \qquad \bar{\Phi} : (\mathbf{1}, +1)\,. \tag{2.1}$$

Similarly, both Ψ_{10d} and ϵ_{10d} decompose into two eight-dimensional components transforming as

$$\Psi_\pm, \epsilon_\pm : \left(S_\pm, \pm\frac{1}{2}\right), \tag{2.2}$$

where we denote by $S_\pm$ the positive and negative chiral spinor representations of $SO(7,1)$.

In the next step, we reduce this system further by decomposing the structure group $SO(7,1)$ of $\mathbb{R}^{7,1}$ into $SO(3,1) \times SO(4)$, corresponding to 7-branes on $\mathbb{R}^{1,3} \times \mathbb{C}^2$. Under this decomposition the spinorial representations behave as

$$\begin{aligned} SO(7,1) \times U(1)_R &\longrightarrow SO(3,1) \times SO(4) \times U(1)_R \\ \left(S_+, +\frac{1}{2}\right) &\longmapsto \left[(\mathbf{2},\mathbf{1}), (\mathbf{2},\mathbf{1}), +\frac{1}{2}\right] \oplus \left[(\mathbf{1},\mathbf{2}), (\mathbf{1},\mathbf{2}), +\frac{1}{2}\right] \\ \left(S_-, -\frac{1}{2}\right) &\longmapsto \left[(\mathbf{2},\mathbf{1}), (\mathbf{1},\mathbf{2}), -\frac{1}{2}\right] \oplus \left[(\mathbf{1},\mathbf{2}), (\mathbf{2},\mathbf{1}), -\frac{1}{2}\right], \end{aligned} \tag{2.3}$$

where we made use of the isomorphism $SO(4) \cong SU(2) \times SU(2)$ and gave the representations in terms of the component $SU(2)$'s, such that by $(\mathbf{2},\mathbf{1})$ we denote the left-handed chiral spinor and by $(\mathbf{1},\mathbf{2})$ the right-handed chiral anti-spinor of $SO(4)$, or equivalently $SO(3,1)$.

Now, clearly we want to generalise this to arbitrary four-cycles S. Crucially, however, such submanifolds wrapped by 7-branes are always embedded into a Kähler-manifold — no matter if we are dealing with perturbative D7-branes or 7-branes in an F-theory vacuum. This implies of course, that S inherits a Kähler-structure from its ambient space. Consequently, the structure group of S is not in fact given by $SO(4)$ but instead by $U(2)$. In terms of the spinor representations of $SO(4)$, this means that we further decompose

$$(\mathbf{2},\mathbf{1}) \longmapsto \mathbf{2}_0, \qquad (\mathbf{1},\mathbf{2}) \longmapsto \mathbf{1}_{+1} \oplus \mathbf{1}_{-1}, \tag{2.4}$$

where the subindices are the charges under the central $U(1)$. Applying this to the decomposition of the eight-dimensional spinors $\Psi_\pm$ and $\epsilon_\pm$, we therefore have

$$\begin{aligned} &SO(7,1) \times U(1)_R \longrightarrow SO(3,1) \times U(2) \times U(1)_R \\ &\quad \left(S_+, +\frac{1}{2}\right) \longmapsto \left[(\mathbf{2},\mathbf{1}), \mathbf{2}_0, +\frac{1}{2}\right] \oplus \left[(\mathbf{1},\mathbf{2}), \mathbf{1}_{+1}, +\frac{1}{2}\right] \oplus \left[(\mathbf{1},\mathbf{2}), \mathbf{1}_{-1}, +\frac{1}{2}\right] \\ &\quad \left(S_-, -\frac{1}{2}\right) \longmapsto \left[(\mathbf{2},\mathbf{1}), \mathbf{1}_{+1}, -\frac{1}{2}\right] \oplus \left[(\mathbf{2},\mathbf{1}), \mathbf{1}_{-1}, -\frac{1}{2}\right] \oplus \left[(\mathbf{1},\mathbf{2}), \mathbf{2}_0, -\frac{1}{2}\right]. \end{aligned} \tag{2.5}$$

For phenomenological reasons we are interested in theories that preserve $\mathcal{N} = 1$ in four dimensions and such theories have four supercharges that are charged under the $U(1)_R$-symmetry of the 4d theory and should be scalars in the internal space. In principle this 4d R-symmetry can be a linear combination of the central $U(1)_J$ coming from the Kähler structure and the 8d R-symmetry, such that we should perform a basis change in $U(1)_J \times U(1)_R$ to a basis where two scalar representations are charged under one $U(1)$ factor giving the 4d R-symmetry and uncharged under the second factor. This change of basis is called *topological twisting*. From (2.5) we see that there are two ways to do this corresponding to

$$J_{\text{top}} = J \pm 2R, \tag{2.6}$$

where we denoted the generator of $U(1)_J$ by J and that of the $U(1)_R$ by R. A priori these two embeddings might result in a differently charged spectrum under the 4d R-symmetry. In the case at hand, however, one may confirm that the result is the same for the two of them

$$
\begin{aligned}
&[(\mathbf{2},\mathbf{1}),\mathbf{2}_{+1}]\oplus[(\mathbf{1},\mathbf{2}),\mathbf{2}_{-1}] \\
&\oplus[(\mathbf{1},\mathbf{2}),\mathbf{1}_{+2}]\oplus[(\mathbf{1},\mathbf{2}),\mathbf{1}_{0}]\oplus[(\mathbf{2},\mathbf{1}),\mathbf{1}_{0}]\oplus[(\mathbf{2},\mathbf{1}),\mathbf{1}_{-2}]\,.
\end{aligned} \tag{2.7}
$$

The four supercharges of the 4d theory are then given by

$$
[(\mathbf{1},\mathbf{2}),\mathbf{1}_{0}]\oplus[(\mathbf{2},\mathbf{1}),\mathbf{1}_{0}]\,. \tag{2.8}
$$

Under the same embedding the complex scalar Φ then transforms as $[(\mathbf{1},\mathbf{1}),\mathbf{1}_{-2}]$. So in summary fermionic fields are uncharged under the central $U(1)$ of the structure group of S whereas the complex scalar is charged under it. In other words they transform as sections of some exterior power of the holomorphic tangent bundle of the four-cycle S, in particular Φ transforms as $\Omega^2_S \cong K_S$, where by K_S we denote the canonical bundle.

Of course all of these fields are also charged under the gauge group of the 7-brane stack. In consequence, we conclude that the two bosonic fields of the eight dimensional theory transform as

$$
A \in \Omega^1(S,\mathrm{End}(V)), \qquad \Phi \in \Omega^0(S,\mathrm{End}(V)\otimes K_S), \tag{2.9}
$$

where by V we denoted the associated vector bundle of the gauge bundle.

Knowing the field content of the worldvolume theory, the next step is to find the BPS equations, which can be derived by use of the variational principle on the action. For the sake of brevity we do not derive them here, and only state the result, referring the interested reader to the appendices of [14]. The external space components of the equations of motions read

$$
F_{\mu\nu} = F_{\mu m} = F_{\mu\bar{m}} = 0 \tag{2.10a}
$$

$$
\mathrm{D}_\mu \Phi = \mathrm{D}_\mu \bar{\Phi} = 0, \tag{2.10b}
$$

where we denoted by F the flux associated to A. In words, the flux has no external legs and the complex scalar does not vary over the four-dimensional spacetime. The internal part of the flux may be written as $F = \overline{\partial}_A A + \partial_A A^\dagger - i[A, A^\dagger]$,[1] where by A we denote the holomorphic component of the gauge field,

[1]The commutator for Lie-algebra valued forms $\eta \in \Omega^p \otimes \mathfrak{g}, \gamma \in \Omega^q \otimes \mathfrak{g}$ is given by $[\eta,\gamma] = \eta\wedge\gamma - (-1)^{pq}\gamma\wedge\eta$.

that is its $(0,1)$-part, and conversely by $A^\dagger$ its hermitian conjugate $(1,0)$-form. In terms of these fields, the internal components of the equations of motion can be split into two parts, according to their role in the four-dimensional effective theory. There are two F-term equations, given by

$$F^{2,0} = 0 \tag{2.11a}$$

$$\overline{\partial}_A \Phi = 0. \tag{2.11b}$$

Notice that these two equations imply that the two holomorphic fields Φ, A are closed with respect to the covariant derivative $\overline{\partial}_A$. Moreover, shifting A by an exact form is just a gauge transformation, such that the solutions to the F-term equations are counted by

$$A \in H^{0,1}_{\overline{\partial}_A}(S, \mathrm{End}(V)) \tag{2.12}$$

$$\Phi \in H^{0,0}_{\overline{\partial}_A}(S, \mathrm{End}(V) \otimes K_S) \cong H^{2,0}_{\overline{\partial}_A}(S, \mathrm{End}(V)), \tag{2.13}$$

In the following we will refer to Φ as a two-form unless otherwise indicated. We will see in a moment that we may drop the subindex in $\overline{\partial}_A$ and that indeed the standard bundle cohomologies with respect to the Dolbeault operator count F-term solutions. On top of these two F-term equations, there is a D-term equation, given by

$$\omega \wedge F + \frac{1}{2}[\Phi, \Phi^\dagger] = 0, \tag{2.14}$$

where we denote by ω the Kähler form of S. Notice, that the F-term equations (2.11a)(2.11b) depend only on the holomorphic fields $A^{0,1}$ and Φ, whereas the D-term equation (2.14) mixes holomorphic with antiholomorphic fields $A^\dagger$, $\Phi^\dagger$. Moreover, the D-term equation depends explicitly on the Kähler-form of the four-cycle S, which in turn implies that it receives α'-corrections, while the F-term equations are valid at all orders in α'. These two properties of the D-term equations make them much harder to solve than the F-term equations.

For some properties of the resulting low-energy theory it is fortunately not necessary to solve the D-term equations explicitly. The massless spectrum around a supersymmetric background $\langle A \rangle, \langle \Phi \rangle$, for instance, may be computed without doing so. This is due to a property of many supersymmetric theories, that may be summarised as

$$\frac{\text{D- \& F-term solutions}}{\mathfrak{g}} \cong \frac{\text{F-term solutions}}{\mathfrak{g}_{\mathbb{C}}}. \tag{2.15}$$

In words, the space of solutions of D- and F-term equations modulo gauge transformations is isomorphic to the space of solutions of the F-term equations modulo complexified gauge transformations

To see in detail how to compute the massless spectrum, we split the fields into a background and a fluctuation piece

$$A = \langle A \rangle + a \tag{2.16}$$

$$\Phi = \langle \Phi \rangle + \varphi, \tag{2.17}$$

where we will drop the $\langle \cdot \rangle$ immediately. Recall that the action and therefore the equations of motion, that is D- and F-terms, are invariant under gauge transformations

$$\Phi \longrightarrow B^{-1} \cdot \Phi \cdot B \tag{2.18a}$$

$$iA \longrightarrow iB^{-1} \cdot A \cdot B - B^{-1} \cdot (\overline{\partial} B) \tag{2.18b}$$

for some $B \in g$. If we take $B \equiv e^{i\chi}$ for some $\chi \in \mathfrak{g}$ we may write infinitesimally

$$a \longrightarrow a + \overline{\partial}_A \chi \tag{2.19a}$$

$$\varphi \longrightarrow \varphi + [\Phi, \chi]. \tag{2.19b}$$

Equivalently, we may take the gauge parameter in the complexified algebra $\chi \in \mathfrak{g}_{\mathbb{C}}$ and work on the right hand side of the isomorphism (2.15). Using (2.18), we may therefore adopt the following strategy: We pass to a gauge where $A^{0,1} \equiv 0$, such that in particular all covariant derivatives are normal exterior derivatives $\overline{\partial}_A = \overline{\partial}$. This is the so-called *holomorphic gauge*. In this gauge we expand the F-term equations (2.11) in fluctuations and background and consider the linear piece

$$\overline{\partial} a = 0 \tag{2.20a}$$

$$\overline{\partial} \varphi = 0. \tag{2.20b}$$

We see that solutions to the F-term equations are given by closed one- and two-forms. However, this is not the physical spectrum as we are overcounting all the gauge-equivalent solutions; we still need to mod out by complexified gauge transformations, which are given as

$$a \longrightarrow a + \overline{\partial}\chi \tag{2.21a}$$

$$\varphi \longrightarrow \varphi + [\Phi_{\mathrm{h}}, \chi] \tag{2.21b}$$

for some $\chi \in \mathfrak{g}_{\mathbb{C}}$. Note, that by $\Phi_{\mathrm{h}} \neq \Phi$, we denoted the scalar background in holomorphic gauge. In the following sections we will subsequently consider increasingly complicated classes of backgrounds and derive the spectrum for some examples

2.2 Intersecting brane models

The simplest class of non-trivial backgrounds is given by a Higgs-field vev satisfying

$$[\Phi, \Phi^{\dagger}] = 0, \tag{2.22}$$

which implies that Φ can be taken to lie within the Cartan of $\mathfrak{g}$ by some unitary gauge transformation. Such a vev is simple to deal with because the D-term equations (2.14) only require the worldvolume flux F to be primitive. What is the unbroken gauge symmetry of such a background? As has been shown in [4], any generator of $\mathfrak{g}$ that commutes with Φ also commutes with A. This means that the unbroken gauge group for a non-trivial background is given by the commutant of Φ. Consider for instance the case of a rank r Lie-algebra and give Φ independent vevs along all of its Cartan generators. Clearly, the symmetry is broken down to $U(1)^r$ in this case — the symmetry group of r independent 7-branes. Indeed, we may understand giving a vev to Φ as taking the 7-branes of a stack apart from each other. Doing so renders strings stretching between different branes massive, implying that at massless level the symmetry is broken to that of individual branes.

Consider for instance two 7-branes on a 4-cycle S. For a trivial background this would give a $U(2)$ gauge group and we may discard the centre of mass for now, leaving us with $SU(2)$. Take the vector bundle carried by these branes to be $V = \mathcal{L} \oplus \mathcal{L}^{-1}$ for some holomorphic line bundle $\mathcal{L}$. If we give a vev to the scalar field as

$$\Phi_{\rm h} = \begin{pmatrix} a & \\ & -a \end{pmatrix} \tag{2.23}$$

The only generator commuting with this background is the Cartan itself, such that the gauge group is broken to $U(1)$. Not surprisingly, we can also see this in the spectrum of the theory. From our previous discussion we see that the gauge inequivalent fluctuations in φ around this background are given by $H^0(S, \mathrm{End}(V) \otimes K_S)$ modulo gauge transformations (2.21). In the case at hand we have

$$\begin{aligned} & H^0(S, \mathrm{End}(V) \otimes K_S) \\ & \cong H^0(S, K_S) \oplus H^0(S, \mathcal{L}^2 \otimes K_S) \oplus H^0(S, \mathcal{L}^{-2} \otimes K_S), \end{aligned} \tag{2.24}$$

corresponding to Cartan, upper-right corner and lower-left corner mode of φ. We denote the generators of $\mathfrak{su}(2)_{\mathbb{C}}$ by

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.25}$$

and expand $\varphi = \varphi_P\, P + \varphi_+\, E_+ + \varphi_-\, E_-$. If we make a gauge transformation $\chi \equiv -\frac{1}{2}\varphi_+\, E_+ + \frac{1}{2}\varphi_-\, E_-$ as in (2.21), we see that indeed $\varphi \to \varphi_P\, P$, that is, only the Cartan mode is a gauge inequivalent solution.

Note, that so far we have assumed that the section $a \in H^0(S, K_S)$ has no zeros. Generically, this is not true however. Instead, it has zeros along the self-intersection two-cycle $\mathcal{C} = S \cap S$. What happens along these loci? From the field theory point of view Φ vanishes along $\mathcal{C}$ and therefore the generators of the full $\mathfrak{su}(2)_{\mathbb{C}}$ commute. By the same argument as before we therefore find that along $\mathcal{C}$ we may not gauge away any of the fluctuations contained in φ and correspondingly the gauge inequivalent fluctuations contained in the scalar are given in total by one scalar on S and two more on $\mathcal{C}$

$$H^0(S, K_S) \oplus H^0(\mathcal{C}, (\mathcal{L}^2 \otimes K_S)|_{\mathcal{C}}) \oplus H^0(\mathcal{C}, (\mathcal{L}^{-2} \otimes K_S)|_{\mathcal{C}}). \tag{2.26}$$

The intuition behind this is simply that we are describing two 7-branes wrapping the cycle class S, but are displaced by the section a. If S has a non-trivial self-intersection, it means that the two branes intersect along this curve and correspondingly strings stretching between them become massless along the locus $\mathcal{C}$. From the worldvolume theory perspective we see this as a gauge enhancement.

The correspondence between Higgs-field and 7-brane loci can be made explicit by defining the spectral polynomial

$$P_\Phi(n) = \det\left(n \cdot \mathbf{1} - \Phi\right), \tag{2.27}$$

where by n we label a section whose zeros define the 4-cycle S within its ambient space. Note, that $P_\Phi(n)$ is invariant under complexified gauge transformations such that we may evaluate it in holomorphic gauge $\overline{\partial}\Phi = 0$. The loci of the individual 7-branes are now given by zeros of $P_\Phi(n)$; take for instance the background in (2.23), which has $P_\Phi(n) = (n-a)(n+a)$, meaning that there are two individual branes along the loci $n + a = 0$ and $n - a = 0$. Such that at generic loci on S we should find the massless spectrum of two separate branes and if the intersection $a = -a$ exists, that is, if the section a has zeros, we should find an enhancement along this intersection locus; just as we did in the explicit computation of the spectrum above.

The transformation properties of the additional matter fields along intersection curves can be understood from a group theory perspective: Each set of coinciding 7-branes carries some gauge group G_1, G_2 and the bosonic fields transform in their respective adjoint representations. Along intersection curves this gauge group enhances to $G_S \supset G_1 \times G_2$ and we may decompose

$$\mathrm{ad}(G_S) = \mathrm{ad}(G_1) \oplus \mathrm{ad}(G_2) \oplus \left(\bigoplus_j U_j \otimes U_j' \right) \tag{2.28}$$

for irreducible representations U_j, U_j' of the two gauge groups G_1, G_2. For special unitary groups this will be the bifundamentals. Clearly the first two terms in this decomposition give the bosonic fields living in the worldvolume of the two sets of 7-branes, whereas the last term contains the additional matter fields we found at the intersection. Let us be more explicit by considering the phenomenologically most interesting case of a low-energy theory resulting in an $G_1 = SU(5)$ gauge group, which from the 7-brane perspective means five coinciding 7-branes. Now, to find extra matter, these coinciding branes would need to be intersected by additional branes. If we consider the case of just one brane intersecting the stack, there are two possibilities corresponding to the decompositions

$$SU(6) \longrightarrow SU(5) \times U(1) \tag{2.29}$$

$$SO(10) \longrightarrow SU(5) \times U(1), \tag{2.30}$$

under which the adjoint according to (2.28) decompose as

$$\mathbf{35} \longrightarrow \mathbf{24}_0 \oplus \mathbf{1}_0 \oplus \mathbf{5}_1 \oplus \bar{\mathbf{5}}_{-}1 \tag{2.31}$$

$$\mathbf{45} \longrightarrow \mathbf{24}_0 \oplus \mathbf{1}_0 \oplus \mathbf{10}_2 + \bar{\mathbf{10}}_{-2}. \tag{2.32}$$

So we read off, that depending on the enhancement, the additional matter we find transforms in the **5** or **10** of $SU(5)$ and their conjugates. From the perspective of GUT-phenomenology this leads to the natural question of whether Yukawa-couplings of down-type $\mathbf{10} \times \bar{\mathbf{5}} \times \bar{\mathbf{5}}$ and up-type $\mathbf{10} \times \mathbf{10} \times \mathbf{5}$ may form in compactifications that contain both kins of enhancements (2.29),(2.30). Indeed, this is possible: As we discussed, we find this additional matter on complex curves within the 4-cycle S, implying that the wavefunctions of these fields localise sharply along them. Intuitively a coupling can be formed at loci along which both fields are localised. Put differently we expect these couplings at the intersection points of the corresponding curves. Intuitively these are triple intersection points of 7-branes, which from the group theory point of view, correspond to further enhancements to $SO(12)$ or E_6, respectively. In terms of the worldvolume field theory of 7-branes these couplings come from the superpotential corresponding to the F-terms (2.11), given by

$$W = m_*^4 \int_S \mathrm{Tr}\,(F \wedge \Phi)\,, \tag{2.33}$$

where m_* is the characteristic scale of either F-theory or IIB. Since the superpotential depends only on holomorphic quantities, it is invariant under complexified gauge transformations, implying that we may read off the structure of Yukawa-couplings in holomorphic gauge. It is only when we want to compute the actual value of the couplings, that we need to pass to physical, unitarity gauge [2]. Note moreover, that the localisation of Yukawa couplings in points means that their characteristics can be computed in an ultra-local approach, treating the environment of the points as a patches of flat space, since they do not depend on far-away geometry. This implies in particular that Yukawa-couplings are particularly general quantities in the sense that the same kind of Yukawa coupling may be embedded into a plethora of 7-brane models.

[2]The name *unitarity gauge* refers to the fact, that in this (complex!) gauge, the hermitian bundle metric is simply the identity, whereas in all other gauges it takes a more complicated form.

The intersecting brane models we may construct by simply giving a vev to Φ along the Cartan of $\mathfrak{g}$ are all of the type that all 7-branes contained in the stack wrap a 4-cycle in the same homology class. How can we describe a system of a stack of 7-branes on a 4-cycle class S intersecting a second stack on a different 4-cycle class S'? From our previous discussions it is clear that both stacks host an eight-dimensional $\mathcal{N}=1$ SYM and from our physical intuition we expect additional massless degrees of freedom to appear along the intersection curve $\Sigma = S \cap S'$. As has been argued in [14] these new massless excitations can be described by a six-dimensional defect field theory coupled to the eight-dimensional theories. We denote by G_S and $G_{S'}$ the respective gauge groups carried by the two stacks. From a group theory perspective this is no different to our previous discussion, in that we expect a gauge enhancement along Σ to some larger group

$$G_\Sigma \supset G_S \times G_{S'} \tag{2.34}$$

and correspondingly the adjoint representation decomposes as

$$\mathrm{ad}(G_\Sigma) = \mathrm{ad}(G_S) \oplus \mathrm{ad}(G_{S'}) \oplus \left(\bigoplus_j U_j \otimes U'_j \right) \tag{2.35}$$

for irreducible representations U_j, U'_j of the two gauge groups. It is this last summand that once again contains the additional matter we find along Σ. The additional fields are described by a six-dimensional hypermultiplet coupled to the gauge fields of the two 8d SYM. Such a configuration has been described in [21]. The six-dimensional hypermultiplet contains in particular two bosons $\sigma, \bar{\sigma}^c$. Moreover, its supersymmetry generator ϵ transforms in the $\mathbf{4}' \otimes \mathbf{2}$ of $SO(5,1) \times SU(2)_R$, where the latter factor is an additional R-symmetry that the most general action exhibits. Similar as before we decompose these representations under the reduction to find the necessary topological twisting in order to preserve four-dimensional $\mathcal{N}=1$ supersymmetry. Reducing to four dimensions, corresponds to the decomposition

$$SO(5,1) \longrightarrow SO(3,1) \times U(1) \tag{2.36}$$

$$\mathbf{4}' \longmapsto \left((\mathbf{2},\mathbf{1}), -\frac{1}{2} \right) \oplus \left((\mathbf{1},\mathbf{2}), +\frac{1}{2} \right). \tag{2.37}$$

Once again we need to embed the central $U(1)_R$ of $SU(2)_R$ into the $U(1)$ structure group of Σ. To see how the supersymmetry generator ϵ is charged under it, we decompose

$$SU(2)_R \longrightarrow U(1)_R \tag{2.38}$$

$$\mathbf{2} \longmapsto \mathbf{1}_{+1} \oplus \mathbf{1}_{-1}. \tag{2.39}$$

Now, an embedding of $U(1)_R$ into the structure group of Σ is specified by $J_{\text{top}} = J + \alpha\, R$, where once again R, J denote the generators of the two $U(1)$'s and α is some real number. If such an embedding is to preserve four-dimensional $\mathcal{N} = 1$ supersymmetry, four of the supersymmetries generated by ϵ need to transform as scalars such that we are left with

$$J_{\text{top}} = J \pm \frac{1}{2} R, \tag{2.40}$$

which corresponds to the decomposition

$$(\mathbf{4}', \mathbf{2}) \longmapsto (\mathbf{2}, \mathbf{1})_0 \oplus (\mathbf{2}, \mathbf{1})_{-1} \oplus (\mathbf{1}, \mathbf{2})_0 \oplus (\mathbf{1}, \mathbf{2})_1. \tag{2.41}$$

Without loss of generality we now take the twisting given by $J_{\text{top}} = J - \frac{1}{2} R$ under which the two scalars transform as[3]

$$\sigma \in \Gamma(K_\Sigma^{1/2} \otimes \mathcal{U} \otimes \mathcal{U}') \tag{2.42}$$

$$\bar{\sigma}^c \in \Gamma(\bar{K}_\Sigma^{1/2} \otimes \bar{\mathcal{U}}^* \otimes (\bar{\mathcal{U}}')^*), \tag{2.43}$$

for the associated vector bundles $\mathcal{U}, \mathcal{U}'$. By $\bar{\cdot}$ we denote the corresponding anti-holomorphic bundle and by $\cdot^*$ the dual bundle. Instead of dealing with $\bar{\sigma}^c$ it will be more convenient to consider its conjugate

$$\sigma^c \in \Gamma(K_\Sigma^{1/2} \otimes \mathcal{U}^* \otimes (\mathcal{U}')^*), \tag{2.44}$$

Note, that this ties in naturally with our previous discussions: Namely, if Σ is the self-intersection curve $S \cap S$, we have by adjunction that $K_\Sigma^{1/2} = K_S|_\Sigma$, such that the fields σ, σ^c have the right transformation properties to be understood in terms of a larger Φ incorporating them as components. We will make this correspondence much more explicit in chapter 7.

[3] They arise from fields $(\mathbf{1}, \pm 1)$ under $SO(5,1) \times U(1)_R$, such that following the decomposition to $SO(3,1) \times U(1) \times U(1)_R$ and the topological twisting, they are mapped to $\big((\mathbf{1}, \mathbf{1}), \pm\frac{1}{2}\big)$.

Now, that we understand the bosonic matter appearing along intersection curves Σ, we need to understand how it couples to the eight-dimensional theories and which backgrounds are stable. Once again we limit ourselves to give the BPS-equations and refer to the original reference [14] for details. Both of the scalars satisfy a six-dimensional F-term condition

$$\overline{\partial}_{A+A'}\sigma = \overline{\partial}_{A+A'}\sigma^c = 0. \tag{2.45}$$

Moreover, they couple to both F- and D-term equations of the eight-dimensional theories by introducing source terms with delta-function support along Σ

$$\overline{\partial}_A \Phi = \delta_\Sigma \langle\langle \sigma^c, \sigma \rangle\rangle_S \tag{2.46}$$

$$\omega \wedge F + \frac{1}{2}[\Phi, \Phi^\dagger] = \frac{1}{2}\omega\wedge\delta_\Sigma \left(\mu_S\left(\bar{\sigma}, \sigma\right) - \mu_S\left(\bar{\sigma}^c, \sigma^c\right)\right), \tag{2.47}$$

where by δ_Σ we denote the Poincaré dual $(1,1)$-form of the intersection 2-cycle Σ. We will be quite explicit in defining the outer product $\langle\langle \cdot, \cdot \rangle\rangle_S$ and the moment map $\mu_S(\cdot, \cdot)$ in the following. Feel free to skip ahead to the next paragraph if you are not interested in these details. We denote the natural outer product between the vector bundle $\mathcal{U}$ and its dual by

$$\langle \cdot, \cdot \rangle_{\mathcal{U}} : \, \mathcal{U}^* \otimes \mathcal{U} \longrightarrow \mathcal{O}, \tag{2.48}$$

where $\mathcal{O}$ is the trivial bundle. If we label the generators of $\mathfrak{g}_S$ by T^a_S, they act as linear operators mapping to the adjoint of $\mathfrak{g}_S$

$$\begin{aligned} T_S : \mathcal{U} &\longrightarrow \mathrm{End}(V) \otimes \mathcal{U} \\ \mathcal{U}^* &\longrightarrow \mathrm{End}(V) \otimes \mathcal{U}^*. \end{aligned} \tag{2.49}$$

Locally, this may be understood as taking the generators T_S in the representation U — typically the fundamental — and acting on the vectors in $\mathcal{U}$ by matrix multiplication, $\sigma^c_i \left(T^a\right)^i_j \sigma^j$. For a split-bundle this intuition holds also globally. Using (2.48) and (2.49), we may compose the outer product

$$\begin{aligned} &\langle\langle \cdot, \cdot \rangle\rangle_S : \left(K_\Sigma^{1/2} \otimes \mathcal{U}^* \otimes (\mathcal{U}')^*\right) \oplus \left(K_\Sigma^{1/2} \otimes \mathcal{U} \otimes \mathcal{U}'\right) \longrightarrow K_\Sigma \otimes \mathrm{End}(V) \\ &\langle\langle \cdot, \cdot \rangle\rangle_{\mathrm{ad}(P)} = \langle T \cdot, \cdot \rangle_{\mathcal{U}} \otimes \langle \cdot, \cdot \rangle_{\mathcal{U}'}. \end{aligned} \tag{2.50}$$

Now from this prescription, we have that

$$\langle\langle \sigma^c, \sigma \rangle\rangle \in H^{0,0}\left(\Sigma, \mathrm{End}(V) \otimes K_\Sigma\right) \cong H^{1,0}\left(\Sigma, \mathrm{End}(V)\right). \tag{2.51}$$

The moment map may be composed from the three metrics

$$H : \bar{\mathcal{U}} \otimes \mathcal{U} \longrightarrow \mathcal{O} \tag{2.52a}$$

$$H' : \bar{\mathcal{U}}' \otimes \mathcal{U}' \longrightarrow \mathcal{O} \tag{2.52b}$$

$$h_\Sigma^{1/2} : \bar{K}_\Sigma^{1/2} \otimes K_\Sigma^{1/2} \longrightarrow \mathcal{O} \tag{2.52c}$$

as follows

$$\mu_S : \left(\bar{K}_\Sigma^{1/2} \otimes \bar{\mathcal{U}} \otimes \bar{\mathcal{U}}'\right) \oplus \left(K_\Sigma^{1/2} \otimes \mathcal{U} \otimes \mathcal{U}'\right) \longrightarrow \mathrm{End}(V) \tag{2.53}$$

$$\mu_S = \langle h_\Sigma^{-1/2} \cdot, \cdot \rangle_{K_\Sigma^{1/2}} \langle TH \cdot, \cdot \rangle_{\mathcal{U}} \langle G \cdot, \cdot \rangle_{\mathcal{U}'}. \tag{2.54}$$

In all of the following discussions we will drop the clumsy S in the subscript as we will only every refer to one of the two relevant 4-cycles. See 7 for more details.

2.3 T-branes

In the last section we started out by describing backgrounds with the property $[\Phi, \Phi^\dagger] = 0$ motivated by making an ansatz with simple BPS-conditions. We then argued that the intuition behind this class of vacua is in fact that of 7-branes on a 4-cycle class S that intersect each other. Clearly, this kind of vev for Φ is not the most general one, as there are many vevs for which we have

$$[\Phi, \Phi^\dagger] \neq 0. \tag{2.55}$$

Such a background is called a *T-brane*.[4] Clearly, their BPS-conditions are much more intricate because the D-terms (2.14) will require the presence of non-primitive flux precisely cancelling this contribution. Since also Φ implicitly depends on the bundle metric via the F-term equations (2.11), this boils down to solving a set of coupled partial differential equations on the compact 4-cycle S — or at least in a relevant patch of flat space, as we will see in the following. The nature of this new class of 7-brane vacua is different from the simple backgrounds we considered in the last section: As we have seen, intersecting brane configurations on a 4-cycle S amount to giving a vev to Φ within the Cartan,

[4] In the original reference only upper triangular configurations have been called *T-brane* and this is in fact the origin of the term itself.

that is, in the end we are dealing with an Abelian configuration. Contrastingly, a T-brane configuration is an inherently non-Abelian bound state.

Why should we care about this kind of 7-brane vacua? First of all 7-brane vacua form one of the richest set-ups for realistic phenomenology and as such we should aim to have an as broad as possible understanding not only of its simple configurations but also seemingly more complicated ones. On a related matter, it has indeed been shown that T-branes provide an elegant mechanism to give a realistic hierarchy to Yukawa couplings [3, 4, 7, 22–29]. Lastly, as we will later review, the dictionary between a local 7-brane model as we discuss it here and the global F-theory picture is far from clear and indeed T-branes contain data whose role in global F-theory compactifications is still puzzling, albeit considerable progress has been made recently [9–11].

This section is structured as follows: First we will introduce a number of different T-brane vevs in order to highlight their phenomenologically distinct behaviour as compared to intersecting branes, following [4]. In 2.3.1, we will show why the fate of T-branes in global F-theory compactifications is less clear from that of intersecting branes. Lastly, in 2.3.2 we will review the mechanism of realistic ranks for Yukawa couplings relying on the presence of a T-brane bound state.

Let us begin by considering the simplest T-brane background given in unitarity gauge by

$$\Phi = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \tag{2.56}$$
$$\Rightarrow [\Phi, \Phi^\dagger] = \begin{pmatrix} m \wedge \bar{m} & 0 \\ 0 & -m \wedge \bar{m} \end{pmatrix}.$$

for a vector bundle, once again given by $V = \mathcal{L} \oplus \mathcal{L}^{-1}$ in terms of some holomorphic line bundle $\mathcal{L}$. From (2.56) we read off that Φ does not commute with any of the generators of $\mathfrak{su}(2)$, implying that the gauge group is broken completely on generic loci, while it is restored along the curve $m = 0$. Note, that this information would not have been visible by just considering the worldvolume flux: From the D-term equation (2.14), we see that F needs to have a non-primitive component along the Cartan of $\mathfrak{su}(2)$. So by just considering the flux, one would expect the symmetry to be broken down to its Cartan $U(1)$. Note moreover, that the spectral polynomial of (2.56) is simply given by $P_\Phi(n) = n^2$, i.e. it is the same as for a trivial Higgs vev $\Phi = 0$. That is, from the spectral polynomial the configuration at hand seems to describe two coinciding 7-branes, but its gauge group is completely broken by a non-trivial bound state. To make matters more curious, we find a gauge enhancement along a curve that does not seem to correspond to an intersection curve of two 7-branes. In summary, we see that neither F nor $P_\Phi(n)$ carry the whole information about this vacuum, but instead one needs to consider the full Φ. To better understand the situation at hand, we also compute the massless spectrum. As before, we parametrise $\varphi = \varphi_P\, P + \varphi_+\, E_+ + \varphi_-\, E_-$ and observe that on generic loci we may gauge this to $\varphi\Big|_{\text{gen}} = \varphi_-\, E_-$ by making an infinitesimal gauge transformation $\chi \equiv -\frac{\varphi_P}{m}\, P + \frac{\varphi_+}{2m}\, E_+$. Clearly, this gauge transformation is ill-defined along the curve $m = 0$ and indeed we recover the full fluctuations of $\mathfrak{su}(2)$ along this locus. Since, this background is the simplest example of a T-brane, we also take the opportunity to show how the non-commutativity of Φ renders the task of solving the BPS-equations much more complicated. To be explicit, let us pass to a patch of flat space $\mathbb{C}^2$ with local coordinates (x, y) and assume without loss of generality that $m\big|_{\mathbb{C}^2} \equiv x\ \mathrm{d}x{\wedge}\mathrm{d}y$ and $\omega = \frac{i}{2}\,(\mathrm{d}x{\wedge}\mathrm{d}\bar{x} + \mathrm{d}y{\wedge}\mathrm{d}\bar{y})$. In holomorphic gauge the F-term equations simply require Φ_h to be a function only of the holomorphic coordinates $\Phi_\mathrm{h}(x, y)$. However, if we are interested in the specific wavefunction profiles, we need to pass to unitarity gauge. Even though we are guaranteed the existence of such a gauge transformation by the isomorphism (2.15), it may be difficult to find in practice. In the case at hand we make the ansatz[5]

$$B \equiv \begin{pmatrix} e^{\frac{f}{2}} & \\ & e^{-\frac{f}{2}} \end{pmatrix} \tag{2.57}$$

for the gauge transformation introduced in (2.18), where $f \equiv f(x, \bar{x})$ is some

[5]The form of this ansatz can be inferred from the fact that $[\Phi, \Phi^\dagger]$ lies in the Cartan and therefore so does the flux.

real function yielding gauge- and Higgs-field in unitarity gauge as

$$\Phi = \begin{pmatrix} 0 & me^{-f} \\ 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} \frac{i\overline{\partial} f}{2} & 0 \\ 0 & -\frac{i\overline{\partial} f}{2} \end{pmatrix} \tag{2.58}$$

$$\Rightarrow F^{1,1} = \begin{pmatrix} i\partial\overline{\partial} f & 0 \\ 0 & -i\partial\overline{\partial} f \end{pmatrix}. \tag{2.59}$$

Since the F-terms are invariant under complexified gauge transformations, we only need to solve the D-terms (2.14), which are given as

$$\omega\wedge i\partial\overline{\partial} f\, P + \frac{1}{2} e^{-2f} m\wedge\bar{m}\, P = 0 \tag{2.60}$$

Plugging in for m and ω, this gives

$$\partial_x \overline{\partial}_{\bar{x}} f = |x|^2 e^{-2f}, \tag{2.61}$$

such that it becomes clear that even in a local patch it may be challenging to solve the D-terms explicitly.

The background we just discussed already exhibits most of what we are going to need in the following chapters. In order to motivate the topic a bit better, let us however, consider some more examples of T-branes to highlight some further interesting aspects. To keep matters simple we consider them in flat space $\mathbb{C}^2$ with coordinates (x, y).

Another interesting behaviour can be found for instance for the background

$$\Phi = \begin{pmatrix} x & 1 \\ 0 & -x \end{pmatrix}, \tag{2.62}$$

which by considering the spectral equation $P_\Phi(z) = (z-x)(z+x)$ seems to encode two 7-branes intersecting along the curve $x = 0$. We would therefore expect to find $U(1)^2$ at generic loci, with an enhancement to $SU(2)$ and the corresponding matter fields along $x = 0$. If we consider, however, the full background Φ instead of its spectral equation, not only do we see that the symmetry is broken completely along generic loci, and that along $x = 0$ only a $U(1)$ factor is restored, but in addition we do not find any localised matter along this curve! So our intersecting brane intuition fails completely to describe the background adequately.

As a last example consider the background

$$\Phi = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \tag{2.63}$$

with spectral equation $P_\Phi(z) = z^2 - xy$. Along generic loci this background breaks the underlying $SU(2)$ gauge group completely, while it enhances to the full $SU(2)$ at $x = y = 0$. More curiously, the mantra to find gauge groups in codimension one, matter fields in codimension two and finally couplings in codimension three, does not apply to the background at hand as we find no additional matter on the mentioned curves, but instead at their intersection in codimension two. So once again we recover behaviour that is completely invisible from the spectral data.

2.3.1 F-theory

So far we have dealt with 7-branes in a local way, in the sense that we have described them in terms of their worldvolume theory on a (compact) 4-cycle S. From this perspective it did not matter whether we are describing a stack of D7-branes in perturbative IIB string-theory or whether in fact we are describing a stack of more general 7-branes in F-theory [15]. The only difference in our analysis up to this point would have been that F-theory allows for general ADE-type gauge groups whereas this is not possible in perturbative IIB string theory. The formalism itself is however, independent of this distinction. Recall, that in global F-theory, the information about gauge groups, matter curves and couplings is encoded in singularities of the elliptic fibration along loci of codimension one, two and three in the base.[6] Correspondingly, the information contained in the 7-brane Higgs-field Φ about symmetry breaking related to brane intersection patterns should translate to the singularity structure of the elliptic fibration in a global F-theory model. Clearly, the worldvolume theory does not contain all information about the vacuum it is embedded in, such that this dictionary should be understood locally around the 4-cycle S.

To make contact between the field theoretic description of the worldvolume theory of 7-branes and global F-theory, let us first review how a global elliptic

[6] An introduction to F-theory would exceed the scope of this thesis, and we refer the reader to [30–32] instead.

fibration with singularities along a codimension one locus $w = 0$ can be translated to a local ALE-fibration, which we will then link to the worldvolume vev Φ, following [33]. Recall, that an elliptic fibration $T^2 \hookrightarrow B$ can be described as the solutions to the Weierstraß polynomial

$$y^2 = x^3 + fx + g, \tag{2.64}$$

where x, y are coordinates in the fibre ambient space and f, g are sections of appropriate line bundles over the base. The vanishing order of f, g and the discriminant $\Delta = 27g^2 + 4f^3$ along a divisor $z = 0$ then indicates the gauge symmetry corresponding to the 7-branes wrapping this locus. If we are only interested in a local neighbourhood of such a divisor, we may express the elliptic fibration as [14]

A_n	$y^2 = x^2 + z^{n+1}$
D_n	$y^2 = x^2 z + z^{n-1}$
E_6	$y^2 = x^3 + z^4$
E_7	$y^2 = x^3 + xz^3$
E_8	$y^2 = x^3 + z^5$

. (2.65)

How does this relate to our local worldvolume perspective? Let us focus on A_n, that is $SU(n+1)$ for ease of exposition and refer to the original references [14, 33] for the general cases. From an intuitive point of view, the Higgs field Φ breaks the gauge symmetry, such that its components should translate to the deformations of the local Weierstraß polynomial. Indeed this relation is given in terms of the Casimir operators of Φ

$$s_2 = -\frac{1}{2}\text{Tr}(\Phi^2), \qquad s_k = -\frac{1}{k!}\text{Tr}(\Phi^k), \qquad s_{n+1} = \det(\Phi) \tag{2.66}$$

parametrising the most general deformation of an A_n singularity as

$$y^2 = x^2 + w^{n+1} + \sum_{k=2}^{n+1} s_k w^{n+1-k}. \tag{2.67}$$

Now, consider for simplicity the case for $SU(2)$ given by

$$y^2 = x^2 + w^2 + s_2. \tag{2.68}$$

If we parametrise a generic Higgs vev as

$$\Phi = \begin{pmatrix} v & m \\ p & -v \end{pmatrix}, \tag{2.69}$$

the Casimir is given as $s_2 = mp + v^2$. As we can immediately read off an intersecting brane type vev along v is contained in the Casimir. If, however, we consider a T-brane of the type

$$\Phi \equiv \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}, \tag{2.70}$$

we have $s_2 = 0$. That is, the information about the symmetry breaking is not visible from the Casimirs and therefore the elliptic fibration (2.68). So while intersecting brane configurations can be translated to a local F-theory model via the dictionary we presented, this is not the case for T-branes. Much of the recent interest in T-branes originates in this puzzling behaviour and indeed there has been made some progress in understanding how the local information of this symmetry breaking should be translated to global F-theory [9–11]. We will come back to this issue in 4.1.

2.3.2 The Rank of Yukawa-Couplings

In 2.2 we have already discussed the potential phenomenological applications of 7-brane models to engineer viable $SU(5)$-GUT models and how additional matter and their Yukawa couplings arise. From experiment we know that there are two light families of quarks and one heavy family. Such that at leading order in perturbation theory Yukawa couplings should be of rank one, rendering one family heavy. Indeed this statement has been made precise in [26], in the form of the *rank theorem*, stating that for smooth matter curves intersecting transversely in a point, the rank of the Yukawa coupling is at most one. Subsequently it has been shown that higher order corrections may give mass to the remaining two families by making the Yukawa of rank three at subleading orders. Indeed, it has been shown [22–24, 26] that for the down-like Yukawa coupling $\mathbf{10} \times \bar{\mathbf{5}} \times \bar{\mathbf{5}}$ a rank one structure can be generated if there is only one triple intersection point generating this coupling. Subsequently it was shown in [25, 29] that D3-brane instantons or gaugino condensates on a second 4-cycle may generate non-perturbative corrections leading to a Yukawa couplings with fermion mass hierarchy $(1, \epsilon, \epsilon^2)$.

For up-type Yukawa couplings $\mathbf{10} \times \mathbf{10} \times \mathbf{5}$ the situation is more complicated in that a coupling of rank one at leading order requires additional ingredients apart from the restriction that there should be only one point giving rise to the coupling at hand. There are two known mechanisms to generate such a rank coupling, one relying on 7-brane monodromy [27], the other one on a T-brane background [3, 4]. For the latter one it has later been shown [7], that a rank three structure at subleading order can once again be generated by instanton effects, in analogy to the down-like coupling. Clearly, this application makes T-branes also interesting from a phenomenological point of view and while the physics of Yukawa couplings can largely be computed in an ultra-local approach, they rely on the existence of a T-brane background on some cycle. As we have seen in the previous paragraphs the BPS-stability conditions for this class of backgrounds are very delicate and we will see in later chapters that strong no-go theorems may be formulated, rendering T-branes unstable on certain 4-cycles with potential implications for the existence of such Yukawa couplings in a given vacuum.

Chapter 3

Tachyon Condensation

The description of 7-brane vacua we gave in the last chapter is intrinsically local as it is based in the worldvolume theory on the 4-cycle S. Here we will introduce a complementary picture that is fully global in the setting of IIB string theory. This alternative picture is based on the realisation that the natural mathematical tool to describe D-branes globally is provided by *sheaves* and that correspondingly physical information such as the spectrum can be extracted from them. In the following we will largely follow the reviews [34, 35]. For mathematical details we refer to [36], for instance. Let us begin by introducing the mathematical objects we deal with.

3.1 Sheaves

From the description of gauge theories, we are used to the language of vector bundles and how a stack of branes on some p-cycle hosts such a vector bundle. This description seems somewhat artificial, because we are dealing with some threefold compactification, but the vector bundles we are dealing with are only well-defined on given subloci. So the natural question is whether we can generalise vector bundles to something that is well-defined over the whole threefold. We will see that a subclass of sheaves provides precisely this correspondence, while sheaves of different kind may still describe valid D-brane configurations.

We start by defining a *presheaf* $\mathcal{S}$ on a topological space X as the association of a group, ring or field to every open subset $U \subset X$ which we will denote by

$\mathcal{S}(U)$ and call elements of it *sections*. Additionally, $\mathcal{S}$ has a restriction map for every subset $V \subseteq U \subset X$, acting as $\rho_{UV} : \mathcal{S}(U) \longrightarrow \mathcal{S}(V)$, which is transitive for chains of subsets $W \subseteq V \subseteq U$. That is, we have $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ and moreover $\rho_{UU} = \text{id}$. If it is clear from the context, the restriction map is typically denoted as $\big|_V = \rho_{UV}$.

Now, a *sheaf* satisfies extra *glueing conditions*, giving it more structure than a presheaf. These additional conditions relate the sections of different subsets $U, V \subset X$. Firstly, for any two sections agreeing on the overlap $U \cap V$, there is a section on the union $U \cup V$ restricting to them. That is, for any $\sigma \in \mathcal{S}(U)$ and $\tau \in \mathcal{S}(V)$ that satisfy $\sigma\big|_{U\cap V} = \tau\big|_{U\cap V}$, there is a $\rho \in \mathcal{S}(U \cup V)$ with $\rho\big|_U = \sigma$ and $\rho\big|_V = \tau$. Secondly, for any $\sigma \in \mathcal{S}(U \cup V)$ with $\sigma\big|_U = \sigma\big|_V = 0$ we have $\sigma = 0$.

A simple example of a sheaf is for instance given by the holomorphic functions over X. Let us denote this sheaf by $\mathcal{O}_X$. One can confirm that $\mathcal{O}_X$ indeed forms a sheaf and we will henceforth call it the *structure sheaf* of X. Any sheaf that is just given as a direct sum $\bigoplus_{i=1}^n \mathcal{O}_X$, is called *free* and any sheaf that at least has this structure locally, is called *locally free*.[1] Now, clearly the sections of any line bundle locally look like $\mathcal{O}_X$ and the sections of any arbitrary vector bundle of dimension n locally look like a direct sum $\bigoplus_{i=1}^n \mathcal{O}_X$. Put differently, the set of vector bundles over X is isomorphic to the set of locally free sheaves over X. Clearly, these sheaves will therefore play an important role in the following.

A less trivial example of a sheaf is the so-called *skyscraper sheaf*, which only has support over some point $p \in X$. It is defined by

$$\mathcal{S}(U) = \begin{cases} \mathbb{C}, & \text{if } p \in U \\ 0, & \text{otherwise} \end{cases}. \tag{3.1}$$

So in particular, the dimension of the sections of sheaves may jump over subloci, as opposed to the fibre dimension of vector bundles. Something we would naturally expect from a global description of D-branes on subloci. Indeed we can make this concept more precise by introducing the so-called *torsion sheaf*: Given

[1] We follow 4.1.3 of [34] here.

a sheaf $\mathcal{S}$ on Y and an embedding $i : Y \longrightarrow X$, the torsion sheaf is defined by

$$i_*\mathcal{S}(U) \equiv \mathcal{S}\left(i^{-1}(U)\right). \tag{3.2}$$

Consider for instance a stack of 7-branes on a 4-cycle Y, supporting a vector bundle $\mathcal{V}$ embedded into a Calabi-Yau threefold X. Above we learnt that the sections of $\mathcal{V}$ form a sheaf on Y and now we also have a way to lift it to a sheaf on X. How does $i_*\mathcal{S}$ look like? Spelling out (3.2) gives

$$i_*\mathcal{S}(U) = \begin{cases} \emptyset, & \text{if does not intersect the image of } Y \\ S(V), & \text{for the "biggest" } V \subseteq Y \text{ s.t. } i(V) \subseteq U \end{cases} \tag{3.3}$$

That is, the torsion sheaf is just given by the sections of the original sheaf $\mathcal{S}$ for a subset U that overlaps with the image of Y and has no sections otherwise. So in summary we have seen that we can simply lift any vector bundle on a submanifold and get its corresponding sheaf over the full space. [2] Clearly, there are many sheaves that do not come from vector bundles over submanifolds. This leads to the question what constitutes a "physical" sheaf.

3.2 Physical Sheaves

From the previous section we have seen that vector bundles over subloci correspond to locally free sheaves and moreover how to extend them to the full space via the construction of torsion sheaves. Physically, we expect these sheaves to form the building blocks out of which we construct any other physical sheaves. Indeed, this notion has been made precise in the statement that physical sheaves in IIB string theory are given by objects in the category of *coherent sheaves*. We call a sheaf $\mathcal{S}$ coherent, if there are finitely many locally free sheaves $\mathcal{E}_i$, such that the following sequence is exact

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{E}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{S} \longrightarrow 0. \tag{3.4}$$

A sequence (3.4) is called *locally free resolution* and is non-unique in general.

In intuitive terms (3.4) tells us, that every physical brane configuration can be described by a set of vector bundles ($\cong$ locally free sheaves) over the whole

[2] Actually we need to perform an additional twisting by K_Y^*, related to the Freed-Witten anomaly.

spacetime. It is natural to associate these vector bundles to D9- and anti-D9-branes then. So the locally free resolution can be understood as the recipe to reach a given brane configuration via D9-anti-D9-brane tachyon condensation, where the maps in the locally free resolution correspond to open strings stretching between the constituent stacks. Let us consider some examples to understand this better. As a starting point take

$$0 \longrightarrow \mathcal{E} \xrightarrow{T} \mathcal{F} \longrightarrow \mathcal{S} \longrightarrow 0 \tag{3.5}$$

for two locally free sheaves $\mathcal{E}, \mathcal{F}$. Now, clearly the sheaf at hand is just given by $\mathcal{S} = \mathrm{coker}(T) = \mathcal{F}/\mathrm{Im}(T)$. As we pointed out, intuitively, T plays the role of a vev for the strings stretching between a stack of D9's $\mathcal{E}$ and (relative) anti-D9-brane stack $\mathcal{F}$. If this map is trivial $T = \mathbf{1}$, we see that $\mathrm{Im}(T) = \mathcal{F}$ and therefore $\mathcal{S} = \emptyset$. So two equally large stacks of D9's and anti-D9's without any non-trivial vev annihilate completely, as they should. Let us consider a slightly more interesting case and take $\mathcal{E} = \mathcal{F} = \mathcal{O}_X \oplus \mathcal{O}_X$ and

$$T = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \tag{3.6}$$

for some holomorphic function n. Now, for any open subset $U \subseteq X$ in which n has no zeros, we have $\mathcal{S}(U) = \emptyset$ by the same argument as in the trivial case. Along a zero of n, however, $\mathrm{Im}(T)(\{n = 0\}) = \emptyset$ and therefore $\mathcal{S}(\{n = 0\}) = \mathcal{O}_X \oplus \mathcal{O}_X$. So we are describing a sheaf that is trivial on generic loci in the threefold X and looks like a rank two vector bundle over a codimension one locus. Clearly something like that corresponds to two coinciding D7-branes. We may switch on more complicated vevs to reach T-branes or consider longer resolutions to construct D5-brane or D3-brane configurations, but we leave it here for the moment and provide more examples in chapter 4. Let us instead bring this language to use and try to extract information from it.

3.3 Spectrum between two sheaves

If we believe that any stack of (topological) B-branes can be described as an object in the derived category of coherent sheaves, then one of the first questions is clearly how we can extract the spectrum of open strings stretching between to such stacks. It has been argued that it is counted by the so-called Ext-groups. In order to understand what they are and how they can be computed we will need to make quite a long mathematical digression. Consider two coherent sheaves $\mathcal{S}, \mathcal{T}$ and assume we know their locally free resolutions

$$\begin{array}{c} 0 \longrightarrow \mathcal{E}_n \xrightarrow{\mathrm{d}^{\mathcal{E}}_{n-1}} \mathcal{E}_{n-1} \xrightarrow{\mathrm{d}^{\mathcal{E}}_{n-2}} \cdots \xrightarrow{\mathrm{d}^{\mathcal{E}}_{0}} \mathcal{E}_0 \longrightarrow \mathcal{S} \longrightarrow 0 \\ 0 \longrightarrow \mathcal{F}_n \xrightarrow{\mathrm{d}^{\mathcal{F}}_{n-1}} \mathcal{F}_{n-1} \xrightarrow{\mathrm{d}^{\mathcal{F}}_{n-2}} \cdots \xrightarrow{\mathrm{d}^{\mathcal{F}}_{0}} \mathcal{F}_0 \longrightarrow \mathcal{T} \longrightarrow 0. \end{array} \tag{3.7}$$

Now, we are interested in inequivalent ways to switch on a component for a string stretching between the two sheaves or equivalently its locally free resolutions. Put differently, we are interested in inequivalent maps $f_\bullet$ from $\mathcal{E}_\bullet$ to $\mathcal{F}_\bullet$. This is much easier to understand diagrammatically

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{E}_k & \longrightarrow & \mathcal{E}_{k-1} & \longrightarrow & \cdots \\ & \swarrow^{h_k} & \downarrow f_k & \swarrow^{h_{k-1}} & \downarrow f_{k-1} & & \\ \cdots & \longrightarrow & \mathcal{F}_{k-1} & \longrightarrow & \mathcal{F}_{k-2} & \longrightarrow & \cdots \end{array} . \tag{3.8}$$

We are looking for maps $f_\bullet$ that cannot be composed out of some diagonal map $h_\bullet$ and the differentials of the complexes. That is we are interested in the set of inequivalent maps under

$$f_\bullet \sim f_\bullet + h_\bullet \circ \mathrm{d}^{\mathcal{E}} + \mathrm{d}^{\mathcal{F}} \circ h_\bullet . \tag{3.9}$$

This set is called $\mathrm{Ext}^1\left(\mathcal{E}_\bullet, \mathcal{F}_\bullet\right)$. Now clearly, strings may also stretch to other positions in the locally free resolution, so we should consider them as well. We denote therefore by $\mathcal{F}_\bullet[n]$ the complex $\mathcal{F}_\bullet$ shifted n places to the left and define

$$\mathrm{Ext}^q\left(\mathcal{E}_\bullet, \mathcal{F}_\bullet[n]\right) = \mathrm{Ext}^{q+n}\left(\mathcal{E}_\bullet, \mathcal{F}_\bullet\right), \tag{3.10}$$

that is, in particular $\mathrm{Ext}^p(\mathcal{E}_\bullet, \mathcal{F}_\bullet) = \mathrm{Ext}^1\left(\mathcal{E}_\bullet, \mathcal{F}_\bullet[p-1]\right)$.[3] Moreover, for $n = -1$, we have $\mathrm{Ext}^0(\mathcal{E}_\bullet, \mathcal{F}_\bullet) = \mathrm{Hom}(\mathcal{E}_\bullet, \mathcal{F}_\bullet)$. Lastly, one may work out that on a threefold the highest such group is Ext^3. We have worked out which mathematical objects count the physical spectrum in the language of sheaves, but clearly the definition is not very helpful when it comes to actually computing them. To do so, we need to dive deeper in our mathematical digression and in doing so

[3] Note, that the definition given here agrees with the more mathematical definition in terms of the right derived functor of Hom.

we will also derive some other interesting relations. The aim of the following two sections is to first introduce sheaf cohomology and then show how we can relate the aforementioned Ext-groups in terms of them.

3.3.1 Sheaf Cohomology

Before we start let us point out that we will actually introduce Čech-cohomology, which is a less general form to define a cohomology for sheaves than later developed. However, both notions of cohomology agree on the spaces of interest to us, and the former has a more intuitive formulation. So we proceed with Čech cohomology and call it sheaf cohomology interchangeably in the following.

Given a sheaf $\mathcal{S}$ on a topological space X and an open cover $\{U_\alpha\}$ of X, we define cochains of degree n as follows

$$\begin{aligned} C^0(\mathcal{S}) &= \Pi_\alpha \mathcal{S}(U_\alpha) \\ C^1(\mathcal{S}) &= \Pi_{\alpha\neq\beta}\ \mathcal{S}(U_\alpha \cap U_\beta) \\ &\vdots \\ C^n(\mathcal{S}) &= \Pi_{\alpha_0\neq\cdots\neq\alpha_n}\ \mathcal{S}(U_{\alpha_0}\cap\cdots\cap U_{\alpha_n}). \end{aligned} \tag{3.11}$$

Now, we introduce a boundary operator

$$\begin{aligned} \delta : C^p(\mathcal{S}) &\longrightarrow C^{p+1}(\mathcal{S}) \\ \sigma \longmapsto (\delta\sigma)_{i_0\ldots i_{p+1}} &= \sum_{j=0}^{p+1}(-1)^j \sigma_{i_0,\ldots,\hat{i}_j,\ldots i_{p+1}}\Big|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}}, \end{aligned} \tag{3.12}$$

where by the hatted index $\hat{i}_j$ we indicate that it is omitted. Note, that by $(-1)\sigma$ we mean the inverse element with respect to the relevant group operation — it is not necessarily addition. Armed with these two ingredients, we can go ahead and define Čech cocycles as closed cochains $\delta\sigma = 0$ and correspondingly Čech coboundaries as exact cochains $\sigma = \delta\tau$. Note, that cocycles are skew-symmetric under permutation of indices. We are now ready to define Čech cohomology $H^n(X, \mathcal{S})$ as cocycles modulo coboundaries. Crucially, it has been shown that the resulting cohomologies do not depend on the choice of open covering. We refer to the maths literature for a proof of this statement. Note moreover, that for constant sheaves as for instance $\mathbb{Z}, \mathbb{R}, \mathbb{C}$, the notions of sheaf cohomology and cohomology with respect to the exterior derivative, agree.

Let us try to get some intuition on these definitions by spelling them out and by considering examples. The zeroth cohomology group $H^0(X, \mathcal{S})$ is just given by the δ-closed 0-chains, i.e. an element $\sigma \in H^0(X, \mathcal{S})$ satisfies $0 = \delta\sigma = (\sigma_\alpha - \sigma_\beta)\big|_{U_\alpha \cap U_\beta}$. By the glueing conditions, this just means that σ is a global section. So $H^0(X, \mathcal{S})$ counts the global sections of $\mathcal{S}$.

To gain an understanding of the first cohomology, consider as an example the sheaf $C^\infty(U(1))$. By the definitions above an element $g \in H^1(X, C^\infty(U(1)))$ is a collection of nowhere-vanishing holomorphic functions defined on overlaps of a collection of open sets. Since the group operation on $C^\infty(U(1))$ is multiplication, we have that a δ-closed Čech one-cochain satisfies

$$\mathbf{1} = \delta g = g_{\beta\gamma} g_{\alpha\gamma}^{-1} \gamma_{\alpha\beta} = g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha}. \tag{3.13}$$

To get the cohomology we need to mod out by δ-exact cochains, which is just the statement that $\{g_{\alpha\beta}\}$ and $\{f_{\alpha\beta}\}$ are equivalent if there is a $\phi \in C^0(X, C^\infty(U(1)))$, such that

$$f_{\alpha\beta} = \phi_\alpha g_{\alpha\beta} \phi_\beta^{-1}. \tag{3.14}$$

What is the point? Functions with the property (3.13) define transition functions for $U(1)$-bundles and (3.14) is just the statement that two different sets of transition functions define the same bundle. So $H^1(X, C^\infty(U(1)))$ classifies all inequivalent $U(1)$ bundles on X.

Let us consider a final example giving a well-known result as a byproduct. We want to consider the cohomology of the sheaf $C^\infty(\mathbb{R})$ on X with open covering $\{U_\alpha\}$. Now, $C^\infty(\mathbb{R})$ has the property that we can always find a partition of unity ρ_α on X such that $\sum \rho_\alpha = 1$. Sheaves with this property are called *fine* and behave particularly nice, because of the following: Take a p-cocycle $\sigma \in Z^p(U, C^\infty(\mathbb{R}))$ and define

$$\begin{aligned} \tau &\in C^{p-1}\left(\{U_\alpha\}, C^\infty(\mathbb{R})\right) \\ \tau_{\alpha_0 \cdots \alpha_{p-1}} &= \sum_\beta \rho_\beta \sigma_{\beta, \alpha_0, \ldots, \alpha_{p-1}}. \end{aligned} \tag{3.15}$$

From this follows immediately, that $\delta\tau = \sigma$ and so in conclusion all δ-closed forms are also δ-exact for this sheaf. Correspondingly, we have

$$H^p(X, C^\infty(\mathbb{R})) = 0, \qquad \text{for } p > 0. \tag{3.16}$$

We could have replaced $C^\infty(\mathbb{R})$ by (r,s)-forms $\Omega^{r,s}\left(X, C^\infty(\mathbb{R})\right)$ or by any other fine sheaf in the previous lines without changing the argument.

Apart from the fact that we wanted to get some intuition on sheaf cohomology, we have not chosen the last examples randomly, but instead they actually tie together as follows: Consider the following exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\text{inclusion}} C^\infty(\mathbb{R}) \xrightarrow{e^{i\cdot}} C^\infty(U(1)) \longrightarrow 0 \ , \tag{3.17}$$

which induces the following long exact sequence in cohomology

$$\begin{array}{ccccc} \cdots \longrightarrow H^1(X, C^\infty(\mathbb{R})) & \longrightarrow & H^1(X, C^\infty(U(1))) & & \\ & \swarrow & & & \\ H^2(X,\mathbb{Z}) & \longrightarrow & H^2(X, C^\infty(\mathbb{R})) & \longrightarrow \cdots \end{array} . \tag{3.18}$$

Now, we have just argued why the first and the last cohomology in (3.18) vanish and moreover we previously learnt that $H^1(X, C^\infty(U(1)))$ classifies inequivalent transition functions of $U(1)$-bundles. Finally, the quantity $H^2(X,\mathbb{Z})$ should be familiar to the reader as the group of Chern-classes and from the above we see that

$$H^1(X, C^\infty(U(1))) \cong H^2(X,\mathbb{Z}), \tag{3.19}$$

that is, Chern-classes classify inequivalent line bundles.

Before we proceed, let us make two useful comments. First, it has been shown that sheaf cohomology can be understood as differential forms with special coefficients if and only if the sheaf is locally free. Secondly, for a holomorphic vector bundle $\mathcal{E}$ we may relate sheaf and Dolbeault cohomology as

$$H^n(X,\mathcal{E}) = H^{0,n}(X,\mathcal{E}), \tag{3.20}$$

from which it follows that the highest degree n for a non-vanishing cohomology is $\dim_{\mathbb{C}} X$.

3.3.2 Spectra

We can finally come back to our actual goal to compute physical spectra. To do so, we aim to express the Ext-groups we introduced in 3.3 in terms of the more intuitive sheaf cohomologies of the last section. In spirit this will consist of two steps: First, we compute the *local Ext*, which are sheaves denoted by $\underline{\text{Ext}}^n(\mathcal{S},\mathcal{T})$ and can be thought of as the "fibre-by-fibre" equivalent of the Ext-groups. Secondly, we use the so-called *local-to-global* spectral sequence to obtain the Ext-groups. We will proceed with examples afterwards.

To start with the first part, recall that $\text{Hom}(\mathcal{S},\mathcal{T})$ are the globally defined homomorphisms from $\mathcal{S}$ to $\mathcal{T}$. In contrast, we denote by $\underline{\text{Hom}}(\mathcal{S},\mathcal{T})$ the sheaf of homomorphisms from $\mathcal{S}$ to $\mathcal{T}$ by which we mean that to any open subset $U \subseteq X$ we assign the homomorphisms that are well defined on U. Consequently, the global sections of $\underline{\text{Hom}}(\mathcal{S},\mathcal{T})$ is the group $\text{Hom}(\mathcal{S},\mathcal{T})$. We take $\mathcal{S},\mathcal{T}$ to be coherent, such that they enjoy a locally free resolution as in (3.7). This induces an exact sequence in the Hom-sheaves

$$\underline{\text{Hom}}(\mathcal{E}_0,\mathcal{T}) \longrightarrow \underline{\text{Hom}}(\mathcal{E}_1,\mathcal{T}) \longrightarrow \cdots \longrightarrow \underline{\text{Hom}}(\mathcal{E}_n,\mathcal{T}) \longrightarrow 0\ , \tag{3.21}$$

where the maps in this sequence are just composition with the derivatives of the $\mathcal{E}_\bullet$-resolution. Take for instance $f \in \underline{\text{Hom}}(\mathcal{E}_0,\mathcal{T})(U)$ and compose it to $f \circ d_0^{\mathcal{E}} \in \underline{\text{Hom}}(\mathcal{E}_1,\mathcal{T})(U)$. Note the reverse ordering of (3.21) as compared to (3.7). Now, we define the cohomology of this complex as the $\underline{\text{Ext}}$-sheaves

$$\underline{\text{Ext}}^k(\mathcal{S},\mathcal{T}) = h^k\left(\underline{\text{Hom}}(\mathcal{E}_\bullet,\mathcal{T})\right), \tag{3.22}$$

where by $h^k(\cdot)$ we mean the cohomology computed at the k-th position of (3.21).

The second step now consists of applying the local-to-global spectral sequence

$$E_2^{p,q} = H^p\left(X, \underline{\text{Ext}}^q(\mathcal{S},\mathcal{T})\right) \qquad \Rightarrow \qquad \text{Ext}^{p+q}(\mathcal{S},\mathcal{T}). \tag{3.23}$$

We refer to the maths-literature for more information on this. Typically, the sheaves $\mathcal{S}$ and $\mathcal{T}$ will have support only over subloci S of X, such as for instance a four-cycle in the case of D7-branes. If this is the case and $TX\big|_S$ splits holomorphically into TS and NS, then the spectral sequence (3.23) terminates after the initial leaf. The same is true for vanishing worldvolume flux [35]. Indeed in many cases of interest the spectral sequence is trivial and $E_2^{p,q}$ is the final leaf. Unfortunately, this is not true for generic T-brane configurations, making it challenging to compute their open string spectrum with these methods.

We are finally ready to compute spectra. Let us start with the simplest example two D9-branes, that is two locally free sheaves $\mathcal{E}, \mathcal{F}$.[4] So the induced sequence of $\underline{\mathrm{Hom}}$-sheaves is simply

$$0 \xrightarrow{i} \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{F}) \xrightarrow{0} 0 \tag{3.24}$$

and therefore we compute the only $\underline{\mathrm{Ext}}$-sheaf as

$$\underline{\mathrm{Ext}}^0(\mathcal{E}, \mathcal{F}) = \ker(0)/\mathrm{Im}(i) = \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{F}) = \mathcal{E}^V \otimes \mathcal{F}. \tag{3.25}$$

As we have explained the spectral sequence terminates at $E_2^{p,q}$ and therefore the Ext-groups can be computed as

$$\mathrm{Ext}^0(\mathcal{E}, \mathcal{F}) = H^0(X, \mathcal{E}^V \otimes \mathcal{F}) \tag{3.26a}$$

$$\mathrm{Ext}^1(\mathcal{E}, \mathcal{F}) = H^1(X, \mathcal{E}^V \otimes \mathcal{F}) \tag{3.26b}$$

$$\mathrm{Ext}^2(\mathcal{E}, \mathcal{F}) = H^2(X, \mathcal{E}^V \otimes \mathcal{F}) \tag{3.26c}$$

$$\mathrm{Ext}^3(\mathcal{E}, \mathcal{F}) = H^3(X, \mathcal{E}^V \otimes \mathcal{F}). \tag{3.26d}$$

This seems to give the correct results, but let us try to look at a less trivial example.

Take two D3-branes in flat space, which we take to lie at the origin $p = (0, 0, 0)$. That is we consider the spectrum between two skyscraper sheaves with support at p. As one may check the following is a locally free resolution

$$0 \longrightarrow \mathcal{O} \xrightarrow{M_1} \mathcal{O}^3 \xrightarrow{M_2} \mathcal{O}^3 \xrightarrow{M_3} \mathcal{O} \longrightarrow \mathcal{O}_p \longrightarrow 0 \tag{3.27}$$

for

$$M_1 = \begin{pmatrix} -x \\ y \\ -z \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & -z & -y \\ -z & 0 & x \\ y & x0 & \end{pmatrix}, \quad M_3 = \begin{pmatrix} x & y & z \end{pmatrix}. \tag{3.28}$$

As before, this induces an exact sequence in the $\underline{\mathrm{Hom}}$-sheaves, which is given by

$$0 \to \underline{\mathrm{Hom}}(\mathcal{O}, \mathcal{O}_p) \overset{M_3}{\to} \underline{\mathrm{Hom}}(\mathcal{O}^3, \mathcal{O}_p) \overset{M_2}{\to} \underline{\mathrm{Hom}}(\mathcal{O}^3, \mathcal{O}_p) \overset{M_1}{\to} \underline{\mathrm{Hom}}(\mathcal{O}, \mathcal{O}_p) \to 0\ . \tag{3.29}$$

We may now compute the $\underline{\mathrm{Ext}}$-sheaves by computing the cohomology at each step of this sequence, keeping in mind that $\mathcal{O}_p$ has support only over p and by

[4]In relation to our previous notation, $\mathcal{E}$ can be seen as its own locally free resolution.

extension the same is true also for $\underline{\text{Hom}}(\cdot, \mathcal{O}_p)$

$$\underline{\text{Ext}}^0(\mathcal{O}_p, \mathcal{O}_p) = \underline{\text{Hom}}(\mathcal{O}, \mathcal{O}_p) = \mathcal{O}^V \otimes \mathcal{O}_p = \mathcal{O}_p \tag{3.30a}$$

$$\underline{\text{Ext}}^1(\mathcal{O}_p, \mathcal{O}_p) = \underline{\text{Hom}}(\mathcal{O}^3, \mathcal{O}_p) = \left(\mathcal{O}^3\right)^V \otimes \mathcal{O}_p = \mathcal{O}_p^3 \tag{3.30b}$$

$$\underline{\text{Ext}}^2(\mathcal{O}_p, \mathcal{O}_p) = \underline{\text{Hom}}(\mathcal{O}^3, \mathcal{O}_p) = \mathcal{O}_p^3 \tag{3.30c}$$

$$\underline{\text{Ext}}^3(\mathcal{O}_p, \mathcal{O}_p) = \underline{\text{Hom}}(\mathcal{O}, \mathcal{O}_p) = \mathcal{O}_p. \tag{3.30d}$$

Since once again the spectral sequence terminates at $E_2^{p,q}$ the global Ext-groups are given by

$$\text{Ext}^0(\mathcal{O}_p, \mathcal{O}_p) = H^0(X, \mathcal{O}_p) = H^0(\{p\}, \mathcal{O}) = \mathbb{C} \tag{3.31a}$$

$$\text{Ext}^1(\mathcal{O}_p, \mathcal{O}_p) = H^(X, \mathcal{O}_p) \oplus H^0(X, \mathcal{O}_p^3) = H^0(\{p\}, \mathcal{O}^3) = \mathbb{C}^3 \tag{3.31b}$$

$$\text{Ext}^2(\mathcal{O}_p, \mathcal{O}_p) = \mathbb{C}^3 \tag{3.31c}$$

$$\text{Ext}^3(\mathcal{O}_p, \mathcal{O}_p) = \mathbb{C}, \tag{3.31d}$$

where we used again that $\mathcal{O}_p$ has support only over $\{p\}$ to reduce the cohomology groups over X to those over $\{p\}$. Moreover, $H^1(\{p\}, \mathcal{O}) = \emptyset$ because a point is zero-dimensional. Note, how (3.31a) encodes the degrees of freedom of a complex scalar field — the D3 Higgs field — and (3.31b) those of a vector, while (3.31c),(3.31d) encode their antiparticles.

As a second example let us compute the spectrum between two intersecting D7-branes. So take two T-branes given by the complexes

$$\mathcal{L}_1 \otimes \mathcal{P}_1^{-1} \xrightarrow{P_1} \mathcal{L}_1 \tag{3.32}$$

$$\mathcal{L}_2 \otimes \mathcal{P}_2^{-1} \xrightarrow{P_2} \mathcal{L}_2, \tag{3.33}$$

where $\mathcal{L}_i$ and $\mathcal{P}_i$ are (coherent) sheaves and P_i are sections of $\mathcal{P}_i$. The flux one the branes – and therefore the bundle carried by them – can be computed as [37]

$$F_1 = c_1(\mathcal{L}_1) - \frac{1}{2}c_1(\mathcal{P}_1) \tag{3.34}$$

$$F_2 = c_1(\mathcal{L}_2) - \frac{1}{2}c_1(\mathcal{P}_2). \tag{3.35}$$

We have argued in the last subsection that the physical spectrum is counted by the extension groups. We start out by resolving the first complex and make use that P_1 lifts to a map between the complexes to obtain the exact sequence

$$0 \longrightarrow \underline{\text{Hom}}\left(\mathcal{L}_1, \mathcal{L}_2\Big|_{P_2=0}\right) \xrightarrow{P_1} \underline{\text{Hom}}\left(\mathcal{L}_1 \otimes \mathcal{P}_1^{-1}, \mathcal{L}_2\Big|_{P_2=0}\right) \longrightarrow 0. \tag{3.36}$$

We proceed to compute the $\underline{\mathrm{Ext}}$-sheaves

$$\underline{\mathrm{Ext}}^0 = \ker(P_1)/\{0\} = 0 \tag{3.37}$$

$$\underline{\mathrm{Ext}}^1 = \ker(0)/\mathrm{Im}(P_1) = \mathcal{L}_2 \otimes \mathcal{L}_1^{-1} \otimes \mathcal{P}_1\Big|_{\mathcal{C}:\{P_1=P_2=0\}} \tag{3.38}$$

$$\underline{\mathrm{Ext}}^2 = 0, \tag{3.39}$$

where $\mathcal{C}$ is the intersection curve of the two branes. Once again we evaluate the local-to-global sequence,which gives the physical spectrum as

$$\mathrm{Ext}^0 = 0 \tag{3.40}$$

$$\mathrm{Ext}^1 = H^0\left(\mathcal{L}_2 \otimes \mathcal{L}_1^{-1} \otimes \mathcal{P}_1\Big|_{\mathcal{C}}\right) \tag{3.41}$$

$$\mathrm{Ext}^2 = H^1\left(\mathcal{L}_2 \otimes \mathcal{L}_1^{-1} \otimes \mathcal{P}_1\Big|_{\mathcal{C}}\right) \tag{3.42}$$

$$\mathrm{Ext}^3 = 0. \tag{3.43}$$

Now that we know how to compute the open string spectrum between two brane stacks, we should come to the next point: So far we have just assumed that the stacks we are dealing with, describe stable brane configurations. But in analogy to the last section we only expect a subset of possible brane vacua to actually be stable.

3.4 Stability

To describe configurations of D-branes in IIB string theory in terms of coherent sheaves and how to compute their spectra can also be motivated from a more rigorous point of view from topological field theory as is done in the references given at the beginning of this chapter, [34, 35]. From this perspective it is also clear that our expressions inherently capture the holomorphic data of the vacuum. That is to say, in the language of chapter 2: The F-terms are satisfied by construction and moreover, it is clear why we could compute the spectrum of open strings stretching between two branes without considering the equations of motion. However, as we recall from the last chapter the BPS-conditions consist not only of the F-terms, but also the D-terms, which mix holomorphic and anti-holomorphic data and depend on the position in Kähler moduli space. Clearly, we have not taken this into account in our description in terms of coherent sheaves, so far. So, while we might be able to compute the spectrum or tell if two different complexes encode the same brane, we cannot be sure if a brane is actually stable. In this section we aim to remedy this by introducing a suitable notion of stability. We will do so in two steps. First, we need to find which decay processes are *potentially* possible and secondly, we introduce a condition of stability to see whether this decay actually happens, depending on the position in Kähler moduli space. The first task proves to be mathematically quite involved.

What we aim to do in the following is to put an extra structure on the category of B-branes[5] in the form of so-called *distinguished triangles*, which intuitively simply encode the notion that two branes may potentially bind to a third one. To make things more accessible we will present physical intuition and mathematical definitions side by side. We turn the category of B-branes $\mathcal{C}$ into a *triangulated category* by introducing two extra ingredients: Firstly, the shift-functor $A[n]$ introduced in section 3.3, which denotes the complex A shifted n places to the left. Secondly, a set of distinguished triangles of objects A, B, C of $\mathcal{C}$

$$\begin{array}{ccc} & C & \\ {\scriptstyle c[1]}\swarrow & & \nwarrow{\scriptstyle b} \\ A & \xrightarrow{\quad a \quad} & B \end{array} , \tag{3.44}$$

which may equivalently be written as the exact sequence

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} A[1] \ . \tag{3.45}$$

Crucially, the objects in these sheaves are branes, that is we may think of the objects A, B, C as complexes like (3.4). The way to read these diagrams physically is as follows: Brane B can potentially decay into branes A and C. Vice versa A and C can bind via the string c to form the bound state B. Such distinguished triangles can be constructed using the so-called *mapping cone*, which we will introduce at a later point in 3.4.1. In the next chapter, in 4.1.1, we will see a specific example of a D7-brane and an anti-D5 brane binding to a non-trivial bound state. On top of the two ingredients we just introduced, a couple of axioms need to be met: (i) For any object A, the following is a

[5]Mathematically, this is the derived category of coherent sheaves.

distinguished triangle

$$\begin{array}{ccc} & 0 & \\ {\scriptstyle [1]}\swarrow & & \nwarrow \\ A & \xrightarrow{\mathbf{1}} & A \end{array} \quad . \tag{3.46}$$

This is just the trivial statement A may decay to A and nothing else. (ii) A triangle that is isomorphic to a distinguished triangle, is also distinguished, which just means that irrespective of the (non-unique) complex in which we choose to represent a D-brane, the decay conditions are always the same. (iii) Any morphism $a : A \longrightarrow B$ gives rise to a distinguished triangle as in (3.44) for some C, making formal the notion that if an open string stretches from A to B, they may form a bound state C. (iv) If we have a distinguished triangle (3.44), the following is automatically distinguished as well

$$\begin{array}{ccc} & C & \\ {\scriptstyle c}\swarrow & & \nwarrow{\scriptstyle b} \\ A[1] & \xrightarrow{a[1]} & B \end{array} \quad , \tag{3.47}$$

which is again more intuitive written as an exact sequence

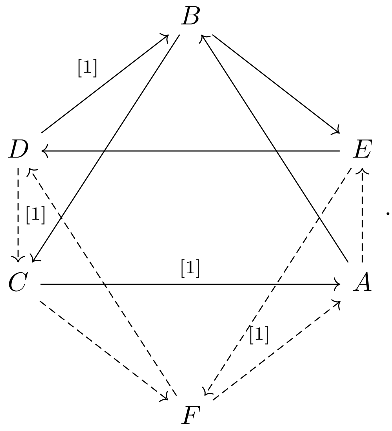
$$B \xrightarrow{\ b\ } C \xrightarrow{\ c\ } A[1] \xrightarrow{\ a\ } B[1] \ . \tag{3.48}$$

In plain English, this just translates to the assertion that if B can potentially decay into A and C, then also C may potentially decay into B and anti-A, which is $A[1]$. (v) Given two distinguished triangles in A, B, C and A', B', C' and two morphisms f, g as in

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array} \quad , \tag{3.49}$$

then a map h as in the diagram can be constructed from them, meaning that if open strings may stretch between the branes A and A' on the one hand and B and B' on the other hand, then there may also stretch an open string between their bound states C and C'. (vi) The last axiom is a criterion for associativity, in the following sense. Assume that there are two distinguished triangles BED

and ABC, as the bold arrows in



. (3.50)

We see from the diagram that by composition there are strings stretching from D to C and from A to E and from axiom (iii) we know that this implies the existence of an object F for each of the two dashed distinguished triangles. The last axiom is just the condition that these two objects are the same, as indicated in (3.50). We have now constructed a framework telling us which bound states may be formed. However, if this bound state is actually stable, depends on the position in Kähler moduli space as we stressed before.

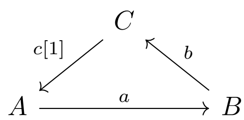
What we are missing then is a condition that tells us which side of the triangle constitutes the stable part, the bound state, or the two component states. This condition is called *π-stability*. The way to derive it, is to use mirror symmetry to translate the IIA expression. Since we will not be dealing with any more IIA physics, we skip this derivation here and refer the interested reader to the references given, instead. Here we will just give the results: At leading order in α', the central charge of a D-brane can be computed as

$$Z(\mathcal{E}_\bullet) = \int_X e^{-B-iJ} \mathrm{ch}(\mathcal{E}_\bullet)\sqrt{\mathrm{td}(X)}. \tag{3.51}$$

We define furthermore the "angle"

$$\xi(\mathcal{E}_\bullet) = \frac{1}{\pi} \arg\left(Z(\mathcal{E}_\bullet)\right), \tag{3.52}$$

which has the property $\xi\left(\mathcal{E}_\bullet[n]\right) = \xi\left(\mathcal{E}_\bullet\right) + n$. The stability condition may now be formulated as follows: Given a distinguished triangle



(3.53)

with A and C stable branes, then B is stable with respect to the decay $B \longrightarrow A + C$ if and only if $\xi(A) < \xi(C)$. Clearly, we are not making the statement that B is stable; there might exist other distinguished triangles including B, that allow for a decay and to be sure, we need to check all of them. Secondly, one might wonder how we can know that A and C are stable. The π-stability condition does not give us a set of stable branes, but instead only stability relations between branes, given such a set. Nevertheless, we need to provide this extra data at one point in moduli space to make meaningful statements about stability. However, since we know that the worldvolume description from chapter 2 is exact in the extreme large volume limit, we have such a point in which we can give the set of all stable branes. For completeness and future reference we introduce aforementioned mapping cone construction. The reader not interested in this, may skip to the next chapter.

3.4.1 The mapping cone construction

The *mapping cone construction* is a mathematical tool to formally add two coherent sheaves to form a third one. Put differently it is a means to explicitly construct bound states of constituent branes. Given two coherent sheaves $\mathcal{S}, \mathcal{T}$ with locally free resolutions $\mathcal{E}_\bullet, \mathcal{F}_\bullet$, we may form a bound state as

$$\cdots \longrightarrow \begin{matrix} \mathcal{E}_{i+1} \\ \oplus \\ \mathcal{F}_{i+1} \end{matrix} \xrightarrow{\begin{pmatrix} d_{\mathcal{E}} & 0 \\ f_i & d_{\mathcal{F}} \end{pmatrix}} \begin{matrix} \mathcal{E}_{i} \\ \oplus \\ \mathcal{F}_{i} \end{matrix} \xrightarrow{\begin{pmatrix} d_{\mathcal{E}} & 0 \\ f_{i-1} & d_{\mathcal{F}} \end{pmatrix}} \begin{matrix} \mathcal{E}_{i-1} \\ \oplus \\ \mathcal{F}_{i-1} \end{matrix} \longrightarrow \cdots , \tag{3.54}$$

where the maps $f_\bullet$ are the open strings stretching between the two branes. We will give examples of the use of this construction in the next chapter.

Chapter 4

Progress in understanding T-branes

In this paper we want to give a brief overview over the progress in understanding T-branes in recent years. Firstly, there has been considerable effort to improve our comprehension of T-branes from the point of view of global F-theory models, in which their role was previously unknown. We will discuss the two proposals [9, 11] on how to encode this data in a global F-theory compactification in 4.1. Secondly, we would like to discuss various constructions relying on T-brane vacua, such as the proposal to construct a de Sitter uplift using a T-brane background [38], which is especially relevant in light of the recent discussions about de Sitter solutions of string theory vacua in general [39] and moreover field theory applications in of T-branes.

4.1 T-branes under Dualities

As we have described in 2.3.1, the dictionary from a local 7-brane model in which we may define a T-brane background, to a global F-theory model is far from clear. This is due to the fact that only part of the information contained in the worldvolume Higgs-field Φ is mapped to the F-theory singularity structure whereas part of it is not. However, since also this second part breaks the gauge symmetry, the information contained in it needs to be encoded in different structures in the global F-theory picture. Recently, two proposals as to how to package this *T-brane data* have been made. On the one hand side, in [9] a suitable generalisation of the intermediate Jacobian has been constructed that holds in singular limits. It is argued that this structure contains the missing data, at least in compactifications to six dimensions. This approach has later been confirmed to work also in the presence of defects [40]. Secondly, it has been argued in [10] that T-branes in 7-brane vacua may be understood as bound states of 7-branes with lower dimensional branes when applying the language of Sen's tachyon condensation of chapter 3. In a companion paper the authors then propose how to lift this language to global F-theory vacua using so-called Eisenbud matrix factorisations [11].

4.1.1 Tachyon Maps and Matrix Factorisations

As we have reviewed in chapter 3, brane configurations in IIB string theory may be described as coherent sheaves. Furthermore, we have outlined how one can understand certain configurations as bound states of different branes taking advantage of additional structure on the category of physical branes, that is induced by the so called mapping cone. In [10] it has been argued that this language may be used to show that many T-brane configurations can in fact be understood as bound states of 7-branes with lower-dimensional branes or even simpler configurations. Recall, that any 7-brane configuration represented by a coherent sheaf $\mathcal{S}$ can be described as a two-term complex

$$0 \longrightarrow \mathcal{E} \xrightarrow{T} \mathcal{F} \longrightarrow \mathcal{S} \longrightarrow 0 \ , \tag{4.1}$$

where $\mathcal{E}, \mathcal{F}$ are locally free sheaves and correspondingly we may identify $\mathcal{S} = \text{coker}(T)$. This complex is called locally free resolution of $\mathcal{S}$. However, as we pointed out before, a locally free resolution is not unique and moreover $\mathcal{S}$ and the complex may be (quasi-)isomorphic to seemingly different complexes. If we find such a second resolution,we may understand $\mathcal{S}$ not only as the bound state indicated by (4.1), but also as a bound state indicated by this second resolution. Let us show how this works for a number of examples taken from the reference given. In the following we will always work in flat space to keep matters simple, where we denote by $S = \mathbb{C}[x, y, z]$ the coordinate ring.

Consider as a first example the simplest T-brane given by a constant nilpotent Higgs-vev

$$\Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \tag{4.2}$$

which breaks down the $U(2)$ gauge group to the centre of mass $U(1)$, as we explained in chapter 2. In the tachyon condensation language this corresponds to the locally free resolution[1]

$$\begin{array}{ccc} S^{\oplus 2} & \xrightarrow{T=\begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}} & S^{\oplus 2} \\ \circlearrowleft & & \circlearrowleft \\ g_{\bar{D}9} & & g_{D9} \end{array} , \tag{4.3}$$

where we also indicated the automorphisms, i.e. the gauge transformations, acting on the stack of $D9$ and anti-$D9$, respectively. Crucially, these automorphisms are independent of each other which allows for a larger class of transformations $T \longrightarrow g_{D9} \cdot T \cdot g_{\bar{D}9}^{-1}$. Indeed we may use these transformations to show that

$$T \longrightarrow \tilde{T} = \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} \cdot T \cdot \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix} , \tag{4.4}$$

where now clearly, the second line yields no contribution to $\text{coker}(\tilde{T})$, such that we have shown that (4.3) is equivalent to the complex

$$S \xrightarrow{\hat{T}=z^2} S \ . \tag{4.5}$$

This is the complex of a single 7-brane on the locus $z^2 = 0$. From this perspective the initial statement that the gauge group is broken down to the centre of mass $U(1)$, is obvious, because this is precisely the gauge group we would expect from a single brane.

As a second example consider, the background

$$\Phi = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} , \tag{4.6}$$

[1] Note, that in the following we omit the trivial parts of the exact sequence for brevity That is, all sequences start in $0 \to \cdots$ and end in $\cdots \to \mathcal{S} \to 0$, where $\mathcal{S}$ is the cokernel-sheaf of the preceding map.

for which we found matter localised on the curve $\{x = 0\}$ in chapter 2, invisible to the spectral polynomial $P_\Phi(z)$. From the tachyon condensation picture, this corresponds to the sequence

$$S^{\oplus 2} \xrightarrow{T=\begin{pmatrix} z & x \\ 0 & z \end{pmatrix}} S^{\oplus 2} \longrightarrow \mathrm{coker}(T) \ . \tag{4.7}$$

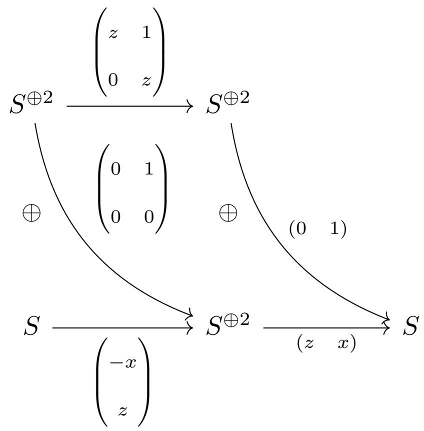
Already in this form we may read off that the dimension of the cokernel sheaf enhances to one on the location of the brane stack $\{z = 0\}$ and enhances to two on the sublocus $\{z = x = 0\}$ as expected from the Higgs-picture. Since this enhancement is invisible to the spectral polynomial, it cannot be related to the intersection of 7-branes, such that it is natural to expect a $D5$ or anti-$D5$ brane to play a role, purely on dimensional grounds. Indeed we consider a 7-brane given by

$$S^{\oplus 2} \xrightarrow{\begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}} S^{\oplus 2} \tag{4.8}$$

and an relative anti-$D5$ brane given by

$$S \xrightarrow{\begin{pmatrix} -x \\ z \end{pmatrix}} S^{\oplus 2} \xrightarrow{\begin{pmatrix} z & x \end{pmatrix}} S \ . \tag{4.9}$$

We reviewed in chapter 3, how bound states of branes may be constructed from their components using the mapping cone, by giving a vev to the open strings stretching between the two stacks. Indeed, we consider the bound state, specified by the following diagram [2]



. (4.10)

Adding up the individual components and maps, we recast this into the single complex

$$S^{\oplus 3} \xrightarrow{\begin{pmatrix} z & 1 & 0 \\ 0 & z & 0 \\ 0 & 1 & -x \\ 0 & 0 & x \end{pmatrix}} S^{\oplus 4} \xrightarrow{(0 \quad 1 \quad z \quad x)} S \ , \tag{4.11}$$

which can be simplified even further by applying automorphisms on the three terms of the sequence and subsequently omitting trivial components as in the previous example. The final result of which is the sequence in (4.7), such that we have shown that we may see this D7-brane bound state also as a bound state of a D7-brane with an anti-D5-brane.

Note, that in the language of distinguished triangles, introduced in section 3.4, the complex (4.10) or equivalently (4.11) represent the object B and (4.8), (4.9) represent objects A, C. It is in this way that we can make sense of the earlier statement that distinguished triangles can be constructed using the mapping cone.

We refer the interested reader to the original reference, for further examples as well as a classification of T-brane backgrounds in these terms. In the following

[2] The relative position of the two complexes and hence the maps stretching between them determines that we are binding to a relative D5. That is to say, if we were to shift the complex by one place, we would form a bound state between a D7 and a D5. The reason for this is, that we a making a choice which of the locally free sheaves in the complex are D9 branes and which are anti-D9 branes.

we will review the authors proposal [11] on how to lift this language from IIB to F-theory.

As we have pointed out in previous sections, the dictionary from the Higgs-bundle of a 7-brane stack to the geometry of an F-theory compactification is blind to certain moduli in the Higgs-vev. This missing information has been dubbed T-brane data in the past. Correspondingly, the full information about an F-theory vacuum should consist of the geometry itself as well as some additional structure holding this data. The proposal of [11] is that this information can be represented in the form of so-called Eisenbud matrix factorisations, which can be related to the language of tachyon condensation in IIB string theory as well as the theory of non-commutative crepant resolutions.

Consider once again an arbitrary D7-brane bound state in IIB string theory given as (4.1). Now, the sheaf $\mathcal{S} = \mathrm{coker}(T)$ is only non-trivial over the D7-brane stack with potential enhancements over subloci. Correspondingly, if we denote the locus of the stack by $\{P_{D7} = 0\}$ for some polynomial, then any section $s \in \mathcal{S}(U)$ satisfies $s \circ P_{D7} = 0$. That is to say, at the level of the cohomology of the complex P_{D7} is the zero-endomorphism and thereby pure homotopy. As a consequence we may always construct a map $\tilde{T}$, such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\;T\;} & F \\ {\scriptstyle P_{D7}}\downarrow & \overset{\tilde{T}}{\swarrow} & \downarrow{\scriptstyle P_{D7}} \\ E & \xrightarrow[\;T\;]{} & F \end{array} \, . \tag{4.12}$$

In equations, this is the requirement

$$T \cdot \tilde{T} = \tilde{T} \cdot T = P_{D7} \cdot \mathbf{1}_2 . \tag{4.13}$$

A pair of matrices $(T, \tilde{T})$ with the property (4.13) is called an *Eisenbud matrix factorisation* of the polynomial P_{D7}. Note, that such matrix factorisations are not unique. So as we have seen any D7-brane tachyon map automatically implies the existence of a second map $\tilde{T}$ to form a matrix factorisation of the locus of its D7-brane stack.

Recall, now that an F-theory vacuum is given as a hypersurface in some ambient space, defined as the zero locus of a Weierstraß-polynomial $P = 0$. Now, the proposal of [11] is, that this polynomial needs to be supplemented by an adequate matrix factorisation, which will contain the T-brane data. Put differently, two F-theory vacua with distinct spectrum may be defined by the same P and only differ in the choice of a matrix factorisation. The crucial claim is, that one may compute the spectrum directly from this matrix factorisation without passing to a smooth geometry either by resolution or deformation, similar to how one computes the open string spectrum in IIB using a tachyon map. The intuitive difference between these two approaches is that performing a resolution amounts physically to passing to the Coulomb branch from which T-brane vacua are inaccessible, such that instead we need to deal with the singular manifold directly to describe these vacua. Let us discuss this in more detail.

Given the definition of a matrix factorisation we have two basic goals: First of all to identify which matrix factorisations are inequivalent and secondly to compute the massless spectrum from them, focussing in particular on chiral and anti-chiral matter. The findings of these computations should be compared to cases with known weak coupling limit as a test. From the definition in (4.13) it is clear that we can enlarge any matrix factorisation by the pairs $(1, P)$ or $(P, 1)$ and it will later be clear that this does not change any physics, such that we should consider matrix factorisations equivalent if they are the same up to direct summands of this form. So we define the *stable category of matrix factorisations* $\underline{\mathrm{MF}}(P)$ as any arbitrary matrix factorisation up to such direct summands $(P, 1)$. The second part of the question on how to compute the massless spectrum and compare it to the weak coupling limit is harder to tackle. This is true in particular, because so far we have given no prescription on how to translate an F-theory matrix factorisation into the weak coupling limit.

The proposal of [11] is that, given an F-theory matrix factorisation(ϕ, ψ) of the Weierstraß-polynomial P, we may define their cokernels $M \equiv \mathrm{coker}(\phi)$ and $\tilde{M} \equiv \mathrm{coker}(\psi)$ in terms of which the chiral and anti-chiral spectrum can be computed as

$$\begin{aligned} \mathrm{Ext}^1\left(M_{\mathrm{tot}}, M_{\mathrm{tot}}\right) = &\, \mathrm{Ext}^1(M, M) \oplus \mathrm{Ext}^1(\tilde{M}, \tilde{M}) \\ &\oplus \underbrace{\mathrm{Ext}^1(M, \tilde{M})}_{\text{chiral matter}} \oplus \underbrace{\mathrm{Ext}^1(\tilde{M}, M)}_{\text{anti-chiral matter}} \;. \end{aligned} \tag{4.14}$$

Checking the results of this claim by comparing to weak coupling computations can be achieved in some cases by a mathematical theorem known as Knörrer's periodicity, which assures: Given a polynomial $P \in S$ for some ring S and a second polynomial $(P + uv) \in S[u, v]$ by which we mean the ring S enlarged by the two coordinates u, v, the stable categories of matrix factorisations are isomorphic

$$\underline{\mathrm{MF}}(P) \cong \underline{\mathrm{MF}}(P + uv). \tag{4.15}$$

Given a matrix factorisation $(\phi, \psi) \in \underline{\mathrm{MF}}(P)$ of size n, we may construct a matrix factorisation in $\underline{\mathrm{MF}}(P + uv)$ of size $2n$ as

$$\left(\begin{pmatrix} \phi & -u \cdot \mathbf{1}_n \\ v \cdot \mathbf{1}_n & \psi \end{pmatrix}, \begin{pmatrix} \psi & u \cdot \mathbf{1}_n \\ -v \cdot \mathbf{1}_n & \phi \end{pmatrix} \right). \tag{4.16}$$

With this statement at hand let us now consider an example.

Take IIB on $X \times \mathbb{R}^{1,3}$, where X is given by the non-compact Calabi-Yau

σ_1	σ_2	z_1	z_2
1	1	-1	-1

,

such that we read off that the intersection curve $z_1 = z_2 = 0$ is a $\mathbb{P}^1$. Now, let us put a D7-brane each on $z_1 = 0$ and $z_2 = 0$ by giving a vev to the tachyon as

$$\begin{array}{ccc} \mathcal{O}(n_1+1) & & \mathcal{O}(n_1) \\ \oplus & \xrightarrow{T = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}} & \oplus \\ \mathcal{O}(n_2+1) & & \mathcal{O}(n_2) \end{array}, \tag{4.17}$$

such that the D7-branes lie on the locus $P_{D7} = z_1 z_2$. The line bundles allow for flux on each of the branes. As we have shown in chapter 3, the chiral and anti-chiral part of the spectrum can be computed as

$$H^0(\mathbb{P}^1, \mathcal{O}(n_2 - n_1 - 1)) \oplus H^1(\mathbb{P}^1, \mathcal{O}(n_2 - n_1 - 1)). \tag{4.18}$$

Now, the F-theory uplift is given by the hypersurface

$$Y^2 = X^3 + X^2 Z^2 - z_1 z_2 Z^6 \quad \subset \mathbb{C}^2 \times \mathbb{P}^2_{2,3,1}, \tag{4.19}$$

which is singular along the locus $Y = X = z_1 = z_2 = 0$. In vicinity of the singularity we may discard the cubic term in X and use the projective rescaling to fix $Z \equiv 1$, such that we get the local form

$$Y^2 = X^2 - z_1 z_2, \tag{4.20}$$

which in turn can be brought to a manifestly conifold form by defining $u = Y+X$ and $v = Y - X$, giving

$$uv + z_1 z_2 = 0 \quad \in \mathbb{C}[z_1, z_2, u, v]. \tag{4.21}$$

So we found that in this local approximation of the example at hand, the Weierstraß-polynomial P and the D7-brane locus are related by

$$P = P_{D7} + uv. \tag{4.22}$$

This is clearly the scenario in which we can apply Knörrer's periodicity (4.15),(4.16) to get a matrix factorisation of P as

$$\phi \equiv \begin{pmatrix} z_1 & -u \\ v & z_2 \end{pmatrix}, \qquad \psi \equiv \begin{pmatrix} z_2 & u \\ -v & z_1 \end{pmatrix}, \tag{4.23}$$

which are maps

$$\underbrace{\begin{matrix} \mathcal{O}(n_1+1) \\ \oplus \\ \mathcal{O}(n_1+1) \end{matrix}}_{=:V_1} \xrightarrow{\phi} \underbrace{\begin{matrix} \mathcal{O}(n_1) \\ \oplus \\ \mathcal{O}(n_1) \end{matrix}}_{=:V_0} \quad \text{and} \quad \underbrace{\begin{matrix} \mathcal{O}(n_2+1) \\ \oplus \\ \mathcal{O}(n_2+1) \end{matrix}}_{\tilde{V}_1} \xrightarrow{\psi} \underbrace{\begin{matrix} \mathcal{O}(n_2) \\ \oplus \\ \mathcal{O}(n_2) \end{matrix}}_{\tilde{V}_0} . \tag{4.24}$$

Now we are interested in the groups $\mathrm{Ext}^1(M, \tilde{M})$ and $\mathrm{Ext}^1(\tilde{M}, M)$ hosting chiral and anti-chiral matter. From (3.8), we know that we may understand them as the morphisms $F \sim F + \psi \circ H_1 + H_0 \circ \phi$ and $\tilde{F} \sim \tilde{F} + \phi \circ \tilde{H}_1 + \tilde{H}_0 \circ \psi$ in the diagram

$$\begin{array}{ccccc} & V_1 & \xrightarrow{\phi} & V_0 & \\ \swarrow^{H_1} & \downarrow F & \swarrow^{H_0} & & \\ \tilde{V}_1 & \xrightarrow{\psi} & \tilde{V}_0 & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & \tilde{V}_1 & \xrightarrow{\psi} & \tilde{V}_0 & \\ \swarrow^{\tilde{H}_1} & \downarrow \tilde{F} & \swarrow^{\tilde{H}_0} & & \\ V_1 & \xrightarrow{\phi} & V_0 & & \end{array} . \tag{4.25}$$

Indeed one may work out, that the inequivalent F and $\tilde{F}$ are then counted by

$$H^0(\mathbb{P}^1, \mathcal{O}(n_2 - n_1 - 1)) \oplus H^1(\mathbb{P}^1, \mathcal{O}(n_2 - n_1 - 1)), \tag{4.26}$$

which is indeed the same as in the IIB case.

The interested reader may find more involved examples, in particular also T-branes, in the original reference, where it is also explained how the theory of Eisenbud matrix factorisations is related to so called *non-commutative crepant resolutions* of singularities. For the sake of brevity we will not delve into further details here, however.

4.1.2 Abelian Duals

Another interesting approach to understand the nature of T-branes in the light of string dualities has been provided by [41,42]. It has been known for some time that non-Abelian Dp-brane configurations with three non-commuting worldvolume scalars allow for a dual Abelian description in terms of a Dp-brane [43]. Such a description is however, not available for T-brane vacua where only two of the worldvolume scalars are non-commuting. The analysis of [41] instead shows that non-Abelian T-branes vacua of D7-branes have a dual description in terms of a single D7-brane with non-trivial worldvolume curvature. Where the former perspective is valid for small field vevs in units of the string scale and the latter is valid for large field vevs — corresponding to a small worldvolume flux in the dual description. For large numbers of branes N both descriptions become increasingly valid and their overlap enlarges. The basic idea to this conclusion may be summarised as follows

$$\begin{array}{ccc} N\text{ non-ab. D7 w/ T-brane} & & \text{abelian D7} \\ \big\downarrow{\scriptstyle\text{T-duality}} & & {\scriptstyle\text{T-duality}}\big\uparrow \\ N\text{D6 bound state} & \xrightarrow{\text{polarisation}} & \text{abelian D8} \end{array} \quad . \tag{4.27}$$

In [42] this was generalised to T-branes of D2-branes.

4.2 Other Developments

Recently, the question whether stable de Sitter vacua can be constructed in string theory has generated lots of attention following [39]. It is therefore particularly interesting to review a proposal [38] to construct such vacua using a T-brane background for a D7-brane stack. In this reference a de Sitter uplifting term at the right scales is induced by the presence of non-trivial three-form fluxes $G_3 = F_3 - \tau H_3$ in conjunction with a T-brane of the type

$$\Phi = \begin{pmatrix} 0 & \varphi \\ 0 & 0 \end{pmatrix} . \tag{4.28}$$

The basic mechanisms relies on the fact that the expansion of DBI- and Chern-Simons-action contains a term

$$\mathcal{L} \supset -2g_s|G|^2 \mathrm{Tr}|\Phi|^2 = 2g_s|G|^2 \varphi \wedge \bar{\varphi} \tag{4.29}$$

$$|G|^2 \equiv \frac{1}{12} g^{mm'} g^{nn'} g^{kk'} G_{3|mnk} G_{3|m'n'k'}, \tag{4.30}$$

which is indeed positive definite and may therefore lift an AdS vacuum to a dS one. Clearly, for this mechanism to work, stable T-branes need to exist on the four-cycle wrapped by the D7-brane stack. As we will see in later chapters, this is not always possible.

Moreover, T-branes have also attracted some attention from the field theory community, claiming that certain field theories are in fact related to T-brane configurations in F-theory. In [44] for instance, it has been claimed that the Higgsing of six-dimensional SCFT's can be understood as a T-brane vev for 7-branes in F-theory via the duality to M-theory with M5-branes.

In [12, 13] T-brane configurations for D6-branes have been explored using probe D2-branes. By using three-dimensional mirror symmetry it is shown how the T-brane data is mapped to the singular geometry of the corresponding M-theory compactification. As a byproduct of this analysis a new class of 3d $\mathcal{N} = 2$ field theories is introduced.

[41, 42] [40]

Chapter 5

T-branes and α'-corrections

5.1 Introduction

So far all of our introductory discussions have assumed that we were in a scenario in which the four-cycle carrying the D7-brane stack is sufficiently large and and only weakly curved, such that effects at higher orders in α' or l_s are irrelevant. However, as we pointed out in the introduction this is not true in general. Instead, the worldvolume theory and by extension its supersymmetry conditions receive corrections in α'. In the case of multiple D7-branes such α'-corrections are encoded in the non-Abelian DBI+CS actions, and their effect can in principle be extracted directly from there. In practice it is however simpler to see how these corrections modify the BPS equations for multiple D7-branes, and then analyse the configurations that solve the corrected equations. The purpose of this chapter is to apply this strategy to analyse α'-corrections in T-brane systems of D7-branes, including all those ingredients that appear in F-theory GUT model building.

Since D7-branes wrapping holomorphic four-cycles are examples of B-branes, we expect that α'-corrections do not modify their F-term equations and only affect their D-term BPS equations. In other words, if we describe the corrected BPS equations as a Hitchin system, the holomorphic 7-brane data will remain unaffected and α'-corrections will only modify the stability condition [45]. This result, which we review from the viewpoint of [46, 47], allows to solve for α'-

corrected T-brane backgrounds with the same strategy used in [4]: we first define their holomorphic data and then solve the D-term equation in terms of a complexified gauge transformation acting on Φ and A. We will then see that α'-corrections will not only change the initial T-brane profile quantitatively, but also qualitatively.

Indeed, a standard class of T-brane configurations features a Higgs field Φ along a set of non-commuting generators E_i and a non-primitive worldvolume flux of the form

$$F = -i\partial\bar{\partial} f \, P \tag{5.1}$$

that solves the classical D-term equation. Here P is a Cartan generator of the gauge group G, while f is a function of the 7-brane coordinates that solves a certain differential equation and that also enters in the profile for Φ [4]. While non-trivial, this Abelian profile for F is relatively simple, in the sense that it could involve several, non-commuting generators of G. In this chapter we will consider the α'-corrected version of this class of systems. As a general result we find that several things can happen:

i) In the most simple example of this setup, which preserves eight supercharges, the same background is also a solution of the α'-corrected D-term equations.

ii) We may lower the amount of supersymmetry to four supercharges by

  a) modifying the Higgs background as $\Phi \to \Phi + \Delta\Phi$, with $[\Phi, \Delta\Phi] = [F, \Delta\Phi] = 0$,

  b) introducing a primitive worldvolume flux H that commutes with Φ and F.

  Ignoring α'-corrections *a*) and *b*) do not modify the T-brane piece of the background. However, taking α'-corrections into account the profile for the function f is modified.

iii) If we perform *a*) and *b*) simultaneously while preserving four supercharges then, in general, (5.1) may not solve the α'-corrected D-term equations and the non-primitive flux F will have to develop new components along the non-Cartan generators E_i. The T-brane profile for Φ will also become more involved.

Interestingly, a) and b) are standard features that one needs to implement in local F-theory GUTs in order to engineer realistic 4d chiral models [7, 8, 48]. One may therefore expect that, in general, the description of T-brane systems leading to realistic F-theory models will be qualitatively modified when taking into account the effect of α'-corrections, at least at the level of non-holomorphic data.

The chapter is organised as follows. In section 5.2 we derive how α'-corrections enter systems of multiple D7-branes, and in particular how they modify their D-term equations. In section 5.3 we solve such α'-corrected D-term equations for system of intersecting D7-branes, relating the corrections to the pull-back on each individual D7-brane embedding. Then, in section 5.4, we turn to solve the α'-corrected D-term equations for simple T-brane backgrounds, which already illustrate the three cases described above. In section 5.5 we discuss how to solve α'-corrected D-term equations in more general T-brane systems and how the same phenomena arise in there. In section 5.6 we briefly comment on the implications of our findings for some local F-theory GUT models.

Several technical details have been relegated to the Appendices. Appendix A contains an alternative derivation of the α'-corrected D-term equations by means of the non-Abelian Chern-Simons action. Appendix B shows that α'-corrections are trivial for certain T-brane systems with globally nilpotent Higgs field. Appendix C shows how adding non-Cartan flux backgrounds can solve the corrected D-term equations in the T-brane backgrounds of section 5.4 that correspond to case iii), at least to next-to-leading order in the α'-expansion. Appendix D shows the results of the analysis of section 5.4 applied to further $SU(2)$ T-brane backgrounds.

5.2 D7-branes, D-terms and their α'-corrections

Let us consider type IIB string theory compactified on a Calabi-Yau threefold X_3, and then quotiented by an orientifold action such that the presence of O3/O7-planes is induced. In order to cancel the related RR charge of these orientifold content one may add different stacks of D3-branes and D7-branes, the latter wrapping four-cycles $\mathcal{S}_a \subset X_3$ in the internal space and with internal worldvolume fluxes F switched on along $\mathcal{S}_a$.

In the simplest configuration that one may consider, each stack would only involve a single D7-brane, wrapping a collection of different, isolated four-cycles $\{\mathcal{S}_a\}$. For each of these D7-branes one can check if the energy is minimised by looking at its BPS conditions, which amount to require that the four-cycle $\mathcal{S}$ is holomorphic that the worldvolume flux threading it is a primitive (1,1)-form in $\mathcal{S}$ [49–51].[1] These BPS conditions are captured by the following functionals [52, 53]

$$W = \int_{\Sigma_5} \mathrm{P}\left[\Omega_0 \wedge e^{-B}\right] \wedge e^{\lambda F} \tag{5.2}$$

$$D = \int_{\mathcal{S}} \mathrm{P}\left[\mathrm{Im}\, e^{iJ} \wedge e^{-B}\right] \wedge e^{\lambda F} \tag{5.3}$$

that in 4d are respectively interpreted as a superpotential and D-term for each D7-brane. Here J is the Kähler form and $\Omega_0 = e^{\phi/2}\Omega$ a holomorphic (3,0)-form in X_3, normalised such that $\frac{1}{6}J^3 = -\frac{i}{8}\Omega \wedge \bar{\Omega}$. In addition, B is the internal B-field, $F = dA$ the worldvolume flux and $\lambda = 2\pi\alpha'$. Finally, Σ_5 is a five-chain describing the deformations of the four-cycle $\mathcal{S}$, which infinitesimally can also be parametrised by the complex position coordinates Φ^i, and $P[\ldots]$ stands for the pull-back on the D7-brane worldvolume, namely

$$\mathrm{P}\left[V_\mu \mathrm{d}z^\mu\right]_\alpha = V_\alpha + \lambda V_i \mathrm{p}_\alpha \Phi^i \tag{5.4}$$

with α a coordinate in $\mathcal{S}$.

More generally, one would consider configurations involving stacks of several 7-branes, with non-Abelian bundles on them and wrapping four-cycles that intersect each other. On a given patch of the internal manifold one can describe such configurations in terms of an 8d twisted super Yang-Mills theory with a given non-Abelian symmetry group G [14, 54–56]. The bosonic field content of this theory is given by a gauge field A and a Higgs-field Φ transforming in the adjoint of G, and whose background profiles will break G to a smaller gauge symmetry group. In this chapter we are interested in configurations in which

[1] In our conventions $\mathcal{S}$ is calibrated by $-J^2$ and so a BPS worldvolume flux is self-dual $F = *_{\mathcal{S}} F$.

the profile for Φ is intrinsically non-Abelian, and more precisely in the kind of profiles considered in [3–5, 9–11] and dubbed T-branes in [4].

Just like in the Abelian case, the non-Abelian profiles for Φ and A need to satisfy certain equations of motion that are captured by 7-brane functionals. In order to describe the non-Abelian generalisation of (5.2) and (5.3) one may proceed as follows [46, 47].[2] First one uses the equations of motion of the background to locally write $\Omega_0 \wedge e^B = d\gamma$, and so rewrite the integral in (5.2) as $\int_{\mathcal{S}} P[\gamma] \wedge e^{\lambda F}$. Then one observes that, since both W and D have both the form of the D7-brane Chern-Simons action, their non-Abelian generalisation should go along the same lines as described in [43]. More specifically, we replace the derivatives in the pull-back by gauge-covariant ones and symmetrise over the gauge indices. We finally obtain

$$W = \int_{\mathcal{S}} \mathrm{STr}\Big\{\mathrm{P}\left[e^{i\lambda\iota_\Phi\iota_\Phi}\gamma\right] \wedge e^{\lambda F}\Big\} \tag{5.5}$$

$$D = \int_{\mathcal{S}} \mathrm{S}\Big\{\mathrm{P}\left[e^{i\lambda\iota_\Phi\iota_\Phi}\mathrm{Im}\, e^{iJ} \wedge e^{-B}\right] \wedge e^{\lambda F}\Big\}. \tag{5.6}$$

where ι_Φ stands for the inclusion of the complex Higgs field Φ, and S for symmetrisation over gauge indices. Just like eqs.(5.2) and (5.3), these functionals describe the D-brane BPS equations whenever the approximations leading to the D-brane DBI + CS actions hold, namely internal volumes with are large and slowly varying profiles for Φ and F in string length units. In this regime the D-term functional (5.6) should take into account all the α'-corrections to the BPS equations for a non-Abelian system of D7-branes.[3]

In order to bring these expressions to a more familiar form let us introduce local complex coordinates x, y, z and take the four-cycle $\mathcal{S}$ along the locus $\{z =$

[2] See [45] for a previous, alternative derivation of these equations.

[3] That is, if we neglect higher derivative corrections of the Riemann tensor. After taking such curvature corrections into account one expects a non-Abelian D-term of the form [45]

$$D = \int_{\mathcal{S}} \mathrm{P}\left[\mathrm{Im}\, e^{iJ} \wedge e^{-B}\right] \wedge e^{\lambda F'} \wedge \sqrt{\hat{\mathcal{A}}(\mathcal{T})/\hat{\mathcal{A}}(\mathcal{N})}$$

with $\hat{\mathcal{A}}$ the A-roof genus of the tangent $\mathcal{T}$ and normal $\mathcal{N}$ bundles, and $F' = F - \frac{1}{2}F_{\mathcal{N}}$ with $F_{\mathcal{N}}$ the normal bundle curvature [57–61]. Here $\sqrt{\hat{\mathcal{A}}(\mathcal{T})/\hat{\mathcal{A}}(\mathcal{N})} = 1 - \frac{1}{48}[p_1(\mathcal{T}) - p_1(\mathcal{N})] + \ldots$ with p_1 the real four-form given by the first Pontryagin class. Note that this correction does not affect the Abelian D-term but it is non-trivial in the non-Abelian case. In the following we will consider a local patch in which the Kähler metric is locally flat, and therefore take $p_1 = 0$ and $F' = F$. It would be interesting to see if our results could change qualitatively when these curvature corrections become important.

$0\}$ – that is x and y are the coordinates of $\mathcal{S}$. In this local description the Higgs field is given by

$$\Phi \equiv \phi \frac{\mathrm{p}}{\mathrm{p}z} + \overline{\phi} \frac{\mathrm{p}}{\mathrm{p}\overline{z}}. \tag{5.7}$$

where ϕ is a matrix in the complexified adjoint representation of G and $\overline{\phi}$ its Hermitian conjugate. Locally, we may also take $\gamma \equiv z\,\mathrm{d}x{\wedge}\mathrm{d}y$, such that in particular we have $\iota_\Phi \gamma = 0$. Performing a normal coordinate expansion of γ and plugging it into (5.5) then gives

$$W = \lambda^2 \int_{\mathcal{S}} \mathrm{Tr}\,\{\phi\, dx{\wedge}dy{\wedge}F\} \;=\; \lambda^2 \int_{\mathcal{S}} \mathrm{Tr}\,\{\iota_\Phi \Omega{\wedge}F\}\,. \tag{5.8}$$

which is the 7-brane superpotential considered in [14, 54, 55].[4] Crucially, the integrand does not depend on λ, which implies that the F-term conditions are entirely topological and receive no α'-corrections.

We will now see that this is not the case for the D-terms (5.6), which are evaluated as

$$\begin{aligned} D = \int_{\mathcal{S}} \mathrm{S}\Big\{ & \lambda P[J]{\wedge}F - \frac{i\lambda}{6}\iota_\Phi\iota_\Phi J^3 + \frac{i\lambda^3}{2}\iota_\Phi\iota_\Phi J{\wedge}F{\wedge}F \\ & - \mathrm{P}[J{\wedge}B] - i\lambda^2\iota_\Phi\iota_\Phi(J{\wedge}B){\wedge}F + \frac{i\lambda}{2}\iota_\Phi\iota_\Phi(J{\wedge}B^2)\Big\}, \end{aligned} \tag{5.9}$$

where we have kept terms of all orders in λ in this expansion.[5] In our local patch we may take the flat space Kähler form to be

$$J = \underbrace{\frac{i}{2}\mathrm{d}x{\wedge}\mathrm{d}\overline{x} + \frac{i}{2}\mathrm{d}y{\wedge}\mathrm{d}\overline{y}}_{=:\omega} + 2i\mathrm{d}z{\wedge}\mathrm{d}\overline{z}, \tag{5.10}$$

decompose the background B-field as $B \equiv B\big|_{\mathcal{S}} + B_{z\overline{z}}\mathrm{d}z{\wedge}\mathrm{d}\overline{z}$ and write $\mathcal{F} = \lambda F - B\big|_{\mathcal{S}}$, yielding

$$\begin{aligned} D = \int_{\mathcal{S}} \mathrm{S}\Big\{ & \mathrm{P}[J]{\wedge}\mathcal{F} + \frac{i\lambda}{2}\left(\iota_\Phi\iota_\Phi J\right)\left(\mathcal{F}^2 - \omega^2\right) \\ & - i\lambda\left(\iota_\Phi\iota_\Phi B\right)\omega{\wedge}\mathcal{F} - \omega{\wedge}\mathrm{P}[B_{z\overline{z}}\mathrm{d}z{\wedge}\mathrm{d}\overline{z}]\Big\}. \end{aligned} \tag{5.11}$$

[4]Notice that in these references the two-form $\iota_\Phi\Omega$ is denoted by Φ.

[5]Including curvature corrections there would be an extra term of the form $\frac{i\lambda}{48}\iota_\Phi\iota_\Phi J[p_1(\mathcal{T}) - p_1(\mathcal{N})]$.

Here we defined the Abelian pull-back ω to $\mathcal{S}_4$ as indicated in (5.10), such that we have

$$\iota_\Phi \iota_\Phi J = 2i[\phi, \overline{\phi}]$$
$$\iota_\Phi \iota_\Phi J^3 = 6i[\phi, \overline{\phi}]\omega^2.$$

To proceed we note that $2i[\phi, \overline{\phi}]$ is a zero-form and secondly, that $6i[\phi, \overline{\phi}]\omega^2$ has no transverse legs to $\mathcal{S}$. That is, in both cases the pull-back P acts trivially. Lastly, one may compute

$$\mathrm{P}[J] = \omega + 2i\lambda^2(\mathrm{D}\phi)\wedge(\bar{D}\overline{\phi}). \tag{5.12}$$

so at the end we have that the D-term equations amount to $D = 0$ with

$$D = \int_{\mathcal{S}} \mathrm{S}\Big\{ \omega\wedge\mathcal{F} + \lambda^2 \mathrm{D}\phi\wedge\overline{\mathrm{D}\phi}\wedge\left(2i\mathcal{F} - B_{z\bar{z}}\omega\right) \tag{5.13}$$
$$+ \lambda\left[\phi, \overline{\phi}\right]\left(\omega^2 - \mathcal{F}^2 - iB_{z\bar{z}}\omega\wedge\mathcal{F}\right)\Big\}.$$

For vanishing B-field, this simplifies to

$$D = \lambda \int_{\mathcal{S}} \mathrm{S}\Big\{ \omega\wedge F + 2i\lambda^2 \mathrm{D}\phi\wedge\overline{\mathrm{D}\phi}\wedge F + \left[\phi, \overline{\phi}\right]\left(\omega^2 - \lambda^2 F^2\right)\Big\}. \tag{5.14}$$

These expressions reproduce those found in [45], and can be recovered by analysing the non-Abelian Chern-Simons action of a stack of D7-branes, as discussed in Appendix A.

Note that both terms at leading order in λ, namely $\omega\wedge F + \left[\phi, \overline{\phi}\right]\omega^2$, are purely algebra valued. Crucially, this is not the case anymore when we include higher orders, because these additional terms contain products of generators. From the original formula in (5.3) it is clear that these products have to be understood in the same way as in the exponentiation map, which implies that for matrix algebras $\mathfrak{g} \subset GL(n, \mathbb{C})$ they are simply the matrix products in the fundamental representation of said algebra. Taking into account the symmetrisation procedure, we end up considering terms of the form

$$\mathrm{S}\big\{T_1 \dots T_n\big\} = \frac{1}{\#\text{ of perm.}} \sum_{\text{all perm. } \sigma} T_{\sigma_1} \cdots T_{\sigma_n}. \tag{5.15}$$

Formally speaking, including higher order corrections in λ means that the D-terms are valued in the universal enveloping algebra $U(\mathfrak{g})$ rather than $\mathfrak{g}$ itself.

5.3 α'-corrections for intersecting branes

To get some intuition on the meaning of the α' corrections on D-terms, let us first consider the case where the Higgs field ϕ and the gauge flux F can be diagonalised, as is for the case of intersecting D7-brane backgrounds. Then the D-term equations amount to

$$D = \lambda \int_{\mathcal{S}} \mathrm{P_{a}}_b[J] \wedge F \;=\; \lambda \int_{\mathcal{S}} \left(\omega + 2i\lambda^2 \mathrm{p}\phi \wedge \overline{\partial\phi}\right) \wedge F, \tag{5.16}$$

that is to say the α'-corrections are given entirely by the Abelian pull-back of the Kähler-form J to $\mathcal{S}$, $\mathrm{P_{ab}}[J] \equiv \left(\omega + 2i\lambda^2 \mathrm{p}\phi \wedge \overline{\partial\phi}\right)$. This implies that flux needs to be primitive with respect to this pull-back rather than with respect to $\omega \equiv J|_{\mathcal{S}} = \frac{i}{2}\left(\mathrm{d}x \wedge \mathrm{d}\overline{x} + \mathrm{d}y \wedge \mathrm{d}\overline{y}\right)$, the difference being the α' corrections to the D-term.

Let us be more specific and consider the background

$$\phi = \begin{pmatrix} \mu^2 x & 0 \\ 0 & -\mu^2 x \end{pmatrix} \tag{5.17}$$

and a flux F that commutes with ϕ. Namely we have

$$F \;=\; F_{x\bar{x}}\, dx \wedge d\bar{x} + F_{y\bar{y}}\, dy \wedge d\bar{y} + F_{x\bar{y}}\, dx \wedge d\bar{y} + F_{y\bar{x}}\, dy \wedge d\bar{x} \tag{5.18}$$

where $F = F^\dagger$ imposes $F_{x\bar{y}} = F_{y\bar{x}}$ and a reality condition for $F_{x\bar{x}}$, $F_{y\bar{y}}$. In particular, due to our Ansatz these components must be of the form $i(a\sigma_3 + b\mathbf{1})$, with a, b real functions.

Imposing that $dF = 0$ and the leading order D-term condition $\omega \wedge F = 0$ sets these functions to be constant and such that $F_{x\overline{x}} = -F_{y\overline{y}}$, while $F_{x\overline{y}}$ is constant but otherwise unconstrained. The latter is also true for the α'-corrected D-term constraint, while the relation between $F_{x\overline{x}}$ and $F_{y\overline{y}}$ is modified to

$$F_{x\overline{x}} = -(1 + 4\lambda^2 |\mu|^4) F_{y\overline{y}}, \tag{5.19}$$

Notice, that this condition reduces to the naive primitivity condition $F_{x\overline{x}} + F_{y\overline{y}} = 0$ in the limit $\lambda \to 0$, while for finite λ it gives a correction that grows with the complex parameter $\mu \in \mathbb{C}$, $[\mu] = L^{-1}$.

Physically, the α'-corrected D-term condition is quite easy to understand. Indeed, notice that the Higgs-field vev in (5.19) describes an $SU(2)$ gauge theory which is broken completely over generic loci, and in particular there is no D7-brane on the naive gauge theory locus $\{z = 0\}$. Instead we may compute the D7-brane loci via the discriminant $\det(z \cdot \mathbf{1} - \lambda \cdot \phi) = (z - \lambda\mu^2 x)(z + \lambda\mu^2 x)$, which indicates that the system contains two D7-branes located at $\{z = \pm\lambda\mu^2 x\}$ and μ^2 is their intersection slope. A more suitable description can be obtained by passing to a new system of coordinates

$$u \equiv z + \lambda\mu^2 x \tag{5.20}$$

$$v \equiv z - \lambda\mu^2 x \tag{5.21}$$

$$w \equiv y, \tag{5.22}$$

in which the branes loci are given by $\{u = 0\}$ and $\{v = 0\}$, and then analysing each of the D7-branes individually in term of their Abelian D-terms. For instance, to have primitive flux along the D7-brane located at $\{u = 0\}$ translates into

$$0 = J|_{\{u=0\}} \wedge F \tag{5.23}$$

$$\Rightarrow F_{v\bar{v}} = -\left(\frac{1}{4\lambda^2|\mu|^4} + 1\right) F_{w\bar{w}} \tag{5.24}$$

$$\Rightarrow F_{x\bar{x}} = -\left(1 + 4\lambda^2|\mu|^4\right) F_{y\bar{y}} \tag{5.25}$$

and similarly for $\{v = 0\}$. This is precisely the result we obtained earlier in (5.19) from the perspective of the gauge theory on $\{z = 0\}$. So intuitively the D-term equations in this description just tells us that the flux should be primitive along the actual brane world-volumes, rather than the locus $\mathcal{S}$ from which we describe the parent gauge theory.

5.4 α'-corrections in simple T-brane backgrounds

After seeing the effect of α' corrections for intersecting D7-branes, let us investigate which types of effects we receive for T-brane backgrounds. In general, these backgrounds are such that $[\phi, \bar{\phi}] \neq 0$ and so a non-primitive flux F, satisfying $F \wedge \omega \neq 0$, is needed to solve the D-term equations at leading order [4].

In order to find BPS solutions for these backgrounds one may apply the strategy outlined in [4]. Namely, one first defines the T-brane Higgs background in a unphysical holomorphic gauge [23, 26]

$$A^{(0,1)} = 0 \qquad \bar{\partial}\phi_{\rm hol} = 0 \tag{5.26}$$

and then rotate these fields by a complexified gauge transformation of the symmetry group G

$$A^{(0,1)} \to A^{(0,1)} + ig\bar{\partial}g^{-1} \qquad \phi \to g\phi g^{-1} \tag{5.27}$$

in order to attain a unitary gauge in which the D-term condition is satisfied. In the following we will apply this same strategy to solve for the α'-corrected D-term equations. We will consider two simple examples in which the leading order non-primitive flux lies in the Cartan subalgebra of the symmetry group G, as this also simplifies the Ansatz to solve the D-term equations at higher order in α'.

5.4.1 A simple SU(2) background

Let us first analyse a simple $SU(2)$-background already considered in [4] where the Higgs field profile in the holomorphic gauge reads

$$\phi_{\rm hol} = m \begin{pmatrix} 0 & 1 \\ ax & 0 \end{pmatrix} = -imE^+ + imaxE^- \tag{5.28}$$

where $m, a \in \mathbb{C}$ and $[m] = [a] = L^{-1}$, and the generators $E^\pm$ are defined in Appendix C. This time the discriminant gives the D7-brane locus $z^2 = \lambda^2 am^2 x$. Moreover, since we have $\det \phi_{\rm hol} = -m^2 ax$, we see that this is a reconstructible brane background according to the definition given in [4]. To solve the D-term equations we proceed as above and pass to a unitarity gauge via a complexified gauge transformation in SU(2). More precisely we take

$$g = e^{\frac{f}{2}\sigma_3} \tag{5.29}$$

which implies that in the unitarity gauge the D7-brane backgrounds reads

$$\phi = m \begin{pmatrix} 0 & e^f \\ axe^{-f} & 0 \end{pmatrix}, \tag{5.30}$$

$$F = -i\partial\bar{\partial} f \cdot \sigma_3 \,. \tag{5.31}$$

At leading order in λ the D-term equations read

$$\left(\mathrm{p}_x\overline{\partial}_{\overline{x}} + \mathrm{p}_y\overline{\partial}_{\overline{y}}\right) f\,\sigma_3 = [\phi,\bar{\phi}] \quad \Rightarrow \quad \left(\mathrm{p}_x\overline{\partial}_{\overline{x}} + \mathrm{p}_y\overline{\partial}_{\overline{y}}\right) f = |m|^2\left(e^{2f} - |ax|^2 e^{-2f}\right). \tag{5.32}$$

Finding f at this level amounts to solve a partial differential equation of Painlevé III type on the radial coordinate $|x|$, as has been already discussed in [4]. More precisely, we may solve it by making the Ansatz $f = f(|x|)$ and parametrise $x \equiv re^{i\theta}$, yielding

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right) f = |m|^2\left(e^{2f} - |a|^2 r^2 e^{-2f}\right). \tag{5.33}$$

Redefining $e^{2f(r)} \equiv r|a|e^{2j(r)}$ further simplifies this to

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right) j = |a||m|^2 r\sinh(2j). \tag{5.34}$$

Finally we define $s \equiv \frac{2}{3}\sqrt{2|a||m|^2 r^3}$ such that we are left with

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}s}\right) j = \sinh(2j), \tag{5.35}$$

which is the standard expression for a particular kind of Painlevé III equation analysed in [62]. Finally, we may directly solve (5.32) asymptotically near $|x| = 0$ by

$$f = f_0(x,\overline{x}) = \log c + c^2|mx|^2 + \frac{|m|^2|x|^4}{4c^2}\left(2|m|^2c^6 - |a|^2\right) + \ldots \tag{5.36}$$

with c an arbitrary dimensionless parameter whose value should be close to 0.73 if we want to avoid poles for large values of $|x|^2$ [7].

Let us now consider the α'-corrected D-term equation. Applying (5.14) to this setup we obtain the following equation

$$\left(\mathrm{p}_x\overline{\partial}_{\overline{x}} + \mathrm{p}_y\overline{\partial}_{\overline{y}}\right) f = |m|^2\left(e^{2f} - |ax|^2 e^{-2f}\right)\left(1 + 4\lambda^2 Q_f\right) + \lambda^2 R[f,f], \tag{5.37}$$

where

$$Q_f = (\mathrm{p}_x\overline{\partial}_{\overline{x}}f)(\mathrm{p}_y\overline{\partial}_{\overline{y}}f) - (\mathrm{p}_x\overline{\partial}_{\overline{y}}f)(\mathrm{p}_y\overline{\partial}_{\overline{x}}f) \tag{5.38}$$

$$R[f,g] = |m|^2\left[\left(4\mathrm{p}f\wedge\overline{\partial}fe^{2f} + |a|^2e^{-2f}(\mathrm{p}x - 2x\mathrm{p}f)\wedge(\overline{\partial}\overline{x} - 2\overline{x}\overline{\partial}f)\right)\wedge\mathrm{p}\overline{\partial}g\right]_{x\overline{x}y\overline{y}}$$

describe the new operators that appear due to the α'-corrections. Notice however that by keeping the Ansatz $f \equiv f(x,\bar{x})$ both Q_f and $R[f,f]$ vanish identically and we are back to eq.(5.32). Therefore, the solution to the corrected

D-term still amounts to $f = f_0(x, \bar{x})$ and the above T-brane background does not suffer any modification due to α'-corrections. Notice that in this case the T-brane background preserves 1/4 of the supercharges of flat space. Further examples of T-brane systems preserving eight supercharges are analysed in Appendix B, again obtaining the result that α'-corrections do not modify the background.

The analysis becomes more interesting if we consider a more general flux background, with a new component which will lower the amount of preserved supersymmetry. As usual we may consider adding such fluxes along generators that commute with the T-brane background. For instance we may add a worldvolume flux along the identity generator of $\mathfrak{u}(2)$, which could arise either from the D7-brane itself or form the pull-back of a bulk B-field. We first consider the case where this flux is

$$H_1 \,=\, \mathrm{Im}\,(\kappa\, dx \wedge d\bar{y})\, \mathbf{1} \tag{5.39}$$

with $\kappa \in \mathbb{C}$ and $[\kappa] = L^{-2}$ parametrising the local flux density. At leading order in α', the vanishing D-term condition would allow for an arbitrary κ without modifying the T-brane background, as the above flux is primitive. Its α'-corrected counterpart, however, has non-trivial components along the generators σ_3 and $\mathbf{1}$, implying two independent D-term equations. Namely

$$\begin{aligned} \left(\mathrm{p}_x \overline{\partial}_{\overline{x}} + \mathrm{p}_y \overline{\partial}_{\overline{y}}\right) f &= |m|^2 \left(e^{2f} - |ax|^2 e^{-2f}\right)\left(1 + 4\lambda^2 Q_f + \lambda^2 |\kappa|^2\right) \\ &\quad + \lambda^2 R[f,f] \\ 0 &= \mathrm{Re}\Big(|a|^2 e^{-2f} \kappa x \mathrm{p}_y f \left(2\overline{x}\overline{\partial}_{\overline{x}} f - 1\right) + 2e^{2f}\kappa \mathrm{p}_y f \overline{\partial}_{\overline{x}} f\Big) \end{aligned} \tag{5.40}$$

with the second line corresponding to the D-term constraint along the identity generator. Such equation is automatically satisfied if we again impose the Ansatz $f \equiv f(x, \bar{x})$, while the first one becomes

$$\left(\mathrm{p}_x \overline{\partial}_{\overline{x}} + \mathrm{p}_y \overline{\partial}_{\overline{y}}\right) f = |m|^2 \left(e^{2f} - |ax|^2 e^{-2f}\right)\left(1 + \lambda^2 |\kappa|^2\right). \tag{5.41}$$

Hence, we are back to eqs.(5.32) and (5.36) with the replacement

$$m \,\to\, m' = m\sqrt{1 + \lambda^2 |\kappa|^2}. \tag{5.42}$$

Finally, let us consider the case where the flux background on the identity is

$$H \,=\, H_1 + H_2 - i\mathrm{p}\bar{\mathrm{p}}h\, \mathbf{1} \tag{5.43}$$

where H_1 is again given by (5.39), and H_2 is an different piece of primitive constant flux

$$H_2 \;=\; \rho\, i\, (dx \wedge d\bar{x} - dy \wedge d\bar{y})\, \mathbf{1} \tag{5.44}$$

with $\rho \in \mathbb{R}$ and $[\rho] = L^{-2}$. In addition, we consider $h \equiv h(x, \overline{x}, y, \overline{y})$ to be an arbitrary function that we may expand around the origin as a polynomial, starting at quadratic order. In addition, we write the gauge transformation (5.29) as the following expansion

$$f \;=\; f_0(x, \bar{x}) + \sum_{i=1}^{\infty} (\lambda\rho)^{2i}\, f_i(x, \overline{x}, y, \overline{y}) \tag{5.45}$$

with $f_0(x, \bar{x})$ the solution found for $\rho = 0$, which near the origin behaves as (5.36) with the replacement (5.42).

In this case solving the D-term equations becomes more challenging, but one may perform a perturbative expansion on the dimensionless parameter $\lambda\rho$ and keep the terms up to $\mathcal{O}((\lambda\rho)^2)$ in order to simplify them. On the one hand, for the D-term constraint along the generator σ_3 we find

$$(\mathrm{p}_x \overline{\partial}_{\overline{x}} + \mathrm{p}_y \overline{\partial}_{\overline{y}}) f\, \sigma_3 = [\phi, \overline{\phi}]\left(1 + 4\lambda^2 Q_H\right), \tag{5.46}$$

where now

$$Q_H = \left(\mathrm{p}_x \overline{\partial}_{\overline{x}} h - \rho\right)\left(\mathrm{p}_y \overline{\partial}_{\overline{y}} h + \rho\right) - \left(\mathrm{p}_x \overline{\partial}_{\overline{y}} h - \frac{i}{2}\kappa\right)\left(\mathrm{p}_y \overline{\partial}_{\overline{x}} h - \frac{i}{2}\overline{\kappa}\right). \tag{5.47}$$

On the other hand, for the constraint along the identity we have

$$\begin{aligned}(\mathrm{p}_x \overline{\partial}_{\overline{x}} + \mathrm{p}_y \overline{\partial}_{\overline{y}}) h = \lambda^2 \big[& 4|m|^2 \left(e^{2f} - |ax|^2 e^{-2f}\right) \mathrm{p}_x \overline{\partial}_{\overline{x}} f \left(\mathrm{p}_y \overline{\partial}_{\overline{y}} h + \rho\right) \\ & - 2R[f, h + \rho |y|^2]\big]\end{aligned}$$

with R defined as in (5.38). We find the following solutions for h at lowest orders in $\lambda\rho$ and near the origin

$$\begin{aligned}h = \lambda^2 \rho |mx|^2 &\left(|m'x|^2 \left(|a|^2 + 2c^6 |m'|^2\right) - \frac{2}{c^2}\left(|a|^2 - 2c^6 |m'|^2\right)\right) \\ &+ \mathcal{O}(\lambda^3 \rho^3)\end{aligned} \tag{5.48}$$

while from (5.46) we find that the leading correction to f_0 is

$$\begin{aligned}f_1 = {}& 2|mx|^2 \left(8\lambda^2 |m|^2 |m'|^2 c^6 - 2c^2 - 4\lambda^2 |am|^2 - 2c^2 |m'x|^2\right) + \\ & + 2|mx|^4 \Big(\frac{2|am|^2}{c^2} + \frac{\lambda^2 |a|^4}{c^4} + 48\lambda^2 |m|^4 |m'|^4 c^8 - 16\lambda^2 |am|^2 |m|^2 |m'|^2 c^2\Big)\end{aligned} \tag{5.49}$$

where we have again Taylor-expanded around $x = 0$.

To summarise we find that, if we add a primitive constant flux H_1 that commutes with the Higgs background and of the form (5.39), the D-terms equations can be solved by an appropriate choice of gauge transformation (5.29), that induces a non-primitive flux along the $\mathfrak{su}(2)$ generator σ_3. When we also include the constant primitive flux H_2 of the form (5.44) the same is essentially true, but now we must also add a non-primitive flux $\mathrm{p}\bar{\mathrm{p}}h$ along the identity generator of $\mathfrak{u}(2)$ to solve the D-term constraints.

5.4.2 A simple SU(3) background

Let us now consider a slightly more complicated SU(3) T-brane background, again preserving four supercharges. The Higgs field profile in the holomorphic gauge is given by

$$\phi_{\text{hol}} = m \begin{pmatrix} \mu y & 1 & 0 \\ ax & \mu y & 0 \\ 0 & 0 & -2\mu y \end{pmatrix} \equiv -im\, E^+ + imax\, E^- + m\mu y\, Q, \tag{5.50}$$

where the form of the generators $E^{\pm}$, Q and $P \equiv [E^+, E^-]$ is detailed in Appendix C.

As before, we may solve for the D-terms equations by performing a gauge transformation of the form (5.27). Because $[\phi, \bar{\phi}] \propto P$, the natural choice is now $g = \exp(\frac{f}{2}P)$ and so in the unitary gauge we have a background given by

$$\begin{aligned} \phi &= -ime^f\, E^+ + imaxe^{-f}\, E^- + m\mu y\, Q \\ F &= -i\mathrm{p}\overline{\partial} f\, P, \end{aligned} \tag{5.51}$$

With this Ansatz there is only one non-trivial D-term constraint, corresponding to the generator P. The α' corrections complicate the form of this equation with respect to the leading order counterpart, and we obtain

$$\begin{aligned} \left(\mathrm{p}_x\overline{\partial}_{\overline{x}} + \mathrm{p}_y\overline{\partial}_{\overline{y}}\right) f = |m|^2 \left(e^{2f} - |ax|^2 e^{-2f}\right) \left(1 + 4\lambda^2 Q_f\right) \\ - \frac{2}{3}\lambda^2 R[f,f] - 4\lambda^2 |m|^2|\mu|^2 \mathrm{p}_x\overline{\partial}_{\overline{x}} f \end{aligned} \tag{5.52}$$

By using the Ansatz $f = f(x, \overline{x})$ this expression simplifies to

$$\mathrm{p}_x\overline{\partial}_{\overline{x}} f = \frac{|m|^2}{1 + 4\lambda^2|m|^2|\mu|^2} \left(e^{2f} - |ax|^2 e^{-2f}\right) \tag{5.53}$$

which is asymptotically solved by (5.36) with the replacement

$$m \to \tilde{m} = \frac{m}{\sqrt{1 + 4\lambda^2|m|^2|\mu|^2}} \tag{5.54}$$

Let us now add further worldvolume flux to this background. For simplicity we will add it along generators that commute with the $\mathfrak{su}(2)$ subalgebra generated by $\{E^{\pm}, P\}$. Namely we consider the following generators

$$T = \begin{pmatrix} \mathbf{1}_{2\times 2} & \\ & 0 \end{pmatrix} \tag{5.55}$$

$$B = \begin{pmatrix} \mathbf{0}_{2\times 2} & \\ & 1 \end{pmatrix}, \tag{5.56}$$

Notice that an arbitrary combination of these generators does not belong to $\mathfrak{su}(3)$ but rather to its central extension $\mathfrak{u}(3)$. Indeed, only if we consider a worldvolume flux satisfying $F^B + 2F^T = 0$ we will have an SU(3) background.

Similarly to the SU(2) example one may first consider a flux that commutes with the generators of the T-brane background, namely of the form

$$H_1 = \mathrm{Im}\,(\kappa\, dx{\wedge}d\bar{y})\; T \tag{5.57}$$

$$G = M\,(\mathrm{d}x{\wedge}\mathrm{d}\bar{x} + \mathrm{d}y{\wedge}\mathrm{d}\bar{y})\; B + N\,(\mathrm{d}x{\wedge}\mathrm{d}\bar{x} - \mathrm{d}y{\wedge}\mathrm{d}\bar{y})\; B + \mathrm{Im}\,(O\,\mathrm{d}x{\wedge}\mathrm{d}\bar{y})\; B \tag{5.58}$$

where $M, N \in \mathbb{R}$ and $\kappa, O \in \mathbb{C}$. We may also generalise the Ansatz to $f \equiv f(x, \bar{x}, y, \bar{y})$. The corrected D-term equations then read:

$$\begin{aligned}
0 &= 8\lambda^2|m\mu|^2(M+N) + N \\
\left(\mathrm{p}_x\overline{\partial}_{\bar{x}} + \mathrm{p}_y\overline{\partial}_{\bar{y}}\right) f &= |m|^2\left(e^{2f} - |ax|^2e^{-2f}\right)\left(1 + \lambda^2|\kappa|^2 + 4\lambda^2 Q_f\right) \\
&\quad - \frac{2}{3}\lambda^2 R[f,f] - 4\lambda^2|m|^2|\mu|^2\mathrm{p}_x\overline{\partial}_{\bar{x}}f \\
0 &= \lambda^2\mathrm{Re}\Big(|a|^2e^{-2f}\kappa x\mathrm{p}_y f\left(2\bar{x}\overline{\partial}_{\bar{x}}f - 1\right) + 2e^{2f}\kappa\mathrm{p}_y f\overline{\partial}_{\bar{x}}f\Big) \\
&\quad + \left(e^{2f} - |ax|^2e^{-2f}\right)\mathrm{Re}\Big(\kappa\mathrm{p}_y\overline{\partial}_{\bar{x}}f\Big) \\
0 &= \lambda^2\kappa|m\mu|^2\left(|a|^2e^{-2f}\,|1 - 2x\mathrm{p}_x f|^2 - 4e^{2f}|\mathrm{p}_x f|^2\right)
\end{aligned} \tag{5.59}$$

Here the first equation correspond to the generator B and it is identical to the D-term constraint found in (5.19) for the case of intersecting 7-branes. It fixes the relation between M and N and decouples from the rest of the equations, that will not depend on M, N, O. The second equation corresponds to the D-term along the generator P and it is again given by (5.52). The third and fourth equations are new, and correspond to the D-term constraints along the generators T and $E^{\pm}$, respectively. From the last one we see that the only way to have a non-vanishing flux κ is to take the limit $\mu \to 0$, which would essentially take us to the previous $SU(2)$ example.

Despite this result, one is able to accommodate a background flux along the generator T by considering a slightly different Ansatz. Indeed, let us proceed as in the previous $SU(2)$ example and generalise the above flux Ansatz to

$$\begin{aligned} H &= H_1 + H_2 - i\mathrm{p}\bar{\mathrm{p}}h\,T \\ H_2 &= \rho\, i\,(dx \wedge d\bar{x} - dy \wedge d\bar{y})\; T \\ h &\equiv h(x, \bar{x}, y, \bar{y}). \end{aligned} \tag{5.60}$$

while returning to the Ansatz $f \equiv f(x, \bar{x})$ for the flux along P. The corrected D-term equations now read:

$$0 = 8\lambda^2 |m\mu|^2 (M + N) + N \tag{5.61}$$

$$\left(1 + 4\lambda^2 |m\mu|^2\right) \mathrm{p}_x \overline{\partial}_{\overline{x}} f = |m|^2 \left(e^{2f} - |ax|^2 e^{-2f}\right)\left(1 + 4\lambda^2 Q_H\right) \tag{5.62}$$

and

$$\begin{aligned} \left(\mathrm{p}_x \overline{\partial}_{\overline{x}} + \mathrm{p}_y \overline{\partial}_{\overline{y}}\right) h &= 4\lambda^2 |m|^2 |\mu|^2 \left(\rho - \mathrm{p}_x \overline{\partial}_{\overline{x}} h\right) \\ &+ 2\lambda^2 \left(\rho + \mathrm{p}_y \overline{\partial}_{\overline{y}} h\right) \left(4|m|^2 e^{2f} |\mathrm{p}_x f|^2 + 2[\phi, \overline{\phi}] - |am|^2 e^{-2f} \left|2x\mathrm{p}_x f - 1\right|^2\right) \end{aligned} \tag{5.63}$$

$$0 = \lambda^2 |m\mu|^2 \left(2\mathrm{p}_x \overline{\partial}_{\overline{y}} h + \kappa\right) \left(|a|^2 e^{-2f} \left|2x\mathrm{p}_x f - 1\right|^2 - 4e^{2f} |\mathrm{p}_x f|^2\right) \tag{5.64}$$

with Q_H again given by (5.47). Notice the last equation now imposes $2\mathrm{p}_x \overline{\partial}_{\overline{y}} h + \kappa = 0$, which essentially requires that the effective flux of the form (5.57) vanishes. Naively, this seems to imply that α'-corrected D-terms do impose constraints on worldvolume fluxes commuting with the Higgs field T-brane background, contrary to what happens at leading order in α'. Nevertheless, one can show that a non-trivial κ is allowed if one generalises the gauge transformation Ansatz $g = \exp(\frac{f}{2}P)$ to include complexified transformations along the non-Cartan generators $E^{\pm}$ as well. We leave the somewhat technical proof of this statement to Appendix C, where such generalised transformations are studied in more detail.

If for simplicity we set $\kappa = 0$, make the Ansatz (5.45) and solve again perturbatively in $\lambda\rho$ we find the following asymptotic solutions around $x = 0$:

$$f_0 = \log c + c^2|\tilde{m}x|^2 + \frac{|\tilde{m}|^2|x|^4}{4c^2}\left(2c^6|m|^2 - |a|^2\left(1 + 4\lambda^2|m\mu|^2\right)\right) \tag{5.65}$$

$$\begin{aligned} f_1 = {} & 4|\tilde{m}x|^2\left(2\lambda^2|m|^2\left(|a|^2 - 2c^4\right) + c^2\right) \\ & + \frac{|\tilde{m}|^4|x|^4}{c^4}\Big(\frac{|a|^2}{|m|^2}c^2 + 2\left(\lambda^2\left(|a|^4 - 4|a|^2c^2\left(c^2 - |\mu|^2\right)\right) - 2c^8\right) \\ & + 8\lambda^2|m|^4\left(4\lambda^4|\mu|^4\left(|a|^4 - 4|a|^2c^4\right) + 2c^6\lambda^2|\mu|^2\left(|a|^2 + 6c^4\right) + c^{12}\right) \\ & + 4\lambda^2|m|^2\left(-4|\mu|^2\left(c^8 - \lambda^2|a|^4\right) + |a|^2c^2\left(4\lambda^2\left(|\mu|^4 - 4c^2|\mu|^2\right) + c^4\right) + 6c^{10}\right)\Big) \end{aligned} \tag{5.66}$$

and

$$\begin{aligned} h = {} & \frac{2\lambda^2|\tilde{m}x|^2\rho\left(2\left(c^4 + c^2|\mu|^2\right) - |a|^2\right)}{c^2} \\ & + \frac{\lambda^2|\tilde{m}x|^4\rho}{c^2\left(1 + 4\lambda^2|m\mu|^2\right)}\Big(\frac{|a|^2}{|\tilde{m}|^2}\left(|m|^2\left(3c^2 - 4\lambda^2|\mu|^2\right) - 1\right) \\ & + 2c^6|m|^2\left(\left(c^2 + 4\lambda^2|\mu|^2\right) + 1\right)\Big) \end{aligned} \tag{5.67}$$

To summarise, in this more complicated $SU(3)$ background that preserves four supercharges we also find different kinds of solutions for the α'-corrected D-term equations. One first class of corrections comes from the intersection slope μ that appears in ϕ_{hol}, and which corresponds to a generator Q commuting with the T-brane $\mathfrak{su}(2)$ subalgebra $\{E^\pm, P\}$. Such corrections are relatively easy to take into account, as they only modify the parameters of the Painlevé III equation. Further non-trivial corrections come from adding worldvolume fluxes commuting with the Higgs background. One the one hand, adding some of these primitive fluxes require a modification of the non-primitive flux $\text{p}\overline{\partial}f$ along P and adding one of the form $\text{p}\overline{\partial}h$ along T. On the other hand, adding some other components requires a more drastic change: to generalise the standard gauge transformation g to also include non-Cartan generators $E^\pm$. In the next section we will analyse from a more general viewpoint when each of these two cases occurs.

5.5 More general backgrounds

With the two examples of the previous section in mind, let us describe how α' corrections affect the D-term equations for more general kinds of T-branes. As before we will take the simplifying assumption that, given the gauge group G and its corresponding Lie algebra $\mathfrak{g}$, the leading order D-term equations can be solved via a complexified gauge transformation (5.27) of the form

$$g = e^{\frac{f_i}{2}P_i} \tag{5.68}$$

where $f_i = f_i(x, \overline{x}, y, \overline{y})$ and P_i belong to the Cartan subalgebra of $\mathfrak{g}$. We then write the Higgs field profile in the holomorphic gauge in the block diagonal form

$$\phi_{\text{hol}} = m \begin{pmatrix} \psi^1_{\text{hol}} & & & \\ & \psi^2_{\text{hol}} & & \\ & & \ddots & \\ & & & \psi^n_{\text{hol}} \end{pmatrix}, \tag{5.69}$$

with $[m] = L^{-1}$, and where the entry ψ^i_{hol} is an $n \times n$ matrix of holomorphic functions on x, y. One simple example of such structure is the SU(3) example of section 5.4.2, which contained a 1×1 and a 2×2 block. As discussed below eqs.(5.59), the α'-corrected D-term equations do not couple one block to the other. The same statement holds for the more general T-brane structure with the block-diagonal form (5.69): for the purposes of analysing α'-corrections we can focus on each individual block ψ^i_{hol} at a time, an forget about the rest.

In the case that ψ^i_{hol} is a 1×1 block, the effects of α'-corrections will be similar to the ones studied in section 5.3. As in there, the α'-corrections will impose primitivity with respect to the standard pull-back of J on the spectral surface

$$z = \lambda\, m\, \psi^{1\times 1}_{\text{hol}}(x, y)\,. \tag{5.70}$$

More interesting is the case where ψ^i_{hol} is a 2×2 block, as these contain the T-brane nature of the background. As we have already seen in section 5.4 for these cases the α'-corrected D-term equations may become rather involved to solve, specially when we add additional primitive worldvolume fluxes. In general, within that block we will have a holomorphic Higgs field profile of the form

$$\psi^{2\times 2}_{\text{hol}} = u_0\mathbf{1} + u_1\sigma_1 + u_2\sigma_2 + u_3\sigma_3 = u_0\mathbf{1} - iu_+E^+ + iu_-E^- + u_3\sigma_3 \tag{5.71}$$

where u_i, $u_\pm$ are complex functions on x, y, $[u_i] = [u_\pm] = L^0$. Near the origin, we can approximate such functions up to their linear behaviour, so each of them is characterised by three independent complex numbers. However, we may absorb three numbers in constant shifts of the local coordinates x, y, z. More precisely, by a shift in z we may remove the constant term in u_0, rendering it a linear function in x, y. Similarly, by shifts on x and y we may remove the constant pieces in u_3 and u_-. Then we are left with only one function, namely u_- that may contain a constant term, and therefore with essentially two different possibilities

$$\psi_{\text{hol}}^{2\times 2}\big|_{x=y=0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \psi_{\text{hol}}^{2\times 2}\big|_{x=y=0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.72}$$

Examples of backgrounds of the first kind are those analysed in section 5.4, while several of the second kind are studied in Appendix D. In both cases the holomorphic Higgs background is parametrised by eight dimension-full parameters, namely

$$\begin{aligned} u_0 &= \mu_{0,x}x + \mu_{0,y}y & u_3 &= \mu_{3,x}x + \mu_{3,y}y \\ u_- &= \mu_{-,x}x + \mu_{-,y}y & u_+ &= \mu_{+,x}x + \mu_{+,y}y + \epsilon \end{aligned} \tag{5.73}$$

where $[\mu_{i,\alpha}] = L^{-1}$ and $\epsilon = 0, 1$ describes the two cases in (5.72). Imposing that the leading order D-term equation is solved by (5.68) means that at $\lambda \to 0$ we need a complexified gauge transformation of the form

$$g^{2\times 2} = e^{\frac{1}{2}(f\sigma_3 + h\mathbf{1}_2)} \tag{5.74}$$

for solving the 2×2 block which we are analysing. In practice, this is only possible if $[\psi_{\text{hol}}^{2\times 2}, \overline{\psi_{\text{hol}}^{2\times 2}}] \in$ Cartan, which requires $\mu_{3,x} = \mu_{3,y} = 0$. We then have that in our setup

$$\psi_{\text{hol}}^{2\times 2} = \begin{pmatrix} \mu_{0,x}x + \mu_{0,y}y & \mu_{+,x}x + \mu_{+,y}y + \epsilon \\ \mu_{-,x}x + \mu_{-,y}y & \mu_{0,x}x + \mu_{0,y}y \end{pmatrix}. \tag{5.75}$$

One may now wonder if taking into account α'-corrections will drastically change the form of the complexified gauge transformation (5.74) solving for the D-term equation. For this we observe that

- If no background fluxes along $\mathbf{1}_2$ are present, then the Ansatz (5.74) remains invariant (with $h \equiv 0$), although α'-corrections may vary the specific form of f with respect to its leading order value.

- If we switch a background flux H along $\mathbf{1}_2$ then, for a generic $\psi^{2\times 2}_{\rm hol}$, some components of H will preserve the Ansatz (5.74), while others will force to consider a gauge transformation including non-Cartan generators $E^{\pm}$, as discussed in Appendix C.

Let us be more precise on the last point, since adding non-Cartan generators to (5.74) implies having a non-Abelian flux background that will complicate the T-brane system. By inspection (see e.g., Appendix C) one quickly realises that the relevant D-term equations for this problem are those along the non-Cartan components $E^{\pm}$, which may or may not have solution for the Ansatz (5.74). If there is no solution, one needs to generalise this Ansatz to include the generators $E^{\pm}$ and therefore a non-Abelian gauge background appears through (5.27).

Due to the symmetrisation procedure, the D-term equations along $E^{\pm}$ receive contributions only from the middle term in eq.(5.14). More precisely, assuming the Ansatz (5.74) we have that

$$\begin{aligned} {\rm D}\psi^{2\times 2} &\equiv ({\rm D}\psi)_{\mathbf{1}}\, \mathbf{1}_2 + ({\rm D}\psi)_+ E^+ + ({\rm D}\psi)_- E^- \\ &= (\mu_{0,x}{\rm d}x + \mu_{0,y}{\rm d}y)\ \mathbf{1}_2 \\ &\quad + (\mu_{+,x}{\rm d}x + \mu_{+,y}{\rm d}y + 2{\rm p}f\,(\mu_{+,x}x + \mu_{+,y}y + \epsilon))\ e^{f}E^+ \\ &\quad + (\mu_{-,x}{\rm d}x + \mu_{-,y}{\rm d}y - 2{\rm p}f\,(\mu_{-,x}x + \mu_{-,y}y))\ e^{-f}E^-, \end{aligned} \tag{5.76}$$

and that the D-term equations along $E^{\pm}$ read

$$0 = D_{\pm} = 2i\lambda^2\Big(({\rm D}\psi)_{\pm}\wedge\overline{({\rm D}\psi)_{\mathbf{1}}} + ({\rm D}\psi)_{\mathbf{1}}\wedge\overline{({\rm D}\psi)_{\mp}}\Big)\wedge H\,. \tag{5.77}$$

From here we see that these equations are non-trivial only if the Higgs-vev $\psi^{2\times 2}_{\rm hol}$ has components simultaneously along the identity and a (non-Cartan) generator of $\mathfrak{su}(2)$, which will be generically the case. Moreover, the total background flux H along the identity (including the piece $-i{\rm p}\bar{\rm p}h$) must be non-vanishing for this equation to be non-trivial. Let us discuss how this condition constrains the background flux H. Recall that H must satisfy the corrected primitivity condition

$$0 = \omega\wedge H + \lambda^2\left(2i({\rm D}\psi)_{\mathbf{1}}\wedge\overline{({\rm D}\psi)_{\mathbf{1}}} - 2{\rm Im}[({\rm D}\psi)_+\wedge\overline{({\rm D}\psi)_-}] - {\rm Tr}([\phi,\bar{\phi}]F)\right)\wedge H \tag{5.78}$$

and satisfy the Bianchi identity $dH = 0$. Then we find that only some profiles for H may satisfy the complex equations (5.77) and the real equation (5.78) simultaneously. Those profiles that satisfy (5.78) but fail to satisfy (5.77) will not be compatible with the initial Ansatz (5.74) and therefore will require the presence of a non-Cartan flux background at $\mathcal{O}(\lambda^2)$.

In practice one may find by inspection which profiles for H are compatible with the Abelian Ansatz (5.74), although in some simple cases one may be more specific. In particular, let us consider the cases where

- $(\mathrm{D}\psi)_+ \wedge (\mathrm{D}\psi)_- = 0$

  Or equivalently $(\mathrm{D}\psi)_+ = \gamma(\mathrm{D}\psi)_-$ for some complex function γ. In this case one finds that all fluxes H of the form

$$H \propto i(\mathrm{D}\psi)_{\mathbf{1}} \wedge \overline{(\mathrm{D}\psi)_{\mathbf{1}}} \quad (5.79)$$

$$H \propto i(\mathrm{D}\psi)_- \wedge \overline{(\mathrm{D}\psi)_-} \quad (5.80)$$

  satisfy eq.(5.77). Moreover if $\bar{\gamma} \equiv \gamma^{-1}$ then both equations in (5.77) become the same. In particular for $\gamma \equiv \eta = \pm 1$ they become a real condition and

$$H \propto \mathrm{Re}\left[\sqrt{\eta}\,(\mathrm{D}\psi)_- \wedge \overline{(\mathrm{D}\psi)_{\mathbf{1}}}\right] \quad (5.81)$$

  also becomes a solution to (5.77). Any combination of these allowed components satisfying $dH = 0$ and (5.78) will not require a non-Abelian flux background, while the rest will.

- $(\mathrm{D}\psi)_\pm \wedge (\mathrm{D}\psi)_{\mathbf{1}} = 0$

  Or equivalently $(\mathrm{D}\psi)_\pm = \gamma(\mathrm{D}\psi)_{\mathbf{1}}$ for a complex function γ. In this case again both equations in (5.77) becomes conjugate to each other and

$$H \propto i(\mathrm{D}\psi)_{\mathbf{1}} \wedge \overline{(\mathrm{D}\psi)_{\mathbf{1}}} \quad (5.82)$$

$$H \propto \mathrm{Im}\left[\gamma(\mathrm{D}\psi)_\mp \wedge \overline{(\mathrm{D}\psi)_{\mathbf{1}}}\right] + \frac{i}{2}(\mathrm{D}\psi)_\mp \wedge \overline{(\mathrm{D}\psi)_\mp} \quad (5.83)$$

  automatically satisfy (5.77). Again, a combination of those satisfying (5.78) and $dH = 0$ will be compatible with an Abelian flux background.

- $(\mathrm{D}\psi)_+ \wedge (\mathrm{D}\psi)_- = (\mathrm{D}\psi)_+ \wedge (\mathrm{D}\psi)_{\mathbf{1}} = (\mathrm{D}\psi)_- \wedge (\mathrm{D}\psi)_{\mathbf{1}} = 0$

In this case we have that (5.77) will be solved by

$$H \propto i(\mathrm{D}\psi)_{\mathbf{1}}\wedge\overline{(\mathrm{D}\psi)_{\mathbf{1}}} \tag{5.84}$$

$$H \propto \mathrm{Im}\,[\gamma(\mathrm{D}\psi)_{\mathbf{1}}\wedge\bar{\eta}] \tag{5.85}$$

for arbitrary complex function γ and one-form $\eta \in \Omega^{(1,0)}$. Such that we have more freedom to satisfy primitivity condition and Bianchi identity than in the previous cases.

One can check that this general discussion reproduces the results found in the two simple examples of section 5.4. On the one hand, for the $SU(2)$ example of section 5.4.1 we have that $(\mathrm{D}\psi)_{\mathbf{1}} = 0$. Hence (5.77) is trivially satisfied and so non-Cartan fluxes are absent in the corrected solution. On the other hand, in the $SU(3)$ example of section 5.4.2, the 2×2 T-brane block is such that

$$(\mathrm{D}\psi)_+, (\mathrm{D}\psi)_- \propto dx\,, \qquad\qquad (\mathrm{D}\psi)_{\mathbf{1}} \propto dy \tag{5.86}$$

We are then in the case $(\mathrm{D}\psi)_+ = \gamma(\mathrm{D}\psi)_-$, with γ a complicated function. It is then easy to see that

$$H = \rho\, i\,(dx\wedge d\bar{x} - dy\wedge d\bar{y}) + \mathcal{O}(\lambda^2), \qquad \rho\in\mathbb{R} \tag{5.87}$$

is a linear combination of the two-forms (5.79) and (5.80) which satisfies the Bianchi identity and the primitivity condition at leading order. This is precisely the flux component denoted as H_2 in section 5.4.2, explicitly shown to be compatible with the Abelian Ansatz (5.74) therein. On the contrary, a flux of the form (5.57) is shown to be incompatible with such an Ansatz, and non-Cartan flux generators need to be added as described in Appendix C. This again matches our general discussion, as for some choices of κ the flux (5.57) can be made of the form (5.81). But since in this example $\gamma \neq \pm 1$ such a flux is incompatible with the naive Abelian Ansatz, and non-Cartan generators need to be included.

5.6 Applications to local F-theory models

The T-brane backgrounds that we considered in the previous section are very similar to those used to generate phenomenological Yukawa hierarchies in F-theory GUTs [7, 8, 48], with the main difference that there Φ and F are valued in the Lie algebra of the exceptional groups E_6, E_7 and E_8. Nevertheless, in order to build models of SU(5) unification the Higgs background is embedded in unitary subalgebras of these exceptional groups and, at least naively, one may use this fact to apply our results.

Let us for instance consider the E_6 T-brane background constructed in [7]

$$\phi = m\left(e^f E^+ + mxe^{-f}E^-\right) + \mu^2(bx-y)Q\,, \tag{5.88}$$

where the generators $E^\pm$ generate a $\mathfrak{su}(2)$ subalgebra via $[E^+, E^-] = P$ and Q a commuting $\mathfrak{u}(1)$ subalgebra, see [7] for precise definitions. This background is quite similar to the one considered in section 5.4.2, as one can see from acting with ϕ on the doublet sector $(\mathbf{10}, \mathbf{2})_{-1}$ within the adjoint of $\mathfrak{e}_6$ [7]

$$[\phi, R_+E_{\mathbf{10}^+} + R_-E_{\mathbf{10}^-}] = \begin{pmatrix} -\mu^2(bx-y) & m \\ m^2x & -\mu^2(bx-y) \end{pmatrix} \begin{pmatrix} R_+E_{\mathbf{10}^+} \\ R_-E_{\mathbf{10}^-} \end{pmatrix}. \tag{5.89}$$

Naively, this action can be identified with a 2×2 Higgs block $\psi^{2\times 2}$ of the sort discussed in section 5.5. In fact, it is identical to the 2×2 block that arises from eq.(5.51) if there we perform the replacements

$$y \to y - bx\,, \qquad a \to m\,, \qquad m\mu \to \mu^2\,. \tag{5.90}$$

One can now apply the analysis of the previous section to this case. As in the SU(3) example of section 5.4.2, we are in the case $(\mathrm{D}\psi)_+ = \gamma(\mathrm{D}\psi)_-$ for $\gamma \neq \pm 1$. Therefore, primitive fluxes of the kind $H_{\rm nc}\mathbf{1}_{2\times 2}$ with a component of the form

$$H_{\rm nc} \propto \mathrm{Re}\big((\mathrm{D}\psi)_- \wedge \overline{(\mathrm{D}\psi)_\mathbf{1}}\big) \propto \mathrm{Re}\big(\mathrm{d}x\wedge(\bar{b}\mathrm{d}\overline{x} - \mathrm{d}\overline{y})\big) \tag{5.91}$$

are not allowed at order λ^2 without adding further non-Cartan fluxes. Interestingly, for the case $b = 1$ used in [7] to compute physical Yukawas, we have that such problematic flux reads

$$H_{\rm nc} \overset{b=1}{\propto} \mathrm{Re}\big(\mathrm{d}x\wedge\mathrm{d}\overline{y}\big)\,, \tag{5.92}$$

which allows for some primitive fluxes. In fact, the worldvolume primitive fluxes considered in [7] were of the form

$$F_p = iQ_R(dy\wedge d\overline{y} - dx\wedge d\overline{x}) + iQ_S(dx\wedge d\overline{y} + dy\wedge d\overline{x}) \tag{5.93}$$

with Q_R, Q_S some Cartan generators that reduce to the identity for the sector of interest. Therefore, according to our naive analysis the presence of these primitive fluxes may modify the non-primitive Abelian flux Ansatz given by $g = \exp(\frac{1}{2} f P)$ with $f = f(x, \overline{x})$, but it will not require the presence of non-Cartan generators in the flux background. Hence it seems that the computation of physical Yukawas made in [7] may be affected by α' corrections but not drastically, in the sense that the Ansatz for the T-brane background taken there survives at the next-to-leading order in α'. This will change as soon as the worldvolume flux (5.93) is chosen more general or b is chosen such that $\text{Im}\, b \neq 0$.

Chapter 6

Compact T-branes

Since initial interest in T-branes comes from the construction of realistic Yukawa points, most analyses have been carried out in local patches of flat space. While this is sufficient to capture much of the relevant data for the rank structure of the Yukawa couplings, it is blind to most information. This chapter aims to make progress in by analysing the conditions to construct T-branes with a compact embedding. That is, we analyse D7-branes with a non-Abelian profile for its worldvolume scalar Φ, globally well-defined over a compact Kähler four-cycle S and without any poles. We dub such configurations as *compact T-branes*, and analyse them by inspecting the related Hitchin system of equations over S. We therefore extend previous analysis of this sort, which so far have been essentially performed only at a local level.[1]

As usual, obstructions may be found when trying to extend a local solution globally. In our case we find that constructing compact T-brane solutions crucially depends on the Ricci curvature of the surface S, and more precisely on its cohomology class. Indeed, we find obstructions to the existence of compact T-branes over complex four-cycles of vanishing or positive-definite curvature, like K3 or del Pezzo surfaces. On surfaces of negative-definite curvature, instead, solutions can always be constructed, generalising the result of Hitchin for Riemann surfaces of genus $g > 1$ [63]. Finally, for surfaces of indefinite

[1]An alternative treatment is via tachyon condensation techniques, particularly suitable for T-branes defined over 7-brane intersections. In this case a global analysis can also be carried out, as shown in [10].

curvature the construction will depend on the particular region of the Kähler moduli space where we sit.[2] This latter case raises the question of the fate of T-branes when we move in Kähler moduli space and, in particular, when we pass from one region to another by crossing stability walls. In this respect, we find that a T-brane is either converted into a different BPS object as it crosses the wall, or it splits into non-mutually-BPS constituents. As could be expected, the T-brane's fate will ultimately depend on its topological data, and we analyse several interesting cases in terms of them.

The chapter is organised as follows. In section 6.1 we specify the class of T-branes that we will be studying, with special emphasis on their global description in terms of a compact four-cycle. We then turn to discuss solutions to the BPS equations, first the analogous of the original Hitchin solution and then generalisations thereof. In section 6.2 we prove a topological obstruction to building compact T-brane solutions: they cannot be hosted by four-cycles of vanishing or positive-definite Ricci curvature class. Finally, in section 6.3 we analyse the stability of the allowed T-brane constructions as we move in large volume Kähler moduli space, and in particular their fate after crossing a stability wall.

Some technical details are relegated to the appendices. In appendix E we give a four-dimensional interpretation of the non-harmonicity of the worldvolume flux in T-brane solutions. In appendix F we construct several explicit examples of the stability-wall transitions discussed in section 6.3.

6.1 Global aspects of T-branes

Consider a stack of 7-branes wrapping a compact Kähler surface S. Following [14, 54–56], the 7-brane configuration and degrees of freedom can be characterised in terms of an eight-dimensional action on $\mathbb{R}^{1,3} \times S$ with a non-Abelian symmetry group G. In particular, such data are encoded in terms of two two-forms on S: the field strength $\mathbb{F} = d\mathbb{A} - i\mathbb{A} \wedge \mathbb{A}$ of the 7-branes gauge boson $\mathbb{A}$, and the (2,0)-form Higgs field Φ, whose eigenvalues describe the 7-brane transverse geometrical deformations. Both $\mathbb{A}$ and Φ transform in the adjoint of

[2] More precisely, we find that, if ρ is the Ricci form of S and J its Kähler form, then compact T-branes can be constructed when $\int_S \rho \wedge J < 0$.

the initial gauge group G, which is nevertheless broken to a subgroup due to their non-trivial profile. Finally, such profiles need to satisfy certain equations of motion, which in the case of supersymmetric configurations are given by

$$\bar{\partial}_{\mathbb{A}}\Phi = 0 \tag{6.1a}$$

$$\mathbb{F}^{(0,2)} = 0 \tag{6.1b}$$

$$J \wedge \mathbb{F} + \frac{1}{2}[\Phi, \Phi^{\dagger}] = 0\,, \tag{6.1c}$$

where J is the Kähler two-form of S. These equations are a generalisation of the celebrated Hitchin system [63] to a four-manifold. Upon dimensional reduction to four dimensions, the first two equations ensure the vanishing of the F-terms, while the third equation ensures the vanishing of the D-terms.

In this chapter we will analyse 7-brane backgrounds with non-commuting expectation values for the worldvolume scalar Φ, namely such that $[\Phi, \Phi^{\dagger}] \neq 0$, also known as T-branes in the string theory literature. We will restrict to those T-brane configurations that are globally well-defined over a compact Kähler surface S and such that the Higgs field profile is absent of poles.[3] We dub such T-brane configurations as *compact T-branes*, in the sense that the spectral equation for Φ describes a compact surface. Notice that poles are naturally associated to field-theory defects originating from additional 7-branes intersecting the stack, so we may interpret a compact T-brane as a stack of 7-branes in isolation from the others. In other words, we may see them as basic building blocks of BPS 7-brane configurations in type IIB/F-theory compactifications. We will moreover focus on solutions of equations (6.1) involving an Abelian profile for the gauge field. Said differently, in our backgrounds the source of non-commutativity of the 7-brane system will come entirely from Φ.

In order to describe the essential features of compact T-branes, in this section we will focus on the simplest possible example, namely a stack of two D7-branes. This case allows to generalise the original example of Hitchin on a Riemann surface [63] to a compact complex four-cycle. From there one may generalise the T-brane Ansatz in a number of ways, finding backgrounds with a non-harmonic worldvolume flux. As we will see, the departure from harmonicity is governed by certain non-linear differential equations, and this will allow to connect our

[3] See [40] for a recent account of Hitchin systems with poles.

constructions with the literature of T-brane solutions in flat space.

6.1.1 T-branes and non-harmonic fluxes

Let us focus on a stack of two 7-branes wrapping S, and therefore on a super-Yang-Mills theory on $\mathbb{R}^{1,3} \times S$ with symmetry group $G = SU(2)$. We will always assume that S is simply-connected, i.e. $\pi_1(S) = 0$. This will simplify our analysis considerably because it implies, in particular, that holomorphic line bundles on S have their topology completely specified by the first Chern class. As mentioned, we will also restrict attention to a rank-two gauge bundle $\mathbb{V}$ on S of split type, i.e.

$$\mathbb{V} = \mathcal{L} \oplus \mathcal{L}^{-1}\,, \tag{6.2}$$

where $\mathcal{L}$ is a line bundle whose curvature we denote by F. The F-term (6.1b) of the eight-dimensional super-Yang-Mills theory forces F to be a differential form of Hodge-type $(1,1)$, which gives $\mathcal{L}$ a holomorphic structure. Moreover, since F is closed, using the Hodge decomposition, we can uniquely write it as

$$F = F^{\mathrm{h}} + \mathrm{d}\alpha\,, \tag{6.3}$$

where the superscript $^{\mathrm{h}}$ denotes the harmonic representative and α is a globally well-defined one-form. Note that the absence of non-trivial first-cohomology classes on S, following from its simply-connectedness, forbids harmonic representatives for α. We can thus always choose (globally) a gauge that kills the exact part of α, such that we can write

$$\alpha = -\frac{\mathrm{d}^c\, g\,(\mathbf{x}, m\bar{f}x)}{2}\,, \tag{6.4}$$

where $g(\mathbf{x}, m\bar{f}x)$ is a globally well-defined real function on S (with local complex coordinates collectively denoted by $\mathbf{x}$) such that $\int_S g\, \mathrm{dv}ol_S = 0$, and $\mathrm{d}^c = i(\bar{\partial} - \partial)$. Using that S is Kähler, it is easy to see that the co-differential operator $\delta = -*\mathrm{d}*$ annihilates the expression (6.4), and hence α is co-closed. In this way, the gauge field strength becomes

$$F = F^{\mathrm{h}} - i\partial\bar{\partial} g\,. \tag{6.5}$$

The function g, or equivalently α, will play a key rôle in the sequel. It will be the unknown of the non-linear partial differential equation governing T-brane backgrounds, which arises from the equation (6.1c) of the eight-dimensional super-Yang-Mills theory. In an ordinary intersecting-brane background, where Φ is diagonalisable, this equation forces F to be primitive. By a standard result in Kähler geometry (see e.g. [?]), every primitive (1,1)-form on a Kähler two-fold is anti-self-dual with respect to the Hodge-star operator. Since F is closed, this implies then that F is also co-closed, and hence harmonic. Now, reversing the argument, a T-brane supersymmetric configuration will involve a gauge field strength which is closed but not anti-self-dual, and therefore F will not necessarily be given by the harmonic representative of a certain cohomology class. This departure from harmonicity is described by g.

As we will see, the information that g encodes is lost in the four-dimensional effective theory. It can only be recovered when we include the D7-brane Kaluza-Klein modes into the four-dimensional description, as we discuss in appendix E. In other words, g determines the microscopic details of the T-brane background, which only the eight-dimensional theory is sensitive to.

In order to determine g let us for convenience define the global real function

$$\varphi(\mathbf{x}, m\bar{f}x)\sigma_3 \equiv *[\Phi, \Phi^\dagger]\,, \tag{6.6}$$

where, compatibly with our choice of gauge bundle $\mathbb{V}$, we restrict our attention to commutators proportional to the third Pauli matrix σ_3. Then one can see that $\varphi \geq 0$ all over S and that equation (6.1c) reads

$$F \wedge J = -\frac{\varphi}{4}\, J^2\,. \tag{6.7}$$

Using the Lefschetz decomposition of harmonic forms, we can write

$$F^{\mathrm{h}} = \frac{c}{4}\, J + F^{\mathrm{h}}_{\mathrm{p}}\,, \tag{6.8}$$

where c is a constant, $F^{\mathrm{h}}_{\mathrm{p}}$ is primitive and the numerical factor is for later convenience. Of course this splitting depends on the Kähler moduli of our string compactification, and the periods of the two summands are generally real (moduli-dependent) numbers which must add up to (half-)integer numbers to satisfy the quantization condition for F.[4]

[4]Recall that, in cohomology, $\frac{1}{2\pi}[F] = c_1(\mathcal{L})$.

Using that S is Kähler, one can show that $2i\partial\bar{\partial}g \wedge J = *\Delta g$, where Δ is the Laplace operator in real coordinates. This leads us to an elegant rewriting of equation (6.7):

$$\Delta g(\mathbf{x}, m\bar{f}x) = c + \varphi(\mathbf{x}, m\bar{f}x) \,. \tag{6.9}$$

At this point, one fixes an hermitian metric on S, and solves equation (6.9) for g, or equivalently for the unitary connection A on $\mathcal{L}$. Notice that a necessary requirement to solve this equation is that its r.h.s. integrates to zero, i.e.

$$c = -\frac{1}{2\mathrm{V}ol(S)}\,\mathrm{Tr}\int_S [\Phi, \Phi^\dagger]\,\sigma_3 \,, \tag{6.10}$$

which is nothing but the condition for vanishing D-term potential in the four-dimensional low-energy effective theory.

Practically, equation (6.9) can only be solved analytically in few situations, because in general φ will depend non-linearly on g. Nevertheless this equation is always of elliptic type [63] and, as such, on a compact manifold it admits a unique smooth solution if the input function φ is smooth and provided that (6.10) is satisfied [64].

The most convenient and adopted [4, 5] approach to formulate the problem is to fix the holomorphic structure of $\mathcal{L}$ such that $A^{0,1} = 0$, which turns the anti-holomorphic covariant derivative of equation (6.1a) into the simple Dolbeault operator $\bar{\partial}$. In this frame, equation (6.1c) (or else (6.9)), becomes an equation for the hermitian metric h on $\mathcal{L}$, which appears in the gauge field strength. The latter is indeed the curvature of the associated Chern connection $A^{1,0} \sim h^{-1}\partial h$, i.e. locally $F = -i\partial\bar{\partial}\log h$. Given that we can locally write $F^{\mathrm{h}} = -i\partial\bar{\partial}\log h_0$ and that F and F^{h} are in the same cohomology class, we see that the unknown function g is globally-well defined and enters the metric h as a conformal factor, i.e. $h = h_0\, e^g$.

For concreteness, let us consider a nilpotent Higgs field profile

$$\Phi = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \tag{6.11}$$

where $m \in H^{2,0}(S, \mathcal{L}^2)$. Equivalently, we can also see m as a scalar holomorphic section of the line bundle $\mathcal{M} \equiv \mathcal{L}^2 \otimes K_S$, with K_S the canonical bundle of S. By a slight abuse of notation, in the following we will describe both kinds of object with the same symbol, being clear from the context which one we are referring to. As it stands, this profile is a solution of equation (6.1a) in the holomorphic gauge. However, equation (6.1c) contains the adjoint $\Phi^\dagger$, which depends on the metric as

$$\Phi^\dagger = H^{-1}\Phi^+ H \,, \tag{6.12}$$

where the superscript $^+$ indicates complex conjugation and matrix transposition, and $H = \mathrm{d}iag(h, h^{-1})$. This brings a non-linearity in the partial differential equation (6.9), which can now be written as

$$\Delta g = c + \frac{h_0^2\,|m|^2}{h^S}\, e^{2g} \,, \tag{6.13}$$

where h^S, the determinant of the fixed hermitian metric on S, appears because of applying the Hodge-star operator on a four-form. This is a rather non-trivial equation that reduces to a Liouville-like equation when m is constant and h^S is the flat metric [4]. Nevertheless, there is a particularly nice setup in which (6.13) simplifies even further, as we discuss explicitly in the next subsection.

As a side remark, note that, for the split-type configurations (6.2) we consider in this chapter, the stability-based algebro-geometric criterion [?] for existence and unicity of solutions of the non-Abelian BPS equations (6.1) is trivially satisfied. For instance, it is immediate to see that the only sub-bundle of $\mathbb{V}$ preserved by the Higgs field (6.11) (i.e. $\mathcal{L}$) has negative J-slope, as enforced by the D-term equation (6.10).

6.1.2 The Hitchin Ansatz

The most emblematic class of Higgs-bundle configurations is probably the one originally studied by Hitchin in the case of Riemann surfaces [63]. One can straightforwardly extend this Ansatz to the present context of complex surfaces, as first suggested in [64]. This would correspond to taking the nilpotent Higgs field (6.11) such that the line bundle $\mathcal{M}$ is the trivial one, which amounts to demanding that[5]

$$\mathcal{L} \simeq K_S^{-1/2} \,. \tag{6.14}$$

[5]At weak coupling this is made compatible with cancellation of the Freed-Witten anomalies of the individual branes by considering a suitably-quantised primitive flux associated to the centre-of-mass $U(1)$.

Since S is compact, this choice implies that the quantity m in (6.11) can only be a constant. Notice also that equation (6.14) only fixes the cohomology class of the gauge curvature in terms of that of S, but not its actual representative. Therefore, let us write the Ricci form of S as

$$\rho = \rho^{\mathrm{h}} - 2i\partial\bar{\partial}s(\mathbf{x}, m\bar{f}x)\,, \tag{6.15}$$

where s is another globally well-defined smooth real function on S such that $\int_S s\,\mathrm{dv}ol_S = 0$, and the factor of 2 is for later convenience. Then, eq.(6.14) states that $F^{\mathrm{h}} = \rho^{\mathrm{h}}/2$, or equivalently, using (6.5), that[6]

$$F = \frac{\rho}{2} - i\partial\bar{\partial}(g - s)\,. \tag{6.16}$$

Loosely speaking, e^{g-s} is the conformal factor needed to rescale the hermitian metric on the surface S to get the hermitian metric on the line bundle $\mathcal{L}$. More precisely we have

$$h_0 = \sqrt{h^S}e^{-s}\,. \tag{6.17}$$

Using the above relation, our partial differential equation (6.13) becomes

$$\Delta g = c + |m|^2 e^{2(g-s)}\,, \tag{6.18}$$

where, as said, in this Hitchin set of solutions m is a complex number. Let us now analyse two possible sub-cases of this setup.

Kähler-Einstein metric

The easiest possible situation is analogous to the one originally considered by Hitchin in the case of Riemann surfaces [63]. This arises when $g = s$. Taking into account the D-term condition (6.10), which now simply says that $c = -|m|^2$, equation (6.18) reads

$$\Delta g(\mathbf{x}, m\bar{f}x) = 0\,, \tag{6.19}$$

whose unique solution on S is $g(\mathbf{x}, m\bar{f}x) = 0$. This, in turn, means that also $s = 0$, and thus that both the gauge flux F and the Ricci form ρ are harmonic. If in particular $h^{1,1}(S) = 1$, then $F^{\mathrm{h}}_{\mathrm{p}} = 0$ in equation (6.8) and therefore we have

$$\rho = -\frac{|m|^2}{2}J\,. \tag{6.20}$$

[6] Recall that, in cohomology, $\frac{1}{2\pi}[\rho] = c_1(K_S^{-1})$.

Thus the metric on our surface S is Kähler-Einstein with Einstein constant $-|m|^2/2$, that is it has constant negative Ricci curvature.

We can reverse the above argument and get a more useful statement. If we fix the metric on S to be Kähler-Einstein, then $\rho = kJ$ with k a real constant, which in particular means that $s = 0$ in equation (6.15). Equation (6.13) now reads

$$\Delta g = |m|^2 \left(e^{2g} - \frac{1}{\mathrm{V}ol(S)} \int_S e^{2g} \mathrm{dv}ol_S \right) , \tag{6.21}$$

where we substituted the value of c fixed by the D-term (6.10). The above equation automatically implies that $g(\mathbf{x}, m\bar{f}x) = 0$, because it admits a unique smooth solution. Therefore we conclude that, if we fix a (negatively curved) Kähler-Einstein metric on S, the vacuum solution for a constant nilpotent Higgs field involves a non-primitive, but still harmonic gauge flux.

Beyond Kähler-Einstein

If instead we consider a non-Kähler-Einstein metric on S, the vacuum profile of the gauge flux will necessarily depart from the harmonic representative, and will be uniquely fixed by the equation

$$\Delta g = |m|^2 \left(e^{2g-2s} - \frac{1}{\mathrm{V}ol(S)} \int_S e^{2g-2s} \mathrm{dv}ol_S \right) . \tag{6.22}$$

As before, there will be a unique smooth solution for g. Note that this extension beyond Kähler-Einstein is also possible in the case of Riemann surfaces, thus directly generalising the type of solution discussed in [63].

6.1.3 Generalising the Ansatz

There are a few ways of generalising the above simple set of solutions, namely by considering Higgs field profiles that are non-nilpotent and by considering line bundles $\mathcal{L}$ that do not meet the topological condition (6.14). In the following we will consider and combine both generalisations, comparing the resulting equations for the function g with the local T-brane solutions in the literature.

Non-nilpotent Higgs field

Let us first consider the case of four-cycles where the condition (6.14) is met, but now we have a non-nilpotent profile for the Higgs field. Namely we consider it to be of the form

$$\Phi = \begin{pmatrix} 0 & m \\ p & 0 \end{pmatrix} \tag{6.23}$$

where $p \in H^{2,0}(S, \mathcal{L}^{-2})$, or equivalently a scalar holomorphic section of the line bundle $\mathcal{P} \equiv \mathcal{L}^{-2} \otimes K_S$. Notice that due to (6.14) we have that $\mathcal{P} \simeq K_S^2$. Such a bundle will have sections in many four-cycles of negative curvature, like for instance in those where K_S also does. In this case eq. (6.13) generalises to

$$\Delta g = c + h_S^{-1} \left(|m|^2 h_0^2 e^{2g} - |p|^2 h_0^{-2} e^{-2g} \right) , \tag{6.24}$$

and so, using eq. (6.17), we arrive to

$$\Delta g = c + \left(|m|^2 e^{2g} - h_0^{-4} |p|^2 e^{-2g} \right) e^{-2s} . \tag{6.25}$$

As before, $|m|^2$ is a constant, while $h_0^{-4}|p|^2$ is a globally well-defined smooth function on S. Finally, enforcing the 4d D-term condition implies that c is given by

$$c = -\frac{1}{\mathrm{V}ol(S)} \int_S \left(|m|^2 e^{2g} - h_0^{-4} |p|^2 e^{-2g} \right) e^{-2s} \mathrm{dv}ol_S , \tag{6.26}$$

so that eq. (6.25) has a (unique) solution.

Notice that now g will not vanish in the Kähler-Einstein case $s = 0$. Instead, eq. (6.25) will become a complicated non-linear equation for g. Near the locus where $p = 0$ we can Taylor expand the function $h_0^{-4}|p|^2$, and recover an equation very similar to that obtained in the local T-brane $\mathbb{Z}_2$ background of [4]. As pointed out in there, such an equation can be rewritten as a Painlevé III differential equation. Hence one would expect that, at least in a local patch near $p = 0$, the profile for g can be expressed in terms of solutions to that equation. Finally, one may depart from a Kähler-Einstein metric by considering $s \neq 0$. This will modify the (unique) solution for g, which will depend on the profiles of the functions $|m|e^{-s}$ and $h_0^{-2}|p|e^{-s}$.

Non-trivial bundle $\mathcal{M}$

Let us now consider relaxing the topological condition (6.14), or in other words assume that $\mathcal{M} \equiv \mathcal{L}^2 \otimes K_S$ is a non-trivial bundle with sections. Given its definition, we can express the hermitian metric on $\mathcal{M}$ as

$$h_{\mathcal{M}} = h_S^{-1} h_0^2 \, e^{2g} = h_{\mathcal{M},0} \, e^{2(g-s)} , \tag{6.27}$$

where $h_{\mathcal{M},0}$ corresponds to the metric with curvature $2F^{\rm h} - \rho^{\rm h}$ and s is again defined by (6.15). We can then express (6.13) as

$$\Delta g = c + \|m\|^2_{\mathcal{M}} \, e^{2(g-s)} \,, \qquad \|m\|^2_{\mathcal{M}} \equiv h_{\mathcal{M},0}|m|^2 \,, \tag{6.28}$$

with $\|m\|_{\mathcal{M}}$ a globally well-defined, smooth function on S that vanishes over the same locus as m. This corresponds to an obvious generalisation of eq. (6.18), where now the input function that determines g is given by $e^{-s}\|m\|_{\mathcal{M}}$. Since $\|m\|_{\mathcal{M}}$ is non-constant, g will be non-trivial even in the Kähler-Einstein case $s = 0$, and so the gauge flux F will depart from harmonicity.

Finally, one may combine a non-trivial bundle $\mathcal{M}$ with a non-nilpotent Higgs field (6.23), again assuming that $\mathcal{P} \equiv \mathcal{L}^{-2} \otimes K_S$ has sections. In that case, we may express the metric for this bundle as

$$h_{\mathcal{P}} = h_S^{-1} h_0^{-2} \, e^{-2g} = h_{\mathcal{P},0} \, e^{-2(g+s)} \,, \tag{6.29}$$

with $h_{\mathcal{P},0}$ the metric of curvature $-2F^{\rm h} - \rho^{\rm h}$. We then consider the globally well-defined, vanishing smooth function on S given by $\|p\|^2_{\mathcal{P}} \equiv h_{\mathcal{P},0}|p|^2$. Together with the above definition for $\|m\|^2_{\mathcal{M}}$, we obtain an equation for g of the form

$$\Delta g = c + \left(\|m\|^2_{\mathcal{M}} e^{2g} - \|p\|^2_{\mathcal{P}} e^{-2g} \right) e^{-2s} \,. \tag{6.30}$$

While arising from a more general setup, this new differential equation is in fact very similar to (6.25), with the new functions that determine g now given by $e^{-s}\|m\|_{\mathcal{M}}$ and $e^{-s}\|p\|_{\mathcal{P}}$.

6.2 A no-go theorem

The simple examples discussed in the previous section suggest that it is relatively easy to construct global T-brane configurations on four-manifolds with negative Ricci curvature. While it may seem that this preference comes from imposing the Hitchin Ansatz or generalisations thereof, there is in fact a deeper reason behind. Indeed, in the following we will see that compact T-brane configurations with Abelian gauge bundles cannot be implemented on four-manifolds of vanishing or positive Ricci curvature. We will first show this no-go result for the configuration with symmetry group $G = SU(2)$ and split gauge bundle of the type (6.2), and then generalise it to groups of higher rank.

The case of SU(2)

In order to investigate the possible obstructions to the construction of compact T-branes, let us first consider the stack of two D7-branes wrapping a simply-connected Kähler surface S, and with split gauge bundle $\mathbb{V} = \mathcal{L} \oplus \mathcal{L}^{-1}$. As before, we may start considering the T-brane background given by the nilpotent Higgs vev

$$\Phi = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}, \tag{6.31}$$

where $m \in H^0(S, \mathcal{M})$. Now, the very fact that an holomorphic section m exists implies that the divisor associated to $\mathcal{M} \equiv \mathcal{L}^2 \otimes K_S$ is effective. That is, for J in the Kähler cone we have

$$\int_S J \wedge c_1(\mathcal{M}) = \int_S J \wedge (2c_1(\mathcal{L}) + c_1(K_S)) \geq 0 \tag{6.32}$$

with the equality holding if and only if $\mathcal{M}$ is trivial.[7] Moreover, the 4d D-term condition (6.10), or equivalently

$$\int_S [\Phi, \Phi^\dagger] = -2 \int_S J \wedge F \cdot \sigma_3 \,, \tag{6.33}$$

for a Higgs field of the form (6.31) implies that

$$2 \int_S J \wedge c_1(\mathcal{L}) < 0 \,, \tag{6.34}$$

where we just used that $F/2\pi$ represents $c_1(\mathcal{L})$ in cohomology. Subtracting the l.h.s. of (6.34) to the middle expression in (6.32), we get the statement that we can construct such a T-brane in a region of Kähler moduli space where

$$\int_S J \wedge c_1(K_S) > 0 \,. \tag{6.35}$$

This conditions forbids S to be K3 or a manifold with positive-definite Ricci curvature. Indeed, if it were positive definite, the canonical class, which is represented by minus the Ricci form, would necessarily have a negative volume everywhere in Kähler moduli space. Kähler surfaces with negative-definite Ricci curvature certainly satisfy the necessary requirement (6.35), but surfaces with

[7] We will always be at large volume, so in particular well away from boundaries of the Kähler cone.

indefinite curvature may also do so. The second inequality we get from (6.32) and (6.34) is

$$\int_S J \wedge c_1(\mathcal{M}) < \int_S J \wedge c_1(K_S)\,, \tag{6.36}$$

which simply states that the volume of the holomorphic curve $\{m = 0\}$ must be strictly smaller than the one of the self-intersection curve of S.[8] As a result, given a surface of non-positive curvature and a point in Kähler moduli space, (6.36) selects a subset of the lattice of bundles $[\mathcal{L}]$ that one can use to build a T-brane background.

As an example, take the case where S has only one Kähler modulus, i.e. $h^{1,1}(S) = 1$. Together with the fact that S is simply-connected, this implies that every gauge "line bundle" $\mathcal{L}$ on S is of the form $\mathcal{L} \simeq K^{-n/2}$, for some non-zero integer n. Then, the two conditions (6.32) and (6.34) boil down to $n \leq 1$ and $n > 0$ respectively, which are both solved only by the choice $n = 1$. This is nothing but the generalisation of Hitchin's class of solutions to a four-manifold, as already analysed in [64] .

Let us now consider the most general Higgs vev compatible with a split rank-two gauge bundle, namely

$$\Phi = \begin{pmatrix} 0 & m \\ p & 0 \end{pmatrix}\,, \tag{6.37}$$

where now $m \in H^0(S, \mathcal{M})$ and $p \in H^0(S, \mathcal{P})$, with $\mathcal{P} \equiv \mathcal{L}^{-2} \otimes K_S$. Suppose now, without loss of generality, that the Fayet-Iliopoulos (FI) term in (6.33) is positive, namely condition (6.34) is satisfied. Then we obtain the following inequalities among the areas of the various curves involved

$$0 \leq \int_S J \wedge c_1(\mathcal{M}) < \int_S J \wedge c_1(K_S) < \int_S J \wedge c_1(\mathcal{P})\,, \tag{6.38}$$

where again the first inequality (with equality if and only if $\mathcal{M}$ is trivial) comes from requiring that $\mathcal{M}$ admits at least one holomorphic section, as otherwise equation (6.33) with positive FI term would be violated. Conversely, if the FI is negative, we get the same statement (6.38) with $\mathcal{M}$ and $\mathcal{P}$ swapped. In other words, the modes determining the sign of the D-term define the curve with the smallest volume. In any of these cases we have that (6.35) must be satisfied,

[8] Note that such a curve needs not be holomorphic.

which again obstructs the construction of compact T-brane configurations on four-manifolds of vanishing or positive-definite Ricci curvature.

Incidentally, notice that the product mp transforms as a section of $H^0(S, K_S^2)$, and it appears in the spectral equation for the Higgs field. Therefore for the background (6.37) one could have guessed the obstruction to realise it on del Pezzo surfaces from a more standard, spectral-surface-based reasoning, see e.g. [65]. Nevertheless, our analysis provides more detailed information about the obstruction, like for instance the inequalities (6.38) that select a subset of possible line bundles $[\mathcal{L}]$.

Higher rank groups

Let us now consider a general simple Lie group G, of Lie algebra $\mathcal{G}$ specified by a Cartan subalgebra H_i and the set of roots E_ρ. In the canonical basis, they satisfy the following set of relations

$$\begin{array}{rcl} [H_i, E_\rho] & = & \rho^i E_\rho \\ [E_\rho, E_\rho^\dagger] & = & \sum_i \rho^i H_i \end{array} \qquad i = 1, \ldots, \mathrm{rank}(\mathcal{G}) \,. \tag{6.39}$$

For our purposes it is more convenient to instead consider the algebra in the so-called Chevalley basis. The latter is specified with respect to a chosen set of simple roots:

$$\begin{array}{rcl} [h_i, e_j] & = & C_{ji} e_j \\ [e_i, e_j^\dagger] & = & \delta_{ij} h_j \end{array} \qquad i, j = 1, \ldots, \mathrm{rank}(\mathcal{G}) \,, \tag{6.40}$$

where h_i are the Cartan generators and e_i the generators associated to the simple roots in this basis. Finally, C_{ij} the Cartan matrix, that can always be decomposed as

$$C = D\, S \,, \qquad D_{ij} = \frac{2\delta_{ij}}{\alpha_j \cdot \alpha_j} \,, \qquad S_{ij} = \alpha_i \cdot \alpha_j \,, \tag{6.41}$$

where α_i stand for the simple-root vectors in the canonical basis (6.39). There, a general root vector can be decomposed as

$$\rho = \sum_i v_\rho^i \alpha_i \qquad v_\rho^i \in \mathbb{Z} \,, \tag{6.42}$$

and then for its corresponding generator in the Chevalley basis we have that

$$[h_i, e_\rho] = q_\rho^i e_\rho \,, \qquad q_\rho^i = \sum_j v_\rho^j C_{ji} \,. \tag{6.43}$$

In this setup, let us take the following Ansatz for our T-brane background

$$\frac{F}{2\pi} = \sum_i \omega_i h_i = \sum_i c_1(\mathcal{L}_i) h_i \tag{6.44}$$

and

$$\Phi = \sum_{\gamma \in R'} m^\gamma e_\gamma \,, \tag{6.45}$$

where $m^\gamma \in H^{2,0}(\otimes_i (\mathcal{L}_i)^{q^i_\gamma})$ and γ runs over a root subset R' such that

$$[e_\gamma, e^\dagger_\beta] = \delta_{\gamma\beta} \sum_i \gamma^i h_i \,, \qquad \forall \gamma, \beta \in R' \,. \tag{6.46}$$

As a result we have

$$[\Phi, \Phi^\dagger] = \sum_{\gamma, i} m^\gamma \wedge \bar{m}^\gamma \, \sigma^i_\gamma \, h_i \,, \tag{6.47}$$

with

$$\sigma^i_\gamma \; = \; \sum_j D_{ij} v^j_\gamma \,. \tag{6.48}$$

Given this background, the fact that m^γ are holomorphic sections implies

$$\int_S \left(\sum_i q^i_\gamma \, c_1(\mathcal{L}_i) + c_1(K_S) \right) \wedge J \geq 0 \qquad \forall \gamma \in R' \,. \tag{6.49}$$

In addition, the D-term condition implies that

$$\int_S c_1(\mathcal{L}_i) \wedge J = - \sum_\gamma \sigma^i_\gamma \, \|m^\gamma\|^2 \tag{6.50}$$

where we have defined

$$\|m^\gamma\|^2 \equiv \frac{1}{2} \int_S m^\gamma \wedge \bar{m}^\gamma \,. \tag{6.51}$$

Therefore

$$\sum_i q^i_\gamma \int_S c_1(\mathcal{L}_i) \wedge J = - \sum_{i,\beta} q^i_\gamma \, \sigma^i_\beta \, \|m^\beta\|^2 = - \sum_{\beta \in R'} v^t_\gamma \, DSD \, v_\beta \, \|m^\beta\|^2 \qquad \forall \gamma \in \; R'. \tag{6.52}$$

Now, notice that the matrix

$$A_{\gamma\beta} \; = \; v^t_\gamma \, DSD \, v_\beta \; = \; \sigma^t_\gamma \, S \, \sigma_\beta \tag{6.53}$$

is semi-definite positive, and definite positive when the set of vectors $\{v_\gamma\}$, $\{\sigma_\gamma\}$ or $\{q_\gamma\}$, $\gamma \in R'$ are linearly independent. Therefore, when $\{v_\gamma\}$ are not linearly independent there are zero modes of $A_{\alpha\beta}$ that correspond to D-flat directions.[9] Going along them one can switch off the necessary number of vevs in the subset of roots R' such that it gets reduced to R'', that corresponds to a set of linearly independent vectors. For this new subset R'' we have that $A_{\gamma\beta}$ is positive definite, and then we have that

$$\sum_{\gamma,i} \|m^\gamma\|^2 \int_S q^i_\gamma\, c_1(\mathcal{L}_i) \wedge J = -\sum_{\gamma,\beta} A_{\gamma\beta}\|m^\gamma\|^2\|m^\beta\|^2 \,<\, 0 \tag{6.55}$$

where now $\gamma, \beta \in R''$. As a result

$$\int_S c_1(K_S)\wedge J > \int_S \left(\frac{\sum_{\gamma,i}\|m^\gamma\|^2 q^i_\gamma c_1(\mathcal{L}_i)}{\sum_\gamma \|m^\gamma\|^2} + c_1(K_S)\right)\wedge J \geq 0\,. \tag{6.56}$$

where in the second inequality we have made use of (6.49). Notice that when we have only one γ this equation reduces to

$$\int_S c_1(K_S)\wedge J > \int_S \left(\sum_i q^i_\gamma c_1(\mathcal{L}_i) + c_1(K_S)\right)\wedge J > 0 \tag{6.57}$$

familiar from the $SU(2)$ case.

6.3 T-branes and stability walls

Starting from a T-brane configuration, we now want to study its stability when we move in the moduli space of Kähler structures. Changes are expected to arise simply because the r.h.s. of the D-term equation (6.33) depends on the Kähler form. In particular, if S has more than one Kähler modulus, there will generically be real codimension-one loci in the Kähler moduli space where the r.h.s vanishes, possibly resulting in a decay of the T-brane, or in its transmutation into a different type of supersymmetric vacuum. In this section, we would like to make a systematic study of what may happen to the T-brane background as we cross such stability walls. We will first consider the sort of T-brane configurations considered in section 6.1, and then extend our analysis to a system of two D7-branes intersecting at a curve.

[9]Moreover, in this case one is able to form a product of sections of the form

$$m^{\gamma_1} m^{\gamma_2} \ldots m^{\gamma_n} \in H^0(K_S^n) \tag{6.54}$$

which cannot exists in a positive curvature four-cycle. Therefore, in positive curvature four-cycles one can consider the $\{\rho_\alpha\}$ to be linearly independent

6.3.1 Coincident branes

Let us consider two D7-branes wrapping a simply-connected Kähler surface S, holomorphically embedded in a Calabi-Yau threefold. As in section 6.1 we consider a split rank-two gauge bundle of the form (6.2), specified by a line bundle $\mathcal{L}$ of curvature F. We moreover consider a Kähler structure compatible with a T-brane of the nilpotent type (6.31). Because of the D-term (6.33), the size of the vev $\langle m \rangle$ is controlled by the FI term $\int F \wedge J$, and thus it is proportional to the distance from the wall, which is defined by the condition $\int F \wedge J = 0$. There we get a vanishing vacuum expectation value for Φ and therefore a standard system of two coincident D7-branes with a worldvolume flux along the Cartan. We are now interested in studying the open-string moduli space in a region around the origin

$$\Phi = 0\,, \tag{6.58}$$

and to see how the D7-brane system evolves when the FI term is switched back on, at the other side of the wall.

To carry such an analysis one may first consider the spectrum of light open-string modes at the wall, where the effective theory has a $U(1) \times U(1)$ gauge group and a set of bifundamental chiral fields charged under the relative $U(1)$, associated to the Cartan. By standard results [66] (see also [67]), the full spectrum of charged massless fields is provided by the appropriate sheaf extension groups. More precisely, as in section 6.1, let us define the two line bundles $\mathcal{M} \equiv \mathcal{L}^2 \otimes K_S$ and $\mathcal{P} \equiv \mathcal{L}^{-2} \otimes K_S$, with K_S the canonical bundle of S. Then one has

$$\begin{aligned} (+) \quad &\in \quad \mathrm{E}xt^1(i_*\mathcal{L}^{-1}, i_*\mathcal{L}) \simeq \underbrace{H^0(S,\mathcal{M})}_{m} \oplus \underbrace{H^1(S,\mathcal{P})}_{a_+}\,, \\ (-) \quad &\in \quad \mathrm{E}xt^1(i_*\mathcal{L}, i_*\mathcal{L}^{-1}) \simeq \underbrace{H^0(S,\mathcal{P})}_{p} \oplus \underbrace{H^1(S,\mathcal{M})}_{a_-}\,, \end{aligned} \tag{6.59}$$

where the signs on the left indicate the relative-$U(1)$ charge and i is the embedding map of S in the Calabi-Yau threefold. Here the H^0 parts correspond to massless off-diagonal fluctuations of the Higgs field, whereas the H^1 parts correspond to off-diagonal components of the non-Abelian gauge field living on

S. Notice that a non-vanishing vacuum expectation value for the latter would correspond to a non-Abelian gauge bundle, and so the vevs for such fields $a_\pm$ were assumed to vanish in the T-brane configurations of section 6.1. We must however take them into account in the following, to study how the D-brane configuration may react as we cross a stability wall.

On top of the charged modes there are also uncharged zero modes, which however only appear as fluctuations of Φ and not of the gauge field, because we are taking S to be simply-connected. Such fields originate from open strings with endpoints on the same D7-brane and thus corresponding to its normal deformations inside the ambient Calabi-Yau manifold. Here we only focus on relative deformations of the two branes wrapping S, and ignore the movements of their centre of mass. Therefore, these deformations appear in the Higgs-field fluctuation as

$$\delta\Phi|_{neutral} = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}, \qquad v \in H^0(S, K_S)\,. \tag{6.60}$$

Note that these vevs were also set to vanish in the T-brane configurations of section 6.1.

Finally, the absence of modes with negative norm (ghosts) for the strings connecting the two branes [54] leads to the following important requirements

$$H^0(S, \mathcal{L}^2) = H^0(S, \mathcal{L}^{-2}) = 0\,. \tag{6.61}$$

These conditions are automatically satisfied if the FI term vanishes and we are inside the Kähler cone.

Given the above spectrum one may analyse how the system behaves at both sides of the wall. For simplicity, we will first consider the case where the modes (6.60) are absent. Then, in a sufficiently small region in Kähler moduli space around the wall, and upon dimensional reduction to 4d, the D-term condition (6.1c) becomes[10]

$$\sum_m |m|^2 + \sum_{a_+} |a_+|^2 - \sum_p |p|^2 - \sum_{a_-} |a_-|^2 = \xi\,, \tag{6.62}$$

[10] We use the same symbol for the eight-dimensional fields and the corresponding four-dimensional zero modes, and suppress the symbol $\langle\cdot\rangle$ to indicate the vev. We moreover work in units of α'.

which is nothing but the vanishing of the 4d D-term scalar potential. By assumption, on one side of the wall we have a supersymmetric configuration where only m-type zero modes have a non-vanishing vev, and so there $\xi > 0$. Then we reach the wall by moving in the Kähler-structure moduli space. After crossing the wall the FI term flips sign, so

$$\xi \equiv -2 \int_S J \wedge c_1(\mathcal{L}) < 0\,. \tag{6.63}$$

Therefore from equation (6.62) it is manifest that if $H^0(S, \mathcal{P}) = H^1(S, \mathcal{M}) = 0$, there is no solution for the D-term equation as we cross the wall. Microscopically, this means that the T-brane we started with disappears as we cross the wall, by decaying into its D7-brane constituents, which are not mutually supersymmetric.[11]

Interestingly, by using the index theorem we are able to formulate a practical *necessary* criterion for such a decay to occur. In particular, applying the index theorem to the line bundle $\mathcal{P}$, we get

$$h^0(S, \mathcal{P}) - h^1(S, \mathcal{P}) = \int_S \mathrm{c}h(\mathcal{P}) \wedge \mathrm{T}d(S)\,, \tag{6.64}$$

where the symbol h^i indicates the dimension of the corresponding group H^i, "ch" is the total Chern character and "Td" is the Todd class.[12] In (6.64) we have used that $h^2(S, \mathcal{P}) = h^0(S, \mathcal{L}^2) = 0$, where the first equality comes from Serre duality, and the second from equation (6.61). Likewise, the index theorem for the line bundle $\mathcal{M}$ means that

$$h^0(S, \mathcal{M}) - h^1(S, \mathcal{M}) = \int_S \mathrm{c}h(\mathcal{M}) \wedge \mathrm{T}d(S)\,, \tag{6.65}$$

where again we used that $h^2(S, \mathcal{M}) = h^0(S, \mathcal{L}^{-2}) = 0$, because of Serre duality and equation (6.61) respectively. By subtracting equation (6.65) to equation (6.64), with some trivial algebra we get to the chiral index of the theory:

$$I = \#(+) - \#(-) = 2 \int_S c_1(\mathcal{L}) \wedge c_1(K_S)\,, \tag{6.66}$$

[11] Note that we are considering the D7-brane stack in isolation, neglecting other D-branes that may yield further chiral zero modes charged under the Cartan $U(1)$. One clearly needs to take into account the full brane content of the compactification to see if crossing the wall really breaks supersymmetry.

[12] For a line bundle $\mathcal{F}$, $\mathrm{c}h(\mathcal{F}) = 1 + c_1(\mathcal{F}) + c_1^2(\mathcal{F})/2$, and for a surface S one has $\mathrm{T}d(S) = 1 - c_1(K_S)/2 + (c_1(K_S)^2 + c_2(S))/12$.

where the symbol $\#(\pm)$ denotes the number of zero modes with $U(1)$-charge $\pm$. Finally, from equation (6.66) we obtain the following implication

$$I = 2\int_S c_1(\mathcal{L}) \wedge c_1(K_S) \;\leq\; 0 \qquad \Longrightarrow \qquad \text{N}o\ T\text{-}brane\ decay\,, \tag{6.67}$$

because if there were no negatively-charged modes available to turn the T-brane into another supersymmetric system, the integral on the l.h.s. would necessarily be positive.

On the contrary, if conditions are met for some negatively-charged modes to exist, the T-brane simply turns into a different supersymmetric state on the other side of the wall.[13] The latter could be another T-brane, if just the p-type modes get a vev, a non-Abelian bundle configuration (T-bundle) if just the a_--type modes get a vev, or a more complicated mixed object. The indices of the individual bundles, quoted in equations (6.64) and (6.65), can turn useful to guess what type of object the T-brane may turn into, although most of the times they cannot give definite answers. In practice, one may compute the cohomology groups in (6.59) case by case, as illustrated in appendix F, to find out the fate of the T-brane at the other side of the wall. There are however a few classes of constructions where a more general statement can be made, as we discuss in the following.

The Hitchin Ansatz

An interesting case of T-branes is the one constructed using what we have dubbed the Hitchin Ansatz, namely when $\mathcal{M}$ is trivial, or equivalently $\mathcal{L} \simeq K_S^{-1/2}$. One important remark regarding this case is that, if the Ricci curvature of S is negative definite, then there will be no stability walls. Indeed, for $\mathcal{L} \simeq K_S^{-1/2}$ we have that the FI term becomes

$$\xi = \int_S J \wedge c_1(K_S)\,, \tag{6.68}$$

which for negative curvature cannot be taken to zero while moving inside the Kähler cone.

[13] One particular case is when $I = 0$, which in the literature corresponds to a wall of threshold stability. Indeed, by looking at the definition (6.66) one realises that $-I$ corresponds to the intersection product used in [68] to classify stability walls.

Let us then consider the case where the Ricci curvature of S is indefinite. This in particular implies absence of holomorphic sections for the canonical bundle (thus S is rigid) and for any power thereof (positive and negative). Therefore no p-type modes are available and, since by assumption S is simply-connected, no a_--type modes are available either. Hence, in this class of configurations, our T-brane is forced to decay into a non-supersymmetric vacuum when the wall is crossed.

A simple instance of a Kähler surface with the above properties can be obtained as follows. Consider $\mathbb{P}^4$ with homogeneous coordinates $x_1, \ldots, x_5$, blown up along a four-cycle, e.g. $\{x_1 = x_2 = 0\}$. The toric weights of this manifold are

x_1	x_2	x_3	x_4	x_5	w
1	1	0	0	0	−1
0	0	1	1	1	1

(6.69)

where $E : \{w = 0\}$ corresponds to the exceptional divisor, homeomorphic to $\mathbb{P}^2 \times \mathbb{P}^1$. In this ambient manifold, we consider the Calabi-Yau threefold CY_3 given by the zero-locus of a smooth polynomial of bi-degree $(1, 4)$, and the D7-brane stack wrapped on $S : E \cap \mathrm{CY}_3$. It is easy to show that this surface is rigid (as a consequence of the rigidity of the exceptional divisor), and moreover has indefinite Ricci curvature, because e.g.

$$\int_{S\cap\{x_1=0\}} c_1(K_S) = 4\,, \qquad \int_{S\cap\{x_3=0\}} c_1(K_S) = -3\,. \tag{6.70}$$

By using the Hirzebruch-Riemann-Roch theorem, we can also easily show that this surface has no cohomologically non-trivial one-forms

$$h^{0,1}(S) = 1 - \frac{1}{12}\int_S c_1^2(K_S) + c_2(S) = 0\,, \tag{6.71}$$

where we used that $h^{0,2}(S) = 0$. If we label $H : \{x_1 = 0\}$ and expand the Kähler form in this basis, $J \equiv v_H\, H + v_E\, E$, we may compute the Fayet-Iliopoulos term as $\xi = 5(4v_H - 7v_E)$, which can indeed acquire both positive and negative values within the Kähler cone.

Negative curvature

Let us now consider the case where the Ricci curvature of the surface S is negative definite. Note that this does not necessarily imply that S can be holomorphically deformed, a subcase to be considered momentarily. By the observation made above, in the negative curvature case we must consider a T-brane whose m-type mode transforms under a non-trivial bundle $\mathcal{M}$. The fact that $\mathcal{M}$ is effective and non-trivial, together with the ampleness of K_S due to the negative curvature, implies that

$$I > -\int_S c_1^2(K_S)\,, \tag{6.72}$$

where the r.h.s. is a negative integer number. Applying the same reasoning to the bundle $\mathcal{P}$, we have that the existence of p-type modes implies that $I \leq \int_S c_1^2(K_S)$, and so whenever

$$I > \int_S c_1^2(K_S) > 0 \tag{6.73}$$

there will be no such p-modes. Notice that imposing (6.73) implies (6.72). Therefore, if we consider a case where (6.73) is satisfied and $h^1(S,\mathcal{M}) = 0$ (see appendix F for an example), then there will be a T-brane decay. Alternatively, if $h^1(S,\mathcal{M}) > 0$ then the T-brane will turn into a supersymmetric non-Abelian bundle configuration on the other side of the wall.

One particular case of a negative curvature four-cycle is when S can be holomorphically deformed, namely when the modes (6.60) exist. Then there is a self-intersection curve defined by $\mathcal{C} \equiv \{v = 0\}$ and with a genus g such that

$$\int_S c_1^2(K_S) = g - 1\,. \tag{6.74}$$

Note that by the adjunction formula one finds that $g = 1 + [S]^3$, where $[S]$ stands for the divisor class of S in the Calabi-Yau. Since $\int_S c_1^2(K_S) > 0$, we have that $[S]^3$ is a positive number and so $g \geq 2$.

In this particular case there is the open-string field v defined in (6.60), which is a modulus along the wall. One may then wonder what happens when the wall is crossed with a non-vanishing Higgs-field vev, namely at

$$\Phi = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}. \tag{6.75}$$

In this case, by dimensionally reducing the D7-brane superpotential

$$W = \int_S \mathrm{Tr}\,(\mathbb{F} \wedge \Phi)\,, \tag{6.76}$$

one obtains Yukawa couplings of the form

$$W \supset d_{ijk}\, v^i a_-^j a_+^k \,, \tag{6.77}$$

which generically give an F-term mass to the negative-chirality modes a_-. Now, if we impose (6.73) and cross the wall at (6.75), for $h^1(S,\mathcal{M}) > 0$ there will be an F-term potential that will make (6.75) vanish and take the system to the supersymmetric configuration of coincident D7-branes with a non-Abelian bundle created by the vev of a_-.

Notice that at (6.75) we have a system of two homotopic D7-branes intersecting at a curve $\mathcal{C}$, with opposite worldvolume fluxes. This is nothing but a particular case of a more general configuration, made of two intersecting D7-branes with arbitrary worldvolume fluxes. As we will now see, one can formulate the T-brane wall-crossing conditions for this more interesting case as well.

6.3.2 Intersecting branes

Let us consider two D7-branes wrapping different simply-connected Kähler surfaces S_1, S_2, holomorphically embedded in a Calabi-Yau threefold. Let $\mathcal{L}_1, \mathcal{L}_2$ be the holomorphic gauge line bundles on each of the two branes, with fluxes F_1, F_2 respectively. As in the coincident case, the four-dimensional effective theory has a $U(1)\times U(1)$ gauge group and bifundamental chiral fields charged under the relative combination. The 4d D-term condition that controls the vacuum expectation values of their scalar components is now given by

$$\sum_{m\in(+,-)} |m|^2 - \sum_{p\in(-,+)} |p|^2 \;=\; \int_{S_2} J\wedge F_2 - \int_{S_1} J\wedge F_1 \;=\; \xi \,, \tag{6.78}$$

where the two sums extend over zero modes with $U(1)\times U(1)$-charges $(+,-)$ and $(-,+)$ respectively. They correspond to open strings stretching from brane 2 to brane 1 and to strings going the opposite way respectively. Assuming that the intersection curve $\mathcal{C} \equiv S_1 \cap S_2$ is connected, such zero modes are counted by the following sheaf extension groups [66] (see also [67]):

$$\begin{aligned} (+,-) \;&\in\; \mathrm{E}xt^1(i_{2*}\mathcal{L}_2, i_{1*}\mathcal{L}_1) \simeq H^0(\mathcal{C}, \mathcal{L}_2^{-1}|_{\mathcal{C}} \otimes \mathcal{L}_1|_{\mathcal{C}} \otimes K_{\mathcal{C}}^{1/2}) \,, \\ (-,+) \;&\in\; \mathrm{E}xt^1(i_{1*}\mathcal{L}_1, i_{2*}\mathcal{L}_2) \simeq H^0(\mathcal{C}, \mathcal{L}_2|_{\mathcal{C}} \otimes \mathcal{L}_1^{-1}|_{\mathcal{C}} \otimes K_{\mathcal{C}}^{1/2}) \,, \end{aligned} \tag{6.79}$$

with $K_{\mathcal{C}}$ its canonical bundle, and i_1, i_2 the embedding maps of branes $1, 2$ respectively.

In this case the wall is defined by the Kähler structure slice where $\int F_1 \wedge J = \int F_2 \wedge J$. There we have a system of two intersecting D7-branes, and thus the spectrum of massless fluctuations is given by equation (6.79). Notice that, unlike in the coincident case, now the spectrum of zero modes is only counted by modes of the Higgs field. We now assume that there is at least one of these two types of modes, say a m-type mode with charge $(+,-)$, so that, at one side of the wall $(\xi > 0)$, there is a supersymmetric bound state with a T-brane profile localised at $\mathcal{C}$. As we cross the wall to the other side, either this T-brane turns into a different kind of T-brane or, if no p-type mode is available, the T-brane decays into the two mutually non-supersymmetric constituents.[14]

Since in this case the spectrum of charged zero modes is simpler, we are able to formulate a *sufficient* criterion for our T-brane to decay across the wall. First, notice that the chiral index of the theory is given by

$$I \equiv \deg \mathcal{L}_1|_{\mathcal{C}} - \deg \mathcal{L}_2|_{\mathcal{C}} \;=\frac{1}{2\pi}\int_{\mathcal{C}} F_1 - F_2\,. \tag{6.80}$$

Let us for now assume that the surfaces S_1, S_2 do not have holomorphic deformations or, if they do, that none of them will split the intersection curve into multiple connected components. Then, calling g the genus of $\mathcal{C}$ and using the Riemann-Roch theorem, the existence of the m-type mode we began with implies that

$$I \;\geq\; 1-g\,, \tag{6.81}$$

with the equality holding if and only if m is constant, which is the analogue of the Hitchin Ansatz for a system of intersecting D7-branes. This relation comes from the fact that the degree of a line bundle on a curve coincides with the number of zeros minus the number of poles of any of its rational sections. Moreover, we have the analogue of (6.67), with the index theorem adapted to this case

$$I \leq\; 0 \qquad \Longrightarrow \qquad \text{No } \textit{T-brane decay}\,. \tag{6.82}$$

Finally, by the same reasoning, if the condition

$$I \;>\; g-1 \tag{6.83}$$

[14] This decay process has been discussed in [10].

is satisfied, there are no p-type modes to form a T-brane on the side of the wall where the FI term is negative. Therefore, we readily see that, if the two D7-branes intersect on a sphere, the fate of our T-brane is to decay when we cross the wall. The same statement holds true when $\mathcal{C}$ is a two-torus and $\int_{\mathcal{C}} F_1 \neq \int_{\mathcal{C}} F_2$. We therefore obtain a simple picture for the decay possibilities of intersecting D7-branes, summarised in figure 6.1.

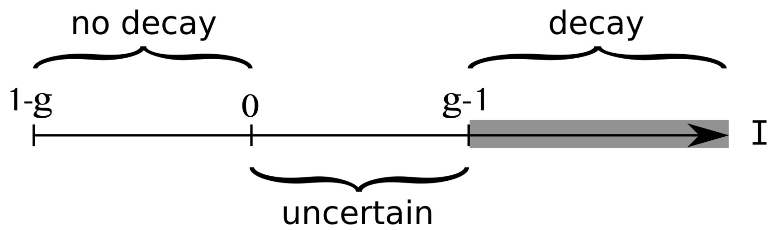


Figure 6.1: Different possibilities of decay into non-BPS constituents as a T-brane constructed from two intersecting D7-branes crosses a stability wall.

If on the other hand the surfaces S_1, S_2 contain holomorphic deformations such that $\mathcal{C}$ splits into multiple components, the wall-crossing picture just described may change. Indeed, when the matter curve $\mathcal{C} = \cup_a \mathcal{C}_a$ is disconnected, one needs to apply (6.79) separately to each individual component $\mathcal{C}_a$ to obtain the massless spectrum. While then the relations (6.81) and (6.82) continue to hold,[15] the sufficient condition for decay (6.83) gets replaced by a significantly weaker one. This is because it is enough to find at least a p-mode localised on any of the connected components of $\mathcal{C}$, in order for the two branes to bind back again into a supersymmetric system across the wall. In other words, decay will only occur when all the available holomorphic deformations of S_1 and S_2 split $\mathcal{C}$ in such a way that on every component $\mathcal{C}_a$ one has $I_a > g_a - 1$.

[15] More precisely, (6.81) should be written in terms of topological invariants as $I \geq h^{0,0}(\mathcal{C}) - h^{0,1}(\mathcal{C})$.

Chapter 7

Compact T-branes with Defects

7.1 T-branes with defects

7.1.1 Compact T-brane systems

Let us consider F-theory on $\mathbb{R}^{1,3} \times \mathcal{M}$, with $\mathcal{M}$ a Calabi-Yau four-fold, and a stack of 7-branes wrapping a compact Kähler surface S of the three-fold base of $\mathcal{M}$. In general, the precise 7-brane configuration and its lightest degrees of freedom can be specified by an eight-dimensional super-Yang-Mills theory on $\mathbb{R}^{1,3} \times S$ with symmetry group G_S [14, 54–56]. The two objects defining such an action are the field strength $\mathbb{F} = d\mathbb{A} - i\mathbb{A} \wedge \mathbb{A}$ of the 7-branes gauge boson $\mathbb{A}$, and the (2,0)-form Higgs field Φ, whose eigenvalues describe the 7-brane transverse geometrical deformations. Both $\mathbb{A}$ and Φ transform in the adjoint of the symmetry group G_S and, whenever they have a non-trivial profile, they break the gauge group to a subgroup of G_S. The 7-brane configurations that preserve supersymmetry correspond to those solving the following set of conditions

$$\bar{\partial}_{\mathbb{A}}\Phi = 0\,, \tag{7.1a}$$

$$\mathbb{F}^{(0,2)} = 0\,, \tag{7.1b}$$

$$J \wedge \mathbb{F} + \frac{1}{2}[\Phi, \Phi^\dagger] = 0\,, \tag{7.1c}$$

with J the Kähler two-form of S. From the four-dimensional viewpoint, the first two equations imply vanishing F-terms, while the third one ensures vanishing D-terms.

In [20] we analysed supersymmetric 7-brane backgrounds with non-commuting expectation values for the Higgs field Φ. In other words we considered configurations solving (7.1) and such that $[\Phi, \Phi^\dagger] \neq 0$, also known as T-branes in the string theory literature [2, 4, 5, 65]. We imposed that such T-brane configurations are globally well-defined over S and that the Higgs field profile is absent of poles. In the remainder of this subsection we will review some of the main results obtained in [20], and in the next one we will see how these results are modified when we allow for the presence of poles.

The simplest T-brane configuration that one may construct is based on the symmetry group $G_S = SU(2)$, which in applications to F-theory GUTs one may identify with the $\mathfrak{su}(2)$ factor in (??) or some other subgroup transverse to G_{GUT}. In this case, the simplest non-trivial gauge bundle that one may consider on S is of rank two and split type,[1] namely $\mathbb{V} = \mathcal{L} \oplus \mathcal{L}^{-1}$. Due to the BPS equation (7.1b), $\mathcal{L}$ is endowed with a holomorphic structure. Then, if $\{T_+, T_-, T_3\}$ with $[T_+, T_-] = T_3$, are the generators of $\mathfrak{sl}(2)$ this translates into a flux background of the form $\mathbb{F} = F\,T_3$, which in the fundamental representation reads

$$\mathbb{F} = \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}, \qquad F = F^{\rm h} - i\partial\bar{\partial}g\,, \tag{7.2}$$

with $F^{\rm h}$ a harmonic (1,1)-form and g real function, both globally well-defined on S. Fixing the holomorphic structure of $\mathcal{L}$ such that $A^{0,1} = 0$, a choice usually dubbed holomorphic gauge [23, 26], allows to rewrite everything in terms of the

[1] As in [20], we will always assume that S is simply-connected, i.e. $\pi_1(S) = 0$. This implies that holomorphic line bundles on S have their topology completely specified by their first Chern class.

hermitian metric h on $\mathcal{L}$. In particular locally we can write $F = -i\partial\bar{\partial}\log h$, with $h = h_0\, e^g$ and h_0 the metric that corresponds to $F^{\rm h}$.

One may pair up this flux background with a nilpotent Higgs background of the form $\Phi = mT_+$, or

$$\Phi = \begin{pmatrix} 0 & \mathbf{m} \\ 0 & 0 \end{pmatrix} . \tag{7.3}$$

where, in the holomorphic gauge, $\mathbf{m} \in H^{2,0}(S, \mathcal{L}^2)$. Equivalently, we can also see $\mathbf{m}$ as a scalar holomorphic section m of the line bundle $\mathcal{M} \equiv \mathcal{L}^2 \otimes K_S$, with K_S the canonical bundle of S.[2] The existence of m implies that $\mathcal{M}$ is effective in S, and therefore

$$\int_S J \wedge c_1(\mathcal{M}) = \int_S J \wedge (2c_1(\mathcal{L}) + c_1(K_S)) \geq 0 \tag{7.4}$$

with the equality holding if and only if $\mathcal{M}$ is trivial and m is constant. On the other hand, the D-term condition (7.1c) for this background reads

$$\int_S J \wedge c_1(\mathcal{L}) = -\frac{1}{8\pi} \mathrm{Tr} \int_S [\Phi, \Phi^\dagger] T_3 < 0 \, . \tag{7.5}$$

Taking both equations into account one obtains the following set of inequalities

$$\int_S J \wedge c_1(K_S) > \int_S J \wedge c_1(\mathcal{M}) \geq 0 \, , \tag{7.6}$$

which constrains the viable choices for the bundle $\mathcal{L}$ and forbids S to be K3 or a manifold with positive-definite Ricci curvature. One may complicate the Higgs field background and replace (7.3) by

$$\Phi = \begin{pmatrix} 0 & \mathbf{m} \\ \mathbf{p} & 0 \end{pmatrix} , \tag{7.7}$$

where $\mathbf{p} \in H^{2,0}(S, \mathcal{L}^{-2})$ defines an element of $H^0(S, \mathcal{P})$, with $\mathcal{P} \equiv \mathcal{L}^{-2} \otimes K_S$. Now $\mathcal{P}$ also needs to be effective and, as discussed in [20], from the D-term equation one recovers a hierarchy of curve areas that imposes $\int_S J \wedge c_1(K_S) > 0$ and restricts the possible choices for $[\mathcal{L}]$. This is consistent with the fact that mp transforms as a section of $H^0(S, K_S^2)$, and as such implies a holomorphic deformation for the surface S that is forbidden in the case of, e.g., del Pezzo surfaces [65].

[2]Throughout the text, boldface quantities like $\mathbf{m}$ will denote holomorphic (2,0)-forms, while the same letter in italic will stand for their scalar counterpart through the canonical isomorphism $H^{2,0}(\mathcal{L}) \simeq H^0(K_S \otimes \mathcal{L})$ for an arbitrary bundle $\mathcal{L}$.

The no-go for $SU(3)$

This no-go result generalises to arbitrary T-brane backgrounds with higher rank gauge group G, as long as the worldvolume fluxes lie along its Cartan subalgebra. For concreteness here we will only review the case of $G_S = SU(3)$, which will become useful at the end of this section for understanding how the no-go result can fail in the presence of defects. We refer the reader to [20] for the general proof of the no-go theorem.

Let us consider $\mathbb{F}$ and Φ taking values in the complexification of the SU(3) algebra, with their profiles expressed in terms of the Chevalley basis $\{\eta_1, \eta_2, \epsilon_1, \epsilon_2, \epsilon_{12}, \theta_1, \theta_2, \theta_{12}\}$ of $\mathfrak{sl}(3)$ (c.f. Appendix G for explicit expressions). By assumption we have a worldvolume flux valued along the Cartan subalgebra. That is

$$\mathbb{F} = F_1\,\eta_1 + F_2\,\eta_2\,. \tag{7.8}$$

In addition, we have a Higgs field profile valued outside of the Cartan subalgebra,[3] but such that the commutator $[\Phi, \Phi^\dagger]$ lies within it in order to satisfy the D-term equations. This condition restricts the possible profiles for Φ, allowing it to have non-vanishing components only up to three independent roots. For instance, one may consider the profile $\Phi = m_1\eta_1 + m_2\eta_2 + p_{12}f_{12}$. That is, in the fundamental representation of $\mathfrak{sl}(3)$ we have the profiles

$$\mathbb{F} = \begin{pmatrix} F_1 & 0 & 0 \\ 0 & F_2 - F_1 & 0 \\ 0 & 0 & -F_2 \end{pmatrix}, \qquad \Phi = \begin{pmatrix} 0 & \mathbf{m}_1 & 0 \\ 0 & 0 & \mathbf{m}_2 \\ \mathbf{p}_{12} & 0 & 0 \end{pmatrix}, \tag{7.9}$$

where $F_{1,2}$ are closed (1,1)-forms such that $[F_i] = 2\pi c_1(\mathcal{L}_i)$ and, in the holomorphic gauge, $\mathbf{m}_1 \in H^{2,0}(\mathcal{L}_1^2 \otimes \mathcal{L}_2^{-1})$, $\mathbf{m}_2 \in H^{2,0}(\mathcal{L}_1^{-1} \otimes \mathcal{L}_2^2)$, $\mathbf{p}_{12} \in H^{2,0}(\mathcal{L}_1^{-1} \otimes \mathcal{L}_2^{-1})$, with some of these sections possibly vanishing. The D-term equation implies that[4]

$$4\pi \int_S c_1(\mathcal{L}_1) \wedge J = -\|m_1\|^2 + \|p_{12}\|^2 \tag{7.10a}$$

$$4\pi \int_S c_1(\mathcal{L}_2) \wedge J = -\|m_2\|^2 + \|p_{12}\|^2. \tag{7.10b}$$

[3] Notice that deformations along the Cartan subalgebra are forbidden in positive curvature manifolds.

[4] Given a split bundle metric $H_{\mathfrak{su}(3)} = \mathrm{d}iag(h_1, h_1^{-1}h_2, h_2^{-1})$, we define $\|m_1\|^2 = \int_S h_1^2 h_2^{-1}\mathbf{m}_1 \wedge \bar{\mathbf{m}}_1$, $\|m_2\|^2 = \int_S h_1^{-1}h_2^2\mathbf{m}_2 \wedge \bar{\mathbf{m}}_2$ and $\|p_{12}\|^2 = \int_S h_1^{-1}h_2^{-1}\mathbf{p}_{12} \wedge \bar{\mathbf{p}}_{12}$.

In addition, each non-vanishing holomorphic section implies an effectiveness constraint.

$$m_1 \to A(\mathcal{M}_1) = \int_S (2c_1(\mathcal{L}_1) - c_1(\mathcal{L}_2) + c_1(K_S)) \wedge J \geq 0\,, \tag{7.11a}$$

$$m_2 \to A(\mathcal{M}_2) = \int_S (-c_1(\mathcal{L}_1) + 2c_1(\mathcal{L}_2) + c_1(K_S)) \wedge J \geq 0\,, \tag{7.11b}$$

$$p_{12} \to A(\mathcal{P}_{12}) = \int_S (-c_1(\mathcal{L}_1) - c_1(\mathcal{L}_2) + c_1(K_S)) \wedge J \geq 0\,. \tag{7.11c}$$

Notice that the product $m_1 m_2 p_{12}$ transforms as a section of $H^0(S, K_S^3)$ and so, if we would like to consider surfaces of positive curvature at least one of these three sections should vanish.[5] Without loss of generality let us take $p_{12} = 0$. Then by using (7.10) one has that

$$\begin{aligned} &\|m_1\|^2 \int_S (2c_1(\mathcal{L}_1) - c_1(\mathcal{L}_2)) \wedge J + \|m_2\|^2 \int_S (-c_1(\mathcal{L}_1) + 2c_1(\mathcal{L}_2)) \wedge J \\ &= -\frac{1}{4\pi} \sum_{i,j} C_{ij} \|m_i\|^2 \|m_j\|^2 < 0. \end{aligned} \tag{7.12}$$

where C_{ij} is the Cartan matrix of $\mathfrak{su}(3)$, and where the last inequality follows from its positive definiteness. Finally, one can derive the set of inequalities

$$\int_S c_1(K_S) \wedge J > \frac{\|m_1\|^2 A(\mathcal{M}_1) + \|m_2\|^2 A(\mathcal{M}_2)}{\|m_1\|^2 + \|m_2\|^2} \geq 0\,, \tag{7.13}$$

with the first inequality following from (7.12) and the second from (7.11a) and (7.11b). It is easy to check that a similar result is obtained for any other choice of holomorphic profile for Φ such that $[\Phi, \Phi^\dagger]$ lies within the Cartan subalgebra. These inequalities are the generalisation of (7.6) for the $SU(2)$ T-brane background and, as in there, they constrain the allowed choices for the bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ and forbid a positive curvature for S.

7.1.2 Introducing defects

Let us now consider coupling a defect theory to the super-Yang-Mills theory on S, following [14]. Such a theory will be localised on $\mathbb{R}^{1,3} \times \Sigma$, where $\Sigma = S \cap S'$ arises from the intersection with a surface $S' \in \mathcal{M}$ wrapped by a second stack of 7-branes. If the symmetry group of that second stack is $G_{S'}$, then in general

[5] If one is interested in surfaces where the product $m_1 m_2 p_{12}$ can exist, one can see that the D-term equation has a flat direction that allow to reach points in which one of these components vanishes.

there will be matter fields transforming under irreducible representations of $G_S \times G_{S'}$ and localised at the intersection locus $\mathbb{R}^{1,3} \times \Sigma$. The lowest component of such multiplets are complex scalars (σ, σ^c) whose internal profile is determined by sections on Σ, namely

$$\sigma \in \Gamma\left(K_\Sigma^{1/2} \otimes \mathcal{U} \otimes \mathcal{U}'\right), \qquad \sigma^c \in \Gamma\left(K_\Sigma^{1/2} \otimes \mathcal{U}^* \otimes (\mathcal{U}')^*\right). \tag{7.14}$$

Here $\mathcal{U}$, $\mathcal{U}'$ are the vector bundles associated via the corresponding irrep to the principal bundles on the 7-brane stacks on S, S', respectively, and restricted to the curve Σ. Non-trivial vevs for such 4d fields correspond to localised sources for our previous Hitchin system describing the internal 7-brane background. More precisely, from the viewpoint of the 7-brane theory on S we have that the BPS equations (7.1) are deformed to

$$\bar{\partial}_{\mathbb{A}} \Phi = \delta_\Sigma \wedge \langle\!\langle \sigma^c, \sigma \rangle\!\rangle_{\mathfrak{g}_S}, \tag{7.15a}$$

$$\mathbb{F}^{(0,2)} = 0, \tag{7.15b}$$

$$J \wedge \mathbb{F} + \frac{1}{2}[\Phi, \Phi^\dagger] = -\frac{1}{2} J \wedge \delta_\Sigma\, \mu \tag{7.15c}$$

Here δ_Σ is the two-form on S with delta-function support along Σ and which represents the Poincaré dual of its cohomology class. Multiplying it appear the complex outer product $\langle\!\langle \sigma^c, \sigma \rangle\!\rangle_{\mathfrak{g}_S}$ and the real moment map μ. Both quantities are bilinear in the defect fields, and in the case that $\mathcal{U}'$ is a line bundle they locally read[6]

$$\langle\!\langle \sigma^c, \sigma \rangle\!\rangle_{\mathfrak{g}_S} = \sigma^c_j (T^I)^j{}_i \sigma^i\, \mathfrak{t}_I, \tag{7.16}$$

$$\mu = h_\Sigma^{-1/2}\left[h'^{-1} \bar{\sigma}^{\bar{k}} H_{\bar{k}j} (T^I)^j{}_i \sigma^i - h' \sigma^c_i (T^I)^i{}_j H^{j\bar{k}} \bar{\sigma}^c_{\bar{k}}\right] \mathfrak{t}_I, \tag{7.17}$$

with $\mathfrak{t}_I$ the generators of $\mathfrak{g}_S = \text{Lie}(G_S)$ and T^I the representation under which the defect fields σ transform. In addition h_Σ is the hermitean metric on the defect curve, and H, h' are the metrics of the bundles $\mathcal{U}$ and $\mathcal{U}'$, respectively, that in eq.(7.17) have been restricted to Σ. Finally, the defect fields satisfy the following equations of motion

$$\bar{\partial}_{\mathbb{A}+\mathbb{A}'} \sigma = \bar{\partial}_{\mathbb{A}+\mathbb{A}'} \sigma^c = 0 \tag{7.18}$$

[6]In the particular case where $\mathcal{U}$ is split, as will be the case in our disucussion below, eqs.(7.15a)-(7.15c) are in fact globally well-defined.

where $\mathbb{A}$, $\mathbb{A}'$ act on the appropriate representation and are restricted from S, S' to Σ, respectively. An important point is that, as a consequence of (7.14), $\langle\langle\sigma^c, \sigma\rangle\rangle_{\mathfrak{g}_S}$ can be considered as a $\mathfrak{g}_S$-valued (1,0)-form on Σ, as it is implicitly assumed in (7.15a). On the other hand μ is a real scalar. We refer to Appendix H and [14] for more details.

A simple setup

Rather than describing the most general configuration involving defects, let us focus in a simple setup that already shows the new possibilities that adding them brings. Consider type IIB string theory compactified on a Calabi-Yau three-fold orientifold and a pair of holomorphic four-cycles S and S' within it. The divisor S is wrapped by two D7-branes, therefore hosting a symmetry group $G_S = U(2)$, while S' is wrapped by a single D7-brane and hosts the group $G_{S'} = U(1)$. At their intersection $\Sigma = S \cap S'$, the symmetry group enhances to $G_\Sigma = U(3)$, and as a consequence Σ localises matter fields σ, σ^c transforming in the bifundamental representations of $\mathfrak{u}(2) \times \mathfrak{u}(1)$. From the viewpoint of S we will have a 6d defect theory on $\mathbb{R}^{1,3} \times \Sigma$ coupled to the $U(2)$ theory on $\mathbb{R}^{1,3} \times S$. The existence and internal profile for such defect fields will depend on the worldvolume fluxes threading the four-cycles S an S', restricted to the curve Σ. By analogy with the setup of section 7.1.1 let us consider a $U(2)$ split gauge bundle $\mathbb{V} = \mathcal{L} \otimes \mathcal{Q} \oplus \mathcal{L}^{-1} \otimes \mathcal{Q}$ threading S, and a line bundle $\mathcal{N}$ threading S'. It is easy to see that the defect fields are only sensitive to following combination of restricted worldvolume bundles

$$\hat{\mathcal{L}}_3 := \mathcal{L}\big|_\Sigma \qquad\qquad \hat{\mathcal{L}}_8 := \mathcal{Q}\big|_\Sigma \otimes \mathcal{N}^{-1}\big|_\Sigma \tag{7.19}$$

together with the canonical bundle on Σ. In particular we have that [14, 67]

$$\sigma_1 \in \Gamma\left(K_\Sigma^{1/2} \otimes \hat{\mathcal{L}}_3 \otimes \hat{\mathcal{L}}_8\right), \qquad \sigma_2 \in \Gamma\left(K_\Sigma^{1/2} \otimes \hat{\mathcal{L}}_3^{-1} \otimes \hat{\mathcal{L}}_8\right) \tag{7.20}$$

$$\sigma_1^c \in \Gamma\left(K_\Sigma^{1/2} \otimes \hat{\mathcal{L}}_3^{-1} \otimes \hat{\mathcal{L}}_8^{-1}\right), \qquad \sigma_2^c \in \Gamma\left(K_\Sigma^{1/2} \otimes \hat{\mathcal{L}}_3 \otimes \hat{\mathcal{L}}_8^{-1}\right). \tag{7.21}$$

In terms of the enhancement group $G_\Sigma = U(3)$, the bundles (7.19) can be related to the canonical generators of the $\mathfrak{su}(3)$ Cartan subalgebra or its complexification $\mathfrak{sl}(3)$, see eq.(G.3). In this sense, one can arrange the different defect fields as entries of the fundamental representation of the $\mathfrak{sl}(3)$ algebra, namely

$$\begin{pmatrix} 0 & m & \sigma_1 \\ p & 0 & \sigma_2 \\ \sigma_1^c & \sigma_2^c & 0 \end{pmatrix} . \tag{7.22}$$

For completeness, we have also added the modes m and p that extend along the bulk of S, and that correspond to elements of $\Gamma(S, \mathcal{M})$ and $\Gamma(S, \mathcal{P})$, respectively, with $\mathcal{M} \equiv \mathcal{L}^2 \otimes K_S$ and $\mathcal{P} \equiv \mathcal{L}^{-2} \otimes K_S$ defined as above. Notice that there is no field charged under the trace of $\mathfrak{u}(3)$, which completely decouples from the remaining degrees of freedom and will not play any rôle in the following. In this sense, one may treat this system as a bulk theory with $G_S = U(2)$, coupled to a defect theory with enhanced symmetry group $G_\Sigma = SU(3)$.

The complex outer product reads

$$\begin{aligned} \langle\!\langle \sigma^c, \sigma \rangle\!\rangle_{\mathfrak{u}(2)} &= \sigma_1 \sigma_2^c \, T_+ + \sigma_2 \sigma_1^c \, T_- \qquad (7.23) \\ &+ \frac{1}{2} \left(\sigma_1 \sigma_1^c - \sigma_2 \sigma_2^c \right) T_3 + \frac{1}{2} \left(\sigma_1 \sigma_1^c + \sigma_2 \sigma_2^c \right) \mathbf{1}_2 \\ &= \begin{pmatrix} \sigma_1 \sigma_1^c & \sigma_1 \sigma_2^c \\ \sigma_2 \sigma_1^c & \sigma_2 \sigma_2^c \end{pmatrix} \qquad (7.24) \end{aligned}$$

where in the second line we have expressed it in the fundamental representation of $\mathfrak{sl}(2)$. Recall that each of these entries will generate a pole for the Higgs field along the corresponding $\mathfrak{u}(2)$ generator. Having poles along the diagonal entries would correspond to a recombination between the two stacks of D7-branes, and would depart from the $SU(2)$ T-brane profiles of section 7.1.1. Therefore, in order to reproduce $SU(2)$ T-branes we are left with four possibilities:

$$\sigma_1 = \sigma_2 = 0 \, , \qquad \sigma_1^c = \sigma_2^c = 0 \, , \tag{7.25}$$

$$\sigma_1 = \sigma_2^c = 0 \, , \qquad \sigma_1^c = \sigma_2 = 0 \, . \tag{7.26}$$

Notice that for either choice in (7.25), the product (7.24) vanishes identically. As a result, in the holomorphic gauge, the Higgs field Φ needs to have a holomorphic profile, just as in the absence of defects. On the contrary, for either choice in (7.26) Φ will develop a pole in one of its off-diagonal entries. As a consequence, Φ should be described by a meromorphic profile with a pole on top of the defect locus Σ. In the following we will discuss each of these two possibilities separately, and see how either of them may give rise to an $SU(2)$ T-brane background on S, even when S is a four-cycle of positive curvature.

The holomorphic scheme

Let us first consider the case (7.25), with the particular choice $\sigma_1^c = \sigma_2^c = 0$. The BPS equations on the four-cycle S are given by

$$\overline{\partial}_{\mathbb{A}}\Phi_S = 0\,, \tag{7.27}$$

$$\mathbb{F}_S^{(0,2)} = 0\,, \tag{7.28}$$

$$J\wedge\mathbb{F}_S + \frac{1}{2}[\Phi_S, \Phi_S^\dagger] = -\frac{1}{2}J\wedge\delta_\Sigma\,\mu\,. \tag{7.29}$$

where the real moment map expressed in the fundamental representation of $\mathfrak{sl}(2)$ is

$$\mu = h_\Sigma^{-1/2} h_8 \begin{pmatrix} h_3|\sigma_1|^2 & h_3\bar{\sigma}_1\sigma_2 \\ h_3^{-1}\bar{\sigma}_2\sigma_1 & h_3^{-1}|\sigma_2|^2 \end{pmatrix}\,. \tag{7.30}$$

Here $h_3 = h_{\mathcal{L}}|_\Sigma$ and $h_8 = h_{\mathcal{Q}}h_{\mathcal{N}}^{-1}|_\Sigma$ are the metrics for the bundles $\hat{\mathcal{L}}_3$ and $\hat{\mathcal{L}}_8$ in (7.19), defined from the restriction of the metrics $h_{\mathcal{Q}}$, $h_{\mathcal{L}}$ and $h_{\mathcal{N}}$ of the line bundles $\mathcal{Q}$, $\mathcal{L}$ and $\mathcal{N}$, respectively. Since we are assuming a split bundle $\mathbb{V}$ on S, the lhs of (7.29) has vanishing off-diagonal elements, and the same must be true for its rhs. From (7.30) we see that this can be achieved by either setting $\sigma_1 = 0$ or $\sigma_2 = 0$. In manifolds of positive curvature, the appropriate choice is linked to the profile for Φ_S.

Indeed, let us consider that Φ_S is given by (7.3). Then, if we write $\mathbb{F}_S = F_0\mathbf{1}_2 + F_3T_3$, the D-term equation (7.29) amounts to

$$\begin{aligned} J\wedge(F_0+F_3) &= -\frac{1}{2}h_{\mathcal{L}}^2\mathbf{m}\wedge\bar{\mathbf{m}}\,, & (7.31)\\ J\wedge(F_0-F_3) &= \frac{1}{2}h_{\mathcal{L}}^2\mathbf{m}\wedge\bar{\mathbf{m}} - \frac{h_8}{2h_3h_\Sigma^{1/2}}|\sigma_2|^2 J\wedge\delta_\Sigma\,, & (7.32)\end{aligned}$$

where we have set $\sigma_1 = 0$. On the other hand, the BPS conditions on S' read

$$\overline{\partial}\Phi_{S'} = 0\ , \tag{7.33}$$

$$F_{S'}^{(0,2)} = 0\ , \tag{7.34}$$

$$J\wedge F_{S'} = \frac{h_8}{2h_3h_\Sigma^{1/2}}|\sigma_2|^2 J\wedge\delta_\Sigma\,. \tag{7.35}$$

As an immediate consequence of these equations we have that

$$2\int_S J\wedge F_0 + \int_{S'} J\wedge F_{S'} = 0\,, \tag{7.36}$$

and so Fayet-Iliopoulos term for the center-of-mass $U(1)$, the one that would correspond to the trace of $\mathfrak{u}(3)$, vanishes identically. This is consistent with the fact that there is no field charged under this $U(1)$, as pointed out before. In other words, eqs. (7.31), (7.32) and (7.35) can be understood as D-term equations for the pair of Cartan generators of $\mathfrak{su}(3)$. They can be understood as a Laplace equation as follows. We consider the linear combination of the two equations that determines the flux F_3, which is given by

$$J\wedge F_3 = -\frac{1}{2}h_{\mathcal{L}}^2\,\mathbf{m}\wedge\bar{\mathbf{m}} + \frac{h_8}{4h_3h_\Sigma^{1/2}}|\sigma_2|^2 J\wedge\delta_\Sigma. \tag{7.37}$$

Now, similarly to (7.2) we may decompose the flux as

$$F_3 \equiv F_p^{\rm h} + \frac{c}{4}J - i\mathrm{p}\overline{\partial}g, \tag{7.38}$$

where $F_p^{\rm h}$ is primitive and harmonic, c is a constant and g is a function. We can now make use of the identity $2i\mathrm{p}\overline{\partial}g\wedge J = *\Delta g$, and the Poincaré-Lelong formula $\delta_\Sigma = \mathrm{dd}^c \log|n'|^2$, with n' the embedding of S' into S, to rewrite (7.37) as

$$\Delta g = c + \frac{h_{\mathcal{L}}^2}{h_S}|m|^2 - \frac{h_8}{h_3h_\Sigma^{1/2}}|\sigma_2|^2\Delta\log|n'|. \tag{7.39}$$

where $|m|^2 = h_S * \mathbf{m}\wedge\bar{\mathbf{m}}$. In terms of integrals, their solution is given by

$$\xi_3 = \int_S J\wedge F_3 \;=\; -\|m\|^2 + \frac{1}{2}\|\sigma_2\|^2\,, \tag{7.40}$$

$$\xi_0 = \int_S J\wedge F_0 \;=\; -\frac{1}{2}\|\sigma_2\|^2\,, \tag{7.41}$$

where

$$\|m\|^2 = \frac{1}{2}\int_S h_{\mathcal{L}}^2\,\mathbf{m}\wedge\bar{\mathbf{m}}\,, \qquad \|\sigma_2\|^2 = \frac{1}{2}\int_\Sigma \frac{h_8}{2h_3h_\Sigma^{1/2}}|\sigma_2|^2 J\,. \tag{7.42}$$

Notice that, whenever S has positive curvature, the existence of the holomorphic section m implies that the lhs of (7.40) must be positive. Then, by appropriately tuning the vev of the defect field σ_2, one can find a solution for this system even for this case. Had we chosen instead that $\sigma_2 = 0$, the above solution would be replaced by one of the form $\int_S J\wedge F_3 = -\|m\|^2 - \frac{1}{2}\|\sigma_1\|^2$ and $\int_S J\wedge F_0 =$

$-\frac{1}{2}\|\sigma_1\|^2$, and there would be no actual solution in positive curvature manifolds. The rôles of σ_1 and σ_2 are reversed if we choose the profile $\Phi_S = pT_-$. Finally, as similar set of solutions can be achieved if in (7.25) we choose that $\sigma_1 = \sigma_2 = 0$. In particular, σ_1^c, σ_2^c play the rôle of σ_2, σ_1 in the above discussion, respectively.

It is interesting to compare the set of solutions with only $\sigma_2 \neq 0$ to the $SU(3)$ T-brane model discussed in section 7.1.1. Indeed, from a $SU(3)$ viewpoint we are turning on vevs for a pair of fields (m_1, m_2 in one case and m, σ_2 in the other) with exactly the same charges, as can be seen from comparing (7.9) with (7.22). From a 4d viewpoint, this implies that these fields have the same charges under the $U(1) \times U(1)$ that survives as a gauge symmetry when worldvolume fluxes are primitive. As a result their D-term potential is the same, as can be seen explicitly by rewriting (7.40) and (7.41) as

$$\xi_3 + \xi_0 = -\|m\|^2 \qquad\qquad 2\xi_0 = -\|\sigma_2\|^2 \tag{7.43}$$

which corresponds to (7.10) with $p_{12} = 0$.

Despite their similarity, in one case we have a no-go theorem preventing S to have positive curvature, while in the other this obstruction is absent. From the viewpoint of the no-go proof for $SU(3)$, the difference relies on the effectiveness constraints (7.11), which are modified in the holomorphic scheme. Indeed, while the analogue of (7.11a) is still valid in this defect scheme, due to the existence of the bulk mode m, eq.(7.11b) is dramatically modified. Instead of a positivity condition on S we will have a condition on the degree of the corresponding bundle on the defect curve Σ. Indeed, the fact that the defect fields satisfy the F-term equation

$$\bar{\partial}_{\mathbb{A}+\mathbb{A}'}\sigma = 0 \tag{7.44}$$

implies that, in the holomorphic gauge, σ_2 is a holomorphic section of Σ. Its existence then imposes the following condition

$$deg\, \hat{\mathcal{L}}_8 - deg\, \hat{\mathcal{L}}_3 \geq 1 - g_\Sigma\,, \tag{7.45}$$

where g_Σ is the genus of Σ. When going from an $SU(3)$ configuration to the above holomorphic defect scheme, eq.(7.11b) is replaced by (7.45). Since in general the latter is neither related to the Fayet-Iliopoulos terms (7.43) nor to the canonical bundle of S, one cannot deduce the first inequality in (7.13), and the no-go theorem is evaded. An explicit example of a surface S with positive curvature and endowed with bundles satisfying the effectiveness conditions (7.45) and the analogue of (7.11a) is constructed in Appendix I.

Of course, in the case where S' is homotopic to S, Σ is a self-intersection curve and (7.45) and (7.11b) can be related. Indeed, one can always see the bundles under which the defect fields are charged as bundles in S restricted to the curve Σ, namely

$$K_\Sigma^{1/2} \simeq K_S|_\Sigma \,, \qquad \hat{\mathcal{L}}_3 = \mathcal{L}_3|_\Sigma \,, \qquad \hat{\mathcal{L}}_8 = \mathcal{L}_8|_\Sigma \,, \tag{7.46}$$

with $\mathcal{L}_3$, $\mathcal{L}_8$ defined on S. In terms of the bundles $\mathcal{L}_1$, $\mathcal{L}_2$ defined below (7.9) we have that $\mathcal{L}_3 \simeq \mathcal{L}_1 \otimes \mathcal{L}_2^{-1/2}$ and $\mathcal{L}_8 \simeq \mathcal{L}_2^{3/2}$, so considering the bundle

$$K_S \otimes \mathcal{L}_3^{-1} \otimes \mathcal{L}_8 \simeq K_S \otimes \mathcal{L}_1 \otimes \mathcal{L}_2^{-2} \tag{7.47}$$

we may assume that σ_2 is the restriction of one of its sections. This promotes the condition (7.45) to the stronger one (7.11b), and so the inequality (7.13) must be satisfied. We will analyse in greater detail the relation between homotopic four-cycles and defects in self-intersection curves in section 7.2.

The meromorphic scheme

Let us now turn to the case (7.26), and for concreteness take the choice $\sigma_1 = \sigma_2^c = 0$. Now the BPS equations on the four-cycle S are given by

$$\overline{\partial}_{\mathbb{A}} \Phi_S = \delta_\Sigma \wedge \sigma_2 \sigma_1^c T_- \,, \tag{7.48}$$

$$\mathbb{F}_S^{(0,2)} = 0 \,, \tag{7.49}$$

$$J \wedge \mathbb{F}_S + \frac{1}{2}[\Phi_S, \Phi_S^\dagger] = -\frac{1}{2} J \wedge \delta_\Sigma \, \mu \,. \tag{7.50}$$

with the real moment map given by

$$\mu = h_\Sigma^{-1/2} h_3^{-1} \begin{pmatrix} -h_8^{-1}|\sigma_1^c|^2 & 0 \\ 0 & h_8|\sigma_2|^2 \end{pmatrix} \,. \tag{7.51}$$

again expressed in the fundamental representation of $\mathfrak{sl}(2)$. Notice that in this case keeping both defect fields with a non-trivial vev is compatible with a split $U(2)$ bundle, and in particular a split $SU(2)$ bundle if we restrict ourselves to $h_8^{-1}|\sigma_1^c| = h_8|\sigma_2|$. On the other hand, the non-trivial source term for the Higgs field F-term suggests that Φ_S has to be of the form (7.7), with at least $\mathbf{p} \neq 0$. Since we have a split bundle $\mathbb{V}$, this mode needs to satisfy the F-term equation

$$\bar{\mathrm{p}}_A \mathbf{p} \;=\; \delta_\Sigma \wedge \sigma_2 \sigma_1^c \tag{7.52}$$

and so, in the holomorphic gauge, has the profile of a meromorphic section of $\mathcal{P} \equiv \mathcal{L}^{-2} \otimes K_S$. As such, $\mathcal{P}$ does not necessarily need to be effective. Instead, the only requirement that needs to satisfy is containing meromorphic sections with poles of some order. More precisely, if v is the divisor function of Σ on S, then we have the identity

$$\bar{\mathrm{p}} \left(\frac{\mathrm{p} v}{v^l} \right) = \frac{2\pi i}{v^{l-1}} \delta_\Sigma \tag{7.53}$$

from which we infer that the pole must be of first order. Therefore, in the absence of holomorphic sections for $\mathcal{P}$, the effectiveness constraint corresponding to the existence of p is given by

$$\int_S J \wedge c_1(\mathcal{P}) + A(\Sigma) = \int_S J \wedge (c_1(K_S) - 2c_1(\mathcal{L})) + A(\Sigma) \geq 0 \tag{7.54}$$

with $A(\Sigma) = \int_\Sigma J$. Finally, if the mode m in (7.7) exists, it must correspond to a holomorphic section, and so the effectiveness constraints (7.4) applies. Notice that, as before, the product mp transforms as a section of $H^0(S, K_S^2)$, but now the fact that it is meromorphic is not in conflict with S being a manifold of positive curvature.

To build the meromorphic scheme, we will assume that $\mathcal{P}$ only has meromorphic sections, so that (7.54) applies, and that $\mathcal{M}$ may contain holomorphic sections, in which case (7.4) would apply. Notice that this implies that both defect fields σ_2 and σ_1^c have a non-trivial vev. Writing again $\mathbb{F}_S = F_0 \mathbf{1}_2 + F_3 T_3$, the D-terms that correspond to this scenario are

$$J \wedge F_3 \;=\; -\frac{1}{2} h_{\mathcal{L}}^2 \mathbf{m} \wedge \bar{\mathbf{m}} + \frac{1}{2} h_{\mathcal{L}}^{-2} \mathbf{p} \wedge \bar{\mathbf{p}} + \frac{1}{4 h_3 h_\Sigma^{1/2}} \left(h_8 |\sigma_2|^2 + h_8^{-1} |\sigma_1^c|^2 \right) J \wedge \delta_\Sigma \tag{7.55}$$

$$J \wedge F_0 \;=\; -\frac{1}{4 h_3 h_\Sigma^{1/2}} \left(h_8 |\sigma_2|^2 - h_8^{-1} |\sigma_1^c|^2 \right) J \wedge \delta_\Sigma \,, \tag{7.56}$$

and the BPS conditions on S' read

$$\overline{\partial} \Phi_{S'} = 0 \;, \tag{7.57}$$

$$F_{S'}^{(2,0)} = 0 \;, \tag{7.58}$$

$$J \wedge F_{S'} = \frac{1}{2 h_3 h_\Sigma^{1/2}} \left(h_8 |\sigma_2|^2 - h_8^{-1} |\sigma_1^c|^2 \right) J \wedge \delta_\Sigma \,. \tag{7.59}$$

As in the holomorphic scheme, we have that the relation (7.36) holds. In this case the solution to the two independent D-term equations is given by

$$\xi_3 = \int_S J \wedge F_3 \quad = \quad \|p\|^2 - \|m\|^2 + \frac{1}{2}\left(\|\sigma_1^c\|^2 + \|\sigma_2\|^2\right) \ , \tag{7.60}$$

$$\xi_0 = \int_S J \wedge F_0 \quad = \quad \frac{1}{2}\left(\|\sigma_1^c\|^2 - \|\sigma_2\|^2\right) \ , \tag{7.61}$$

with the definitions (7.42) and

$$\|p\|^2 = \frac{1}{2}\int_S h_{\mathcal{L}}^{-2}\mathbf{p} \wedge \bar{\mathbf{p}} \ , \qquad \|\sigma_1^c\|^2 = \frac{1}{2}\int_\Sigma \frac{|\sigma_1^c|^2 J}{2h_3 h_8 h_\Sigma^{1/2}} \, . \tag{7.62}$$

Clearly, the simplest set of solutions correspond to those where $\|\sigma_1^c\| = \|\sigma_2\|$ and $m = 0$, so that necessarily $\xi_3 > 0$. Notice that such a FI sign, together with the effectiveness constraints (7.4) and (7.54) imply that

$$\int_S J \wedge c_1(\mathcal{M}) > \int_S J \wedge c_1(K_S) > -A(\Sigma) \, , \tag{7.63}$$

which are in principle compatible with manifolds of positive curvature. In general, we expect to find solutions satisfying (7.63) for values of the defect fields σ_1^c, σ_2 and m such that ξ_3 is positive and not excessively large. Since the product $\sigma_1^c\sigma_2$ sources the meromorphic profile for p, one would presume that its contribution to the D-term is fixed by their value. The analysis of sections 7.2 and 7.3 will provide a more precise picture to this expectation. Finally, when compared to the the $SU(3)$ T-brane model discussed in section 7.1.1 we get a very similar set of D-term equations

$$\xi_3 + \xi_0 = -\|m\|^2 + \|\sigma_1^c\|^2 + \|p\|^2 \qquad 2\xi_0 = -\|\sigma_2\|^2 + \|\sigma_1^c\|^2 \tag{7.64}$$

which is essentially (7.10) with the dictionary $(m_1, m_2, p_{12}) \to (m, \sigma_2, \sigma_1^c)$ and the addition of the contribution from p. One can check that adding a contribution of this form to (7.10) would not change the results below, in the sense that (7.13) would still be valid and positive curvature manifolds excluded. Again, the fact that we may construct T-brane backgrounds with S of positive curvature using the meromorphic scheme is due to the different effectiveness constraints imposed by this class of constructions. These would be (7.4), (7.54) and those related to σ_1^c, σ_2 being holomorphic sections of Σ

$$deg\,\hat{\mathcal{L}}_3 \leq g_\Sigma - 1 \, , \qquad deg\,\hat{\mathcal{L}}_8 \leq g_\Sigma - 1 - \deg\hat{\mathcal{L}}_3 \, . \tag{7.65}$$

As before, these conditions are unrelated to the values of ξ_3 and ξ_8, except in some specific cases like when Σ is the self-intersection curve of S, and we assume that σ_2 and σ_1^c are the restriction of holomorphic sections of the corresponding bundles on S. An explicit construction of positive curvature surface satisfying (7.65) can be found in Appendix I.

7.2 Defects and Hitchin systems

As described in the previous section, defect fields arise at the intersection of two stacks of 7-branes wrapping holomorphic four-cycles S and S'. One particular case is when S has an effective canonical bundle K_S, and S' is a homotopic deformation of S. The curve Σ hosting the defects is, in this case, the self-intersection curve of S, which represents the Poincaré dual of $c_1(K_S)$. Interestingly, the enhancement group $G_\Sigma \supset G_S \times G_{S'}$ over this curve can now be extended to the whole of S, in the sense that this is the symmetry group of the system when S and S' coincide. Therefore the information of the whole system, including the defects, should be contained in a Hitchin system with group G_Σ, and the BPS equations with defects discussed above should be recovered in the limit in which the intersection fields are ultra-localised at Σ.

In the following we would like to explore the dictionary between Hitchin systems on self-intersecting curves and systems with defects in further detail, in order to understand how to recover the latter from the former. To simplify our discussion we will consider a setup with enhanced gauge group $G_\Sigma = SU(3)$, in which two D7-branes wrap S and a third one its homotopic deformation S'. This will allow to easily connect with the simple defect setup analysed in the previous section, and in particular with the holomorphic and meromorphic schemes discussed there. As we will see, from the viewpoint of the $SU(3)$ Hitchin system these two configurations are not that different.

7.2.1 The meromorphic scheme

Let us consider a Hitchin system with gauge group $SU(3)$, defined on a surface S with effective canonical bundle K_S. We introduce a Higgs field which in the holomorphic gauge reads

$$\Phi^{\rm h} = \frac{1}{3}\begin{pmatrix} \mathbf{v} & 0 & 0 \\ 0 & \mathbf{v} & 0 \\ 0 & 0 & -2\mathbf{v} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{s}_2 \\ \mathbf{s}_1^c & 0 & 0 \end{pmatrix} . \tag{7.66}$$

On the one hand, $\mathbf{v} \in H^{(2,0)}(S)$ corresponds to a holomorphic deformation of the cycle S. The piece of Φ proportional to $\mathbf{v}$ has the effect of separating the initial $SU(3)$ stack into two stacks of two and one 7-branes, each wrapping surfaces homotopic to S and intersecting at the curve $\Sigma = \{\mathbf{v} = 0\} \subset S$. On the other hand, the (2,0)-forms $\mathbf{s}_2$, $\mathbf{s}_1^c$ can be considered to be sections of line bundles on S. Indeed, notice that in the limit of coincident 7-branes $\mathbf{v} \to 0$ the system reduces to the $SU(3)$ system in (7.9), upon the identifications $\mathbf{m}_2 \leftrightarrow \mathbf{s}_2$, $\mathbf{p}_{12} \leftrightarrow \mathbf{s}_1^c$. There we may consider a split gauge bundle with a corresponding worldvolume flux of the form

$$\mathbb{F} = F_3\, H_3 + F_8\, H_8 = \begin{pmatrix} F_3 + \frac{1}{3}F_8 & 0 & 0 \\ 0 & -F_3 + \frac{1}{3}F_8 & 0 \\ 0 & 0 & -\frac{2}{3}F_8 \end{pmatrix} , \tag{7.67}$$

where the $\mathfrak{su}(3)$ Cartan generators H_3, H_8 are defined in terms of the canonically normalised ones in (G.3) as $H_3 = 2H_1$ and $H_8 = \frac{2}{\sqrt{3}}H_2$. As usual, the $(1,1)$-forms F_i, $i = 3, 8$ are related to the corresponding line bundles as $[F_i] = 2\pi c_1(\mathcal{L}_i)$. The particular choice of flux in (7.67) allows to relate the corresponding bundles with the pair $\hat{\mathcal{L}}_3$, $\hat{\mathcal{L}}_8$ that appear in the defect schemes of subsection 7.1.2, or more precisely to identify them with the extensions $\mathcal{L}_3$, $\mathcal{L}_8$ introduced around eq.(7.46). Using the relation specified there with the bundles $\mathcal{L}_1$, $\mathcal{L}_2$ that correspond to the flux in (7.9) one finds that

$$\mathbf{s}_2 \in H^{2,0}\left(\mathcal{L}_3^{-1} \otimes \mathcal{L}_8\right) , \qquad \mathbf{s}_1^c \in H^{2,0}\left(\mathcal{L}_3^{-1} \otimes \mathcal{L}_8^{-1}\right) . \tag{7.68}$$

As we turn on the four-cycle deformation $\mathbf{v}$, the flux (7.67) will no longer yield a solution to the D-term equation (7.1c), and we will need to consider a non-split bundle. In general, for non-split bundles one may not identify individual entries of Φ as sections of line bundles as done above. However, as in our case the no-split bundle is continuously connected to a split one in the limit $\mathbf{v} \to 0$, one may impose (7.68) for arbitrary values of $\mathbf{v}$.

The information of the non-split bundle is encoded in the complexified gauge transformation that allows to solve the D-term equations. Let us take it to be of the form

$$B = \begin{pmatrix} e^{f_3/2+f_8/6} & 0 & 0 \\ 0 & e^{-f_3/2+f_8/6} & 0 \\ 0 & 0 & e^{-f_8/3} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2}\xi_1^c\xi_2 & 1 & -\xi_2 \\ \xi_1^c & 0 & 1 \end{pmatrix} \tag{7.69}$$

with $\xi_1^c \in \Gamma(\mathcal{L}_3^{-1} \otimes \mathcal{L}_8^{-1})$ and $\xi_2 \in \Gamma(\mathcal{L}_3^{-1} \otimes \mathcal{L}_8)$. The unitary gauge Higgs field then is

$$\Phi = B\Phi^{\rm h}B^{-1} = \frac{1}{3}\mathbf{v}\mathbb{I}_3 + \begin{pmatrix} 0 & 0 & 0 \\ -e^{-f_3}\left(\mathbf{s}_1^c\xi_2 + \xi_1^c\mathbf{s}_2 + \mathbf{v}\xi_1^c\xi_2\right) & 0 & e^{-\frac{f_3}{2}+\frac{f_8}{2}}\left(\mathbf{s}_2 + \mathbf{v}\xi_2\right) \\ e^{-\frac{f_3}{2}-\frac{f_8}{2}}\left(\mathbf{s}_1^c + \mathbf{v}\xi_1^c\right) & 0 & -\mathbf{v} \end{pmatrix}, \tag{7.70}$$

whose individual entries are globally well-defined (2,0)-forms in S. One the one hand, one expects that the sections ξ_1^c, ξ_2 vanish in the limit $\mathbf{v} \to 0$. On the other hand, as we increase the vev of the deformation $\mathbf{v}$, they should implement the localisation of the unitary profile for the fields $\mathbf{s}_1^c$, $\mathbf{s}_2$ along Σ. We find that an appropriate choice to reproduce both features is

$$\xi_1^c = \frac{s_1^c}{v}\left(e^{-\lambda|v|^2} - 1\right), \qquad \xi_2 = \frac{s_2}{v}\left(e^{-\lambda|v|^2} - 1\right), \tag{7.71}$$

where v, s_1^c, s_2 are the scalar holomorphic sections that correspond to $\mathbf{v}$, $\mathbf{s}_1^c$, $\mathbf{s}_2$. In addition, λ is of the form

$$\lambda = \frac{\lambda_\star}{\sqrt{|g_S|}} \tag{7.72}$$

with $\lambda_\star$ a globally well-defined function of S that, for most purposes of the discussion below, can be considered to be a constant. Notice that away from the self-intersection locus $\Sigma = \{v = 0\}$ the exponential factor in (7.71) can be neglected, and ξ_1^c, ξ_2 become the entries that take (7.66) into its Jordan canonical form. Near Σ the exponential becomes relevant and renders ξ_1^c, ξ_2 regular. In fact, they both vanish at $v = 0$, so their effect on $\Phi^{\rm h}$ will be irrelevant near this locus. Indeed, plugging (7.71) into (7.70) one obtains

$$\Phi = \frac{1}{3}\mathbf{v}\mathbb{I}_3 + \begin{pmatrix} 0 & 0 & 0 \\ e^{-f_3}\frac{s_1 s_2^c}{v^2}\left(1 - e^{-2\lambda|v|^2}\right)\mathbf{v} & 0 & e^{-\frac{f_3}{2}+\frac{f_8}{2}}e^{-\lambda|v|^2}\mathbf{s}_2 \\ e^{-\frac{f_3}{2}-\frac{f_8}{2}}e^{-\lambda|v|^2}\mathbf{s}_1^c & 0 & -\mathbf{v} \end{pmatrix}, \tag{7.73}$$

which displays a clear localisation of the fields s_1^c, s_2 around the self-intersection locus via the exponential factor $e^{-\lambda|v|^2}$. In fact, the entries for such fields corresponds to the wavefunction profile along the Higgs field component that one would expect for the fluctuation fields at the intersection of two 7-branes, cf. [23, 24, 26, 28]. The remaining off-diagonal entry is also localised around Σ but unexpected from the viewpoint of such a wavefunction analysis, which only detects up to a linear dependence on intersection fields. We will however see below that it corresponds to the appearance of a pole in the meromorphic defect scheme.

In this unitary gauge the $\mathfrak{su}(3)$ gauge connection is given by

$$i\mathbb{A}^{(0,1)} = -B\,\bar{\mathrm{p}}\,B^{-1} = \begin{pmatrix} \frac{1}{2}\bar{\mathrm{p}}f_3 + \frac{1}{6}\bar{\mathrm{p}}f_8 & 0 & 0 \\ 0 & -\frac{1}{2}\bar{\mathrm{p}}f_3 + \frac{1}{6}\bar{\mathrm{p}}f_8 & 0 \\ 0 & 0 & \frac{1}{3}\bar{\mathrm{p}}f_8 \end{pmatrix} \tag{7.74}$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ e^{-f_3}\frac{1}{2}\left(\xi_1^c\bar{\mathrm{p}}\xi_2 - \xi_2\bar{\mathrm{p}}\xi_1^c\right) & 0 & e^{-\frac{f_3}{2}+\frac{f_8}{2}}\bar{\mathrm{p}}\xi_2 \\ -e^{-\frac{f_3}{2}-\frac{f_8}{2}}\bar{\mathrm{p}}\xi_1^c & 0 & 0 \end{pmatrix},$$

which after plugging the Ansatz (7.71) becomes

$$i\mathbb{A}^{(0,1)} = \frac{1}{2}H_3\bar{\mathrm{p}}f_3 + \frac{1}{2}H_8\bar{\mathrm{p}}f_8 \tag{7.75}$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -e^{-\frac{f_3}{2}+\frac{f_8}{2}}s_2 \\ e^{-\frac{f_3}{2}-\frac{f_8}{2}}s_1^c & 0 & 0 \end{pmatrix} e^{-\lambda|v|^2}\bar{\mathrm{p}}\left(\lambda\bar{v}\right).$$

Notice that as expected the $\mathfrak{su}(3)$ bundle is not split but, due to this particular Ansatz, we recover a split bundle if we restrict ourselves to the $\mathfrak{u}(2)$ subalgebra that contains H_3. This is in agreement with the simple defect setup discussed in section 7.1.2, and in particular with the meromorphic scheme that we are trying to reproduce. In addition, note that the off-diagonal entries in (7.75) reproduce the expected wavefunction profile along the gauge boson components for the fluctuations of fields localised at 7-brane intersections.

Given the profiles for Φ and $\mathbb{A}$ in the unitary gauge, the next step is to introduce them into the D-term equation (7.1c) to find a solution in terms of f_3, f_8 and λ. For simplicity, let us consider the particular case where s_1^c and s_2 differ by a constant phase, so that we can rewrite the D-term equations in terms of a section s such that $s = e^{-i\varphi_1}s_1^c = e^{-i\varphi_2}s_2$. This can only be a solution if the bundle $\mathcal{L}_8$ is trivial, so we may take $f_8 = 0$. Then, one can see that the following structure is recovered

$$J \wedge \mathbb{F} + \frac{1}{2}[\Phi, \Phi^\dagger] = \begin{pmatrix} C & 0 & e^{-i\varphi_1}\bar{D} \\ 0 & -C & e^{i\varphi_2}D \\ e^{i\varphi_1}D & e^{-i\varphi_2}\bar{D} & 0 \end{pmatrix} \tag{7.76}$$

and the D-term equation reduces to two independent differential equations $C = D = 0$ with unknowns f_3 and λ. The off-diagonal components of (7.76) vanish if one imposes

$$\begin{aligned} 0 = {} & 2iJ\wedge\mathrm{p}\left(s\, e^{-f_3-\lambda|v|^2}\bar{\mathrm{p}}\,(\lambda\bar{v})\right) \\ & + e^{-f_3-\lambda|v|^2}\left(\mathbf{s}\wedge\bar{\mathbf{v}} + e^{-f_3}\mathbf{s}\wedge\bar{\mathbf{s}}\frac{s}{v}\left(e^{-2\lambda|v|^2}-1\right)\right), \end{aligned} \tag{7.77}$$

while the vanishing of the diagonal components amounts to

$$\begin{aligned} 0 = {} & iJ\wedge\mathrm{p}\bar{\mathrm{p}}f_3 + e^{-f_3-2\lambda|v|^2}\left(\frac{1}{2}\mathbf{s}\wedge\bar{\mathbf{s}} + |s|^2 iJ\wedge\mathrm{p}(\lambda v)\wedge\bar{\mathrm{p}}(\lambda\bar{v})\right) \\ & + e^{-2f_3}\frac{1}{2}\mathbf{s}\wedge\bar{\mathbf{s}}\left|\frac{s}{v}\right|^2\left(e^{-2\lambda|v|^2}-1\right)^2. \end{aligned} \tag{7.78}$$

Although they look quite formidable, one can simplify these equations in certain limits. For instance, if we consider eq.(7.77) for small values of s we can neglect the cubic term in the lhs and recover

$$e^{-f_3-\lambda|v|^2}\left[2iJ\wedge\left(\mathrm{p}\bar{\mathrm{p}}(\lambda s\bar{v}) - s\mathrm{p}(f_3+\lambda|v|^2)\wedge\bar{\mathrm{p}}(\lambda\bar{v})\right) + \mathbf{s}\wedge\bar{\mathbf{v}}\right] = 0\,. \tag{7.79}$$

This is nothing but the linearised D-term equation $J\wedge\mathrm{p}_{\langle A\rangle}a - \frac{1}{2}[\langle\Phi\rangle^\dagger, \varphi] = 0$ imposed in the literature to solve for the internal wavefunction profile of fields at matter curves, with $\langle\Phi\rangle$, φ the pieces of (7.73) at zeroth and linear order in $\mathbf{s}$, respectively, and similarly for $\langle A\rangle$, a in (7.75). Notice that the prefactor in (7.79) essentially localises the equation along $\Sigma = \{v = 0\}$, so we may focus on a tubular neighbourhood around the self-intersection locus, as done in local wavefunction computations. Note as well that (7.77) is a complex equation, so together with (7.78) we have three real equations for the two real unknowns f_3 and λ. One may see this as a limitation of our Ansatz (7.69) and (7.71), that could be generalised to solve for the most general set of equations. Nevertheless, one may still find solutions with this Ansatz if near Σ one imposes

$$J\wedge\mathrm{p}\left(se^{-f_3}\lambda\bar{\mathrm{p}}\bar{v}\right) = 0\,, \tag{7.80}$$

after which (7.77) becomes a real equation. In fact, under these assumptions the dependence of s disappears from (7.79), and one obtains a much simpler equation. In particular one may connect with the ultra-local wavefunction results by considering a neighbourhood around a point $p \in \Sigma$, and approximating the metric on S to be flat and the 7-brane worldvolume flux to be constant on that neighbourhood. More precisely, if locally we have $v = m_x x$, $J = \frac{i}{2}(dx \wedge d\bar{x} + dy \wedge d\bar{y})$ and $F_3 = \frac{i}{2}(M_x dx \wedge d\bar{x} + M_y dy \wedge d\bar{y})$, (7.79) is solved by a constant λ of the form

$$|m_x|^2 \lambda = -\frac{M_x}{2} \pm \sqrt{\left(\frac{M_x}{2}\right)^2 + |m_x|^2}\,, \tag{7.81}$$

which reproduces the corresponding local wavefunction solutions, cf. eqs.(2.27) and (2.29) of [28]. Notice that in this particular case a constant λ implies, through the first condition in (7.80), that s only depends on the coordinate y of Σ. In this sense, this simple local setup reproduces one of the assumptions of the defect schemes of last section. In the following we will see how to make this connection more precise and how, by taking the appropriate limit, one can connect the Hitchin D-term equation (7.78) with the defect D-term equation (7.55).

7.2.2 The defect limits

As it is clear from the unitary gauge profile for Φ and $\mathbb{A}$, the $SU(3)$ Hitchin system above localises the charged fields s_1^c and s_2 along the self-intersection curve Σ. In a limit in which such localisation can be approximated by a delta function, one would expect that a defect system should be recovered, and the BPS equations of the $SU(3)$ Hitchin system should become the BPS equations of the meromorphic scheme. In general, one would expect that such a limit is obtained when the intersection slope of the two 7-branes becomes infinite. As we will now see, there are in fact two ways to attain such a limit and recover the defect system. One of them corresponds to increase the vev of the holomorphic deformation field v, and the other to decrease the volume of the four-cycle S.

The small volume limit

Let us assume that we have found a solution for the above $SU(3)$ Hitchin system and consider its behaviour under the following rescaling of the four-cycle metric:

$$|g_S| \to a^2 |g_S| \tag{7.82}$$

with $a \in \mathbb{R}$. As we perform this rescaling the wavefunction profiles for Φ and $\mathbb{A}$ are modified, since

$$\lambda|v|^2 = \frac{\lambda_\star |v|^2}{\sqrt{|g_S|}} \longrightarrow \frac{1}{a}\frac{\lambda_\star |v|^2}{\sqrt{|g_S|}} = \frac{1}{a}\lambda|v|^2 \,. \tag{7.83}$$

Taking the limit $a \to 0$ one for instance finds [4]

$$e^{-\lambda|v|^2} \stackrel{a\to 0}{\longrightarrow} 2\left[1 - H(|v|^2)\right] \equiv 1 - H_\Sigma \,, \tag{7.84}$$

where H is the Heaviside step function, using the half-maximum convention in which $H(0) = \frac{1}{2}$, and we have assumed that $\lambda \neq 0$ everywhere in S. By (7.84), H_Σ is a function that vanishes on Σ and is equal to 1 everywhere else on S. As a consequence

$$\begin{aligned}
\Phi_{a\to 0} \quad &= \quad \frac{1}{3}\begin{pmatrix} \mathbf{v} & 0 & 0 \\ 0 & \mathbf{v} & 0 \\ 0 & 0 & -2\mathbf{v} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \frac{s_1 s_2^c}{v^2}\mathbf{v} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} e^{-f_3} H_\Sigma \\
&+ \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{-\frac{f_3}{2}+\frac{f_8}{2}}\mathbf{s}_2 \\ e^{-\frac{f_3}{2}-\frac{f_8}{2}}\mathbf{s}_1^c & 0 & 0 \end{pmatrix} [1 - H_\Sigma] \,.
\end{aligned} \tag{7.85}$$

Notice that only the first line of (7.85) survives away from Σ, while the second line is fully localised on top of Σ as the corresponding, defect fields in the meromorphic scheme. The surviving off-diagonal component is very suggestive in the sense that, again away from Σ, corresponds to the naive solution to the meromorphic defect equation (7.48).

Now, considering the gauge field in this limit, we have that

$$e^{-\lambda|v|^2}\bar{\mathrm{p}}\,(\lambda\bar{v}) \longrightarrow e^{-\frac{\lambda}{a}|v|^2}\bar{\mathrm{p}}\left(\frac{\lambda}{a}\bar{v}\right) \stackrel{a\to 0}{\longrightarrow} \bar{\mathrm{p}}\bar{v}\,\pi\delta^{(2)}(v) \,, \tag{7.86}$$

with $\delta^{(2)}(v)$ the two-dimensional Dirac delta function with support in Σ. One then finds

$$i\mathbb{A}^{(0,1)}_{a\to 0} = \frac{1}{2}H_3\bar{\partial}f_3 + \frac{1}{2}H_8\bar{\partial}f_8 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -e^{-\frac{f_3}{2}+\frac{f_8}{2}}s_2 \\ e^{-\frac{f_3}{2}-\frac{f_8}{2}}s_1^c & 0 & 0 \end{pmatrix} \bar{\partial}\bar{v}\,\pi\delta^{(2)}(v)\ , \tag{7.87}$$

again finding that the profile for the fields s_1^c, s_2 is localised on top of Σ, now in the form of a δ function. Putting these two result together and using the identities

$$\frac{1}{v}\bar{\partial}H(|v|^2) = \bar{\partial}\left(\frac{1}{v}\right) = \pi\delta^{(2)}(v)\bar{\partial}\bar{v}\,, \tag{7.88}$$

one can see that the F-terms vanish identically. This is to be expected, since the field space direction that we are taking to reach this limit does not affect the F-term equations of the Hitchin system. The correct way to extract the F-term (7.48) is to look at the $\mathfrak{su}(3)$ Hitchin system from the viewpoint of the $\mathfrak{su}(2)$ subalgebra of the corresponding defect scheme. Indeed, one may always rewrite (7.73) and (7.75) as

$$\Phi = \frac{1}{3}\mathbf{v}\mathbb{I}_3 + \begin{pmatrix} \Phi_{\mathfrak{su}(2)} & 0 \\ 0 & -\mathbf{v} \end{pmatrix} + \Phi_{def} \tag{7.89}$$

and

$$i\mathbb{A}^{(0,1)} = \begin{pmatrix} i\mathbb{A}_{\mathfrak{su}(2)} & 0 \\ 0 & -\frac{1}{3}\bar{\partial}f_8 \end{pmatrix} + i\mathbb{A}_{def}\ , \tag{7.90}$$

where, after the rescaling (7.82),

$$\Phi_{\mathfrak{su}(2)} = e^{-f_3}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\frac{s_1 s_2^c}{v^2}\mathbf{v}\left(1 - e^{-2\frac{\lambda}{a}|v|^2}\right) \tag{7.91a}$$

$$i\mathbb{A}^{(0,1)}_{\mathfrak{su}(2)} = \frac{1}{2}\bar{\partial}f_3 T_3 + \frac{1}{6}\bar{\partial}f_8\mathbf{1}_2 \tag{7.91b}$$

and

$$\Phi_{def} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{-\frac{f_3}{2}+\frac{f_8}{2}}\mathbf{s}_2 \\ e^{-\frac{f_3}{2}-\frac{f_8}{2}}\mathbf{s}_1^c & 0 & 0 \end{pmatrix} e^{-\frac{\lambda}{a}|v|^2} \tag{7.92a}$$

$$i\mathbb{A}_{def} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -e^{-\frac{f_3}{2}+\frac{f_8}{2}}s_2 \\ e^{-\frac{f_3}{2}-\frac{f_8}{2}}s_1^c & 0 & 0 \end{pmatrix} e^{-\frac{\lambda}{a}|v|^2}\bar{\partial}\left(\frac{\lambda}{a}\bar{v}\right) \tag{7.92b}$$

In terms of these quantities, the F-term (7.1a) of the $\mathfrak{su}(3)$ Hitchin system reads

$$\bar{\mathrm{p}}_{\mathbb{A}}\Phi = \begin{pmatrix} \bar{\mathrm{p}}_{\mathbb{A}_{\mathfrak{su}(2)}}\Phi_{\mathfrak{su}(2)} & 0 \\ 0 & 0 \end{pmatrix} - \Phi_{def} \wedge \bar{\mathrm{p}}\left(\frac{\lambda}{a}\bar{v}\right) v \tag{7.93}$$
$$- i\left[\mathbb{A}_{def}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & -\mathbf{v} \end{pmatrix}\right] - i[\mathbb{A}_{def}, \Phi_{def}]$$

One can check that

$$\Phi_{def} \wedge \bar{\mathrm{p}}\left(\frac{\lambda}{a}\bar{v}\right) v + i\left[\mathbb{A}_{def}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & -\mathbf{v} \end{pmatrix}\right] \equiv 0\,, \tag{7.94}$$

and that

$$i[\mathbb{A}_{def}, \Phi_{def}] = \begin{pmatrix} 0 & 0 & 0 \\ \Xi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \qquad \Xi = 2 s_1^c \mathbf{s}_2 \wedge \bar{\mathrm{p}}\left(\frac{\lambda}{a}\bar{v}\right) e^{-f_3 - 2\frac{\lambda}{a}|v|^2}\,. \tag{7.95}$$

Therefore, satisfying the F-terms for the $\mathfrak{su}(3)$ Hitchin system amounts to impose that

$$\bar{\mathrm{p}}_{\mathbb{A}_{\mathfrak{su}(2)}}\Phi_{\mathfrak{su}(2)} = \Xi\,, \tag{7.96}$$

and taking the limit $a \to 0$ one obtains

$$\Xi \overset{a\to 0}{\longrightarrow} \pi e^{-f_3}\frac{s_1^c s_2}{v}\mathbf{v} \wedge \bar{\mathrm{p}}\bar{v}\,\delta^{(2)}(v)\,. \tag{7.97}$$

The defect F-term (7.48) is recovered from (7.96) and (7.97) upon the identifications

$$\sigma_1^c = e^{-\frac{1}{2}f_3} s_1^c \qquad \sigma_2 = e^{-\frac{1}{2}f_3} s_2 \qquad \ldots \tag{7.98}$$

The large angle limit

Even if the small volume limit reproduces the F-terms of the meromorphic scheme, the D-term equation (7.1c) cannot be trusted in the regime where it applies. One may nevertheless conceive a second limit, which amounts to increase the vev of the intersection field vev v

$$v \to b\,v \tag{7.99}$$

with $b \in \mathbb{R}$, while keeping the four-cycle metric fixed. Taking $b \to \infty$ will ultra-localise the fields at the self-intersection Σ, and so one would expect to recover again the delta function behaviour of the defect scheme. This time, because we are at large volume, it makes sense to try to solve the D-terms as we vary b. In fact, one should impose that the off-diagonal D-terms in (7.76) are identically satisfied as we move along (7.99), because these correspond to the D-term potential for massive fields at the self-intersection. Since such fields are assumed to be very massive and completely integrated out in the regime where the defect picture is valid, one would never attain the defect limit unless one sets their D-terms to zero. For doing so, let us take the simplifying assumption $|s| = |s_1^c| = |s_2|$ that takes us to (7.76) and assume that we have a configuration such that can find a solution of both D-term equations $C = D = 0$ with our Ansatz (7.71). Performing the rescaling (7.99), eq.(7.77) transforms as

$$\begin{aligned} 0 = & \, 2iJ \wedge \big(b\mathrm{p}\bar{\mathrm{p}}(\lambda s \bar{v}) - s\mathrm{p}(f_3 + b^2 \lambda v \bar{v}) \wedge b\bar{\mathrm{p}}(\lambda \bar{v})\big) \\ & + b\mathbf{s} \wedge \bar{\mathbf{v}} + e^{-f_3} \mathbf{s} \wedge \bar{\mathbf{s}} \frac{s}{bv} \left(e^{-2\lambda b^2 |v|^2} - 1\right), \end{aligned} \tag{7.100}$$

where we have discarded overall exponential factors. In the limit $b \to \infty$, we will be able to find a solution only if λ also scales with b in the following form

$$\lambda \to b^{-1} \lambda \tag{7.101}$$

where this should be interpreted as a rescaling of the function $\lambda_\star$ in (7.72) and not of the metric factor therein. Notice that the rescalings (7.99) and (7.101) have the same combined effect on $\lambda |v|^2$ as in (7.83), with the replacement $a^{-1} \to b$, so as in the previous limit we expect a strong localisation for the intersection fields as we reach $b \to \infty$. This time, however, we also need to consider the behaviour of non-holomorphic data like kinetic terms. Indeed, the kinetic term integrand for the intersection fields scale like

$$\begin{aligned} & iJ \wedge [\mathbb{A}_{def}, \mathbb{A}_{def}^\dagger] + \frac{1}{2} [\Phi_{def}, \Phi_{def}^\dagger] \\ & = - \left(i|s|^2 J \wedge \mathrm{p}(\lambda v) \wedge \bar{\mathrm{p}}(\lambda \bar{v}) + \frac{1}{2} \mathbf{s} \wedge \bar{\mathbf{s}} \right) e^{-f_3 - 2\lambda |v|^2} \\ & \to - \left(i|s|^2 J \wedge \mathrm{p}(\lambda v) \wedge \bar{\mathrm{p}}(\lambda \bar{v}) + \frac{1}{2} \mathbf{s} \wedge \bar{\mathbf{s}} \right) e^{-f_3 - 2b\lambda |v|^2} , \end{aligned}$$

and so the kinetic terms will vanish in the limit $b \to \infty$. This can be fixed by rescaling the normalisation factor of the fields at the intersection, which in practice amounts to

$$s \to b^{1/2} s \,. \tag{7.102}$$

Notice that, compared to (7.82), the effect of the combined rescaling (7.99), (7.101) and (7.102) on Φ and $\mathbb{A}$ is slightly different. Nevertheless, the effect on (7.95) is similar, and so we recover the same limiting behaviour (7.97) that reproduces the F-terms of the meromorphic scheme.

Let us now consider the D-term equation, and in particular the non-Cartan components of (7.76). After taking the limit $b \to \infty$ most of its terms vanish automatically, except one proportional to

$$J \wedge \mathrm{p} \left(s e^{-f_3} \lambda \bar{\mathrm{p}} \bar{v} \right)\big|_\Sigma \,. \tag{7.103}$$

As pointed out before, the vanishing of this quantity is what allows to convert $D = 0$ into a real equation and to find solutions for the D-term equations within the Ansatz (7.71). As we are using such an Ansatz to connect with the defect scheme it seems reasonable that, by consistency, we should restrict to configurations where (7.103) vanishes.

Finally, the diagonal component of (7.76) scales as

$$\begin{aligned} & iJ \wedge \mathrm{p}\bar{\mathrm{p}} f_3 + e^{-f_3 - 2b\lambda|v|^2} b|s|^2 \left(\frac{J^2}{4\sqrt{|g_S|}} + iJ \wedge \mathrm{p}(\lambda v) \wedge \bar{\mathrm{p}}(\lambda \bar{v}) \right) \\ & + e^{-2f_3} \frac{1}{2} \mathbf{p} \wedge \bar{\mathbf{p}} \left(e^{-2b\lambda|v|^2} - 1 \right)^2 , \end{aligned} \tag{7.104}$$

where we have defined $\mathbf{p} = \frac{s^2}{v^2} \mathbf{v}$. Taking the limit $b \to \infty$, and assuming that in a neighbourhood of Σ the following relation holds

$$2i\lambda^2 \sqrt{|g_S|} J \wedge \mathrm{p} v \wedge \bar{\mathrm{p}} \bar{v} = \frac{1}{2} J^2 \,, \tag{7.105}$$

we recover the following D-term equation

$$-iJ \wedge \mathrm{p}\bar{\mathrm{p}} f_3 = e^{-2f_3} \frac{1}{2} \mathbf{p} \wedge \bar{\mathbf{p}} \, H_\Sigma + \lambda_\star \frac{e^{-f_3}}{\sqrt{|g_S|}} |s|^2 \, 2\pi \delta_\Sigma \wedge J \,, \tag{7.106}$$

where we have used the identification

$$\delta_\Sigma = \frac{i}{2} \delta^{(2)} \mathrm{p} v \wedge \bar{\mathrm{p}} \bar{v} \,. \tag{7.107}$$

We then see that we recover the D-term equations of the meromorphic scheme (7.55), upon identifying $h_{\mathcal{L}} = h_3 = e^{f_3}$, $\sqrt{|g_S|} = h_\Sigma^{1/2}$ and $4\pi\lambda_\star |s|^2 = |\sigma|^2$.

In fact, strictly speaking we only reproduce the defect equations away from the self-intersection locus Σ, due to the appearance of H_Σ in (7.106). This is nevertheless consistent with the regimes in which the $\mathfrak{su}(3)$ Hitchin system and the $\mathfrak{su}(2)$ system with defects are reliable descriptions.

Indeed, the $\mathfrak{su}(3)$ Hitchin system description that we are using should only be valid in regions of S where $|v|$ is small compared to the string scale, and beyond that the Hitchin description should only be strictly valid for the $\mathfrak{su}(2)$ sector. The degrees of freedom that are left out from the Hitchin system are those outside of $\mathfrak{su}(2)$, and in particular the non-Cartan entries of Φ and $\mathbb{A}$ that include the fields localised at the self-intersection curve $\Sigma = \{v = 0\}$ and their massive replicas. As we increase the vev of v though the rescaling (7.99), this region of validity narrows down as a tubular region around Σ. This limits the computation of certain non-holomorphic 4d couplings by dimensional reduction, namely those whose integrand does not converge sufficiently fast in that region. This does not seem to be a problem for the kinetic terms of the light localised modes s_1, s_2^c if we perform the rescaling (7.102), but it should affects the kinetic terms of massive modes in the same sector that have a mass comparable to the string scale. In order to correctly integrate out these massive modes one needs to solve their corresponding D-term equations, which are encoded in the non-Cartan D-term equation (7.77). Remarkably, solving this equations at an intermediate stage of the large angle limit implies imposing the relations (7.80) and (7.105) in the corresponding tubular neighbourhood.

This region of validity is somewhat opposite for the defect description. For instance, let us look at the entry of Φ that gives rise to the F-term pole, namely at the piece

$$\frac{s}{v}\mathbf{s}\left(1 - e^{-2b\lambda|v|^2}\right) . \tag{7.108}$$

Whenever $|g_S|^{-1/2}|v|^2 \gg (\lambda_\star b)^{-1}$ this piece reduces to the meromorphic (2,0)-form $\frac{s}{v}\mathbf{s}$, so at this distance from Σ it looks like the $\mathfrak{su}(2)$ 7-brane sector develops a pole. In fact, as disucssed above, at this distance the Hitchin system is only good to describe the $\mathfrak{su}(2)$ subsector of $\mathfrak{su}(3)$. Therefore, it is more useful to think of the non-Cartan fields s as a separate sector, as the defect picture does. As we enter the region $|g_S|^{-1/2}|v|^2 \leq (\lambda_\star b)^{-1}$ the Hitchin system description starts being reliable to describe the $\mathfrak{su}(3)$ system. Then we see that the pole-like behaviour $\frac{s^2}{v}$ starts being softened by the exponential, and that the (2,0)-form (7.108) actually vanishes at $v=0$. The norm of (7.108) looks like a volcano-shaped profile: from far away it seems to develop a pole at $v=0$, but close to Σ there is a turning point that makes the function go down to zero. In the limit $b \to \infty$ this becomes the function $\left|\frac{s^2}{v}\right|^2 H_\Sigma$ that appears in (7.106). The $\mathfrak{su}(2)$ modes whose profile is mostly outside of this region will see a pole, because their coupling is given by an integral that does not care much about the interior of the volcano. It is for those modes that the defect picture is useful. In the strict limit $b \to 0$ this set amounts to essentially all $\mathfrak{su}(2)$ modes, in agreement with the fact that H_Σ is a function of measure zero and its presence does not affect the integrals that give rise to the 4d D-term potential.

7.2.3 The holomorphic scheme

Let us now consider the $SU(3)$ Hitchin system that is related to the holomorphic scheme in the self intersecting curve S. As many of the ingredients are similar to the meromorphic scheme, our discussion will be more sketchy for this case. We start from the following holomorphic Higgs field

$$\Phi^{\rm h} = \frac{1}{3}\begin{pmatrix} \mathbf{v} & 0 & 0 \\ 0 & \mathbf{v} & 0 \\ 0 & 0 & -2\mathbf{v} \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{m} & 0 \\ 0 & 0 & \mathbf{s}_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7.109}$$

with $\mathbf{s}_2 \in H^{2,0}\left(\mathcal{L}_3^{-1} \otimes \mathcal{L}_8\right)$ and $\mathbf{m} \in H^{2,0}\left(\mathcal{L}_3^2\right)$. We choose a complexified gauge transformation of the form

$$B = \begin{pmatrix} e^{f_3/2+f_8/6} & 0 & 0 \\ 0 & e^{-f_3/2+f_8/6} & 0 \\ 0 & 0 & e^{-f_8/3} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -\xi_2\xi_m \\ 0 & 1 & -\xi_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.110}$$

where ξ_2 is given by (7.71) and

$$\xi_m = \frac{m}{v}\left(e^{-\mu|v|^2} - 1\right), \tag{7.111}$$

with $\mu = |g_S|^{-1/2}\mu_\star$ and $\mu_\star$ a function on S. The Higgs field in the unitary frame is now given by

$$\Phi = \frac{1}{3}\mathbf{v}\mathbb{I}_3 + \begin{pmatrix} 0 & e^{f_3}\mathbf{m} & -e^{\frac{f_3}{2}+\frac{f_8}{2}}e^{-\mu|v|^2}\left(e^{-\lambda|v|^2}-1\right)\frac{s_2}{v}\mathbf{m} \\ 0 & 0 & e^{-\frac{f_3}{2}+\frac{f_8}{2}}e^{-\lambda|v|^2}\mathbf{s}_2 \\ 0 & 0 & -\mathbf{v} \end{pmatrix}, \tag{7.112}$$

while the gauge connection is given by

$$i\mathbb{A}^{(0,1)} = H_1\bar{\mathrm{p}}f_3 + \frac{1}{\sqrt{3}}H_2\bar{\mathrm{p}}f_8 - e^{-\frac{f_3}{2}+\frac{f_8}{2}}e^{-\lambda|v|^2}\bar{\mathrm{p}}\,(\lambda\bar{v})\,s_2\,\epsilon_2 \tag{7.113}$$
$$- e^{\frac{f_3}{2}+\frac{f_8}{2}}\left[e^{-\lambda|v|^2}\left(e^{-\mu|v|^2}-1\right)\bar{\mathrm{p}}\,(\lambda\bar{v}) + e^{-\mu|v|^2}\left(e^{-\lambda|v|^2}-1\right)\bar{\mathrm{p}}\,(\mu\bar{v})\right]\frac{ms_2}{v}\,\epsilon_{12}\,.$$

Where we have used the notation of Appendix G for the algebra generators $\{H_1, H_2, \epsilon_2, \epsilon_{12}\}$. These two expressions simplify considerably in the small volume limit:

$$\Phi_{a\to 0} = \frac{1}{3}\mathbf{v}\mathbb{I}_3 + \begin{pmatrix} 0 & e^{f_3}\mathbf{m} & 0 \\ 0 & 0 & e^{-\frac{f_3}{2}+\frac{f_8}{2}}\left[1-H_\Sigma\right]\mathbf{s}_2 \\ 0 & 0 & -\mathbf{v} \end{pmatrix}, \tag{7.114}$$

$$i\mathbb{A}^{(0,1)}_{a\to 0} = \frac{1}{2}H_3\bar{\mathrm{p}}f_3 + \frac{1}{2}H_8\bar{\mathrm{p}}f_8 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -e^{-\frac{f_3}{2}+\frac{f_8}{2}}s_2 \\ 0 & 0 & 0 \end{pmatrix}\bar{\mathrm{p}}\bar{v}\,\pi\delta^{(2)}(v)\ , \tag{7.115}$$

again displaying a split bundle for the $\mathfrak{su}(2)$ subalgebra and field localisation for s_2. The main difference with respect to the meromorphic case is that now the dependence on the defect field s_2 is completely localised on Σ, and a consequence no pole arises. Indeed, performing the split of eqs.(7.89) and (7.90) and repeating the computation below them, one again finds the result (7.96), but now with $\Xi = 0$ due to the absence of a vev for s_1^c.

Let us now analyse the D-terms, whose structure in this case is general

$$J\wedge\mathbb{F} + \frac{1}{2}[\Phi,\Phi^\dagger] = \begin{pmatrix} C_1 & F & E \\ \bar{F} & C_2 & D \\ \bar{E} & \bar{D} & -C_1-C_2 \end{pmatrix}. \tag{7.116}$$

The D-term equation then amounts to three complex and two real equations, while our Ansatz contains four unknown functions: $\{f_3, f_8, \lambda, \mu\}$. To solve the D-term equations within this Ansatz one then needs to make further assumptions. For instance, let us consider the condition $D = 0$, which reads

$$2iJ\wedge\mathrm{p}\left(s_2 e^{-f_3+f_8-\lambda|v|^2}\bar{\mathrm{p}}\,(\lambda\bar{v})\right) \tag{7.117}$$
$$+ e^{-f_3+f_8}\left(e^{-\lambda|v|^2}\mathbf{s}_2\wedge\bar{\mathbf{v}} + e^{-2f_3-\mu|v|^2}\mathbf{m}\wedge\bar{\mathbf{m}}\frac{s_2}{v}\left(e^{-2\lambda|v|^2}-1\right)\right) = 0\,.$$

If again we impose (7.80) (now with the replacement $f_3 \to f_3 - f_8$) in a neighborhood of Σ, this complex equation becomes a real one. In fact, upon performing the rescaling

$$v \to bv\,, \qquad \lambda \to b^{-1}\lambda\,, \qquad \mu \to b^{-1}\mu\,, \qquad s_2 \to b^{1/2}s_2\,, \qquad m \to m\,, \tag{7.118}$$

and taking the large angle limit $b \to \infty$, satisfying (7.117) amounts to impose (7.80) on top of Σ, in analogy with the corresponding non-Cartan equation in the meromorphic scheme. Regarding the condition $E = 0$, which is equivalent to

$$\begin{aligned}
&2iJ\wedge\mathrm{p}\left[\frac{ms_2}{v}e^{f_3+f_8}\left(e^{-\lambda|v|^2}\left(e^{-\mu|v|^2}-1\right)\bar{\mathrm{p}}\left(\lambda\bar{v}\right)+e^{-\mu|v|^2}\left(e^{-\lambda|v|^2}-1\right)\bar{\mathrm{p}}\left(\mu\bar{v}\right)\right)\right] \\
&+ e^{f_3+f_8}e^{-\mu|v|^2}\left(e^{-\lambda|v|^2}-1\right)\frac{s_2}{v}\mathbf{m}\wedge\bar{\mathbf{v}} = 0\,,
\end{aligned} \tag{7.119}$$

one can see that all the terms vanish as we take the large angle limit. Something similar happens for the condition $F = 0$:

$$\begin{aligned}
&2iJ\wedge|s_2|^2\frac{m}{v}e^{f_8-\lambda|v|^2}\mathrm{p}\left(\lambda v\right) \\
&\wedge\left[e^{-\lambda|v|^2}\left(e^{-\mu|v|^2}-1\right)\bar{\mathrm{p}}\left(\lambda\bar{v}\right)+e^{-\mu|v|^2}\left(e^{-\lambda|v|^2}-1\right)\bar{\mathrm{p}}\left(\mu\bar{v}\right)\right] \\
&= e^{f_8-(\mu+\lambda)|v|^2}\left(e^{-\lambda|v|^2}-1\right)\frac{s_2}{v}\mathbf{m}\wedge\bar{\mathbf{s}}_2\,,
\end{aligned} \tag{7.120}$$

Indeed, one can check that both sides of the equation vanish as we take the limit $b \to \infty$. Finally, we have two D-term equations corresponding to the Cartan generators of $\mathfrak{su}(3)$. The condition $C_1 = 0$ amounts to impose

$$\begin{aligned}
&2iJ\wedge\mathrm{p}\bar{\mathrm{p}}\left(\frac{1}{3}f_8+f_3\right)+2ie^{f_3+f_8}\left|ms_2\right|^2J\wedge\zeta\wedge\bar{\zeta} \\
&= e^{2f_3}\mathbf{m}\wedge\bar{\mathbf{m}}+e^{f_3+f_8}\left(e^{-\lambda|v|^2}-1\right)^2e^{-\mu|v|^2}\left|\frac{s_2}{v}\right|^2\mathbf{m}\wedge\bar{\mathbf{m}}
\end{aligned} \tag{7.121}$$

where

$$\zeta = \frac{1}{\bar{v}}\left[e^{-\lambda|v|^2}\left(e^{-\mu|v|^2}-1\right)\mathrm{p}\left(\lambda v\right)+e^{-2\mu|v|^2}\left(e^{-\lambda|v|^2}-1\right)\mathrm{p}\left(\mu v\right)\right]\,. \tag{7.122}$$

The equation $C_2 = 0$ reads in turn

$$\begin{aligned}
&2iJ\wedge\mathrm{p}\bar{\mathrm{p}}\left(\frac{1}{3}f_8-f_3\right) \\
&-2ie^{-f_3+f_8}e^{-2\lambda|v|^2}\left|s_2\right|^2J\wedge\mathrm{p}\left(\lambda v\right)\wedge\bar{\mathrm{p}}\left(\lambda\bar{v}\right) \\
&= e^{-f_3+f_8}e^{-2\lambda|v|^2}\mathbf{s}_2\wedge\bar{\mathbf{s}}_2-e^{2f_3}\mathbf{m}\wedge\bar{\mathbf{m}}\,.
\end{aligned} \tag{7.123}$$

Upon taking the large angle limit ζ vanishes, as does the second term in the rhs of (7.121). In addition, (7.123) simplifies after using the relation (7.105) on top of Σ. We are finally left with

$$-iJ \wedge \mathrm{p}\bar{\mathrm{p}} \left(\frac{1}{3} f_8 + f_3 \right) = -\frac{1}{2} e^{2f_3} \mathbf{m} \wedge \bar{\mathbf{m}} \,, \tag{7.124}$$

$$-iJ \wedge \mathrm{p}\bar{\mathrm{p}} \left(\frac{1}{3} f_8 - f_3 \right) = \frac{1}{2} e^{2f_3} \mathbf{m} \wedge \bar{\mathbf{m}} - \lambda_\star \frac{e^{-f_3+f_8}}{\sqrt{|g_S|}} |s_2|^2 2\pi \delta_\Sigma \wedge J \ , \tag{7.125}$$

and so we recover (7.31) and (7.32) upon the identifications $h_{\mathcal{L}} = h_3 = e^{f_3}$, $\sqrt{|g_S|} = h_\Sigma^{1/2}$ and $4\pi\lambda_\star |s_2|^2 = |\sigma_2|^2$.

7.3 The 4d perspective

In this section we would like to take a four-dimensional perspective on the meromorphic scheme introduced in section 7.1. In particular, we will analyse the space of F-flat directions around the origin of moduli space (i.e. $\langle \Phi \rangle = 0$) and deduce how the vevs of the various 4d fields are constrained. As a warm-up, we first work in the absence of defects, where, as expected, we will find that the massive KK modes of all fields must all have vanishing vev. Then we will add the defect contribution to the superpotential [14] and look for solutions to the F-term equations with two of the intersection fields having non-vanishing vev. As in section 7.1, we will focus, for definiteness, on the fields σ_1^c and σ_2, i.e. those responsible for creating a pole for Φ as in (7.48). In this case, we find two important results. On the one hand, from integrating such an F-term equation, truncated at the zero-mode level, we get a necessary condition for solving it. On the other hand, expanding in a basis of KK modes, we realise that the singular profile for Φ can be understood as a sum of non-trivial vevs for massive KK replicas, rather than that of a zero mode.[7]

Schematically, upon dimensional reduction on a four-cycle S of positive curvature (or simply without holomorphic deformations) we find a superpotential of the form

$$W = \sum_\alpha \mu_\alpha \Psi_\alpha \Phi_\alpha - c_\alpha \Psi_\alpha \sigma_1^c \sigma_2 \tag{7.126}$$

[7]This in turn is analogous to what happens to the KK modes of the Cartan vector field in the presence of a non-primitive flux (see Appendix B of [20]).

where Ψ_α and Φ_a run over the KK modes of (0,1) and (2,0)-forms, respectively, of mass μ_α and σ_1^c, σ_2 are the 4d fields corresponding to the massless defect modes. As a vev is given to the defect fields σ_1^c, σ_2 the F-term for Φ_α implies that

$$\langle\Phi_\alpha\rangle = \frac{c_\alpha}{\mu_\alpha}\langle\sigma_1^c\sigma_2\rangle \tag{7.127}$$

and so the massive (2,0)-forms develop a vev due to their coupling to defects. When such a vev is combined with their wavefunction along S, one obtains a profile that reproduces the pole of the meromorphic scheme. Finally, the couplings of the form $c_0\Psi_0\sigma_1^c\sigma_2$, with Ψ_0 a zero mode, provide an obstruction to give a vev to the product $\sigma_1^c\sigma_2$ and so to realise the meromorphic scheme. When the S has holomorphic deformations, extra Yukawa coupling involving Ψ_0 must be added to (7.126), modifying the corresponding F-term for Ψ_0 and relaxing the above obstructions. In the following we will sketch the main idea of this computation, deferring all technical details of this discussion to Appendix J, where we also give a complete presentation of the F-term constraints.

The spectrum of bulk KK modes

We start with som preliminary material which will allow us to perform the dimensional reduction from 8d to 4d. The Hodge duality operation can be defined to act as follows on the space of (p,q) forms of the internal Kähler surface:

$$* : \Omega^{(p,q)} \longrightarrow \Omega^{(2-q,2-p)}\,. \tag{7.128}$$

This allows us to define the adjoint of the Dolbeault operator

$$\bar{\mathrm{p}}^\dagger = -*\mathrm{p}* \tag{7.129}$$

with respect to the hermitian product on S

$$\int_S \alpha\wedge *\bar{\beta}\,, \tag{7.130}$$

for any two (p,q)-forms α,β. In these conventions the $(2,0)$ forms, which are all primitive, are self-dual [?]. Hence, the holomorphic entries of the Higgs field Φ are all harmonic, self-dual forms. For the purpose of this section, however, we will need to take into account the non-zero modes of Φ too, and thus the space of exact $(2,0)$-forms. Let $\{\chi_A\}_A$ be a complete basis for the space of $(2,0)$-forms, normalized such that

$$\frac{1}{V_S}\int_S \chi_A \wedge *\bar{\chi}^B = \frac{1}{V_S}\int_S \chi_A \wedge \bar{\chi}^B = \delta_A^B\,, \tag{7.131}$$

where, in the first step, we applied self-duality. We take each of the elements of this basis to be eigenstates of the Laplacian operator $\Delta \equiv \bar{\mathrm{p}}\bar{\mathrm{p}}^\dagger + \bar{\mathrm{p}}^\dagger\bar{\mathrm{p}}$. For future convenience, let us split the collective index A as (α_0,α), to divide the basis into zero and non-zero modes. That is, we have that $\Delta\chi_{\alpha_0} = 0$ and $\Delta\chi_\alpha = -k_\alpha^2\chi_\alpha$. In order to expand the KK modes of the vector field, we also pick a complete basis of $(0,1)$-forms, $\{\psi^I\}_I$ normalised such that

$$\frac{1}{V_S}\int_S \psi^I \wedge *\bar{\psi}_J = \delta_J^I\,. \tag{7.132}$$

As before, we take them to be eigenstates of the Laplacian, and separate the zero-modes indicated with the index i_0 from the non-zero modes indexed by i, so that $\Delta\psi^{i_0} = 0$ and $\Delta\psi^i = -l_i^2\psi^i$. It turns out that the subspace of *exact* $(2,0)$-forms is isomorphic to the subspace of *coexact* $(0,1)$-forms. They are mapped into one another by a pair of invertible matrices $\mu, \tilde{\mu}$ as follows

$$\bar{\mathrm{p}}\psi^i = i\mu_\alpha^i\bar{\chi}^\alpha\,, \tag{7.133}$$

$$\bar{\mathrm{p}}^\dagger\bar{\chi}^\alpha = i\tilde{\mu}_i^\alpha\psi^i\,. \tag{7.134}$$

By applying Δ to any of the above equations, and using that $[\Delta,\bar{\mathrm{p}}] = [\Delta,\bar{\mathrm{p}}^\dagger] = 0$, one easily finds that the eigenvalues k_α^2 and l_i^2 must be equal to each other, and in this sense the indices α and i can be identified. Moreover, by applying $\bar{\mathrm{p}}^\dagger$ to the first equation and $\bar{\mathrm{p}}$ to the second one, we get the following set of equations respectively:

$$\begin{aligned} \mu_\alpha^i\tilde{\mu}_j^\alpha &= l_j^2\delta_j^i\,, \\ \tilde{\mu}_i^\alpha\mu_\beta^i &= k_\beta^2\delta_\beta^\alpha\,, \end{aligned} \tag{7.135}$$

with no sum on the rhs. This gives $\mu, \tilde{\mu}$ the meaning of (complex) mass matrices.

Note that all the (p,q)-forms we will deal with are bundle-valued. Since we consider a split $SU(2)$ bundle, this amounts to having three different basis of $(2,0)$-forms $\{\chi_{\bullet A}\}_A$ and three different basis of $(0,1)$-forms $\{\psi_\bullet^I\}_I$, where $\bullet = \{+,-,3\}$ runs over the generators of $\mathfrak{sl}(2)$ (cf. Appendix G) and indicates positive, negative and zero charge respectively. Each of these basis will satisfy orthonormality relations of the form (7.131) and (7.132). Accordingly, we will have to consider three different pairs of mass matrices in (7.135).

F-terms without defects

Let us start with the case where no defects are present. As is well known [14,54], the holomorphic sector is regulated by the following 4d superpotential

$$W = \int_S \mathrm{T}r\, \mathbb{F} \wedge \Phi\,, \tag{7.136}$$

which imposes that $\bar{\mathrm{p}}_{\mathbb{A}}\Phi = \mathbb{F}^{(0,2)} = 0$. We are now interested in studying the space of infinitesimal fluctuations for an $SU(2)$ Hitchin system, around a BPS background such that $\langle\Phi\rangle = 0$, $\langle\mathbb{A}\rangle = AT_3$ and $\mathbb{F} \wedge J = 0$. By working in the holomorphic gauge, we can simply ignore the vacuum profile for A, and hence have

$$\begin{aligned} \bar{\mathrm{p}}_{\mathbb{A}}\Phi &= \bar{\mathrm{p}}\,\delta\Phi + [\delta\mathbb{A}^{(0,1)}, \delta\Phi]\,, = 0 && (7.137)\\ \mathbb{F}^{(0,2)} &= \bar{\mathrm{p}}\,\mathrm{d}\mathbb{A}^{(0,1)} - \frac{i}{2}[\delta\mathbb{A}^{(0,1)}, \delta\mathbb{A}^{(0,1)}] = 0\,, && (7.138) \end{aligned}$$

where we have defined the matrices of fluctuations

$$\delta\Phi = \begin{pmatrix} \mathbf{v} & \mathbf{m} \\ \mathbf{p} & -\mathbf{v} \end{pmatrix}, \qquad \delta\mathbb{A}^{(0,1)} = \begin{pmatrix} 0 & a_+ \\ a_- & 0 \end{pmatrix}. \tag{7.139}$$

We did not consider the fluctuation of the gauge field along the Cartan because, due to the simply-connectedness of S, it does not admit zero-modes, and we will focus on solutions where its KK modes have vanishing expectation values (see Appendix J where the latter are taken into account).

It is immediate to see that, from the off-diagonal components of (7.137) and from the diagonal component of (7.138) we get respectively the three F-term equations

$$\begin{aligned} \bar{\mathrm{p}}\mathbf{m} &= 2a_+ \wedge \mathbf{v}\,, && (7.140)\\ \bar{\mathrm{p}}\mathbf{p} &= -2a_- \wedge \mathbf{v}\,, && (7.141)\\ 0 &= a_+ \wedge a_-\,. && (7.142) \end{aligned}$$

Assuming that zero-modes for a_- exists, we may wedge both sides of eq.(7.140) with each of them, namely with the basis $\{\psi_-^{i_0}\}_{i_0}$. Since in the holomorphic gauge each of the elements of this basis has a holomorphic profile, the left-hand-side is a total derivative and its integral over S vanishes. Similarly, we may wedge eq.(7.140) with the zero modes of a_+, $\{\psi_+^{i_0}\}_{i_0}$, and eq.(7.142) with $\{\chi_{3\alpha_0}\}_{\alpha_0}$, and integrate over S. Using the expansions $a_\pm = a_{\pm I}\psi_\pm^I$ and $\mathbf{v} = v^A\chi_{3A}$, such integrals give us the following constraints

$$\Lambda_{3\,A}^{i_0 j_0} a_{+\,j_0} v^{\alpha_0} = \Lambda_{3\,A}^{i_0 j_0} a_{-\,j_0} v^A = \Lambda_{3\,\alpha_0}^{i_0 j_0} a_{+\,i_0} a_{-\,j_0} = 0\,, \tag{7.143}$$

where the indices i_0, j_0 and α_0 run over the subspace of zero-modes, and we have defined the Yukawa couplings

$$\Lambda_{3A}^{IJ} = \int_S \psi_+^I \wedge \psi_-^J \wedge \chi_{3A}\,. \tag{7.144}$$

From equations (7.143) it is clear that at least two among the three sets of massless modes $\{a_{+,i_0}\}_{i_0}$, $\{a_{-,i_0}\}_{i_0}$ and $\{v^{\alpha_0}\}_{\alpha_0}$ must attain trivial vacuum expectation values. As one would expect, the F-terms also constrain their massive KK replicas to a zero vev, see Appendix J for details.

F-terms with defects

Let us now introduce defects and see how equations (7.143) are modified, inducing for non-trivial vevs for non-zero KK modes. As anticipated, this will be the 4d counterpart of the meromorphic profile introduced in section 7.1.

Defects are localized on the curve $\Sigma \subset S$ and affect the holomorphic sector through the superpotential [14]

$$W_\Sigma = \int_S \delta_\Sigma \wedge \langle\!\langle \sigma^c, \bar{\mathrm{p}}_{\mathbb{A}}\sigma \rangle\!\rangle_{\mathfrak{g}_S}\,. \tag{7.145}$$

For definiteness, let us consider non-trivial vevs for the defect fields σ_1^c and σ_2, which, as seen in section 7.1, generate a first-order pole for the Higgs field along $\Phi = \mathbf{p}T_-$, see eq.(7.52). Hence, while equations (7.140) and (7.142) remain unmodified and can be both satisfied by just setting all modes of a_+ to zero vev, equation (7.141) becomes

$$\bar{\mathrm{p}}\mathbf{p} = -2a_- \wedge \mathbf{v} + \delta_\Sigma \wedge \sigma_1^c \sigma_2\,. \tag{7.146}$$

Since the bilinear $\sigma_1^c\sigma_2$ behaves as a $(1,0)$ form, we expand it as $\sigma_1^c\sigma_2 = (\sigma_1^c\sigma_2)^{i_0}\bar{\psi}_{+\,i_0}$, assuming non-vanishing vevs only for their zero-modes. Again, in the hypothesis that zero-modes for a_+ exist, we wedge the above equation by a complete set of them $\{\psi_+^{i_0}\}_{i_0}$, and integrate over S. The left-hand-sides vanish because they are total derivatives, and we get the equations

$$\Gamma^{i_0}_{+\,j_0}(\sigma_1^c\sigma_2)^{j_0} - 2\Lambda_{3\,\alpha_0}^{i_0 j_0} a_{-\,j_0} v^{\alpha_0} = 0\,. \tag{7.147}$$

In the above we have assumed non-vanishing vevs only for the zero-modes of $\mathbf{v}$ and a_-. Hence all indices run over the subspaces of *holomorphic* $(0,1)$- and $(2,0)$-forms. Moreover, we have defined the pairing

$$\Gamma^I_{+J} = \int_\Sigma \psi_+^I \wedge \bar{\psi}_{+\,J} \;\simeq \int_S \delta_\Sigma \wedge \langle\!\langle \sigma^c, \sigma \rangle\!\rangle^{j_0}_{\mathfrak{g}_S} \wedge \psi^I\,. \tag{7.148}$$

The 4d F-term constraints (7.147) are necessary conditions for the existence of solutions of (7.146), and play an analogous rôle of the 4d D-terms equations obtained in section 7.1, see e.g. eqs.(7.40) and (7.41). But now we get something more, by considering the expansion of equation (7.146) along the non-zero modes of the basis of negatively-charged $(2,1)$-forms, i.e. along $\{*\bar{\psi}_{-\,i}\}_i$. Using the self-duality of $\chi_{-\,\alpha}$ and the definition (7.129), we can take the complex conjugate of (7.134) and get

$$\bar{\mathrm{p}}\chi_{-\,\alpha} = i\bar{\bar{\mu}}^i_{-\,\alpha} * \bar{\psi}_{-\,i}\,. \tag{7.149}$$

Expanding the profile for $\mathbf{p}$ in non-zero modes as $\mathbf{p} = p^\alpha \chi_{-\,\alpha}$, and using (7.149), eq.(7.146) leads to

$$p^\alpha = -\frac{i}{k^2_{-\,\alpha}}\bar{\mu}^\alpha_{-\,i}\left(\Gamma^i_{+\,j_0}(\sigma_1^c\sigma_2)^{j_0} - 2\Lambda_{3\,\beta_0}^{ij_0} a_{-\,j_0} v^{\beta_0}\right)\,. \tag{7.150}$$

Recall that the indices α, i run over the subspace of *non-zero* modes, such that we could invert the mass matrix and make use of (7.135). As a consequence, those appearing in parenthesis in the above equation are generally *different* combinations than the ones appearing in (7.147).

Let us now consider that a single zero-mode of a_- is switched on, say $a_{-\,0}$ and a single zero-mode of $\mathbf{v}$, say v^0. We may then pick one of the equations in (7.147), solve for $a_0 v^0$ and plug the result in (7.150), such that we get

$$p^\alpha = -\frac{i}{k^2_{-\alpha}}\bar{\mu}^\alpha_{-i} c^i_{j_0}(\sigma_1^c\sigma_2)^{j_0}\,, \tag{7.151}$$

where we have defined the coefficients

$$c^i_{j_0} = \Gamma^i_{+\,j_0} - \Lambda^{i\,0}_{3\,0}\frac{\Gamma^0_{+\,j_0}}{\Lambda^{0\,0}_{3\,0}}\,. \tag{7.152}$$

To summarise, the presence of defects oblige certain non-zero KK modes (corresponding to the 8d field **p** in this example) to attain non-vanishing vacuum expectation values, inversely proportional to their mass.

Chapter 8

Conclusions

8.1 English

In this thesis we have analysed BPS-stability for T-brane configurations of 7-branes, putting emphasis on curvature corrections and on the effects of working on compact 4-cycles with potential defects instead of local environments. The results of this work are therefore organised in three chapters: Chapter 5 has been dedicated to the role of α'-corrections in T-brane systems, while chapters 6 and 7 have been dedicated to obstructions to BPS stability when putting T-branes on compact 4-cycles. We will therefore draw conclusions separately.

In chapter 5 we have analysed the effect of α'-corrections on BPS systems of multiple D7-branes, with special emphasis on T-brane configurations. Our main strategy has been to compute how α'-correction modify the D-term BPS condition, solve for the new background profiles for Φ and A, and compare them with the previous leading-order D-term solution. Since α'-corrections do not enter holomorphic D7-brane data, this comparison can be made in terms of the complexified gauge transformation (5.27) in terms of which we solve the D-term equations.

In D7-brane T-brane systems, solving the D-term equation is quite involved already at leading order, which renders our analysis somewhat technical. Nevertheless, we have drawn several lessons from the cases that we have analysed:

- When the Higgs background takes a block-diagonal form (5.69), α'-corrections can be analysed block by block, as they do not couple different blocks.

- For system of intersecting D7-branes α'-corrections have a simple interpretation in terms of the pull-back of the Kähler form on the actual D7-brane embedding. It would be interesting to see if T-brane systems allow for a similar interpretation.

- In all the examples that preserve eight supercharges, α'-corrections do not modify the background. The classical solution also solves the corrected D-term equations. A trivial example of this are intersecting D7-branes without fluxes.

- One may lower the amount of supersymmetry to four supercharges by modifying the Higgs field by a constant slope $\Delta\Phi$ or by adding a constant primitive flux H, both commuting with the group generators involved the T-brane background. At leading order these additions do not modify the T-brane background at all. When α'-corrections are taken into account the T-brane background is modified, but there are several degrees of complexity at which this may happen

    i) In the simplest case α'-corrections only modify the dimensionful parameters which enter the differential equation for the non-primitive flux background (5.1) and the related complexified gauge transformation (5.27), as in eqs.(5.42) and (5.54). Hence they can be typically absorbed into a coordinate redefinition.

    ii) In slightly more complicated cases we need to generalise the complexified gauge transformation to

$$g = e^{\frac{1}{2}(fP+h\mathbf{1})} \tag{8.1}$$

    to absorb the effect of some primitive flux H. The corresponding non-primitive flux is therefore still Abelian, with f being modified from the leading-order expression. The equations governing f and h are rather complicated, but one may solve them by performing a perturbative expansion in α'-suppressed parameters. More precisely we have assumed the following hierarchy

$$\alpha'\rho_i \ll \alpha' m_j^2 \ll 1 \tag{8.2}$$

to find solutions to next-to-leading order in α'. Here ρ_i are primitive flux density parameters and m_j T-brane slope parameters.

iii) In the most complex case the Abelian Ansatz (8.1) is not sufficient to solve the corrected D-term equations, which develop non-trivial components along non-Cartan generators (in particular those which the holomorphic T-brane data depends on). One then needs to consider a complexified gauge transformation that depends on such generators, as in Appendix C. The analysis for these corrected backgrounds is even more involved and one again needs to resort to a perturbative expansion to find solutions to next-to-leading order in α'.

- This last, more complicated case contains all the ingredients that are generic in the construction of 4d chiral local F-theory GUT models, so one may speculate that α'-corrections could change qualitatively the description of these configurations, as we have briefly discussed. In any event, the holomorphic data of these models will not be affected by α'-corrections. In particular the holomorphic Yukawa hierarchies of [7,8,48], which only depend on such holomorphic data, will still be present after α'-corrections are taken into account.

Based on these results, one may conceive of several directions to pursue the analysis of α'-corrections in T-brane systems. First, it would be interesting to extend our background solutions to higher orders in the α' expansion and beyond the limit (8.2). Second, it would be interesting to see if the interpretation of α'-corrections for the intersecting D-brane case can be incorporated in some form for T-brane backgrounds. Moreover, it would be interesting to verify our naive analysis of α'-corrections in F-theory local models based in exceptional groups, and compute how α'-corrections modify the normalisation of chiral mode wavefunctions in realistic models. Finally, it would be interesting to see the consequences of our findings for the recent proposal to use T-branes in the construction of de Sitter vacua [38].

In chapter 6 we have analysed global aspects of T-branes in type IIB/F-theory compactifications. Recall, that in this context T-branes were first presented as interesting configurations that allow for hierarchical Yukawas in F-theory GUTs. Since the computation of Yukawas can be essentially done within a local patch of the four-cycle S_{GUT}, only a local description of the T-brane background is needed to realise this property. Nevertheless, this local picture inevitably misses some crucial features of T-branes, including possible obstructions to their existence, that can only be revealed by a global analysis.

In this spirit we have given a global description of such T-brane configurations from the viewpoint of the Kähler four-cycle S where they are defined. We have focused on T-branes with a pole-free holomorphic Higgs field Φ, and an Abelian gauge flux F, which we have dubbed compact T-branes. We have observed several general features that mainly depend on the topology of S and the pull-back of the threefold Kähler form J. Namely we have found that:

- In general, the worldvolume flux F lies in a non-harmonic representative of its cohomology class. The departure from harmonicity is codified in a globally well-defined function g on S satisfying certain non-linear PDEs. In local patches, such equations reproduce the ones already found in the T-brane literature.

- There is an obstruction to building these T-brane backgrounds on surfaces where the Ricci curvature class vanishes or is positive definite. In the remaining surfaces the existence of T-branes depends on the classes $[\rho], [F] \in H^2(S)$ of the Ricci form and the worldvolume flux, respectively, as well as on the point in Kähler moduli space. For instance, in the simplest case, the following condition needs to be satisfied:

$$0 \leq \int_S J \wedge (2F - \rho) < -\int_S J \wedge \rho \,. \tag{8.3}$$

  Hence, given a four-cycle S and a point in Kähler moduli space, only the subset of quantised fluxes F satisfying (8.3) will be suitable to construct a compact T-brane. Notice that whenever the Ricci form has a negative sign when projected into the Kähler form, one may choose $[F] = [\rho]/2$ (i.e. the Hitchin Ansatz) to satisfy (8.3).

- In those regions of Kähler moduli space where $0 < \xi\alpha' = -\frac{1}{\pi\alpha'}\int_S F \wedge J \ll 1$, we may interpret our T-brane background as a 7-brane bound state obtained after switching on a Fayet-Iliopoulos term ξ, and see the slice

$\xi = 0$ as a T-brane stability wall. The fate of the system as the wall is crossed to the region $\xi < 0$ again depends on the T-brane topological data, and in particular on the two classes $[\rho]$ and $[F]$. A similar statement holds for a T-brane built at the intersection of two 7-branes.

In chapter 7 we have analysed the role of defects for the stability of T-branes. Defects arise due to the presence of further 7-branes on different four-cycles that intersect the T-brane stack and give rise to additional degrees of freedom localised on the intersection curves. Since these new fields couple to the eight-dimensional SYM theory of the T-brane, they modify the BPS-equations. In section 7.1.2 we have shown that these modifications allow for T-branes on four-cycles that possess topological obstructions to stability in the absence of defects. In doing so, we showed that there are two distinct mechanisms to do so. The so-called *holomorphic scheme* leaves the 8d F-term equations unchanged, but introduced contributions of opposite sign to the D-terms, while the *meromorphic scenario* introduces a source term in the F-terms thereby modifying the effectiveness constraint. This source term in particular induces poles in the Higgs field.

In section 7.3, we have investigated the meromorphic scheme from a four-dimensional point of view to give a complementary picture of the pole. Indeed we have shown that the pole can be seen as defect-zero-modes coupling to higher order KK-modes from the Higgs-field. By acquiring a vev, these defect fields in turn impose a vev for the KK-modes of Φ.

Lastly, in section 7.2 we consider the case of a self-intersecting four-cycle. This set-up allowed us to identify both the T-brane locus as well as the intersecting 7-brane locus with the same cycle class. In such a setting we have two analyses available: On the one hand side we may apply the defect theory formalism discussed in the paragraphs above, but on the other hand we may also embed the whole system into a larger gauge algebra and identify the defect fields as components of this larger Higgs-field. We were able to carry out this dictionary in detail for both holomorphic as well as meromorphic scenario.

These general results already suggest many avenues for further investigation. The most pressing question is perhaps what are the implications of our findings for concrete F-theory GUT models. We may for instance consider a model where

S_{GUT} hosts an exceptional symmetry group like $G = E_{6,7,8}$ and a T-brane sector within a subalgebra of G, as it is the case for local models of Yukawas [4,6–8,48]. Then our no-go result implies that either *a*) S_{GUT} cannot be del Pezzo or *b*) the T-brane sector contains some poles. In the latter case, one might interpret such poles as being sourced by further 7-branes intersecting S_{GUT} on matter curves, and it would be interesting to engineer compactifications that reproduce such a setup.

An additional generalisation would be to look at T-brane backgrounds where the gauge bundle is not of the split form (6.2). One simple way of obtaining non-split bundles is by switching on any of the bundle moduli a_+, a_- in (6.59) on top of a T-brane background near the stability wall. Obviously, the no-go result of section 6.2 still holds for these more complicated configurations. In general, for any non-split bundle that can be taken to the split form by moving in open-string moduli space the no-go result will apply, and equation (6.35) should be satisfied. It would be therefore very interesting to analyse the structure of the open-string moduli space around general T-brane backgrounds.

Another direction would be to examine how α' corrections modify the T-brane constructions considered in this chapter. At moderate volumes of the compactification one may in principle apply the same strategy as in [19] to see how such corrections affect the differential equations of section 6.1, that govern the 7-brane background. However, as these corrections do not affect the holomorphic T-brane data and are sufficiently mild not to flip the FI-term sign, the no-go theorem of section 6.2 should still hold.

Finally, as the necessary conditions for the existence of compact T-branes depend on the point in the Kähler moduli space of the compactification, it would be interesting to see if our results could have any implications for Kähler moduli stabilisation.

In summary, as argued in the introduction, our findings can be seen as one further step in the classification of the full set of BPS branes in type IIB/F-theory compactifications. As such, they should have direct consequences for the model-building applications that triggered the recent study of T-branes in this context, and it would be interesting to fully explore such implications. In any event, we expect that having a good understanding of global T-brane configurations will give rise to new insights in the comprehension of string theory vacua.

8.2 Español

En esta tesis hemos analizado la estabilidad BPS de configuraciones de T-branas de 7-branas incluyendo correcciones de curvatura y trabajando en 4-ciclos compactos con posibles defectos. Los resultados de este trabajo están entonces organizados en tres capítulos: El capítulo 5 ha sido dedicado al papel de correcciones α' en sistemas de T-branas, mientras los capítulos 6 y 7 han sido dedicados a las obstrucciones a la estabilidad BPS que ocurren si construimos T-branas en 4-ciclos compactos. Presentamos las conclusiones capítulo por capítulo.

En capítulo 5 hemos analizado el efecto de correcciones α' del sistema BPS de varias D7-branas con énfasis especial en configuraciones de T-branas. Nuestra estrategia principal ha sido calcular las correcciones α' modificando las soluciones de los términos D de las condiciones BPS, resolviendo para el nuevo perfil de fondo en Φ y A, y compararlos con la solución de los términos D a primer orden. Como las correcciones α' no entran en los datos holomorfos de las D7-branas, se puede hacer esa comparación en términos de transformaciones complexificadas (5.27), en función de las cuales resolvemos las ecuaciones de los términos D.

En sistemas de T-branas con D7-branas, resolver las ecuaciones de los términos D es considerablemente complicado ya a primer orden, lo cual hace nuestro análisis algo técnico. No obstante, hemos obtenido varios resultados de los casos que hemos analizado:

- Si el perfil de fondo del Higgs tiene forma de bloque diagonal (5.69), las correcciones α' pueden ser analizadas bloque por bloque, puesto que diferentes bloques no se acoplan.

- Para sistemas de D7-branas intersecantes, las correcciones α' tienen una interpretación sencilla en términos del pull-back de la forma de Kähler en el encaje de las D7-branas. Sería interesante analizar si sistemas de T-branas permiten una interpretación parecida.

- En todos los ejemplos que preservan ocho supercargas, las correcciones α' no modifican el fondo. La solución clásica resuelve también las ecuaciones corregidas de los términos D. Un ejemplo trivial de esto son D7-branas intersecantes sin flujo.

- Se puede bajar la cantidad de supersimetría a cuatro supercargas modificando el campo de Higgs con una elevación constante $\Delta\Phi$ o añadiendo un flujo primitivo constante H para ambos generadores del grupo involucrados en la T-brana. A primer orden estas adiciones no modifican el perfil de la T-brana. Es solo cuando se incluyen correcciones α' que la T-brana está modificada en distintos grados de complejidad

  i) En el caso más sencillo, las correcciones α' solo modifican los parámetros dimensionales que entran en la ecuación diferencial para el perfil del flujo non-primitivo (5.1) y las transformaciones gauge complexificadas relacionadas (5.27), véase eqs.(5.42) y (5.54). Correspondientemente se pueden absorber los cambios en una redefinición de las coordenadas.

  ii) En casos modestamente más complicados, tenemos que generalizar las transformaciones gauge a

$$g = e^{\frac{1}{2}(fP+h\mathbf{1})} \tag{8.4}$$

  absorbiendo el efecto de parte del flujo primitivo H. El flujo no-primitivo correspondiendo a eso, sigue siendo entonces Abeliano, con f modificado de la expresión a primer orden. Las ecuaciones que determinan f y h son más bien complicadas, pero se pueden resolver expandiendo pertubativamente en parámetros suprimidos con α'. Más precisamente hemos supuesto la siguiente jerarquía

$$\alpha'\rho_i \ll \alpha' m_j^2 \ll 1 \tag{8.5}$$

  para encontrar soluciones a segundo orden en α'. Aquí ρ_i son parámetros de densidades de flujo primitivo, mientras que m_j son parámetros de paso de la T-brana.

  iii) En el caso más complejo, el ansatz Abeliano (8.4) no es suficiente para resolver las ecuaciones de los términos D corregidas, desarrollando componentes no-triviales a lo largo de los generadores no-Cartan (en particular de los que dependen los datos holomorfos de la T-brana). En estos casos, se necesita considerar las transformaciones gauge complexificadas dependiendo de estos generadores, como en el apéndice C. Este análisis para los perfiles corregidos es todavía más complejo y es necesario limitarse a una expansión pertubativa para encontrar soluciones a segundo orden en α'.

- Este último caso más complicado contiene todos los ingredientes que son genéricos en la construcción de modelos quirales locales en 4d de GUTs en teoría F, tanto que se puede especular que las correcciones α' puedan cambiar cualitativamente la descripción de estas configuraciones, como hemos discutido brevemente. En todo caso, los datos holomorfos de estos modelos no serán afectados de correcciones α'. En particular las jerarquías de los acoplamientos holomorfos Yukawa de [7,8,48], las cuales solo dependen de estos datos holomorfos, van a estar presentes incluyendo correcciones α'.

Basándose en estos resultados, se pueden concebir varias direcciones de futuro para el análisis de correcciones α' en sistemas de T-branas. Primero, sería interesante extender nuestras soluciones a ordenes más altos en α' y más allá del limite (8.5). Segundo, sería interesante investigar si la interpretación de correcciones α' para el caso de D-branas intersecantes se puede incorporar de alguna forma para sistemas de T-branas. Además sería interesante verificar nuestra análisis de correcciones α' en modelos locales de teoría F basados en grupos excepcionales, calculando cómo las correcciones α' modifican la normalización de las funciones de onda quirales en modelos realistas. Por último, sería interesante observar las consecuencias de nuestros resultados en las propuestas recientes de utilizar T-branas en la construcción de vacíos de Sitter [38].

En capítulo 6 hemos analizado aspectos globales de T-branas en compactificaciones de tipo IIB y teoría F. Recordamos que en este contexto las T-branas han sido presentadas primero como configuraciones interesantes permitiendo jerarquías Yukawa en GUTs de teoría F. Como se puede hacer el cálculo de Yukawas alrededor de un punto del 4-ciclo S_{GUT}, solo la descripción local del perfil de la T-brana es necesaria para esa propiedad. Sin embargo, esta perspectiva local inevitablemente es insensitiva a unas propiedades esenciales de las T-branas, incluyendo posibles obstrucciones a su existencia, las cuales solo son visibles desde un punto de vista global.

Con esa intención hemos dado una descripción general de tales configuraciones de T-branas desde la perspectiva del 4-ciclo Kähler S donde están definidas. Nos hemos enfocado en T-branas con un perfil holomorfo, sin polos del campo Higgs Φ, y un flujo Abeliano F, las cuales hemos nombrado T-branas compactas. Hemos observado varias características que, principalmente, dependen de la topología de S y el pull-back de la forma de Kähler J del threefold. Más específicamente, hemos encontrado:

- En general, el flujo en el worldvolume F es parte de un representante no-armónico de su clase de cohomología. El desvío de la armonicidad está codificada en una función g globalmente bien definida en S, la cual satisface ciertas PDEs non-lineares. En entornos locales, tales ecuaciones reproducen las conocidas en la literatura de T-branas.

- Existe una obstrucción a construir dichos fondos de T-branas en superficies con curvatura de Ricci cero o definida positivo. En las otras superficies, la existencia de T-branas depende de las clases $[\rho], [F] \in H^2(S)$ de la forma de Ricci y del flujo del worldvolume, respectivamente, tanto como del punto en el espacio de Kähler moduli. Por ejemplo, en el caso más simple, las siguientes condiciones tienen que ser satisfechas:

$$0 \leq \int_S J \wedge (2F - \rho) < -\int_S J \wedge \rho \,. \tag{8.6}$$

  Por lo tanto, para un 4-ciclo S y un punto en el espacio de Kähler moduli, solo el subconjunto de flujos cuantizados F que satisfacen (8.6) va a ser adecuado para la construcción de T-branas compactas. Obsérvese que, siempre que la proyección de la forma de Ricci en la forma de Kähler tenga signo negativo, se puede elegir $[F] = [\rho]/2$ (es decir el Hitchin Ansatz) para satisfacer (8.6).

- En las regiones de espacio de Kähler moduli en las cuales $0 < \xi\alpha' = -\frac{1}{\pi\alpha'}\int_S F \wedge J \ll 1$, podemos interpretar nuestro fondo de T-brana como un estado ligado de 7-branas obtenido después de encender un término de

Fayet-Ilioupoulos ξ, y vemos el locus $\xi = 0$ como un pared de estabilidad de T-branas. El destino del sistema cruzando esta pared para $\xi < 0$ de nuevo depende de los datos topológicos de la T-brana y, en particular, de las dos clases $[\rho]$ y $[F]$. Una observación parecida está satisfecha para una T-brana construida en la intersección de dos 7-branas.

En el capítulo 7 hemos analizado el papel de defectos para la estabilidad de T-branas. Los defectos aparecen por la presencia de 7-branas adicionales en cuatro-ciclos distintos, intersecando el locus de la T-brana y dan lugar a nuevos grados de libertad localizados en la curva de intersección. Dado que estos nuevos campos se acoplan a la acción del 8d SYM en la T-brana, modifican las ecuaciones BPS. En la sección 7.1.2 hemos demostrado que estas modificaciones permiten T-branas en cuatro-ciclos que poseen obstrucciones topologicas en la ausencia de defectos. Hemos demostrado que eso puede suceder de dos formas distintas: el *esquema holomorfo* deja invariante las ecuaciones de los términos F en 8d introduciendo contribuciones de signo opuesto a los términos D, mientras que el *esquema meromorfo* introduce un término de fuente en los términos F modificando la condición de efectividad. En consecuencia ese término de fuente induce polos en el campo de Higgs.

En la sección 7.3 hemos investigado el esquema meromorfe desde un punto de vista de cuatro dimensiones para dar una perspectiva complementaria al polo. Hemos comprobado que se puede entender el polo como modos-cero de defectos acoplando con modos KK más altos del campo Higgs. Obteniendo un vev, estos campos defectos entonces imponen un vev a los modos KK de Φ.

Por último, en la sección 7.2 hemos considerado el caso de un cuatro-ciclo auto-intersecante. Este escenario nos permite identificar tanto el locus de la T-brana como de la 7-brana intersecando con la misma clase. En una compactificación así, podemos hacer dos análisis distintos. Por un lado podemos aplicar el formalismo de la teoría de defectos, y por otro lado podemos entender el sistema completo en términos de una álgebra gauge más amplia identificando los campos de defectos como componentes del este campo de Higgs más grande. Hemos hecho este diccionario en detalle tanto para el esquema holomorfo como meromorfo.

Estos resultados generales ya sugieren varias vías de futura investigación. Las dos preguntas más urgentes son quizás i) cómo generaliza todo si permitimos polos en nuestros sistemas de T-branas y ii) qué son las implicaciones para modelos GUT específicos en teoría F. Podemos, por ejemplo, considerar un modelo en el que $S_{\rm GUT}$ soporte un grupo simétrico excepcional como $G = E_{6,7,8}$ y un sector de T-branas en una subalgebra de G, tal como es en el caso de modelos locales de Yukawas [4, 6–8, 48]. Entonces nuestro resultado del no-go implica que o a) $S_{\rm GUT}$ no puede ser del Pezzo o b) que el sector de la T-brana contiene polos. En el último caso, se puede interpretar el origen de dichos polos como 7-branas adicionales cruzando $S_{\rm GUT}$ en curvas de materia y sería interesante construir compactificaciones reproduciendo una configuración así.

Otra generalización adicional sería investigar fondos T-branas con fibrados gauge que no sean de la forma "split" como en (6.2). Una manera sencilla de obtener fibrados no-split es encender uno de los moduli a_+, a_- en (6.59) encima de un fondo de T-brana cerca de la pared de estabilidad. Obviamente el resultado no-go de la sección 6.2 todavía está en vigor para estas configuraciones más complicadas. En general, para cualquier fibrado no-split que puede ser relacionado con la forma split moviéndose en espacio de moduli de cuerdas abiertas, el no-go aplica y la ecuación (6.35) debe ser satisfecha. Sería entonces muy interesante analizar la estructura del espacio de moduli de cuerdas abiertas para T-branas más generales.

Otra dirección sería examinar cómo las correcciones α' modifican las construcciones de T-branas consideradas en este capítulo. A volúmenes moderados de la compactificación se puede, en principio, seguir la misma estrategia que en [19] para ver cómo dichas correcciones afectan a las ecuaciones diferenciales de la sección 6.1, las cuales determinan el fondo de las 7-branas. Como esas correcciones no afectan a los datos holomorfos de T-branas y son suficientemente suaves para no cambiar el signo del término FI, el teorema no-go de la sección 6.2 está satisfecho.

Por fin, como las condiciones necesarias para la existencia de T-branas compactas dependen del punto en espacio de Kähler moduli de la compactificación, sería interesante ver como nuestros resultados pueden tener implicaciones para el espacio de Kähler moduli.

En resumen, como hemos argumentado en la introducción, se pueden considerar nuestros resultados como un paso adelante en la clasificación completa de branas BPS en compactificaciones de tipo IIB/teoría F. En consecuencia, deben tener consecuencias directas para aplicaciones de model-building, las cuales motivaron el estudio reciente de T-branas en ese contexto, y sería interesante explorar estas implicaciones. En todo caso, esperamos que un mejor conocimiento de configuraciones globales de T-branas dé lugar a nuevas revelaciones en la compresión de los vacíos de teoría de cuerdas.

Appendix A

D-terms from the Chern-Simons action

In section 5.2 we discussed how to derive the D-terms for non-Abelian stacks of D7-branes in IIB orientifolds with O3/7-planes via their generalised calibration conditions. As we will now show, one can reach the same result by considering the 4d couplings that arise from the Chern-Simons action. Indeed, as was argued in [69], the D-terms of the four dimensional effective action are related by supersymmetry to terms of the form $\int \tilde{B}_2 \wedge F$, where $\tilde{B}_2$ is a 4d two-form dual to an axion and F the field strength of a gauge group generator. As in other D-brane setups here the two-forms $\tilde{B}_2$ arise from RR p-forms, and so such couplings will be contained in the D-brane Chern-Simons action.

The non-Abelian Chern-Simons action for a stack of D7-branes is given by [43]

$$S_{\mathrm{CS}} = \mu_p \int_{\mathbb{R}^{1,3}\times\mathcal{S}} \mathrm{STr}\left(\mathrm{P}\left[e^{i\lambda\iota_\Phi\iota_\Phi} \sum C^{(n)} \wedge e^{-B}\right] \wedge e^{\lambda F}\right), \tag{A.1}$$

where we will use the same parametrisation for the Higgs-field as in the main text

$$\Phi = \phi \frac{\mathrm{p}}{\mathrm{p}z} + \overline{\phi}\frac{\mathrm{p}}{\mathrm{p}\overline{z}}. \tag{A.2}$$

For simplicity, let us assume that the odd cohomology groups of the compactification manifold $H^2_-(X_3) = H^4_-(X_3)$ vanish. Then the harmonic components of the internal B-field are projected out, and the same applies to the 4d two-forms that could arise from the dimensional reduction of the RR forms C_2 and C_6. The only relevant 4d two-forms and their axion duals arise from the expansion of the orientifold-even RR forms

$$C^{(4)} = c_2^a \omega_a + \rho_a \tilde{\omega}^a + \dots \tag{A.3}$$

$$C^{(8)} = e_2\, \omega_6 + \dots \tag{A.4}$$

where ω_a, $\tilde{\omega}^a$ run over the bases of integer two- and four-forms in the internal space, respectively (such that $J = e^{\phi_{10}/2} v^a \omega_a$) and $\omega_6 = d\mathrm{vol}_X/\sqrt{g_X}$ is the unique harmonic six-form with unit integral over X_3. Plugging this into (A.1) gives

$$S_{\rm CS} \supset \lambda^2 \mu_p \int_{\mathbb{R}^{1,3}\times\mathcal{S}} \mathrm{STr}\Bigg\{ F_{4d} \wedge \Bigg[e_2 \wedge i\iota_\Phi \iota_\Phi \omega_6 \tag{A.5}$$
$$+ c_2^a \wedge \Bigg(P[\omega_a] \wedge F + \frac{i\lambda^2}{2} \iota_\Phi \iota_\Phi \left(\omega_a\right) F^2 \Bigg) \Bigg] \Bigg\},$$

where F_{4d} stands for the components of the D7-brane field strength with legs on $\mathbb{R}^{1,3}$, and we have imposed the absence of internal B-field.

The two-forms coupling to F_{4d} have as 4d duals

$$dc_2^a = \frac{g^{ab}}{4\mathcal{K}^2} *_{\mathbb{R}^{1,3}} d\rho_b \qquad\qquad de_2 = e^{2\phi_{10}} *_{\mathbb{R}^{1,3}} dC_0 \tag{A.6}$$

where $\tau = C_0 + ie^{-\phi_{10}}$ is the type IIB axio-dilaton, $\mathcal{K} = \frac{1}{6}\mathcal{K}_{abc} v^a v^b v^c$ with $\mathcal{K}_{abc}$ the triple intersection numbers of X_3, and g^{ab} is the inverse of $g_{ab} = \frac{1}{4\mathcal{K}} \int_{X_3} \omega_a \wedge *\omega_b$. Such duality relations tells us how a vector multiplet coupling to c_2^a and e_2 enters the type IIB Kähler potential. Let us start from the usual expression

$$K_{IIB} = -\log(S+\bar{S}) - \log(\mathcal{K}^2) - \log\left(\int \Omega \wedge \bar{\Omega}\right) \tag{A.7}$$

where $S = -i\tau$. Here $\mathcal{K}^2$ should be seen as a function of $\mathrm{Re}\, T_a$, with $T_a = -\frac{1}{2}\mathcal{K}_{abc} v^a v^b - i\rho_a$. Then a vector multiplet V_i coupling to these axions via a Stückelberg coupling Q_α^i should enter the Kähler potential (A.7) through the replacements

$$S + \bar{S} \;\to\; S + \bar{S} - Q_0^i V_i\,, \qquad\qquad T_a + \bar{T}_a \;\to\; T_a + \bar{T}_a - Q_a^i V_i\,. \tag{A.8}$$

Finally, the Fayet-Iliopoulos term corresponding to V_i will be given by

$$\xi_i \propto \left(\frac{\text{p}K}{\text{p}V_i}\right)_{V=0} . \tag{A.9}$$

This prescription has been applied in [70] to reproduce the D-terms of intersecting D6-brane models, which automatically include the α' corrections of mirror type IIB setups. The latter have been analysed from this viewpoint in the Abelian case in [71]. In the following we will see that it can also be used to reproduce the D-terms of α'-corrected non-Abelian D7-brane systems.

Indeed, we may apply the above prescription generator by generator of the non-Abelian gauge group of the D7-brane stack, extracting the Stückelberg charges Q^i_α from the couplings $\int_{\mathbb{R}^{1,3}} \tilde{C}_2^\alpha \wedge F_i$. At the end we obtain that the above prescription amounts to perform the following replacement in (A.5)

$$e_2 \;\rightarrow\; e^{\phi_{10}} , \qquad\qquad c_2^a \;\rightarrow\; -\frac{v^a}{\mathcal{K}} , \tag{A.10}$$

that is, to trade the two forms by their partners in the corresponding linear multiplet. We then finally obtain a non-Abelian D-term proportional to

$$\lambda^2 \mu_p \int_{\mathcal{S}} \text{S}\left\{ P[J] \wedge F + \frac{i\lambda^2}{2}(\iota_\Phi \iota_\Phi J) F^2 - \frac{i}{6} \iota_\Phi \iota_\Phi J^3 \right\}.$$

where we have used that $J = e^{\phi_{10}/2} v^a \omega_a$. Hence we precisely recover the expression as in (5.14). Finally, a similar analysis can be done for the case of non-vanishing internal B-field to recover (5.13).

Appendix B

Globally nilpotent T-brane backgrounds

In [41] it was recently shown that certain non-Abelian D7-brane vacuum solutions may be described in terms of a single curved D7-brane. More specifically, these vacua are compactifications of IIB string theory on $\mathbb{R}^{1,5} \times \mathbb{C}^2$ with a globally nilpotent Higgs-vev in $SU(N)$. Taking (x, z) to parametrise the $\mathbb{C}^2$-factor, the D7-brane stack on $\{z = 0\}$ is described by

$$\Phi = \begin{pmatrix} 0 & \phi_1 & & & \\ & 0 & \phi_2 & & 0 \\ & & \ddots & \ddots & \\ & 0 & & 0 & \phi_{N-1} \\ & & & & 0 \end{pmatrix}, \qquad \phi_a = \sqrt{a(N-a)}\, e^{C_{ab} f_b/2}, \tag{B.1}$$

where C_{ab} is the Cartan matrix of $SU(N)$ and the $\{f_a\}$ are functions of the D7-brane world-volume coordinates $(x, \overline{x})$. The flux is given as

$$F = -\mathrm{p}\overline{\partial} f_a C_a, \tag{B.2}$$

where the C_a are the Cartan generators of $SU(N)$. In this reference, explicit solutions $\{f_a\}$ to the D-term equations have been computed at leading order in α'. This leading order solution was then used to provide a description of this system in terms of a single, curved D7-brane. The latter description is in principle valid whenever the field vevs are large compared to α', but the authors of [41] noted that their solution should also be valid in regions where such vevs are small, due to the characteristic of their solution.

In the following we will take a complementary viewpoint and analyse the above background via the non-Abelian Hitchin system, better suited for for small field vevs. We will compute their α'-corrections explicitly and see that, just like in other T-brane backgrounds preserving eight supercharges, the classical solution is still valid after α'-corrections are taken into account. This implies that the classical analysis encodes all the information of the system, and that the dictionary built in [41] is not affected by α'-corrections.

Indeed, from eq.(5.14), we know that the corrected D-term equations are of the form $D = D_0 + \lambda^2 D_2 = 0$, with D_0 the leading order D-term and D_2 given by

$$D_2 = \int_{\mathcal{S}} \mathrm{S}\left\{ 2i\mathrm{D}\phi\wedge\overline{\mathrm{D}\phi}\wedge F - \left[\phi, \overline{\phi}\right] F^2 \right\}. \tag{B.3}$$

However in this background F, $\mathrm{D}\phi$ and $\bar{D}\overline{\phi}$ only have legs along $\mathrm{d}x$ and $\mathrm{d}\overline{x}$, and therefore D_2 vanishes identically. Hence, the whole system is insensitive to α'-corrections irrespective of how large the values for $\langle\phi\rangle$, $\langle D\phi\rangle$ and $\langle F\rangle$ are.

Appendix C

Non-Capital flux backgrounds

When analysing non-Abelian D-term equations in section 5.4, we have always made the Ansatz that the gauge transformation g that defines the non-primitive flux lies entirely within the Cartan subalgebra of the gauge group G. However, when analysing α'-corrected D-terms, the gauge derivatives generically introduce contributions to the D-terms also along the non-Cartan generators. Hence, it is natural to wonder whether adding worldvolume fluxes along non-Cartan generators may provide new solutions to the D-term equations.

In general, introducing non-Cartan fluxes via a gauge transformation leads to very involved BPS equations. For the setup at hand we may, however, follow a simple approach. Since we know that at leading order in λ no such flux is required to solve the D-term equations, we may assume that it is purely a λ-correction. This suggests that we capture the relevant physics if we perform an infinitesimal gauge transformation

$$\phi \longrightarrow \phi + [\delta g, \phi] \tag{C.1}$$

$$\overline{A} \longrightarrow \overline{A} + i\bar{\partial}\delta g, \tag{C.2}$$

with δg proportional to some small parameter $\lambda^2\alpha$, $[\alpha] = L^{-4}$. In the following we will implement this strategy for the two T-brane backgrounds analysed in section 5.4.

SU(2) example

Let us consider the $SU(2)$ background analysed in subsection 5.4.1, which we reproduce here for convenience

$$\phi = m \begin{pmatrix} 0 & e^f \\ axe^{-f} & 0 \end{pmatrix}, \tag{C.3}$$

$$F = -i\mathrm{p}\overline{\partial} f\,\sigma_3 - i\mathrm{p}\overline{\partial} h\,\mathbf{1}\,. \tag{C.4}$$

On top of this background we perform a gauge transformation of the form

$$\delta g \equiv \lambda^2 \left(\frac{\alpha}{2} E^+ + \frac{\overline{\alpha}}{2} E^- \right), \tag{C.5}$$

where

$$E^+ = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \qquad E^- = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix}. \tag{C.6}$$

Notice that the relation between the gauge parameters multiplying $E^\pm$ is necessary for the resulting flux to satisfy the Bianchi identity. Acting on the above background such gauge transformation gives

$$\delta\phi = -\frac{i\lambda^2 m}{2} \left(\alpha a x e^{-f} + \overline{\alpha} e^f \right) \sigma_3 \tag{C.7}$$

$$\delta F = -i\lambda^2 \mathrm{p}\overline{\partial} \left(\alpha\, E^+ - \overline{\alpha}\, E^- \right). \tag{C.8}$$

We then plug this into the D-term equations and consider the linear terms induced by this infinitesimal transformation

$$\begin{aligned} \omega\wedge\delta F + \omega^2 \left([\phi, \overline{\delta\phi}] + [\delta\phi, \overline{\phi}] \right) = \; & \frac{\lambda^2}{2} \left(\mathrm{p}_x \overline{\partial}_{\overline{x}} + \mathrm{p}_y \overline{\partial}_{\overline{y}} \right) \left(\alpha\, E^+ + \overline{\alpha}\, E^- \right) + \\ + \frac{\lambda^2 |m|^2}{2} & \left(2\overline{\alpha a x} + \alpha e^{2f} + \alpha |ax|^2 e^{-2f} \right) E^+ \\ + \frac{\lambda^2 |m|^2}{2} & \left(2\alpha a x + \overline{\alpha} e^{2f} + \overline{\alpha} |ax|^2 e^{-2f} \right) E^- . \end{aligned} \tag{C.9}$$

Interestingly the infinitesimal gauge transformation only introduces components in $E^\pm$, which means these new contributions are entirely decoupled from the D-term equations within the main text and may be considered independently.

From (C.9) we read off, that the parts in E^+ and E^- are conjugate to each other, and so we only need to satisfy one new D-term equation:

$$\left(\mathrm{p}_x \overline{\partial}_{\overline{x}} + \mathrm{p}_y \overline{\partial}_{\overline{y}} \right) \alpha = -2\overline{\alpha a x} |m|^2 - \alpha |m|^2 \left(e^{2f} + |ax|^2 e^{-2f} \right). \tag{C.10}$$

which we may solve asymptotically near the origin by plugging in the solution for f given in (5.36)

$$\alpha = \gamma \left(1 - c^2|mx|^2 - \frac{|mx|^4}{4c^2}\left(c^6 + \frac{|a|^2}{|m|^2}\right)\right) , \tag{C.11}$$

where $\gamma \in \mathbb{C}$ and $[\gamma] = L^{-4}$. We may interpret this one-parameter solution as a massless deformation to the T-brane background allowed at the infinitesimal level by the F- and D-terms. As pointed out in [4], this $SU(2)$ background contains one zero mode precisely along the generators $E^{\pm}$. Therefore it is natural to relate the parameter γ with the vev of this zero mode.

SU(3) example

Let us now apply this strategy to the $SU(3)$ background of subsection 5.4.2, more precisely we act with the infinitesimal gauge transformation

$$\delta g \equiv \lambda^2 \left(\frac{\alpha}{2}E^+ + \frac{\overline{\alpha}}{2}E^-\right) , \tag{C.12}$$

on the background (5.51). Now

$$E^+ = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad E^- = \begin{pmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so this transformation takes us to

$$\tilde{\phi} = \phi + \frac{\lambda^2}{2} m \left(\alpha m x e^{-f} - \overline{\alpha} e^f\right) P \tag{C.13}$$

$$\overline{\tilde{A}} = \overline{A} + \frac{i\lambda^2}{2}\overline{\partial}\left(\alpha E^+ + \overline{\alpha} E^-\right) , \tag{C.14}$$

so that we get new contributions to the D-term equations given by

$$\begin{aligned} \delta D = & -i\lambda^2 \omega \wedge \mathrm{p}\overline{\partial}\alpha\, E^+ + \lambda^2 \overline{mx}|m|^2 \overline{\alpha}\, E^+ \\ & - \frac{\lambda^2}{2}\alpha |m|^2 \left(e^{2f} + |mx|^2 e^{-2f}\right) E^+ + \text{h.c.} \end{aligned} \tag{C.15}$$

again exclusively along the non-Cartan generators $E^{\pm}$. This time the D-term equations have already some components along such non-Cartan generators.[1]

[1] More precisely, the fourth equation in (5.64) is a linear combination of those in the generators E^+ and E^- — which are conjugate to each other. The equation in E^+ reads

$$D^+ = -i\lambda^2|m|^2 \left(2\overline{\mu}e^f \mathrm{p}_x f \left(2\mathrm{p}_y \overline{\partial}_{\overline{x}} h + \overline{\kappa}\right) - \mu\overline{a}e^{-f}\left(2\overline{x}\overline{\partial}_{\overline{x}} f - 1\right)\left(2\mathrm{p}_x \overline{\partial}_{\overline{y}} h + \kappa\right)\right) \tag{C.16}$$

Recall from the discussion in the main text that it is precisely this equation that forced to set $\kappa = 0$. Therefore one may wonder if these new contributions proportional to α may allow for a non-trivial κ. Indeed, one can confirm that a gauge transformation given by

$$\alpha = \overline{x}\alpha_0 + |x|^2\alpha_1 + \overline{x}|x|^2\alpha_2 + \dots, \tag{C.17}$$

where the constant coefficients α_i depend intricately on $\kappa, f, \dots$ is such a solution. For instance we have that

$$\begin{aligned}
\alpha_0 =& \frac{4c\overline{\kappa\mu}^2}{m^*\left(4|m\mu|^2\lambda^2+1\right)\left(5c^6+4\lambda^2\left(|\kappa|^2c^6+\left(c^6+2\right)|m\mu|^2\right)+2\right)} \\
&\times\Bigg(-32\lambda^4|\mu|^4|m|^6+4\lambda^2\mu\left(|\kappa|^2\lambda^2c^6+c^6-4\right)\overline{\mu}|m|^4 \\
&+|m|^2\left(\lambda^2\left(4\lambda^2|\kappa|^4+9|\kappa|^2\right)c^6+5c^6-2\right)-|a|^2\left(|\kappa|^2\lambda^2+1\right)\left(4|m\mu|^2\lambda^2+1\right)\Bigg) \\
\alpha_1 =& -\frac{2\kappa\mu^2\left(m^*\right)^2}{c}.
\end{aligned} \tag{C.18}$$

Appendix D

Further SU(2) T-brane backgrounds

We have analysed in section 5.4 two different cases of T-brane backgrounds, whose non-commuting Higgs field generators lie entirely within an $\mathfrak{su}(2)$ subalgebra of the Lie group. As discussed in section 5.5, whenever that is the case one may focus on such $\mathfrak{su}(2)$ subalgebra when solving for the α'-corrected D-term equations, as the equations corresponding to other generators decouple. In this appendix we will apply the analysis of section 5.4 to further $SU(2)$ T-brane backgrounds, which are also examples of the 2×2 T-brane blocks discussed in section 5.5. Unlike the examples in section 5.4, here none of the backgrounds will be associated to a monodromy. In general we find that the presence or absence of monodromy does not really affect the behaviour of α'-corrections in T-brane systems.

In general we will follow the strategy of subsection 5.4.1 when analysing the backgrounds below. First we consider an Ansatz with a gauge transformation of the form (5.29) with $f \equiv f(x, \overline{x}, y, \overline{y})$ and a worldvolume flux of the form (5.39). In general we find that the Ansatz for the gauge transformation can be reduced to $f \equiv f(x, \overline{x})$. Moreover the effect of κ can be absorbed in the parameter m' defined in (5.42) in some cases, like in the T-brane examples 1 and 2, while others like T-brane example 3 seem to require a vanishing κ or a non-Cartan gauge transformation (c.f. Appendix C). Second we generalise our flux background to the form (5.43) and consider the expansion (5.45) for the gauge transformation, which in practice result in functions f and h that only depend on $(x, \overline{x})$, at least at lowest order in the expansion parameter $\lambda\rho$. As the procedure is identical for all the cases we present our results in a sketchy way, displaying the independent D-term equations for each Ansatz and the asymptotic solutions near the origin for the second one. All of the following examples satisfy $[\phi, \overline{\phi}] \equiv C\sigma_3$ for some C depending on the Higgs-vev, which we will use to abbreviate the following expressions. We will compute the D-term equations for the same two Ansätze as in 5.4. That is, on the one hand for a flux consisting of the two components

$$\begin{aligned} F &= -i\mathrm{p}\overline{\partial} f \cdot \sigma_3 \qquad\qquad f \equiv f(x, \overline{x}, y, \overline{y}) \\ H &= \mathrm{Im}\,(\kappa\, dx \wedge d\overline{y})\ \mathbf{1}, \end{aligned} \tag{D.1}$$

henceforth called Ansatz 1, and on the other hand for

$$\begin{aligned} F &= -i\mathrm{p}\overline{\partial} f \cdot \sigma_3 \\ H &= \mathrm{Im}\,(\kappa\, dx \wedge d\overline{y})\ \mathbf{1} + \rho\, i\,(dx \wedge d\overline{x} - dy \wedge d\overline{y})\ \mathbf{1} - i\mathrm{p}\overline{\mathrm{p}} h\,\mathbf{1} \\ f &\equiv f(x, \overline{x})\, h \equiv h(x, \overline{x}, y, \overline{y}), \end{aligned} \tag{D.2}$$

called Ansatz 2 in the following.

T-brane 1

$$\phi_{\mathrm{hol}} = m \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{D.3}$$

Ansatz 1:

$$\begin{aligned} &(\mathrm{p}_x\overline{\partial}_{\overline{x}} + \mathrm{p}_y\overline{\partial}_{\overline{y}}) f = C\left(1 + \lambda^2|\kappa|^2 + 4\lambda^2 Q_f\right) \\ &- \frac{8}{3}\lambda^2|m|^2 e^{2f}\left(\mathrm{p}_y\overline{\partial}_{\overline{y}} f\overline{\partial}_{\overline{x}} f\mathrm{p}_x f - \mathrm{p}_y f\overline{\partial}_{\overline{x}} f\mathrm{p}_x\overline{\partial}_{\overline{y}} f + \overline{\partial}_{\overline{y}} f\left(\mathrm{p}_y f\mathrm{p}_x\overline{\partial}_{\overline{x}} f - \mathrm{p}_y\overline{\partial}_{\overline{x}} f\mathrm{p}_x f\right)\right) \end{aligned}$$

Ansatz 2:

$$\begin{aligned} \mathrm{p}_x\overline{\partial}_{\overline{x}} f &= C\left(1 + 4\lambda^2 Q_H\right) \\ (\mathrm{p}_x\overline{\partial}_{\overline{x}} + \mathrm{p}_y\overline{\partial}_{\overline{y}}) h &= -8\lambda^2|m|^2 e^{2f}\overline{\partial}_{\overline{x}} f\mathrm{p}_x f\left(\mathrm{p}_y\overline{\partial}_{\overline{y}} h + \rho\right) + 4C\lambda^2\mathrm{p}_x\overline{\partial}_{\overline{x}} f(\rho + \mathrm{p}_y\overline{\partial}_{\overline{y}} h) \end{aligned}$$

Asymptotic solution

$$f_0 = \log c + c^2|m'x|^2 + \frac{1}{2}c^4|m'x|^4$$
$$f_1 = -4|mx|^2\left(4c^6\lambda^2|m|^2|m'|^2 + c^2\right) - 4c^4|m'|^2|m|^2x|^4\left(10c^4\lambda^2|m'|^2|m|^2+1\right)$$
$$h = -4c^4\lambda^2\rho|m|^2|m'x|^2 - 6c^6\lambda\rho|m|^2|m'x|^4$$

T-brane 2

$$\phi_{\text{hol}} = m\begin{pmatrix} 0 & ax \\ 0 & 0 \end{pmatrix} \tag{D.4}$$

Ansatz 1:

$$\begin{aligned}(\mathrm{p}_x\overline{\partial}_{\overline{x}} + \mathrm{p}_y\overline{\partial}_{\overline{y}})f = C\left(1+\lambda^2|\kappa|^2+4\lambda^2 Q_f\right) \\ -\frac{2}{3}\lambda^2|ma|^2e^{2f}\Big(\mathrm{p}_y\overline{\partial}_{\overline{y}}f\,|2x\mathrm{p}_xf+1|^2 + 4|x|^2|\mathrm{p}_yf|^2\mathrm{p}_x\overline{\partial}_{\overline{x}}f \\ -4\mathrm{Re}\left(x\mathrm{p}_yf\left(2\overline{x}\overline{\partial}_{\overline{x}}f+1\right)\mathrm{p}_x\overline{\partial}_{\overline{y}}f\right)\Big)\end{aligned}$$

Ansatz 2:

$$\begin{aligned}&\mathrm{p}_x\overline{\partial}_{\overline{x}}f = C\left(1+4\lambda^2Q_H\right) \\ &(\mathrm{p}_x\overline{\partial}_{\overline{x}} + \mathrm{p}_y\overline{\partial}_{\overline{y}})h = \\ &-2\lambda^2|ma|^2e^{2f}\,|2x\mathrm{p}_xf+1|^2\left(\mathrm{p}_y\overline{\partial}_{\overline{y}}h+\rho\right) + 4\lambda^2C\mathrm{p}_x\overline{\partial}_{\overline{x}}f(\rho+\mathrm{p}_y\overline{\partial}_{\overline{y}}h)\end{aligned}$$

Asymptotic solution

$$f_0 = \log c + \frac{1}{4}c^2|m'a|^2|x|^4$$
$$f_1 = -c^2|am|^2|x|^4\left(2\lambda^2c^2|am|^2+1\right)$$
$$h = -2\lambda^2\rho c^2|amx|^2$$

T-brane 3

$$\phi_{\text{hol}} = m\begin{pmatrix} by & ax \\ 0 & by \end{pmatrix} \tag{D.5}$$

Ansatz 1:

$$\left(1+4\lambda^2|mb|^2\right)\mathrm{p}_x\overline{\partial}_{\overline{x}}f-\mathrm{p}_y\overline{\partial}_{\overline{y}}f=C\left(1+\lambda^2|\kappa|^2+4\lambda^2 Q_f\right)$$
$$-\frac{2}{3}\lambda^2|ma|^2e^{2f}\Big(\mathrm{p}_y\overline{\partial}_{\overline{y}}f\,|2x\mathrm{p}_xf+1|^2+4|x|^2|\mathrm{p}_yf|^2\mathrm{p}_x\overline{\partial}_{\overline{x}}f$$
$$-2\mathrm{Re}\left(x\mathrm{p}_yf\left(2\overline{x}\overline{\partial}_{\overline{x}}f+1\right)\mathrm{p}_x\overline{\partial}_{\overline{y}}f\right)\Big)$$
$$0=-i\lambda^2a\overline{b}|m|^2\overline{\kappa}e^f\left(2x\mathrm{p}_xf+1\right)$$

Ansatz 2:

$$\mathrm{p}_x\overline{\partial}_{\overline{x}}f\left(1+4\lambda^2|mb|^2\right)=C\left(1+4\lambda^2Q_H\right)$$
$$(\mathrm{p}_x\overline{\partial}_{\overline{x}}+\mathrm{p}_y\overline{\partial}_{\overline{y}})h=-2\lambda^2|m|^2\left(|a|^2e^{2f}\,|2x\mathrm{p}_xf+1|\left(\mathrm{p}_y\overline{\partial}_{\overline{y}}h+\rho\right)+2|b|^2\left(\mathrm{p}_x\overline{\partial}_{\overline{x}}h-\rho\right)\right)$$
$$+4\lambda^2C\mathrm{p}_x\overline{\partial}_{\overline{x}}f(\rho+\mathrm{p}_y\overline{\partial}_{\overline{y}}h)$$
$$0=(2x\mathrm{p}_xf+1)\left(2\mathrm{p}_y\overline{\partial}_{\overline{x}}h+\overline{\kappa}\right)$$

Asymptotic solution

$$f_0=\log c+\frac{|am|^2|x|^4c^2}{16\lambda^2|bm|^2+4}$$
$$f_1=-c^2|am|^2|x|^2\left(2\lambda^2c^2|ma|^2+1\right)$$
$$h=-\frac{2\lambda^2\rho|mx|^2\left(c^2|a|^2-2|b|^2\right)}{4\lambda^2|bm|^2+1}$$

T-brane 4

$$\phi_{\mathrm{hol}}=m\begin{pmatrix}0&ax\\by&0\end{pmatrix}\tag{D.6}$$

Ansatz 1:

$$
\begin{aligned}
&(\mathrm{p}_x\overline{\partial}_{\overline{x}} + \mathrm{p}_y\overline{\partial}_{\overline{y}})f = C\left(1 + \lambda^2|\kappa|^2 + 4\lambda^2 Q_f\right) \\
&-\frac{2}{3}\lambda^2|ma|^2 e^{2f}\Bigg(\mathrm{p}_y\overline{\partial}_{\overline{y}}f\left|2x\mathrm{p}_x f + 1\right|^2 + 4|x|^2|\mathrm{p}_y f|^2\mathrm{p}_x\overline{\partial}_{\overline{x}}f \\
&-4\mathrm{Re}\left(x\mathrm{p}_y f\left(2\overline{x}\overline{\partial}_{\overline{x}}f + 1\right)\mathrm{p}_x\overline{\partial}_{\overline{y}}f\right)\Bigg) \\
&-\frac{2}{3}\lambda^2|mb|^2 e^{-2f}\Bigg(\mathrm{p}_x\overline{\partial}_{\overline{x}}f\left|2y\mathrm{p}_y f - 1\right|^2 + 4|y|^2|\mathrm{p}_x f|^2\mathrm{p}_y\overline{\partial}_{\overline{y}}f \\
&+4\mathrm{Re}\left(y\left(1 - 2\overline{y}\overline{\partial}_{\overline{y}}f\right)\mathrm{p}_y\overline{\partial}_{\overline{x}}f\mathrm{p}_x f\right)\Bigg) \\
&0 = |a|^2 e^{2f}\mathrm{Re}\left(\kappa x\mathrm{p}_y f\left(2\overline{x}\overline{\partial}_{\overline{x}}f + 1\right)\right) \\
&+|b|^2 e^{-2f}\mathrm{Re}\left(\kappa\overline{y}\overline{\partial}_{\overline{x}}f\left(2y\mathrm{p}_y f - 1\right)\right)
\end{aligned}
$$

Appendix E

4d interpretation of flux non-harmonicity

In section 6.1.1 we defined $\mathrm{d}\alpha = -i\partial\bar{\partial}g$ to be the exact part of the worldvolume flux that typically appears in T-brane solutions. For intersecting branes, a non-harmonic exact flux profile would break supersymmetry, and it would be seen as turning a non-vanishing vev for a Kaluza-Klein mode for the gauge vector field. If we consider a T-brane in the vicinity of a stability wall of the sort analysed in section 6.3.1, this correspondence between non-harmonic fluxes and Kaluza-Klein modes remains to a good extent accurate. Therefore, it is natural to interpret α as a set of KK modes that got a vacuum expectation value when the 4d Fayet-Iliopoulos term was switched on and the system evolved to a T-brane background. In the following we would like to give a more precise description of this intuition, in terms of the 4d effective gauge theory.

Let us begin with the D-term part of the 8d action, which is given by [14]

$$S \supset \int_{\mathbb{R}^{1,3}\times S} \mathrm{Tr}\left(\mathcal{D}\wedge *\mathcal{D}\right) \tag{E.1}$$

$$\begin{aligned}\mathcal{D} &= -*\left(J\wedge F + \frac{1}{2}[\Phi,\Phi^\dagger]\right) \\ &= *\left(-\frac{c}{4}J\wedge J - J\wedge \mathrm{d}\alpha - \frac{1}{2}*\varphi\right)\sigma_3,\end{aligned} \tag{E.2}$$

where we have applied the general Ansatz of section 6.3.1 and in particular made use of eqs. (6.6) and (6.8). To convert this to a 4d action, we need to expand the relevant fields in eigenbasis of the Laplacian, and then perform dimensional reduction. More precisely, we denote by ψ_n a real 0-form basis of the Laplacian, normalised as

$$\Delta_0 \psi_n \equiv -c_n^2 \psi_n \tag{E.3}$$

$$\frac{1}{V_S}\int_S \psi_n \wedge *\psi_m \equiv \delta_{nm}\,, \tag{E.4}$$

where V_S stands for the volume of the four-cycle S. As said before, α should contain the eigenmodes of the gauge vector field A. Now, given the relation (6.4) and the fact that $[\Delta, \mathrm{d}^c] = 0$, if the function g is an eigenmode of the Laplacian so will be α. Therefore, one naturally expands α as

$$\alpha = \frac{2}{V_S}\sum_{n\neq 0} a_n(x)\, \mathrm{d}^c \frac{\psi_n}{c_n}\,, \tag{E.5}$$

where $a_n(x)$ are interpreted as canonically-normalised 4d fields, which are eventually going to acquire a vev. Additionally, we can interpret the function φ defined in (6.6) in terms of the internal profile of the Higgs-field zero mode. More precisely, near the wall of stability we have that

$$\varphi = |\phi(x)|^2 \frac{1}{V_S}\sum_n m_n \psi_n\,, \tag{E.6}$$

where $m_n \in \mathbb{R}$ and $\phi(x)$ is the 4d charged field whose vev generates a T-brane profile of the form (6.11). On the one hand, the fact that ϕ is canonically normalised translates into $m_0 = 1$. On the other hand, the fact that we obtain a finite quartic coupling for this field when we plug (E.6) into (E.1) translates to the fact that the sum $\sum_n m_n^2$ must converge. Finally, one may easily extend this decomposition to a more general non-nilpotent-Higgs-field profile. Here for simplicity we will focus on the nilpotent case.

Plugging both expansions in the above action we obtain

$$S \supset \frac{1}{2V_S}\int_{\mathbb{R}^{1,3}} \mathrm{d}^4x \left(\left(cV_S + |\phi|^2\right)^2 + \sum_{n\neq 0}\left(4c_n a_n - m_n|\phi|^2\right)^2 \right), \tag{E.7}$$

which is nothing but eq. (6.9) expanded in a basis of eigenmodes of the Laplacian. In other words, we have that at the wall there are cubic couplings of the form $a_n|\phi|^2$. If now $c \neq 0$ and ϕ develops a vev to cancel the first term, that is the usual 4d D-term, the Kaluza-Klein modes of the gauge vector field must also do so. In particular we have that

$$< a_n > = \frac{m_n}{4c_n} |\phi|^2 \,. \tag{E.8}$$

As the m_n are bounded from above, these vev's for the KK modes will typically decrease as their mass c_n increases.

Appendix F

Examples of wall crossing for coincident branes

As a proof of existence, we will construct different examples of 4-cycles inside a compact Calabi-Yau showing the properties discussed in section 6.3.1. Consider the toric ambient space $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_2$, where we label coordinates and divisor classes as given in table F.1. Using the Stanley-Reisner ideal, we can read off

x_1	x_2	x_3	x_4	x_5	x_6	x_7
1	1	0	0	0	0	0
0	0	1	1	0	0	0
0	0	0	0	1	1	1
$\uparrow$		$\uparrow$		$\uparrow$		
H_1		H_2		H_3		

Table F.1: Ambient space $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_2$.

that the only non-vanishing intersection product in the ambient space is given by $H_1 \cdot H_2 \cdot H_3^2 = 1$. We define a Calabi-Yau 3-fold X inside this ambient space by the zero locus of the most general polynomial in the class $[X] = 2H_1 + 2H_2 + 3H_3$. One may check that X is non-singular. Using Lefshetz hyperplane theorem we know that $H^{1,1}(\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_2) \cong H^{1,1}(X)$, such that X inherits the Kähler form

$$J = v_1\, H_1 + v_2\, H_2 + v_3\, H_3\,, \qquad v_i \geq 0 \tag{F.1}$$

from the ambient space. Similarly, we have $H^{0,1}(X) = H^{0,1}(\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_2) = 0$. In the following we will show different wall-crossing phenomena present on three 4-cycles inside the Calabi-Yau.

Decay

First, consider the 4-cycle S defined by the vanishing locus $S = \{x_5 + x_6 + x_7 = 0\}$. Using the adjunction formula, we compute its total Chern class as

$$\begin{aligned} c(S) &= \frac{c(X)}{[S]} = \frac{c(\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_2)}{[X]\,[S]} \\ &= 1 - H_3 + \cdots , \end{aligned} \tag{F.2}$$

from which we can read off in particular that S is negatively curved, $\mathcal{R} = -c_1(K_S) = c_1(S) = -H_3$. In the notation of section 6.3.1, we take

$$\mathcal{M} = H_1 \tag{F.3}$$

$$\Rightarrow \mathcal{P} = \mathcal{M}^{-1} \otimes {\mathcal{K}_S}^2 = 2H_3 - H_1 , \tag{F.4}$$

where we can identify line bundles and their Chern classes, because $h^{0,1} = 0$ and therefore $\mathrm{Pic}(S) \cong H^{1,1}(S) \cap H^2(S, \mathbb{Z})$. To determine the physical spectrum of the coincident branes we need to compute the zeroth and first cohomologies of $\mathcal{M}$ and $\mathcal{P}$. We can simply read off the zeroth cohomologies from the toric data, where wee see, in particular, that $\mathcal{M}$ is effective whereas $\mathcal{P}$ is not. To determine the first cohomology groups we use *cohomCalg* [72,73], and in summary we have

$$h^\bullet(\mathcal{M}) = \big(2, 0, 0\big) \tag{F.5}$$

$$h^\bullet(\mathcal{P}) = \big(0, 0, 0\big) . \tag{F.6}$$

From here we see that T-branes can only be stable on one side of the wall. Moreover, from

$$\begin{aligned} \xi &= -2 \int_S c_1(\mathcal{L}) \wedge J = -\frac{1}{2} \int_S \Big(c_1(\mathcal{M}) - c_1(\mathcal{P})\Big) \wedge J \\ &= 2v_1 - v_2 - 2v_3 , \end{aligned} \tag{F.7}$$

we see that the Fayet-Ilioupoulos term can indeed acquire both signs depending on the position in Kähler moduli space. Notice that $\int_S c_1^2(K_S) = 0$ and $I = 2$, in agreement with the necessary condition of section 6.3.1 for a decay.

T-brane to T-brane crossing

Let us repeat the analysis of the last subsection for the different combination of 4-cycle S and line bundle $\mathcal{M}$ given by

$$[S] = 2H_1 + 3H_3 \tag{F.8}$$

$$\mathcal{M} = H_1 + 4H_3 \tag{F.9}$$

$$\Rightarrow \mathcal{P} = 3H_1 + 2H_3\,, \tag{F.10}$$

where S should be defined for instance by the most general polynomial in the given class in order to be non-singular. The line bundle cohomologies are given by

$$h^\bullet(\mathcal{M}) = \big(30, 0, 0\big) \tag{F.11}$$

$$h^\bullet(\mathcal{P}) = \big(24, 0, 0\big) \tag{F.12}$$

and the Fayet-Ilioupoulos

$$\xi = -6v_1 - 3v_2 + 2v_3\,. \tag{F.13}$$

From the above we read off that the Fayet-Ilioupoulos term can acquire both signs, and T-branes are stable on both sides, due to the condensation of either the modes of $\mathcal{M}$ or of $\mathcal{P}$.

T-brane to T-brane or bound state of gauge field

Last, consider

$$[S] = 2H_1 + 2H_3 \tag{F.14}$$

$$\mathcal{M} = 3H_3 \tag{F.15}$$

$$\Rightarrow \mathcal{P} = 2H_1 + H_3\,, \tag{F.16}$$

where the bundle cohomologies are given by

$$h^\bullet(\mathcal{M}) = \big(10, 1, 0\big) \tag{F.17}$$

$$h^\bullet(\mathcal{P}) = \big(9, 0, 0\big)\,, \tag{F.18}$$

and the Fayet-Ilioupoulos is given by

$$\xi = -4v_1 - v_2 + 2v_3\,, \tag{F.19}$$

which can acquire both signs depending on the position in Kähler moduli space. We read off that on one side of the wall T-branes are stable, whereas at the other side we may either have T-brane bound states, non-Abelian gauge profiles or a combination of the two.

Appendix G

Lie algebra conventions

In section 7.1.1 we made use of both the generators of the complexified Lie-algebra $\mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}(2)$ as well as $\mathfrak{su}(3)_{\mathbb{C}} = \mathfrak{sl}(3)$. Let us therefore summarise the conventions used for the generators here.

$\mathfrak{sl}(2)$ generators

As already indicated in 7.1.1, we use the following conventions

$$T_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad T_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{G.1}$$

satisfying the commutation relations

$$[T_+, T_-] = T_3, \qquad , [T_3, T_+] = 2T_+, \qquad [T_3, T_-] = -2T_- . \tag{G.2}$$

$\mathfrak{sl}(3)$ generators

In the main text our examples were constructed in an $\mathfrak{su}(3)$ background, where we made use both of the generators in the Cartan-Weyl basis as well as in the Chevalley basis, that has only integer structure constants. For convenience we give both bases explicitly here.

We denote the generators in the Cartan-Weyl basis by capital letters. Two elements of the Cartan are given by

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad H_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{G.3}$$

whereas the simple and highest roots are given by

$$E_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} E_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{G.4}$$

$$E_{12} = [E_1, E_2] = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Correspondingly, the negative roots are

$$\Theta_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Theta_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{G.5}$$

$$\Theta_{12} = [\Theta_2, \Theta_1] = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Conversely, we denote the generators in the Chevalley basis by lower-case letters. The Cartan elements are

$$\eta_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \eta_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{G.6}$$

while we denote simple and highest roots as

$$\epsilon_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \epsilon_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{G.7}$$

$$\epsilon_{12} = [\epsilon_1, \epsilon_2] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the negative roots correspondingly as

$$\theta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \theta_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{G.8}$$

$$\theta_{12} = [\theta_2, \theta_1] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Appendix H

BPS equations with defects

For convenience, here we spell out in detail the notation concerning the defect-BPS equations used in the main text, based on [14]. The setting we are interested in, is a 7-brane stack hosting an 8d SYM, which is coupled to defects localised at the interesection with another 7-brane stack. Take S and S' to be these two 4-cycles intersecting in a complex curve $\Sigma \equiv S \cap S'$, which we take to be irreducible and smooth for simplicity. If we denote the two gauge groups as $G_S, G_{S'}$, the matter content of the theory can the be decomposed as

$$\mathrm{ad}(G_\sigma) = \mathrm{ad}(G_S) \oplus \mathrm{ad}(G_{S'}) \oplus \left(\bigoplus_j U_j \otimes U'_j \right) , \tag{H.1}$$

where the last part corresponds to additional matter localised on Σ transforming in bifundamental representations U, U' of the two gauge groups and G_σ denotes the gauge enhancement found along this locus. In particular the defect theory contains a pair of complex scalars (σ, σ^c) transforming as

$$\sigma \in \Gamma \left(K_\Sigma^{1/2} \otimes \mathcal{U} \otimes \mathcal{U}' \right) \tag{H.2a}$$

$$\sigma^c \in \Gamma \left(K_\Sigma^{1/2} \otimes \mathcal{U}^* \otimes (\mathcal{U}')^* \right) , \tag{H.2b}$$

where we denoted by $\mathcal{U}, \mathcal{U}'$ the associated vector bundles of U, U', which are determined by restricting the principal bundles on the 7-brane stacks to Σ.

In the following we will denote by $\langle \cdot, \cdot \rangle_{\mathcal{U}}$ the natural product between $\mathcal{U}$ and its dual bundle $\mathcal{U}^*$ and accordingly for $\mathcal{U}'$. This product induces a map to the Lie-algebra $\mathfrak{g}_S$ of G_S. If we denote the action of the generators of $\mathfrak{g}_S$ in $\mathcal{U}$ by

T, it is given by

$$T : \mathcal{U}^* \oplus \mathcal{U} \longrightarrow \mathfrak{g}_S \qquad \text{(H.3)}$$
$$(u, v) \mapsto \langle T\cdot, \cdot\rangle_{\mathcal{U}}.$$

Note, moreover that the bundles $\mathcal{U}, \mathcal{U}'$ and $K_\Sigma^{1/2}$ are all hermitian and therefore equipped with a metric

$$H : \mathcal{U} \longrightarrow \bar{\mathcal{U}} \qquad \text{(H.4a)}$$
$$H' : \mathcal{U}' \longrightarrow \bar{\mathcal{U}}' \qquad \text{(H.4b)}$$
$$h_\Sigma^{-1/2} : K_\Sigma^{1/2} \longrightarrow \bar{K}_\Sigma^{1/2}. \qquad \text{(H.4c)}$$

With these maps at hand, we may now construct the product and moment map introduced in 7.15. Recall that they are maps

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathfrak{g}_S} : \quad \left(K_\Sigma^{1/2} \otimes \mathcal{U}^* \otimes (\mathcal{U}')^*\right) \quad \oplus \quad \left(K_\Sigma^{1/2} \otimes \mathcal{U} \times \mathcal{U}'\right) \quad \longrightarrow \quad K_\Sigma \otimes \mathfrak{g}_S \qquad \text{(H.5)}$$
$$\mu : \quad \left(\bar{K}_\Sigma^{1/2} \otimes \bar{\mathcal{U}} \otimes \bar{\mathcal{U}}'\right) \quad \oplus \quad \left(K_\Sigma^{1/2} \otimes \mathcal{U} \otimes \mathcal{U}'\right) \quad \longrightarrow \quad \mathfrak{g}_S, \qquad \text{(H.6)}$$

the first of which can now be composed out of the natural product of $\mathcal{U}'$ and H.3 as

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathfrak{g}_S} = \langle T\cdot, \cdot\rangle_{\mathcal{U}} \otimes \langle \cdot, \cdot\rangle_{\mathcal{U}'}, \qquad \text{(H.7)}$$

while the second also involves the hermitian bundle metrics $G, H, h_\Sigma^{-1/2}$ as

$$\mu := \langle h_\Sigma^{-1/2} \cdot, \cdot\rangle_{K_\Sigma^{1/2}} \langle TH\cdot, \cdot\rangle_{\mathcal{U}} \langle H'\cdot, \cdot\rangle_{\mathcal{U}'}. \qquad \text{(H.8)}$$

Locally, we may therefore write

$$\langle\!\langle \sigma^c, \sigma \rangle\!\rangle_{\mathfrak{g}_S} = \sigma_j^c (T^I)^j{}_i \sigma^i \, \mathfrak{t}_I \qquad \text{(H.9)}$$
$$\mu = h_\Sigma^{-1/2} \left[H'^{-1}_{\bar{A}B} \bar{\sigma}_{\bar{A}}^{\bar{k}} H_{\bar{k}j} (T^I)^j{}_i \sigma_B^i - H'^{B\bar{A}} \sigma_i^{c\,B} (T^I)^i{}_j H^{j\bar{k}} \bar{\sigma}_{\bar{k}}^{c\,\bar{A}} \right] \mathfrak{t}_I, \qquad \text{(H.10)}$$

where we denoted $\mathfrak{t}_I$ the generators of $\mathfrak{g}_S = \mathrm{Lie}(G_S)$ and by T^I their action on $\mathcal{U}$. Note, that this equation holds globally on Σ in the case that both $\mathcal{U}$ and $\mathcal{U}'$ are split bundles.

Appendix I

Examples of holomorphic and meromorphic scenario

In this section we will construct an explicit example of a stable T-brane on a positive curvature 4-cycle intersecting a second 4-cycle in a curve. To keep it short, we present an example that allows both for the holomorphic as well as the meromorphic mechanism. Of course not all combinations of intersecting 4-cycles allow for this, in general. More specifically, we will construct a del-Pezzo surface S embedded into a Calabi-Yau 3-fold X_3 intersecting a second 4-cycle S'. We consider the toric ambient space Y_4 given in tab. I.1. We may cut out a CY-3 fold X_3 given in class by $[X_3] = 12D_2 - 4D_1 + 5D_3 + 6D_4$, where the divisor classes D_i are associated to $\{x_i = 0\}$. This Calabi-Yau has previously been contructed in [74]. An explicit, non-singular representative may be found

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
1	1	0	0	0	4	1	1
0	1	0	0	1	6	2	2
0	0	1	0	0	2	1	1
0	0	0	1	0	3	1	1

Table I.1: Toric ambient space Y_4 with Stanley-Reissner ideal of Y_4 is given by $\{x_1x_2, x_2x_5, x_1x_3, x_1x_4, x_4x_6, x_3x_7x_8, x_5x_6x_7x_8\}$.

but is omitted here for brevity.

We reproduce also the intersection polynomial from [74] as

$$I_{X_3} = 2D_1^3 - 2D_2D_3^2 + 2D_3^3 + 3D_2D_3D_4 - 2D_2D_4^2 - 6D_3D_4^2 + 8D_4^3. \quad \text{(I.1)}$$

The KD"ahler cone is given by

$$J = \sum_{i=1}^{5} v_i G_i, \qquad 0 < v_i, \quad \text{(I.2)}$$

where we used

$$G_1 = D_2, \qquad G_2 = 2D_2 + D_3 + D_4, \quad \text{(I.3)}$$

$$G_3 = -2D_1 + 6D_2 + 2D_3 + 3D_4, \quad G_4 = -2D_1 + 6D_2 + 3D_3 + 3D_4, \quad \text{(I.4)}$$

$$G_5 = 6D_2 + 2D_3 + 3D_4. \quad \text{(I.5)}$$

Given this ambient 4-fold and Calabi-Yau 3-fold data, we may consider the two 4-cycles $S, S' \subset X_3$ given as

$$[S] = D_4, \qquad [S'] = 3D_2, \quad \text{(I.6)}$$

both of which have non-singular representatives and intersect transversally. Note, that S is a rigid $\mathrm{dP}_1 = \mathbb{P}^1 \times \mathbb{P}^1$. Since all of our computations will be on S from now on, we also give the intersection polynomial of S as

$$I_S = 3D_2D_3 - 2D_2D_4 - 6D_3D_4 + 8D_4^2. \quad \text{(I.7)}$$

In order to construct examples for the holomorphic and meromorphic scenario respectively, we will neeed to make different bundle choices, such that we deal with the two cases independently.

Holomorphic Scenario

We take the two bundles as

$$c_1(\mathcal{L}_3) = -2D_1 - 2D_2 - D_3 - 2D_4 \quad \text{(I.8)}$$

$$c_1(\mathcal{L}_8) = 2D_1 + 2D_2 + 2D_3 + D_4. \quad \text{(I.9)}$$

Finally, we have fixed all the necessary data to define the holomorphic scenario and may check if all requirements are ment. First of all $\mathcal{M}$ is indeed effective, since we have

$$\int_S J \wedge c_1(\mathcal{M}) = 2v_2 + 6v_4 \geq 0. \tag{I.10}$$

Moreover, because of

$$\deg_\Sigma \left(\hat{\mathcal{L}}_3^{-1} \otimes \hat{\mathcal{L}}_8 \otimes K_\Sigma^{1/2} \right) = 6, \tag{I.11}$$

holomorphic sections σ_2 exist.

Meromorphic Scenario

Let us repeat this exercise for the meromorphic scenario. We define the two bundles in question as

$$c_1(\mathcal{L}_3) = -2D_1 - 2D_2 - 2D_3 - 4D_4 \tag{I.12}$$

$$c_1(\mathcal{L}_8) = 2D_1 + 2D_2 + D_3 + 2D_4. \tag{I.13}$$

Indeed this choice satisfies the effectiveness constraint for $\mathcal{M} \otimes [\Sigma]$, since we may compute

$$\int_S J \wedge (c_1(\mathcal{M}) + [\Sigma]) = 2v_1 + 13v_2 + 39v_4 \geq 0. \tag{I.14}$$

And moreover, because of

$$\deg_\Sigma \left(\hat{\mathcal{L}}_3^{-1} \otimes \hat{\mathcal{L}}_8 \otimes K_\Sigma^{1/2} \right) = 30 \tag{I.15}$$

$$\deg_\Sigma \left(\hat{\mathcal{L}}_3^{-1} \otimes \hat{\mathcal{L}}_8^{-1} \otimes K_\Sigma^{1/2} \right) = 18, \tag{I.16}$$

holomorphic σ_2 and σ_1^c modes exist.

Appendix J

4d reduction and massive modes

In section 7.3, we discussed the four-dimensional picture related to the previous sections, but restricted ourselves to the most relevant subcase and omitting many technical details. As stated in the main text, we will give these details here. We have organised this appendix in the same way as section 7.3 to make the comparison as simple as possible. Before we begin with the physical analysis, let us discuss the different form-eigenbases of the Laplacian we will need for the computation as well as some mathematical conventions. Note that we sum over repeated indices, except for the dummy-index $\bullet$.

The spectrum of bulk KK modes

Let us quickly review the notation we use with respect to the notation with respect, to Hodge star, scalar product and adjoint operators. We denote by $*$ the map

$$* : \Omega^{(p,q)} \longrightarrow \Omega^{(2-q,2-p)}, \tag{J.1}$$

which induces a scalar product

$$\langle \alpha, \beta \rangle \equiv \frac{1}{V_S} \int_S \alpha \wedge * \bar{\beta} \tag{J.2}$$

and it is with respect to this scalar product, how we define the adjoint differential operators

$$\langle \overline{\partial}\alpha, \beta \rangle = \langle \alpha, \overline{\partial}^{\dagger}\beta \rangle \tag{J.3}$$

$$\Rightarrow \overline{\partial}^{\dagger} = - * \mathrm{p} * . \tag{J.4}$$

Recall, that our T-brane example form the main text is given in $\mathfrak{su}(2)$, such that all forms each are valued in three different bundles, corresponding to the three generators of $\mathfrak{su}(2)$. We, therefore denote by ψ_3^I and $\psi_\pm^I$ these three $(0,1)$-form eigenbases of the Laplacian

$$\Delta_{\overline{\partial}_\bullet} \psi_\bullet^I \equiv -(l_\bullet^I)^2 \psi_\bullet^I \tag{J.5}$$

and accordingly the $(2,0)$-form bases as χ_3^α and $\chi_\pm^\alpha$

$$\Delta_{\overline{\partial}_\bullet} \chi_\bullet^A = -(k_\bullet^A)^2 \chi_\bullet^A, \tag{J.6}$$

where there is no summation over the repeating indices. Moreover, we take both bases to be orthornormal. That is

$$\delta^{AB} = \frac{1}{V_S} \int_S \chi_\bullet^A \wedge \bar{\chi}_\bullet^B \tag{J.7}$$

$$\delta^{IJ} = \frac{1}{V_S} \int_S \psi_\bullet^I \wedge * \bar{\psi}_\bullet^J . \tag{J.8}$$

Recall the gauge covariant derivative and its Laplacian

$$\overline{\partial}_A^{\dagger} = - * \mathrm{p}_A * \tag{J.9}$$

$$\Delta_{\overline{\partial}_A} = \overline{\partial}_A \overline{\partial}_A^{\dagger} + \overline{\partial}_A^{\dagger} \overline{\partial}_A , \tag{J.10}$$

and let us define its action on the one-form bases as

$$\overline{\partial}\psi_3^I \equiv i \mu_{3A}^I \bar{\chi}_3^A \tag{J.11}$$

$$\left(\overline{\partial}_A \psi^I\right)_\pm \equiv i \mu_{\pm,A}^I \bar{\chi}_\mp^A , \tag{J.12}$$

Note, that this equation gives us a relation between the eigenvalues under the Laplacian for the two bases. Namely, by acting with the Laplacian on both sides of the equation we get

$$\Rightarrow \overline{\partial}\psi_3^I \equiv i \frac{(k_3^A)^2}{(l_3^I)^2} \mu_{3A}^I \bar{\chi}_3^A , \tag{J.13}$$

such that for a given pair (i, α) eqs.(J.11) and (J.13) may only be satisfied if either $\mu^I_{3A} = 0$ or $(k^A_3)^2 = (l^I_3)^2$.

As we will see below, the superpotential couples one- and two-forms in the Yukawa-couplings and therefore we will need to give relations between the $(0,1)$-form and the $(2,0)$-form basis, such that we define the set of constants Λ as

$$\Lambda^{IJ}_{3A} = \int_S \psi^I_+ \wedge \psi^J_- \wedge \chi_{3A} \tag{J.14}$$

$$\Lambda^{IJ}_{+A} = \int_S \psi^I_3 \wedge \psi^J_+ \wedge \chi_{3A} \tag{J.15}$$

$$\Lambda^{IJ}_{-A} = \int_S \psi^I_3 \wedge \psi^J_- \wedge \chi_{3A}. \tag{J.16}$$

Lastly, to integrate the 6d superpotential on Σ, we need introduce a set of constants parametrising the norm on Σ

$$\Gamma^{IJ} \equiv \int_\Sigma \psi^I_\bullet \wedge \bar{\psi}^J_\bullet. \tag{J.17}$$

F-terms without defects

We will start by computing the four dimension superpotential from 8d SYM and then in a second step compute the additional contributions induced by defects. Recall that

$$W_S = \int_S \mathrm{Tr}\, \Phi \wedge \mathbb{F}. \tag{J.18}$$

As in section 7.3, we will work in the case of an $SU(2)$ split-bundle and are now interested to study infinitesimal fluctuations around the background $\langle \Phi \rangle = 0$ and $\mathbb{A} = A\, T_3$, such that $\mathbb{F} \wedge J = 0$. We denote the fluctuations by

$$\delta A^{(0,1)} \equiv \begin{pmatrix} a_3 & a_+ \\ a_- & -a_3 \end{pmatrix}, \qquad \delta \Phi \equiv \begin{pmatrix} v & m \\ p & -v \end{pmatrix}. \tag{J.19}$$

Let us now pass to four dimensions by expanding the modes in suitable basis. To this end, recall that the relevant fields transform as

$$a_3 \in \Omega^{0,1}(S, \mathcal{O}), \qquad a_\pm \in \Omega^{0,1}(S, \mathcal{L}^{\pm 2}), \tag{J.20}$$

$$v \in \Omega^{2,0}(S, \mathcal{O}), \qquad m \in \Omega^{2,0}(S, \mathcal{L}^2), \qquad p \in \Omega^{2,0}(S, \mathcal{L}^{-2}). \tag{J.21}$$

Each of these six spaces needs to be expanded in its own basis, defined in eqs. (J.5)and (J.6).

$$v \equiv v_A \chi_3^A, \qquad \dots, \qquad a_\bullet \equiv a_{\bullet I} \psi_\bullet^I. \tag{J.22}$$

Plugging this into (J.18) gives

$$\begin{aligned} W_{4d} &= i\mu^I_{3A} v_A a_{3I} - 2i v_A a_{+I} a_{-J} \Lambda^{IJ}_{3A} \\ &+ i\mu^I_{-A} m_A a_{-I} + 2i m_A a_{3I} a_{-J} \Lambda^{IJ}_{-A} + i\mu^I_{+A} p_A a_{+I} - 2i p_A a_{3I} a_{+J} \Lambda^{IJ}_{+A} \end{aligned} \tag{J.23}$$

We may read off the equations of motions for u, v, m and p from each line, while those for $a_\bullet$ are given by

$$0 = v_A \mu^I_{3\,A} + m_A a_{-J} \Lambda^{IJ}_{-\,\alpha} - p_A a_{+\,J} \Lambda^{IJ}_{+\,\alpha}, \tag{J.24}$$

$$0 = m_A \mu^I_{-\,\alpha} + 4 v_A a_{+\,J} \Lambda^{IJ}_{3A} - 2 m_A a_{3\,j} \Lambda^{IJ}_{-\,\alpha}, \tag{J.25}$$

$$0 = p_A \mu^I_{+\,\alpha} - 4 v_A a_{-\,J} \Lambda^{IJ}_{3A} + 2 p_A a_{3\,j} \Lambda^{IJ}_{+\,\alpha}, \tag{J.26}$$

where we have used that $\Lambda^{JI} = -\Lambda^{IJ}$. Note, that the $\mu^I_{\bullet,\alpha}$ vanish if either α or i are harmonic forms. Restricting to the remaining forms, μ is in fact invertible and we may express the equations of motion with respect to derivation by the u, v, m and p as

$$\mu^\alpha_{3I} a_{3I} = 2 a_{+K} a_{-J} \Lambda^{KJ}_{3A} \tag{J.27}$$

$$0 = \left(\mu^I_{-A} - 2 a_{3J} \Lambda^{IJ}_{-A}\right) a_{-I} \tag{J.28}$$

$$0 = \left(\mu^I_{+A} + 2 a_{3J} \Lambda^{IJ}_{+A}\right) a_{+I}. \tag{J.29}$$

Zero-modes

Let us focus for a moment on zero-modes, that is all the μ's vanish. Let us moreover assume that a_3 contains no zero-modes because S is simply-connected so in particular the e.o.m with respect to it does not exist for zero-modes. Then the zero-modes need to satisfy

$$0 = a_{+I} a_{-J} \Lambda^{IJ}_{3A} \tag{J.30}$$

$$0 = v_A a_{-J} \Lambda^{IJ}_{3A} \tag{J.31}$$

$$0 = v_A a_{+J} \Lambda^{IJ}_{3A}. \tag{J.32}$$

Apart from pathological cases, this implies that for each set of indices two of the three fields vanish.[1]

Massive modes

For the massive modes the μ are diagonal, invertible matrices. We can see from eq. (J.27) - (J.29) that one solution is given by $a_3 = a_\pm = 0$ and from (7.142) - (J.26) that this allows for $v = p = m = 0$. This is the solution we are after. In general there are no other solutions.[2]

F-terms with defects

Let us now compute the additional defect contributions to the superpotential. The six-dimensional defect superpotential is given by

$$\begin{aligned} W_\Sigma &= \int_\Sigma \langle \sigma^c, \overline{\partial}_{\mathbb{A}} \sigma \rangle \\ &= \int_\Sigma \langle \sigma^c, \overline{\partial}_{\langle \mathbb{A} \rangle} \sigma - i\delta\mathbb{A}(\sigma) \rangle, \end{aligned}$$

where by $\delta\mathbb{A}(\sigma)$ we denoted the action of $\delta\mathbb{A}$ in the fundamental representation of $\mathfrak{su}(2)$ on σ. Recall from eqs. (7.20) and (7.21) that the fields σ^c, σ transform as sections of $K_\Sigma^{1/2}$, whereas the product $\langle \sigma^c \sigma \rangle$ transforms as a section of K_Σ, such that it is much more convenient to expand the product of the two fields in

[1] If for every α, the matrices Λ^{IJ}_{3A} regarded as a map to $\mathbb{C}$ have the same non-trivial kernel, then these conditions do not imply the vanishing of the individual fields in the above equations. I.e. The double sum over i, j might allow for cancellations.

[2] Assume we want to find a solution with $a_- = 0$, then to solve (J.28) a_3 needs to acquire a vev in order to cancel the term in parenthesis. Correspondingly, (J.27) then forces a_+ to acquire a non-vanishing vev $a_+ \neq 0$, which in turn implies that (J.29) can only be solved if the term in parenthesis vanishes. This is generically not possible as there is no relation between the $\mu_\pm$'s and $\Lambda_\pm$'s. To be more precise: For such a cancellation to happen, either we need to have $\mu^I_{+,\alpha} = A\,\mu^I_{-,\alpha}$ and $\Lambda^{IJ}_{+,\alpha} = A\,\Lambda^{IJ}_{+,\alpha}$ for some $A \in \mathbb{C}$ or that $\mathrm{supp}(\Lambda_+) \subseteq \ker(\Lambda_-)$ and vice-versa.

our previous bases $\{\psi_3, \psi_+, \psi_-\}$, than to expand the individual fields [3]

$$\sigma_1^c \sigma_1 = (\sigma_1^c \sigma_1)_i \, \bar{\psi}_3^I, \qquad \sigma_2^c \sigma_2 = (\sigma_2^c \sigma_2)_i \, \bar{\psi}_3^I \tag{J.33}$$
$$\sigma_1^c \sigma_2 = (\sigma_1^c \sigma_2)_i \, \bar{\psi}_+^I, \qquad \sigma_2^c \sigma_1 = (\sigma_2^c \sigma_1)_i \, \bar{\psi}_-^I.$$

Pluggin all of this into (J.33), gives

$$\begin{aligned} W_\Sigma = {} & i \, (\sigma_2^c \sigma_2)_I \, \Gamma^{IJ} a_{3J} - i \, (\sigma_1^c \sigma_1)_I \, \Gamma^{IJ} a_{3J} \\ & - i \, (\sigma_1^c \sigma_2)_I \, \Gamma^{IJ} a_{+J} - i \, (\sigma_2^c \sigma_1)_I \, \Gamma^{IJ} a_{-J}. \end{aligned} \tag{J.34}$$

With both superpotential contributions eqs. (J.23) and (J.34) at hand, we may now compute the equations of motion. Those for the component fields of $\delta\Phi$ can be easily read off from eq. (J.23) and do not depend on the defect fields. On the other hand those for the components of $\delta\mathbb{A}$ are given by

$$\Gamma^{IJ} \left((\sigma_1^c \sigma_1)_j - (\sigma_2^c \sigma_2)_j \right) = v_A \mu_{3A}^I + m_A a_{-J} \Lambda_{-,\alpha}^{IJ} - p_A a_{+J} \Lambda_{+,\alpha}^{IJ} \tag{J.35a}$$
$$\Gamma^{IJ} \, (\sigma_2^c \sigma_1)_j = m_A \mu_{-,\alpha}^I + 2 v_A a_{+J} \Lambda_{3A}^{IJ} - m_A a_{3,j} \Lambda_{-,\alpha}^{IJ} \tag{J.35b}$$
$$\Gamma^{IJ} \, (\sigma_1^c \sigma_2)_j = p_A \mu_{+,\alpha}^I - 2 v_A a_{-J} \Lambda_{3A}^{IJ} + p_A a_{3,j} \Lambda_{+,\alpha}^{IJ}, \tag{J.35c}$$

while those for the defect fields themselves are given as

$$0 = \sigma_2^l \left(M^{12} \right)_{kl}^I \Gamma^{IJ} a_{+J} \tag{J.36a}$$
$$0 = \sigma_1^l \left(M^{21} \right)_{kl}^I \Gamma^{IJ} a_{-J} \tag{J.36b}$$
$$0 = \sigma_2^{c\,K} \left(M^{21} \right)_{kl}^I \Gamma^{IJ} a_{-J} \tag{J.36c}$$
$$0 = \sigma_1^{c\,K} \left(M^{12} \right)_{kl}^I \Gamma^{IJ} a_{+J}. \tag{J.36d}$$

Let us now try to solve them, at least partially. We proceed separately for zero modes and massive modes.

Zero-modes

Let us focus for a moment on zero-modes, that is all the μ's vanish. Let us moreover assume that a_3 contains no zero-modes because S is simply-connected

[3] At the cost of more notation, one might also expand the the fields individually and then define a set of additional coefficients, that relate the basis of 0-forms valued in $K_\Sigma^{1/2} \otimes \mathcal{L}^\pm$ to those of $(1,0)$-forms valued in $\mathcal{L}^\pm$ and $\mathcal{O}$.

so in particular the e.o.m with respect to it does not exist for zero-modes. Then the zero-modes need to satisfy

$$0 = a_{+K} a_{-J} \Lambda_{3A}^{KJ} \tag{J.37a}$$

$$\Gamma^{IJ} \left(\sigma_2^c \sigma_1\right)_J = 2 v_A a_{+J} \Lambda_{3A}^{IJ} \tag{J.37b}$$

$$\Gamma^{IJ} \left(\sigma_1^c \sigma_2\right)_J = -2 v_A a_{-J} \Lambda_{3A}^{IJ}. \tag{J.37c}$$

The same comments regarding a non-trivial kernel of the Λ's as in the case without defects also apply here. Note, that the index I runs only over zero-modes whereas in particular α may run over massive modes!

Massive modes

For the massive modes the story is more interesting. First, under the same caveats as for the non-defect case, we have $a_3 = a_\pm = 0$ — so we may have only non-trivial vevs in the zero-mode part of the $a_\pm$ modes. Let us therefore denote by $\hat{a}$ the zero-modes of a to emphasise that the massive modes vanish. In this notation the equations of motion for the massive modes in v, m, p read

$$\begin{aligned} v = &\left(\mu^I_{3\,A}\right)^{-1} \Gamma^{IJ} \left(\left(\sigma_1^c \sigma_1\right)_j - \left(\sigma_2^c \sigma_2\right)_j\right) \\ &- \left(\mu^I_{3\,A}\right)^{-1} \left(m_\beta \hat{a}_{-\,J} \Lambda^{IJ}_{-\,\beta} - p_\beta \hat{a}_{+\,J} \Lambda^{IJ}_{+\,\beta}\right) , \end{aligned} \tag{J.38a}$$

$$m = \left(\mu^I_{+\,\alpha}\right)^{-1} \Gamma^{IJ} \left(\sigma_2^c \sigma_1\right)_j - 2 \left(\mu^I_{-\,\alpha}\right)^{-1} v_\beta \hat{a}_{+\,J} \Lambda^{IJ}_{3\,\beta} \,, \tag{J.38b}$$

$$p_A = \left(\mu^I_{-,\alpha}\right)^{-1} \Gamma^{IJ} \left(\sigma_1^c \sigma_2\right)_j + 2 \left(\mu^I_{+\,\alpha}\right)^{-1} v_\beta \hat{a}_{-\,J} \Lambda^{IJ}_{3\,\beta} \,. \tag{J.38c}$$

These are the generalised version of the constraints found in eqs.(7.150).

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