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FINITE YANG-MILLS THEORIES

AND THE BJORKEN-JOHNSON-LOW LIMIT*

Ahmed Ali

The Stevens Institute of Technology
Hoboken, New Jersey 07030

and

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305†

and

Jeremy Bernstein

The Stevens Institute of Technology
Hoboken, New Jersey 07030

ABSTRACT

We consider the Bjorken-Johnson-Low limit for the propagator in massless Yang-Mills theories. The significance of our result in terms of imposing an eigenvalue on the theory so as to render it finite is discussed.

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I. INTRODUCTION

Baker and Johnson[1] have made the observation that finite massless QED either has free electron and photon propagators or noncanonical commutation relations. For the sake of completeness we present their argument before discussing the Yang-Mills situation. For simplicity we first give the discussion for fields of zero spin, $\phi(\vec{x}, t)$ which we suppose to be coupled in some fashion with a dimensionless, unrenormalized coupling constant g_0. The details of the coupling do not matter so long as the Bjorken-Johnson-Low[2] limiting procedure is valid. We suppose, to begin with, that the $\phi(\vec{x}, t)$ have a mass m and then pass to the mass-zero limit. The object to be discussed is the unrenormalized ϕ propagator $D(q^2)$;

$$\begin{aligned} D(q^2) &= -i \int d^4x < 0 | \left(\phi(\vec{x}, t)\phi(0)\right)_+ | 0 > e^{iq\cdot x} \\ &\equiv d\left(\frac{q^2}{\Lambda^2}, \frac{m^2}{q^2}, g_0\right) / q^2 + m^2 \end{aligned} \tag{1}$$

Our notation is as follows:

$$\begin{aligned} q\cdot x &= \vec{q}\cdot\vec{x} - q_0 t \\ \left(A(\vec{x}, t)B(0)\right)_+ &= \theta(t)\, A(\vec{x}, t)\, B(0) + \theta(-t)\, B(0)\, A(\vec{x}, t) \end{aligned} \tag{2}$$

and $d\left(\frac{q^2}{\Lambda^2}, \frac{m^2}{q^2}, g_0\right)$ is a Lorentz-scalar, dimensionless, form factor which is, in general, a function of the ultraviolet cutoff Λ. (There may be an infrared cutoff whose dependence is not included explicitly since it does not alter the discussion.) We now pass to the BJL[2] limit in Eq. (1) by taking

$$\vec{q}^{\,2} = \text{fixed}$$

and

$$q_0 \sim +\infty$$

with the appropriate analyticity we can write

$$D(q^2) = -\frac{i}{q_0}\int d^3\vec{x}\, e^{i\vec{q}\cdot\vec{x}} \sum_n \left\{ \frac{<0|\phi(\vec{x},0)|n><n|\phi(0)|0>}{1+\left(\frac{p_0}{q_0}\right)^n} - \frac{<0|\phi(0)|n><n|\phi(\vec{x},0)|0>}{1-\left(\frac{p_0}{q_0}\right)^n} \right\}$$

$$\simeq -\frac{i}{q_0}\int d^3\vec{x}\, e^{i\vec{q}\cdot\vec{x}} \left\{ <0|[\phi(\vec{x},0),\phi(0)]|0> + \frac{1}{q_0} <0|[\dot{\phi}(\vec{x},0),\phi(0)]|0> + O\left(\frac{1}{q_0^2}\right) \right\}$$

$$= -\frac{1}{q_0^2}\, d(-\infty, 0, g_0) \tag{3}$$

If the ϕ obeys canonical commutation relations at equal times we conclude that

$$d(-\infty, 0, g_0) = 1 \quad . \tag{4}$$

Now, in the massless case, $m = 0$, we have by dimensional analysis

$$d = d\left(\frac{q^2}{\Lambda^2}, g_0\right) \tag{5}$$

and if the theory is to be ultraviolet finite for some "eigen" g_0 say

$$g_0 = g \tag{6}$$

we have

$$d = d(0, g) < \infty \tag{7}$$

at the eigenvalue. Thus, as $q_0 \to +\infty$

$$D(q^2) \to -\frac{1}{q_0^2}\, d(0, g) \tag{8}$$

so the equal time commutation relation is given by[3]

$$\left[\phi(\vec{x},0),\, \dot{\phi}(0)\right] = i\ d(0,g)\ \delta^3(\vec{x}) \tag{9}$$

which is only canonical if

$$d(0,g) = 1 \quad . \tag{10}$$

This is the Baker-Johnson argument.

This discussion can be extended straight forwardly to the following cases:

Spinor Fields coupled to:

a) Massless Abelian vector mesons, or

b) Massive Abelian vector mesons

The massive non-Abelian case deserves a special discussion as a preliminary to the work in the next section. We confine ourselves, for illustrative purposes, to the self-coupled SU(2) Yang-Mills case in which the vector meson $\vec{b}_\mu(x)$ is an isotopic triplet. The key observation is that $\dot{\vec{b}}_\mu(x)$ is not canonical to $\vec{b}_\mu(x)$. Indeed if

$$\mathscr{L}(x) = -\frac{1}{4}\left(\frac{\partial}{\partial x^\mu}\vec{b}_\nu(x) - \frac{\partial}{\partial x^\nu}\vec{b}_\mu(x) + g_0\ \vec{b}_\mu(x)\times\vec{b}_\nu(x)\right)^2 - \frac{1}{2}m_0^2\,\vec{b}_\mu(x)\cdot\vec{b}^\mu(x) \tag{11}$$

then the canonical momenta are given by

$$\pi_0(x) = 0 \tag{12}$$

and

$$\vec{\pi}_i(x) = \dot{\vec{b}}_i(x) - g_0\vec{b}_i(x)\times\vec{b}_0(x) + \frac{\partial}{\partial x_i}\ \vec{b}_0(x) \quad . \tag{13}$$

With canonical commutation relations among the $\vec{b}_\mu$ and $\vec{\pi}$ this leads to the following equal time commutation relations[4]

$$\left[b_i(\vec{x},0)_s,\ b_j(0)_t\right] = 0 \tag{14}$$

and

$$\left[\dot{b}_i(\vec{x},0)_s, b_j(0)_t\right] = -i\left[\delta_{ij} - \frac{1}{m_0^2}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\right]\delta^3(\vec{x})\,\delta_{st}$$

$$+ i\,\epsilon_{str}\, b_i(\vec{x},t)_r \frac{\partial}{\partial x_j}\delta^3(\vec{x})\frac{g_0}{m_0^2}$$

$$- i\frac{g_0^2}{m_0^2}\epsilon_{sr\ell}\,\epsilon_{trm}\, b_i(\vec{x},0)_\ell b_j(0)_m\,\delta^3(\vec{x}) \quad . \tag{15}$$

Here, s and t are isotopic indices. We may ask, how does this set of commutation relations modify the discussion above?[5] Let us consider for $i, j = 1, 2, 3$

$$D_{ij}^{st}(q^2) = \left(\delta_{ij} - \frac{q_i q_j}{m_0^2}\right) d\left(\frac{q^2}{\Lambda^2}, \frac{m^2}{q^2}, g_0\right) / q^2 + m^2$$

$$= -i \int d^4x \;<0|\left(b_i(\vec{x},0)_s\, b_j(0)_t\right)_+|0>\, e^{iq\cdot x}$$

$$\simeq -\frac{1}{q_0^2}\left[\delta_{ij} - \frac{q_i q_j}{m_0^2} - \frac{g_0^2}{m_0^2}\epsilon_{sr\ell}\,\epsilon_{trm}\int d^3x\, e^{i\vec{q}\cdot\vec{x}}\,\delta^3(\vec{x})\right.$$

$$\left. <0|b_i(\vec{x},0)_\ell\, b_j(0)_m|0>\right] . \tag{16}$$

The key question is what is $<0|b_\mu(\vec{x},0)_\ell\, b_\nu(\vec{x},0)_m|0>$? On the grounds of positivity of the metric in Hilbert space and Lorentz covariance one may argue[6] that this vacuum expectation value must vanish; i.e., from positivity it must be positive for all $\mu=\nu$, while from Lorentz covariance the $\mu=\nu=0$ terms must have the opposite sign to $\mu=\nu=i$. Hence this infinite expression must be regulated in such a way that it vanishes. Thus for the massive non-Abelian case we still have

$$d(-\infty, 0, g_0) = 1 \quad . \tag{17}$$

We have been working in a formalism in which the m=0 limit cannot be taken directly. But note that in the massless case there is no gauge in which positivity in the Hilbert space metric and Lorentz covariance can be maintained simultaneously. This gives us a clue that the massless non-Abelian theory may yield something new. In fact, as we shall see, the Baker-Johnson argument no longer obtains.

II. THE MASSLESS NON-ABELIAN CASE

In the canonical theory we still have Eq. (13) and Eq. (14) in the $m_0=0$ case. We must, however, recompute

$$\left[b_i(\vec{x},0)_s,\ \dot{b}_j(0)_t\right]$$

using Eq. (13). Since

$$\vec{\pi}_0 = 0 \tag{18}$$

$\vec{b}_0$ is not a dynamical variable and must be eliminated. Hence one must solve the equations of motion in some gauge. It is convenient to do this in the Coulomb gauge with

$$\vec{\nabla}\cdot\vec{b}(\vec{x},t) = 0 \tag{19}$$

especially in view of the fact that in this gauge Schwinger[7] has computed the commutator we are interested in, in a closed form. All we need to do is to expand Schwinger's result to order g_0^2. The details are straightforward but tedious, and we find

$$\begin{aligned}\left[b_i(\vec{x},0)_s,\ \dot{b}_j(0)_t\right] &= -i\ \delta_{st}\ \delta^{tr}_{ij}(\vec{x}) \\ &\quad -i\,\frac{g_0^2}{4\pi}\ \epsilon_{s\ell m}\,\epsilon_{mtn}\ b_i(\vec{x},0)_\ell \int \frac{d^3x'}{|\vec{x}'-\vec{x}|} \\ &\quad \times \delta^{tr}_{kj}(\vec{x})\ b_k(\vec{x}',0)_n\end{aligned} \tag{20}$$

where

$$\delta_{ij}^{tr}(\vec{x}) = \frac{1}{(2\pi)^3} \int d\vec{k}\ e^{i(\vec{k}\cdot\vec{x})} \left[\delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2}\right] \tag{21}$$

When we take the Fourier transform of the vacuum expectation value of Eq. (20), which we call $O_{ij}(\vec{q})_{st}$, the term of order g_0^2 diverges as $\log\left(\frac{\Lambda^2}{|\vec{q}|^2}\right)$ where Λ is an ultraviolet cutoff. Using rotational covariance and transversality we can write,

$$O_{ij}(\vec{q})_{st} = \left(\delta_{ij} - \frac{q_i q_j}{|\vec{q}|^2}\right) \pi(|\vec{q}|^2)\ \delta_{st} \tag{22}$$

where $\pi(|\vec{q}|^2)$ is a dimensionless form factor. Thus[8]

$$O_{ij}(\vec{q})_{st} = \left(\delta_{ij} - \frac{q_{ij}}{|\vec{q}|^2}\right) \delta_{st} \times \left[1 - \frac{g^2}{12\pi^2}\left(5 + 4\ \ln \frac{\Lambda}{|\vec{q}|}\right)\right] \tag{23}$$

Hence our computation has given Z_3 in the Coulomb gauge to order g_0^2, i.e.,

$$Z_3 = 1 - \frac{g^2}{3\pi^2}\ \ln\left(\frac{\Lambda}{\mu}\right) \tag{24}$$

where μ is any $|\vec{q}| = \mu$.

In general

$$Z_3 = Z_3\left(\frac{\Lambda}{\mu}, g_0\right) \tag{25}$$

which at the eigenvalue, if there is one, will take the form

$$Z_3 = Z_3(g) \tag{26}$$

In the massless Yang-Mills theories we may have both an eigenvalue condition and canonical commutation relations.

III. DISCUSSION

In this section we discuss the relation between the BJL limit for d_c, given above, and the results of the same limit taken by means of the Callan-Symanzik equations.[9] Since d_c is gauge dependent we confine our remarks to the Coulomb gauge in which we have been working. The question is under what circumstances are these two analyses compatible? Recalling that manifest covariance is lost in the Coulomb gauge we write for the BJL limit.

$$d_c\left(\frac{|\vec{q}|^2}{\mu^2}, \frac{q_0^2}{\mu^2}, g\right) \underset{\substack{|\vec{q}|^2 \text{ fixed} \\ q_0^2 \to +\infty}}{\sim} F\left(\frac{|\vec{q}|^2}{\mu^2}, g\right) < \infty \quad ; \tag{27}$$

i.e., the entire q_0 dependence at infinity is in the $\frac{1}{q^2}$ of the propagator which we have factored out. Now

$$d_c\left(\frac{|\vec{q}|^2}{\mu^2}, \frac{q_0^2}{\mu^2}, g\right) Z_3\left(\frac{\Lambda^2}{\mu^2}, g\right) = d\left(\frac{|\vec{q}|^2}{\Lambda^2}, \frac{q_0^2}{\Lambda^2}, g\right) \tag{28}$$

with

$$g = \frac{Z_3^{3/2}}{Z_1} g_0 \quad . \tag{29}$$

We have two CS equations; i.e.,

$$\left(\mu \frac{\partial}{\partial\mu} + \beta(g) \frac{\partial}{\partial g} + \gamma(g)\right) d_c = 0 \tag{30}$$

and

$$\left(\mu \frac{\partial}{\partial\mu} + \beta(g) \frac{\partial}{\partial g} - \gamma(g)\right) Z_3 = 0 \tag{31}$$

where

$$\gamma(g) = \frac{1}{Z_3} \mu \frac{d}{d\mu} Z_3 \tag{32}$$

and

$$\beta(g) = \mu \frac{dg}{d\mu} \quad . \tag{33}$$

Thus, solving,

$$d_c\left(\frac{|\vec{q}|^2}{\mu^2}, \lambda^2 \frac{q_0^2}{\mu^2}, g\right) = d_c\left(\frac{|\vec{q}|^2}{\lambda^2\mu^2}, \frac{q_0^2}{\mu^2}, \bar{g}(\tau, g)\right) \quad \exp\left(+ \int_0^\tau \gamma\left(\bar{g}(g,x)\right) dx\right) \tag{34}$$

where

$$\tau = \frac{1}{2} \log \lambda^2 \tag{35}$$

$$\bar{g} = \bar{g}(g, \tau); \qquad \bar{g}(g, 0) = g \tag{36}$$

and

$$\frac{d\bar{g}}{d\tau} = \beta(\bar{g}(g, \tau)) \quad . \tag{37}$$

Letting

$$\lambda = \frac{\Lambda}{\mu} \tag{38}$$

we have

$$Z_3(\lambda^2, g) = Z_3(1, \bar{g}) \exp\left(- \int_0^\tau \gamma\left(\bar{g}(g,x)\right) dx\right) \tag{39}$$

To make contact with the BJL limit we make the assumption that

$$\begin{aligned} &\lim_{|\vec{q}| \to \infty} \lim_{q_0 \to \infty} d_c\left(\frac{|\vec{q}|}{\mu}, \frac{q_0}{\mu}, g\right) \\ &= \lim_{\lambda \to \infty} d_c\left(\lambda \frac{|\vec{q}|}{\mu}, \lambda \frac{q_0}{\mu}, g\right)_{BJL} = d_c\left(\lambda \frac{|\vec{q}|}{\mu}, g\right)\Big|_{\lambda \to \infty} \end{aligned} \tag{40}$$

Now in our example, to order g^2

$$d_c\left(\lambda \frac{|\vec{q}|}{\mu}, g\right) \sim \log\left(\lambda \frac{|\vec{q}|}{\mu}\right) \underset{\lambda\to\infty}{\sim} \infty \quad . \tag{41}$$

If this behavior persists to all orders, i.e., if $d_c \to \infty$ there is nothing much to be said. However, if the logs sum to suitable powers so that

$$\lim_{\lambda\to\infty} d_c\left(\lambda \frac{|\vec{q}|}{\mu}, g\right) < \infty \tag{42}$$

for some range of values of g then we may draw some interesting conclusions. In this case we would have

$$\lim_{\lambda\to\infty} \int_0^{\tau} \gamma(\bar{g}(g,x))\, dx < \infty \tag{43}$$

$$\cdots$$

i.e.,

$$0 < \lim_{\lambda\to\infty} Z_3(\lambda^2, g) < \infty \quad . \tag{44}$$

But this is only possible if there exists a g' such that at g'

$$\beta(g') = 0 \tag{45}$$

and

$$\gamma(g') = 0 \quad . \tag{46}$$

Hence for the analyses of the limit to be compatible β must have at least <u>three</u> zeros. The reason for this is that the zero of β at g' must be associated with a negative slope for this to be a "stagnation point"; i.e.,

$$\lim_{\lambda\to\infty} \bar{g} = g' \tag{47}$$

so that

$$\lim_{\lambda\to\infty} Z_3(1,\bar{g}) \to Z_3(1,g') < \infty \quad . \tag{48}$$

The zero of β at the origin is associated with a negative slope so the curve of β starts down and if there is a second zero it will be reached with a positive slope. It is the third zero, if there is one, which will have the desired negative slope. The physical coupling constant need not be at this zero, only in the domain of attraction of this zero.

We see, therefore, that the consistency of these two approaches to the BJL limit places very strong constraints on β and γ. Since these functions have only been computed to very low orders in perturbation theory around the origin we do not know if the theory can meet these conditions, or if some proof can be found that they cannot be met. The introduction of fermions complicates the analysis still further. But we have seen that the self-coupled massless Yang-Mills theory in isolation is, already, a very intriguing, nontrivial situation.

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3. There are several discussions of "anomalous" equal time commutation relations in the literature. See, for example, R. Jackiw and G. Preparata, Phys. Rev. 185, 1748 (1969); and

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4. See, for example, T. D. Lee, S. Weinberg and B. Zumino, Phys. Rev. Letters 18, 1029 (1967).

5. R. Jackiw and G. Preparata (op. cit.) have considered the BJL limit of this theory.

6. This covariance argument is stated in Lee et al. (op. cit.). Jackiw and Preparata (op. cit.) note the ambiguity of this term, which ambiguity must be resolved by covariance.

7. J. Schwinger, Phys. Rev. 125, 1043 (1962). We would like to thank R. Treat for checking Eq. (20).

8. One may include couplings to fermions currents which modify the commutation relations and hence Z_3. These currents must be defined by point-splitting which is what causes the modified commutation relations. We would like to thank Lowell Brown for helpful discussion of this limiting process.

9. See, for example, H. D. Politzer, Phys. Rev. Letters 30, 1346 (1973); and

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Shu-Yuan Chu, to be published, has speculated on some of the consequences of assuming eigenvalue conditions for Yang-Mills theories.

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