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INSTABILITY OF A BUNCHED BEAM WITH SYNCHROTRON-FREQUENCY SPREAD

YONGHO CHIN,† KOHTARO SATOH, and KAORU YOKOYA

National Laboratory for High Energy Physics Oho-machi, Tsukuba-gun, Ibaraki-ken, 305, Japan

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The synchrotron-frequency spread due to nonlinearity of the rf voltage can stabilize a beam by Landau damping even in the presence of a resistive impedance. In this paper, a generalized dispersion relation is derived for transverse and longitudinal instabilities of a Gaussian bunch from Sacherer's integral equation including both azimuthal mode couplings and the spread in the synchrotron frequency. The method allows calculation of the stability limit for any radial mode for each azimuthal mode. Consistency with Besnier's method with the dispersion matrix is shown numerically. The generalized dispersion relation is applied with some crude assumptions to the transverse instability due to mode couplings in a double rf system adjusted to provide zero slope of the rf wave. The result shows that as the bunch current increases, the growth rate rises gradually without a clear threshold, in contrast to the case of no synchrotron-frequency spread.

1. INTRODUCTION

The interaction of a bunched beam with its surroundings causes transverse and longitudinal instabilities of coherent oscillations. Recently the fast transverse instability has been observed in large electron storage rings such as PETRA[1] and PEP,[2] and restricted the stored currents. Its growth time is very short and less than one synchrotron oscillation period. In the TRISTAN electron-positron storage ring,[3] cures for this instability would be important to increase the current limit.

For analysis of bunched-beam instabilities, Sacherer's integral equation has been formulated and solved for several types of bunch distributions.[4–8] Formulations with mode couplings[9–11] for the fast instability have also been derived by generalizing Sacherer's integral equation. In the standard derivation, the synchrotron frequency is assumed to be the same for all particles in a bunch. Strictly speaking, this assumption is not true because space charge forces and rf nonlinearity produce a synchrotron-frequency spread. Sacherer[4] has used a dispersion relation for the analysis with a spread and has shown that the imaginary part of the coherent oscillation frequency vanishes, i.e., a bunch is stable inside a certain stability limit. This mechanism is Landau damping. In the usual treatment of transverse instabilities, the incoherent betatron frequency spread is considered as a source of Landau damping. However, recently a higher-harmonic rf system has been discussed as a useful cure for the fast transverse instability;[12] this rf system produces a highly nonlinear potential in the synchrotron phase space and then leads to a longer bunch and a greatly increased spread in the synchrotron frequency.[13] In this case, the effect of the synchrotron-frequency spread becomes more important.

† On leave from Department of Physics, Faculty of Science, The University of Tokyo, Bunkyo-ku, Tokyo, 113.

Besnier[14] and recently Zotter[15] have developed an eigenvalue formulation by expanding the radial function in orthogonal polynomials. The effect of the synchrotron-frequency spread is described by adding a "dispersion matrix" to the usual "interaction matrix".[7] Though this method is a satisfactory one, exponential solutions appear everywhere and no stability limit can be obtained. To calculate Landau damping, we have to return to the usual dispersion relation by solving the initial-value problem. Careful consideration is necessary in dealing with coherent oscillations and in expanding the radial function in orthogonal polynomials inside the stability limit.

In this paper, we will derive a formulation for the transverse and longitudinal instability of a Gaussian bunch with the synchrotron-frequency spread by generalizing the usual dispersion relation. For longitudinal oscillation, we start from Sacherer's integral equation without mode couplings. The method of Besnier is reviewed for comparison and discussion. For transverse oscillation, we derive a formulation with mode couplings and discuss the effect of the synchrotron-frequency spread on the fast transverse instability. We also discuss, with some simplifying and crude assumptions, the characteristics of the transverse instability due to mode couplings under a double (main plus higher-harmonic) rf system adjusted to provide zero slope of the rf wave. Coherent oscillations inside a stability limit and their mathematical treatment are investigated in the Appendix.

2. SACHERER'S INTEGRAL EQUATION WITH SYNCHROTRON-FREQUENCY SPREAD

Sacherer's integral equation[4] can be derived when the trajectories of the particles in synchrotron phase space $(\theta, \dot{\theta}/\omega_S)$ are circular and the unperturbed distribution function g_0 in the Vlasov equation is a function only of the phase-space amplitude r. Here we use the angular position $\theta = r\cos\phi$ in the ring as the longitudinal phase-space coordinate. In order to use Sacherer's integral equation, we assume here that the synchrotron frequency ω_S is a function of r, $\omega_S = \omega_S(r)$ and its spread is very small compared with ω_S. We express the dispersion in the synchrotron frequency as

$$\omega_S(r) = \omega_{SC} + D(r), \tag{2.1}$$

where $\omega_{SC} = \nu_{SC}\omega_0$ is the synchrotron frequency at the center of bunch. We also still assume $g_0 = g_0(r)$, neglecting the small dependence of g_0 on phase space angle ϕ.

For longitudinal oscillations, Sacherer's integral equation for the perturbed distribution function (or radial function) g_m without mode coupling[15] becomes

$$[\Delta\omega - mD(r)]\, g_m(r) = w(r) \int_0^\infty G(r, r')\, g_m(r') r'\, dr', \tag{2.2}$$

where

$$\Delta\omega = \omega - m\omega_{SC} \tag{2.3}$$

is the coherent frequency shift for the azimuthal mode m and

$$w(r) = \frac{1}{r}\frac{dg_0}{dr} \tag{2.4}$$

is the weight function. The kernel $G(r, r')$ is given by

$$G(r, r') = +im\omega_S\xi_S \sum_{p=-\infty}^{\infty} \frac{Z_L(\omega_p)}{p} J_m(pr)J_m(pr'), \tag{2.5}$$

where

$$\xi_S = \frac{N\alpha e\omega_0{}^2}{T\omega_S{}^2 E/e} \tag{2.6}$$

is the scaling factor and $Z_L(\omega_p)$ is the longitudinal coupling impedance at frequency

$$\omega_p = (pM + n)\omega_0 + m\omega_S.$$

The other notations are as follows:

ω_0 = revolution angular frequency
M = number of bunches
n = coupled-bunch mode number
N = number of particles in a bunch
α = momentum compaction factor
e = elementary charge
E = beam energy
$T = 2\pi/\omega_0$ = revolution period.

Since we assume the synchrotron-frequency spread is small, we shall keep the first-order term in the dispersion $D(r)$ in Eq. (2.2) and take $\omega_S \approx \omega_{SC}$ in the kernel $G(r, r')$.

If we can find a complete set of normalized functions $f_k(r)$ $(k = 0, 1\ 2, \ldots)$, which have the orthogonality relation

$$\int_0^{\infty} w(r) f_k(r) f_l(r)\, r dr = \delta_{kl}, \tag{2.7}$$

we can expand g_m and $G(r, r')$ as

$$g_m(r) = w(r) \sum_{k=0}^{\infty} c_k f_k(r), \tag{2.8}$$

$$G(r, r') = \sum_{k,l} M_{kl} f_k(r) f_l(r'), \tag{2.9}$$

where

$$M_{kl} = \int_0^{\infty} r\, dr \int_0^{\infty} r' dr'\, G(r, r') w(r) f_k(r) w(r') f_l(r') \tag{2.10}$$

are the elements of the interaction matrix.

Equation (2.2) can be rewritten as

$$g_m(r) = \frac{w(r)\displaystyle\int_0^\infty G(r, r')g_m(r')r'dr'}{\Delta\omega - mD(r)}$$

$$= \frac{w(r)\sum_{k,l} M_{kl}f_k(r)}{\Delta\omega - mD(r)} \times \int_0^\infty f_l(r')g_m(r')r'dr'$$

$$= \frac{w(r)\sum_{k,l} M_{kl}f_k(r)}{\Delta\omega - mD(r)} c_l, \tag{2.11}$$

where the denominator is assumed not to vanish.
Multiplying Eq. (2.11) by $rf_s(r)$ and integrating with r, we obtain an infinite set of linear equations

$$c_s = \sum_{k,l} \alpha_{ks}^{(m)} M_{kl}c_l, \tag{2.12}$$

where

$$\alpha_{ks}^{(m)} = \int_0^\infty \frac{w(r)f_k(r)f_s(r)\, r\, dr}{\Delta\omega - mD(r)}. \tag{2.13}$$

Thus Sacherer's integral equation becomes equivalent to solving the following equation

$$\det\left|\delta_{sl} - \sum_{k=0}^{\infty} \alpha_{ks}^{(m)} M_{kl}\right| = 0. \tag{2.14}$$

This formulation is easily found to be the generalization of the well-known dispersion relation using the synthetic kernel approximation.[4] If we take only the lowest radial mode, the expansion function $f_0{}^{(m)}(r)$ is given by

$$f_0{}^{(m)}(r) = \sqrt{N_m}r^m, \tag{2.15}$$

with normalization constant N_m.
Then Eq. (2.14) leads to the dispersion relation

$$1 = N_m M_{00} \int \frac{w(r)r^{2m+1}\, dr}{\Delta\omega - mD(r)}. \tag{2.16}$$

Another method of analysis has been studied by Besnier,[14] who introduced the "dispersion matrix" for the usual eigenvalue equations. We review here the formulation in order to compare with our method. Numerical comparison and careful investigation of Landau damping will be done in Sec. 4. The product of the dispersion $D(r)$ and the radial function g_m is also expanded in orthogonal polynomials $f_k(r)$

$$D(r)g_m(r) = w(r)\sum_{k=0}^{\infty} b_k f_k(r). \tag{2.17}$$

The coefficient b_k can be expressed with c_k in Eq. (2.8)

$$b_k = \sum_{l=0}^{\infty} N_{kl}c_l, \tag{2.18}$$

where

$$N_{kl} = \int_0^{\infty} D(r)w(r)f_k(r)f_l(r)\, rdr \tag{2.19}$$

are the elements of the dispersion matrix.
Thus Sacherer's integral equation can be written as

$$\Delta\omega \sum_k c_k f_k(r) - m\sum_{k,l} N_{kl}c_l f_k(r) = \sum_{k,l} M_{kl}c_l f_k(r). \tag{2.20}$$

The effect of the dispersion of the synchrotron frequency is described by adding the dispersion matrix N to the interaction matrix M and the eigenvalue equation

$$\det(\Delta\omega I - M - mN) = 0. \tag{2.21}$$

3. DISPERSION RELATION INCLUDING MODE COUPLINGS

For transverse motion, Sacherer's integral equation without mode couplings is given by Chao and Yao in Ref. 6 as

$$\left[-i(\omega - \omega_\beta) + \omega_s \frac{\partial}{\partial\phi}\right] g_m(r)\, e^{im\phi} + \frac{ce}{2TE\nu_\beta} g_0(r) \sum_{p=-\infty}^{\infty} Z_T(p+\nu)\tilde{\rho}_m\left(p + \nu - \frac{\xi}{\alpha}\right) \exp\left[i\left(p + \nu - \frac{\xi}{\alpha}\right)\theta\right] = 0, \tag{3.1}$$

where

$$\tilde{\rho}_m(p) = \frac{1}{2\pi}\iint d\theta\, d\dot{\theta}\, \exp(-ip\theta) g_m(r)\, e^{im\phi}, \tag{3.2}$$

$\omega = \nu\omega_0$ is the coherent angular frequency and $g_m(r)$ is the perturbed distribution function for the azimuthal mode number m in the synchroton phase space. The other quantities are $Z_T(\omega)$, the transverse coupling impedance, $\omega_\beta = \nu_\beta\omega_0$, the betatron-oscillation angular frequency, c, the velocity of light, and ξ, the chromaticity.
We deal here only with a rigid dipole motion of the transverse coherent oscillation. To

derive this equation, they assumed the azimuthal dependence of the perturbed distribution function $g(r, \phi)$ as

$$g(r, \phi) \approx g_m(r)\, e^{im\phi}.$$

A more general form is obtained by replacing $g_m(r)e^{im\phi}$ by $g(r, \phi)$ in Eqs. (3.1) and (3.2). Furthermore, the ν_β appearing in the denominator of the second term should be replaced by β/R as pointed out by Kohaupt,[1] where β is the betatron function at the place where transverse forces are generated and R is the average radius. We thus start from the integral equation including mode couplings

$$\left[-i(\omega - \omega_\beta) + \omega_s(r)\frac{\partial}{\partial\phi}\right] g(r, \phi)$$
$$+ \frac{ec\omega_0\beta}{4\pi RE} \sum_{p=-\infty}^{\infty} Z_T(p + \nu)\, \tilde{\rho}\left(p + \nu - \frac{\xi}{\alpha}\right) \exp\left[i\left(p + \nu - \frac{\xi}{\alpha}\right)\theta\right] g_0(r) = 0, \quad (3.3)$$

where

$$\tilde{\rho}(p) = \frac{1}{2\pi}\int\int d\theta\, d\dot{\theta}\, \exp(-ip\theta)\, g(r, \phi). \quad (3.4)$$

This equation is derived more in detail in Ref. 11.
When the coordinate of synchrotron oscillation is transformed from $(\theta, \dot{\theta})$ to (r, ϕ) in Eq. (3.4) according to

$$\theta = r\cos\phi, \qquad \dot{\theta} = -\omega_s(r)\, r\sin\phi,$$

the volume element $d\theta\, d\dot{\theta}$ transforms with Jacobian determinant to a new volume element

$$\left[\omega_s(r) + r\sin^2\phi\,\frac{d\omega_s(r)}{dr}\right] r\, dr\, d\phi. \quad (3.5)$$

Since we assumed that the dispersion $D(r)$ is small compared with ω_{sc} and we keep the first-order term in $D(r)$ in Eq. (3.3), we ignore this correction and take an approximation in the volume element (3.5)

$$\left[\omega_s(r) + r\sin^2\phi\,\frac{d\omega_s(r)}{dr}\right] \approx \omega_{sc}. \quad (3.6)$$

It will be found later that the volume element (3.5) appears also in the normalization of $g_0(r)$ and the effect of the correction is almost cancelled in the final result.

Solutions of Eq. (3.3) have to be periodic in ϕ with period 2π; thus we can expand $g(r, \phi)$ with the azimuthal mode number m

$$g(r, \phi) = \sum_{m=-\infty}^{\infty} g_m(r)\exp(im\phi). \quad (3.7)$$

Substitution of Eq. (3.7) into Eq. (3.3) yields

$$-i\sum_m g_m[\omega - \omega_\beta - m\omega_s(r)]\exp(im\phi) + \frac{ec\omega_0\beta}{4\pi RE}\sum_p Z_T(p+\nu)\tilde{\rho}\left(p+\nu-\frac{\xi}{\alpha}\right)\exp\left[i\left(p+\nu-\frac{\xi}{\alpha}\right)\theta\right]\cdot g_0(r) = 0. \quad (3.8)$$

Multiplying by $\exp(-in\phi)$ and integrating over ϕ from 0 to 2π, we obtain

$$[\omega - \omega_\beta - n\omega_s(r)]g_n = -i\frac{ec\omega_0\beta}{4\pi RE}\sum_p Z_T(p+\nu)\,\tilde{\rho}\left(p+\nu-\frac{\xi}{\alpha}\right)(i)^n J_n\left(\left(p+\nu-\frac{\xi}{\alpha}\right)r\right)g_0, \quad (3.9)$$

and then

$$g(r,\phi) = -i\frac{ec\omega_0\beta}{4\pi RE}\sum_{m=-\infty}^{\infty}\sum_p Z_T(p+\nu)\,\tilde{\rho}\left(p+\nu-\frac{\xi}{\alpha}\right) \times (i)^m\frac{g_0 J_m\left(\left(p+\nu-\frac{\xi}{\alpha}\right)r\right)}{\omega-\omega_\beta-m\omega_s(r)}\exp(im\phi). \quad (3.10)$$

By use of a Fourier transformation defined by Eq. (3.4), Eq. (3.10) becomes

$$\tilde{\rho}(q) = -i\frac{ec\omega_0\beta}{4\pi RE}\sum_{m=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} Z_T(p+\nu)\,\tilde{\rho}\left(p+\nu-\frac{\xi}{\alpha}\right) \times \omega_{sc}\int\frac{g_0(r)J_m\left(\left(p+\nu-\frac{\xi}{\alpha}\right)r\right)J_m(qr)}{\omega-\omega_\beta-m\omega_s(r)}\,r\,dr. \quad (3.11)$$

For a Gaussian bunch, $g_0(r)$ is given by

$$g_0(r) = \frac{Ne}{2\pi\omega_{sc}\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right), \quad (3.12)$$

where ω_{sc} comes from the normalization

$$\iint g_0(r)\,d\theta\,d\dot{\theta} = Ne, \quad (3.13)$$

and σ is the bunch length in θ unit, i.e. $\sigma = \sigma_z/R$, when σ_z is the standard deviation of the bunch length.

When no dispersion exists, we could easily find orthogonal polynomials $e_k^{(m)}(x)$ which satisfy

$$\int_0^\infty \exp\left(-\frac{t^2}{2}\right)t^{2m+1}e_k^{(m)}(t^2)e_l^{(m)}(t^2)\,dt = \delta_{kl}, \qquad (m \geqslant 0). \quad (3.14)$$

Using the polynomials $e_k^{(m)}(t^2)$, we can expand the Bessel function as

$$J_m(p\sigma t) = \sum_{k=0}^{\infty} C_{mk}(p\sigma)t^m e_k^{(m)}(t^2), \tag{3.15}$$

$$J_{-m}(p\sigma t) = (-1)^m J_m(p\sigma t), \qquad (m \geqslant 0)$$

and then we have an expansion formula

$$\int_0^{\infty} \exp\left(-\frac{r^2}{2\sigma^2}\right) J_m(qr)\, J_m(pr)\, r\, dr = \sigma^2 \sum_{k=0}^{\infty} C_{|m|k}(q\sigma)C_{|m|k}(p\sigma). \tag{3.16}$$

The orthogonal polynomials assure good convergence of the expansion formula and make the required number of the radial modes much smaller for approximate calculation. With dispersion, the weight function changes from $\exp(-t^2/2)$ to $\exp(-t^2/2)/(\omega - \omega_\beta - m\omega_s(r))$ and we can no longer find such orthogonal polynomials in general. If, however, the dispersion is small, we can still use the same orthogonal polynomials as those in the case of no dispersion, which does not lose the good convergence of the expansion formula.

The polynomials which satisfy Eq. (3.14) are given by[8]

$$e_k^{(m)}(t^2) = D_k^{(m)}\, L_k^{(m)}\left(\frac{t^2}{2}\right), \tag{3.17}$$

where $L_k^{(m)}$ are the generalized Laguerre polynomials[16] and

$$D_k^{(m)} = \left(\frac{k!}{2^m(m+k)!}\right)^{1/2}. \tag{3.18}$$

The integral in Eq. (3.11) can be expressed as a sum

$$\int \frac{g_0(r)J_m((p+\nu-\xi/\alpha)r)\, J_m(qr)}{\omega - \omega_\beta - m\omega_s(r)}\, r\, dr$$

$$= \sigma^2 \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \alpha_{kl}^{(m)}\, C_{|m|k}\left(\left(p+\nu-\frac{\xi}{\alpha}\right)\sigma\right) C_{|m|l}(q\sigma), \tag{3.19}$$

where

$$\alpha_{kl}^{(m)} = \int \frac{\exp(-t^2/2)\, t^{2|m|+1}\, e_k^{(|m|)}(t^2)\, e_l^{(|m|)}(t^2)}{\omega - \omega_\beta - m\omega_s(\sigma t)}\, dt, \tag{3.20}$$

and

$$C_{|m|k}(p\sigma) = \frac{1}{\sqrt{(|m|+k)!k!}} \exp\left(-\frac{p^2\sigma^2}{2}\right)\left(\frac{p\sigma}{\sqrt{2}}\right)^{|m|+2k}. \tag{3.21}$$

We obtain from Eq. (3.11)

$$\tilde{\rho}(q) = -iK \sum_p Z_T(p+\nu)\, \tilde{\rho}\left(p+\nu-\frac{\xi}{\alpha}\right)$$

$$\times \sum_{m=-\infty}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \alpha_{kl}^{(m)}\, C_{|m|k}\left(\left(p+\nu-\frac{\xi}{\alpha}\right)\sigma\right) C_{|m|l}(q\sigma), \tag{3.22}$$

with

$$K = \frac{Ne^2\beta c\omega_0}{8\pi^2 RE}. \tag{3.23}$$

Since the $C_{|m|k}(p\sigma)$ have a recursion relation

$$\begin{aligned} C_{|m|k}(p\sigma) &= \sqrt{\frac{(|m| + 2k)!}{(|m| + k)!k!}}\, C_{|m|+2k,0}\,(p\sigma) \\ &= \sqrt{\frac{h!}{(h - k)!k!}} \end{aligned} \tag{3.24}$$

with $h = |m| + 2k$, we can transform the triple summation with m, k and l in Eq. (3.22) into a double summation

$$\begin{aligned} \tilde{\rho}(q) = &-iK\sum_{p} Z_T(p + \nu)\, \tilde{\rho}\left(p + \nu - \frac{\xi}{\alpha}\right) \\ &\times \sum_{h=0}^{\infty}\sum_{u=0}^{\infty} \beta_{hu}F_h\left(\left(p + \nu - \frac{\xi}{\alpha}\right)\sigma\right) F_u(q\sigma), \end{aligned} \tag{3.25}$$

where β_{hu} is a symmetrical matrix with elements:

$$\begin{aligned} &\beta_{00} = \alpha_{00}^{(0)}, \quad \beta_{01} = 0, \quad \beta_{11} = \alpha_{00}^{(-1)} + \alpha_{00}^{(1)}, \\ &\beta_{02} = \sqrt{2}\alpha_{10}^{(0)} = 0, \quad \beta_{12} = 0, \quad \beta_{22} = \alpha_{00}^{(-2)} + \alpha_{00}^{(2)} + 2\alpha_{11}^{(0)}, \\ &\beta_{03} = 0, \quad \beta_{13} = \sqrt{3}(\alpha_{01}^{(-1)} + \alpha_{01}^{(1)}), \quad \beta_{23} = 0, \\ &\beta_{33} = \alpha_{00}^{(-3)} + \alpha_{00}^{(3)} + 3(\alpha_{11}^{(-1)} + \alpha_{11}^{(1)}), \ldots \end{aligned} \tag{3.26}$$

We now expand the Fourier transform of the line density in terms of $F_u(q\sigma)$

$$\tilde{\rho}(q) = \sum_{u=0}^{\infty} a_u F_u(q\sigma). \tag{3.27}$$

Substitution of this equation into Eq. (3.25) leads to linear equations

$$a_u = -iK \sum_{s=0}^{\infty}\sum_{h=0}^{\infty} \beta_{hu}M_{hs}a_s, \tag{3.28}$$

where

$$M_{hs} = \sum_{p=-\infty}^{\infty} Z_T(p + \nu)F_h\left(\left(p + \nu - \frac{\xi}{\alpha}\right)\sigma\right) F_s\left(\left(p + \nu - \frac{\xi}{\alpha}\right)\sigma\right). \tag{3.29}$$

Thus we have to look for eigenvalues of the equation

$$\det|\delta_{us} + iK\sum_{h=0}^{\infty} \beta_{hu}M_{hs}| = 0. \tag{3.30}$$

When mode couplings can be neglected, we take one particular azimuthal mode m in the sum of Eq. (3.22) and expand the line mode in $C_{ml}(q\sigma)$

$$\tilde{\rho}(q) = \sum_{l=0}^{\infty} d_l C_{ml}(q\sigma). \tag{3.31}$$

Substituting this equation into Eq. (3.22), we obtain the linear equations

$$d_l = -iK \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \alpha_{kl}^{(m)} M_{ks} d_s, \tag{3.32}$$

where

$$M_{ks} = \sum_{p=-\infty}^{\infty} Z_T(p+\nu) C_{mk}\left(\left(p+\nu-\frac{\xi}{\alpha}\right)\sigma\right) C_{ms}\left(\left(p+\nu-\frac{\xi}{\alpha}\right)\sigma\right). \tag{3.33}$$

When mode couplings are included, we can no longer expand the radial function $g_m(r)$ in the phase space as in Eq. (2.8), using the same orthogonal polynomials $t^m e_k{}^{(m)}(t^2)$ as for negligible mode couplings, because the orthogonality relation (3.14) is not available between different azimutual mode numbers. Instead of them, we need to take a linearly independent complete orthonormal set by rearranging $t^m e_k{}^{(m)}(t^2)$ according to Gram-Schmidt orthogonalization process,[17] whose results are difficult to express with $t^m e_k{}^{(m)}(t^2)$ in general. On the other hand, analysis of Sacherer's integral equation in the frequency domain can make it easy to find such linearly independent bases, because the expansion functions $C_{mk}(p\sigma)$ already depend only on the sum $m + 2k$, except for their coefficients [see Eq. (3.24)]. Due to this reason, we can obtain the simple formulation for the generalization of Sacherer's integral equation including mode couplings.

4. LANDAU DAMPING

The synchrotron-frequency spread is mainly produced by space-charge forces and nonlinearity of the rf voltages. The space-charge contribution is proportional to the beam current and hence changes the growth rates of coherent oscillations, but does not give a threshold current.[15] On the other hand, the spread due to the rf nonlinearity is independent of the beam current and thus will give a threshold. Since we are interested in a stability limit, we shall consider only the contribution of the rf nonlinearity.

The dispersion in the synchrotron frequency can be approximated by

$$D(r) = -S\frac{r^2}{\sigma^2}, \tag{4.1}$$

where S is the spread in ω_s between the center and one standard deviation σ of the bunch.

For longitudinal oscillations of a Gaussian bunch without mode couplings, we evaluate Eq. (2.13) for the dipole mode $m = 1$ and obtain

$$\alpha_{00}^{(1)} = \frac{1}{2S}\left[1 + \frac{z}{2}\exp\left(-\frac{z}{2}\right)E_1\left(-\frac{z}{2}\right)\right],$$

$$\alpha_{01}^{(1)} = \frac{1}{2\sqrt{2S}}\left\{1 - \frac{z}{2} + \frac{z}{2}\exp\left(-\frac{z}{2}\right)E_1\left(-\frac{z}{2}\right)\left[2 - \frac{z}{2}\right]\right\},$$

$$\alpha_{11}^{(1)} = \frac{1}{2S}\left\{1 - \frac{3}{4}z + \frac{z^2}{8} + \frac{z}{2}\exp\left(-\frac{z}{2}\right)E_1\left(-\frac{z}{2}\right)\left[2 - z + \frac{z^2}{8}\right]\right\}, \tag{4.2}$$

where

$$z = -\frac{\omega - \omega_{sc}}{S} \tag{4.3}$$

and $E_1(z)$ is the exponential integral.[16]
The elements of the dispersion matrix in Eq. (2.19) can also be calculated by using the recurrence relations for the generalized Laguerre polynomials; we get

$$N_{kl} = 2S[(m + 2k + 1)\delta_{kl} - A_k{}^2\delta_{k-1,l} - A_{k+1}^2\delta_{k+1,l}], \tag{4.4}$$

with

$$A_k{}^2 = k(m + k). \tag{4.5}$$

Figure 1 shows the stability diagram calculated for the lowest radial mode $k = 0$ for the case of $m = 1$ mode, using the synthetic kernel approximation (2.16). When the higher radial-mode effects are included, we cannot express the impedance simply by a single matrix element M_{00}. We calculate here the complex frequency shift as a function of the beam current I for a given impedance. Figure 2(a) and (b) show the coherent tune Re ν ($=$ Re ω/ω_0) and the growth rate τ^{-1} ($=$ Imω), respectively, for $\{m = 1, \bar{k} = 0\}$ mode in the presence of the spread

$$S = \frac{\omega_{sc}}{16}\sigma_\phi{}^2$$

for the case of a narrowband resonant impedance. Here $\sigma_\phi = h\sigma_z/R$ is the standard deviation of the bunch length measured in rf phase angle. The numerical values of the parameters used in this calculation are

$E = 30$ GEV,	$\nu_{sc} = 0.1$,
$\omega_0 = 100$ KHz $\times\ 2\pi$,	$\alpha = 10^{-3}$,
h = harmonic number = 5,000,	$\sigma_z = 2$ cm,
f_r = resonant frequency	750.015 MHz
R_s = shunt impedance	0.1 GΩ
Q = quality factor	15,000.

It is found that a threshold exists, above which the coherent tune starts to appear and simultaneously the growth rate starts to rise abruptly as the current increases. For coherent oscillations below the threshold current (or inside the stability limit), a more careful analysis is necessary in dealing with the singularity of the perturbation function in Eq. (2.11). We discuss this problem in the Appendix in detail. The results show that the coherent oscillation can be generally expressed by a superposition of stationary oscillations and the mode frequency spectrum is pure real and continuous for each

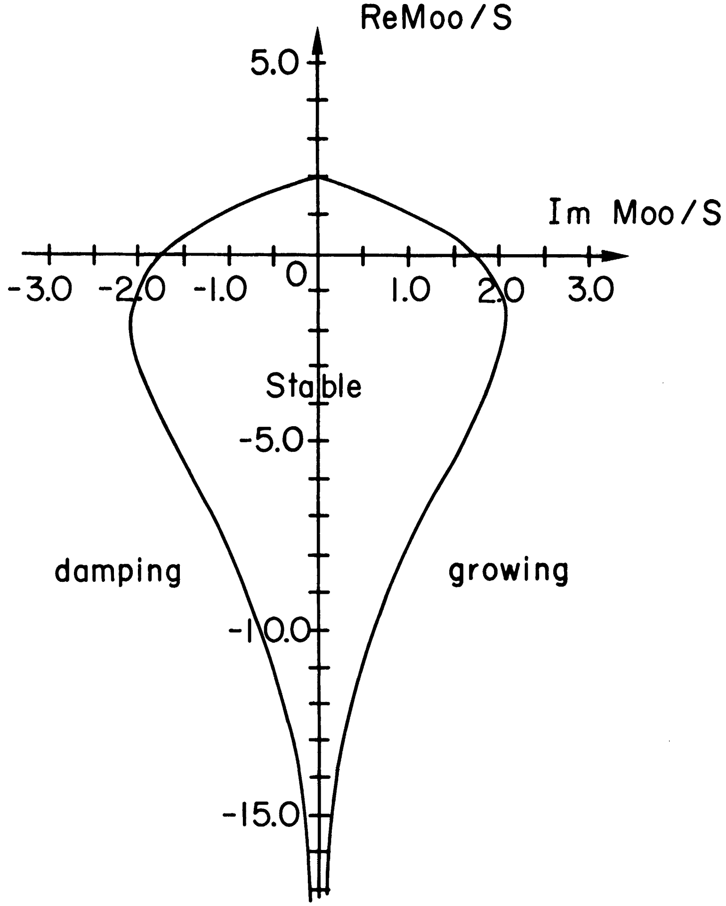

FIGURE 1 Stability diagram for the lowest radial mode for the case of $m = 1$ mode.

radial mode. We can no longer determine one frequency uniquely for each coherent mode inside the stability limit. Above a threshold current, a discrete line spectrum moves out from the continuous band and the corresponding coherent oscillation grows (or damps) exponentially. This phenomenon is Landau damping.

The same results for the growth rate can be obtained by Besnier's method with the dispersion matrix. Figures 3(a) and (b) show the coherent tunes Re ν and the growth rates τ^{-1} respectively, which are calculated with use of the 10×10 matrix. In this method, exponentially growing solutions appear for any currents and it seems that no strictly defined threshold can be obtained. However, as is evident from Fig. 3(b), the growth rate of the radial mode $k = 0$ is weakened by couplings between other radial modes due to the dispersion for lower currents. At the limit of an infinite number of radial modes, no exponentially growing solution could exist below a given threshold current. (See Fig. 4)

We identify the radial mode number for each eigenmode by counting the number of nodes of the real part of the corresponding radial eigenfunction. This method can be used only for eigenmodes outside the stability limit. Since the radial function becomes discontinuous inside the stability limit, it can not be expressed correctly by expansion

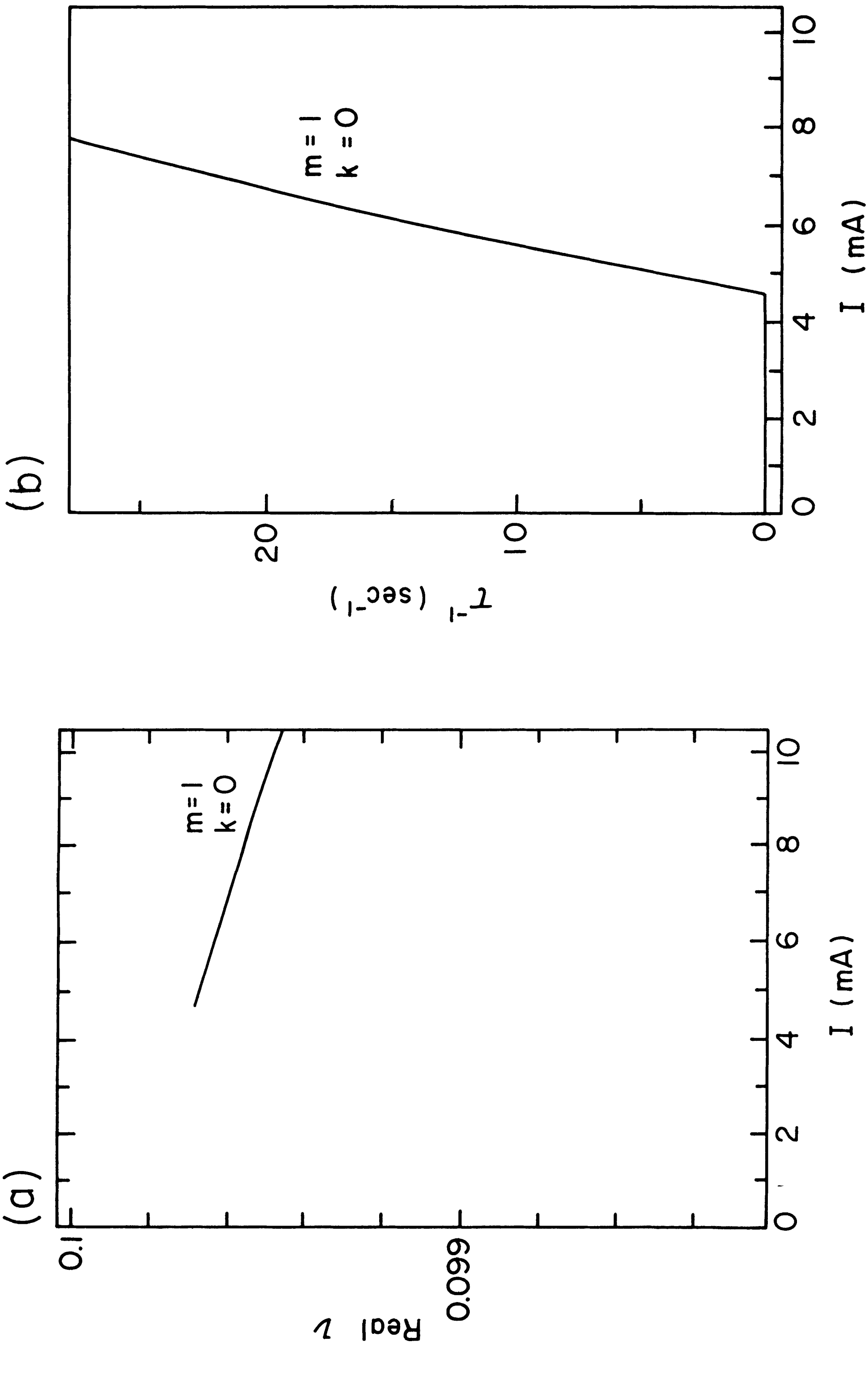

FIGURE 2 (a) Coherent tune Re $\nu(= \mathrm{Re}\, \omega/\omega_0)$ and (b) growth rate τ^{-1} $(= \mathrm{Im}\, \omega)$ as a function of the beam current I for spread $S = \omega_{sc}\sigma_\Phi{}^2/16$.

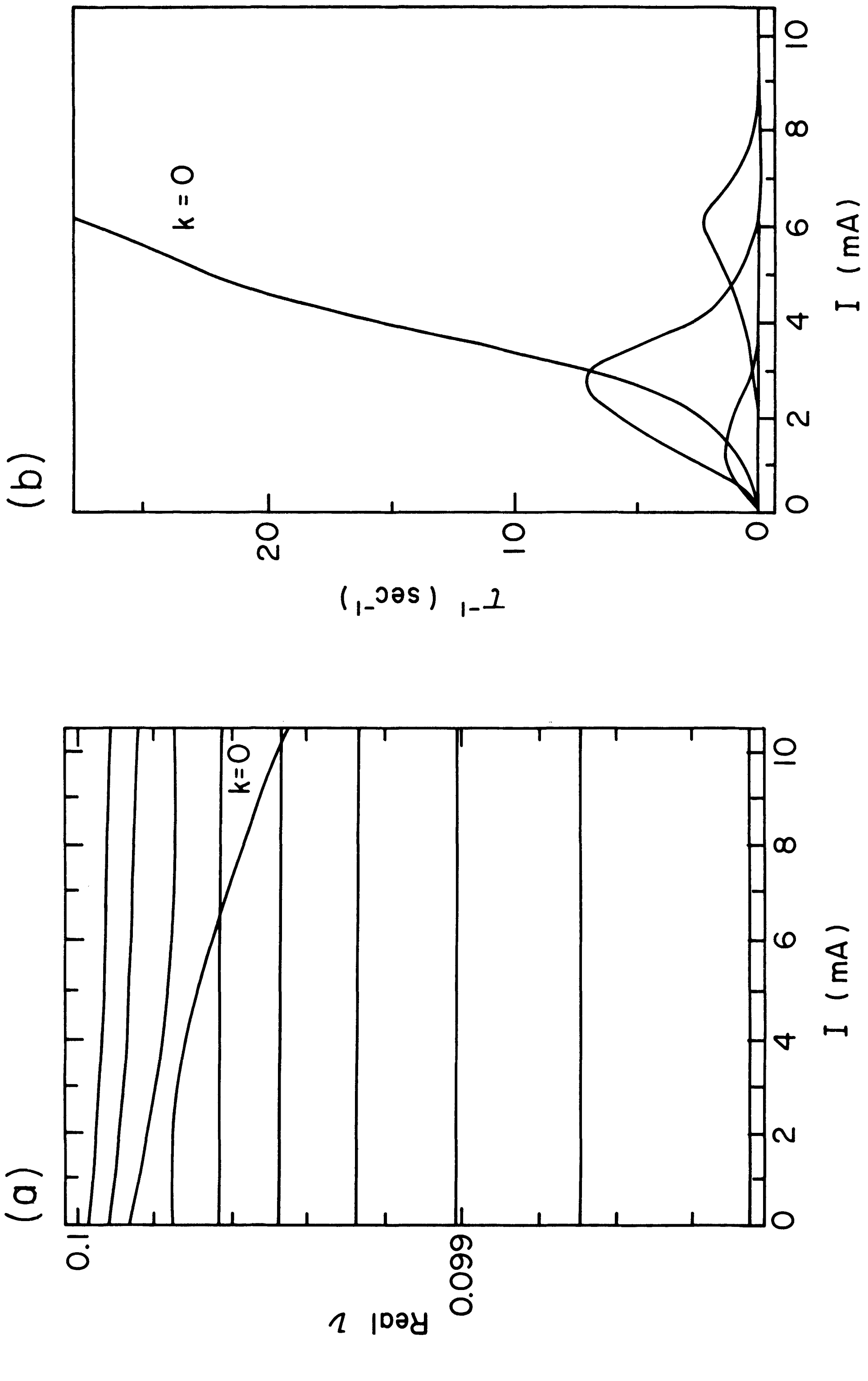

FIGURE 3 (a) Coherent tunes RE ν and (b) growth rates τ^{-1} as a function of the beam current I for spread $S = \omega_{sc}\sigma_{\phi}^{2}/16$ calculated with use of the 10×10 dispersion matrix.

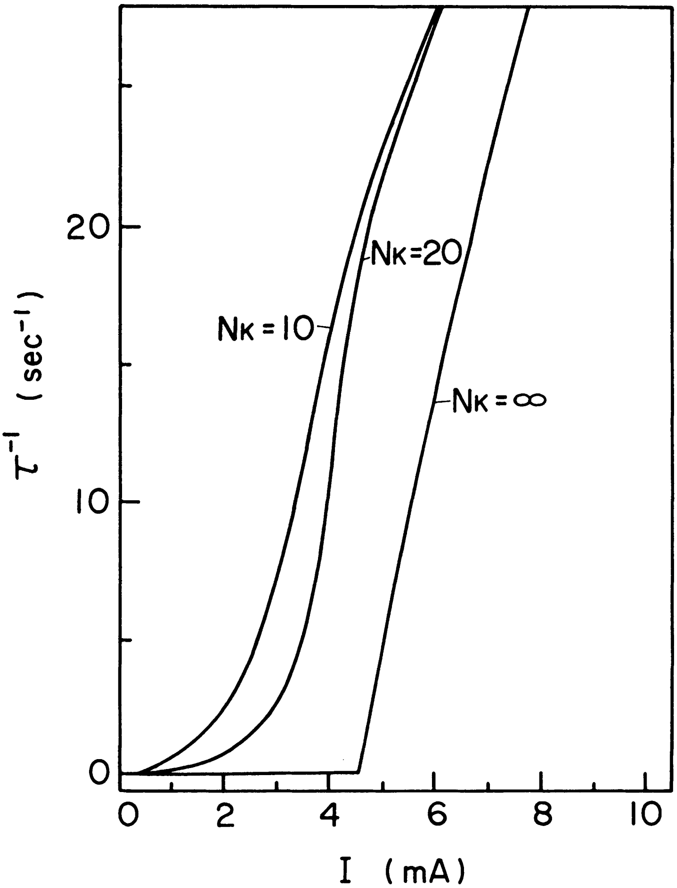

FIGURE 4 Growth rate τ^{-1} of the lowest radial mode for various dispersion matrix size N_k.

in orthogonal polynomials. If we insist on expanding it, the resulting eigenfunctions become very different from the polynomials themselves and thus we cannot determine the radial mode numbers for them. For example, we could not find the radial eigenfunction with no node corresponding to the lowest radial mode below the threshold current ~ 5 mA. Above that, an eigenfunction with no node appears and the radial mode number shown in Fig. 3(b) becomes meaningful.

The radial modes except $k = 0$ mode are still stable above this threshold, i.e., they are still inside each stability limit. Figure 5 shows the coherent tunes calculated with use of a 20×20 matrix. The eigenvalue of each eigenmode does not agree with the result of the corresponding mode calculated with use of the 10×10 matrix shown in Fig. 3(a). Even if we increase the number of radial modes, each eigenvalue does not converge. These results show that a unique solution does not exist for each eigenmode inside the stability limit. With no spread, each mode frequency forms a discrete line spectrum and can be calculated to a high convergence with a moderate matrix size. However, with spread, the coherent-mode frequency has a continuous spectrum and the eigenmode can take various eigenvalues. Only the eigenmode outside the stability limit takes a unique eigenvalue. This characteristic of mode frequencies inside the stability limits is consistent with the results in the Appendix.

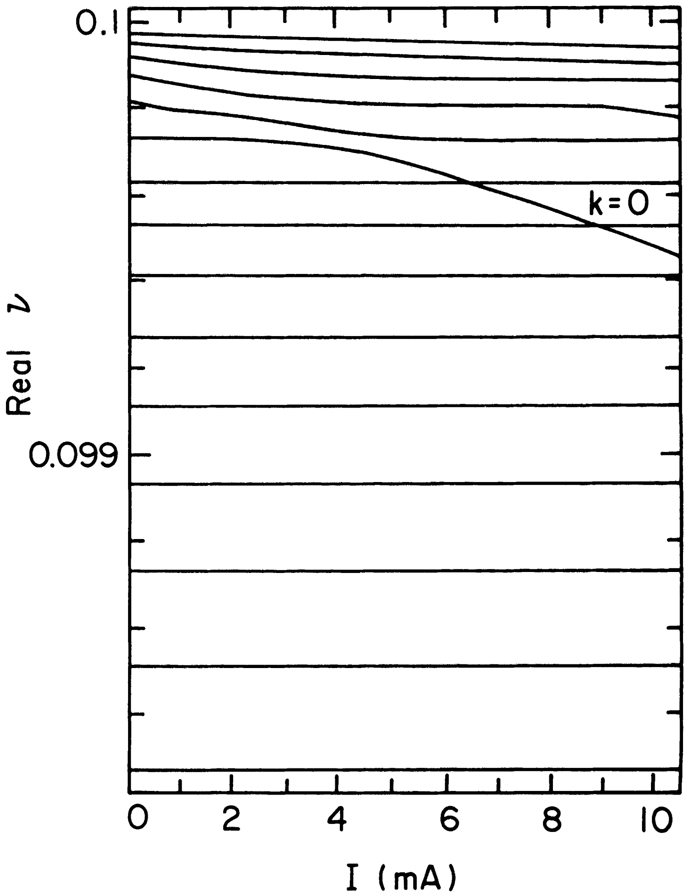

FIGURE 5 Coherent tunes Re ν calculated with use of the 20 × 20 dispersion matrix.

5. TRANSVERSE MODE COUPLINGS

In the preceding section, we have considered the longitudinal instability in a narrow-band impedance which generally yields both real and imaginary frequency shifts. On the other hand, for the fast transverse instability due to mode couplings, we have to take a broadband impedance,[11] because it is induced by a short-range force.[1,2] For a Gaussian bunch, the imaginary part of the interaction matrix becomes negligible in a broad-band impedance, which thus only leads to a real frequency shift.[11] The growth time of the fast instability is very short and less than one synchrotron-oscillation period. Even if the fast instability occurs inside the incoherent frequency band, its growth could be hardly stabilized by Landau damping due to a small spread in the synchrotron frequency. However, we can expect that the change of magnitude of a real frequency shift before mode coupling due to the presence of synchrotron-frequency spread will improve the threshold. In general, a first mode coupling occurs between $\{m = 0, k = 0\}$ and $\{m = -1, k = 0\}$ modes for a short bunch.[11] Since the mode $m = 0$ does not include synchrotron motion, its frequency shift is hardly affected by the synchrotron-frequency spread. The dispersion would mainly contribute to change of the real frequency shift for $m = -1$ mode.

We assume as in Sec. 4 that the synchrotron-frequency dispersion can be approximated by

$$D(r) = -S\frac{r^2}{\sigma^2}, \tag{5.1}$$

where S is independent of the beam current.

Integrating Eq. (3.20) with Eq. (5.1), we have

$$\begin{aligned}
\beta_{00} &= \frac{1}{\omega - \omega_\beta}, \qquad \beta_{01} = 0,\\
\beta_{11} &= -\frac{1}{2S}\left[1 + \frac{x}{2}\exp\left(-\frac{x}{2}\right)E_1\left(-\frac{x}{2}\right)\right]\\
&\quad + \frac{1}{2S}\left[1 + \frac{y}{2}\exp\left(-\frac{y}{2}\right)E_1\left(-\frac{y}{2}\right)\right],\\
\beta_{02} &= 0, \qquad \beta_{12} = 0, \ldots
\end{aligned} \tag{5.2}$$

where

$$x = \frac{\omega - \omega_\beta + \omega_{sc}}{S}$$

and

$$y = -\frac{\omega - \omega_\beta - \omega_{sc}}{S}.$$

Since we here consider the coupling between $m = 0$ and $m = -1$ modes, we neglect the second term in β_{11}

$$\beta_{11} \approx -\frac{1}{2S}\left[1 + \frac{x}{2}\exp\left(-\frac{x}{2}\right)E_1\left(-\frac{x}{2}\right)\right].$$

$\underline{S < 0}$

The relative coherent tunes $\mathrm{Re}\Delta\nu/\nu_{sc}$ ($= \mathrm{Re}(\omega - \omega_\beta)/\omega_{sc}$) and the growth rate within one synchrotron oscillation period, T_s/τ ($= 2\pi\,\mathrm{Im}\,(\omega - \omega_\beta)/\omega_{sc}$) for $S = -\omega_{sc}\sigma_\phi{}^2$ in a broad-band resonant impedance are shown by the broken lines in Fig. 6(a) and (b), respectively. The following parameters are used:

$$\begin{aligned}
E &= 14.5\ GeV, & \nu_{sc} &= 0.044,\\
\nu_\beta &= 21.25, & \omega_0 &= 136.4\ \mathrm{KHz} \times 2\pi,\\
\beta &\sim 160\ \mathrm{m}, & \omega_{rf} &\sim 350\ \mathrm{MHz} \times 2\pi,\\
f_r &= 1.3\ \mathrm{GHz}, & R_s &= 0.675\ \mathrm{M\Omega/m},\\
Q &= 1, & \sigma_z &= 2\ \mathrm{cm}.
\end{aligned}$$

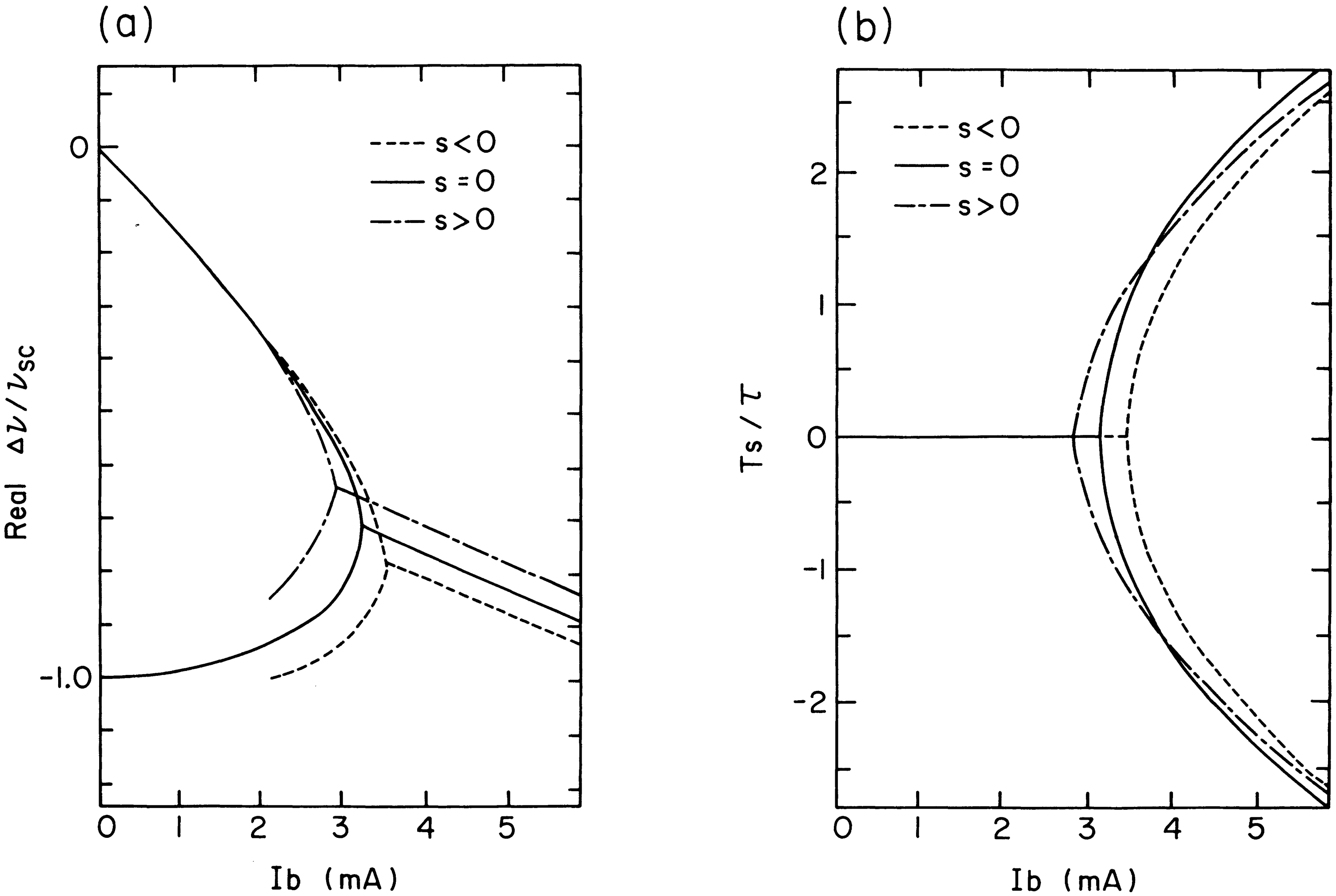

FIGURE 6 (a) Relative coherent tunes Re $\Delta\nu/\nu_{sc}(=\mathrm{Re}(\omega-\omega_\beta)/\omega_{sc})$ and (b) growth rates within one synchrotron oscillation period, T_s/τ, versus the bunch current I_b for various spreads in the synchrotron frequency.

We used a large value for the spread S to clarify its effect on the threshold current. Inside the incoherent frequency band, pure real eigenvalues can not be found, because they have singularities for the denominator in Eq. (3.20). The real relative tune appears outside the band. The solid lines in Fig. 6(a) and (b) show the relative coherent tunes Re $\Delta\nu/\nu_{sc}$ and the growth rates T_s/τ for no spread. It is found that the threshold current becomes a little larger when the incoherent frequency band spreads in the opposite direction of $m = 0$ mode, i.e. $S < 0$. Even if the coupled mode enters into the frequency band, the change of the growth rate is very small.

$S > 0$

The relative coherent tunes Re $\Delta\nu/\nu_{sc}$ and the growth rates T_s/τ for $S = \omega_{sc}\,\sigma_\phi{}^2$ are shown by the dot-dash lines in Fig. 6(a) and (b), respectively. Contrary to the case $S < 0$, the threshold current becomes a little smaller. Strictly speaking, the real relative coherent tune for $m = -1$ mode cannot exist within the incoherent frequency band before the bunch current exceeds the fast instability limit and $\Delta\nu$ comes to contain an imaginary part. This is due to the impractical assumption that the incoherent frequency band spreads infinitely. In Fig. 6(a), the real relative tunes whose determinants in Eq. (3.30) are small enough are shown as the eigenvalues.

Finally we investigate roughly the characteristic of the transverse mode couplings under a double (main plus higher-harmonic) rf system. A double rf system makes it possible to control the synchrotron frequency, the spread in the synchrotron frequency and the bunch length. Adjustment of the higher-harmonic system so as to reduce the slope of the rf waveform to zero at the bunch center yields a large bunch length and a large spread in the synchrotron frequency (bunch-lengthening mode).[13] In particular, a large bunch length is considered as one of the useful cures for the fast transverse instability.

The particle trajectories in the phase space are no longer circular and the unperturbed distribution function depends on both the amplitude r and the angle ϕ. The phase motion is described by the Jacobian elliptic function cn and the product $sndn$.[13] Since it is very difficult to treat this problem rigorously, we here make a crude assumption that we can still use the formulation obtained in Sec. 3, although the synchrotron-frequency spread is no longer small enough to allow this crude assumption.

For the bunch-lengthening mode, the synchrotron frequency at the center of bunch ω_{sc} becomes zero. Our formulation in a broad-band impedance has the dependence on ω_{sc} only in the integral (3.20), since the ω_{sc} appearing in the transformation of the volume element (3.5) is cancelled by the normalization of g_0. We thus can take $\omega_{sc} \approx 0$ without producing any singularity. For simplicity, we approximate the dispersion by $\omega_s(r) = Sr^2/\sigma^2$ so that the integral (5.2) is usable. Figures 7(a) and (b) show the relative coherent tunes Re $\Delta\nu/\Delta\nu_{sc}^\circ$ and the growth rates T_s°/τ, respectively, where we take only the first two terms in Eq. (3.30). We used $\sigma_z = 6$ cm as the increased bunch length and the spread $S = \omega_{sc}^\circ\sigma_\phi{}^2$ with the synchrotron tune $\nu_{sc}^\circ = \omega_{sc}^\circ/\omega_0 = 0.044$ in the single rf system. In order to compare them with the results for no spread, we show the relative values divided by the synchrotron tune ν_{sc}° and period $T^\circ = 1/\nu_{sc}^\circ f_0$. Since the mode $m = -1$ has a continuous mode frequency spectrum over the incoherent frequency band, the coupling between $m = 0$ and $m = -1$ modes occurs at any bunch current. We can no longer obtain a well-defined threshold and the growth rate rises gradually as the bunch current increases. These instabilities have to be damped by a transverse feedback system[18] or radiation damping. In this case, the threshold current would be determined at the point where the growth rate exceeds the damping rate due to these systems.

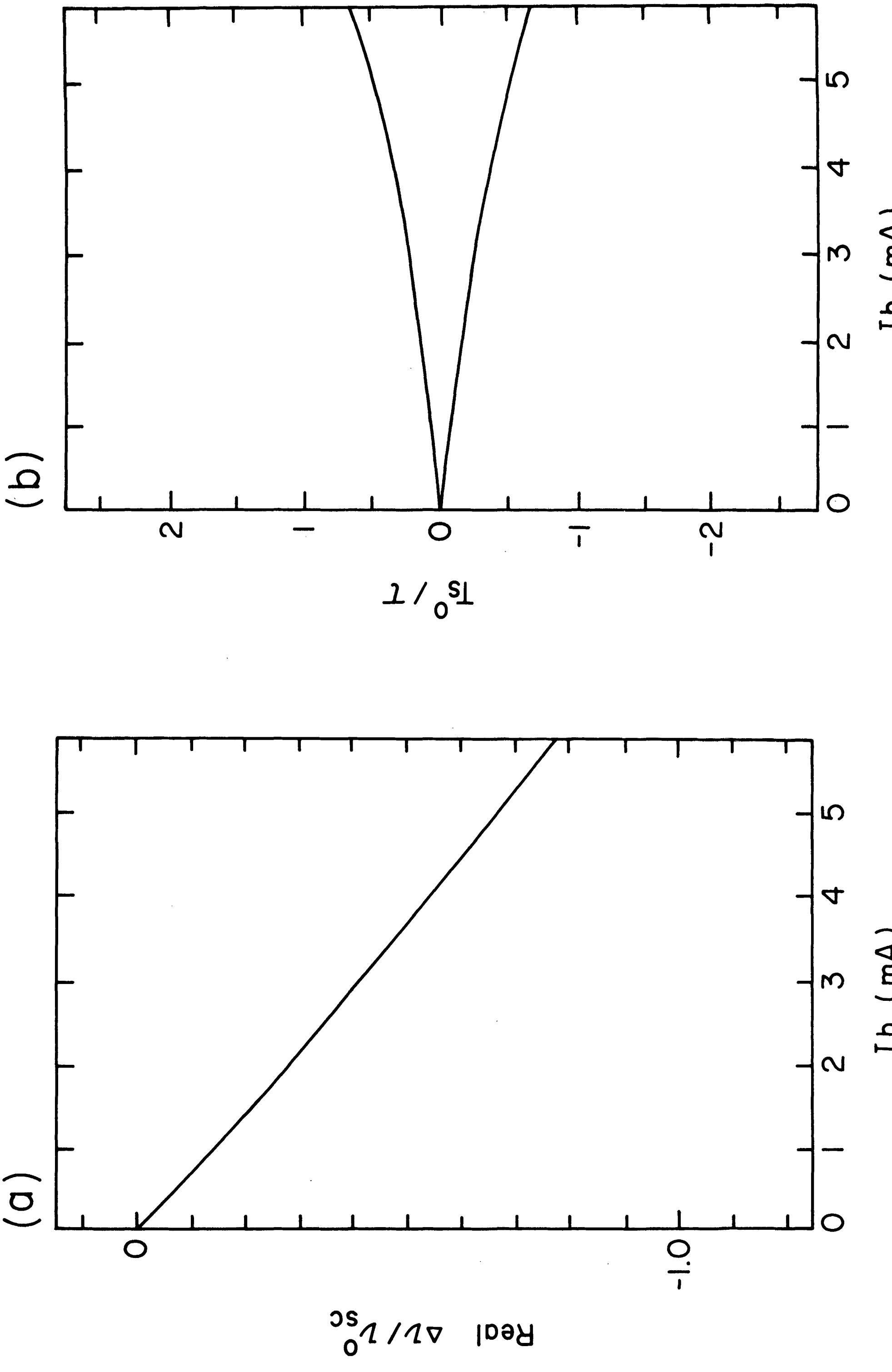

FIGURE 7 (a) Relative coherent tunes Re $\Delta\nu/\nu^\circ_{sc}$ and (b) growth rates T°_s/τ versus the bunch current I_b in a double rf system of bunch lengthening mode. The relative values divided by the synchrotron tune ν°_{sc} and period $T^\circ_s = 1/\nu^\circ_{sc} f_0$ in the single rf system are shown for comparison.

6. CONCLUSIONS

We have derived a generalized dispersion relation for the transverse and longitudinal coherent oscillations, starting from Sacherer's integral equation including both mode couplings and the synchrotron-frequency spread. The stability limit due to Landau damping can be calculated for any radial mode for each azimutual mode. Numerical results show that Besnier's method becomes approximately consistent with the generalized dispersion relation in the limiting case of expansion of the radial function in an infinite number of orthogonal polynomials. The threshold current due to the transverse fast instability can be improved little by the presence of a small synchrotron-frequency spread.

We applied the method with some crude assumptions to the transverse instability due to mode couplings in a double rf system adjusted to provide zero slope of the rf wave. The result shows that as the bunch current increases, the growth rate rises gradually without a clear threshold, in contrast to the case of no synchrotron-frequency spread. A threshold current would be determined by the competition with any damping systems.

Inside the stability limits, mode frequencies of coherent oscillations have pure real continuous spectra. We can no longer determine uniquely a coherent frequency for each mode.

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APPENDIX

Coherent Oscillations Inside a Stability Limit

The dispersion relation (2.14) provides a one-to-one mapping from the complex impedance Z plane to the complex coherent-frequency ω plane outside a stability limit. The area, except for the real axis in the ω plane, is completely covered by this mapping, so that the stability region in Z plane must be related to real ω axis. However, when we try to find stationary solutions, a mathematical difficulty arises that the denominator in Eq. (2.11) vanishes for real ω and the integral is singular. The proper treatment of this difficulty has been developed by van Kampen[19] for stationary oscillations in a plasma and leads to a complete set of stationary solutions.

The distribution g_m is meaningful only in its integral after multiplication with other functions. Instead of Eq. (2.11), the solution of g_m can be generally expressed by

$$g_m = \left[\frac{P.V.}{\Delta\omega - mD(r)} + \lambda\, \delta(\Delta\omega - mD(r)) \right] F(r), \tag{A.1}$$

where

$$F(r) = w(r) \sum_{k,l} M_{kl}\, f_k(r)\, c_l,$$

$P.V.$ means principal value of the integral and λ is the complex constant to be determined according to the way of dealing with the pole. The choice of $\lambda = \pm i\pi$ corresponds to adding an infinitesimal imaginary part to ω. In general the procedure of integrating across the pole can not be determined a priori, so that the variable λ is an arbitrary parameter. At the limit of vanishing current, the distributions g_m have only the second term of RHS in Eq. (A.1). Each of them consists of a δ-function or a shell distribution and has coherent frequency $\omega(r) = \omega_{sc} + D(r)$ determined by the amplitude r in phase space. As a current increases, the shells interact each other by the transverse self force and the distribution g_m gets finite width. The first term in Eq. (A.1) shows this effect.

Multiplying Eq. (A.1) by $rf_s(r)$ and integrating with r, we obtain similar linear equations with Eq. (2.12). We now have to look for eigenvalues $\Delta\omega$ (real), Re λ and Im λ instead of $\Delta\omega$ (complex) for a given impedance Z(complex) such that the determinant (2.14) becomes zero. Since the degree of freedom increases by unity, one of the variables $\Delta\omega$, Re λ and Im λ is a redundant parameter and a continuous range of eigenvalues for real $\Delta\omega$ can exist. For example, if we take $\Delta\omega$ (or amplitude r) as a free parameter, we can relate any real $\Delta\omega$ to complex impedance Z by choosing λ appropriately as to satisfy the equation (2.14). Thus the mode frequency of coherent oscillation forms a continuous spectrum inside the stability limit.

The most general solution for g_m is a superposition of stationary solutions (A.1). The integration of the product of g_m and a suitable function can express any given initial distribution. Owing to the fact that the stationary solutions in this superposition all oscillate with different frequencies, the initial perturbation is eventually smeared out. The proof of completeness of these stationary solutions is shown in Ref. 19.

Just at the stability limit, we have to take the Cauchy principal value and use $\lambda = \pm i\pi$, where the sign depends on whether we want to know the growing solutions or the damping solutions. The corresponding mode frequency moves out from the continuous spectrum and the coherent oscillation grows (or damps) exponentially outside the stability limit.