String Theory on Five Dimensional Anti de Sitter Space-Times: Fundamental Aspects and Applications.

Diego M. Hofman

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Abstract

In this thesis we study basic properties and applications of String Theory on AdS_5 backgrounds. We do this in the framework of the AdS/CFT Correspondence and use our results to learn about four dimensional Conformal Field Theories.

The first part of this work deals fundamentally with the problem of solving the exact spectrum of anomalous dimensions of planar $\mathcal{N} = 4$ Super Yang Mills theory for all values of the 't Hooft coupling λ. We study the problem for operators of large $SO(6)$ charge J and identify the string configurations dual to magnons in the spin chain picture of the gauge theory. We name these states Giant Magnons. Furthermore we study their interactions and discuss the implications of the spectrum of states on the analytic structure of the exact scattering matrix of the theory. It is found that BPS states account for all the poles present in the full S-matrix. We also study the spectrum of Giant Magnons attached to D3-branes (Giant Gravitons). The dual operators in $\mathcal{N} = 4$ SYM are long strings of $SO(6)$ scalars connected to baryonic operators constructed of order N fields. The problem turns out to be mapped to solving the mulitparticle spectrum of a spin chain with non trivial boundary conditions. We study the properties of the boundary reflection matrix in detail and write equations that determine the associated phase factor.

The second part of this work deals with applications of this type of string theories to the collider physics of conformal theories. We study infrared safe observables in the CFT given by energy correlation functions. We discuss the short distance behavior of these objects and explain that this physics is controlled by non local light ray operators. We find the dual String Theory description of these observables and use these results to study the strong coupling physics of conformal theories. We also describe the precise string states dual to the light ray operators. We argue that the energy operators that account for the energy measured at a calorimeter in a collider experiment should always be positive in any UV complete Quantum Field Theory. This fact has consequences in the higher derivative terms in the gravity action of the dual description. Finally, we discuss a proposed bound for the central charges of CFTs that is a consequence of the energy positivity condition.

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Contents

Chapter 1

Introduction

In this Chapter we provide a short introduction to the topic under consideration in this thesis. We discuss why it is of physical relevance to study String Theory and give a very short introduction to the the general area and the specific frame in which this work is contained: the AdS/CFT Correspondence. Finally we present the lines of research pursued in what follows. We first argue on the importance of understanding (and maybe solving completely) the simplest examples of the Correspondence. Then, we discuss the possible applications of gauge/gravity duality in different areas of physics and why these application can help us understand Quantum Gravity in a more profound way.

1.1 Why String Theory?

Before going into a detailed explanation of the topics that we will develop in the course of this work, it makes sense to pause for a minute and ask what is the motivation to do research in this area.

The 20th century was a very prolific one in terms of advancements in fundamental physics. Most notably, there were two major revolutions in the way we understand our universe.

The first of these revolutions is a conceptual framework known as Quantum Mechanics. The main lesson of Quantum Mechanics is that the microscopic world works in quite a different way from what we experience macroscopically. Even though the rules of Quantum Mechanics might seem strange, they have passed experimental tests with flying colors and are at the core of the most predictive physical theories available to man (the calculation of the gyromagnetic factor in QED being the most common example). The fundamental constant that determines the onset of quantum behavior is Planck's constant $\hbar$. When the action associated with a mechanical system is of order $\hbar$ we can't approximate dynamics by the classical equations of motion and we are driven into the quantum realm.

The other fundamental development of the 20th century, we know by the name of General Relativity. In simple terms, this is our current theory of gravity at macroscopic distances. Although this is a very successful theory, it is now understood that this can't be a complete theory of gravity at arbitrarily high energy scales. Einstein's theory as it stands has resisted attempts to quantize it following similar rules as the ones used with other fundamental interactions. We know, of course, that a full quantum theory of Gravity is needed if we are to understand phenomena at energy scales beyond the Planck mass $M_{Planck} = \sqrt{\frac{\hbar c}{G_N}} \sim 10^{19} GeV$ (or distances smaller than $\ell_{Planck} = \sqrt{\frac{\hbar G_N}{c^3}} \sim 10^{-35} m$). In this regime the gravitational action becomes of the order of $\hbar$ and we can't understand the system without a quantum theory of gravity.

String Theory is our current paradigm in which, we believe, Quantum Gravity can be understood. It is important to stress, at this point, that String Theory was really born in the 1960's as a theory of the strong interactions responsible for holding the atomic nucleus together. One of the main reasons why it made sense to consider String Theory as a theory of the strong force is that the meson spectrum is organized in Regge trajectories in the same way as the free spectrum of String Theory, namely

$m^2 = a + b\,J$, where J is the angular momentum of the state and a and b were experimentally measured constants. The way we now understand the strong interaction through Quantum Chromodynamics (QCD) allows us to understand the matching of the spectrum: we can think of a meson as a bound state of two quarks connected by a *flux tube of SU(3) glue* which can be modeled as a relativistic string. After the success of QCD it was found that closed string theories always possessed a spin 2 massless state with the properties of the graviton, the quantum of the gravitational theory. It was then understood that String Theory was an ideal candidate for a complete theory of gravity. The main idea being that the *softer* interaction between string-like objects would tame the divergences found in previous attempts to quantize point-like gravitons.

It seemed interesting then, as it is now, that the same theory could be understood both as a theory of gravity and strong nuclear interactions. Several positive results in the last 20 years have led researchers in the field to take the theory seriously, not only as a framework for gravity but, also, for the whole of fundamental interactions. In these developments it was fundamental to understand that String Theory not only possessed stringy degrees of freedom, but that higher dimensional solitonic objects (D-branes) were also available [1]. It was then possible to embed gauge theories of the type of QCD living in these D-branes inside of the full of String Theory. But the real surprise came about at the end of 1997.

In December of 1997 Maldacena published results [2] that made the relation between gauge theory and gravity precise and allowed for an understanding of the curiosities discussed in the previous paragraphs. This work, together with the construction of a precise dictionary between these seemingly unrelated theories in [3] and [4] laid the foundations of what came to be known as the AdS/CFT correspondence. In short, the basic statement of AdS/CFT is that gravitational theories on a particular space-time (Anti de Sitter) are dual to Conformal Field Theories (CFT).

In many cases, these theories turn out to be gauge theories, of a similar type to QCD. Duality means that these are just two different descriptions of the same physics. One of the nicer aspects of this duality is that calculations in String Theory are easy (see 1.2) when the gauge theory becomes strongly coupled. This regime is particularly hard when studied by the conventional methods of Quantum Field Theory.

These results have changed the way in which we understand Quantum Gravity and Gauge Theories in a fundamental way. More than 10 years after the discovery of the AdS/CFT Correspondence its implications have not been completely explored and its applications to real world physics have just surfaced recently. It is the purpose of the work presented in this thesis to add to this knowledge by both exploring fundamental aspects of the duality as well as its possible applications.

1.2 The AdS/CFT Correspondence

As it is impossible to give a full review of String Theory (or even a very comprehensive one of the AdS/CFT Correspondence) in a few pages, we will restrict the discussion to a few arguments that motivate the AdS/CFT correspondence and also some basic remarks on the dictionary that connects calculations in the gravitational theory to equivalent ones in the field theory. Good references on the basics of String Theory are [5–9]. As far as the AdS/CFT Correspondence goes, there are many good reviews. Great examples are [10–12].

Although the idea of finding a string theory dual to field theories goes back to 't Hooft [13] and Polyakov [14, 15], the concrete ideas we will discuss shortly were realized in [2–4].

One way to motivate the Correspondence is to think of the low energy dynamics of the degrees of freedom living on a D3-brane. We choose D3-branes as our starting point as it will generate the most popular and well understood version of gauge/gravity duality. The main idea is that we can describe low energy degrees of freedom in two equivalent ways: either by considering the massless open string (gauge) modes living on the brane or by considering the closed string (gravitational) modes living in the near horizon geometry generated by the back reaction of the brane on the flat space background.

Let us start by looking at the D-brane. A D-brane is an object where open strings can end. As such it is possible to define a low energy theory of massless modes living on it, given by the least excited open string states. Because String Theory is really a quantum theory of gravity, this brane must be allowed to fluctuate and become fully dynamical. It makes sense then, that some of the massless modes in the brane must describe the space time fluctuations of this object. A D3-brane is a 3+1 dimensional object embedded in 9+1 dimensional type IIB Superstring Theory. Therefore, there must be 6 scalar modes living on the brane that account for (local) translations in the transverse directions. Translational symmetry guarantees these are massless excitations. Also, we know that D-branes are 1/2 BPS objects. That means we need to complete the six scalars with the other fields in the supermultiplet of $\mathcal{N} = 4$ Super Yang Mills. If we consider N D-3 branes on top of each other (we will see why this is a good idea shortly), our low energy theory is given by $U(N)$ $\mathcal{N} = 4$ SYM.

We can now consider the background given by the back reaction of N D3-branes on the geometry. The solution is very similar to a black hole solution when the horizon is scaled down to zero size (extremal solution):

$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-\frac{1}{2}} \left(-dt^2 + dx^i dx^i\right) + \left(1 + \frac{L^4}{r^4}\right)^{\frac{1}{2}} \left(dr^2 + r^2 d\Omega_5^2\right) \tag{1.2.1}$$

where $i = 1, 2, 3$ and L is related to the tension of a single D3-brane T_3 by $L^4 \sim$

NG_NT_3 with G_N being the 10 dimensional Newton constant. If we are only interested in the low energy dynamics on the brane we only have to look at the near horizon limit of (1.2.1), $r \to 0$:

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + dx^i dx^i + dz^2\right) + L^2 d\Omega_5 \tag{1.2.2}$$

where $z = \frac{L^2}{r}$. This is just $AdS_5 \times S^5$ spacetime. In this case, the dynamics of the system are encoded in the gravity modes that live in this near horizon space. We are led, then, to identify these two theories: $\mathcal{N} = 4$ SYM in four dimensions is dual to (quantum) gravity in $AdS_5 \times S^5$ background. Notice that the gravitational theory lives in one more (non-compact) dimension than our field theory. This is a generic feature of the duality and explains why strings in four dimension were not an exact description of the strong force. In general dimensions we will have an AdS_{d+1}/CFT_d Correspondence. It is very interesting to keep track of the symmetries of these theories. The conformal group in d dimensions translates exactly to the isometry group of AdS_{d+1}. This leads to the fact the we are geometrizing the energy scale of our theory.

Let us try to understand how the parameters of these two theories are related and what were the specific limits considered in the naive description above. From the point of view of the gravity theory, the only parameter is the length scale L which gives the radius of AdS and the sphere. Because the metric is sourced by N D3-branes we argued that (we neglect order 1 multiplicative factors in the following discussion):

$$L^4 \sim NG_NT_3 \tag{1.2.3}$$

In String Theory the (ten dimensional) Newton constant is given by $G_N \sim g_{st}^2 \alpha'^4$. It is easy to explain this equality. $G_N^{\frac{1}{2}}$ is just the coupling associated with a 3 graviton tree level graph. On the other hand the genus zero (tree level) diagram in String

Theory contributes g_{st}^{-2} for the Riemann surface and g_{st} for every graviton vertex. Matching these results implies $G_N \sim g_{st}^2$. Dimensional analysis implies the α' dependence as that is the only scale available. What about T_3?

The string tension of a single D-brane is given by its coupling to gravity. That is given by disc diagram in String Theory. In this case the topology is different from the sphere and the Riemann surface contributes g_{st}^{-1}. Once again, dimensional analysis fixes the α' dependence. The final result is

$$T_3 \sim g_{st}^{-1} \alpha'^{-2} \tag{1.2.4}$$

Using (1.2.3) we obtain

$$L^4 \sim N g_{st} \alpha'^2 \tag{1.2.5}$$

We also know that the open string coupling constant g_{YM} is related to the closed string coupling g_{st} by $g_{YM} = g_{st}^{\frac{1}{2}}$ [7]. Therefore our final expression is

$$L^4 \sim \lambda \alpha'^2 \qquad \text{where} \quad \lambda = N g_{YM}^2 \tag{1.2.6}$$

From this result we can consider what are the important limits that we have considered. In our discussion we have neglected the excited stringy modes. For this approximation to be valid we need to consider theories with small curvatures compared with the string scale. Therefore, we need

$$L \gg \sqrt{\alpha'} \qquad \rightarrow \lambda \gg 1 \tag{1.2.7}$$

This requirement means that only field theory modes are important. Equation (1.2.7) is actually great news. It means we can trust our gravity description when field theoretic methods fail at large 't Hooft coupling ($\lambda \gg 1$). A useful picture is to imagine one theory with a coupling λ that we can dial. At small λ we have a field theory description while at strong coupling we have a string theory description. Notice that at large, but finite, λ we can have string theory degrees of freedom besides the massless modes. In this general form it is hard to keep the theory under control as there will be quantum corrections in the string theory that come as powers of g_{st}. It is therefore of great interest to consider the large N limit where we keep λ fixed. In that case $g_{st} \to 0$ and tuning λ we explore a parameter space where on the one hand ($\lambda \ll 1$) we have a planar field theory (only graphs that can be drawn in a plane contribute) and on the other ($\lambda \gg 1$) we have classical String Theory. At $\lambda \to \infty$ we are left with classical gravity, no string degrees of freedom and no loops. In this regime our D3-brane solution (1.2.1) is perfectly valid (notice that we have used that the tension of a D-brane goes as g_{st}^{-1} (1.2.4) which is the correct expression at large N, $g_{st} \to 0$). Although this is the regime in which the duality is best understood, it is believed to be correct for all values of (N, λ).

Once we have understood this case, we can try to repeat the story for other solitonic objects similar to the D3-brane and obtain other versions of the AdS/CFT Correspondence. It turns out that this is possible for the M2 and M5 theories where the dilaton decouples and we can obtain a smooth geometry as in the case of the D3-brane. Other D-branes do not have this smooth limit and more complicated systems of branes are necessary to derive other versions of the duality (AdS_3 needs a system that contains both D1-branes and D5-branes) where we find a CFT. Nonetheless it is proposed that the AdS/CFT Correspondence is valid for all dimensions, regardless of the brane construction that help us figure out the specific dynamics. The simplicity of the D3-brane solution is at the core of our particularly advanced understanding of this example.

Let us now study the specific dictionary that allows us to compare computations on both theories. We will develop this machinery in the large N limit, where we have a classical gravity theory. The main observable that is available in a field theory is the set of vacuum correlation functions. It would be of use, then, to be able to calculate these observables in our gravity dual. In order to arrive at the dictionary it is important to stress that AdS spaces posses a boundary. The meaning of this is that while there is an infinite distance to the boundary, signals can bounce back from it and be back in the bulk in finite coordinate time. In the coordinates presented in (1.2.2) the boundary is located at $z = 0$. It is convenient to think that our field theory (its UV degrees of freedom, really) lives at the boundary of AdS spacetime while the gravity dual lives in the bulk. In this setup is very natural to understand that deformations of our theory correspond to changing the boundary conditions on our gravity fields. In that case, turning on an operator in the boundary fixes boundary conditions for the specific field that couples to such operator.

The concrete proposed dictionary is [3] and [4]:

$$Z_{bulk}[\phi(x^i, z = 0) = \phi_0(x^i)] = \left\langle e^{\int d^4x \phi_0(x)\mathcal{O}(x)} \right\rangle \tag{1.2.8}$$

where Z_{bulk} is the partition function of our gravity theory. ϕ is the dual gravity mode to the operator $\mathcal{O}$ in the CFT. ϕ_0 just sets the boundary condition for our gravity mode. Notice that by taking derivatives with respect to ϕ_0 we can obtain the n point function in the field theory. How do we calculate Z_{bulk}? Although this is difficult for a generic quantum theory of gravity, in the large N limit at large enough λ we will have classical general relativity (coupled to local fields) and the partition function will be dominated by a saddle point approximation. Therefore,

$$e^{-N^2 S_{classical}[\phi_0]} = \left\langle e^{\int d^4x \phi_0(x)\mathcal{O}(x)} \right\rangle \tag{1.2.9}$$

Of course, there will be subleading corrections in λ to this expression. The recipe, in this limit, is very simple then. We solve classical equations of motion for our field living in curved space and we calculate their action evaluated in this saddle point. From this we can calculate n point functions.

This prescription relates the conformal weight of the operator $\mathcal{O}$ to the mass of our field living in AdS. This can be obtained purely from the asymptotic behavior of our bulk fields. For example, a scalar field of mass m in AdS_5 presents asymptotic solutions that decay as $z^{\Delta_\pm}$ for $z \to 0$, with $\Delta_\pm = 2 \pm \sqrt{4+m^2}$. If we renormalize the source ϕ_0 in such a way that the boundary condition remains finite at $z \to 0$ it is possible to see that this implies the dual operator has conformal weight[1] $\Delta = \Delta_+$ [12].

It is not hard to argue, from the relation of m^2 and Δ, that computing the spectrum of string excitations (for finite but large λ) is the dual problem to solving for the spectrum of dimensions of operators in the CFT.

1.3 Can we solve String Theory / Gauge Theory?

In the present work we will focus mainly on the 5 dimensional example discussed above. There are two main reasons to do this. The first one is that, as a consequence of the previous discussion, the $AdS_5 \times S^5$ String Theory / $\mathcal{N} = 4$ Super Yang Mills Gauge Theory duality is the most studied and better understood. Even more than that, this case turns out to be simple enough (but still very rich!) that it is thought to be completely solvable in the large N limit (more about this below). This is the statement that the theory is integrable. Giving a short explanation of what this means is the purpose of this section. The work contained in chapters 2, 3 and 4 contains research that focus on this feature of the theory.

The second reason to study String Theory on AdS_5 is that its dual conformal theory lives in $3+1$ dimensions. Since this is the dimensionality of our physical spacetime it is a great tool to study real world physics. There are many phenomena

[1] Actually, for the scalar case it is possible to consider dual operators with both $\Delta = \Delta_\pm$ as long as the satisfy the unitarity bound.

that can be studied in this way and we will review this line of research in the next section. Chapters 5 and 6 focus on this aspect.

So, what is integrability? Let's consider a system with a finite number of degrees of freedom. For example, the classical mechanics of a finite number of particles. A system as such is said to be integrable if we have as many independently conserved quantities as degrees of freedom. If that is the case, then we can integrate the equations of motion and solve the system explicitly. The same type of definition is valid for quantum mechanical systems where we require n commuting conserved operators for a system with n degrees of freedom. This definition can be generalized to field theories where we have an infinite number of degrees of freedom. Most examples of solvable theories that we know are integrable: harmonic oscillators, central force problems, Sine-Gordon Model, etc. Integrable systems are rare but provide great lamp posts to understand more complicated theories. The fact that we can understand the harmonic oscillator problem exactly helps us understand other theories that are some type of perturbation of this simpler one. If we are to understand field theory beyond perturbation theory (that is, for finite coupling λ) it would help to have such a non trivial lamp post. It turns out that planar ($N \to \infty$) $\mathcal{N} = 4$ $SU(N)$ SYM is believed to be integrable for all values of the 't Hoof coupling λ. This has been proved for several subsectors of the theory perturbatively up to three loops [16–19] and at strong coupling by studying the dual classical string theory [20]. It seems, therefore, that $\mathcal{N} = 4$ SYM might be the harmonic oscillator of four dimensional field theories.

In practice, for our purposes, the property of integrability means that the theory might be exactly solvable and that we should spend our time working on it as this could be an example in which we can understand and test the AdS/CFT Correspondence exhaustively. What do we mean by solving the theory? From the point of view of $\mathcal{N} = 4$ SYM, this means obtaining all the correlation functions of all operators in the theory. This is certainly a complicated task that is out of reach at this point. A subset of this problem is finding the complete spectrum of anomalous dimensions of the theory. From the AdS/CFT dictionary presented in the previous section, this amounts to solving the spectrum of string states on $AdS^5 \times S^5$. This is a very interesting and yet non trivial problem.

In fact, the problem of solving the spectrum of planar $\mathcal{N} = 4$ SYM at strong coupling is very hard and very little progress had been made outside of computations that involved operators protected by supersymmetry until a new idea was proposed. The key insight was presented in [21], where it was realized that the problem might be tractable for operators of very large charges under the internal symmertry group of $\mathcal{N} = 4$ SYM, $SO(6)$. In that case, the dual string theory could be quantized exactly in certain scaling limit: the BMN limit. In this limit, excitations of the string worldsheet have momentum that scale as $p = \frac{n}{J}$ with n fixed (and representing the Fourier mode excited in the string) and J the large $SO(6)$ charge, $J \to \infty$. It was also understood here that the dual gauge theory operators are *long* single trace operators with large J charge.

On the gauge theory side, systematic perturbative calculations became available after the remarkable work in [16] (see [22] for a review) where it was realized that the problem could be mapped to the diagonalization of an integrable spin chain hamiltonian. In this setup the fundamental quasi particles in the spectrum were equivalent to magnons in the spin chain picture. Later, it was understood in [23] that the exact spectrum of one particle excitations could be calculated. The exact expression in [23] is

$$\Delta - J = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \tag{1.3.10}$$

where Δ is the dimension of the operator and J its charge. The problem at weak coupling could be understood at finite p from the work in [16] and its higher loop generalizations. For example, at one loop $\Delta - J = 1 + \frac{\lambda}{2\pi^2} \sin^2 \frac{p}{2}$, in accordance with

(1.3.10). On the other hand the result at strong coupling was available in the BMN limit and read $\Delta - J = \sqrt{1 + \frac{\lambda n^2}{4\pi^2 J^2}}$, with $n = pJ$. As can be seen, at this point it was not identified what the dual degrees of freedom to magnons were in the string theory at finite p. The identification of these objects is the topic that is discussed in Chapter 2.

Since the theory is integrable, it is possible (in principle) to obtain the full spectrum of the theory, at least at large J, by a technique called the Bethe Ansatz (see [22] and references therein) from the one particle spectrum and the S-matrix of two such objects. If one solves for the S-matrix exactly, one can claim to have solved the theory completely at large J. At present there is a proposal for the exact form of the S-matrix [24,25]. The study of the analytic properties of this matrix by studying the dynamics of magnons is the main topic that is discussed in Chapter 3.

Once single trace operators are better understood, it is of interest to study baryonic operators as well. These are written as determinants of a large number (J of order N) of fields and are dual to giant graviton configurations in the bulk [26]. The excitations of these operators can be understood, once again, as a spin chain problem where there is a non trivial boundary condition and we need to compute a reflection matrix to solve the problem. In String Theory, this corresponds to attaching open strings to a giant graviton. These systems were studied at low momentum in the String Theory and perturbatively in the Gauge Theory [27,28]. A complete analysis, including equations that determine the full reflection matrix, following the lessons of [23] and Chapters 2 and 3, is conducted in Chapter 4.

It is important to stress that while String Theory on $AdS_5 \times S^5$ is very special and many of the things we might learn from this example can't be easily extended to other cases[2], it is still a setup in which we might understand the AdS/CFT Correspondence fully. Furthermore, it is a unique opportunity for us to attempt to solve a four

[2]One case that seems to be similar and also integrable is given by String Theory on $AdS_4 \times CP^3$, [29,30].

dimensional gauge theory completely at all values of the coupling constant. If we are to understand more complicated theories, it seems only reasonable to first fully understand the harmonic oscillator of Quantum Field Theories in detail. The work presented in Chapters 2, 3 and 4 corresponds to research in this direction.

1.4 Where do we go from here?

Even if we understand String Theory on $AdS_5 \times S^5$ exactly, what do we do with it?

As we mentioned in the last section, there are *phenomenological* reasons to study String Theory on AdS_5. When it comes to the specific example dual to $\mathcal{N} = 4$ SYM, it makes sense that, since this is, in some way, the simplest possible 4 dimensional theory, we can find many situations in which we can approximate more complicated behavior by this better understood theory. One such example in which String Theory computations have resulted particularly accurate and useful is in the context of physics of the Quark Gluon Plasma, studied experimentally at the Relativistic Heavy Ion Collider (RHIC). By studying the theory at finite temperature we break conformality and supersymmetry explicitly and $\mathcal{N} = 4$ SYM becomes a reasonable approximation to QCD.

Of great importance in this direction has been the study of the universal behavior of the viscosity to entropy density ratio ($\frac{\eta}{s}$) in theories with a gravity dual [31,32]. The study of this quantity, of phenomenological character, had some interesting feedback into the fundamental properties of field theories and gravity theories. It was first proposed that all field theories might satisfy $\frac{\eta}{s} \geq \frac{1}{4\pi}$. Although further work confirmed that this is not the case [33], it was understood that fundamental questions related to causality and positivity of energy were related to this problem. This problem is addressed, as a consequence of a different sort of application of AdS/CFT that we will discuss shortly, in Chapter 5 and, in depth, in Chapter 6.

Another application of importance, with the era of the Large Hadron Collider (LHC) just around the cornenr, is to the area of collider physics. The importance of understanding CFTs , asymptotically free theories and QCD, in particular, has been made manifest in the last few years as the largest particle collider ever built is about to be turned on in Geneva, Switzerland. Hopefully, new physics will be discovered in this experiment. The conformal field theories that are dual to String Theory on AdS_5 spacetimes have been proposed as candidates for the new type of physics that might be found at the LHC [34]. With this motivation the study of such type of applications is discussed in Chapter 5. By describing natural observables in a CFT from the point of view of collider physics it is also possible to gain insight into the structure of the quantum theory of gravity and the basic properties of the field theory, as it was commented on above.

Lastly, although this topic won't be discussed in this thesis, we should stress that a lot of interest has been developing in the last few years on possible applications of the AdS/CFT Correspondence to Condensed Matter systems [35,36]. This is another clear example of the great spectrum of applications of gauge/gravity duality. It is fair to say that, after more than 10 years, all the implications of this great advance in theoretical physics have not been fully worked out yet.

1.5 Road map to this thesis

Here we give a short summary of what is contained in each Chapter of this thesis.

Chapters 2, 3 and 4 deal with fundamental aspects of the spectrum of String Theory on $AdS_5 \times S^5$.

In Chapter 2 we identify the fundamental states in the String Theory on $AdS_5 \times S^5$ dual to the magnon excitations in the spin chain picture of the Gauge Theory. We call these classical string configurations Giant Magnons. It is found that their spectrum matches the prediction coming from symmetry arguments [23]. Furthermore, the S-matrix of these objects is discussed and found in agreement with a previous proposal coming from indirect methods [37].

In Chapter 3 we study the analytic structure of the magnon s-matrix. In particular, we account for the poles present in the S-matrix by a precise matching to physical states corresponding to (and only) BPS magnons. It is shown that all classical string solutions found at strong coupling fit nicely in this picture where only BPS states are present. Finally we check that the exact proposal in [24, 25] is consistent with our results.

In Chapter 4 we discuss the inclusion of boundary conditions in the spin chain picture of $\mathcal{N} = 4$ SYM. This situation is dual to the study of excitations of giant gravitons in the dual String Theory. The reflection matrix of this system is studied and found to be consistent with integrability. Furthermore, boundary degrees of freedom are identified and their spectrum is computed. Equations for the scalar factor of the reflection matrix are also proposed.

Chapters 5 and 6 deal with applications of the AdS/CFT Correspondence and their implications for the basic properties of the underlying field theories.

In Chapter 5 we study applications of the Ads/CFT Correspondence to the collider physics of CFTs. Energy correlation functions are studied and defined at strong coupling. Their short distance behavior is also studied and related to the presence of non local operators in the operator product expansion of energy operators. The dual states to these non local operators are identified in the String Theory. Also, it is found that positivity of the energy measured in calorimeters in an ideal collider experiment implies bounds on the central charges a and c in supersymmetric CFTs. Finally, the jet structure of CFTs is discussed. It is found that the jet patterns found at weak coupling tends to disappear at strong coupling.

In Chapter 6 we study the restrictions imposed on higher derivative theories of gravity by causality of their dual CFTs. It is shown that bounds on the couplings of the gravity action found in this way are directly related to the positivity of energy condition discussed in Chapter 5. Bounds on the central charges of the theory are understood this way. Finally, we present a field theoretic argument why the energy operators defined in Chapter 5 should have a positive spectrum in any UV complete Quantum Field Theory.

Finally we present our conclusions and outlook in Chapter 7.

Chapter 2

The Spectrum of String Theory on $AdS_5 \times S^5$: Giant Magnons

Studies of $\mathcal{N} = 4$ super Yang Mills operators with large R-charge have shown that, in the planar limit, the problem of computing their dimensions can be viewed as a certain spin chain. These spin chains have fundamental "magnon" excitations which obey a dispersion relation that is periodic in the momentum of the magnons. This result for the dispersion relation was also shown to hold at arbitrary 't Hooft coupling. Here we identify these magnons on the string theory side and we show how to reconcile a periodic dispersion relation with the continuum worldsheet description. The crucial idea is that the momentum is interpreted in the string theory side as a certain geometrical angle. We use these results to compute the energy of a spinning string. We also show that the symmetries that determine the dispersion relation and that constrain the S-matrix are the same in the gauge theory and the string theory. We compute the overall S-matrix at large 't Hooft coupling using the string description and we find that it agrees with an earlier conjecture.

The work in this chapter is contained in [38]. This article was coauthored with Juan Maldacena.

2.1 Preliminaries

String theory in $AdS_5 \times S^5$ should be dual to $\mathcal{N} = 4$ Yang Mills [2–4]. The spectrum of string states should be the same as the spectrum of operators in the Yang Mills theory. One interesting class of operators are those that have very large charges [21]. In particular, we consider operators where one of the SO(6) charges, J, is taken to infinity. We study states which have finite $E - J$. The state with $E - J = 0$ corresponds to a long chain (or string) of Zs, namely to the operator $Tr[Z^J]$. We can also consider a finite number of other fields W that propagate along this chain of Zs. In other words we consider operators of the form

$$O_p \sim \sum_l e^{ilp}(\cdots ZZZWZZZ\cdots) \tag{2.1.1}$$

where the field W is inserted at position l along the chain. On the gauge theory side the problem of diagonalizing the planar Hamiltonian reduces to a type of spin chain [16, 39, 40], see [41–43] for reviews and further references. In this context the impurities, W, are "magnons" that move along the chain.

Using supersymmetry, it was shown that these excitations have a dispersion relation of the form [23]

$$E - J = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}} \tag{2.1.2}$$

Note that the periodicity in p comes from the discreteness of the spin chain. The large 't Hooft coupling limit of this result is

$$E - J = \frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p}{2}\right| \tag{2.1.3}$$

Since this is a strong coupling result, it should be possible to reproduce it on the string theory side. At first sight it would seem that such a dispersion relation would require the string worldsheet to be discrete. In fact, this is not the case. We will show how to recover (2.1.3) on the string theory side with the usual strings moving in $AdS_5 \times S^5$. The key element is that p becomes a geometrical angle which will explain the periodic result. Thus we are able to identify the elementary excitations of the spin chain on the string theory side in an explicit fashion. The identification of these magnons allows us to explain, from the gauge theory side, the energy spectrum of the string spinning on S^5 which was considered in [44].

We will discuss the presence of extra central charges in the supersymmetry algebra which match the gauge theory analysis in [23]. Having shown that the two algebras match, then the full result (2.1.2) follows. Moreover, as shown in [23] the symmetry algebra constrains the $2 \to 2$ S matrix for these excitations up to an overall phase. This S matrix is the asymptotic S-matrix discussed in [45]. It should be emphasized that these magnons are the fundamental degrees of freedom in terms of which we can construct all other states of the system. Integrability [20, 46, 47] implies that the scattering of these excitations is dispersionless. We check that this is indeed the case classically and we compute the classical time delay for the scattering process. This determines the large 't Hooft coupling limit of the scattering phase. The final result agrees with the large λ limit of [37]. This is done by exploiting a connection with the sine Gordon model [48–50].

2.2 Elementary excitations on an infinite string

2.2.1 A large J limit

Let us start by specifying the limit that we are going to consider. We will first take the ordinary 't Hooft limit. Thus we will consider free strings in $AdS_5 \times S^5$ and planar diagrams in the gauge theory. We then pick a generator $J = J_{56} \subset so(6)$ and consider the limit when J is very large. We will consider states with energies E (or operators with conformal dimension $\Delta = E$) which are such that $E - J$ stays finite in the limit.

We keep the 't Hooft coupling $\lambda \equiv g^2 N$ fixed. This limit can be considered both on the gauge theory and the string theory sides and we can interpolate between them by varying the 't Hooft coupling after having taken the large J limit. In addition, when we consider an excitation we will keep its momentum p fixed. In summary, the limit that we are considering is

$$J \rightarrow \infty , \qquad \lambda = g^2 N = \text{fixed} \tag{2.2.4}$$

$$p = \text{fixed} , \qquad E - J = \text{fixed} \tag{2.2.5}$$

This differs from the plane wave limit [21] in two ways. First, here we are keeping λ fixed, while in [21] it was taken to infinity. Secondly, here we are keeping p fixed, while in [21] $n = pJ$ was kept fixed.

One nice feature of this limit is that it decouples quantum effects, which are characterized by λ, from finite J effects, or finite volume effects on the string worldsheet[1].

Also, in this limit, we can forget about the momentum constraint and think about single particle excitations with non-zero momentum. Of course, when we take J large but finite, we will need to reintroduce the momentum constraint.

2.2.2 Review of gauge theory results

In this subsection we will review the derivation of the formula (2.1.2). This formula could probably have been obtained in [53] had they not made a small momentum approximation at the end. This formula also emerged via perturbative computations [41–43]. A heuristic explanation was given in [54], which is very close to the string picture that we will find below. Here we will follow the treatment in [23] which exploits some interesting features of the symmetries of the problem.

The ground state of the system, the state with $E - J = 0$, preserves 16 supersym-

[1]The importance of decoupling these two effects was emphasized in [51, 52].

metries[2]. These supercharges, which have $E - J = 0$, act linearly on the impurities or magnons. They transform as $(2,1,2,1)+(1,2,1,2)$ under $SU(2)_{S^5,L} \times SU(2)_{S^5,R} \times SU(2)_{AdS_5,L} \times SU(2)_{AdS_5,R}$ where the various $SU(2)$ groups corresponds to the rotations in AdS_5 and S^5 which leave Z invariant. These supercharges are the odd generators of two $SU(2|2)$ groups[3]. The energy $\epsilon \equiv E - J$ is the (non-compact) $U(1)$ generator in each of the two $SU(2|2)$ supergroups. In other words, the two $U(1)$s of the two $SU(2|2)$ groups are identified. A single impurity with $p = 0$ transforms in the smallest BPS representation of these two supergroups. In total, the representation has 8 bosons plus 8 fermions. This representation is BPS because its energy is $\epsilon = E - J = 1$ which follows from the BPS bound that links the energy to the $SU(2)^4$ charges of the excitations. Let us now consider excitations with small momentum p. At small p we can view the dispersion relation as that of a relativistic theory. Note that as we add a small momentum, the energy becomes higher but we still expect to have 8 bosons plus 8 fermions and not more, as it would be the case for representations of $SU(2|2)^2$ with $\epsilon > 1$. What happens is that the momentum appears in the right hand side of the supersymmetry algebra. This ensures that the representation is still BPS. In fact, for finite p there are two central charges [23]. These extra generators add or remove Zs to the left or right of the excitation and they originate from the commutator terms in the supersymmetry transformation laws, namely terms like $\delta W \sim \psi + [Z, \chi]$, see [23]. These extra central charges are zero for physical states with finite J since we will impose the momentum constraint.

The full final algebra has thus three "central" generators in the right hand side, they are the energy ϵ and two extra charges which we call k^1, k^2. Together with the energy these charges can be viewed as the three momenta k^μ of a 2+1 dimensional Poincare superalgebra. This is the same as the 2+1 dimensional Poincare superal-

[2]More precisely it is annihilated by 16 + 8, but the last 8 act non-linearly on the excitations, they correspond to fermionic impurity annihilation operators with zero momentum.

[3]Note that they are not $PSU(2|2)$ groups.

gebra recently studied in [55, 56], we will see below that this is not a coincidence. Notice that the Lorentz generators are an outer automorphism of this algebra but they are not a symmetry of the problem we are considering. See appendix A for a more detailed discussion of the algebra.

As explained in [23] the expression for the "momenta" is $k^1 + ik^2 = h(\lambda)(e^{ip} - 1)$ and similarly for the complex conjugate. This then implies that we have the formula

$$E - J = k^0 = \sqrt{1 + |k_1 + ik_2|^2} = \sqrt{1 + f(\lambda)\sin^2\frac{p}{2}} \tag{2.2.6}$$

The function $f(\lambda)$ is not determined by this symmetry argument. We know that $f(\lambda) = \frac{\lambda}{\pi}$ up to three loops in the gauge theory [16–18, 57, 58] and that it is also the same at strong coupling (where it was checked at small momenta in [21]). [53] claims to show it is exactly $f = \frac{\lambda}{\pi}$ for all values of the coupling, but we do not fully understand the argument[4].

In the plane wave matrix model [21, 59] one can also use the symmetry algebra to determine a dispersion relation as in (2.2.6) and the function $f(\lambda)$ is nontrivial. More precisely, large J states in the plane wave matrix model have an $SU(2|2)$ group (extended by the central charges to a 2+1 Poincare superalgebra) that acts on the impurities.

The conclusion is that elementary excitations moving along the string are BPS under the 16 supersymmetries that are linearly realized. Supersymmetry then ensures that we can compute the precise mass formula once we know the expression for the central charges.

[4]It is not clear to us why in equation (10) in [53] we could not have a function of λ in the right hand side.

2.2.3 String theory description at large λ

We will now give the description of the elementary impurities or elementary magnons at large λ from the string theory side. In this regime we can trust the classical approximation to the string sigma model in $AdS_5 \times S^5$.

In order to understand the solutions that we are going to study, it is convenient to consider first a string in flat space. We choose light cone gauge, with $X^+ = \tau$, and consider a string with large P_-. The solution with $P_+ = 0$ corresponds to a lightlike trajectory with X^- =constant, see figure 2.1(a,c). Now suppose that we put two localized excitations carrying worldsheet momentum p and $-p$ respectively. Let us suppose that at some instant of time these are on opposite points of the worldsheet spatial circle, see figure 2.1(b). We want to understand the spacetime description of such states. It is clear that the region to the left of the excitations and the region to the right will be mapped to the same lightlike trajectories with X^- =constant that we considered before. The important point is that these two trajectories sit at different values of X^-. This can be seen by writing the Virasoro constraint as

$$\partial_\sigma X^- = 2\pi\alpha' T_{\tau\sigma} \ , \qquad \Delta X^- = 2\pi\alpha' \int d\sigma T_{\tau\sigma} = 2\pi\alpha' p \tag{2.2.7}$$

where $T_{\tau\sigma}$ is the worldsheet stress tensor of the transverse excitations and we have integrated across the region where the excitation with momentum p is localized. Thus the final spacetime picture is that we have two particles that move along lightlike trajectories that are joined by a string. At a given time t the two particles move at the speed of light separated by $\Delta X^1|_t = \Delta X^-|_{X^+} = 2\pi\alpha' p$ and joined by a string, see figure 2.1(d). Of course, the string takes momentum from the leading particle and transfers it to the trailing one. On the worldsheet this corresponds to the two localized excitations moving toward each other. As the worldsheet excitations pass through each other the trailing particle becomes the leading one, see figure 2.1(c).

For a closed string X^- should be periodic, which leads to the momentum constraint $p_{total} = 0$.

In the limit of an infinite string, or infinite P_-, we can consider a single excitation with momentum p along an infinite string. Then the spacetime picture will be that of figure 2.2 where we have two lightlike trajectories, each carrying infinite P_-, separated by $\Delta X^- \sim p$ which are joined by a string. There is some P_- being transferred from the first to the second. But since P_- was infinite this can continue happening for ever[5]. The precise shape of the string that joins the two points depends on the precise set of transverse excitations that are carrying momentum p.

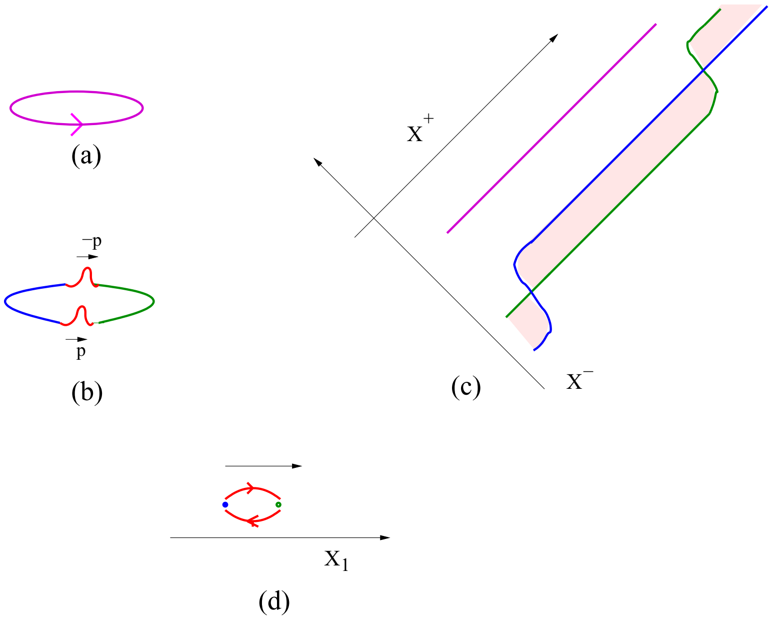


Figure 2.1: Localized excitations propagating along the flat space string worldsheet in light cone gauge. (a) Worldsheet picture of the light cone ground state, with $P_+ = 0$. (b) Worldsheet picture of two localized excitations with opposite momenta propagating along the string. (c) Spacetime description of the configurations in (a) and (b). The configuration in (a) gives a straight line at a constant X^-. The configuration in (b) gives two straight lines at constant X^- when the localized excitations are separated on the worldsheet. When the two excitations in (b) cross each other the lines move in X^-. (d) Snapshot of the spacetime configuration in (b), (c) at a given time t.

Armed with the intuition from the flat space case, we can now go back to the

[5]As a side remark, notice that these lightlike trajectories look a bit like light-like D-branes, which could be viewed as small giant gravitons in the $AdS_5 \times S^5$ case. In this study we take the 't Hooft limit before the large J limit so we can ignore giant gravitons. But it might be worth exploring this further. Strings ending in giant gravitons were recently studied in [28]

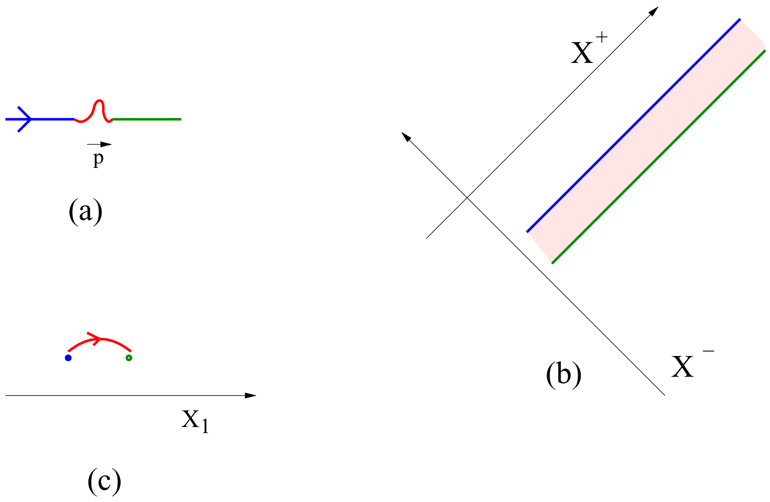


Figure 2.2: Localized excitations propagating on an infinite string. (a) Worldsheet picture of a localized excitation propagating along the string. (b) Spacetime behavior of the state in lightcone coordinates. We have two lightlike lines with a string stretching between them. (c) Snapshot of the state at a given time. The configuration moves to the right at the speed of light.

$AdS_5 \times S^5$ case. We write the metric of S^5 as

$$ds^2 = \sin^2\theta d\varphi^2 + d\theta^2 + \cos^2\theta d\Omega_3^2 \tag{2.2.8}$$

where φ is the coordinate that is shifted by J. The string ground state, with $E-J=0$, corresponds to a lightlike trajectory that moves along φ, with $\varphi - t$=constant, that sits at $\theta = \pi/2$ and at the origin of the spatial directions of AdS_5.

We can find the solution we are looking for in various ways. We are interested in finding the configuration which carries momentum p with least amount of energy $\epsilon = E - J$. For the moment let us find a solution with the expected properties and we will later show that it has the minimum amount of energy for fixed p. We first pick a pair of antipodal points on S^3 so that, together with the coordinate θ and φ they form an S^2. After we include time, the motion takes place in $R \times S^2$. We can now write the Nambu action choosing the parametrization

$$t = \tau \ , \qquad \varphi - t = \varphi' \tag{2.2.9}$$

and we consider a configuration where θ is independent of τ. We then find that the action reduces to

$$S = \frac{\sqrt{\lambda}}{2\pi} \int dt d\varphi' \sqrt{\cos^2\theta \theta'^2 + \sin^2\theta} \tag{2.2.10}$$

It is easy to integrate the equations of motion and we get

$$\sin\theta = \frac{\sin\theta_0}{\cos\varphi'} \,, \qquad -(\frac{\pi}{2} - \theta_0) \leq \varphi' \leq \frac{\pi}{2} - \theta_0 \tag{2.2.11}$$

where $0 \leq \theta_0 \leq \pi/2$ is an integration constant. See figure 2.3. In these variables the string has finite worldsheet extent, but the regions near the end points are carrying an infinite amount of J. We see that for this solution the difference in angle between the two endpoints of the string at a given time t is

$$\Delta\varphi' = \Delta\varphi = 2(\frac{\pi}{2} - \theta_0) \tag{2.2.12}$$

It is also easy to compute the energy

$$E - J = \frac{\sqrt{\lambda}}{\pi} \cos\theta_0 = \frac{\sqrt{\lambda}}{\pi} \sin\frac{\Delta\varphi}{2} \tag{2.2.13}$$

We now propose the following identification for the momentum p

$$\Delta\varphi = p \tag{2.2.14}$$

We will later see more evidence for this relation. Once we use this relation (2.2.13) becomes

$$E - J = \frac{\sqrt{\lambda}}{\pi} \left|\sin\frac{p}{2}\right| \tag{2.2.15}$$

in perfect agreement with the large λ limit (2.1.3) of the gauge theory result (2.1.2). The sign of p is related to the orientation of the string. In other words, $\Delta\varphi$ is the angular position of the endpoint of the string minus that of the starting point and it can be negative.

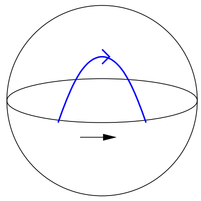


Figure 2.3: Giant magnon solution to the classical equations. The momentum of the state is given by the angular distance between the endpoints of the string. We depicted a configuration with $0 < p < \pi$. A configuration with negative momentum would look the same except that the orientation of the string would be reversed. The string endpoints are on the equator and the move at the speed of light.

In order to make a more direct comparison with the gauge theory it is useful to pick a gauge on the worldsheet in such a way that, for the string ground state (with $E - J = 0$), the density of J is constant[6]. There are various ways of doing this. One specific choice would be the light cone gauge introduced in [60]. We will now do something a bit different which can be done easily for strings on $R \times S^2$ and which will turn out useful for our later purposes. This consists in choosing conformal gauge and setting $t = \tau$. x labels the worldsheet spatial coordinate. In this gauge the previous solution takes the form

$$\begin{aligned} \cos\theta &= \frac{\cos\theta_0}{\cosh\left[\frac{x-\sin\theta_0 t}{\cos\theta_0}\right]} = \frac{\sin\frac{p}{2}}{\cosh\left[\frac{x-\cos\frac{p}{2}t}{\sin\frac{p}{2}}\right]} \\ \tan(\varphi - t) &= \cot\theta_0 \tanh\left[\frac{x-\sin\theta_0 t}{\cos\theta_0}\right] = \tan\frac{p}{2}\tanh\left[\frac{x-\cos\frac{p}{2}t}{\sin\frac{p}{2}}\right] \end{aligned} \tag{2.2.16}$$

where we used (2.2.14). In this case we see that the range of x is infinite. These coordinates have the property that the density of J away from the excitation is constant. This property allows us a to make an identification between the coordinate x and the position l (see (2.1.1)) along the chain in the gauge theory. More precisely,

[6]Note that we only require J to be constant away from the excitations, it could or could not be constant in the regions where $E - J > 0$.

we compute the density of J per unit x in order to relate l and x

$$\frac{dJ}{dx} = \frac{\sqrt{\lambda}}{2\pi} \qquad \text{or} \qquad dl = \frac{\sqrt{\lambda}}{2\pi} dx \tag{2.2.17}$$

This relation allows us to check the identification of the momentum (2.2.14) since the relation between energy and momentum (2.1.3) determines the velocity in the gauge theory through the usual formula

$$v_{gauge} = \frac{dl}{dt} = \frac{d\epsilon(p)}{dp} = \frac{\sqrt{\lambda}}{2\pi} \cos\frac{p}{2} \ , \qquad \text{for} \quad p > 0 \tag{2.2.18}$$

On the other hand we see from (2.2.16) that the velocity is

$$v_{string} = \frac{dx}{dt} = \sin\theta_0 = \cos\frac{\Delta\varphi}{2} \tag{2.2.19}$$

We see that after taking into account (2.2.17) the two velocities become identical if we make the identification (2.2.14).

The solution becomes simpler if expressed in terms of the coordinates introduced in [61]. Those coordinates were specially adapted to describe 1/2 BPS states which carry charge J. So it is not surprising that they are also useful for describing small excitations around such states. The $AdS_5 \times S^5$ metric in those coordinates is a fibration of R, characterizing the time direction, and two S^3s (coming form AdS_5 and S^5) over a three dimensional space characterized by coordinates x_1, x_2, y. The plane $y = 0$ is special because one of the two 3-spheres shrinks to zero size in a smooth way. Thus the plane $y = 0$ is divided into regions (or "droplets") where one or the other S^3 shrinks to zero size. The $AdS_5 \times S^5$ solution contains a single circular droplet of radius R where the S^3 coming from AdS_5 shrinks, see figure 2.4. Particles carrying $E - J = 0$ live on the boundary of the two regions. In fact the circle constituting the boundary of the two regions sits at $\theta = \pi/2$ and it is parameterized by $\varphi' = \varphi - t$ in previous coordinates. We will be only interested in the metric on this special plane at $y = 0$ which takes the form, for $r < 1$,

$$ds^2 = R^2 \left\{ -(1-r^2) \left[dt - \frac{r^2}{(1-r^2)} d\varphi' \right]^2 + \frac{dr^2 + r^2 d\varphi'^2}{(1-r^2)} + (1-r^2) d\Omega_3^2 + \cdots \right\} \tag{2.2.20}$$

where $r^2 = \sin^2\theta = x_1^2 + x_2^2$ and the dots remind us that we are ignoring the y coordinate and the second sphere, which has zero size at $y = 0$ for $r \leq 1$.

In these coordinates the solution is simply a straight line that joins two points of the circle as shown in figure 2.4(a). This can be seen from (2.2.11), which can be rewritten as

$$r \cos\varphi' = x_1 = \text{const} \tag{2.2.21}$$

The energy is simply the length of the string measured with the flat metric on the plane parameterized by x_1, x_2; $ds^2_{flat} = R^2(dx_1^2 + dx_2^2)$. In fact, the picture we are finding here is almost identical to the one discussed from the gauge theory point of view in [54][7]. If we restrict the arguments in [54] to 1/2 BPS states and their excitations we can see how this picture emerges from the gauge theory point of view. Namely, we first diagonalize the matrix Z in terms of eigenvalues. Then the impurity is an off diagonal element of a second matrix W which joins two eigenvalues that are at different points along the circle. The energy formula follows from the commutator term $tr|[Z,W]|^2$ in the gauge theory, see [54] for more details.

These coordinates are very useful for analyzing the symmetries. In particular, we will now explain the appearance of extra central charges and we will match the superalgebra to the one found on the gauge theory side in [23]. Under general considerations we know that the anticommutator of two supersymmetries in 10 dimensional supergravity contains gauge transformations for the NS-B_2 field [62]. These act non-

[7]The difference is that [54] considered an S^5 in R^6 and a string stretching between two points on S^5 through R^6.

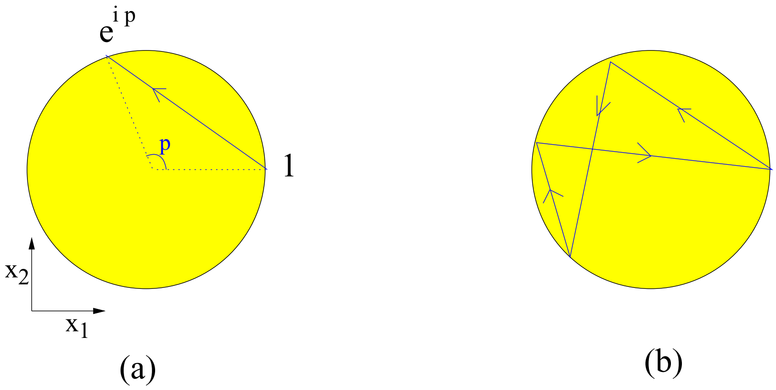


Figure 2.4: Giant magnons in LLM coordinates (2.2.20). (a) A giant magnon solution looks like a straight stretched string. It's momentum p is the angle subtended on the circle. k_1 and k_2 are the projections of the string along the directions $\hat{1}$ and $\hat{2}$. The direction of the string gives the phase of $k_1 + ik_2$, while its length gives the absolute value of the same quantity. (b) A closed string state built of magnons that are well separated on the worldsheet. Notice that the total central charges $k_1,\ k_2$ vanish. Similarly the total angle subtended by the string, which is the total momentum p_{total} also vanishes modulo 2π.

trivially on stretched strings. In flat space this leads to the fact that straight strings are BPS. In fact, inserting the explicit expression of the Killing spinors in [61] into the general formula for the anticommutator of two supercharges [62] it is possible to see that the relevant NS gauge transformations are those with a constant gauge parameter $\Lambda_1,\ \Lambda_2$, $\delta B = d\Lambda$ [8]. Strings that are stretched along the $\hat{1}$ or $\hat{2}$ directions acquire a phase under such gauge transformations. Thus these are the central charges that we are after. Note that in order to activate these central charges it is not necessary to have a compact circle in the geometry. In fact, the string stretched between two separated D-branes in flat space is BPS for the same reason[9].

Actually, the supersymmetry algebra is identical to a supersymmetry algebra in 2+1 dimensions, where the string winding charges, k^1, k^2, play the role of the spatial momenta[10]. See appendix A for more details on the algebra. From the 2+1

[8]The requisite spinor bilinear is closely related to the one in eqn. (A.45) of [61]. Namely, the expression in [62] involves terms of the form $\bar{\epsilon}_1^* \gamma^\mu \epsilon_2$, which becomes $\epsilon^t \Gamma^2 \Gamma^\mu \epsilon$ in the notation of [61].

[9]If we think of the string with $J = \infty$ as a lightlike D-brane, the analogy becomes closer.

[10]All these statements are independent of the shape of the droplets in [61]. This particular statement is easiest to see if we consider droplets on a torus and we perform a T-duality which takes us to a 2+1 dimensional Poincare invariant theory (in the limit the original torus is very small). This is a theory studied in [56, 61]. This also explains why the supersymmetry algebra is the same

dimensional point of view it is a peculiar Poincare super algebra since it has $SO(4)^2$ charges in the right hand side of supersymmetry anti-commutators. Of course, this is the same supersymmetry algebra that appeared in the gauge theory discussion [23]. In conclusion, the symmetry algebra is exactly the same on both sides. The extra central charges are related to string winding charges. We can think of the vector given by the stretched string as the two spatial momenta $k^1,\ k^2$ appearing in the Poincare superalgebra. In other words, we can literally think of the stretched string in figure 2.4 as specifying a vector k^1, k^2 of size

$$k^1 + ik^2 = \frac{R^2}{2\pi\alpha'}(e^{ip} - 1) = i\frac{\sqrt{\lambda}}{\pi}e^{i\frac{p}{2}}\sin\frac{p}{2} \tag{2.2.22}$$

Then the usual relativistic formula for the energy implies (2.2.6), as in the gauge theory. Note that the problem we are considering does *not* have lorentz invariance in 2+1 dimensions. Lorentz invariance is an outer automorphism of the algebra, that is useful for analyzing representations of the algebra, but it is not an actual symmetry of the theory. In particular, in our problem the formula (2.2.6) is not a consequence of boost invariance, since boosts are not a symmetry[11]. It is a consequence of supersymmetry, it is a BPS formula. Note that rotations of $k^1,\ k^2$ are indeed a symmetry of the problem and they correspond to rotations of the circle in figure 2.4. This is the symmetry generated by J. Note also that a physical state with large but finite J will consist of several magnons but the configurations should be such that we end up with a closed string, see figure 2.4(b) . Thus, for ordinary closed strings the total value of the central charges is zero, since there is no net string winding. This implies, in particular, that for a closed finite J string there are no new BPS states other than the usual ones corresponding to operators $Tr[Z^J]$.

in the two problems.

[11]One might wonder whether boosts are a hidden symmetry of the string sigma model. This is not the case because we can increase $|k|$ without bound by performing a boost, while physically we know that $|k|$ is bounded as in (2.2.22).

Notice that the classical string formula (2.2.13) is missing a 1 in the square root as compared to (2.1.2). This is no contradiction since we were taking p fixed and λ large when we did the classical computation. This 1 should appear after we quantize the system. In fact, for small p and λ large, we can make a plane wave approximation and, after quantization, we recover the 1 [21, 63]. But if we did not quantize we would not get the 1, even in the plane wave limit. So we see that in the regime that the 1 is important we indeed recover it by doing the semiclassical quantization. This 1 is also implied by the supersymmetry algebra. The argument is identical to the one in [23] once we realize that the central charges are present and we know the relation between the central charges and the momentum p, as in (2.2.22). Notice that the classical solutions we discussed above break the $SO(4)$ symmetry since they involve picking a point on $S^3 \subset S^5$ where the straight string in figure 2.4(a) is sitting. Upon collective coordinate quantization we expect that the string wavefunction will be constant on this S^3. In addition, we expect to have fermion zero modes. They originate from the fact that the magnon breaks half of the 16 supersymmetries that are left unbroken by the string ground state. Thus we expect 8 fermion zero modes, which, after quantization, will give rise to $2^4 = 16$ states, 8 fermions + 8 bosons. This argument is correct for fixed p and large λ. In fact, all these arguments are essentially the same as the ones we would make for a string stretching between two D-branes. Notice that all magnons look like stretched strings in the S^5 directions, as in figure 2.4(a), including magnons corresponding to insertions of $\partial_\mu Z$ which parameterize elementary excitations along the AdS_5 directions. Of course, here we are considering a single magnon. Configurations with many magnons can have large excursions into the AdS directions.

Since the stretched string solution in figure 2.4(a) is BPS, it is the minimum energy state for a given p.

In fact, we can consider large J string states around other 1/2 BPS geometries, given by different droplet shapes as in [61]. In those cases, we will have BPS configurations corresponding to strings ending at different points on the boundary of the droplets and we have strings stretching between these points. It would be nice to see if the resulting worldsheet model is integrable.

Note that the fact that the magnons have a large size (are “giant”) at strong coupling is also present in the Hubbard model description in [64] [12].

Finally, let us point out that our discussion of the classical string solutions focussed on an $R \times S^2$ subspace of the geometry. Therefore, the same solutions will describe giant magnons in the plane wave matrix model [21,59] and other related theories [55]. Similar solutions also exist in $AdS_{2,3} \times S^{2,3}$ with RR fluxes (NS-fluxes would change the equations already at the classical level).

2.2.4 Spinning folded string

In this subsection we apply the ideas discussed above to compute the energy of a spinning folded string considered in [44]. This is a string that rotates in an S^2 inside S^5. For small angular momentum J this is a string rotating around the north pole. Here we are interested in the limit of large J where the ends of the rotating string approach the equator, see figure 2.5. In this limit the energy of the string becomes [44]

$$E - J = 2\frac{\sqrt{\lambda}}{\pi} \tag{2.2.23}$$

This string corresponds to a superposition of two “magnons” each with maximum momentum, $p = \pi$. Notice that the dispersion relation implies that such magnons are at rest, see (2.2.18). They are equally spaced on the worldsheet. At large J we can ignore the interaction between the “magnons” and compute the energy of the state as a superposition of two magnons. We see that the energy (2.2.23) agrees precisely

[12] We thank M. Staudacher for pointing this out to us.

with the energy of two magnons with $p = \pi$.

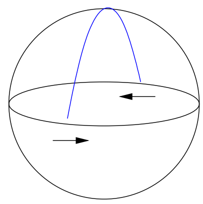


Figure 2.5: Spinning string configuration that corresponds to two magnons with $p = \pi$.

Since the λ dependence of the strings spinning in AdS in [44] is somewhat similar, one might find an argument for that case too. In fact, the solutions we are considering here, such as the one in figure 2.4(b) is the sphere analog of the solutions with spikes considered in [65].

2.3 Semiclassical S-matrix

2.3.1 General constraints on the S-matrix

In this section we consider the S-matrix for scattering of two magnons. On the gauge theory side this is the so called "asymptotic S-matrix" discussed in [45]. In the string theory side it is defined in a similar way: we take two magnons and scatter them. Then, we define the S matrix for asymptotic states as we normally do in 1+1 dimensional field theories. Since the sigma model is integrable [20, 46], we expect to have factorized scattering. It was shown in [47] that integrability still persists in the lightcone gauge (this was shown ignoring the fermions). In fact we will later check explicitly that our magnons undergo classical dispersionless scattering.

As we mentioned above the supersymmetry algebra is the same in the gauge theory and the string theory. We have only shown here that the algebra is the same at the classical level on the string theory side. But it is very natural to think that after quantization we will still have the same algebra. Thus, any constraint coming from

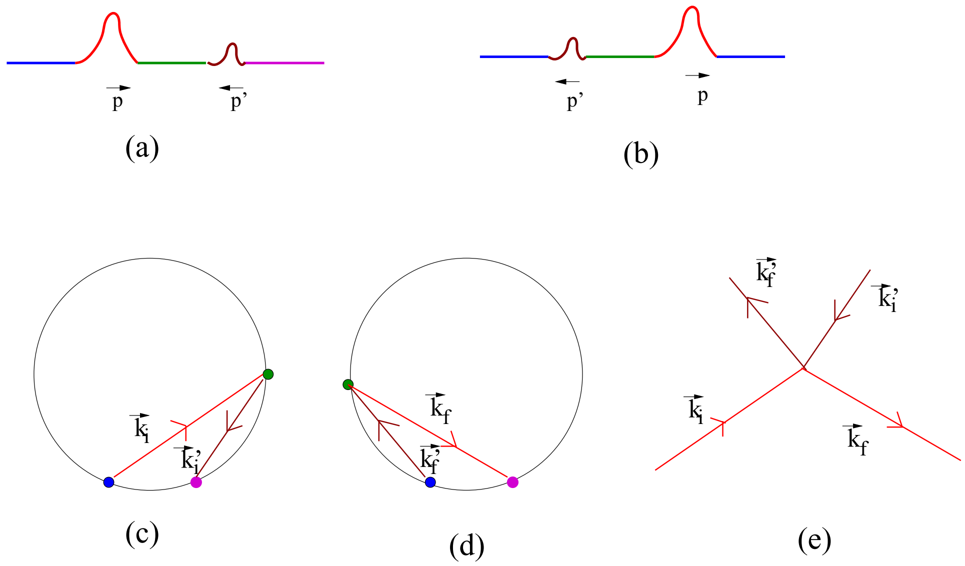


Figure 2.6: Scattering of $2 \to 2$ magnons. (a) Worldsheet picture of asymptotic initial state. (b) Worldsheet picture of asymptotic final state. (c) Initial state in LLM coordinates. (d) Final state in LLM coordinates. (e) Initial and final configuration for the momenta $\vec{k}$ that are relevant for the 2+1 dimensional kinematics of the process.

this algebra is the same. An important constraint for the S matrix was derived by Beisert in [23]. Each of the magnons can be in one of 16 states (8 bosons plus 8 fermions). So the scattering matrix is a $16^2 \times 16^2$ matrix. Beisert showed that this matrix is completely fixed by the symmetry up to an overall phase (and some phases that can be absorbed in field redefinitions). Schematically $S = \hat{S}_{ij} S_0$ where $\hat{S}_{ij}$ is a known matrix and S_0 is an unknown phase. The same result holds then for the string theory magnons. In fact, it was conjectured in [37, 45, 66] that the two S-matrices differ by a phase. Here we are pointing out that this structure is a consequence of the symmetries on the two sides. The fact that the whole S matrix is determined up to a single function is analogous to the statement that the four particle scattering amplitude in $\mathcal{N} = 4$ SYM is fixed up to a scalar function of the kinematic invariants. The reason is that two massless particles with 16 states each give a single massive, non-BPS, representation with $2^8 = 16^2$ states.

A two magnon scattering process has a kinematics that is shown in figure 2.6.

Notice that we can literally think of the straight strings as determining the initial and final momentum vectors of the scattering process as in figure 2.6(e). The orientation of these vectors is important. The constraints on the matrix structure of the S matrix are exactly the same as the constraints that a four particle scattering amplitude in a relativistic 2+1 dimensional field theory with the same superalgebra would have. These constraints are easy to derive in the center of mass frame. And we could then boost to a general frame. Notice that from the 2+1 dimensional point of view fermions have spin, and thus their states acquire extra phases under rotation. In other words, when we label a state by saying what its momentum p is, we are just giving the magnitude of $\vec{k}$, but not its orientation. The orientation of $\vec{k}$ depends on the other magnons. For example, in the scattering process of figure 2.6(a,b) the initial and final states have the same momenta p, p', but the initial vectors $\vec{k}_i$, $\vec{k}'_i$ have different orientation than the final vectors $\vec{k}_f$, $\vec{k}'_f$. When we consider a sequence of scattering processes, one after the other, it is important to keep track of the orientation of $\vec{k}$. In other words, under an overall rotation the S matrix is not invariant, it picks up some phases due to the fermion spins. In [23] these phases are taken into account by extra insertions of the field Z which makes the chain "dynamic".

Note that the constraints on the matrix structure of the scattering amplitude are applicable in a more general context to any droplet configuration of [61]. For example, it constrains the scattering amplitude for elementary excitations in other theories with the same superalgebra. Examples are the massive M2 brane theory [67] or the theories considered in [55, 56].

Note that the existence of closed subsectors is a property of factorized scattering (integrability) and the matrix structure of the $\hat{S}$ matrix, but does not depend on the precise nature of the overall phase. Thus closed subsectors exist on both sides. This argument shows this only in the large J limit where the magnons are well separated and we can use the asymptotic S matrix[13].

We expect that the overall phase, S_0, will interpolate between the weak and strong coupling results. The full interpolating function has not yet been determined[14].

In this section we will compute in a direct, and rather straightforward way, the semiclassical S-matrix for the scattering of string theory magnons. It turns out that the result will agree with the one derived in [37] through more indirect methods.

Notice that at large 't Hooft coupling and fixed momentum, the approximate expression (2.1.3) amounts to a relativistic approximation to the non-relativistic formula (2.1.2). Similarly, in this limit, the matrix prefactor $\hat{S}$ becomes that of a relativistic theory and it is a bit simpler.

Notice that the theory in light cone gauge is essentially massive so that we can define scattering processes in a rather sharp fashion, in contrast with the full covariant sigma model which is conformal, a fact that complicates the scattering picture. Nevertheless, starting from the conformal sigma model can be a useful way to proceed [51, 52].

2.3.2 Scattering phase at large λ and the sine-Gordon connection

In the semiclassical limit where λ is large and p is kept fixed the leading contribution to the S matrix comes from the phase δ in $S_0 = e^{i\delta}$, which goes as $\delta \sim \sqrt{\lambda} f(p, p')$. For fixed momenta, this phase dominates over the terms that come from the matrix structure $\hat{S}$ in the scattering matrix. In this section, we compute this phase, ignoring the matrix prefactor $\hat{S}$ in the S-matrix.

In the semiclassical approximation the phase shift can be computed by calculating

[13] Note that this is not obviously in contradiction with the arguments against closed subsectors on the string theory side that were made in [68], which considered finite J configurations.

[14] There is of course (the very unlikely possibility) that the two phases are different and that AdS/CFT is wrong.

the time delay that is accumulated when two magnons scatter. The computation is very similar to the computation of the semiclassical phase for the scattering of two sine Gordon solitons, as computed in [69]. In fact, the computation is almost *identical* because the magnons we discuss are in direct correspondence with sine Gordon solitons. This uses the relation between classical sine Gordon theory and classical string theory on $R \times S^2$ [48–50][15]. It is probably also possible to obtain these results from [71, 72], but we found it easier to do it using the correspondence to the sine Gordon theory. The map between a classical string theory on $R \times S^2$ and the sine Gordon model goes as follows. We consider the string action in conformal gauge and we set $t = \tau$. Then the Virasoro constraints become

$$1 = \dot{\mathbf{n}}^2 + \mathbf{n}'^2 \ , \qquad \dot{\mathbf{n}}.\mathbf{n}' = 0 \tag{2.3.24}$$

where $\mathbf{n}^2 = 1$ parameterizes the S^2. The equations of motion follow from these constraints. The sine Gordon field is defined via

$$\cos 2\phi = \dot{\mathbf{n}}^2 - \mathbf{n}'^2 \tag{2.3.25}$$

For the "magnon" solution we had above we find that ϕ is the sine Gordon soliton

$$\tan\frac{\phi}{2} = \exp\left[\frac{\cos\frac{p}{2}t - x}{\sin\frac{p}{2}}\right] = e^{-\gamma(x-vt)} \ , \qquad v = \cos\frac{p}{2} \ , \qquad \gamma^{-2} = 1 - v^2 \tag{2.3.26}$$

Notice that the energy of the sine Gordon soliton is inversely proportional to the string theory energy of the excitation (2.2.13)

$$E_{s.g.} = \gamma = \cosh\hat{\theta} \ , \qquad \epsilon_{magnon} = \frac{\sqrt{\lambda}}{\pi}\frac{1}{\gamma} \ , \qquad \cosh\hat{\theta} = \frac{1}{\sin\frac{p}{2}} \tag{2.3.27}$$

[15]As explained in [70] the two theories have different poisson structures so that their quantum versions are different.

where we measure the sine Gordon energy relative to the energy of a soliton at rest and we introduced the sine Gordon rapidity $\hat{\theta}$. Note that a boost on the sine Gordon side translates into a non-obvious *classical* symmetry on the $R \times S^2$ side. Do not confuse this approximate boost symmetry of the sine Gordon theory with the boosts that appeared in our discussion of the supersymmetry algebra. Neither of them is a true symmetry of the problem, but they are not the same!.

We now consider a soliton anti-soliton pair and we compute the time delay for their scattering as in [69]. (If we use a soliton-soliton pair we obtain the same classical answer[16]). Since the x and t coordinates are the same in the two theories, this time delay is precisely the same for the string theory magnons and for the sine Gordon solitons. The Sine Gordon scattering solution in the center of mass frame is

$$\tan \frac{\phi}{2} = \frac{1}{v} \frac{\sinh \gamma v t}{\cosh \gamma x} \tag{2.3.28}$$

The fact that the sine Gordon scattering is dispersionless implies that the scattering of magnons is also dispersionless in the classical limit (of course we also expect it to be dispersionless in the quantum theory).

The time delay is

$$\Delta T_{CM} = \frac{2}{\gamma v} \log v \tag{2.3.29}$$

We now boost the configuration (2.3.28) to a frame where we have a soliton moving with velocity v_1 and an anti-soliton with velocity v_2, with $v_1 > v_2$. Then the time delay that particle 1 experiences as it goes through particle 2 is

$$\Delta T_{12} = \frac{2}{\gamma_1 v_1} \log v_{cm} \tag{2.3.30}$$

[16] In fact, for a given p we have a family of magnons given by a choice of a point on S^3 which is telling us how the string is embedded in S^5. In the quantum problem this zero mode is quantized and the wavefunction will be spread on S^3. In the classical theory we expect to find the same time delay for scattering of two magnons associated to two arbitrary points on S^3.

where v is the velocity in the center of mass frame

$$2\log v_{cm} = 2\log\tanh\left[\frac{\hat{\theta}_1 - \hat{\theta}_2}{2}\right] = \log\left[\frac{1-\cos\frac{p_1-p_2}{2}}{1-\cos\frac{p_1+p_2}{2}}\right] \;, \qquad \text{for} \quad p_1,\ p_2 > 0 \quad (2.3.31)$$

We can now compute the phase shift from the formula

$$\frac{\partial\delta_{12}(\epsilon_1,\epsilon_2)}{\partial\epsilon_1} = \Delta T_{12} \qquad (2.3.32)$$

We obtain

$$\delta = \frac{\sqrt{\lambda}}{\pi}\left\{\left[-\cos\frac{p_1}{2} + \cos\frac{p_2}{2}\right]\log\left[\frac{1-\cos\frac{p_1-p_2}{2}}{1-\cos\frac{p_1+p_2}{2}}\right]\right\} - p_1\frac{\sqrt{\lambda}}{\pi}\sin\frac{p_2}{2} \qquad (2.3.33)$$

Note that, even though the time delay is identical to the sine Gordon one, the phase shift is different, due to the different expression for the energy (2.3.27). This implies, in particular, that the phase shift is not invariant under sine Gordon boosts. The first term in this expression agrees precisely with the large λ limit of the phase in [37][17]. The second term in (2.3.33) looks a bit funny. However, we need to recall that the definition of this S-matrix is a bit ambiguous. This ambiguity is easy to see in the string theory side and was noticed before. For example [73–76] and [77] obtained different S-matrices for the scattering of magnons at low momentum (near plane wave limit). The difference is due to a different choice of gauge which translates into a different choice of worldsheet x variable. In [73–76] the x variable was defined in such a way that the density of J is constant. In [77] it was defined so that the density of $E + J$ is constant. In our case we have defined it in such a way that the density of E is constant, since we have set $\dot{t} = 1$ in conformal gauge. All these choices give the same definition for the x variable when we consider the string ground state. The difference lies in the different length in x that is assigned to the magnons, which have

[17]The phase in [37] contains further terms in a $1/\sqrt{\lambda}$ expansion which we are not checking here.

$E - J \neq 0$. Thus the S matrix computed in different gauges will differ simply by terms of the form $e^{ip_i f(p_j)}$ where $f(p_j)$ is the difference in the length of the magnon on the two gauges. Of course the Bethe equations are the same in both cases since the total length of the chain is also different and this cancels the extra terms in the S matrix. The position variable that is usually chosen on the gauge theory side assigns a length 1 to the impurity. At large λ we can ignore 1 relative to λ and say that the length of the impurity is essentially zero. Thus we can say that the gauge theory computation is using coordinates where the density of J is constant. Using the relation between the gauge theory spatial coordinate l and our worldsheet coordinate x (2.2.17) (which is valid in the region where $E - J = 0$) we get that the interval between of two points separated by a magnon are related by

$$\Delta l = \int dx \frac{dJ}{dx} = \int dx \frac{dE}{dx} - \left(\frac{dE}{dx} - \frac{dJ}{dx}\right) = \frac{2\pi}{\sqrt{\lambda}} \Delta x - \epsilon \tag{2.3.34}$$

where Δl is the interval in the conventions of [37] and Δx is the interval in our conventions. So we see that in our gauge the magnon will have an extra length of order $\epsilon(p)$. Thus $S_{string-Bethe} = S_{ours} e^{ip_1 \epsilon_2}$, where $S_{string-Bethe}$ is the S matrix in the conventions used in [37]. This cancels the last term in (2.3.33). In summary, after expressing the result in conventions adapted to the gauge theory computation we find that for $sign(\sin \frac{p_1}{2}) > 0$ and $sign(\sin \frac{p_2}{2}) > 0$ we get

$$\delta(p_1, p_2) = -\frac{\sqrt{\lambda}}{\pi}\left(\cos \frac{p_1}{2} - \cos \frac{p_2}{2}\right) \log \left[\frac{\sin^2 \frac{p_1 - p_2}{4}}{\sin^2 \frac{p_1 + p_2}{4}} \right] \tag{2.3.35}$$

The cases where $p < 0$ can be recovered by shifting p by a period so that $\sin \frac{2\pi + p}{2} > 0$. The function (2.3.35) should be trusted when $\sin \frac{p_i}{2} > 0$ and it should be defined to be periodic with period 2π outside this range. Note that this function goes to zero when $p_1 \to 0$ with p_2 fixed. When p is small we need to quantize the system. We can check that, after quantization, the S matrix is still trivial for small p_1 and fixed p_2.

This can be done by expanding in small fluctuations around our soliton background. We find that the small excitations propagate freely through the soliton.

The leading answer (2.3.35) vanishes at small p. In fact, at small p it is important to properly quantize the system and the result depends on the polarizations of the states, see [73–76]. For example, in the $SU(1|1)$ sector [73–76] found (see also [77])

$$S_{string} = -1 + i\frac{1}{2}\left(p_1\sqrt{1+\frac{\lambda}{4\pi^2}p_2^2} - p_2\sqrt{1+\frac{\lambda}{4\pi^2}p_1^2} - p_1 + p_2\right) \tag{2.3.36}$$

Corrections to the leading phase (2.3.35) were computed in [78] and some checks were made in [79, 80].

On the gauge theory side the phase is known up to three loop orders in λ [17, 18, 57, 58]. Of course, finding the full interpolating function is an outstanding challenge[18].

Finally, to complete the discussion of scattering states, we comment on the space-time picture of the scattering process. In the classical theory, besides specifying p, we can also specify a point on S^3 for each of the two magnons that are scattering off each other. We do not know the general solution. The sine Gordon analysis we did above applies only if the point on S^3 is the same for the two magnons or are antipodal for the two magnons. In the first case we have a soliton-soliton scattering in the sine-Gordon model and in the second we have a soliton-anti-soliton scattering. Both give the same classical time delay. The soliton anti-soliton scattering with $p_1 = -p_2$ looks initially like loop of string made of two magnons. One of the endpoints has infinite J and the other has finite J. The point in the front, which initially carries a finite amount of J, looses all its J and it moves to the left. The loop becomes a point and then the loops get formed again but with the finite J point to the left, behind the point that carries infinite J. See figure 2.7. The soliton soliton scattering is represented by a doubly folded string that looks initially like a two magnon state. As time evolves the

[18] An all loop guess was made in [81] (see also [64]), but this guess appears to be in conflict with the strong coupling results obtained via AdS/CFT.

point in the front, which carries finite J, detaches from the equator and moves back of the other endpoint which carries infinite J. The final picture is, again, equivalent to the original one with front and back points exchanged. We see that in both cases asymptotic states are well defined and look like individual magnons.

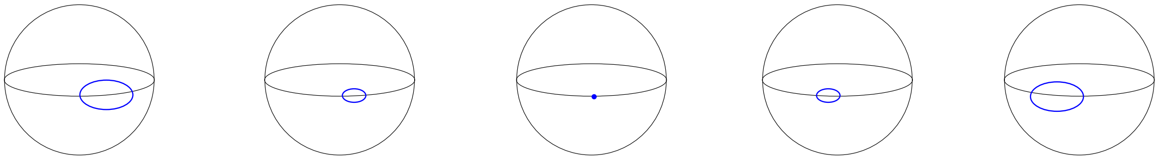


Figure 2.7: Evolution of the soliton anti-soliton scattering state. Time increases to the right. We have chosen a rotating frame on the sphere where the point with infinite J is stationary. At $t = 0$ the string is all concentrated at a point, the point that carries infinite J. The state looks asymptotically as two free magnons on opposite hemispheres and with the same endpoints.

2.4 Discussion

In this chapter we have introduced a limit which allows us to isolate quantum effects from finite volume effects in the gauge theory/spin chain/string duality. In this limit, the symmetry algebra is larger than what is naively expected. This algebra is a curious type of 2+1 superpoincare algebra, without the lorentz generators, which are not a symmetry. The algebra is the same on both sides. In this infinite J limit the fundamental excitation is the "magnon" which is now identified on both sides. The basic observable is the scattering amplitude of many magnons. Integrability should imply that these magnons obey factorized scattering so that all amplitudes are determined by the scattering matrix of fundamental magnons. The matrix structure of this S-matrix is determined by the symmetry at all values of the coupling. So the whole problem boils down to computing the scattering phase [23]. This phase is a function of the two momenta of the magnons and the 't Hooft coupling. At weak coupling it was determined up to three loops [17, 18, 57, 58]. At strong coupling we have the leading order result, computed directly here and indirectly in [37] (see also [73–76]). The one loop sigma model correction to the S-matrix was computed using similar methods in [78]. As in other integrable models, a clever use of crossing symmetry plus a clever choice of variables enables the computation of the phase at all values of the coupling. A crossing symmetry equation was written by Janik in [82]. The kinematics of this problem are a bit different than that of ordinary relativistic 1+1 dimensional theories. In fact, the kinematic configuration has a double periodicity [82]. This is most clear when we define a new variable θ_p as

$$|\vec{k}|^2 = \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2} = \sinh^2 \theta_p \ , \qquad \epsilon = \cosh \theta_p \tag{2.4.37}$$

We have a periodicity in $\theta_p \to \theta_p + 2\pi i$ and in $p \to p + 2\pi$. Crossing is related to the change $\theta_p \to \theta_p + i\pi$. The full amplitude does not need to be periodic in these variables since there can be branch cuts. The compiled information in the references cited here plus the results in this chapter helped the authors of [24, 25] to propose the exact expression for this phase. It has so far passed all consistency checks so far. In the next chapter we study the pole structure of the whole scattering matrix and relate it to the physical excitations given by the BPS magnons discussed (at strong coulpling) in this chapter.

We should finally mention that perhaps it might end up being most convenient to think of the problem in such a way that the magnon will be composed of some more elementary excitations, as is the case in [64][19]. On the other hand, we do not expect these more elementary excitations to independently propagate along the chain. For this reason it is not obvious how to match to something on the string theory side.

[19] The equivalence [64] between the Hubbard model and the gauge theory holds only up to 3 loop orders.

2.5 Appendix A: The supersymmetry algebra

We start with a single $SU(2|2)$ subgroup first. This algebra has two $SU(2)$ generators and a non-compact generator $k^0 \equiv \epsilon = E - J$. The superalgebra is

$$\{Q'^{bs}, Q^{ar}\} = \epsilon^{ba}\epsilon^{sr}k^0 + 2[\epsilon^{ba}J^{sr} - \epsilon^{sr}J^{ba}] \tag{2.5.38}$$

$$\{Q^{bs}, Q^{ar}\} = = 0 \ , \qquad \{Q'^{bs}, Q'^{ar}\} = 0 \tag{2.5.39}$$

(with $\epsilon^{+-} = 1$) where we denote by a, b the first $SU(2)$ indices and by rs the second $SU(2)$ indices. We also have a reality condition $(Q^{ar})^\dagger = \epsilon_{ab}\epsilon_{rs}Q'^{bs}$. The central extensions considered in [23] involve two other central generators k^1, k^2 appearing on the right hand side of (2.5.39), we will arbitrarily choose the normalization of these generators in order to simplify the algebra. In order to write the resulting algebra it is convenient to put together the two generators as

$$q^{\alpha ar} = (Q^{ar}, Q'^{ar}) \ , \text{ or } \quad q^{+ar} = Q^{ar} \ , \qquad q^{-ar} = Q'^{ar} \tag{2.5.40}$$

where α, β will be $SL(2,R) = SO(2,1)$ indices. We introduce the gamma matrices

$$(\gamma_0)_\alpha^{\ \beta} = i\sigma^3 \ , \quad (\gamma_1)_\alpha^{\ \beta} = \sigma^1 \ , \quad (\gamma_2)_\alpha^{\ \beta} = \sigma^2 \ , \quad (\gamma_\mu)^{\alpha\beta} = \epsilon^{\alpha\delta}(\gamma_\mu)_\delta^{\ \beta} \tag{2.5.41}$$

The full anti commutators will now have the form

$$\{q^{\alpha ar}, q^{\beta bs}\} = i\epsilon^{ba}\epsilon^{sr}(\gamma_\mu)^{\alpha\beta}k^\mu - 2\epsilon^{\alpha\beta}[\epsilon^{ba}J^{sr} - \epsilon^{sr}J^{ba}] \tag{2.5.42}$$

The smallest representation of this algebra contains a bosonic doublet and a fermionic doublet transforming as $(1,2)_b + (2,1)_f$ under $SU(2) \times SU(2)$. If we think of these as particles in three dimensions, then we also need to specify the 2+1 spin of the excitation. It is zero for the bosons and $+\frac{1}{2}$ for the fermions. Let us call them

ϕ^r and ψ^a. Notice that this representation breaks parity in three dimensions. Once we combine this with a second representation of the second $SU(2|2)$ factor and the central extensions we obtain the eight transverse bosons and fermions. We then have excitations $\phi^r \tilde{\phi}^{r'}$ which are the bosons in the four transverse directions in the sphere. They have zero spin, which translates into the fact that they have zero J charge. This is as expected for insertions of impurities of the type X^i, $i = 1, 2, 3, 4$ corresponding to the SO(6) scalars which have zero charge under J. We can also form the states $\psi^a \tilde{\psi}^{a'}$. These have spin one from the 2+1 dimensional point of view. These states are related to impurities of the form $\partial_i Z$, which have $J = 1$. States with a boson and a fermion, such as $\phi^r \tilde{\psi}^{a'}$ or $\psi^a \tilde{\phi}^{r'}$ correspond to fermionic impurities, which have $J = \frac{1}{2}$. Notice that the spectrum is not parity invariant, we lack particles with negative spin. This is expected since parity in the x_1, x_2 plane of the coordinates in [61] is not a symmetry.

Chapter 3

Magnon Interactions: Poles in the S-Matrix

Here we investigate the analytic structure of the magnon S-matrix in the spin-chain description of planar $\mathcal{N} = 4$ SUSY Yang-Mills/$AdS_5 \times S^5$ strings. Semiclassical analysis suggests that the exact S-matrix must have a large family of poles near the real axis in momentum space. We show that these are double poles corresponding to the exchange of pairs of BPS magnons. Their locations in the complex plane are uniquely fixed by the known dispersion relation for the BPS particles. The locations precisely agree with the exact proposal in [24, 25]. These poles do not signal the presence of new bound states.

The work in this chapter is contained in [83]. This article was coauthored with Nick Dorey and Juan Maldacena.

3.1 Preliminaries

In 't Hooft's large N limit, a gauge theory is reduced to the sum of planar diagrams. These diagrams give rise to a two dimensional effective theory which is supposed to be the worldsheet of a string. A great deal of effort has been devoted recently to studying planar $\mathcal{N} = 4$ super Yang Mills, culminating in a conjecture for the exact

$\lambda = g^2 N$ dependence of some quantities [24, 25] (see also [84]). This has been done using the assumption of exact integrability plus other reasonable assumptions. The quantity that has been conjectured is the exact S matrix describing the scattering of worldsheet excitations. Let us briefly describe how this quantity is defined in the gauge theory [45]. In $\mathcal{N} = 4$ super Yang mills we have an $SO(6)$ R-symmetry. We pick an $SO(2)$ subgroup generated by $J = J_{56}$. We denote by Z the scalar field that carries charge one under J. We then consider singe trace local operators with very large charge, $J \to \infty$, and conformal dimension, Δ, close to J, so that $\Delta - J$ is finite in the limit. In this limit the local operator contains a large number of fields Z and a finite number of other fundamental fields. These Z fields form a sort of one dimensional lattice on which the other fields propagate. It turns out that we can describe all the other fields in terms of a set of 8 bosonic and 8 fermionic fundamental fields [66] . The 8 bosonic ones are four of the scalar fields that are not charged under J and the four derivatives of Z, $\partial_\mu Z$. We can view the different fields at each site as a generalized spin. Therefore, we are dealing with a generalized spin chain. For this reason the fundamental excitations are often called "magnons". The symmetry algebra acting on an infinite chain is enhanced in such a way that the fundamental excitations (or magnons) are BPS for all values of their momenta [23].

This symmetry also completely constrains the matrix structure of the $2 \to 2$ scattering amplitude [23]. Only the overall phase is undetermined. The phase is constrained by a crossing symmetry equation [80, 82]. Recently, an expression for this phase, sometimes called "dressing factor" [37], was proposed in [24,25]. The proposed phase is a non-trivial function of the 't Hooft coupling λ and is supposed to be valid for all values of this parameter. The function also depends on the two momenta of the particles p_1, p_2 and can be analytically continued to complex values of these variables. As we do so, we encounter poles and branch points. In this chapter we explore the physical meaning of some poles that appear when we perform this analytic continuation. For S matrices in any dimension, simple poles are typically associated to on shell intermediate states. In fact, such simple poles appear in some of the matrix elements of the full matrix S and are independent of the dressing factor. They were interpreted as BPS bound states for single magnon excitations in [85]. It turns out that one can have BPS bound states of n fundamental magnons for any positive integer n. Further poles were necessary in order to account for branch cuts in the scattering of magnons in the classical limit considered in [38]. In the proposal of [24, 25] these turn out to be double poles. Such double poles do not arise from bound states and look a bit puzzling at first sight. However, similar double poles appear in the S matrix for sine Gordon theory and their physical origin was elucidated by Coleman and Thun [86] (see [87] for a recent review on two dimensional S-matrices). In short, they arise from physical processes where the elementary particles exchange pairs of particles rather than a single particle. In higher dimensions such processes give rise to branch cuts, but they give double poles in two dimensions. The precise position of these double poles depends on the dispersion relation for the exchanged particles. If we assume that the exchanged particles are the BPS magnon bound states discussed above we find that the double poles appear precisely where the conjecture of [24, 25] predicts them. Thus, this computation can be viewed a a check of their proposal, since it has singularities where (with hindsight) one would have expected them.

More generally, the existence of the singularities dictated by the spectrum (and no others in the physical region) provides extra constraints on the S-matrix beyond those of unitarity, crossing and factorizability. As in relativistic models [88], these constraints help to remove ambiguities associated with the homogeneous solutions of the crossing equations. Optimistically, one can hope that this type of reasoning might provide a physical basis for selecting the conjectured S-matrix of [25] from the many possible solutions to the crossing equation.

In [38] some non-BPS localized classical solutions were found and it was suggested that they could correspond to non-BPS bound states. If true this would mean that the S-matrix should have single poles rather than double poles. These solutions move in an $R \times S^2$ subspace of the full $AdS_5 \times S^5$ theory and within this subspace they are stable. Here we show that these solutions can decay once they are embedded in $AdS_5 \times S^5$. In fact these solutions can be viewed as a non-BPS superposition of two coincident BPS magnons with large n and the same momentum. The decay of the solutions corresponds to these two BPS magnons moving away from each other. Our results suggest that the only single-particle asymptotic states of the theory are the tower of BPS boundstates described in [85, 89–91].

3.2 S-matrix Singularities

Since the S-matrix is a physical observable, its singularities should have a physical explanation. In fact, S-matrix singularities arise when we produce on shell particles that propagate over long distances or long times. Thus the singularities are interpreted as an IR phenomenon associated to the propagation of particles.

Let us consider a $2 \to 2$ scattering process. The simplest and most familiar example is a single pole. These poles arise when an intermediate on shell particle is produced in the collision. See Figure 3.1(a).

If we have more than one particle becoming on shell we can have other types of singularities. The simplest example is a two particle threshold, where we can start producing a pair of intermediate on-shell particles, see Figure 3.1(b). For our purposes we will need to consider more complicated cases. This problem was studied

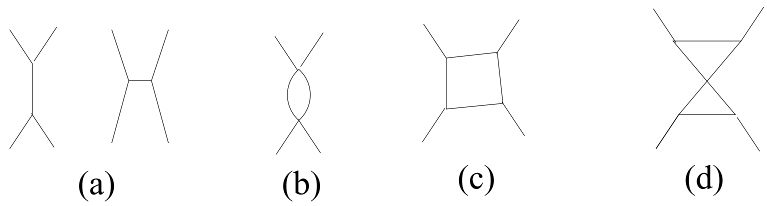


Figure 3.1: Diagrams associated to singularities of the S-matrix.

by Landau [92] who found the general rules for locating the singularities. The singularities correspond to spacetime graphs where the vertices are points in spacetime, representing local interaction regions. Lines are on shell particles with particular momenta p^μ and the spacetime momenta are conserved at the vertices. In addition, the four momentum of the line connecting two vertices obeys $x_2^\mu - x_1^\mu = \alpha p^\mu$, with $\alpha > 0$ (there is one α for each internal line). Thus, the singularity is associated to the physical propagation of these on shell particles. Notice that this condition implies that if the energy p^0 is positive, then $x_2^0 > x_1^0$, so that the particle is going forwards in time. More details can be found in [93, 94]. The type of singularity that we obtain depends on the dimension, since we generally have integrals over loop momenta. We typically encounter branch points[1]. For the box diagram in Figure 3.1(c) we have a D dimensional loop integral and one mass shell delta function for each line. We expect a divergence in $D < 4$ dimensions and branch points for $D \geq 4$. In two dimensions, $D = 2$, we naively get the square of a delta function at zero. A more careful analysis reveals that we get a double pole [86, 94]. Other graphs, such as the two loop diagram in Figure (3.1)(d), also give rise to double poles. In two dimensions we get double poles in diagrams with L loops and N internal lines when $N - 2L = 2$.

So far, we were assuming that the singularities arise for real values of the external momenta. On the other hand, it is often the case that the singularities only arise when we perform an analytic continuation in the external momenta. The spacetime

[1]The discontinuity across the branch cut emanating from it can be evaluated using the Cutkosky rules [93].

points discussed above now live in a complexified spacetime. If we are interested in the "first" singularity that appears as we move away from the physical sheet, it is sufficient to consider momenta which are in "anti-Euclidean" space where p^0 is real and p^i are purely imaginary [94]. As we go through branch cuts we can encounter additional singularities which relax some of the rules discussed above. In particular, we can relax the condition that the α's for each line are positive. Here we will be interested in the first singularity that arises and not on the ones that are on other sheets. In two dimensions, the energy and momentum conservation conditions and the on shell conditions for all internal lines determine the energies and momenta up to some discrete options. At any of these values we will have a singularity in some sheet. If we are interested in the "first" or physical sheet singularities then we need to impose a the further condition that the α_i discussed a above are positive, or equivalently, that the graph in "anti-Euclidean" space closes. This selects a subset of all the discrete solutions to the energy and momentum conservation conditions.

In our case, we do not have a relativistic theory, so we do not know a priory how much we can analytically continue the amplitudes and expect to find a physical explanation for all singularities. However, since the origin of the singularities is an IR phenomenon we expect that the discussion should also hold for spin chains. Understanding precisely the whole physical region is beyond the scope of this study. Here we will perform the analytic continuation only within a very small neighborhood of the real values looking for the "first" singularities. As we explained above, after we impose the momentum and energy conservation conditions we will be left with discrete possible solutions. We should select among them to find the singularities that are on the physical sheet. We are not going to give a general rule, as was given for relativistic theories. Instead we are going to analyze mainly two cases in this chapter. In the first case we will be scattering particles in the near plane wave region where particles are approximately relativistic and we can apply the relativistic rules in order to check whether the singularity is on the first sheet or not. The second case is when the particles we scatter are in the giant magnon region. In this case we can use a slow motion, non relativistic approximation plus some physical arguments regarding the interpretation of the solutions in order to select among different solutions.

3.3 Poles from physical processes

As discussed above, the singularities of the S-matrix correspond to different on-shell intermediate states. In particular, the location and nature of the singularities are essentially determined once the spectrum of the theory is known. To understand the poles of the magnon S-matrix we therefore begin by reviewing the spectrum of the $\mathcal{N}=4$ spin chain.

The fundamental excitations of the spin chain are the magnons themselves, which lie in a sixteen-dimensional BPS representation of the unbroken $SU(2|2)\times SU(2|2)$ supersymmetry. The closure of the SUSY algebra on this multiplet uniquely determines [23] the magnon dispersion relation to be [81] (see also [53, 54]),

$$E = \Delta - J = \sqrt{1 + 16g^2 \sin^2\left(\frac{p}{2}\right)} \tag{3.3.1}$$

where $g = \sqrt{g_{YM}^2 N}/4\pi$. In addition, any number of elementary magnons can form a stable boundstate. The n-magnon boundstate also lies in a BPS representation of supersymmetry (of dimension $16n^2$). The theory therefore contains an infinite tower of BPS states labelled by a positive integer n. The exact dispersion relation for these states is again fixed by supersymmetry to be [85, 91],

$$E = \Delta - J = \sqrt{n^2 + 16g^2 \sin^2\left(\frac{p}{2}\right)} \tag{3.3.2}$$

The existence of these states can be confirmed both in the gauge theory spin-chain and in the string world-sheet theory for appropriate values of the coupling [85]. It remains possible that there are additional boundstates in the theory which are not BPS. However we will see that we have no reason to introduce them, at least if one believes in the recently conjectured S matrix in [24, 25]. In fact, the poles in this proposed result can be accounted for by thinking about physical processes involving the set of BPS states described above.

Kinematics

The dispersion relation (3.3.2) for an arbitrary BPS state is conveniently written in terms of complex spectral parameters,

$$X^{\pm} \quad = \quad e^{\pm ip/2} \, \frac{\left(n + \sqrt{n^2 + 16g^2 \sin^2\left(\frac{p}{2}\right)}\right)}{4g \sin\left(\frac{p}{2}\right)} \tag{3.3.3}$$

which obey the constraint,

$$\left(X^+ + \frac{1}{X^+}\right) - \left(X^- + \frac{1}{X^-}\right) \quad = \quad \frac{in}{g} \tag{3.3.4}$$

In terms of these parameters, the particle energy and momentum are given by,

$$E(X^{\pm}) = \frac{g}{i}\left[\left(X^+ - \frac{1}{X^+}\right) - \left(X^- - \frac{1}{X^-}\right)\right], \qquad p(X^{\pm}) = \frac{1}{i}\log\left(\frac{X^+}{X^-}\right) \tag{3.3.5}$$

These quantities are real provided $X^+ = (X^-)^*$. More generally we will consider an analytic continuation of the kinematic variables where the reality condition is relaxed (but the constraint (3.3.4) is maintained).

For magnons belonging to an $SU(2)$ subsector, the integer n appearing in the constraint (3.3.4) corresponds to a conserved $U(1)$ R-charge. It is convenient to view

(3.3.4) as giving the charge as a function of the spectral parameters $n(X^{\pm})$. The velocity of the particle in appropriately-normalised[2] worldsheet coordinates (x, t) is given as,

$$v = \frac{dx}{dt} = \frac{1}{2g}\frac{dE}{dp} = \frac{X^+ + X^-}{1 + X^+X^-} = \frac{2g\sin(p)}{\sqrt{n^2 + 16g^2\sin^2\left(\frac{p}{2}\right)}} \tag{3.3.6}$$

and we also define the rapidity parameter,

$$\begin{aligned} u(X^{\pm}) &= \frac{1}{2}\left[\left(X^+ + \frac{1}{X^+}\right) + \left(X^- + \frac{1}{X^-}\right)\right] \\ &= \frac{1}{2g}\cot\left(\frac{p}{2}\right)\sqrt{n^2 + 16g^2\sin^2\left(\frac{p}{2}\right)} \end{aligned} \tag{3.3.7}$$

In the strong coupling limit $g \gg 1$ there are two distinct kinematic regimes which will be of interest. The first is the Giant Magnon regime where the conserved momentum p is held fixed as $g \to \infty$. We then have

$$E \simeq 4g\sin\left(\frac{p}{2}\right) \ , \qquad X^+ \sim \frac{1}{X^-} \sim e^{ip/2} \ , \qquad u \sim 2v \sim 2\cos\frac{p}{2} \tag{3.3.8}$$

These excitations correspond to classical solutions of the worldsheet theory.

The second regime of interest is that of the plane wave limit where we take $p \to 0$ as $g \to \infty$ with $k = 2gp$ held fixed. In this case we recover the familiar relativistic relations

$$E \simeq \sqrt{n^2 + k^2} \ , \quad v = \frac{k}{\sqrt{n^2 + k^2}} \ , \qquad X^+ \sim X^- = \frac{n + E}{k} \sim \frac{n + \sqrt{n^2 + k^2}}{k} \tag{3.3.9}$$

These two regimes[3], (3.3.8) and (3.3.9), amount to two ways of solving the constraint (3.3.4) for large g, by setting $X^+ \sim 1/X^-$ or $X^+ \sim X^-$.

[2]The normalisation is chosen so that the velocity of light is unity in the plane wave limit discussed below. With this normalization p generates translations in $x/(2g)$.

[3]The two regimes in question are connected by a third, studied in [95, 96] where $X^{\pm} \sim 1$ (or $X^+ \sim X^- \sim -1$).

S-matrix definitions and conventions

For simplicity, we will consider scattering of two states in the $SU(2)$ subsector, since the scattering in the other sectors is determined once we know the result in the $SU(2)$ subsector [23]. Of course, we will allow intermediate states to be completely general. We can define the S matrix (just a complex number in this case) by looking at the wavefunction for two magnons and writing it as

$$\Psi(x_1, x_2) = e^{ip_1 x_1 + p_2 x_2} + S(p_1, p_2) e^{ip_1 x_2 + ip_2 x_1} \ , \qquad x_1 \ll x_2 \tag{3.3.10}$$

And $\Psi(x_1, x_2) = \Psi(x_2, x_1)$ for $x_1 \gg x_2$, since we have two identical bosons. We see that we can view the first term as the incoming wave and the second as the reflected way if $v_1 > v_2$. We will be interested in analytically continuing in the external momenta p_i. We write then

$$p_1 = p + iq \ , \qquad p_2 = p - iq \tag{3.3.11}$$

We will see that the part of the wavefunction depending on the relative coordinate $x = x_1 - x_2$ has the form $\Psi \sim e^{-qx} + S(1,2) e^{qx}$. Thus we see that the first term diverges as $x \to -\infty$ if $q > 0$. We set boundary conditions on the non-normalizable piece of the wavefunction, by saying that the coefficient of the exponential is one, and we "measure" the coefficient of the other which is the S matrix. We then see that a pole of S (with $q > 0$) can correspond to a bound state.

This S matrix can be written as

$$S\left(X_1^\pm, X_2^\pm\right) = \sigma^{-2}\left(X_1^\pm, X_2^\pm\right) \mathcal{S}_{BDS}^{-1}\left(X_1^\pm, X_2^\pm\right) \tag{3.3.12}$$

where the inverse factors originate from the fact that the conventions for defining the

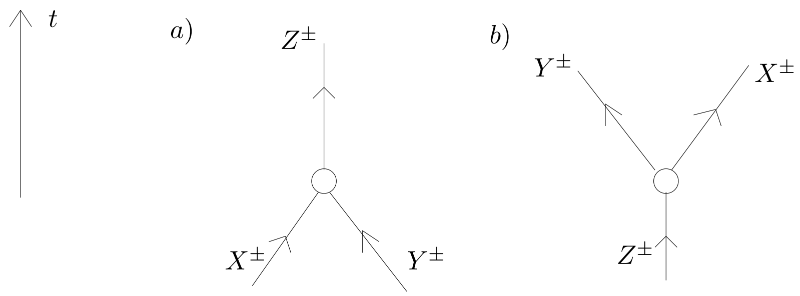


Figure 3.2: Three-point vertices.

S matrix in the recent literature are the opposite from ours [4]. The second factor in (3.3.12) was originally written in [81] and is

$$\mathcal{S}_{BDS}^{-1}\left(X_1^\pm, X_2^\pm\right) = \frac{u(X_1^\pm) - u(X_2^\pm) - i/g}{u(X_1^\pm) - u(X_2^\pm) + i/g} = \frac{(X_1^- - X_2^+)(1 - 1/X_1^- X_2^+)}{(X_1^+ - X_2^-)(1 - 1/X_1^+ X_2^-)} \quad (3.3.13)$$

The quantity $\sigma(X_1^\pm, X_2^\pm)$ is known as the dressing factor and we will discuss the explicit proposal in [24, 25] in Section 5 below. For the moment we will simply use the that fact that the proposed dressing factor has no poles or zeros when $u_1 - u_2 = \pm i$ on the main branch that is closest to physical values.

3.3.1 Simple poles

The singularities of the S-matrix correspond to spacetime diagrams where each particle is on-shell. In general, we will analytically continue to complex values of the momenta and energies, and thus each particle corresponds to a line in complexified Minkowski space. Particle lines can meet at three-point vertices which conserve charge, energy and momentum. We will now analyze the allowed vertices.

We begin by considering a three point vertex corresponding to the creation of a BPS particle with spectral parameters $Z^\pm$ from two others with parameters $X^\pm$ and $Y^\pm$ as shown in Figure 3.2a. If all particles belong to the same $SU(2)$ subsector, then

[4] Our σ is the same as the σ in [24, 25] and similarly, our S_{BDS} is the same S-matrix appearing in [81].

each carries a conserved $U(1)_R$ charge $Q = n$. In this case, the conservation laws for energy, momentum and charge read,

$$\begin{aligned} E(X^\pm) + E(Y^\pm) &= E(Z^\pm) \\ p(X^\pm) + p(Y^\pm) &= p(Z^\pm) \quad \mathrm{mod}(2\pi) \\ n(X^\pm) + n(Y^\pm) &= n(Z^\pm) \end{aligned} \tag{3.3.14}$$

The same equations must also hold for the vertex corresponding to the time-reversed process shown in Figure 3.2b. As the dependence of each conserved quantity on the spectral parameters is of the form $f(X^+) - f(X^-)$ for some function f, it is straightforward to solve the conservation equations. There are two inequivalent solutions,

$$\alpha : \quad \begin{aligned} X^+ &= Z^+ \\ X^- &= Y^+ \\ Y^- &= Z^- \end{aligned} \qquad\qquad \beta : \quad \begin{aligned} X^+ &= Y^- \\ X^- &= Z^- \\ Y^+ &= Z^+ \end{aligned} \tag{3.3.15}$$

The two solutions are related by the interchange of $X^\pm$ and $Y^\pm$.

Combining vertices of the type described above, we obtain the diagram shown in Figure 3.3a corresponding to the scattering of two BPS particles with spectral parameters $X_1^\pm$, $X_2^\pm$. If both particles belong to an $SU(2)$ subsector they carry positive conserved charges $Q_1 = n(X_1^\pm)$ and $Q_2 = n(X_2^\pm)$. The diagram corresponds to the formation of a BPS boundstate of charge Q_1+Q_2 in the s-channel. These charge assignments are shown in Figure 3.3b. For individual vertices the conservation laws can be implemented using either solution α or β described above. However, once the vertices are connected by an internal line, consistency requires the choice of the same solution at both vertices. If we choose the solution β at both vertices, the spectral parameters of the internal line are fixed to be $Z^+ = X_2^+$, $Z^- = X_1^-$ and the diagram

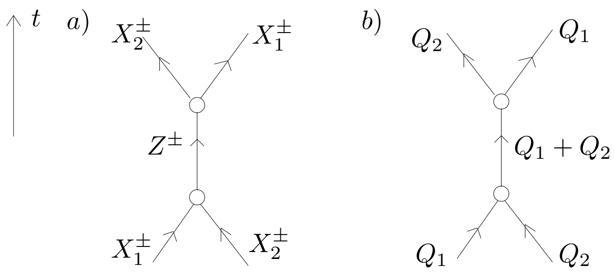


Figure 3.3: Formation of a bound state in the s-channel.

leads to an S-matrix pole when the parameters of the initial (and final) particles obey $X_1^+ = X_2^-$. This pole is present in (3.3.13). In addition, parametrizing the momenta as in (3.3.11) we find that $q > 0$ at this pole. This can be seen most easily by noting that $u_1 - u_2 = -i/g$ and that $\partial_p u(p) < 0$ for real momenta. The other possiblity is to choose the solution α at each vertex which leads to the relation $X_1^- = X_2^+$. This possibility implies that $q < 0$ and would not lead to bound states. The S matrix may or may not have a pole at this position, but we should not interpret it as a bound state. For example, in the $SL(2)$ subsector there is a pole at this position, but we do not have boundstates associated with them.

The three-point vertices shown in Figure 3.2, are the only possible ones if all three particles belong to the same $SU(2)$ sector. In any $SU(2)$ sector, there is a conserved $U(1)$ charge Q such that BPS particles of type n carry positive charge $Q = n$. However, the theory also contains BPS particles of negative charge $Q = -n$ with respect to the same $U(1)$. The particles of negative charge correspond to a different $SU(2)$ sector. For each value of n, the two particles belong to the same $SU(2|2)^2$ multiplet, but have opposite charges under the Cartan generator corresponding to the $U(1)$ in question. The crossing symmetry of the S-matrix suggests that we should also admit vertices for interactions between BPS particles of positive and negative charge. For a BPS particle with spectral parameters $X^\pm$ the crossing transformation is,

$$X^+ \to \tilde{X}^+ = 1/X^+ \qquad X^- \to \tilde{X}^- = 1/X^- \tag{3.3.16}$$

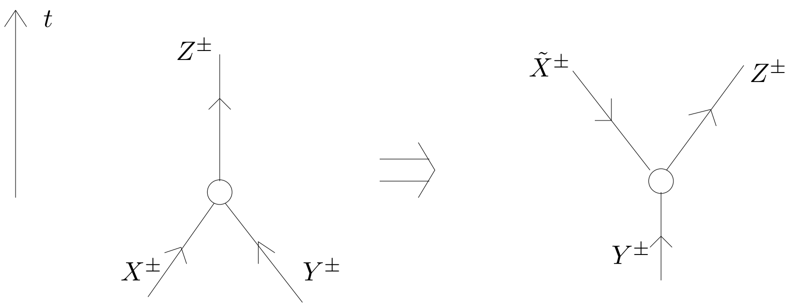


Figure 3.4: Crossing transformation of vertex.

This transformation changes the sign of the energy and momentum, $E(\tilde{X}^\pm) = -E(X^\pm)$, $p(\tilde{X}^\pm) = -p(X^\pm)$ but preserves the form of the constraint (3.3.4) with $n(\tilde{X}^\pm) = n(X^\pm)$. If we apply this transformation to a vertex with an *incoming* state of spectral parameters $X^\pm$ and positive charge $Q = n(X^\pm)$, we obtain a new vertex with an *outgoing* particle of spectral parameters $\tilde{X}^\pm$. For the new vertex to conserve charge, we must interpret the outgoing particle as one of negative charge $\tilde{Q} = -n(\tilde{X}^\pm) = -Q$. We will indicate a particle of negative charge by reversing direction of the arrow appearing on its world line. To illustrate the crossing transformation we will apply it to an incoming leg on the vertex shown in Figure 2a to obtain a new vertex as shown in Figure 3.4. The solutions α and β of the conservation laws for the original vertex yield two solutions for the new vertex,

$$\tilde{\alpha} : \quad \begin{aligned} 1/\tilde{X}^+ &= Z^+ \\ 1/\tilde{X}^- &= Y^+ \\ Y^- &= Z^- \end{aligned} \qquad\qquad \tilde{\beta} : \quad \begin{aligned} 1/\tilde{X}^+ &= Y^- \\ 1/\tilde{X}^- &= Z^- \\ Y^+ &= Z^+ \end{aligned} \tag{3.3.17}$$

A feature of these equations that is worth noting is the following. If particle $Y^\pm$ is in the giant magnon regime, then one of the two other particles has to also be in the giant magnon regime while the last one is in the plane wave regime.

We can now discuss the pole at $X_1^+ = 1/X_2^-$. This arises naturally if we take the s-channel diagram and we cross one of the two particles. We then have a diagram

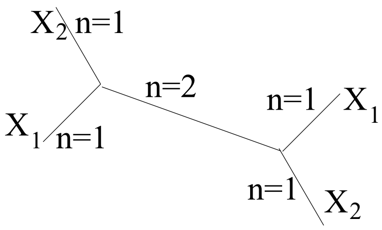


Figure 3.5: t-channel contribution.

that looks like the one in Figure 3.5 by using (3.3.17). Note that the intermediate particle has $U(1)$ charge zero, but this is not a problem since there are such particles in the full $SU(2|2)^2$ multiplet corresponding to a BPS magnon with $n = 2$. When the external particles are in the near plane wave regime, it is clear that this is an allowed process since the theory is nearly relativistic.

If the external particles are both in the giant magnon region, then, as we mentioned after (3.3.17), the intermediate particle is in the plane wave regime. The two giant magnons behave as two heavy particles that are moving at slow relative velocities which are interacting through a potential generated by the exchange of the lighter particle. Let us now remind the reader how these poles arise when we think about the non-relativistic approximation for the two heavy particles. We will have a lagrangian of the form $L \sim M\dot{x}^2 + e^{mx}$ where $mx \ll 0$. When we try to solve the problem in the Born approximation we are lead to

$$\mathcal{A} \sim \langle \Psi_{out}|V|\Psi_{in}\rangle = \int dx e^{-qx} e^{mx} e^{-qx} \tag{3.3.18}$$

which gives a divergence when $q = m/2$. Note that this divergence arises from long distance effects in the quantum mechanics problem and does not depend on the details of the potential at short distances. In addition, this feature is independent of the sign of M. Indeed, when we expand the giant magnon dispersion relation we will find that M is negative. The intermediate particle of mass m ($m > 0$) is carrying momentum

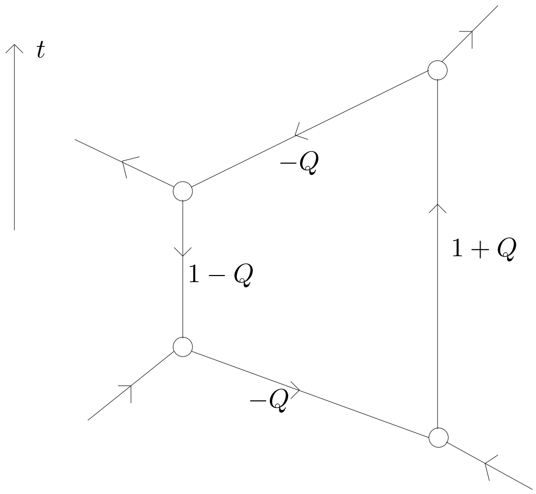


Figure 3.6: The box diagram. $Q > 1$.

$k \sim -im$. In principle, one could have considered a situation where this particle carries momentum $k \sim +im$, but it would have given rise to an unphysical growing potential. We will use similar criteria in more complicated situations below in order to isolate the physical singularities from the unphysical ones. As we mentioned in Section 2, we can have singularities in unphysical sheets from such solutions.

3.3.2 Double poles

Following the discussion in Section 2, we look for diagrams that can give rise to double poles. Let us start by considering the one loop box diagram shown in Figure 3.6, where the external legs are all elementary magnons in the $SU(2)$ sector and carry charge 1. The box diagram[5] represents an s-channel process where the intermediate states consist of two BPS particles of charges $1+Q$ and $1-Q$ where $Q > 1$ is a positive integer. More generally the states going around the loop can correspond to other members of the corresponding BPS boundstate magnon multiplet which is labelled by the positive integer n. What will be important for us is the value of n for

[5]This diagram is different from those considered by Coleman and Thun in the relativistic sine-Gordon case. In that case this diagram can also be considered and give singularities that coincide with the ones found in [86] using different diagrams.

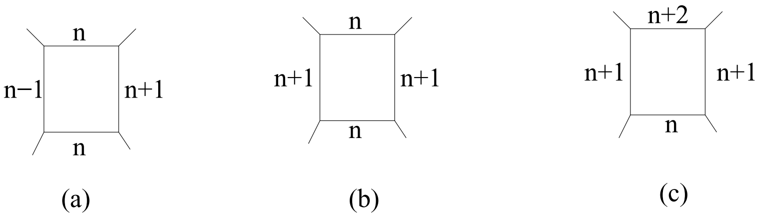


Figure 3.7: Several choices for the $SU(2|2)^2$ representations appearing in the internal lines. In addition we can change the ± 1 to ∓ 1 to find new possibilities.

each of these magnons since it is the parameter appearing in the dispersion relation of the state. When we assign values of n for each of the internal lines we should remember that, since the external lines have $n = 1$, group theory says that at each vertex the values of n should differ by plus or minus one. In figure 3.7 we have written several choices. We will first concentrate on the one in figure 3.7a, which is the one suggested by figure 3.6. The choice in 3.7c gives the same condition as the one we will consider explicitly. The choices in 3.7b lead to inconsistent equations once we impose the condition that the pole is on the physical sheet.

To verify this is an allowed process there are two steps,

1 Implement energy and momentum conservation at each vertex.

2 Impose additional conditions to make sure that the singularity arises on the physical branch.

We will find that after the first step, we will have fixed all the momenta of the intermediate lines, up to some discrete choices. The second condition will rule out some of them. In particular, we can perform approximations when we are evaluating the second condition, since the exact position of the poles is already fixed after the first step.

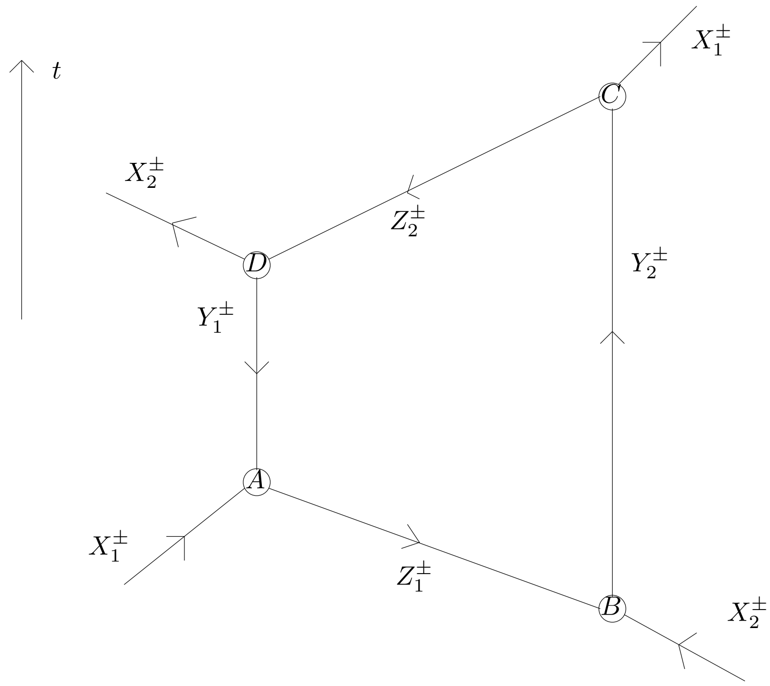


Figure 3.8: Spectral parameters and vertices for the box diagram.

To perform the first step we assign spectral parameters to each line in the diagram as shown in Figure 3.8. Momentum and energy conservation are implemented by the vertex rules described above together with further rules obtained by repeated applications of the crossing transformation. There are two possible choices at each of the four vertices A, B, C and D. For each of these choices we will then have to evaluate the second criterion.

Let us first consider the case where the two particles are in the plane wave regime. There we can use the approximate relativistic formulas and demand that the graphs closes in "anti-Euclidean" space. Each of the vertices involves particles of mass 1 (the external line), Q and $Q-1$ (with $Q>1$). The solution to the energy and momentum conservation condition tells us that the momentum is approximately zero so that the energies are given approximately by 1 and Q and $Q-1$ respectively. Therefore the particles Q and $Q-1$ cannot be both be going into the future at this vertex. It is clear that we cannot obey this condition both at vertices A and B. Thus, none of the graphs gives rise to a singularity in the near plane wave region.

We now consider the case where both incoming particles are giant magnons. As we mentioned above, one of the particles emerging from the vertex has to be giant and the other a plane wave particle. Moreover, the giant particle has energy and momentum similar to the incoming particle. Thus, we expect that it continues to move to the future. Since the particle Z is coming from the past in vertex A or B, we conclude that particle Z must be of the plane wave type. Thus, we see that we have the qualitative conditions $X_1^\pm \sim Y_1^\pm$ and $Z^+ \sim Z^- = X_1^\pm$, where the last sign will depend on the type of solution of the momentum conservation conditions (3.3.15) that we choose. For example, for vertex $\mathbf{A}$ we have the two choices

$$\mathbf{A}: \quad \begin{aligned} X_1^+ &= 1/Y_1^- \\ X_1^- &= 1/Z_1^- \\ Y_1^+ &= Z_1^+ \end{aligned} \qquad\qquad \mathbf{A}': \quad \begin{aligned} X_1^+ &= 1/Z_1^+ \\ X_1^- &= 1/Y_1^+ \\ Y_1^- &= Z_1^- \end{aligned} \tag{3.3.19}$$

Approximately, these to choices amount to saying whether $Z \sim X_1^+$ (for $\mathbf{A}$) or $Z \sim X_1^-$ (for $\mathbf{A}'$). For simplicity, let us analyze this condition around $p = \pi$ (see (3.3.11)). Since this is a maximum in the dispersion relation we can go to "anti-Euclidean space" by setting $p = \pi + iq$. The expansion around this point looks similar to the non-relativistic expansion of a relativistic theory, except for the sign of the mass. Moreover, we will have that $q \ll 1$ and thus we will approximate $X_1^+ \sim i$, $X_2^- \sim -i$. Equating these to Z and parametrizing its momentum as $k = i\kappa$ (see (3.3.9)) we find that[6]

$$\mathbf{A}: \qquad X_1^+ \sim Z_1 \,, \qquad \to i \sim \frac{n \pm \sqrt{n^2 - \kappa^2}}{i\kappa} \,, \qquad \to \kappa \sim -n \tag{3.3.20}$$

$$\mathbf{A}': \qquad X_1^- \sim Z_1 \,, \qquad \to -i \sim \frac{n \pm \sqrt{n^2 - \kappa^2}}{i\kappa} \,, \qquad \to \kappa \sim +n \tag{3.3.21}$$

[6]The choice of sign in the square root corresponds to the sign for the energy.

As we mentioned above in our discussion of t channel contributions, we can interpret the exchange of the Z particle as giving rise to a potential which will go like $e^{-\kappa x}$. In order to get a sensible potential for $x \ll 0$ we must demand that $\kappa < 0$. This selects the first condition, **A**.

This process can be repeated at each vertex and thus we select only one particular combination which is,

$$\mathbf{A}: \quad \begin{array}{c} X_1^- = 1/Z_1^- \\ Z_1^+ = Y_1^+ \\ Y_1^- = 1/X_1^+ \end{array} \qquad \mathbf{B}: \quad \begin{array}{c} X_2^- = 1/Z_1^+ \\ X_2^+ = Y_2^+ \\ Y_2^- = 1/Z_1^- \end{array} \tag{3.3.22}$$

$$\mathbf{C}: \quad \begin{array}{c} X_1^+ = 1/Z_2^- \\ 1/Z_2^+ = Y_2^+ \\ Y_2^- = X_1^- \end{array} \qquad \mathbf{D}: \quad \begin{array}{c} Z_2^+ = 1/X_2^+ \\ Z_2^- = Y_1^- \\ 1/X_2^- = Y_1^+ \end{array} \tag{3.3.23}$$

or, more simply,

$$\begin{array}{cc} X_1^+ = \dfrac{1}{Y_1^-} = \dfrac{1}{Z_2^-} & \qquad X_1^- = Y_2^- = \dfrac{1}{Z_1^-} \\ X_2^+ = Y_2^+ = \dfrac{1}{Z_2^+} & \qquad X_2^- = \dfrac{1}{Y_1^+} = \dfrac{1}{Z_1^+} \end{array} \tag{3.3.24}$$

From these relations as well as the constraints (3.3.4) on each set of spectral parameters we derive a condition on the rapidities of the two incoming particles,

$$u(X_1^\pm) - u(X_2^\pm) = -\frac{i}{g}\, n \ , \qquad n > 1 \tag{3.3.25}$$

for each integer $n > 1$. In particular, the graph shown in Figure 3.6 gives rise to the double pole with $n = Q$. In Section 5 we will find that these are indeed the positions of the poles of the dressing factor σ^{-2}. Note that the other ways of solving the energy and momentum conservation conditions at each vertex, which we have ruled out by the arguments given above, would have led to equations similar to (3.3.25) but the integer in the right hand side would have had a different range.

We should note that when we write the condition of the pole as (3.3.25) we are losing some information since our analysis leading to (3.3.25) used that we were in the giant magnon region. Thus the condition (3.3.25) only applies in the giant magnon region where the real part, r_i of u_i, satisfies $|r_i| < 2$. On the other hand, we have already seen that in the plane wave region where $|r_i| > 2$ there are no poles. Indeed, we will find that this is a feature of the proposed S-matrix in [24, 25].

Other diagrams

In this section we will analyze other possible on-shell diagrams which might contribute. In particular we will consider diagrams of the type originally studied by Coleman and Thun [86] in the context of the sine-Gordon model. Since we have already accounted for all the poles in the S matrix, we expect that these diagrams give the same poles that we have found before. There are distinct graphs corresponding to s- and t-channel processes. A candidate t-channel diagram is shown in Figure 3.9. Relative to the process considered in the previous section, the new feature is that the internal legs cross in the center of the diagram and we must include the appropriate S-matrix element in our evaluation of the diagram. In the cases considered in [86], the central scattering process takes place at generic values of the momenta for which the corresponding S-matrix element is finite (and non-zero). In these cases, the extra factor is a harmless phase which does not affect the analysis of singularities. This needs to be reconsidered in the present case.

To verify the consistency of this diagram we will proceed exactly as in the previous subsection. First we assign spectral parameters to each internal line as shown in Figure 3.10. Conservation laws at each vertex are solved as follows,

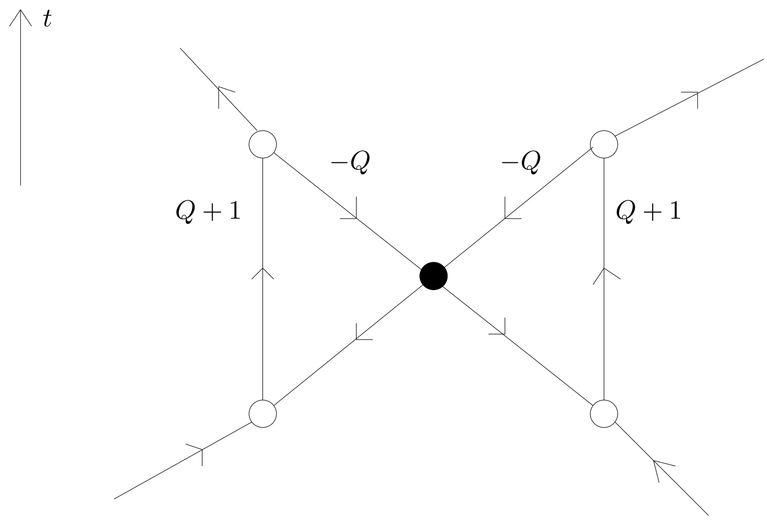


Figure 3.9: The spacetime graph for t-channel process.

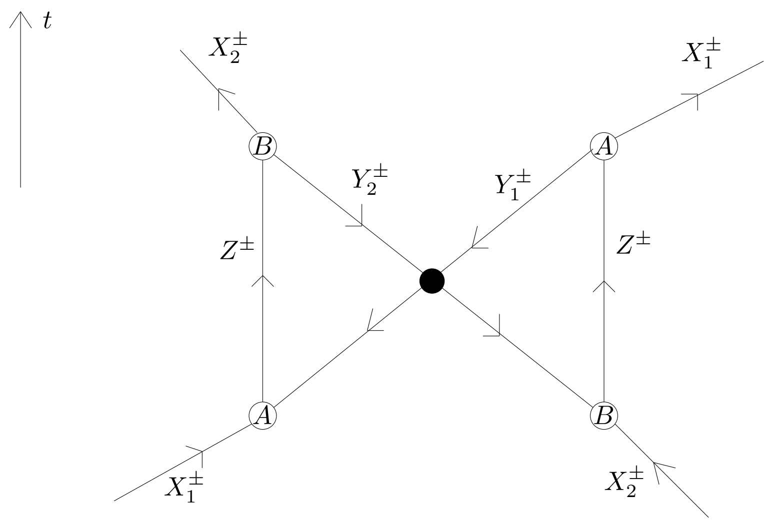


Figure 3.10: Spectral parameters and vertices for the t-channel diagram.

$$
\mathbf{A}: \quad \begin{array}{c} X_1^+ = 1/Y_1^- \\ X_1^- = Z^- \\ 1/Y_1^+ = Z^+ \end{array} \qquad \mathbf{B}: \begin{array}{c} X_2^+ = Z^+ \\ X_2^- = 1/Y_2^+ \\ 1/Y_2^- = Z^- \end{array} \tag{3.3.26}
$$

For the same reasons we discussed above this diagram does not lead to singularities if the external particles are in the plane wave region. Therefore, let us assume that the external particles are in the giant magnon region. As before there are other possible choices for the vertices but only this one is allowed, once we use the condition that the Y lines are in the plane wave region and that they should lead to reasonable potentials in the approximate quantum mechanics problem describing the slow motion of the particles. The relations (3.3.26) lead to the following condition on the rapidities of the incoming particles,

$$
u_1 - u_2 = -\frac{i}{g} n \ , \qquad n > 0 \tag{3.3.27}
$$

where $n = Q$ is a positive integer. Unlike the process considered in the previous subsection the case $n = Q = 1$ is allowed.

Naively the above analysis predicts double poles at the positions given in equation (3.3.27). These coincide with the double poles we found earlier from the diagram in Figure 3.6 except for the case $Q = 1$ which appears to predict a new double pole at $X_1^+ = 1/X_2^-$. However, we still have to consider the contribution of the "blob" in the centre of the diagram shown in Figures 3.10 and 3.9. In the case $Q = 1$ this corresponds to the scattering of two anti-magnons with spectral parameters $Y_1^\pm$ and $Y_2^\pm$. From the vertex conditions (3.3.26) we deduce that these parameters obey the relation $Y_1^- = 1/Y_2^+$. This is clearly a special value of the momentum. If the magnon S matrix has a pole of order D when $X_1^+ = 1/X_2^-$ then, by unitarity, the S-matrix for two anti-magnons (which is the same as that for two magnons by crossing on both legs) will have a zero of degree D when the spectral parameters obey $Y_1^- = 1/Y_2^+$. Instead of simply predicting a double pole via the formula[7]

$$D = N - 2L = 6 - 4 = 2, \tag{3.3.28}$$

the Coleman-Thun analysis then yields a self-consistency equation for the degree D of the pole,

$$D = N - 2L - D = 2 - D \tag{3.3.29}$$

with solution $D = 1$. This indicates a *simple pole* at $X_1^+ = 1/X_2^-$. Thus this diagram gives also a simple pole. This is in agreement with the simple pole that we found from the t channel. It is also in agreement with the exact S-matrix where such a pole arises from the BDS contribution to the $SU(2)$ sector S-matrix. In fact, similar considerations also hold for higher values of Q where the central scattering corresponds to a zero of the boundstate S-matrix obtained consistently by fusion [90]. In these cases the double zero cancels the double pole and the diagram is finite. Thus, we only obtain the single pole that is given by the t channel diagram.

3.4 Classical, semiclassical and approximate results

3.4.1 Localized classical solutions and the absence of non-BPS bound states

In [38], some localized oscillating solutions were found. There, it was proposed that these solutions could correspond to non-BPS bound states. As we have seen here, the singularities of the S-matrix are not interpreted as new particles, but come from physical processes involving the known BPS particles. In this section we discuss the solutions in [38] and explain why they do not give rise to bound states. What

[7] As above N is the number of internal lines and L is the number of loops in the diagram.

happens is that the "breather" solutions in [38] can be split into two BPS magnons with opposite charges. In order to see this, it is not enough to consider solutions in $R\times S^2$ as it was done in [38], but one should consider solutions in a bigger subspace of $AdS_5\times S^5$. It turns out that it is enough to consider solutions in $R\times S^3$. Solutions in this subspace were constructed by Spradlin and Volovich in [97] (see also [98] and appendix 3.8). Their solutions are functions of the worldsheet coordinates $\sigma^{0,1}$ and of several complex parameters which we will denote[8] as $\lambda_i^{\pm}$. If we are to ensure that we have a proper real solution we generically need[9] that the complex conjugates of the set of parameters $\{\lambda_i^+\}$ matches the set of parameters $\{\lambda_i^-\}$. The simplest solution corresponds to a BPS magnon of some charge. This solution is characterized by two parameters $\lambda^{\pm}$ which are taken to be complex conjugates of each other. These parameters can be identified with the kinematic variables for the BPS magnon as $X^{\pm}=\lambda^{\pm}$ when the conserved $U(1)$ charge Q (denoted J_2 in [97]) is positive. When this charge is negative the correct identification is $X^+=1/\lambda^-$, $X^-=1/\lambda^+$

A second, more complicated solution was also considered in [97]. This solution depends on four complex parameters $\lambda_1^{\pm}$, $\lambda_2^{\pm}$. If we identify these parameters as the parameters of two magnons, $X_i^{\pm}=\lambda_i^{\pm}$, the solution describes the scattering of these two solitons. Also, if we take

$$\lambda_1^+=1/\lambda_1^- \ , \quad \lambda_2^+=1/\lambda_2^- \ , \qquad (\lambda_1^+)^*=\lambda_2^- \ , \qquad (\lambda_2^+)^*=\lambda_1^- \tag{3.4.30}$$

with a general complex value for λ_1^+, we recover the "breather" solution of [38] . Note that the solution depends on just one complex parameter which corresponds to the following two real variables: the momentum p of the breather and the excitation number ν. This solution can be viewed as coming from the analytic continuation

[8] They were called λ_i and $\bar{\lambda}_i$ in [97]

[9] With the exception of the degenerate cases where the solution collapses to a single BPS soliton of charge Q (i.e. $\lambda_1^-=\lambda_2^+$).

of the scattering of two zero charge magnons with rapidities $X_i^{\pm} = \lambda_i^{\pm}$. With this identification $X_i^+ \neq (X_i^-)^*$, so it seems we can't view 1 and 2 as physical particles.

There is, however, a very interesting property of the solutions in [97]. They are symmetric under the exchange of the parameters $\lambda_1^- \leftrightarrow \lambda_2^-$, without exchanging[10] the parameters λ_i^+. Thus, we can take the "breather" solution we just mentioned and identify the rapidities of two magnons, instead, as

$$\begin{aligned} \lambda_1^+ &= X_1^+ \qquad & \lambda_2^+ &= X_2^+ \\ \lambda_1^- &= X_2^- \qquad & \lambda_2^- &= X_1^- \end{aligned}$$

We now see from (3.4.30) that $X_i^+ = (X_i^-)^*$. Therefore, the same solution can be viewed as arising from two magnons with kinematic variables corresponding to real physical variables of two BPS solitons with opposite charges, both moving with the same velocity. In this way we clearly see that the "breather" solution in [38] can be separated into a superposition of two physical particles[11]. Therefore, if we we were to quantize the solution in [38] we would have a variable corresponding to the relative position of these two particles and we would not end up with a bound state.

As a side comment, the breather solutions in [38] for very high excitation number $n = 2k+c$ (where c is a constant associated with the energy localized at the boundaries of the excitation in (3.4.31)) look like excitations of the form

$$\cdots ZZZW\bar{Z}\cdots\bar{Z}\bar{W}ZZZ\cdots \tag{3.4.31}$$

[10] Equivalently, we can exchange only $\lambda_1^+ \leftrightarrow \lambda_2^+$.

[11] The sine-Gordon theory also has localized oscillating solutions which do not correspond to new bound states, but are superpositions of the ordinary solitons and breathers (see [99] for example).

where W and $\bar{W}$ are two impurities and there are k $\bar{Z}$s. This can split into two BPS magnon multiplet with charges $n/2$ and $-n/2$, one made of $n/2$ Ws and the other made of $n/2$ $\bar{W}$s.

In [97], another solution was considered which was called a "bound state" solution. It was constructed by performing an analytic continuation of the scattering solution for charged magnons. This solution also has rapidities related as $\lambda_1^+ = (\lambda_2^-)^*$ and $\lambda_1^- = (\lambda_2^+)^*$. We see that under the exchange of the two λ_i^- we recover the kinematic variables of two physical magnons. Moreover, these two magnons have different velocities which implies that this "bound state" solution is simply an ordinary (non localized) scattering solution. It turns out, then, that all possible physical choices for the parameters $\{\lambda_i^\pm\}$ yield a non-localized scattering solution[12]. Thus, we have concluded that there is no solution in the literature that looks like a (non-BPS) bound state. Therefore, there is no reason, from the classical solutions alone, to think there are more states in the spectrum than the BPS bound states.

This conclusion agrees with our picture from the analysis of poles in the previous section that already suggested that we should not be able to construct new non-BPS bound states, at least from the scattering of elementary magnons.

3.4.2 The semiclassical limit of the quantum theory

The processes considered in Section 3, leading to double poles at positions dictated by (3.3.25), can with be compared directly with the classical analysis. In the semiclassical limit, $g \gg 1$, the incoming particles correspond to Giant Magnons of charge[13] $Q = 1 \sim 0$. The corresponding classical solution has complex parameters $X_i^\pm = \lambda_i^\pm$ where $\lambda_i^\pm$ are related as in (3.4.30). On the other hand the intermediate

[12]With the exception, again, of some degenerate cases given by sets that collapse to the one particle BPS state of charge Q (i.e. $\lambda_2^+ = \lambda_2^-$).

[13]By this we mean that the charge $Q = 1$ carried by these states can be neglected in this limit.

state in the s-channel box diagram corresponds to two solitons of charges $1-Q \sim -Q$ and $Q+1 \sim Q$ with complex parameters,

$$\lambda_1^+ = 1/Y_1^- \qquad \lambda_2^+ = Y_2^+$$
$$\lambda_1^- = 1/Y_1^+ \qquad \lambda_2^- = Y_2^-$$

which obey the reality conditions $\lambda_1^+ = (\lambda_1^-)^*$ and $\lambda_2^+ = (\lambda_2^-)^*$. As the corresponding rapidities $u(Y_1^{\pm})$ and $u(Y_2^{\pm})$ vanish and both momenta are equal to π, both the constituent solitons are at rest. Note also that the particles of charge $-Q$, exchanged between the two massive solitons, have vanishingly small energy in the semiclassical limit.

The equalities (3.3.24) imply that the classical solutions corresponding to the incoming and intermediate states are related by the interchange of parameters,

$$\lambda_1^+ \to \lambda_1^+, \qquad \lambda_2^+ \to \lambda_2^+, \qquad \lambda_1^- \to \lambda_2^-, \qquad \lambda_2^- \to \lambda_1^- \tag{3.4.32}$$

As this interchange is a symmetry of the two-soliton solution, the process corresponds to a *single* classical field configuration. This configuration can be interpreted as consisting of two solitons of opposite charge $\pm Q$ which are both at rest or as consisting of two solitons of zero charge with complex momenta $p_1 = \pi + iq$ and $p_2 = \pi - iq$. As already established, there is a space of such solutions with the sine-Gordon breather as a special case.

3.4.3 Scattering of giant magnons and a non-relativistic limit

If we consider a single magnon classical solution, as in [38], we find that the solution has fermionic and bosonic zero modes. These are also called "collective coordinates". The bosonic zero modes parametrize the space $R \times S^3$ where R denotes the position of the magnon and S^3 parametrizes the orientation of the solution inside S^5. In addition we have fermion zero modes [100]. The quantization of all these zero modes should give the whole tower of BPS magnons. Suppose that we start with a given magnon of momentum $\bar{p}$ and consider other magnons with very similar momentum. We can describe them by looking at the quantum mechanics of the collective coordinates. In order to understand what the Hamiltonian is we can expand the expression of the energy around a reference momentum[14], $p = \bar{p} + \frac{k}{2g}$. Assuming $0 < \bar{p} < 2\pi$

$$\begin{aligned} E &= \sqrt{n^2 + 16g^2 \sin^2 \frac{p}{2}} = 4g \sin \frac{p}{2} + \frac{1}{8g \sin \frac{p}{2}} n^2 \\ &= \bar{E} + vk - \frac{1}{8g} |\sin \frac{\bar{p}}{2}| k^2 + \frac{1}{8g |\sin \frac{\bar{p}}{2}|} n^2 \end{aligned} \tag{3.4.33}$$

with

$$v = \frac{dE}{dk} = \cos \frac{\bar{p}}{2} \qquad \bar{E} = 4g |\sin \frac{\bar{p}}{2}| \tag{3.4.34}$$

As we saw above, diagrams giving rise to double poles involve "heavy" giant magnon particles and "light" plane wave particles. Double poles arise when two heavy particles exchange a pair of light particles. In this regime, where the relative motion of the heavy particles is slow, we expect that we should be able to view the problem as non-relativistic motion of the collective coordinates plus a potential that arises from the exchange of the light particles. Since the dispersion relation for the light particles is approximately relativistic the potential will have the usual Yukawa form for particles of mass m, i.e. $V \sim e^{-mr}$ for $mr \gg 1$. However, we should recall that the two magnons are moving with similar velocities which are large, $v_1 \sim v_2 \sim v$. If we denote by x the distance between the two magnons, we find that the potential has

[14]In this normalization k is related to translations in the coordinate x introduced in (3.3.6). This is the normalization where the speed of light in the plane wave region is one.

the boosted form

$$V \sim \sum_n g f_n(\Omega_1, \Omega_2) e^{-n|x|\gamma} , \qquad \gamma^{-2} = 1 - v^2 \tag{3.4.35}$$

where f_n is a function that depends on the angles on both S^3s and the fermion zero modes. We have also indicated how the potential scales with g. This scaling is determined as follows. We know that the classical solutions are independent of g. Thus, classical scattering properties, such as time delays are independent of g. Since the mass is proportional to g (see (3.4.33)), then, the potential should also be proportional to g such that the classical equations of motion are independent of g.

It is now convenient to rescale $x \to \hat{x} = \gamma x$ and $k \to \hat{k} = k/\gamma$. We can also remove the linear term in the momentum in (3.4.33) by defining a new energy $\hat{\epsilon}$ which is a linear combination of the old energy and the momentum, $\hat{\epsilon} = \epsilon - vk$. In these new variables we have a Hamiltonian

$$\hat{H} = \frac{1}{8g|\sin\frac{\bar{p}}{2}|}\left[-\hat{k}_1^2 - \hat{k}_2^2 + n_1^2 + n_2^2 + \sum_n g^2 f_n(\Omega_1, \Omega_2) e^{-n|\hat{x}_1 - \hat{x}_2|}\right] \tag{3.4.36}$$

This is the form for the quantum mechanics of the collective coordinates in the slow relative velocity regime. We expect that the potential should be determined completely by the symmetries plus integrability, but we will not work it out here.

It is worth mentioning that, in these variables, the position of the double poles (see section 3) at $u_1 - u_2 = -in/g$ is

$$\hat{k} = \frac{1}{2}(\hat{k}_1 - \hat{k}_2) = in, \qquad \text{with} \;\; n = 2, 3, 4, ... \tag{3.4.37}$$

where $\hat{k}$ is the relative momentum conjugate to $\hat{x}_1 - \hat{x}_2$.

3.4.4 A toy model for double poles

Since finding the full potential in (3.4.36) is beyond the scope of this study we will consider a toy model which we can solve and is such that it also displays the double poles. Notice that in a non-relativistic quantum mechanics problem with one degree of freedom the scattering matrix can have only single poles [101]. In fact, in the quantum mechanical problem the poles can have different origins. We can have poles corresponding to bound states and we can have poles that arise because we have a potential decaying like e^{-mr} and there is a resonance between this exponential decay and the increase of the wavefunction as we discussed around (3.3.18). It should be emphasized that the pole originates from the tail of the potential and is independent of the details of the potential for small r. In the full relativistic theory, the bound states give rise to poles in the s channel and the tails of the potential to poles in the t channel.

The double poles we are considering also arise from the tails of the potential and are not related to bound states. In order to see them, we should include at least two degrees of freedom. We can include an internal angular coordinate φ which lives on a circle. If we have a potential of the form $V \sim e^{in\varphi}e^{-m_n|x|}$, then we will have double poles which can be interpreted, in a relativistic context, as coming from the exchange of a pair of massive particles.

In our toy model, instead of the full space of magnon collective coordinates, we will have $R \times S^1$. We denote by $\varphi = \varphi_1 - \varphi_2$ the relative angular coordinate. We consider a theory where we only treat the relative coordinates x and φ. We now look for an integrable potential which contains the exchanged particles charged under the $U(1)$ symmetry that shifts φ_i. We recall that the non-relativistic limit considered here has a counterpart in the sine-Gordon theory [88]. In that context, integrable potentials of the form $V \sim \frac{1}{\sinh^2 x}$ arise. With this in mind we can consider a simple generalization of this case that will fit our purposes. We propose to study the following hamiltonian

$$H = \frac{1}{2}\left[-\hat{k}^2 + \ell^2\right] + \frac{v}{4\sinh^2\left(\frac{\hat{x}+i\varphi}{2}\right)} + \frac{v}{4\sinh^2\left(\frac{\hat{x}-i\varphi}{2}\right)}\ , \qquad v = \frac{1}{4} - (m+\frac{1}{2})^2 \quad (3.4.38)$$

where v is a coupling constant (which we have parameterized in terms of another coupling m), $\hat{k}$ is the momentum conjugate to the relative coordinate and ℓ is conjugate to $\varphi = \varphi_1 - \varphi_2$. As advertised, this model is integrable (for reviews on these Calogero-Sutherland systems see [102–104]), has the right classical limit and coupling dependence, as long as we make $v \sim \mathcal{O}(g^2)$. This system is naturally defined on the cylinder $R \times S^1$. Since we are considering the scattering of identical particles the space is really $(R \times S^1)/\mathbb{Z}_2$.

This potential is singular at $z = 0$ and near the singularity it behaves as $\cos(2\theta)/r^2$ where $z = re^{i\theta}$. Unfortunately, since the coupling constant is large, this means that the potential is attractive for some angles, and particles might "fall to the center". This can be avoided by modifying the potential near the singularities. Since the poles we are interested in come from long distances, this should have no effect. We do not know of an easy way to modify the potential while preserving integrability. However, for particular values of the coupling v there exist explicit solutions of the above problem where the wavefunction vanishes at the singularities. Thus, a modification of the potential at the singularities to make the problem well defined should have no effect on these particular wave functions.

In order to solve this problem we define the following quantities

$$z = \frac{1}{2}(x + i\varphi) \qquad \bar{z} = \frac{1}{2}(x - i\varphi) \quad (3.4.39)$$

$$\beta^2 + \bar{\beta}^2 = 4\hat{\epsilon} \qquad \text{with} \qquad H = \hat{\epsilon} = \frac{\ell^2}{2} - \frac{\hat{k}^2}{2} \quad (3.4.40)$$

Therefore the Schrodinger equation we need to solve is

$$\left[\frac{\partial^2}{\partial z^2} - \beta^2 + \frac{v}{\sinh^2 z}\right]\psi = 0 \ , \qquad v = \frac{1}{4} - (m+\frac{1}{2})^2 \tag{3.4.41}$$

and its barred version. Then, the general form of the solutions to this equation (and its bared version) are

$$\psi^\beta(z) = e^{\beta z} F(1+m, -m, 1-\beta, u) \quad \text{with} \quad u = \frac{1}{2}\left(1 - \frac{1}{\tanh(z)}\right) \tag{3.4.42}$$

where F is the ordinary hypergeometric function ${}_2F_1$. The second independent solution is $\psi^{-\beta}(z)$. Therefore, the general form of the solutions to our whole problem is $\Psi^{\beta,\bar\beta}(z,\bar z) = \psi^\beta(z)\psi^{\bar\beta}(\bar z)$ with β, $\bar\beta$ satisfying (3.4.40). We can check that in the large x region the wave function behaves as

$$\Psi^{\beta,\bar\beta}(z,\bar z) \sim e^{\beta z + \bar\beta \bar z} \sim e^{i\hat k x + i\ell\varphi} \ , \qquad x \gg 0 \ , \qquad \beta = \ell + i\hat k \ , \quad \bar\beta = -\ell + i\hat k \tag{3.4.43}$$

where ℓ is integer, since φ is periodic. One important condition is that the wave function remains normalizable in the neighborhood of the singularity located at $z = \bar z = 0$. In fact, we see that two independent solutions of (3.4.41) go like z^{-m} and z^{1+m}. Let us assume that m in integer [15]. We then see that if we demand that the wavefunction is antisymmetric[16] under $z \to -z$ and $\bar z \to -\bar z$ we need to combine solutions so that we have combinations such as $z^{-m}\bar z^{1+m}$ and $z^{1+m}\bar z^{-m}$ near the origin. We see that such combinations vanish as we approach $z \to 0$ so that a small modification of the potential is not expected to affect the solutions in an important way. In terms of the

[15] Curiously, for integer m, these models naturally arise from matrix quantum mechanics [104]. At these values of the coupling the potential $1/\cosh^2 z$ is also reflectionless. We also see that (3.4.43) becomes an order m polynomial of u.

[16] We actually need wavefunctions that have definite parity. It turns out that symmetric ones have a singular behavior at the singularity, so we need to consider antisymmetric ones. Actually, there is a physical interpretation of why only antisymmetric functions survive. As the interaction has a $\cos 2\theta$ term near the origin, only odd angular momentum states are such that the potential averages to zero over an orbit. Symmetric states, thus, fall to the center.

functions (3.4.43) we need to consider the following combination

$$\Psi(z,\bar{z}) = -\Psi(-z,-\bar{z}) = \frac{\Gamma(1+m-\beta)\Gamma(1+m-\bar{\beta})}{\Gamma(1-\beta)\Gamma(1-\bar{\beta})}\Psi^{\beta,\bar{\beta}} - \frac{\Gamma(1+m+\beta)\Gamma(1+m+\bar{\beta})}{\Gamma(1+\beta)\Gamma(1+\bar{\beta})}\Psi^{-\beta,-\bar{\beta}} \tag{3.4.44}$$

The fact that the wavefunction is antisymmetric suggests that it is a natural solution to the problem of scattering of identical fermions, rather than bosons. Using the asymptotic expression (3.4.43), we can read off the S-matrix of this problem from the coefficients in (3.4.44) to be

$$S(\beta,\bar{\beta}) = -\frac{\Gamma(1+m-\beta)\Gamma(1+m-\bar{\beta})}{\Gamma(1-\beta)\Gamma(1-\bar{\beta})}\frac{\Gamma(1+\beta)\Gamma(1+\bar{\beta})}{\Gamma(1+m+\beta)\Gamma(1+m+\bar{\beta})}$$

If we scatter particles 1 and 2 with equal angular momentum J_2, then the relative angular momentum vanishes, $\ell = 0$ and $\beta = \bar{\beta} = i\hat{k}$. In this situation the S-matrix is

$$S = -\frac{(1-i\hat{k})^2(2-i\hat{k})^2(3-i\hat{k})^2...(m-i\hat{k})^2}{(1+i\hat{k})^2(2+i\hat{k})^2(3+i\hat{k})^2...(m+i\hat{k})^2} \tag{3.4.45}$$

This expression has double poles at $\hat{k} = in$ with $n = 1, 2, 3, ..., m$. We discussed in the last subsection that, from dynamical arguments, the coupling of the quantum mechanical model for the collective coordinates needs to be of order $\mathcal{O}(g^2)$. Therefore we need $m \sim \mathcal{O}(g)$. In the large g regime we obtain the infinite series of double poles that we expected from the complete theory. However, we also obtain an extra double pole at $\hat{k} = i$. This is not unexpected as our toy model does not forbid $J_2 = 0$ states for the heavy particles[17]. Note that there is no single pole related to the fact that, in the toy model, the exchanged particles change ℓ. In the true theory there is a single pole at $\hat{k} = i$. One the other hand if we look at the S-matrix for $\ell \neq 0$ we find a simple pole at $\hat{k} = i|\ell|$ representing the t-channel exchange of the particle that is giving rise to the potential in the toy model. In this case, we also retain double poles

[17] Note, however, that they are indeed excluded for the intermediate "light" states. This can be seen by expanding the effective potential at large distances.

for $\hat{k} = ir$ with $r = 1 + |\ell|, 2 + |\ell|, ...$

We conclude that this very simple model presents many of the characteristics associated with the complete S-matrix of the full string theory. Of course, it would be nice to find the correct quantum mechanics theory that describes the full problem in this regime.

3.5 Poles in the Beisert-Eden-Hernandez-Lopez-Staudacher S-matrix

3.5.1 Integral expression for the dressing factor

In [24,25] a conjecture was made for the exact form of the unknown dressing factor that appears in the magnon S-matrix. This dressing factor was expressed as

$$\sigma^2 = \frac{R^2(x_1^+, x_2^+)R^2(x_1^-, x_2^-)}{R^2(x_1^+, x_2^-)R^2(x_1^-, x_2^+)}\,, \qquad R^2(x_1, x_2) = e^{i2\chi(x_1,x_2) - i2\chi(x_2,x_1)} \tag{3.5.46}$$

The function χ was given as a series expansion in powers of $1/x_i$. In order to study its analytic structure it is more convenient to write it as an integral expression. We find it convenient to introduce a new function $\tilde{\chi}(1,2)$, which differs from $\chi(1,2)$ in [25] by terms that are symmetric under $1 \leftrightarrow 2$. Such symmetric terms cancel in (3.5.46) and thus $\tilde{\chi}$ will lead to the same dressing factor if we use it in (3.5.46). In appendix A we derive the following integral expression for $\tilde{\chi}$ from the formulae in [25]

$$\tilde{\chi}(1,2) = -i \oint \frac{dz_1}{2\pi} \oint \frac{dz_2}{2\pi} \frac{1}{(x_1 - z_1)(x_2 - z_2)} \times \log\Gamma(1 + ig(z_1 + \frac{1}{z_1} - z_2 - \frac{1}{z_2})) \tag{3.5.47}$$

where the integral is over the contours with $|z_1| = |z_2| = 1$. With these contours the integral is well defined and there are no singularities on the integration contour if we assume that $|x_1|, |x_2| > 1$, as it is the case for physical particles.

3.5.2 Poles of the dressing factor in the giant magnon region

As we start analytically continuing in x_1 and x_2 we might encounter singularities. Singularities will appear when poles or branch cuts of the integrand pinch the contour. We will study only a subset of all possible singularities. We will set g to be large and then we will focus on the giant magnon region with fixed p, there $x^{\pm}$ are near the unit circle but away from $x^{\pm} = 1$. We will then analytically continue away from physical values of $x^{\pm}$ in this region, but we will still stay near the unit circle and far from $x^{\pm} = 1$. In other words, starting from the original values for p_1 and p_2 we will analytically continue only in a small neighborhood of these values of order $1/g$ around the physical values.

In order to find the poles of $\tilde{\chi}$ in (3.5.47), we view it as a function of three independent variables: g, x_1 and x_2. In order to avoid having to keep track of the branch of the log in (3.5.47) we take a derivative of $\tilde{\chi}$ with respect to g. This will allow us to isolate the singularities of $\tilde{\chi}$ that depend on g. In fact, there are no singularities that are independent of g because we can set $g = 0$ and we see that $\tilde{\chi}$ is identically zero. Thus, taking the derivative and disregarding some terms that do not contribute to the integral we find

$$\begin{aligned} g^2 \partial_g \tilde{\chi} &= \sum_{n=1}^{\infty} g^2 \partial_g \tilde{\chi}_n \\ g^2 \partial_g \tilde{\chi}_n &= -\oint \frac{dz_1}{2\pi} \frac{dz_2}{2\pi} \frac{1}{(x_1 - z_1)(x_2 - z_2)} \times \frac{n}{-i\frac{n}{g} + z_1 + \frac{1}{z_1} - z_2 - \frac{1}{z_2}} \end{aligned} \quad (3.5.48)$$

We now do the integral over z_1 by deforming the contour, which starts out at $|z_1| = 1$, towards the origin of the z_1 plane. The only poles we find come from the last factor in (3.5.48). The pole is at $z_1 = z_n(z_2)$ were the function $z_n(z)$ is defined as the solution to

$$z_n + \frac{1}{z_n} - z - \frac{1}{z} = i\frac{n}{g} \ , \qquad\qquad n > 0 \tag{3.5.49}$$

as in [24] . Out of the two roots of this equation we should pick out the root whose absolute value is less than one. We can select this root unambiguously if $|z| = 1$. So from now on we only talk about this root and its analytic continuation as a function of z. We thus find that the result of the z_1 integral is

$$g^2 \partial_g \tilde{\chi}^n = -i \oint \frac{dz}{2\pi} \frac{1}{(x_2 - z)(x_1 - z_n(z))} \times \frac{n}{1 - \frac{1}{z_n^2(z)}} \tag{3.5.50}$$

where we have set $z = z_2$.

We would like to understand where (3.5.50) has poles when we change x_i. Note that if $|x_i| > 1$ (at it is the case for physical values) then the integral is finite. As we start analytically continuing x_i we could have a situation where the pole at $z_2 = x_2$ moves inside the unit circle. In that case, we might think that all we need to do is to deform the contour so that $|z_2| < 1$, but this could push $z_n(z_2)$ to larger values in such a way that it becomes equal to x_1. In other words, there is pole only if two poles of the integrand pinch the integration contour. We discuss below when this happens. If the contour is indeed pinched then the integral is equal to $-2\pi i$ times the residue at $z_2 = x_2$. This gives

$$\begin{aligned} g^2 \partial_g \tilde{\chi}^n &= -\frac{1}{x_1 - x_n(x_2)} \frac{n}{1 - \frac{1}{x_n^2(x_2)}} \\ \tilde{\chi} &= -i \log(x_1 - x_n(x_2)) \end{aligned} \tag{3.5.51}$$

We then find that $e^{2i\tilde{\chi}} \sim (x_1 - x_n(x_2))^2$ has double zeros.

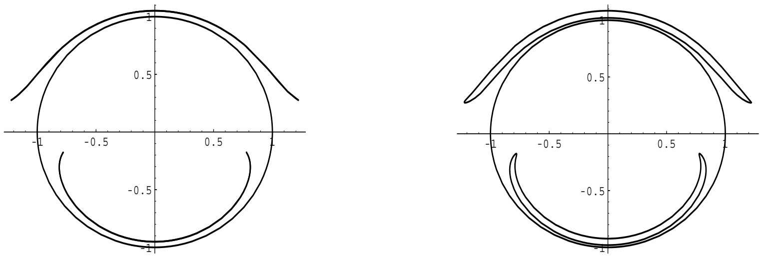


Figure 3.11: On the left we see plots of $z_n(z)$ and $1/z_n(z)$ for z on the unit circle. The circle gets mapped to an interval. z_n is the one inside the unit circle. We have set $n/g = 0.1$. In the plot on the right we have taken $|z| = .98 < 1$.

3.5.3 When do we pinch the contour?

We need to understand some aspects of the function z_n more precisely (see Figure 3.11). The essential feature is the following. If we set $z = e^{i\theta}$ then $z_n \sim 1/z$ for $0 < \theta < \pi$ and $z_n \sim z$ for $-\pi < \theta < 0$. Recall also that in the giant magnon region the physical values of $x^{\pm} \sim e^{\pm ip/2}$. Both are close to the unit circle but x^+ is in the upper half plane and x^- in the lower half plane.

Suppose that we have $x_2 = x_2^+$. This will be on the upper half plane and for finite and physical (real) values of p_2 it will be close to the unit circle, but outside the unit circle. If we keep x_1 fixed and we analytically continue in x_2^+ then we see that z_n will be in the lower half plane when $z \sim x_2^+$, thus we can only pinch the contour if x_1 is also in the lower half plane. This happens when $x_1 = x_1^-$, but not for $x_1 = x_1^+$. Since $z_n \sim 1/z$ for $z \sim x_2^+$ we find that $|z_n|$ increases as $|z|$ decreases (see Figure 3.11). Thus we pinch the contour when $x_2 = x_2^+$ and $x_1 = x_1^-$.

Now suppose that $x_2 = x_2^-$. If $z \sim x_2^-$ we have $z_n \sim z$ so that when we decrease $|z|$ we will also decrease $|z_n|$ and we will not pinch the contour. So the only case where we pinch the contour is when $x_2^+ = z$ and $x_1^- = z_n$. Of course, we have discussed poles from $\tilde{\chi}(1,2)$. When we consider $\tilde{\chi}(2,1)$ we have the same story with $1 \leftrightarrow 2$.

Thus, the final result is that we have poles and zeros of the form

$$\sigma^2 \sim e^{-2i(\tilde{\chi}(x_1^-,x_2^+)-\tilde{\chi}(x_2^-,x_1^+))} \sim \prod_{n>0} \frac{(x_2^- - x_n(x_1^+))^2}{(x_1^- - x_n(x_2^+))^2} \tag{3.5.52}$$

We have only indicated terms that give rise to poles or zeros in the region of interest (large g , $p_1 \sim p_2$ of order one and $p_1 - p_2 \sim \mathcal{O}(g)$). Of course, the expression for the location of poles is exact and we will find poles at these locations in the appropriate branch for any values of p_i and g. [18]

The double zeros of σ^2 (or double poles of σ^{-2}) lie at

$$x_2^- = x_n(x_1^+) \ , \qquad n > 0 \tag{3.5.53}$$

This implies that

$$u_1 - u_2 = -\frac{i(n+1)}{g} = -i\frac{m}{g}, \quad m > 1 \tag{3.5.54}$$

Thus we see that the poles do not start at one. In fact, we do not get poles in σ^{-2} for $m = 1$.[19] Note that equation (3.5.53) contains more information than (3.5.54) since we have specified a particular branch of the function x_n . In fact, if we only looked at (3.5.54) we might incorrectly conclude that there are poles in the near plane wave region with $x_1^\pm \sim x_2^\pm$ and $|x_i^\pm| \gg 1$. On the other hand, we see from (3.5.53) that in this region $x_n(x_1^+) \sim 1/x_1^+$, whose absolute value is not much larger than one. Of course, it is obvious from the integral expression (3.5.47) that there are no poles in the region $|x_i| > 1$ since the integral is explicitly finite there.

In this whole discussion we have assumed that $n \ll g$. If this were not the case, we would have to move by a larger amount from the giant magnon region and we

[18]Note that there are not poles or zeros at $x_1^- = x_n(x_2^-)$ in the branch describing the neighborhood of the giant magnon region. Such poles are probably present on another branch.

[19]Notice, however, that there are poles (or zeros) at that position in the one loop factor answer in [24] (see also [78]). These poles (or zeros) are cancelled by all the higher order contributions.

would have to understand better the whole analytic structure of the function. Finally we note that a related integral representation of the BHL/BES phase has recently been obtained in [105]. Like Eqn (3.5.48) above, the formula (5.11) appearing in this reference is also suggestive of a sum over intermediate states involving BPS particles.

3.6 Discussion

We have understood the physical origin of the poles in the S matrix for the scattering of fundamental magnons. These poles can be cleanly isolated for strong coupling in the giant magnon region. In this case, the poles are far from other singularities of the S matrix. These poles can be explained by the interchange of BPS magnons. The origin of the double poles in the magnon S-matrix is the same as the origin of the double poles in the sine Gordon S-matrix. The position of the poles is determined by the spectrum of BPS particles of the model. These poles are accounted for by considering all the BPS particles that were found to exist on this chain. Therefore there is no reason to think that the spin chain has any other bound states beyond the BPS ones we already know about. In fact, the localized non-BPS classical solutions that were found in [38] were shown to be continuously connected to separated BPS magnon configurations. Thus, those solutions do not correspond to new states. One thing that we have not explored is the physical origin of the branch cuts in the dressing factor. It would be very nice to understand these more precisely.

3.7 Appendix A: Derivation of (3.5.47)

We start from the expression for $\chi(x_1, x_2)$ as in equation (A.9) of [25]

$$
\begin{aligned}
\chi &= -2\sum_{r=2}^{\infty}\sum_{s=r+1}^{\infty}\cos(\frac{\pi}{2}(s-r-1))\frac{1}{x_1^{r-1}x_2^{s-1}}\int_0^\infty \frac{dt}{t}\frac{J_{r-1}(2gt)J_{s-1}(2gt)}{e^t-1} \qquad (3.7.55)\\
\chi &= -2\frac{1}{x_1x_2^2}\sum_{w=0}^{\infty}\sum_{l=0}^{\infty}(-1)^l\frac{1}{(x_1x_2)^w x_2^{2l}}\int_0^\infty \frac{dt}{t}\frac{J_{w+1}(2gt)J_{2+2l}(2gt)}{e^t-1}
\end{aligned}
$$

where $s = r + 1 + 2l$ and $r = w + 2$. We then use the following expression for the Bessel functions

$$
J_n(z) = \int_0^{2\pi}\frac{d\theta}{2\pi}e^{-in\theta+iz\sin\theta} \qquad (3.7.56)
$$

We insert these expressions in the above formulas, we perform the sums and after a simple shift of the integration variables we obtain

$$
\begin{aligned}
\chi &= 2i\frac{1}{x_1x_2^2}\int_0^{2\pi}\frac{d\theta_1}{2\pi}\int_0^{2\pi}\frac{d\theta_2}{2\pi}e^{-i\theta_1-2i\theta_2}\frac{1}{(1-\frac{e^{-2i\theta_2}}{x_2^2})(1-\frac{e^{-i\theta_1-i\theta_2}}{x_1x_2})}\times \qquad (3.7.57)\\
&\quad\times\int_0^\infty\frac{dt}{t}\frac{e^{i(2gt)(-\cos\theta_1+\cos\theta_2)}}{e^t-1}
\end{aligned}
$$

We now use that

$$
\int_0^\infty \frac{dt}{t}\frac{e^{-zt}}{e^t-1} = C_1 + zC_2 + \log\Gamma(1+z) \qquad (3.7.58)
$$

where C_1 and C_2 are divergent constants which do not contribute once we do the integral over θ_i. We then find

$$
\begin{aligned}
\chi &= 2i\frac{1}{x_1x_2^2}\int_0^{2\pi}\frac{d\theta_1}{2\pi}\int_0^{2\pi}\frac{d\theta_2}{2\pi}e^{-i\theta_1-2i\theta_2}\frac{1}{(1-\frac{e^{-2i\theta_2}}{x_2^2})(1-\frac{e^{-i\theta_1-i\theta_2}}{x_1x_2})}\times \qquad (3.7.59)\\
&\quad\times\log\Gamma(1+i2g(\cos\theta_1-\cos\theta_2))
\end{aligned}
$$

This last term looks a bit messy and with many possible branch cuts.

Let us simplify this expression a bit more. First we note that we can write (3.7.59) as

$$\begin{aligned}
\chi &= \int \frac{2e^{-i\theta_2}}{x_2} \frac{1}{1-\frac{e^{-i2\theta_2}}{x_2^2}} f(\theta_1,\theta_2) \qquad (3.7.60)\\
&= \int \left[\frac{1}{1-\frac{e^{-i\theta_2}}{x_2}} - \frac{1}{1+\frac{e^{-i\theta_2}}{x_2}} \right] f(\theta_1,\theta_2)\\
&= \int \frac{1}{1-\frac{e^{-i\theta_2}}{x_2}} \left[f(\theta_1,\theta_2) - f(\theta_2,\theta_1) \right]\\
&= \int \left[\frac{1}{1-\frac{e^{-i\theta_2}}{x_2}} - \frac{1}{1-\frac{e^{-i\theta_1}}{x_2}} \right] f(\theta_1,\theta_2)
\end{aligned}$$

where we used that $f(\theta_1+\pi, \theta_2+\pi) = f(\theta_2,\theta_1)$. We have also relabelled the integration variables.

By forming the antisymmetric combination $\chi_a(1,2) \equiv \chi(1,2) - \chi(2,1)$ it is possible to simplify this expression further. One can see that

$$\begin{aligned}
\chi_a(1,2) &= i \int \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi} e^{-i\theta_1 - i\theta_2} \frac{(x_1-x_2)(e^{-i\theta_2}-e^{-i\theta_1})}{(x_1-e^{-i\theta_1})(x_2-e^{-i\theta_2})(x_1-e^{-i\theta_2})(x_2-e^{-i\theta_1})} \qquad (3.7.61)\\
&\quad \log\Gamma(1+i2g(\cos\theta_1-\cos\theta_2))\\
&= -i \oint \frac{dz_1}{2\pi}\frac{dz_2}{2\pi} \frac{(x_1-x_2)(z_2-z_1)}{(x_1-z_1)(x_2-z_2)(x_1-z_2)(x_2-z_1)} \times\\
&\quad \log\Gamma(1+i2g(\cos\theta_1-\cos\theta_2))\\
&= -i \oint \frac{dz_1}{2\pi} \oint \frac{dz_2}{2\pi} \left(\frac{1}{(x_1-z_1)(x_2-z_2)} - \frac{1}{(x_1-z_2)(x_2-z_1)} \right) \times\\
&\quad \times \log\Gamma(1+i2g(\cos\theta_1-\cos\theta_2))
\end{aligned}$$

where we have set $z_i = e^{-i\theta_i}$. In these expressions, of course, we replace $2\cos\theta_1 = z_1 + \frac{1}{z_1}$, etc. Note that we can view this last expression as arising from $\chi_a(1,2) = \tilde{\chi}(1,2) - \tilde{\chi}(1,2)$ with $\tilde{\chi}$ as in (3.5.47), which is the formula we wanted to derive.

3.8 Appendix B: Classical magnon solutions found by Spradlin and Volovich

For the reader's convenience we reproduce the solutions in [97]. We will use complex target space coordinates $Z_1 = X_1 + iX_2$ and $Z_2 = X_3 + iX_4$ describing an S^3 given by $|Z_1|^2 + |Z_2|^2 = 1$. The one magnon solution, written in terms of the $\lambda^\pm$ and worldsheet $\sigma^{0,1}$ variables, is given by

$$Z_1 = \frac{e^{+i\sigma^0}}{\sqrt{\lambda^+\lambda^-}} \frac{\lambda^+ e^{-2i\mathcal{Z}^+} + \lambda^- e^{-2i\mathcal{Z}^-}}{e^{-2i\mathcal{Z}^+} + e^{-2i\mathcal{Z}^-}} \quad (3.8.62)$$

$$Z_2 = \frac{e^{-i\sigma^0}}{\sqrt{\lambda^+\lambda^-}} \frac{i(\lambda^- - \lambda^+)}{e^{-2i\mathcal{Z}^+} + e^{-2i\mathcal{Z}^-}} \quad (3.8.63)$$

where we defined

$$\mathcal{Z}^\pm = \frac{1}{2}\left(\frac{\sigma^1 - \sigma^0}{\lambda^\pm - 1} + \frac{\sigma^1 + \sigma^0}{\lambda^\pm + 1}\right) \quad (3.8.64)$$

A two magnon (scattering) solution was also presented in [97] and discussed in section 4. The form of this solution is

$$Z_1 = \frac{e^{i\sigma^0}}{2\sqrt{\lambda_1^+\lambda_2^+\lambda_1^-\lambda_2^-}} \frac{N_1}{D} \quad (3.8.65)$$

$$Z_2 = \frac{-i}{2\sqrt{\lambda_1^+\lambda_2^+\lambda_1^-\lambda_2^-}} \frac{N_2}{D} \quad (3.8.66)$$

with

$$
\begin{aligned}
D &= \lambda_{12}^{++}\lambda_{12}^{--}\cosh(u_1+u_2)+\lambda_{12}^{+-}\lambda_{12}^{-+}\cosh(u_1-u_2)+ \\
&\quad \lambda_{11}^{+-}\lambda_{22}^{+-}\cos(v_1-v_2) && (3.8.67) \\
N_1 &= \lambda_{12}^{++}\lambda_{12}^{--}\left[\lambda_1^+\lambda_2^+e^{+u_1+u_2}+\lambda_1^-\lambda_2^-e^{-u_1-u_2}\right]+ \\
&\quad \lambda_{12}^{-+}\lambda_{12}^{+-}\left[\lambda_1^+\lambda_2^-e^{+u_1-u_2}+\lambda_1^-\lambda_2^+e^{-u_1+u_2}\right]+ \\
&\quad \lambda_1^+\lambda_1^-\lambda_{11}^{+-}\lambda_{22}^{+-}e^{i(v_1-v_2)}+\lambda_2^+\lambda_2^-\lambda_{11}^{+-}\lambda_{22}^{+-}e^{-i(v_1-v_2)} && (3.8.68) \\
N_2 &= \lambda_{11}^{+-}e^{iv_1}\left[\lambda_{12}^{++}\lambda_{12}^{-+}\lambda_2^-e^{u_2}+\lambda_{12}^{--}\lambda_{12}^{+-}\lambda_2^+e^{-u_2}\right]+ \\
&\quad \lambda_{22}^{+-}e^{iv_2}\left[\lambda_{21}^{++}\lambda_{21}^{-+}\lambda_1^-e^{u_1}+\lambda_{21}^{--}\lambda_{21}^{+-}\lambda_1^+e^{-u_1}\right] && (3.8.69)
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_{jk}^{\pm\pm} &= \lambda_j^{\pm}-\lambda_k^{\pm} && (3.8.70) \\
u_j &= i\left[\mathcal{Z}_j^+-\mathcal{Z}_j^-\right] && (3.8.71) \\
v_j &= \mathcal{Z}_j^++\mathcal{Z}_j^--\sigma^0 && (3.8.72)
\end{aligned}
$$

These solutions admit the following generalization: $u_j \longrightarrow u_j+a_j$ and $v_j \longrightarrow v_j+b_j$, with a_j and b_j real. Two of these four parameter can be reabsorbed by a worldsheet coordinate redefinition. We are, therefore, left with two parameters corresponding to a relative distance and a relative phase.

The scattering solution presents the symmetry $\lambda_1^+ \leftrightarrow \lambda_2^+$ or, alternatively $\lambda_1^- \leftrightarrow \lambda_2^-$. Also, it collapses to the single magnon solution for $\lambda_j^+=\lambda_j^-$ for $j=1$ or 2.

Chapter 4

Giant Magnons Meet Giant Gravitons

We study the worldsheet reflection matrix of a string attached to a D-brane in $AdS_5 \times S^5$. The D-brane corresponds to a maximal giant graviton and it wraps an S^3 inside S^5. In the gauge theory, the open string is described by a spin chain with boundaries. We study an open string with a large $SO(6)$ charge, which allows us to focus on one boundary at a time and to define an asymptotic boundary reflection matrix. We consider two cases corresponding to two possible relative orientations for the charges of the giant graviton and the open string. Using the symmetries of the problem we compute the boundary reflection matrix up to a phase. These matrices obey the boundary Yang Baxter equation. A crossing equation is derived for the overall phase. We perform weak coupling computations up to two loops and obtain results that are consistent with integrability. Finally, we determine the phase factor at strong coupling using classical solutions.

The work in this chapter is contained in [106]. This article was coauthored with Juan Maldacena.

4.1 Preliminaries

Recently there has been a great deal of progress in understanding planar $\mathcal{N} = 4$ super Yang Mills, see [23–25, 37, 66, 78, 80, 82] and references therein. Planar Yang Mills theories give rise to a two dimensional theory which can be viewed as the worldsheet of a string. From the gauge theory point of view, single trace operators give rise to a closed spin chain, which in turn is related to a two dimensional field theory on a circle. When the charges of the state under consideration are very large one can view the gauge fixed closed string theory [107, 108] as living on a large circle. The limit where the string is infinite is particularly simple [21, 45] and one can solve exactly this problem [23–25, 53, 82]. By "solving" we mean finding the fundamental excitations, their dispersion relation, and their scattering amplitudes on the infinite string for all values of the 't Hooft coupling. It is very useful to consider the symmetries of the problem, which are larger than naively expected [23]. These symmetries determine completely the matrix structure of the two particle scattering matrix [23, 109]. The remaining phase can then be determined by using a crossing symmetry equation [24, 25, 82].

In integrable field theories it is often possible to define the system on a half line, with suitable boundary conditions such that the system remains integrable. A nice example is the boundary Sine-Gordon theory studied in [110]. In this chapter we study some physical problems in $\mathcal{N} = 4$ super Yang Mills that lead to a system with a boundary. From the string theory point of view we expect to have boundaries when we have D-branes. Then the open string excitations are described by a two dimensional field theory with a boundary. Such D-branes can arise in several situations:

- Gauge theories with additional flavors. Open strings correspond to strings with a quark and an anti-quark at the ends.
- Theories with lower dimensional defects, which in some cases can be realized as

D-branes in the bulk [111].

- Certain large charge operators in $\mathcal{N} = 4$ super Yang mills. For example, operators of charge N of the form $\det(Z)$, where Z is one of the complex scalar fields in the theory. We will focus on such operators and their excitations in this chapter [26, 112].

Another case where integrable systems with boundaries arise is when we consider operator insertions along a Wilson loop [113]. This is a situation where, despite the absence of explicit D-branes in the bulk, we end up with a system with a boundary. Of course, we could say that a Wilson line is a an open string which ends on the boundary of AdS_5.

Previous work analyzing open spin chains in $\mathcal{N} = 4$ super Yang Mills or the corresponding open strings with various boundary conditions includes [27, 28, 111, 113–129]. We focus, mainly, on two intimately related cases which consist of giant graviton operators with two possible orientations relative to the open string ground state. We show that in one case we have boundary degrees of freedom, while in the other case we do not.

The central idea in this chapter is a generalization of the analysis by Beisert [23, 109] to the case where we have boundaries. Namely, we will use the symmetries of the system to determine the matrix structure of the boundary scattering matrix. We then proceed to write a crossing equation for the phase factor. Although we have not solved the crossing equation, we have computed the phase factor at weak and strong coupling.

We have also checked that the boundary Yang Baxter equation is obeyed. This follows by an argument similar to the one used in [109]. Furthermore, we performed calculations at two loops in the weak coupling expansion and obtained results compatible with integrability. At strong coupling, this system leads to a classically integrable boundary condition for the string sigma model [124].

When studying the action of the symmetries, it has proven to be useful to have in mind the physical picture for the extra central charges suggested by the classical string theory analysis in [38] (see also [54, 109] for a related picture). Although we explicitly discuss the specific case of giant gravitons, our methods can be extended without too much work to the various cases listed above.

4.2 Giant gravitons, determinants and boundaries

We study open strings attached to maximal giant gravitons [26] in $AdS_5 \times S^5$. These were previously studied at weak coupling at one loop in [27] and at two loops in [126], while a strong coupling classical analysis was carried out recently in [124]. Problems with the integrability of the theory at two loops were pointed out in [126]. We will see, however, that a non trivial extra term coming from a subtle interaction with the boundary will render the theory integrable.

4.2.1 Giant magnons meet giant gravitons

Giant gravitons

Giant gravitons are D3 branes in $AdS_5 \times S^5$ [26]. These D3 branes wrap topologically trivial cycles, but are prevented from collapsing by their coupling to the background fields. We will concentrate on the so called "maximal giant gravitons" which are D3 branes wrapping a maximum size S^3 inside S^5. We can introduce coordinates for the S^5 in terms of $W = \Phi^1 + i\Phi^2$, $Y = \Phi^3 + i\Phi^4$ and $Z = \Phi^5 + i\Phi^6$, with $|Z|^2 + |W|^2 + |Y|^2 = 1$. Maximal giant gravitons are given by a pair of independent linear equations $a^I\Phi^I = b^I\Phi^I = 0$, and are all equivalent up to an $SO(6)$ rotation of the sphere. These configurations preserve half of the supercharges. The particular half that they preserve depends on their orientation inside the S^5.

We are interested in studying open string excitations on the giant gravitons. Our methods work best when the open string carries a large amount of charge. Thus, we also want to single out a special generator, $J = J_{56}$, of $SO(6)$ which generates rotations in the 56 plane. We consider open strings with large charge J. In the field theory such states will involve a large number of insertions of the field Z. Since we are breaking the SO(6) symmetry by selecting a particular generator, J, we find that the explicit open string description depends on the orientation of the giant graviton inside S^5.

We will consider two cases where the D3 brane wraps the following three spheres

- The three sphere given by $Z = 0$. We will call this the $Z = 0$ giant graviton brane. We choose its orientation so that it preserves the same supersymmetries as the field Z in the field theory.

- The three sphere given by $Y = 0$, which we call the $Y = 0$ giant graviton brane. This brane preserves half of the supersymmetries preserved by the field Z in the field theory.

Giant magnons hitting giant gravitons

In what follows we will study open strings with a large amount of charge J. The centrifugal force pushes most of this string to the circle at $|Z| = 1$. We choose a light cone gauge so that a pointlike string moving along this great circle corresponds to the BMN vacuum [21]. In light-cone gauge the string has length J. The ground state of this string preserves half of the spacetime supersymmetries. In particular, it preserves those supercharges with $\Delta - J = 0$, where Δ is the conformal dimension. Furthermore, we can have excitations with momentum p that move along the string. The lowest energy excitation with a given momentum is BPS. It corresponds to an elementary magnon on the corresponding gauge theory spin chain. The state manages

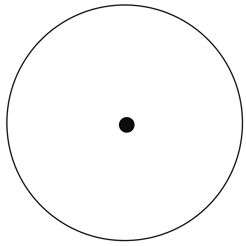


Figure 4.1: $Z = 0$ brane in the Z plane.

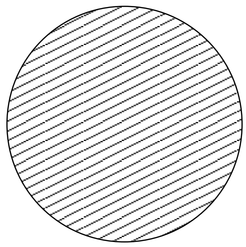


Figure 4.2: $Y = 0$ brane in the Z plane.

to be BPS due to the existence of additional central charges [23]. A convenient picture for the origin of these central charges is the following [38]. We draw the projection of the configuration on the Z plane. This plane is embedded in $AdS_5 \times S^5$ as explained in detail in [61]. The string ground state corresponds to a point on the rim of the circle. An elementary excitation corresponds to a segment that joins two points on the rim. The two central charges correspond to string winding charges along this Z plane [38]. It is now convenient to think about the two branes mentioned above in these coordinates. The $Z = 0$ giant graviton brane is simply a point at $Z = 0$, and it wraps an S^3 inside the S^5, see figure 4.1. The $Y = 0$ giant graviton brane, on the other hand, covers the whole disk, see figure 4.2. At each point of the disk it also wraps an S^1 inside the S^3 that sits at that point. This circle shrinks at the rim of the disk so that we end up with a brane with the S^3 topology.

In the large J limit the string worldsheet is a very long segment, so that when we analyze the effects near one of the boundaries we can forget about the existence of the other boundary and consider the system on a half infinite line. Therefore, we consider first the problem of a giant magnon coming from infinity and bouncing off the boundary back to infinity. In particular, this means that our states interpolate between the usual vacuum of BMN states [21] and the boundary. Furthermore, this implies that one of the ends of the string looks like a "heavy" particle - i.e., there is an infinite amount of J charge at this point - moving at the speed of light in a maximum circle of S^5, see figure 4.3 and [38].

Let us now look at the shape of the corresponding strings on the Z plane. The shape of this string could be complicated at a random point in worldsheet time, but in the asymptotic region (worldsheet time $t \to \pm\infty$) they must look like giant magnons. This means they connect two points on the rim of the disk. This yields no surprise for the $Y = 0$ brane: the asymptotic scattering states for the $Y = 0$ brane are just strings stretched between points on the rim. This might give the impression that the strings are contained within the D-brane. This is not necessarily true; there is an additional $S^3 \subset S^5$ at each point on the disk and the brane and the string could be separated within this S^3.

The $Z = 0$ brane presents an interesting characteristic. In order for the string to interpolate between the correct states we are led to the following picture of the asymptotic scattering configuration, see figure 4.3 (b). We need to have a string that connects the rim of the disk to the center where the $Z = 0$ giant graviton brane sits.

This, in turn, suggests that the $Z = 0$ brane carries a boundary degree of freedom. Even when there is not asymptotic excitation we should have the piece of string connecting the rim of the disk to $Z = 0$, see figure 4.3 (c).

A string lying along a segment in the Z plane carries non vanishing central charges of the worldsheet algebra, since we argued that those central charges correspond to string winding charges on the Z plane.

An important comment at this point is that strings with finite J charge never reach the asymptotic vacuum described above and consequently cannot reach the rim

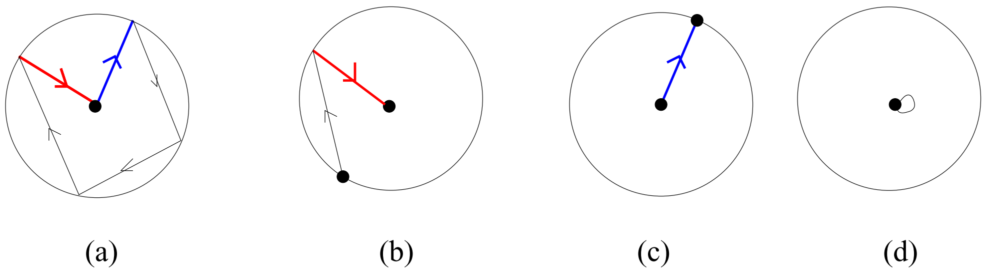


Figure 4.3: (a) Large J open string attached to a $Z = 0$ giant graviton brane. (b) Asymptotic form of the initial condition for the worldsheet scattering of a magnon off the right boundary. The dot on the boundary represents an infinite string in the lightcone ground state. (c) The boundary degree of freedom corresponds to a string going from the brane to the rim of the circle. (d) A string configuration for sufficiently small J does not get close to the boundary of the circle.

of the Z plane. These strings are localized around the brane at the center of the circle.

From the picture presented so far, we are lead to a simple guess for the energy of the boundary state, once we understand the representation of $SU(2|2)^2$ to which it belongs. Let us assume that it belongs to the smallest BPS representation. We will later substantiate this statement by a weak coupling computation where we check that this is indeed the case. Once this is shown for weak coupling, it will be true at all values of the coupling. This implies that the energy is $\epsilon = \sqrt{1 + |k|^2}$ where $\vec{k}$ are the two the central charges. We then notice that the central charge is precisely half the central charge of a magnon with momentum $p = \pi$, which corresponds to a string joining antipodal points on the rim. Therefore,

$$\epsilon_B = \sqrt{1 + 4g^2}\ , \qquad\qquad g^2 = \frac{\lambda}{16\pi^2} \tag{4.2.1}$$

where λ is the 't Hooft coupling. Moreover, since the string in figure 4.3 (c) is sitting at a point in the $S^3 \subset S^5$ we have collective coordinates and their quantization is expected to lead to BPS boundary bound states with higher $SU(2|2)^2$ charges, as we have in the bulk [85, 91]. These states have energy $\epsilon_B(n) = \sqrt{n^2 + 4g^2}$.

These statements do not rely on integrability, only on the symmetries of the theory. Our exact and perturbative calculations presented in the following sections agree precisely with the results discussed above.

4.2.2 Determinants in the gauge theory: the weak coupling description

The coordinates chosen in the previous section make it easy to translate this analysis to the gauge theory side of the story. Here we think of W, Y, Z as the three complex scalars of $\mathcal{N} = 4$ super Yang Mills (and of course we also have their complex conjugates).

Then the $Z = 0$ giant graviton brane, which is the maximal giant graviton given by the equation $Z = 0$, corresponds to the gauge theory operator $\det(Z)$ [112, 130–132]. This is a gauge invariant operator with $J = N$. Of course, the $Y = 0$ giant graviton brane is then obtained by an $SO(6)$ rotation as the operator $\det(Y)$. Both of these operators correspond to the maximal giant gravitons on their ground state. We now want to consider giant gravitons with open strings attached. These are given by replacing one of the entries of the determinant by a chain similar to the one appearing in single trace operators [27, 120, 131, 133–136]. For example, for the $Y = 0$ giant graviton brane we can write

$$\mathcal{O}_Y = \epsilon^{j_1 j_2 \ldots j_{N-1} A}_{i_1 i_2 \ldots i_{N-1} B} Y^{i_1}_{j_1} Y^{i_2}_{j_2} \ldots Y^{i_{N-1}}_{j_{N-1}} (ZZZ \ldots ZZZ)^B_A \tag{4.2.2}$$

where one can make impurities propagate inside the chain of Zs. Thus we consider operators of the form

$$\mathcal{O}_Y(\chi) = \epsilon^{j_1 j_2 \ldots j_{N-1} A}_{i_1 i_2 \ldots i_{N-1} B} Y^{i_1}_{j_1} Y^{i_2}_{j_2} \ldots Y^{i_{N-1}}_{j_{N-1}} (\ldots ZZZ\chi ZZZ \ldots)^B_A \tag{4.2.3}$$

where χ denotes a generic impurity. For the $Z = 0$ giant graviton brane, an operator of the form (4.2.2) with Y replaced by Z would factorize into a determinant and a single trace [120]. This would not describe an open string but a D-brane plus a closed string. Instead we consider excitations of the form

$$\mathcal{O}_Z(\chi, \chi', \chi'') = \epsilon^{j_1 j_2 \ldots j_{N-1} A}_{i_1 i_2 \ldots i_{N-1} B} Z^{i_1}_{j_1} Z^{i_2}_{j_2} \ldots Z^{i_{N-1}}_{j_{N-1}} (\chi ZZ \ldots ZZZ\chi' ZZZ \ldots ZZ\chi'')^B_A \quad (4.2.4)$$

where the impurities χ and χ'' are stuck at the ends of the Z-string. The impurities will reflect when they get to the ends of the string of Zs. Of course, in the large J limit, we only have to worry about one of the ends at a time.

As we mentioned above the two kinds of giant gravitons are related by an $SO(6)$ transformation. Thus, if we start with the $Z = 0$ brane and we add Y impurities so as to completely "fill" the chain we would end up with a state of the form

$$\mathcal{O}' = \epsilon^{j_1 j_2 \ldots j_{N-1} A}_{i_1 i_2 \ldots i_{N-1} B} Z^{i_1}_{j_1} Z^{i_2}_{j_2} \ldots Z^{i_{N-1}}_{j_{N-1}} (YYY \ldots YYY)^B_A \quad (4.2.5)$$

which is simply an $SO(6)$ transform of the state $\mathcal{O}$ in (4.2.2).

4.3 Exact Results for the boundary reflection matrix

Following the work of Beisert [23, 109], it is possible to calculate, up to an overall phase, the reflection matrix associated with the scattering of impurities from the boundaries discussed in the previous section. All we need are the symmetries of the theory and the representations of the states involved. In order to carry out this analysis it is important to understand well the symmetries of the system. Let us first discuss the symmetries of the bulk, before we add the boundaries. As explained in [23,109] we have a centrally extended $SU(2|2)^2$ algebra. We can consider one of these factors at a time. Each factor has eight supercharges $\mathfrak{Q}^\alpha{}_a$ and $\mathfrak{S}^a{}_\alpha$ which transform under $SU(2)\times SU(2)\subset SU(2|2)$. We denote the generators of $SU(2)\times SU(2)$ as $\mathfrak{R}^a{}_b$, $\mathcal{L}^\alpha{}_\beta$ respectively. We follow the notation of [23]. The algebra contains a generator $\mathfrak{C}=\frac{\epsilon}{2}$, where ϵ is the energy of an excitation around the vacuum built with Zs, $\epsilon=\Delta-J_{56}$. In addition we have two extra bosonic generators k and $\bar{k}$ which are the extra central charges which appear in the anti-commutators[1]

$$\{\mathfrak{Q}^\alpha{}_a,\mathfrak{Q}^\beta{}_b\}=\epsilon_{ab}\epsilon^{\alpha\beta}\frac{k}{2}\,,\qquad\qquad \{\mathfrak{S}^a{}_\alpha,\mathfrak{S}^b{}_\beta\}=\epsilon^{ab}\epsilon_{\alpha\beta}\frac{k^*}{2} \tag{4.3.6}$$

These imply that the BPS condition reads $\epsilon^2=1+kk^*$. For the fundamental bulk excitation we also have a relation between k and the momentum

$$|k|^2=16g^2\sin^2\frac{p}{2} \tag{4.3.7}$$

The phase of k is a bit more subtle and we will discuss it later.

The fundamental of $SU(2|2)$ can be split in the following way $\boxslash={}^B\square\oplus{}^F\square$, under $SU(2)\times SU(2)$, where we specified that one doublet is bosonic while the other one is fermionic, i.e. ${}^B\square=(\phi^{\dot{+}},\phi^{\dot{-}})$ and ${}^F\square=(\psi^+,\psi^-)$. We have added a dot to the bosonic $SU(2)$ indices to remind us that they transform under a different $SU(2)$ than the fermions. It is useful to write down the transformation rules for the fundamental multiplet as

$$\begin{aligned}\mathfrak{Q}^\alpha{}_a|\phi^b\rangle &= a\delta^b_a|\psi^\alpha\rangle\,, &\qquad \mathfrak{Q}^\alpha{}_a|\psi^\beta\rangle=b\epsilon^{\alpha\beta}\epsilon_{ab}|\phi^b\rangle\\ \mathfrak{S}^a{}_\alpha|\phi^b\rangle &= c\epsilon_{\alpha\beta}\epsilon^{ab}|\psi^\beta\rangle\,, &\qquad \mathfrak{S}^a{}_\alpha|\psi^\beta\rangle=d\delta^\beta_\alpha|\phi^a\rangle\end{aligned} \tag{4.3.8}$$

where $ad-cb=1$. We find that $\frac{k}{2}=ab$, $\frac{k^*}{2}=cd$ and the energy is $\epsilon=2\,\mathfrak{C}=ad+bc$.

[1] In the notation of [23] $\frac{k}{2}=\mathfrak{P}$ and $\frac{k^*}{2}=\mathfrak{K}$.

We will pick the following parametrization for (a, b, c, d):

$$
\begin{aligned}
a &= \sqrt{g}\eta \\
b &= \frac{\sqrt{g}}{\eta} f \left(1 - \frac{x^+}{x^-}\right) \\
c &= \frac{\sqrt{g} i\eta}{f x^+} \\
d &= \frac{\sqrt{g}}{i\eta} \left(x^+ - x^-\right)
\end{aligned}
\tag{4.3.9}
$$

The momentum of the particle is given by $\frac{x^+}{x^-} = e^{ip}$. The $ad - bc = 1$ condition translates into the mass shell condition

$$
x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g} \tag{4.3.10}
$$

The unitarity of the representation demands that

$$
\eta = \sqrt{ix^- - ix^+} \tag{4.3.11}
$$

up to a phase, which we set to one. Unitarity also requires that f is a phase, which contributes to the phase of the central charge as $k = -2gf(e^{ip} - 1)$. We can think about the central charges in terms of the segment that the magnon describes in the Z plane, by stretching from z_1 to z_2,

$$
z_2 - z_1 = f(e^{ip} - 1) = -\frac{k}{2g} \tag{4.3.12}
$$

Then the phase f represents the orientation of that segment, see figure 4.4. This orientation depends on the sum of the momenta of the magnons that are to the left of the magnon under consideration[2]. Thus f is given by the angle that the magnon is

[2] This corresponds to the non-local parametrization of the problem, as described in [109]. This can also be described by forgetting about f and adding markers $\mathcal{Z}^{\pm}$, see [109] for details.

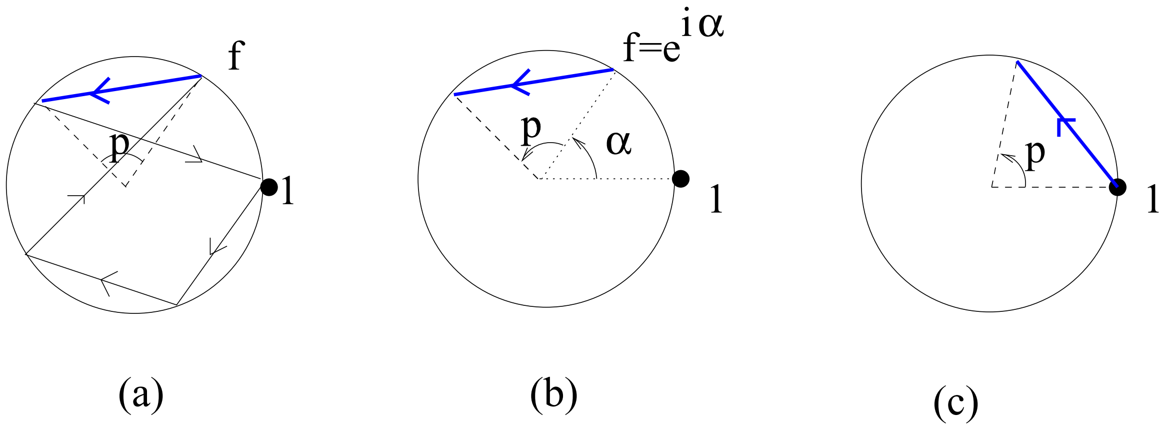


Figure 4.4: (a) We depict a configuration of well separated magnons living on a long string. We choose the point 1 as a reference point and we want to describe the magnon with momentum p. (b) f is the point on the unit circle where the magnon starts and gives the angle required to rotate it to the reference point 1, as in (c).

making in a given state, relative to the magnon with the same momentum that starts at $z_1 = 1$ and goes to $z_2 = e^{ip}$, see figure 4.4. In the case that we have a semi-infinite string it is convenient to take the reference point to coincide with the point where this infinite string is located on the circle.

When we return to the full problem we need to consider two extended $SU(2|2)$ factors and the representation is the product of the fundamental for each, giving a total of 16 states. For example we get

$$Y = \phi^{\dot{-}} \times \tilde{\phi}^{\dot{-}} \ , \qquad W = \phi^{\dot{+}} \times \tilde{\phi}^{\dot{-}} \ , \qquad \overline{W} = \phi^{\dot{-}} \times \tilde{\phi}^{\dot{+}} \ , \qquad \overline{Y} = \phi^{\dot{+}} \times \tilde{\phi}^{\dot{+}} \tag{4.3.13}$$

where the fields $\phi^{\pm}$ and $\tilde{\phi}^{\pm}$ transform under two different $SU(2|2)$ groups. When we consider two extended $SU(2|2)$ factors we get six central charges. However, in this physical problem we require that the central charges for the two factors are equal (we set to zero the difference).

When we consider the $Z = 0$ giant graviton brane we preserve the full symmetry group. Physical states with finite J correspond to strings that start and end on the D-brane that sits at $Z = 0$ and they thus carry zero total central charges $k = k^* = 0$.

On the other hand when we consider the $Y = 0$ giant graviton brane we only preserve the subgroup which is also preserved by the field Y. Let us consider the anticommutator

$$\{\mathfrak{Q}^{\alpha}{}_{a}, \mathfrak{S}^{b}{}_{\beta}\} = \delta_a^b \delta_\beta^\alpha \mathfrak{C} + \delta_\beta^\alpha \mathfrak{R}^{b}{}_{a} + \delta_a^b \mathfrak{L}^{\alpha}{}_{\beta} \tag{4.3.14}$$

and concentrate on the supercharges with a $\dot{+}$ index, $\mathfrak{Q}^{\alpha} \equiv \mathfrak{Q}^{\alpha}{}_{\dot{+}}$ and $\mathfrak{S}_{\alpha} \equiv \mathfrak{S}^{\dot{+}}{}_{\alpha}$. These supercharges annihilate an object with $\mathfrak{J} \equiv \mathfrak{C} + \mathfrak{R}^{\dot{+}}{}_{\dot{+}} = 0$, which is a singlet under the second $SU(2)$, such as a gauge invariant operator made purely with the field Y (notice that an upper $\dot{-}$ index carries $\mathfrak{R}^{\dot{+}}{}_{\dot{+}} = -\frac{1}{2}$). These supercharges, together with $\mathfrak{J}$ and the second $SU(2)$ generators form an $SU(1|2)$ subgroup. The (noncompact) U(1) generator[3], $\mathfrak{J}$, in $SU(1|2)$, which appears in the right hand side of the supersymmetry algebra, is given by $2\mathfrak{J} = \epsilon + 2\mathfrak{R}^{\dot{+}}{}_{\dot{+}} = \Delta - J_{56} - J_{34} - J_{12}$ for one $SU(1|2)$ factor and it is $2\tilde{\mathfrak{J}} = \epsilon + 2\tilde{\mathfrak{R}}^{\dot{+}}{}_{\dot{+}} = \Delta - J_{56} - J_{34} + J_{12}$ for the other.

Let us now study each case in detail.

4.3.1 The $Y = 0$ giant graviton brane or $SU(1|2)^2$ theory

As we mentioned above, the symmetries that commute with the field Y lead to an $SU(1|2)^2$ subgroup. In order to study the problem we first focus on one $SU(1|2)$ subgroup and compute the reflection matrix in this case.

The $SU(1|2)$ algebra arises by restricting all the generators of the $SU(2|2)$ algebra to the ones carrying only $\dot{+}$ indices. As we mentioned above the (non-compact) U(1)

[3] This factor is really non-compact in our problem, hopefully we can continue to call it a $U(1)$ without causing confusion.

generator is $\mathfrak{J} = \mathfrak{C} + \mathfrak{R}^{\dot{+}}{}_{\dot{+}}$ and the non-vanishing commutators are

$$
\begin{aligned}
[\mathfrak{J}, \mathfrak{Q}^\alpha] &= -\frac{1}{2}\mathfrak{Q}^\alpha && (4.3.15)\\
[\mathfrak{J}, \mathfrak{S}_\alpha] &= \frac{1}{2}\mathfrak{S}_\alpha && (4.3.16)\\
\left[\mathcal{L}^\alpha{}_\beta, \mathcal{J}^\gamma\right] &= \delta^\gamma_\beta \mathcal{J}^\alpha - \frac{1}{2}\delta^\alpha_\beta \mathcal{J}^\gamma && (4.3.17)\\
\{\mathfrak{Q}^\alpha, \mathfrak{S}_\beta\} &= \mathcal{L}^\alpha{}_\beta + \delta^\alpha_\beta \mathfrak{J} && (4.3.18)
\end{aligned}
$$

where $\mathcal{J}^\alpha$ is any generator with upper index α. Notice that this algebra is not centrally extended. All central extensions that appeared in the $SU(2|2)$ algebra do not contribute tot he anticommutators of the surviving supercharges have disappeared. In this case a finite J physical open string does not necessarily have zero central charges, but the central charges, $k,\ k^*$ are not preserved by the boundary.

We can find the action of this algebra on the states of the fundamental representation of $SU(2|2)$ from (4.3.8). For completeness we give the action of all generators

$$
\begin{array}{lll}
\mathcal{L}^\alpha{}_\beta|\phi^{\dot\pm}\rangle = 0\ , & \mathcal{L}^\alpha{}_\beta|\psi^\gamma\rangle = \delta^\gamma_\beta|\psi^\alpha\rangle - \frac{1}{2}\delta^\alpha_\beta|\psi^\gamma\rangle & \\
\mathfrak{J}|\phi^{\dot -}\rangle = bc\,|\phi^{\dot -}\rangle\ , & \mathfrak{J}|\phi^{\dot +}\rangle = ad\,|\phi^{\dot +}\rangle\ , & \mathfrak{J}|\psi^\alpha\rangle = \frac{1}{2}(ad+bc)\,|\psi^\alpha\rangle \\
\mathfrak{Q}^\alpha|\phi^{\dot -}\rangle = 0\ , & \mathfrak{Q}^\alpha|\phi^{\dot +}\rangle = a|\psi^\alpha\rangle\ , & \mathfrak{Q}^\alpha|\psi^\beta\rangle = b\epsilon^{\alpha\beta}|\phi^{\dot -}\rangle \\
\mathfrak{S}_\alpha|\phi^{\dot -}\rangle = c\epsilon_{\alpha\beta}|\psi^\beta\rangle\ , & \mathfrak{S}_\alpha|\phi^{\dot +}\rangle = 0\ , & \mathfrak{S}_\alpha|\psi^\beta\rangle = d\delta^\beta_\alpha|\phi^{\dot +}\rangle
\end{array}
$$

with $\alpha, \beta, \gamma = +, -$.

Since the $SU(1|2)$ algebra does not have a central extension, we find that for general momentum we have a non-BPS representation since the charge $\mathfrak{J} = \frac{\epsilon}{2} + \mathfrak{R}^{\dot{+}}{}_{\dot{+}}$ can vary continuously. Thus we expect that the fundamental representation of the extended $SU(2|2)$ transforms irreducibly. In fact, it transforms as the representation of $SU(1|2)$ with the supertableaux $\boxslash\boxslash$. This has the right dimensions as $\boxslash\boxslash = {}^B 1_{\frac{\epsilon}{2}-\frac{1}{2}} \oplus {}^B 1_{\frac{\epsilon}{2}+\frac{1}{2}} \oplus {}^F \square_{\frac{\epsilon}{2}}$, where we have broken the representation in $U(1)\times SU(2)$

multiplets and we have indicated whether we have bosons or fermions. In terms of the degrees of freedom of the $SU(1|2)$ fundamental representation $\boxslash = (\varphi, \chi^{\pm})$ we can represent the corresponding states as $(\chi^+\chi^- - \chi^-\chi^+, \varphi\varphi, \varphi\chi^{\pm} + \chi^{\pm}\varphi)$. We now would like to match these states to the fundamental of the extended $SU(2|2)$ algebra. Matching their bosonic charges we see that

$$\boxslash\boxslash = \begin{pmatrix} \chi^+\chi^- - \chi^-\chi^+ \\ \varphi\varphi \\ \varphi\chi^- + \chi^-\varphi \\ \varphi\chi^+ + \chi^+\varphi \end{pmatrix} = \begin{pmatrix} \phi^{\dot{-}} \\ \phi^{\dot{+}} \\ \psi^- \\ \psi^+ \end{pmatrix} \tag{4.3.19}$$

In the special case of zero momentum $p = 0$, the representation splits into two, one is the identity, given just by $\phi^{\dot{-}}$, and the other three states form the fundamental, BPS representation of $SU(1|2)$ with one bosonic, $\phi^{\dot{+}}$, and two fermionic states. Recall that the field Y is given by $Y = \phi^{\dot{-}} \times \tilde{\phi}^{\dot{-}}$, so it is reasonable that for zero momentum it is a singlet under $SU(1|2)$ since the $SU(1|2)$ subalgebra was found by demanding that all generators annihilate Y. In this chapter we are interested in the case with non-zero momentum where we have a single $SU(1|2)$ non-BPS representation.

The reflection matrix

The $SU(1|2)$ reflection matrix[4] $\mathcal{R}$ can now be calculated by demanding that $[\mathcal{R}, \mathcal{J}] = 0$ for all generators $\mathcal{J}$. The vanishing of the commutators of $\mathcal{R}$ and the bosonic operators imply that $\mathcal{R}$ must be diagonal with equal entries for the fermionic components.

[4] The full reflection matrix of the theory is just the product of two $SU(1|2)$ reflection matrices.

Namely,

$$\mathcal{R} = \begin{pmatrix} r^- & 0 & 0 & 0 \\ 0 & r^+ & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \end{pmatrix} \tag{4.3.20}$$

The commutators with the fermionic operators yield the following conditions:

$$\begin{array}{rcl} ar - a'r^+ & = & 0 \\ br^- - b'r & = & 0 \\ cr - c'r^- & = & 0 \\ dr^+ - d'r & = & 0 \end{array} \quad \longrightarrow \quad \begin{array}{c} r^- = \frac{c}{c'}r = \frac{b'}{b}r \\ \\ r^+ = \frac{a}{a'}r = \frac{d'}{d}r \end{array} \tag{4.3.21}$$

where the primed variables are the quantum numbers of the state after the reflection. These are obtained from the original ones by

$$x^\pm \to x'^\pm = -x^\mp \tag{4.3.22}$$

This follows from conservation of energy, $p \to -p$ and holding $x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}$. Note that η, (4.3.11), is invariant under (4.3.22), so $\eta' = \eta$. The phase f might change as well. f represents the point where the magnon starts in the Z circle, see figure 4.4. When we have a boundary scattering process the values for f for the incoming and the outgoing magnon are related by the geometry of the scattering process in the Z plane. In other words, it is determined by the conservation laws. We represent the relevant conservation laws in figure 4.5 for the scattering from a right boundary and a left boundary.

We see that in the case that we scatter from a boundary on the right, then f does not change, $f' = f$. If the orientation is opposite (boundary on the left), f changes to $f' = f\left(\frac{x^+}{x^-}\right)^2$, see figure 4.5(c,d). Incidentally, (4.3.21) requires $bc = b'c'$

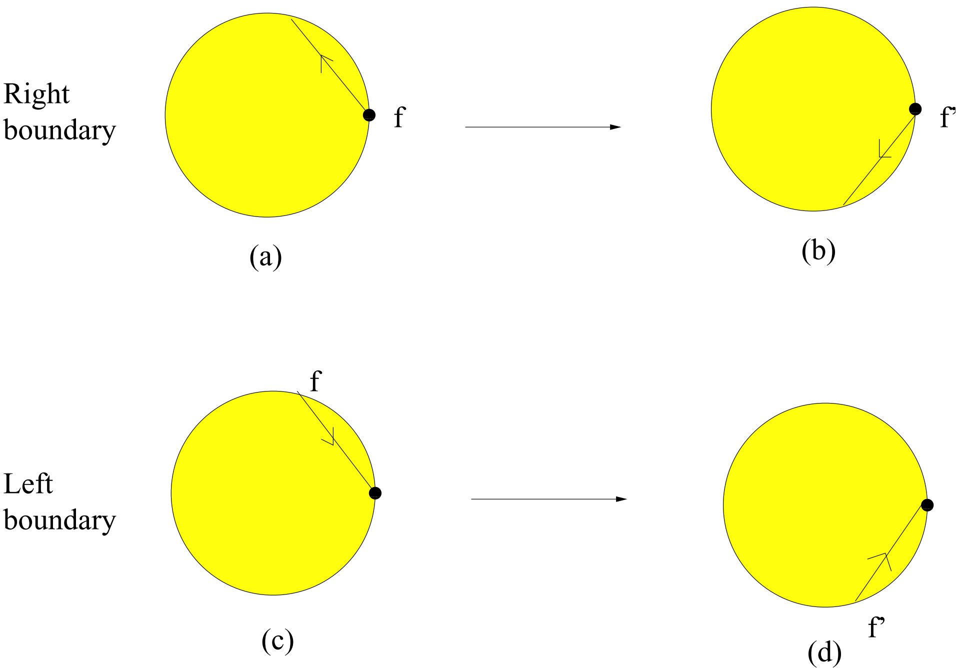


Figure 4.5: We depict several scattering configurations in a situation where we have a semi-infinite string. We choose the infinite region ("heavy" particle / BMN vacuum) to lie at the reference point 1 in the complex plane. We can read off the values of the phase f for the initial and final states from these figures. In (a) and (b) we depict the initial and final configuration for the scattering off a boundary on the right. We can see that in this case $f = f' = 1$. In (c) and (d) we have the initial and final configurations for scattering from a boundary on the left. $f = e^{-ip} \neq f' = e^{+ip}$ in this setup. In all cases we located the point that sets the phase for the incoming state, f, and for the final state, f'. The arrow goes from left to right on the string worldsheet.

and $ad = a'd'$. This follows trivially from conservation of energy $\epsilon = ad + bc$ and the mass shell condition $ad - bc = 1$. Plugging in the values for the quantum numbers yields

$$\mathcal{R}_R = \mathcal{R}_{0R}(p) \begin{pmatrix} -e^{-ip} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{for a right boundary} \tag{4.3.23}$$

and

$$\mathcal{R}_L = \mathcal{R}_{0L}(p) \begin{pmatrix} -e^{ip} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{for a left boundary.} \tag{4.3.24}$$

In these expressions $\frac{x^+}{x^-} = e^{ip}$ and $\mathcal{R}_{0R}(p),\ \mathcal{R}_{0L}(p)$ are arbitrary phases. We see that the two results are consistent with the reflection symmetry that we have in the problem. In fact, if we assume reflection symmetry we can also relate $\mathcal{R}_{0L}(p) = \mathcal{R}_{0R}(-p)$. In addition, unitarity requires $\mathcal{R}_{0L}(-p) = 1/\mathcal{R}_{0L}(p)$, $\mathcal{R}_{0R}(-p) = 1/\mathcal{R}_{0R}(p)$.

The magnons in the full theory are the product of two fundamental magnons of each extended $SU(2|2)$ algebra. Similarly, they are the product of representations for each $SU(1|2)$ subalgebra.

From this result we can predict a ratio of reflection amplitudes. For example the ratio of the amplitudes of scattering a $Y = \phi^{\dot{-}} \times \tilde{\phi}^{\dot{-}}$ and a $W = \phi^{\dot{+}} \times \tilde{\phi}^{\dot{-}}$ is $-e^{\mp ip}$ for R, L boundaries respectively. Remember that in our conventions p is the incoming momentum. If the boundary is placed on the left this momentum is negative. So left and right results are consistent. We will compare this result with explicit calculations

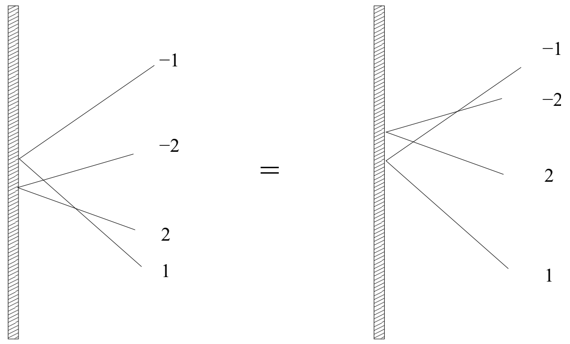


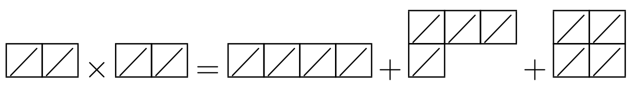
Figure 4.6: The content of the Yang Baxter equation is that these two processes give the same answer.

in the following sections.

Another interesting comment is that this matrix does not contain poles or zeros, unless they are included explicitly in $\mathcal{R}_0(p)$. This means that if there is a bound state in one channel, all channels must have one. In the next section we will check that there is no bound state at weak coupling. We will also compute $\mathcal{R}_0(p)$ perturbatively to two loops at weak coupling and to leading order at strong coupling.

The Yang Baxter equation

We now check that this reflection matrix satisfies the boundary Yang Baxter equation. This equation is represented graphically in figure 4.6 and it states that one can compute the reflection of a pair of particles in two ways. As in the case of the bulk Yang Baxter equation one can check the equation in a simple way using the symmetries [109]. The idea is to look at the Hilbert space of two particles and decompose it in representations of $SU(1|2)$ and then check the equation in each representation. This can be done in a simple way if each representation contains a state that scatters diagonally, so that all scattering amplitudes are simply phases. The intermediate representations of the 2 particle incoming states are:



(4.3.25)

The first representation on the right hand side of (4.3.25) contains the state $\phi_1^{\dot{+}}\phi_2^{\dot{+}}$, the second contains the states $\psi_1^+\psi_2^+$ and $\psi_1^-\psi_2^-$ and the third one contains $\phi_1^{\dot{-}}\phi_2^{\dot{-}}$, which are all states that scatter diagonally.

Let us now check the boundary Yang Baxter equation for two excitations that scatter diagonally. Let us denote by $S(1,2)$ their bulk scattering. $S(1,2)$ is simply a phase by assumption. Similarly, we have the reflection $r(1)$ and $r(2)$ from the boundary which is also a phase. Thus we have

$$S(1,2)r(1)S(-2,1)r(2) = r(2)S(-1,2)r(1)S(-2,-1) \tag{4.3.26}$$

Since we only have phases we see that $r(1)$ and $r(2)$ drop out from the equation and we are only left with a requirement involving the bulk S matrix. This requirement is obeyed if the bulk S matrix is parity invariant, $S(1,2) = S(-2,-1)$. This is an invariance of the bulk S matrix, thus we see that the boundary Yang Baxter equation is satisfied. We have also checked explicitly that the equation is indeed satisfied.

The crossing equation

In order to derive the crossing equation we need to form a singlet state according to the derivation in [109]. This identity state is

$$\mathbf{1}_{(p,\bar{p})} \;\; = \;\; f_p e^{ip/2}(\phi_p^+\phi_{\bar{p}}^- - \phi_p^-\phi_{\bar{p}}^-) + \epsilon_{\alpha\beta}\psi_p^\alpha\psi_{\bar{p}}^\beta \tag{4.3.27}$$

where the subindex p denotes the momentum and energy $\epsilon(p)$ of the first particle and the index $\bar{p}$ denotes the momentum $\bar{p} = -p$ and energy $\bar{\epsilon} = -\epsilon(p)$ of the second, crossed, particle. If we think in terms of the fermionic part of the state we can view the state as a hole, $\psi_+(p)$, and negative energy electron $\psi_-(\bar{p})$. In this case, we clearly see that we get back the original vacuum of the theory. Thus adding this state should

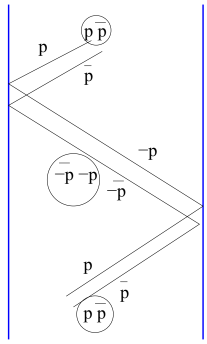


Figure 4.7: We scatter the singlet state $p\bar{p}$ from the right boundary and then from the left boundary in order to come back to the original situation. We demand that this double scattering gives one.

have no effect on the theory. By scattering this two particle state from a third and demanding that the result is invariant one can obtain the crossing equation [82, 109].

If we start with this state and we scatter it from the right boundary we obtain the state $r(p)\mathbf{1}_{(-\bar{p},-p)}$, where $r(p)$ is some reflection phase. We see that we do not get the same state because the particle and antiparticle are in a different order. However, if we have a left boundary and we now scatter the resulting state we get back to the original state (4.3.27), see figure 4.7. We now use that parity invariance implies that the scattering phase we get from the second scattering is the same as the one we got from the first boundary. Thus we find that the total scattering phase is $r(p)^2$. Now it makes sense to demand that the total scattering phase is one, $r(p)^2 = 1$.

So, we get $r(p) = \pm 1$. By considering different boundaries on the two sides we see that the signs should be all plus or all minus, for all boundaries in the theory. We take this sign to be plus. We'll show this in a moment, by looking at the plane wave limit.

When we scatter this state from the boundary we will need the boundary reflection matrix (4.3.23) and the bulk S matrix written in [109].

At the end of the day we obtain

$$\begin{aligned}
\mathbf{1}_{(p,\bar{p})} &= h_b S_0(p,-\bar{p})\mathcal{R}_{0R}(p)\mathcal{R}_{0R}(\bar{p})\left[f_{-p}e^{ip/2}(\phi^+_{-\bar{p}}\phi^-_{-p} - \phi^-_{-\bar{p}}\phi^-_{-p}) + \epsilon_{\alpha\beta}\psi^\alpha_{-\bar{p}}\psi^\beta_{-p}\right] = \\
&= h_b S_0(p,-\bar{p})\mathcal{R}_{0R}(p)\mathcal{R}_{0R}(\bar{p})\mathbf{1}_{(-\bar{p},-p)} \\
h_b &\equiv \frac{\frac{1}{x^-}+x^-}{\frac{1}{x^+}+x^+}
\end{aligned} \tag{4.3.28}$$

where S_0 is the phase factor as defined by Beisert in [109] and $\mathcal{R}_0$ is the phase factor which multiplies the boundary reflection matrix that we had above. Thus the crossing equation has the form

$$\mathcal{R}_{0R}(p)\mathcal{R}_{0R}(\bar{p}) = \frac{1}{h_b}\frac{1}{S_0(p,-\bar{p})} = \frac{\frac{1}{x^+}+x^+}{\frac{1}{x^-}+x^-}\frac{1}{S_0(p,-\bar{p})} \tag{4.3.29}$$

This would be the equation in the case that we had only one $SU(2|2)$ factor. In the full theory, where we have the two $SU(2|2)$ factors we define the full reflection factor to be simply $\mathcal{R}^2_{0R}(p)$, and the bulk phase factor is usually written in terms of a dressing factor σ^2 through the equation [37]

$$S_0(p_1,p_2)^2 = \frac{(x_1^+ - x_2^-)}{(x_1^- - x_2^+)}\frac{(1-\frac{1}{x_1^- x_2^+})}{(1-\frac{1}{x_1^+ x_2^-})}\frac{1}{\sigma^2(p_1,p_2)} \tag{4.3.30}$$

Then the equation for the full theory becomes

$$\mathcal{R}^2_{0R}(p)\mathcal{R}^2_{0R}(\bar{p}) = \frac{1}{h_b^2}\frac{1}{S_0^2(p,-\bar{p})} = \frac{x^+ + \frac{1}{x^+}}{x^- + \frac{1}{x^-}}\sigma^2(p,-\bar{p}) \tag{4.3.31}$$

Notice that in the plane wave limit [21] the right hand side of this equation is just 1. In this limit our theory is non interacting and we know that, in the $SU(2)$ subsector, $\mathcal{R}^2_R(p) = \mathcal{R}^2_R(\bar{p}) = -1$, as this is just a relativistic theory with Dirichlet boundary conditions. From equation (4.3.23) we see that this implies $\mathcal{R}^2_{0R}(p) = \mathcal{R}^2_{0R}(\bar{p}) = -1$. This means that the plus sign is the correct one for the right hand side of equation

(4.3.31).

Finally, we should also mention that unitarity implies

$$\mathcal{R}_{0R}(p)\mathcal{R}_{0R}(-p) = 1 \tag{4.3.32}$$

4.3.2 The $Z = 0$ giant graviton brane or $SU(2|2)^2$ theory

We now study the case of a $Z = 0$ giant graviton brane, which preserves the full $SU(2|2)$ symmetry, see figures 4.1, 4.3. The new feature of this case is the existence of a boundary degree of freedom. We assume that the boundary degree of freedom transforms in the fundamental representation of extended $SU(2|2)^2$. It seems clear that this is the case at weak coupling where we have an impurity stuck between the Z-determinant and the string of Z's producing the large J open string. Then we expect that this should continue to be the case at all values of the coupling. Since the supersymmetry algebra has been extended by the addition of two central charges we need to understand the values of the central charges for the impurity. Here, we will be guided by the string pictures we discussed above, where the central charges are associated to the winding number of the string in the z plane. Thus the central charge vector is simply the vector given by a string going from the brane at $z = 0$ to the rim of the disk, see figure 4.3 (c). We can also view the central charge vector as a complex number. This fixes the absolute value of the central charge vector

$$|k|^2 = 4g^2 \tag{4.3.33}$$

The phase of the central charge depends on the momenta of the other magnons that are in the problem and changes when a magnon scatters from the boundary. Below we will explain how it changes. The conclusion is that the representation of the boundary impurity is again the fundamental of the extended symmetry algebra.

The only difference between the impurity representation and the magnon one is in the relation between the central charges and the momentum (the impurity does not have a momentum quantum number), and in the precise dynamics of the phase of the central charge. It turns out that the problem completely factorizes into each extended $SU(2|2)$ factor. Thus we consider first the case where we have only one $SU(2|2)$ factor.

Let us start by being more specific about the representation properties of the boundary degree of freedom. The transformation properties are as in the bulk case, (4.3.8), but with the following values of a, b, c, d.

$$a_B = \sqrt{g}\eta_B \tag{4.3.34}$$

$$b_B = \frac{\sqrt{g}f_B}{\eta_B} \tag{4.3.35}$$

$$c_B = \frac{\sqrt{g}i\eta_B}{x_B f_B} \tag{4.3.36}$$

$$d_B = \frac{\sqrt{g}x_B}{i\eta_B} \tag{4.3.37}$$

where we have added the subindex B to distinguish these from the bulk case. Unitarity of the representation requires $|\eta_B|^2 = -ix_B$ and that f_B is just a phase. The shortening/mass shell condition implies

$$ad - bc = 1 \quad \longrightarrow \quad x_B + \frac{1}{x_B} = \frac{i}{g}\,, \qquad x_B = \frac{i}{2g}\left(1 + \sqrt{1+4g^2}\right) \tag{4.3.38}$$

where we picked the solution for x_B which leads to positive energy

$$\epsilon = ad + bc = \frac{g}{i}\left(x_B - \frac{1}{x_B}\right) = \sqrt{1+4g^2} \tag{4.3.39}$$

The phase f_B depends on the other magnons in the problem and can be understood most simply by looking at figure 4.8. For a right boundary, f_B is the position of the

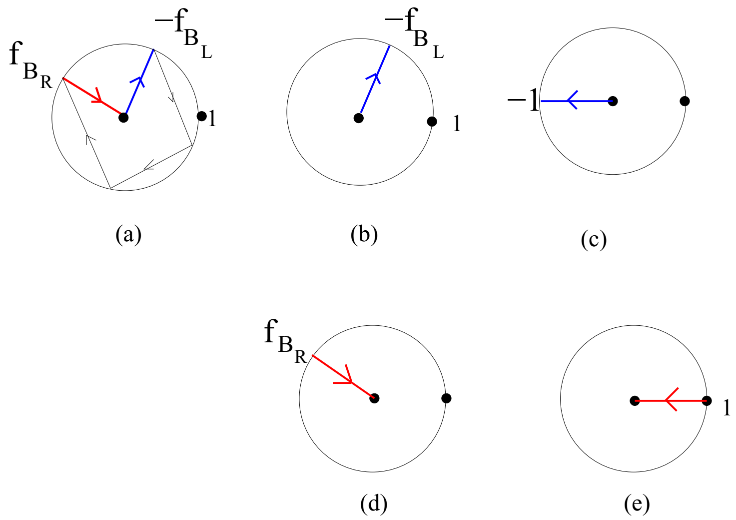


Figure 4.8: In (a) we see a generic open string configuration in the regime that J is very large and the magnons are very well separated. We have denoted by f_{B_L} and f_{B_R} the corresponding parameters of left and right boundaries, respectively. In (b) we isolate the piece of string corresponding to the left boundary impurity. Its phase $-f_B$ is the end point of this string. f_B is also the phase by which the configuration was rotated with respect to the reference configuration in (c). In (d) we isolated the piece of string corresponding to the right boundary. f_B is the starting point of the string on the circle. This phase is also the one by which the configuration was rotated with respect to the reference configuration in (e). These figures can be viewed as the central charge vectors (except for a $-2g$ factor) for the states involved and also as the projections of the physical string configurations to the z plane in the $AdS_5 \times S^5$ geometry.

endpoint of the last magnon on the Z circle. Equivalently it is given by the sum of the momenta of all magnons to the left of the boundary. Since the system ends at the right boundary, this means that $f_B = \prod_j e^{ip_j} f_1$ for all the magnons in the system, where f_1 is the starting point of the first magnon.

We now derive the boundary S matrix for this system. We must first understand how f and f_B change under scattering, see figure 4.9. Let us consider the case of right boundary scattering. In the initial state we have $f_B = e^{ip} f$. In the final state the magnon phase does not change, $f' = f$ and $f'_B = e^{-ip} f = e^{-2ip} f_B = \left(\frac{x^-}{x^+}\right)^2 f_B$, see figure 4.9 a, b. On the other hand, for a left boundary $f_B = -f$, see figure 4.9 c, d. In this case $f' = -f'_B = e^{2ip} f$, or $f'_B = \left(\frac{x^+}{x^-}\right)^2 f_B$. x_B does not change in either case.

Let us now analyze the case with a left boundary in detail. The following equations

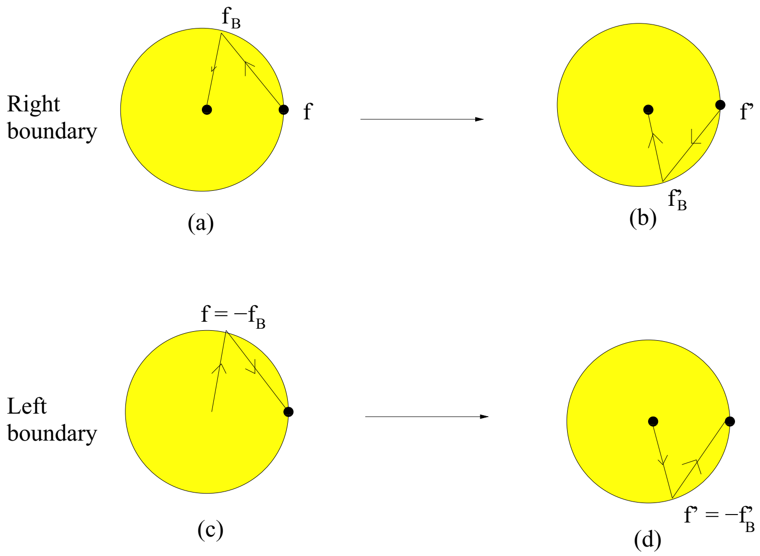


Figure 4.9: In (a) we see the initial state for scattering from a right boundary and in (b) we see the final state. We have indicated the phases of the central charge in both cases. In (c) we see the initial state for right boundary scattering and in (d) we see the final state. These figures can be viewed as the central charge vectors (except for a $-2g$ factor) for the states involved and also as the projections of the physical string configurations to the z plane in the $AdS_5 \times S^5$ geometry.

summarize the quantum numbers of the incoming particle and the boundary and how they change after scattering:

$$
\begin{array}{rclcrcl}
a &=& \sqrt{g}\eta & \qquad & a &=& a \\
b &=& \frac{\sqrt{g}}{\eta} f \left(1 - \frac{x^+}{x^-}\right) & & b' &=& -\frac{x^+}{x^-} b \\
c &=& \frac{\sqrt{g} i\eta}{f x^+} & & c' &=& -\frac{x^-}{x^+} c \\
d &=& \frac{\sqrt{g}}{i\eta}\left(x^+ - x^-\right) & & d' &=& d
\end{array}
\tag{4.3.40}
$$

$$
\begin{array}{rclcrcl}
a_B &=& \sqrt{g}\eta_B & \qquad & a'_B &=& a_B \\
b_B &=& -\frac{\sqrt{g} f}{\eta_B} & & b'_B &=& \left(\frac{x^+}{x^-}\right)^2 b_B \\
c_B &=& -\frac{\sqrt{g} i\eta_B}{x_B f} & & c'_B &=& \left(\frac{x^-}{x^+}\right)^2 c_B \\
d_B &=& \frac{\sqrt{g} x_B}{i\eta_B} & & d'_B &=& d_B
\end{array}
\tag{4.3.41}
$$

In order to calculate the reflection matrix, $\mathcal{R}$, we demand that all commutators of the reflection matrix with the generators of $SU(2|2)$ vanish. In this case the operators act on two particle states, so the computation is more involved that in the last case. In particular, we have to remember that fermionic operators acting on two particle states are defined as $Q = Q_1 \otimes 1 + (-)^F \otimes Q_2$, where F is the fermionic number of particle state 1. The computation is almost identical to the one performed in [23]. Invariance under the bosonic generators implies that the $\mathcal{R}$ matrix can be written as [23] [109]

$$\begin{aligned}
\mathcal{R}|\phi_B^a \phi_p^b\rangle &= A|\phi_B^{\{a}\phi_{-p}^{b\}}\rangle + B|\phi_B^{[a}\phi_{-p}^{b]}\rangle + \frac{1}{2}C\epsilon^{ab}\epsilon_{\alpha\beta}|\psi_b^\alpha \psi_{-p}^\beta\rangle && (4.3.42)\\
\mathcal{R}|\psi_B^\alpha \psi_p^\beta\rangle &= D|\psi_B^{\{\alpha}\psi_{-p}^{\beta\}}\rangle + E|\psi_B^{[\alpha}\psi_{-p}^{\beta]}\rangle + \frac{1}{2}F\epsilon^{\alpha\beta}\epsilon_{ab}|\phi_B^a \phi_{-p}^b\rangle && (4.3.43)\\
\mathcal{R}|\phi_B^a \psi_p^\alpha\rangle &= G|\psi_B^\alpha \phi_{-p}^a\rangle + H|\phi_B^a \psi_{-p}^\alpha\rangle && (4.3.44)\\
\mathcal{R}|\psi_B^\alpha \phi_p^a\rangle &= K|\psi_B^\alpha \phi_{-p}^a\rangle + L|\phi_B^a \psi_{-p}^\alpha\rangle && (4.3.45)
\end{aligned}$$

where a, b represent bosonic indices, $\dot{\pm}$, and α, β are fermionic indices, $\pm$. The (anti) symmetrization symbols are defined with a $\frac{1}{2}$ normalization factor, i.e. $\{ab\} = \frac{ab+ba}{2}$.

It is understood that the states on the right hand side of these equations are out states and, therefore, have primed quantum numbers. In particular, they have primed phases, f' and f'_B. [5]

Acting with the fermionic generators on both sides we get constraints on A, B, C, D, E, F, G, H, K, L that determine them completely up to an overall phase. We

[5]Note that we are working in the so called non-local representation [109]. One can also reintroduce the markers $\mathcal{Z}^\pm$ in a simple way.

get:

$$
\begin{aligned}
A &= \mathcal{R}_0 \frac{x^+(x^+ + x_B)}{x^-(x^- - x_B)} \\
B &= \mathcal{R}_0 \frac{2x^+x^-x_B + (x^+ - x_B)[-2(x^+)^2 + 2(x^-)^2 + x^+x^-]}{(x^-)^2(x^- - x_B)} \\
C &= \mathcal{R}_0 \frac{2\eta\eta_B}{f} \frac{(x^- + x^+)(x^-x_B - x^+x_B - x^-x^+)}{x_B x^-(x^+)^2(x^- - x_B)} \\
D &= \mathcal{R}_0 \\
E &= \mathcal{R}_0 \frac{2[(x^+)^2 - (x^-)^2][-x^+x^- + x_B(x^- - x^+ + x^-(x^+)^2)] - x_B(x^+x^-)^2(x_B - x^-)}{(x^-x^+)^2 x_B(x^- - x_B)} \\
F &= \mathcal{R}_0 \frac{2f}{\eta\eta_B} \frac{(x^+ - x^-)(x^+ + x^-)(x_B x^+ - x_B x^- + x^+x^-)}{(x^-)^3(x^- - x_B)} \\
G &= \mathcal{R}_0 \frac{\eta_B}{\eta} \frac{(x^+ - x^-)(x^+ + x^-)}{(x^- - x_B)x^-} \\
H &= \mathcal{R}_0 \frac{(x^+)^2 - x_B x^-}{x^-(x^- - x_B)} \\
K &= \mathcal{R}_0 \frac{[x_B x^+ + (x^-)^2]}{x^-(x^- - x_B)} \\
L &= \mathcal{R}_0 \frac{\eta}{\eta_B} \frac{(x^+ + x^-)x_B}{x^-(x^- - x_B)}
\end{aligned}
\tag{4.3.46}
$$

Notice that the phase f appears explicitly in C and F. We can eliminate f at the cost of introducing markers, $\mathcal{Z}^{\pm}$, as explained in [109].

The boundary Yang Baxter equation is satisfied by the exactly the same argument used by Beisert in [109], as the symmetries and representations are the same as in the bulk. As in that case, there are two intermediate representations for 3 particle states and each one contains a state that scatters diagonally.

Note also that the boundary scattering in the full theory is given by taking the product of two such reflection matrices, one for each $SU(2|2)$ factor. One could also derive a crossing equation by scattering the identity state (4.3.27) as we did in the $SU(1|2)$ case.

Note that $\frac{A}{D}$ is a prediction for the ratio of amplitudes of $YY \to YY$ scattering in the $SU(2)$ sector to $\psi\psi \to \psi\psi$ in the $SU(1|1)$ sector. In the following section we

Figure 4.10: Pole at $x^- = x_B$

will test the ratio $\frac{A}{B}$ and calculate the phase factor at weak coupling.

Boundary bound states

It is interesting to note that the coefficient A has a pole at $x^- = x_B$. In the full problem, once we take the product of the two reflection matrices we expect that the overall phase factor is such that the scattering in the $SU(2)$ subsector continues to have a single pole at this position. In fact, this will be explicitly checked at weak coupling in section 4.4.3. Thus, we expect to have single pole at all values of the coupling. This pole signals the presence of a bound state, similar to the ones considered in [85]. Following the same rules as in [83] we see that this pole is a generated by the Landau diagram in figure 4.10 that yields a normalizable wave function. Figure 4.10 represents an actual boundary bound state in the s-channel. The incoming fundamental magnon binds to the boundary degree of freedom to form a BPS bound state corresponding to a double box representation of $SU(2|2)^2$. As in the bulk case, we can introduce a new parameter $x_B^{(2)} \equiv x^+$. Once we set $x^- = x_B$, we find that

$$x_B^{(2)} + \frac{1}{x_B^{(2)}} = 2\frac{i}{g} \tag{4.3.47}$$

The energy of the bound state is given by $\epsilon = \frac{g}{i}(x_B^{(2)} - \frac{1}{x_B^{(2)}})$, as in (4.3.39). We can now consider the boundary scattering of another magnon with this new boundary impurity. This can be computed by scattering this second magnon, parametrized by $x_2^{\pm}$, off the bound state made out of the original impurity and the first magnon, parametrized by $x_B^{(2)} = x_1^+,\ x^- = x_B$. This scattering is described a the product of the scattering amplitudes of the second magnon from the first, the reflection matrix, and the scattering of the reflected second magnon with the first. This full amplitude has a pole at $x_2^- = x_B^{(2)}$. Thus we can have a new bound state characterized by $x_B^{(3)} \equiv x_2^+$. Proceeding in this fashion we obtain a structure of bound states very similar to what we had in the bulk [85,90]. An n particle bound state is given by $x_B = x_1^-$, $x_1^+ = x_2^-$, $x_i^+ = x_{i+1}^-$, $x_B^{(n)} = x_{n-1}^+$. Then using the equations for each of the particles one can see that

$$x_B^{(n)} + \frac{1}{x_B^{(n)}} = n\frac{i}{g}\ , \qquad \epsilon_B = \frac{g}{i}(x_B^{(n)} - \frac{1}{x_B^{(2)}}) = \sqrt{n^2 + 4g^2} \tag{4.3.48}$$

These are in the same representation of the extended $SU(2|2)^2$ superalgebra as the bulk magnons [91], except, of course, that the central charges are given by the line going from the center of the disk to the rim of the disk.

4.4 Results at weak coupling

In this section we present some results obtained from weak coupling calculations in the gauge theory. We consider the operators $\mathcal{O}_Y$ and $\mathcal{O}_Z$ described by expressions (4.2.3) and (4.2.4). We study the large J limit, where the chain is infinitely long and we focus on the physics near each of the boundaries. We study $\mathcal{N} = 4$ super Yang Mills at two loops, using the results for the dilatation operator obtained in [18] to calculate the reflection matrices in the $SU(2)$ subsector. Furthermore, we perform some non trivial checks, in the $SU(3)$ subsector, of the ratios of the matrix elements of the exact matrices discussed in the previous section. Finally, in appendix 4.7 we discuss the integrability of the resulting Hamiltonian.

4.4.1 The two loop Hamiltonian at weak coupling in the $SU(2)$ sector

In order to calculate the reflection matrices we first need to calculate the appropriate Hamiltonian including the boundary contributions. This has been calculated at one loop in [27] and at two loops in [126]. We review this calculation and discuss an extra term, relative to [126], that is present at two loops. This term, although subtle, is crucial to make the spin chain integrable.

Our starting point is the general expression for the one and two loop dilatation operator [18] in the SU(2) subsector. This is

$$\begin{aligned} D = -2\frac{g^2}{N} : \mathrm{Tr}[Y,Z][\check{Y},\check{Z}] : -2\frac{g^4}{N^2} : \mathrm{Tr}[[Y,Z],\check{Z}][[\check{Y},\check{Z}],Z] : \\ -2\frac{g^4}{N^2} : \mathrm{Tr}[[Y,Z],\check{Y}][[\check{Y},\check{Z}],Y] : +4\frac{g^4}{N} : \mathrm{Tr}[Y,Z][\check{Y},\check{Z}] : \end{aligned} \qquad (4.4.49)$$

where $\check{X}$ means $\frac{\partial}{\partial X}$.

We can calculate the effective Hamiltonian operating on a $SU(2)$ spin chain from this operator. The bulk part of this Hamiltonian is [18, 126]

$$H_{\text{bulk}} = \sum_i (2g^2 - 8g^4)(I - P_{i,i+1}) + 2g^4 \sum (I - P_{i,i+2}) \qquad (4.4.50)$$

where $P_{i,j}$ is the permutation operator between sites i and j.

Let us discuss the boundary terms that need to be added when we attach our spin chain to a giant graviton. As the interaction has a range of two sites we only need to worry about the first few sites of the chain, assuming a boundary on the left. Let us assume our spin chain starts as

$$\epsilon X_B^{N-1} | \underbrace{X_0}_{0} \underbrace{X_1}_{1} \underbrace{X_2}_{2} \ldots \tag{4.4.51}$$

where X_i are fields that can take the values Y, Z. We have been schematic and have omitted indices in this expression. The $|$ separates the giant graviton from the rest of the chain.

From the site 1 onwards we have the bulk Hamiltonian. At site 0, the Hamiltonian acts differently. To leading order in $1/N$, the determinant cannot have a field of the same flavor next to it [27, 120, 134]. This means that X_0 is always different from X_B. We also have to be careful about this when we operate with the Hamiltonian. If X_1 or X_2 are equal to X_B then the corresponding permutation operator acting on the site 0 will vanish. With these rules in mind, if we consider the action of D (4.4.49) on the chain by applying all derivatives outside the determinant[6], we find that H acts on the first three sites as

$$H_{\text{naive}} = (2g^2 - 8g^4) q_1^{X_B} + 2g^4 q_2^{X_B} \tag{4.4.52}$$

where $q_i^{X_B}$ acts as the identity if $X_i = X_B$ and as zero if it is not. If this was the whole story we would reproduce the results of [126]. However, we still need to consider the possibility of the dilatation operator acting on the determinant and its neighboring sites. It turns out there is only one term in the dilatation operator (4.4.49) that contributes to this extra piece. This term is roughly $\frac{g^4}{N^2} \mathrm{Tr}\left(\check{X}_B X_0 X_B \check{X}_B \check{X}_0 X_B\right)$ with the first derivative acting on the determinant. Naively, this term is suppressed by a factor of N as can be seen from (4.4.49). However, since there are $N-1$ letters inside the determinant, there are $\mathcal{O}(N)$ possible actions of the derivative. All these

[6] This amounts to truncating H_{bulk} at the end of the chain.

subleading terms add up cancelling the $\frac{1}{N}$ suppression. This extra term is[7]

$$H_{\rm det} = 4g^4 q_1^B \tag{4.4.53}$$

The final form of the two loop boundary Hamiltonian in the $SU(2)$ sector is:

$$\begin{aligned} H &= H_{\rm bulk} + H_{\rm naive} + H_{\rm det} = \\ &= (2g^2 - 8g^4)\sum_{i=1}^{\infty}(I - P_{i,i+1}) + 2g^4 \sum_{i=1}^{\infty}(I - P_{i,i+2}) + (2g^2 - 4g^4)q_1^{X_B} + 2g^4 q_2^{X_B} \end{aligned} \tag{4.4.54}$$

Notice that the chain starts effectively at site 1, as the site 0 is fixed by the boundary[8]. This Hamiltonian, with the explicit inclusion of $H_{\rm det}$ (4.4.53), is consistent with integrability. This is suggested in appendix 4.7 by explicitly constructing the perturbative asymptotic Bethe ansatz solution for the two magnon problem.

We can now use this result to calculate scattering amplitudes for different boundaries in the $SU(2)$ subsector.

4.4.2 The $SU(1|2)$ reflection matrix off a $\det(Y)$ boundary

Let us now consider the operators involving an open chain on ending on the operator $\det(Y)$, corresponding to the $Y = 0$ giant graviton brane. We focus on the large J limit, where we have a large number of Zs producing a long open string, and we focus on one end of the chain at a time. In that case one can compute the boundary reflection matrix. Let us start considering the operator $\mathcal{O}_Y$ (4.2.2) that corresponds to the vacuum. Acting with the Hamiltonian (4.4.54) we find

$$H\mathcal{O}_Y(Z) = 0 \tag{4.4.55}$$

[7]This term should also be added to the expressions in [125].

[8]This situation will change when we move to the $SU(3)$ subsector

where we plugged in $X_B = Y$ in the expression (4.4.54). This was expected, since it is a BPS state the vacuum has zero energy. We see that we have no degree of freedom, as the first excitations will be massive. If we place an impurity moving with momentum p far away from the boundary, all boundary terms vanish, and we recover the bulk expression for the energy

$$H\mathcal{O}_Y(Y_p) = \left(8g^2 \sin^2 \frac{p}{2} - 32g^4 \sin^4 \frac{p}{2}\right) \mathcal{O}_Y(Y_p) \tag{4.4.56}$$

for a one particle state with momentum p, $\mathcal{O}_Y(Y_p)$; see equation (4.4.58). The formula for the energy is just the expansion to second order in g^2 of the anomalous part of the magnon energy

$$\epsilon - 1 = \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}} - 1 \sim 8g^2 \sin^2 \frac{p}{2} - 32g^4 \sin^4 \frac{p}{2} \tag{4.4.57}$$

Let us now compute the reflection matrix. We write a wavefunction of the form

$$\mathcal{O}_Y(Y_p) = \sum_{x=1}^{\infty} \Psi(x)\, \mathcal{O}_Y(Y_x) = \sum_{x=1}^{\infty} \left(e^{+ipx} + Re^{-ipx}\right) \mathcal{O}_Y(Y_x) \tag{4.4.58}$$

where $\mathcal{O}_Y(Y_x)$ is an operator of the form given by equation 4.2.3 with the impurity placed at site x. In principle, there can be corrections of order g^2 near $x \sim 0$, as was discussed for the bulk in [45]. This turns out not to be necessary in our case. If we apply the Hamiltonian we see that this is an eigenstate of the right energy, provided we set

$$\Psi(0) = 0 \qquad \Psi(-1) + \Psi(1) = 0 \tag{4.4.59}$$

where we have analytically continued the expression for the wavefuntion, $\Psi(x) = e^{ipx} + Re^{-ipx}$, to negative values of x. Remarkably both equations can be satisfied simultaneously without the inclusion of corrections by setting $R = -1$. In terms of the reflection matrix for each $SU(1|2)$ factor (4.3.24), and recalling the expression for

Y, (4.3.13), we see that

$$-1 = R = \mathcal{R}_{0L}^2 e^{2ip}\ , \qquad \rightarrow \qquad \mathcal{R}_{0L}^2 = -e^{-2ip} \tag{4.4.60}$$

up to two loops. We see that the two loop correction vanishes. It would be interesting to see at what loop order we get the first deviation from this result.

Finally, we notice that there are no poles associated with boundary bound states in this matrix. This confirms, at weak coupling, our assumption that there are no boundary degrees of freedom in this theory.

One loop test for the $SU(1|2)^2$ reflection matrix

In this section we will compare the reflection amplitudes of Y, $\overline{Y}$ and W ($\overline{W}$ should be the same as W) off a boundary that consists of a $Y = 0$ giant graviton brane. These calculations were performed at one loop in [27], where they have an expression for the one loop boundary hamiltonian in the $S0(6)$ sector. In our notation[9] the results they obtain for scattering off a boundary (a $\det(Y)$ boundary) on the left are

$$R_W = e^{-ip} = R_{\overline{W}} \tag{4.4.61}$$

$$R_Y = -1 \tag{4.4.62}$$

$$R_{\bar{Y}} = -e^{-2ip} \tag{4.4.63}$$

Notice the quotients $\frac{R_W}{R_Y} = -e^{-ip}$ and $\frac{R_{\bar{Y}}}{R_Y} = e^{-2ip}$ are the ones predicted by our exact matrix (4.3.24), recalling the expressions (4.3.13) for the impurities. Also, the overall factors are the same as the ones calculated at two loops in this section.

[9] Among other things they define the origin of the chain at site 1 instead of site 0. This introduces some phases.

4.4.3 The $SU(2|2)$ spectrum and reflection matrix off a $\det(Z)$ boundary

Let us now go through a similar calculation for the $SU(2|2)$ reflection matrix, which corresponds to the case that we have an open chain ending on a $\det(Z)$ operator. In this case the ground state is non trivial. As we argued before, the letter placed next to the determinant, $\det(Z)$, cannot be a Z. This means that, at the very least, one field gets trapped in between the vacuum described by a chain of Zs and the D-brane. Our simplest guess for this operator is $\mathcal{O}_Z(Y, \cdots)$, (4.2.4), where the dots represents the other boundary which we are not discussing now. Direct computation shows that this is an eigenstate with energy

$$H\mathcal{O}_Z(Y, \cdots) = (2g^2 - 2g^4)\,\mathcal{O}_Z(Y, \cdots) \tag{4.4.64}$$

This energy is the contribution from one boundary. In the case of the full chain, we have a second impurity at the other end and we have to add the corresponding energy. This energy agrees precisely with the weak coupling expansion of the exact formula (4.3.39),

$$\epsilon_B = \sqrt{1 + 4g^2} \sim 1 + 2g^2 - 2g^4 \tag{4.4.65}$$

This computation tests the boundary term in the Hamiltonian (4.4.54).

Once again, scattering states have the same energy as in the bulk, so the total energy is

$$H\mathcal{O}_Z(Y, Y_p, \cdots) = \Big[(2g^2 - 2g^4) + (8g^2 \sin^2\frac{p}{2} - 32g^4 \sin^4\frac{p}{2})\Big]\,\mathcal{O}_Z(Y, Y_p, \cdots) \tag{4.4.66}$$

In appendix 4.8 we construct explicitly the wavefunction up to two loops, check this expression for the energy, and compute the reflection amplitude to two loops. We find

$$R' = -\frac{1-2e^{ip}}{1-2e^{-ip}} + 2g^2 \frac{e^{-ip}(e^{ip}-1)^3(e^{ip}+1)(1-4e^{ip}+e^{i2p})}{(e^{ip}-2)^2} \tag{4.4.67}$$

This fixes the overall phase $\mathcal{R}_{0L}$ in (4.3.46) at two loops for weak coupling. We would like to write this expression as a function of $x^\pm, x_B$ such that we can make a guess that might be correct to a few higher orders as in [81]. Moreover, writing the expression this way allows for the identification of poles in the reflection matrix in a straightforward way. Notice that the coefficient A in the matrix $\mathcal{R}$ (4.3.46) has the right limit at 1 loop but disagrees with (4.4.67) at two loops. We propose an expression that coincides with (4.4.67) up to two loops.

$$R' = -\frac{(x^+ + x_B)}{(x^- - x_B)} \frac{\left(x^+ + \frac{1}{x_B}\right)}{\left(x^- - \frac{1}{x_B}\right)} \frac{(x^- + x_B)}{(x^+ - x_B)} \frac{\left(x^- + \frac{1}{x_B}\right)}{\left(x^+ - \frac{1}{x_B}\right)} \tag{4.4.68}$$

In checking this it is useful to remember the weak coupling expansions

$$x_B = \frac{i}{2g}\left(1+\sqrt{1+4g^2}\right) \sim \frac{i}{g} + ig + \cdots \tag{4.4.69}$$

$$x^\pm = e^{\pm i\frac{p}{2}} \frac{\left(1+\sqrt{1+16g^2\sin^2\frac{p}{2}}\right)}{4g\sin\frac{p}{2}} \sim e^{\pm i\frac{p}{2}}\left(\frac{1}{g\,2\sin\frac{p}{2}} + 2g\,\sin\frac{p}{2} + \cdots\right) \tag{4.4.70}$$

This expression for R' presents four simple poles. The pole at $x^- = x_B$ is responsible for the singularities of the weak coupling expansion (4.4.67). This is the pole that is already visible at one loop. This pole gives rise to a bound state in the s-channel and corresponds to the BPS boundary bound states that we discussed in section 4.3.2. We do not know if all the other poles of (4.4.67) survive when we add higher order corrections. It should be possible to perform an analysis similar to the one in [83], to determine the presence or absence of the other poles.

We can now also read off the two loop value of $\mathcal{R}_{0L}$ in (4.3.46)

$$\mathcal{R}_{0L}^2 = \frac{R'}{A^2} = -\left(\frac{x^-}{x^+}\right)^2 \frac{(x^- - x_B)}{(x^+ - x_B)} \frac{\left(x^+ + \frac{1}{x_B}\right)}{\left(x^+ - \frac{1}{x_B}\right)} \frac{(x^- + x_B)}{(x^+ + x_B)} \frac{\left(x^- + \frac{1}{x_B}\right)}{\left(x^- - \frac{1}{x_B}\right)} \tag{4.4.71}$$

One loop test for the $SU(2|2)^2$ reflection matrix

We compare our exact results for the reflection matrix, (4.3.46), with the weak coupling results, as we did for the $SU(1|2)^2$ case. Unlike the previous case, this calculation is not available in the literature. We will need to compute the scattering process of a W approaching a $Z = 0$ brane with a Y degree of freedom. At one loop the fermions do not play a role and we can consider the $SU(3)$ sector to be closed. (This can be seen from the expression of C in the exact solution, which is $\mathcal{O}(g)$ while A and B are of order unity). Therefore, our process is

$$|Y_B W_p\rangle \to R'_W |Y_B W_{-p}\rangle + R'_Y |W_B Y_{-p}\rangle \tag{4.4.72}$$

The Hamiltonian at one loop for the $SU(3)$ sector can be obtained by restricting the $SO(6)$ result in [27]. In our notation this is

$$H = 2g^2 \left(\sum_{i=0}^{\infty} (I - P_{i,i+1}) + P_{0,1} q_1^Z \right) \tag{4.4.73}$$

This means that when there is Z in the first (1) site it is the same as in the $SU(2)$ subsector, but the permutation operator does contribute when Y and W occupy the 0 and 1 site as opposed to the $SU(2)$ case. The reason for this is obvious: both Y and W can appear next to the determinant of Zs. We use the following trial eigenstate:

$$\Psi = \sum_{x=1}^{\infty} \left(e^{ipx} + R'_W e^{-ipx}\right) |Y_B W_x\rangle + R'_Y e^{-ipx} |W_B Y_x\rangle \tag{4.4.74}$$

where $|X^1_B X^2_x\rangle$ is a state with an X^1 at the boundary (the site labelled by zero) and an X^2 at position x. In the bulk ($x > 1$) the eigenvalue equation yields the necessary value of the energy for both W and Y states.

$$E = 2g^2 \left(1 + 2 - e^{ip} - e^{-ip}\right) \tag{4.4.75}$$

Let us see what happens for the first site

$$E \begin{pmatrix} \psi_W(1) \\ \psi_Y(1) \end{pmatrix} = \begin{pmatrix} 2\psi_W(1) - \psi_W(2) - \psi_Y(1) \\ 2\psi_Y(1) - \psi_Y(2) - \psi_W(1) \end{pmatrix} \tag{4.4.76}$$

where $\psi_W = e^{ipx} + R'_W e^{-ipx}$ and $\psi_Y = R'_Y e^{-ipx}$. Using the bulk equations we get

$$\begin{pmatrix} \psi_W(1) \\ \psi_Y(1) \end{pmatrix} = \begin{pmatrix} \psi_W(0) - \psi_Y(1) \\ \psi_Y(0) - \psi_W(1) \end{pmatrix} \tag{4.4.77}$$

Plugging the ansatz for the wave function we get

$$R'_W = \frac{e^{2ip} - e^{ip} + 1}{e^{ip} - 2} \tag{4.4.78}$$

$$R'_Y = \frac{e^{2ip} - 1}{e^{ip} - 2} \tag{4.4.79}$$

These values satisfy $|R'_Y|^2 + |R'_W|^2 = 1$ as they should to comply with unitarity. Now we can compare the quotients $\frac{R'_W}{R'}$ and $\frac{R'_Y}{R'}$with the expected values from the exact calculations, (4.3.46). Here R' is the value encountered in the $SU(2)$ sector at one loop (4.4.67). Namely,

$$R' = \frac{2e^{ip} - 1}{1 - 2e^{-ip}} \tag{4.4.80}$$

The resulting quotients are:

$$\frac{R'_W}{R'} = \frac{e^{ip} + e^{-ip} - 1}{2e^{ip} - 1} \, , \qquad \frac{R'_Y}{R'} = \frac{e^{ip} - e^{-ip}}{2e^{ip} - 1} \tag{4.4.81}$$

From the exact result (4.3.46) we have

$$\frac{R'_W}{R'} = \frac{1}{2}\left(1 + \frac{B}{A}\right) \, , \qquad \frac{R'_Y}{R'} = \frac{1}{2}\left(1 - \frac{B}{A}\right) \tag{4.4.82}$$

Expanding A, B, using the first terms in (4.4.69)(4.4.70), we checked that these equations are true. This is a nontrivial one loop check for the bosonic subsector of the reflection matrix. A very easy check is that $R'_Y + R'_W = R'$.

4.5 Results at strong coupling

In this section, we discuss results obtained in the strong coupling regime from string theory. As long as one is interested in the leading terms in g, it is possible to calculate scattering amplitudes by calculating time delays in classical sine Gordon theory [38]. We make use of this possibility to calculate the overall phase of the reflection matrix at strong coupling for both the $Z = 0$ and $Y = 0$ giant graviton branes. To be more precise, at strong coupling there are three regimes, depending on how we scale the momentum. We can keep the momentum fixed and then compute as we mentioned above; this is the giant magnon regime. We could also scale the momentum as $p \sim 1/g$ and this corresponds to the near plane wave limit. Finally we can set $p \sim 1/\sqrt{g}$, see [95]. For the case of bulk scattering it is possible to write a formula which captures the leading order result both in the plane wave and giant magnon regimes [37]. Here we will focus on the giant magnon region. As we briefly discussed in section 4.3.1, the result in the plane wave region is trivial. Some results in the near plane wave region were obtained in [118].

4.5.1 Boundary conditions in the sine Gordon theory

According to the work of Pohlmeyer [48] it is possible to map the problem of a string propagating on $\mathbb{R} \times S^2$ into the classical sine Gordon model, see also [49, 50, 70]. This connection was used in [38] to calculate the strong coupling limit of the bulk scattering phase of string theory on $AdS_5 \times S^5$. We will do the same here.

We use string worldsheet coordinates in which $\dot{t} = 1$. Then, the sine Gordon field, $\phi(x,t)$, is related to the unit vector $\boldsymbol{\eta}$ describing the S^2 as

$$\cos 2\phi = \dot{\boldsymbol{\eta}}^2 - \boldsymbol{\eta}'^2 \tag{4.5.83}$$

where

$$\boldsymbol{\eta}^2 = 1 \ , \qquad \dot{\boldsymbol{\eta}}^2 + \boldsymbol{\eta}'^2 = 1 \ , \qquad \dot{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}' = 0 \tag{4.5.84}$$

We can consider simple cases leading to different boundary conditions for the sine Gordon theory.

1. Scattering off a $Z = 0$ giant graviton brane

2. Scattering off a $Y = 0$ giant graviton brane where we chose the S^2 within brane, e.g. the S^2 given by $|Z|^2 + (\Phi_1)^2 = 1$

3. Scattering off a $Y = 0$ giant graviton brane where we chose the S^2 transverse to the brane, e.g. the S^2 given by $|Z|^2 + (\Phi_3)^2 = 1$

Recall that $Z = \Phi^5 + i\Phi^6$, $Y = \Phi^3 + i\Phi^4$.

In the first case the boundary is fixed at the center of the Z plane. This means that the S^2 boundary condition is $\dot{\boldsymbol{\eta}}|_{\text{Boundary}} = 0$. Therefore, using equations (4.5.83) and (4.5.84), we find the Dirichlet boundary condition $\phi|_{\text{Boundary}} = \frac{\pi}{2}$. This type of boundary conditions were discussed for the classical sine Gordon theory in [137] and the time delay was calculated. Note that $\phi = \frac{\pi}{2}$ corresponds to the maximum of the sine Gordon potential. This implies that the field has to move from the maximum to the minimum and this leads to some energy that is localized near the boundary. This corresponds to the boundary degree of freedom, or boundary impurity, that we discussed above.

The second case represents a string that is entirely contained inside the D-brane that it is attached to. Therefore, the string end point (the one ending on the D-brane) can move freely on the S^2, thus $\boldsymbol{\eta}' = 0$ and this leads to another Dirichlet boundary condition for the sine Gordon field $\phi|_{\text{Boundary}} = 0$. In this case the field is at the minimum of the potential and we have nothing localized at the boundary.

Finally, in the third case the endpoint of the string, which has to lie both on the D-brane and inside the S^2, has to be on the rim of the disk $|z| = 1$, which is the only region common to both. One can then show that this leads to $\phi'|_{\text{boundary}} = 0$.

In this fashion, we see how different physical configurations in $AdS_5 \times S^5$ lead to different boundary problems for the sine Gordon theory. Interestingly enough, all the boundary conditions that were discussed belong to the special class that make the boundary field theory integrable [110]. Incidentally, the string theory setup we are studying was shown to be integrable at large g in [124]. It would be interesting to see if other integrable boundary conditions in the sine Gordon model map to other configurations in the string theory.

We should mention that this description that uses the sine Gordon theory is only an approximation (valid in the classical limit). It is not capturing the fact that there are collective coordinates characterizing the magnon. These arise because the magnon has an S^3 worth of possible orientations inside the S^5. (In addition, we have fermion zero modes [100].) As we quantize these coordinates we get all the BPS bound states with various values of the angular momentum charge n [85, 91]. In particular, the fundamental impurities, such as the fields Y, X, etc, have wavefunctions that are spread over this S^3. Thus, when we talked about solutions that were localized within a given S^2, we were making an approximation where we neglected this motion. One could get a better approximation by considering the solutions in [97, 98], which can be used to describe the classical limit of the scattering of BPS bound states [85] with angular momentum $n \sim \mathcal{O}(g)$ from the boundary. In the case of the $Z = 0$ brane, where we have a boundary impurity, we construct the solution as follows. Con consider a soliton of the bulk theory with momentum $p = \pi$ that is at rest at the origin. This is a solution that obeys the boundary conditions of the boundary theory. Its energy is simply half of the energy of the original soliton. We can similarly consider the generalizations with angular momentum discussed in [91, 97, 98]. In that case both the angular momentum and energy are half of what they were in the bulk. However, in the boundary case, we want to quantize the angular momentum so that it is an integer after dividing by half. Thus we get a formula for the energies that has the form

$$\epsilon_B = \frac{1}{2}\sqrt{(2n)^2 + 16g^2} = \sqrt{n^2 + 4g^2} \tag{4.5.85}$$

where n is an integer. This is in agreement with the exact results (4.3.48).

4.5.2 Time delays and scattering phases

Let us consider first the case where we have a $Y = 0$ giant graviton brane. It is convenient to think about the problem by using a "method of images" where the incoming soliton scatters an antisoliton or a soliton coming from the other side of the boundary, depending on the boundary conditions. From our experience with the sine Gordon model and the bulk calculations in [38], we know the result will be independent of whether the image state is a soliton or an antisoliton. Therefore, we don't need to specify this in our calculations.

When we translate between the sine-Gordon results and the results computed in the conventions that are more natural at weak coupling we need to be careful about the fact that these two different conventions differ in the definition of the spatial coordinate. This was explained in more detail in [38, 90, 138]. In fact, we can work in conventions that coincide with the gauge theory conventions and notice that the classical boundary scattering amplitude has a simple relation to the bulk scattering amplitude once we note that the boundary scattering amplitude can be computed by the "method of images". Let us consider the case where we scatter from a right boundary[10].

For a $Y = 0$ brane, we have two solitons, one with momentum $p_1 = p$ and another with momentum $p_2 = -p$. The bulk scattering phase is related to the time delays

$$\Delta T_{12} = \frac{dp_1}{d\epsilon_1}\partial_{p_1}\delta(p_1, p_2) \ , \qquad\qquad \Delta T_{21} = \frac{dp_2}{d\epsilon_2}\partial_{p_2}\delta(p_1, p_2) \tag{4.5.86}$$

where $\delta(p_1, p_2)$ is the bulk scattering phase computed in [38]

$$\delta(p_1, p_2) = -4g\,(\cos\frac{p_1}{2} - \cos\frac{p_2}{2})\log\left[\frac{\sin^2\frac{p_1-p_2}{4}}{\sin^2\frac{p_1+p_2}{4}}\right] \tag{4.5.87}$$

where $sign(\sin p_i) > 0$. For $p_2 = -p < 0$ we should set $p_2 = 2\pi - p$ in this formula, and this is what we will always mean by $-p$. In the case that $p = p_1 = -p_2$ we find that the two time delays are equal to each other and to the time delay for scattering from the boundary $\Delta T_{12} = \Delta T_{21} = \Delta T_B(p)$. Thus we conclude that the classical

[10]We can obtain the result for left boundaries by a parity transformation $\mathcal{R}_L(p) = \mathcal{R}_R(-p)$.

(right) boundary scattering phase, $\mathcal{R}_R = e^{i\delta_{B,R}}$, is the solution to

$$\begin{aligned} \frac{dp}{d\epsilon}\partial_p \delta_{B,R}(p) &= \Delta T_B(p) = \frac{1}{2}(\Delta T_{12} + \Delta T_{21}) = \\ &= \frac{1}{2}\left(\frac{dp_1}{d\epsilon_1}\partial_{p_1}\delta(p_1,p_2) + \frac{dp_2}{d\epsilon_2}\partial_{p_2}\delta(p_1,p_2)\right)\Big|_{p_1=-p_2=p} \end{aligned} \tag{4.5.88}$$

A solution to this equation is then

$$\delta_{B,R}(p) = \frac{1}{2}\delta(p,-p) = -8g\,\cos\frac{p}{2}\log\cos\frac{p}{2} \tag{4.5.89}$$

where δ is (4.5.87). This describes right-boundary scattering. Note that we get the same answer regardless of the state of the impurity, since the matrix structure of the reflection matrix (4.3.23) is subleading at large g. This also means that this an actual calculation of the overall phase factor $\mathcal{R}_{0R}^2$ at strong coupling and to leading order.

We can check that this result obeys the classical limit of the crossing equation (4.3.31)

$$\delta_{B,R}(p) + \delta_{B,R}(\bar{p}) = -\delta(p,-\bar{p}) + \mathcal{O}(1) \tag{4.5.90}$$

where the $\mathcal{O}(1)$ terms are order one in the $1/g$ expansion. Notice that in order to get the results for $\bar{p}$, we should set $p \to -p$ in (4.5.87) (4.5.89) and, as we mentioned before, to get the results for $-p$ we should set $p \to 2\pi - p$ in (4.5.87) (4.5.89).

This result is valid in the giant magnon regime. We remind the reader that reflection becomes trivial in the plane wave region, as magnons become noninteracting. In that case, we get Dirichlet boundary conditions for the fields Y, $\overline{Y}$ and Neumann for W, $\overline{W}$. This implies that $\mathcal{R}_{0R}^2 = -1$ in the plane wave regime.

In a similar way we can compute the classical limit of the boundary scattering for the $Z=0$ brane. In this case we have a boundary impurity. Using the "method of images" we can represent the boundary impurity as a third soliton, with momentum $p=\pi$ that is sitting at the boundary. This type of solutions was obtained explicitly for the sine Gordon model in [137]. In order to compute right boundary scattering we consider a bulk configuration with three solitons with $p_1 = p$, $p_2 = -p$ and $p_3 = \pi$. Then the time delay is

$$\Delta T(p) = \Delta T_{12} + \Delta T_{13} = \frac{1}{2}(\Delta T_{12} + \Delta T_{21}) + \Delta T_{13} \tag{4.5.91}$$

Writing this as in (4.5.88) we find the large coupling expression for the phase in (4.3.46), $\mathcal{R}^2_{0,R} = e^{i\delta^Z_{B,R}}$,

$$\delta^Z_{B,R}(p) = \frac{1}{2}\delta(p, -p) + \delta(p, \pi) = -4g\,\cos\frac{p}{2}\,\log\left[\cos^2\frac{p}{2}\,\frac{(1-\sin\frac{p}{2})}{(1+\sin\frac{p}{2})}\right] \tag{4.5.92}$$

where

$$\delta(p, \pi) = -4g\,\cos\frac{p}{2}\log\left[\frac{1-\sin\frac{p}{2}}{1+\sin\frac{p}{2}}\right] \tag{4.5.93}$$

The classical limit of the crossing symmetry equation is expected to be similar and it would still be obeyed since (4.5.93) is odd under $p \to -p$ (which is what we should to do to cross $p \to \bar{p}$).

4.6 Discussion

In this chapter we considered some D-brane configurations in $AdS_5 \times S^5$ and considered the worldsheet theory of an open string ending on the D-brane. We focused on the D-branes that correspond to maximal giant gravitons. In the dual field theory, these D-branes correspond to determinant operators of the form $\det(Y)$, $\det(Z)$, where Y, Z are two complex combinations of the scalar fields in $\mathcal{N} = 4$ super Yang Mills. We considered an open string attached to this operator with a large value of J, where J is one of the generators of $SO(6)$. In the dual field theory this corresponds to attaching a long string of Zs to the determinant operator. This can be viewed as a spin chain defined on an interval. We then considered impurities propagating on this chain of Zs. The symmetries of the problem determine completely the single impurity reflection matrix up to an overall phase. These reflection matrices are asymptotic, as in the bulk [45]. Namely, we need to go far away from the boundary to measure it. Thus, the strict mathematical definition of the reflection matrix requires $J = \infty$.

We considered two cases. First the case where the determinant operator was $\det(Y)$. In this case the boundary breaks the bulk symmetry group to an $SU(1|2)^2$ subgroup. Yet, this symmetry is powerful enough to determine the matrix structure of the reflection matrix. In fact, in a natural basis, the reflection matrix is diagonal.

We then considered the case where we have a $\det(Z)$ operator. In this case an impurity gets trapped between the string of Zs describing the open string ground state and the determinant operator. This impurity acts as a boundary degree of freedom. This problem respects the full extended $SU(2|2)^2$ symmetry that we have on the bulk of a chain of Zs, or the bulk of the string in light cone gauge [107, 108]. The boundary impurity transforms in the fundamental representation of the extended $SU(2|2)^2$ algebra and has a (complex) central charge with fixed modulus and a phase that is determined by the momenta of the other particles. This is very similar to the structure we have in the bulk of the string. The algebra determines the energy of the boundary impurity. In this case, the reflection matrix acts on the boundary degree of freedom. The resulting matrix is rather similar to the one describing the bulk scattering of two impurities [23]. Also, the bulk particle can form BPS bound states with the boundary degrees of freedom. Thus, the spectrum of boundary degrees of freedom includes an index n which characterizes the total number of impurities forming the bound state.

Both of reflection matrices obey the boundary Yang Baxter equation, which is a requisite for integrability. In the first case, we derived explicitly the form of the crossing equation by considering the scattering of a particle/hole pair and demanding that the corresponding reflection amplitude is trivial. This derivation could be extended to the second case in a straightforward way.

We then performed computations in the weak coupling regime. Here we checked the integrability of the system up to two loops. We resolved the problems raised in [126] by noticing that there is an extra boundary contribution to the spin chain Hamiltonian. The results we obtain at two loops are consistent with integrability, in the sense that the asymptotic Bethe ansatz for two particles works properly. Nevertheless, we have not proven the full integrability of the system at two loops. We also computed the undetermined phase factor in the reflection matrix up to two loops in the weak coupling expansion. In addition, we checked that the matrix structure obtained by the symmetry arguments was consistent with the explicit weak coupling results.

We also computed the strong coupling limit of the reflection phase. At strong coupling there are two perturbative regimes, the near plane wave regime and the giant magnon regime, depending on the momentum of the impurity. We computed the leading order result for the scattering amplitude in the giant magnon regime. The computation can be carried out in a simple way by using a "method of images", where we view the problem with a boundary in terms of a problem on the full line with the proper symmetry under reflection[11]. This gives the boundary scattering phase in terms of the bulk scattering phase.

Note that our computations of the matrix structure of the reflection matrix are valid also for other systems where we have $SU(2|2)$ symmetry. One such system is the plane wave matrix model [21], where one can study configurations analogous to the ones considered here, even though this particular system appears not to be integrable [139].

[11]This method is useful for the classical theory but it is not appropriate for the full quantum theory.

4.7 Appendix A: integrability at two loops

It was pointed out in [126] that the Bethe ansatz seems to fail at two loops for the problem just studied. We will now show that the problems raised disappear once we consider the correct Hamiltonian (4.4.54). In particular, the problem was found when one tried to construct a two particle state using the original scattering data.

We will consider a wave function of the form $\Psi(x,y) = \Psi_0(x,y) + g^{2|x-y|}\Upsilon(x,y)$ where we will only be concerned with corrections of order g^2 to the standard Bethe ansatz wave function $\Psi_0(x,y)$. This is the asymptotic Bethe ansatz discussed in [45]. Our state is

$$\mathcal{O}_Y(Y_{p_1}Y_{p_2}) = \sum_{0<x<y}^{\infty} \Psi(x,y)\, \mathcal{O}_Y(Y_x Y_y) \tag{4.7.94}$$

The equations we have to satisfy in the bulk are

$$\begin{aligned}
E\Psi(x,y) \quad &= \\
&(2g^2-8g^4)\,(4\Psi(x,y)-\Psi(x-1,y)-\Psi(x+1,y)-\Psi(x,y-1)-\Psi(x,y+1)) \\
&+2g^4\,(4\Psi(x,y)-\Psi(x-2,y)- \\
&\Psi(x+2,y)-\Psi(x,y-2)-\Psi(x,y+2)) \quad \text{for} \quad 2<x<y-2 &(4.7.95)\\
E\Psi(x,x+2) \quad &= \\
&(2g^2-8g^4)\,(4\Psi(x,x+2)-\Psi(x-1,x+2)-\Psi(x+1,x+2) \\
&-\Psi(x,x+1)-\Psi(x,x+3))+2g^4\,(2\Psi(x,x+2)-\Psi(x-2,x+2) \\
&-\Psi(x,x+4)) \quad \text{for} \quad 2<x &(4.7.96)\\
E\Psi(x,x+1) \quad &= \\
&(2g^2-8g^4)\,(2\Psi(x,x+1)-\Psi(x-1,x+1)-\Psi(x,x+2)) \\
&+2g^4\,(4\Psi(x,x+1)-\Psi(x-2,x+1)- \\
&\Psi(x+1,x+2)-\Psi(x-1,x)-\Psi(x,x+3)) \quad \text{for} \quad 2<x &(4.7.97)
\end{aligned}$$

where E is the sum of the one particle energies. These equations specify $\Upsilon(x,y)$ completely, as well as the bulk scattering matrix [45] [66]. In order to obtain information about the reflection matrix we need to check the eigenvalue equation for sites close to the boundary. If we pick sites of the form $(2,x)$ our equations are:

$$\begin{aligned} E\Psi(2,x) &= (2g^2-8g^4)\,(4\Psi(2,x)-\Psi(1,x)-\Psi(3,x)-\Psi(2,x-1)-\Psi(2,x+1)) \\ &\quad +2g^4\,(3\Psi(2,x)-\Psi(4,x)- \\ &\quad \Psi(2,x-2)-\Psi(2,x+2))+2g^4\Psi(2,x) \quad \text{for} \quad 4<x \qquad (4.7.98) \\ E\Psi(2,4) &= (2g^2-8g^4)\,(4\Psi(2,4)-\Psi(1,4)-\Psi(3,4)-\Psi(2,3)-\Psi(2,5)) \\ &\quad +2g^4\,(\Psi(2,4)-\Psi(2,6))+2g^4\Psi(2,4) \qquad (4.7.99) \\ E\Psi(2,3) &= (2g^2-8g^4)\,(2\Psi(2,3)-\Psi(1,3)-\Psi(2,4)) \\ &\quad +2g^4\,(3\Psi(2,3)-\Psi(2,5)-\Psi(3,4)-\Psi(1,2))+2g^4\Psi(2,3) \qquad (4.7.100) \end{aligned}$$

If we use the original equations, these just imply $\Psi(0,x)=\Psi_0(0,x)=0$ for $x>2$. These are the analogous equations to $\Psi(0)=0$ in the single particle case and determine the one particle reflection matrix to be consistent with the Bethe ansatz.

We still have to consider the sites (1,x). These can't introduce any more constraints, as our function is already fully determined. The resulting equations are:

$$\begin{aligned} E\Psi(1,x) &= (2g^2-8g^4)\,(3\Psi(1,x)-\Psi(2,x)-\Psi(1,x+1)-\Psi(1,x-1)) \\ &\quad +2g^4\,(3\Psi(1,x)-\Psi(3,x)-\Psi(1,x-2)-\Psi(1,x+2))+ \\ &\quad (2g^2-4g^4)\Psi(1,x) \quad \text{for} \quad 3<x \qquad (4.7.101) \\ E\Psi(1,3) &= (2g^2-8g^4)\,(3\Psi(1,3)-\Psi(2,3)-\Psi(1,4)-\Psi(1,2)) \\ &\quad +2g^4\,(\Psi(1,3)-\Psi(1,5))+(2g^2-4g^4)\Psi(1,3) \qquad (4.7.102) \\ E\Psi(1,2) &= (2g^2-8g^4)\,(\Psi(1,2)-\Psi(1,3))) \\ &\quad +2g^4\,(2\Psi(1,2)-\Psi(1,4)-\Psi(2,3))+(2g^2-2g^4)\Psi(1,2) \qquad (4.7.103) \end{aligned}$$

Making use of the bulk equations the first of these expressions yields $\Psi(-1,x) + \Psi(1,x) = \Psi_0(-1,x)+\Psi_0(1,x) = 0$ for $x > 3$. These are the analog of $\Psi(1)+\Psi(-1) = 0$ and impose no further constraints, as our wave function satisfies this identity. The second equation gives the same result for $x = 3$. The last of these equations is the one that presented a conflict in [126]. In our case this equation can be written (to order g^4) as

$$2g^4\left(\Psi_0(1,2) + \Psi_0(-1,2)\right) + (2g^2 - 8g^4)\Psi_0(0,2) + 2g^4\Psi_0(0,1) = 0 \qquad (4.7.104)$$

This is satisfied by our Bethe ansatz as $\Psi_0(1,2) + \Psi_0(-1,2) = 0$, $\Psi_0(0,2) = 0$ and $\Psi_0(0,1) = 0$. This shows that the two particle problem can be solved by the asymptotic Bethe ansatz technique, suggesting integrability.

4.8 Appendix B: computation of the $SU(2|2)$ reflection matrix at two loops

The wave function for a one particle state scattering of the boundary should satisfy:

$$\begin{aligned} E\Psi(x) &= (2g^2 - 8g^4)(2\Psi(x) - \Psi(x+1) - \Psi(x-1)) \\ &\quad +2g^4(2\Psi(x) - \Psi(x+2) - \Psi(x-2)) \\ &\quad +(2g^2 - 2g^4)\Psi(x) \quad \text{for} \quad x > 2 \end{aligned} \qquad (4.8.105)$$

for the trial wave function $\Psi(x) = \Psi_0(x) + g^2\delta_{x,1}\Upsilon$. The g^2 correction is just an exponential tail attached to the boundary that accounts for the interactions at two loops. Further corrections are higher order in g^2. $\Psi_0(x)$ is just the reflecting wave solution $\Psi_0(x) = e^{ipx} + R'e^{-ipx}$, where R' has, in principle, g^2 corrections to the 1 loop result. From this expression we check that the energy of this state is indeed (4.4.66).

The equation that determines Υ comes from the coefficient of the Schrodinger equation for site 2. Namely

$$\begin{aligned} E\Psi(2) &= (2g^2-8g^4)(2\Psi(2)-\Psi(3)-\Psi_0(1))-2g^4\Upsilon \\ &\quad +2g^4(\Psi(2)-\Psi(4))+(2g^2-4g^4)\Psi(2) \end{aligned} \tag{4.8.106}$$

Using the bulk equation (4.8.105) we get

$$\Upsilon = \Psi(0)-2\Psi(2) \tag{4.8.107}$$

The equation at site 1 determines the reflection amplitude. This is

$$\begin{aligned} &E\Psi_0(1)+2g^4(3-e^{ip}-e^{-ip})\Upsilon = \\ &(2g^2-8g^4)(\Psi_0(1)-\Psi(2))+2g^4(\Psi_0(1)-\Psi(3))+ \\ &2g^4\Upsilon+2g^4\Psi_0(1) \end{aligned} \tag{4.8.108}$$

where $2g^2(3-e^{ip}-e^{-ip})$ is the one loop energy extracted from (4.4.66). Using the bulk equation we get

$$\begin{aligned} &2g^4(2-e^{ip}-e^{-ip})\Upsilon = \\ &(10g^4-4g^2)\Psi_0(1)+(2g^2-8g^4)\Psi_0(0)+2g^4\Psi_0(-1) \end{aligned} \tag{4.8.109}$$

Plugging in for Υ and the wave function we get

$$\begin{aligned} &2g^4(2-e^{ip}-e^{-ip})-4g^4(2e^{i2p}-e^{i3p}-e^{ip}) \\ &-(10g^4-4g^2)e^{ip}-(2g^2-8g^4)-2g^4e^{-ip} = \\ &-R'[2g^4(2-e^{ip}-e^{-ip})-4g^4(2^{-i2p}-e^{-ip}-e^{-i3p}) \\ &-(10g^4-4g^2)e^{-ip}-(2g^2-8g^4)-2g^4e^{ip}] \end{aligned} \tag{4.8.110}$$

This in turn implies the weak coupling expansion

$$R' = -\frac{1-2e^{ip}}{1-2e^{-ip}} + 2g^2 \frac{e^{-ip}(e^{ip}-1)^3(e^{ip}+1)(1-4e^{ip}+e^{i2p})}{(e^{ip}-2)^2} \qquad (4.8.111)$$

This is the result (4.4.67).

Chapter 5

Conformal Collider Physics

We study observables in a conformal field theory which are very closely related to the ones used to describe hadronic events at colliders. We focus on the correlation functions of the energies deposited on calorimeters placed at a large distance from the collision. We consider initial states produced by an operator insertion and we study some general properties of the energy correlation functions for conformal field theories. We argue that the small angle singularities of energy correlation functions are controlled by the twist of non-local light-ray operators with a definite spin. For $\mathcal{N} = 1$ superconformal theories the one point function for states created by the R-current or the stress tensor are determined by the two parameters a and c characterizing the conformal anomaly. Demanding that the measured energies are positive we get bounds on a/c. We also give a prescription for computing the energy and charge correlation functions in theories that have a gravity dual. The prescription amounts to probing the falling string state as it crosses the AdS horizon with gravitational shock waves. In the gravity approximation the energy is uniformly distributed on the sphere at infinity, with no fluctuations. We compute the stringy corrections and we show that they lead to small, non-gaussian, fluctuations in the energy distribution. Corrections to the one point functions or antenna patterns are related to higher derivative corrections in the bulk.

The work in this chapter is contained in [140]. This article was coauthored with Juan Maldacena.

5.1 Preliminaries

In this chapter we consider conformal field theories and we study physical processes that are closely related to the ones studied at particle colliders. In some sense we will be studying "conformal collider physics". We consider an external perturbation that is localized in space and time near $t \sim \vec{x} \sim 0$. This external perturbation couples to some operator $\mathcal{O}$ of the conformal field theory and produces a localized excitation in the conformal field theory. This excitation then grows in size and propagates outwards. We want to study the properties of the state that is produced. For this purpose we consider idealized "calorimeters" that measure the total flux of energy per unit angle far away from the region where the localized perturbation was concentrated. As a particular example one could have in mind a real world process $e^+e^- \to \gamma^* \to$ hadrons [1], where we produce hadrons via an intermediate off shell photon. We can treat the process to lowest order in the electromagnetic coupling constant and to all orders in the strong coupling constant. The QCD computation reduces to studying the state created on the QCD vacuum by the electromagnetic current j^μ_{em}. From the point of view of QCD this current is simply a global symmetry. In this case the theory is not conformal, but at high enough energies we can approximate the process as a conformal one to the extent that we can ignore the running of the coupling and the details of the hadronization process. In this Chapter we analyze similar processes but in conformal field theories.

Our goal is to describe features of the produced state. For example, at weak coupling we expect to see a certain number of fairly well defined jets. At strong coupling we expect to see a more spherically symmetric distribution [141], [142], [143]. We need suitably inclusive variables which are IR finite. In QCD this is commonly done using inclusive jet observables [144] , see [145] for a review. Here we study

[1] For early work on the applications of scale invariance to strong interactions and, in particular, e^+e^- collisions.

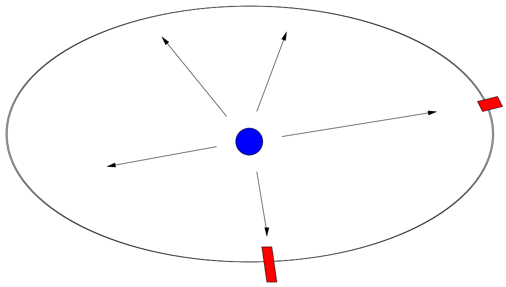


Figure 5.1: A localized excitation is produced in a conformal field theory and its decay products are measured by calorimeters sitting far away.

a particularly simple set of inclusive observables which are the energy correlation functions, originally introduced in [146–148] . They are defined as follows. We place calorimeters at angles $\theta_1, \cdots, \theta_n$ and we measure the total energy per unit angle deposited at each of these angles. We multiply all these energies together and compute the average over all events. These are also inclusive, IR finite observables which one could use to study properties of the produced state. Energy correlation functions for hadronic final states have been measured experimentally and they are one of the ways of making precise determinations of α_s.

A nice feature of energy correlation functions is that they are defined in terms of correlation functions of local gauge invariant operators. They are given in terms of the stress tensor operator [149] . More precisely, consider the expression for the integrated energy flux per unit angle at a large sphere of radius r

$$\mathcal{E}(\theta) = \lim_{r \to \infty} r^2 \int_{-\infty}^{\infty} dt \, n^i T^0_{\ i}(t, r\vec{n}^i) \tag{5.1.1}$$

where n^i is a unit vector in R^3 and it specifies the point on the S^2 at infinity where we have our "calorimeter". If we integrate this quantity over all angles we get the total energy flux which is equal to the energy deposited by the operator insertion. Energy correlation functions are defined as the quantum expectation value of a product of energy flux operators on the state produced by the localized operator insertion

$$\langle \mathcal{E}(\theta_1) \cdots \mathcal{E}(\theta_n) \rangle \equiv \frac{\langle 0 | \mathcal{O}^\dagger \mathcal{E}(\theta_1) \cdots \mathcal{E}(\theta_n) \mathcal{O} | 0 \rangle}{\langle 0 | \mathcal{O}^\dagger \mathcal{O} | 0 \rangle} \tag{5.1.2}$$

where $\mathcal{O}$ is the operator creating the localized perturbation. Note that the operators are ordered as written, they are not time ordered. Notice, also, that the expectation values in the left hand side of (5.1.2) are defined on the particular state created by the operator $\mathcal{O}$ and they are *not* vacuum expectation values. The energy operators are very far away from each other and they commute with each other. This will become more clear below when we think of the operators as acting on null outgoing infinity, sometimes called $\mathcal{J}^+$. Of course, we usually think of the energy deposited at various calorimeters as commuting observables, since we measure them simultaneously. Notice that when we compute an n point function we place calorimeters at n points but we also allow energy to go through the regions where we have not placed calorimeters.

We will assume that we have a conformal field theory. There are several motivations for doing so. First, the conformal case is simpler because it has more symmetry and, at the same time, it allows us to consider theories that are strongly coupled. There are some interesting statements that can be made using conformal symmetry. Second, we could have a theory for new physics beyond the Standard Model which is conformal, as in the Randall-Sundrum II [150] or the unparticle [151, 152] scenarios, or approximately conformal, as in the "hidden valley" scenario [34] . One would like to describe the events in these theories. In order for energy correlations to be observable to us we need some way to transfer the energy from the new sector back to the standard model, as in [34] . Depending on the details, this conformal breaking and conversion process might or might not destroy the energy correlations one computes in the conformal theory. We will not discuss this problem here. A similar issue arises in QCD because of hadronization. The final motivation is a more theoretical one, which is to understand better the AdS/CFT correspondence [2], [3], [4]. Energy correlations are natural observables on the field theory side which one would like to understand using gravity and string theory in AdS. We will see that on the gravity side, energy correlations translate into the probing of a string state, created by the localized perturbation, with a gravitational shock wave as it falls into the AdS horizon. Thus, the problem becomes a high energy scattering calculation in the bulk.

5.2 Energy correlations in conformal field theories

In this section we study energy correlation functions in general conformal field theories. The discussion in this section is valid for any value of the coupling.

5.2.1 Energy correlations in various coordinates systems

The goal of this subsection is to think about energy correlations in various coordinate systems in order to make manifest its various properties and also in order to simplify later computations.

It is interesting to take a step back and think about the energy density as follows. For any generator, G, of the conformal group there is an associated conformal killing vector ζ_G^μ ($x^\mu \to x^\mu + \zeta_G^\mu$). The associated conserved charge can be written as the integral of a conserved current, constructed by contracting ζ_G^μ with the stress tensor, over a spatial hypersurface

$$Q_G = \int_{\Sigma_3} *_4 j_G \ , \qquad\qquad j_G^\mu \equiv T_{\mu\nu}\zeta_G^\nu \tag{5.2.3}$$

where the normalization of the stress tensor is chosen so that $T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}$. This expression of the charges is covariant under conformal transformations. It is also invariant under Weyl transformations of the four dimensional metric[2] $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$,

[2]We are ignoring the conformal anomaly since it only contributes as a c-number, independent of

$T_{\mu\nu} \to \Omega^{-2} T_{\mu\nu}$.

It is convenient to understand clearly the symmetries of the problem. We are interested in measuring the flux of energy at large distances. Thus, we focus our attention on the boundary of Minkowski space $R^{1,3}$. The conformal generators that leave the boundary fixed are the dilatation and the Poincare generators, including the translations P^μ and the $SO(1,3)$ lorentz transformations. In other words, we have the whole conformal group except the special conformal transformations. In order to see that the large r limit in (5.1.1) is well defined, and also to gain some more insight into the problem, it is convenient to perform a conformal transformation from the original coordinates x^μ to new coordinates y^μ. The new coordinates are such that the future boundary of the original Minkowski space is mapped to the null surface $y^+ = 0$. The explicit change of coordinates is

$$y^+ = -\frac{1}{x^+} \ , \qquad y^- = x^- - \frac{x_1^2 + x_2^2}{x^+} \ , \qquad y^1 = \frac{x^1}{x^+} \ , \qquad y^2 = \frac{x^2}{x^+} \tag{5.2.4}$$

where $y^\pm = y^0 \pm y^3$, and similarly for $x^\pm$. The inverse change of coordinates is given by the same expressions with $x \leftrightarrow y$. The advantage of the new coordinates is that now the energy is expressed in terms of an integral over the surface at $y^+ = 0$ and we do not have to take any limit, such as the large r limit in (5.1.1) . Actually, to be more precise, the surface $y^+ = 0$ corresponds to the future lightlike boundary of Minkowski space. The energy correlation function (5.1.1) involves an integral over the past and the future boundaries of Minkowski space. However, in the physical situation we are interested in, where we have the vacuum in the past, there is no contribution from the past light-like boundary and we can focus only on the future boundary. Of course, one could also directly define the energy flux operator in terms of an integral over only the future boundary.

In order to switch between different coordinate systems it is convenient to think

the quantum state of the field theory.

about $R^{1,3}$ as follows. We introduce the six coordinates Z^M subject to the identification $Z^M \sim \lambda Z^M$ and the constraint[3]

$$-(Z^{-1})^2 - (Z^0)^2 + (Z^1)^2 + (Z^2)^2 + (Z^3)^2 + (Z^4)^2 = 0 \tag{5.2.5}$$

The usual coordinates on $R^{1,3}$ are projective coordinates $x^\mu = \frac{Z^\mu}{Z^{-1}+Z^4}$, $\mu = 0, 1, 2, 3$. The metric induced on this surface, (5.2.5) , by the $R^{2,4}$ metric is fixed up to an overall x-dependent factor. We can choose a metric by choosing a "gauge condition" such as $Z^{-1} + Z^4 = 1$. Different "gauge conditions" lead to metrics that differ by a Weyl rescaling. The coordinates y^μ in (5.2.4) correspond to the choice

$$y^0 = -\frac{Z^{-1}}{Z^0 + Z^3}, \qquad y^3 = -\frac{Z^4}{Z^0 + Z^3}\ , \qquad y^1 = \frac{Z^1}{Z^0 + Z^3}\ , \qquad y^2 = \frac{Z^2}{Z^0 + Z^3} \tag{5.2.6}$$

In fact, using (5.2.6) and (5.2.4) we can easily go between the two sets of coordinates. We have $dx^\mu dx_\mu = \frac{dy^\mu dy_\mu}{(y^+)^2}$. We also clearly see that (5.2.4) amounts to a $\frac{\pi}{2}$ rotation in the [-1,0] plane and in the [4,3] plane of $R^{2,4}$, which is an element of the conformal group. The boundary of Minkowski space is the null surface given by $Z^{-1} + Z^4 = 0$. We can think of the various generators of the conformal group as the antisymmetric matrices $M^{[MN]}$ which generate the transformations $\delta Z^N = M^{[NM]} Z_M$. Defining $Z^\pm = Z^{-1} \pm Z^4$, we can see that all the generators that leave the surface $Z^+ = 0$ invariant are all the ones with no $+$ index plus the generator $M^{[+-]}$. In this language the four momentum generators in the x coordinates correspond to $M^{[-\mu]}$, $\mu = 0, 1, 2, 3, 4$. These generators have a particularly simple form at $Z^+ = 0$

$$P_\mu \sim Z_\mu \frac{\partial}{\partial Z^-} - Z_- \frac{\partial}{\partial Z^\mu} \quad \longrightarrow \quad P_\mu|_{Z^+=0} \sim Z_\mu \frac{\partial}{\partial Z^-} \tag{5.2.7}$$

(note that $Z_- = -Z^+/2$). Since the Killing vectors are all proportional to each other,

[3]Note that Z^{-1} is the "minus one" component of the vector Z and it does not denote the inverse of Z. Hopefully, this notation will not cause confusion.

then all four generators involve a single component of the stress energy tensor. Using (5.2.3) , (5.2.6) and (5.2.7) we can write

$$\begin{aligned} P^0_x + P^3_x =& \int dy_1 dy_2\, \mathcal{E}(y_1,y_2) \\ P^0_x - P^3_x =& \int dy_1 dy_2\, (y_1^2+y_2^2)\mathcal{E}(y_1,y_2) \\ P^1_x =& \int dy_1 dy_2\, y^1 \mathcal{E}(y_1,y_2) \\ P^2_x =& \int dy_1 dy_2\, y^2 \mathcal{E}(y_1,y_2) \\ \mathcal{E}(y_1,y_2) \equiv& 2\int_{-\infty}^{\infty} dy^- T_{--}(y^-, y^+=0, y^1, y^2) \end{aligned} \tag{5.2.8}$$

We see that they are all determined by T_{--} thanks to the simple form of the generators at $Z^+ = 0$ (5.2.7) . The conclusion is that we are computing correlation functions of T_{--} and these determine all the components of the energy and the momentum. These expression have the advantage that no limit is involved but they have the disadvantage that the $SO(3)$ rotation symmetry is not manifest. Since no limit is involved, it is clear that the expectation values of (5.2.8) will be finite. In fact, we are considering an external operator insertion which is localized in x space. This implies, in particular, that it is localized near $x^+ \sim 0$ so that it is far enough from $y^+ = 0$ which is the point where we insert the operators (5.2.8) .

We should note that the dilatation symmetry of the original coordinates $x^\mu \to \lambda x^\mu$ becomes a boost in the y^+, y^- plane in the y coordinates (5.2.4) . Similarly the dilatation transformation in the y variables becomes a boost in the $x^\pm$ plane.

An alternative point of view is the following. We write the original coordinates as

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2 = r^2 \left[\frac{-dt^2 + dr^2}{r^2} + d\Omega_2^2 \right] \tag{5.2.9}$$

The original metric and the bracketed metric in (5.2.9) differ by a Weyl transformation, but such a transformation leaves the physics of the CFT invariant. So we can

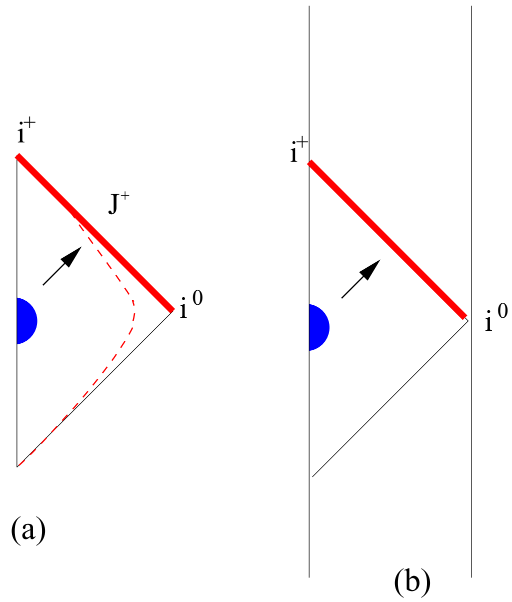


Figure 5.2: (a) Penrose diagram of flat Minkowski space. The doted line is a surface at constant r where we measure the energy flux. In the large r limit this becomes the light-like boundary, $\mathcal{J}^+$, of Minkowski space. We consider only the future part of the boundary. The semicircle represents a localized operator insertion. In (b) we extend the coordinates to the conformal completion of Minkowski space, which gives us $S^3 \times R$. The future boundary of the original space is simply the light-cone of the point at spatial infinity, i^0.

view our CFT as defined on an extremal black hole: $AdS^2 \times S^2$. Then, the boundary of Minkowski space corresponds to the black hole horizon situated at $t, r = \infty$. We see that we can view our measurement as one done at the horizon of an extremal black hole. (Of course we can also consider other coordinates related by Weyl transformations which would suggest other pictures.) By introducing new coordinates we can write the AdS_2 metric in (5.2.9) as

$$ds^2 = \frac{-dt^2 + dr^2}{r^2} = \frac{-d\tau^2 + d\sigma^2}{\sin^2 \sigma} \ , \qquad t = \frac{\sin \tau}{\cos \tau + \cos \sigma} \ , \qquad r = \frac{\sin \sigma}{\cos \tau + \cos \sigma} \tag{5.2.10}$$

The horizon is at $\tau^+ \equiv \tau + \sigma = \pi$. We also define $\tau^- = \tau - \sigma$. We can then write the generators (5.2.8) as

$$
\begin{aligned}
P^0 &= \int d\Omega_2\, \mathcal{E}(\vec{n}) \\
P^i &= \int d\Omega_2\, n^i\, \mathcal{E}(\vec{n}) \\
\mathcal{E}(\vec{n}) &\equiv 2 \int_{\tau^+=\pi} d\tau^- \left(\cos\frac{\tau^-}{2}\right)^2 T_{\tau^-\tau^-}
\end{aligned}
\tag{5.2.11}
$$

where n^i is a unit vector in R^3 and specifies a point on S^2. In these coordinates the $SO(3)$ rotation symmetry is manifest. The fact that the energy flux and the momentum flux is related to the same operator, T_{--}, is indeed what we would naively expect in a theory of massless particles. Namely, if at some point of the sphere we have energy $\mathcal{E}(\theta)$ then we have momentum $P^i = n^i\mathcal{E}(\theta)$. Here we have shown that this also holds for a general interacting CFT. This is due to the simple form of the Killing vector (5.2.7) at $Z^+ = 0$.

Note also that the SO(1,3) Lorentz symmetry acts on the 2-sphere as the $SL(2, C)$ group of conformal transformations of S^2. Our problem however, does not reduce to computing correlators in a 2d CFT, since the state we are considering breaks the $SL(2, C)$ invariance. Under these transformations the operator $\mathcal{E}$ transforms as a dimension three operator. The easiest way to see this it to recall that these $SL(2, C)$ transformations are the ordinary Lorentz transformations of the original coordinates. In particular we have seen that $x^\pm$ boosts become dilatation operators in the y variables. In those variables it is clear that $\int dy^- T_{--}$ has dimension three. In particular, one can find the relation between the operator $\mathcal{E}(y^1, y^2)$ which is defined on a plane to the one on the sphere, $\mathcal{E}(\vec{n})$, by following the coordinate transformation between the plane and the sphere at $Z^+ = y^+ = 0$

$$\begin{aligned} y_1 + iy_2 =& \frac{\sin\theta e^{i\varphi}}{(1+\cos\theta)} = \tan\frac{\theta}{2} e^{i\varphi} \\ dy_1^2 + dy_2^2 =& \frac{d\theta^2 + \sin^2\theta d\varphi^2}{(1+\cos\theta)^2} \equiv \Omega^2 ds^2_{S^2} \\ \mathcal{E}(y_1, y_2) =& \Omega^{-3}\mathcal{E}(\vec{n}) = (1+\cos\theta)^3 \mathcal{E}(\vec{n}) \end{aligned} \tag{5.2.12}$$

Physically, we expect that our idealized calorimeters will measure positive energies. Therefore, the expectation values of $\mathcal{E}(\vec{n})$ should be non-negative. In quantum field theory the expectation value of the stress tensor can be negative in some spacetime region. However, in our case we are integrating the stress tensor along a light like direction. In a free field theory one can show that the expectation value

$$\int dy^- \langle T_{--} \rangle \geq 0 \tag{5.2.13}$$

is positive on any state [153] [4]. We expect that the same should be true in an interacting field theory. In appendix A we recall the argument in free field theories and give a handwaving argument suggesting that this should be true in general. We will later see that this condition implies interesting constraints on certain field theory quantities, so it would be nice to be able to give a more solid argument for the positivity of (5.2.13) than the one we give in the appendix.

Notice that the energy flux operators $\mathcal{E}(\theta)$ commute with each other since operators at different values of θ are separated by spacelike distances. This is most clear when we express the operators in terms of the y coordinates as in (5.2.8) . Thus, we can certainly consider the probability that we measure specific energy functions $\mathcal{E}(\theta) = f(\theta)$ and derive the probability functional that governs the process. Once can also impose some cuts on the energy distribution and compute such probabilities.

[4]The curved space analog of this condition has also been explored for free fields in curved space, since it plays a role in proving singularity theorems in general relativity.

This is done when jet cross sections are computed, as in [144] . In fact, a specific Feynman diagram with n particles coming out at angles $\theta_1, \cdots, \theta_n$ gives a contribution to the case where the energy function $f(\theta)$ is a delta function localized at these points. The energy correlation functions we have defined correspond to average energies where we also allow extra particles that come out and do not go into the calorimeters we are choosing to focus on.

Besides putting a detector at infinity that measures energy we can also put a detector that measures charge. In that case we have the charge flux operator

$$\mathcal{Q}(\vec{n}) = \lim_{r\to\infty} r^2 \int_{-\infty}^{\infty} dt\, n^i j_i(t, r\vec{n}) \tag{5.2.14}$$

where j is the current associated to a global $U(1)$ symmetry of the field theory. In the coordinates (5.2.4) this becomes $\mathcal{Q}(y^1, y^2) = \int dy^- j_-(y^-, y^+ = 0, y^1, y^2))$. Under $SL(2, C)$ transformations $\mathcal{Q}$ transforms as a field of conformal dimension two. We can similarly compute energy and charge correlation functions. One can also easily consider non-abelian global symmetries, and measure the components of various charges, as long as we do not put two charge insertions at the same point.

Now let us make some remarks on the operator ordering. Since the energy flux operators commute with each other for different θ, then, it does not matter how they are ordered. However, it is important that they are inserted between the operator, $\mathcal{O}$, that creates the state and the one annihilating it, as in (5.1.2) . This is the standard ordering when we compute expectation values. If we use perturbation theory to compute them it is important that we do *not* use Feynman propagators since those are for time ordered situations. However, to do perturbation theory it is very convenient to use Fenymann propagators. In such a case we have to be careful to remember that we should use the in-in [154], [155, 156] formalism to evaluate the expectation value. This consists in choosing a contour that starts with the initial state, goes forward in time to the times where the stress tensor operators are evaluated and then goes backwards in time.

In a conformal field theory we could also consider the following. Minkwoski space can be mapped to a finite region of $R \times S^3$. In fact, $R \times S^3$ can be split into an infinite number of regions, each of which is mapped to Minkowski space. In that case we can consider one of the regions as the original Minkowski space and the region immediately to the future as the region parametrizing the part of the Schwinger-Keldysh contour that goes back in time, as long as we transform the wavefunction of the in state in the bra appropriately. We found this picture useful for gaining intuition, but not particularly useful for doing computations.

5.2.2 Small angle singularities and the operator product expansion

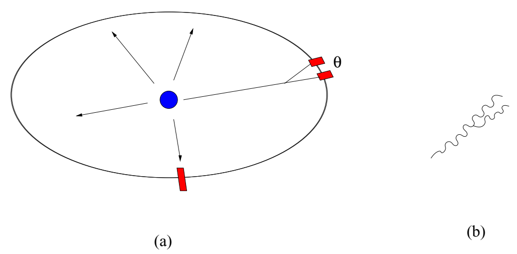


Figure 5.3: (a) Singularities in the energy correlation functions arise when we place two calorimeters very close to each other, at a small angle θ. (b) At the level of Feynman diagrams such singularities come from colinear radiation.

The energy correlation functions develop singularities when two of the energy operators are evaluated at very similar angles $\theta_1 \sim \theta_2$, see figure 5.3a. Such singularities are related to collinear radiation. To leading order in the gauge theory coupling $\lambda = 4\pi\alpha_s N$ the leading singularity goes like $\mathcal{E}(\theta_1)\mathcal{E}(\theta_2) \sim \frac{C\lambda}{\theta_{12}^2}$ [146–148] and it comes from a Feynman diagram like the one shown in figure 5.3b.

It is clear that such a limit should be characterized by some sort of operator product expansion. In this section we will make some remarks on the type of operators that appear in this expansion.

It is simpler to think about the problem in the y^μ coordinates introduced in (5.2.4) . We should, then, compute the OPE of operators of the form

$$\mathcal{E}(y^1, y^2)\mathcal{E}(0,0) \sim \int dy^- T_{--}(y^-, y^+ = 0, \vec{y}) \int dy'^- T_{--}(y'^-, y^+ = 0, \vec{0}) \qquad (5.2.15)$$

The two operators are sitting at two different points in the transverse directions. We have set one at zero for convenience and the other at $\vec{y} = (y_1, y_2)$. Note that the distance between the two stress tensor insertions is $|\vec{y}|$ irrespective of the values of y^-, y'^-. This distance is spacelike, so one expects to be able to perform an operator product expansion when $\vec{y} \to \vec{0}$. Nevertheless, since the two stress tensors are sitting at two very different points in the y^- directions, the operators appearing in the OPE are not local operators. To leading order the operator is specified by two points that are light-like separated [157,158] . Such operators are useful for thinking about many high energy processes in QCD [159–161], [162,163], [164]. They are sometimes called "string operators" or "light ray" operators. Various "parton distribution" functions are defined in terms of matrix elements of such operators, see [162,163] for example. It is important to note that these operators are non-local along one light-like direction but they are perfectly local in all remaining three directions.

In order to characterize these non-local operators it is useful to label them according to their transformation properties under the conformal group [165] (see [166] for a review). Let us define the twist generator to be $T = \Delta - j$, where j is the spin (really a boost generator) in the y^+, y^- plane[5]. More explicitly, the twist transformation is $(y^+, y^-, \vec{y}) \to (\lambda^2 y^+, y^-, \lambda\vec{y})$. The spin is the transformation $(y^+, y^-, \vec{y}) \to (\eta y^+, \eta^{-1} y^-, \vec{y})$. As it is well known, at zero coupling, one can consider twist two

[5]Note that we define j to be the spin in the y^+, y^- plane only, not the total spin. The spin in the transverse directions is another generator which does not appear in the definition of the twist.

operators which correspond to primary operators of higher spin. For example, if we have a scalar field, ϕ, in the adjoint representation, then we can schematically define the operators

$$\mathcal{U}_j = Tr[\phi \overleftrightarrow{\partial}^j_- \phi] \tag{5.2.16}$$

This is schematic because there is a precise combination of derivatives that makes it a conformal primary[6]. Such conformal primaries exist only if j is even. One is sometimes interested in extending the definition of such operators to generic, real or complex, values of j. This problem was considered in detail in [165] . There, it was found that one could start with the operators

$$\mathcal{U}(y^-, y'^-) = Tr[\phi(y^-)W(y^-, y'^-)\phi(y'^-)] = Tr[\phi(y^-)Pe^{\int_{y^-}^{y'^-} A}\phi(y'^-)] \tag{5.2.17}$$

where W is an adjoint Wilson line along a null direction. All operators are inserted at the same values of y^+, y^1 and y^2 (but of course, at different values of y^-). We can also replace ϕ by a fermion or a gluon operator F_{-i}. Under twist transformations y^- remains invariant but the transverse coordinates are rescaled. In the quantum theory this scaling transformation mixes the operator (5.2.17) with operators with other values of y^-, y'^-. By thinking about the action of the collinear conformal group (the $SL(2, R)$ set of transformations of x^-) it is possible to diagonalize the action of the twist generator. To leading order the operators are diagonalized by considering suitable combinations of these light-ray operators [165], [166]. These operators are labeled by their center of mass momentum k_- along the y^- direction and their spin. For our purposes we will be interested only in operators which are integrated over the center of mass position along the y^- direction so that they carry zero momentum along y^-. In that case the operators of arbitrary spin constructed from scalar fields

[6]The precise form is $\mathcal{U}_j = \sum_{k=0}^{j} \frac{(-1)^k}{[k!(j-k)!]^2} Tr[(\partial_-^k \phi)\partial_-^{j-k}\phi]$, where ϕ is a scalar field [167], [168], [169].

can be written as

$$\mathcal{U}_{j-1} = \int_{-\infty}^{\infty} dy^- \int_0^\infty \frac{du}{u^{j+1}} Tr[\phi(y^- + u)W(y^- + u, y^- - u)\phi(y^- - u)] \qquad (5.2.18)$$

The subindex of $\mathcal{U}$ denotes the total spin and j denotes the spin before we do the y^- integration. This is an expression that makes sense for arbitrary complex values of j. When j approaches an even integer we find a pole in j coming from a logarithmic divergence in the integral at small u of the form $\int \frac{du}{u}$. The coefficient of this divergent term contains the ordinary local operator (5.2.16) , see [165], [166] for more details. There are similar expressions for operators constructed from two fermions or two Yang-Mills field strengths. One can compute the value of the twist for these operators and one finds [165] $\tau(j) = 2 + \gamma(j)$, where $\gamma(j)$ is the anomalous dimension. One can also consider higher twist operators which contain more field insertions or extra derivatives with respect to the transverse direction or y^+. In that case, in order to diagonalize the matrix of anomalous dimensions, it is not enough to give the total spin of the operator. Nevertheless, this can be done, see [166] .

The OPE has the schematic form

$$\mathcal{E}(\vec{y})\mathcal{E}(\vec{0}) \sim \int dy^- T_{--}(y^-, \vec{y}) \int dy'^- T_{--}(y'^-, 0) \sim \sum_n |\vec{y}|^{\tau_n - 4} \, \mathcal{U}_{j-1,n}\big|_{j=3} \qquad (5.2.19)$$

where the sum is over all operators which are local in $y^+, \vec{y}$, but not necessarily local in y^-, which have total spin $j-1 = 2$, (or $j = 3$) and twist τ_n. The spin is determined since the total spin of the left hand side is one for each of the two energy insertions. Equation (5.2.19) is schematic because we have not explicitly indicated the fact that the operators in the right hand side could carry spin in the transverse directions. A more precise expression has the form

$$\mathcal{E}(\vec{y})\mathcal{E}(\vec{0}) \sim\sim \sum_{k,n} y^{(i_1} \cdots y^{i_k)} |y|^{\tau_{n,k}-k-4}\, \mathcal{U}_{(i_1\cdots i_k);j-1;n}\Big|_{j=3} \tag{5.2.20}$$

where we have now considered operators that carry spin in the transverse directions, the indices $i_1, \cdots i_k$ are symmetric and traceless.

Among the operators which have twist two at zeroth order there are only a few that have $j = 3$. For example, in QCD there are only two, a bilinear in fermions and a bilinear in the gluon field strength. Thus, for the given spin we are considering ($j = 3$) we will have to diagonalize a finite matrix of anomalous dimensions.

In summary: *The small angle behavior of the energy correlation functions is determined by the spin $j = 3$ non-local operators that appear in the OPE*

$$\langle \mathcal{E}(\theta_1)\mathcal{E}(\theta_2)\cdots\rangle \sim \sum_n |\theta_{12}|^{\tau_n - 4} \langle \mathcal{U}_{3-1,n}(\theta_2)\cdots\rangle \tag{5.2.21}$$

where the dots denote other energy insertions and $|\theta|$ is the angle between the two energy insertions that are getting close to each other. The sum over n runs over all the higher twist operators that can appear. We will see that in $\mathcal{N} = 4$ super Yang Mills these operators develop large anomalous dimensions at strong coupling.

Note that the spin symmetry in the y^+, y^- plane, that we used to select the operators that contribute, is the dilatation symmetry of the original Minkowski space. This symmetry ensures that the energy correlation functions scale properly as we rescale the total energy (or rescale the variables x^μ). In other words, there can be no anomalous dimensions under total energy rescalings since that would conflict with energy conservation. This is the physical reason why we are forced to select particular operators in this OPE.

In the case of QCD the small angle behavior of energy correlation functions was computed a long time ago in [170], [171] using a slightly different language. They also needed to include the effects of the running coupling.

Let us now turn to the case of $\mathcal{N} = 4$ super Yang Mills at weak coupling. The weak coupling computation of the leading twist anomalous dimensions was done in [172], [173–175] (see also [25, 84], [176]). We should consider operators which are invariant under all the symmetries that leave the particular component of the stress tensor in (5.2.20) invariant. These include the $SO(6)$ R-symmetry and a parity symmetry. We can classify the operators according to their transformation properties under the $SO(2)$ group that transforms the transverse coordinates. All operators are made out of a pair of scalars, fermions or gauge field strengths. The local operators with zero transverse spin in $SO(2)$ and spin j (j even) in the $+-$ directions are [172]

$$Tr[\phi \overleftrightarrow{\partial}_-^j \phi] \ , \qquad Tr[F_{-i} \overleftrightarrow{\partial}_-^{j-2} F_{-i}] \ , \qquad Tr[\psi \Gamma_- \overleftrightarrow{\partial}_-^{j-1} \psi] \tag{5.2.22}$$

Supersymmetry relates these three towers of operators. Since supersymmetry carries spin, the various members of the supermultiplet have different spin. However, the anomalous dimension for all the members of the supermultiplet is the same and it is given by a function which has the weak coupling expansion [172], [177]

$$\gamma(j) = \frac{\lambda}{2\pi^2}[\psi(j-1) - \psi(1)] + \cdots \tag{5.2.23}$$

where $\psi = \Gamma'(z)/\Gamma(z)$. This was computed also to two and three loops in [173–175] . The fact that $\gamma(j = 2) = 0$ corresponds to the fact that the stress tensor is not renormalized.

Since we are interested in operators with a definite spin, we conclude that the three operators that diagonalize the anomalous dimension matrix are in three different multiplets. For spin three operators we have the anomalous dimensions [172]

$$\begin{aligned} \tau_1 - 2 =&\gamma(j=3)\ , &\qquad \tau_2 - 2 &= \gamma(j=5)\ , &\qquad \tau_3 - 2 &= \gamma(j=7) \\ \tau_1 - 2 =&\frac{\lambda}{2\pi^2}\ , &\qquad \tau_2 - 2 &= \frac{11\lambda}{12\pi^2}\ , &\qquad \tau_3 - 2 &= \frac{137\lambda}{120\pi^2} \end{aligned} \tag{5.2.24}$$

where we just gave the first order expression. We see from (5.2.24) and (5.2.23) that all three anomalous dimensions in (5.2.24) are positive and $\tau_1 - 2$ is the smallest one which will give us the leading order singularity. However, for weak coupling all three contributions are similar.

In addition to the operators we discussed, we can also have operators which have non-zero transverse spin. At twist two, the only one consistent with the symmetries is the spin two operator

$$\mathcal{U}_{(il);j} = Tr[F_{-(i} \overleftrightarrow{\partial}_{-}^{j-2} F_{-l)}] \tag{5.2.25}$$

where the indices $i, l = 1, 2$ are symmetrized and traceless. In the $\mathcal{N} = 4$ theory these operators are in the same supermultiplet as the ones considered above [39] . For this reason their anomalous dimension is also given in terms of the same formula

$$\tilde{\tau}_j - 2 = \gamma(j+2)\ , \qquad \tilde{\tau}_3 = 2 + \frac{11\lambda}{12\pi^2} \tag{5.2.26}$$

Thus we expect to have a small angle singularity of the form

$$\langle \mathcal{E}(\vec{y})\mathcal{E}(0)\cdots\rangle \sim \sum_{a=1}^{3} |y|^{-2+(\tau_a-2)} c_a \langle \mathcal{U}_a \cdots\rangle + y^{(i}y^{l)}|y|^{-4+(\tilde{\tau}_3-2)}\tilde{c}\langle \mathcal{U}_{(il)}\cdots\rangle \tag{5.2.27}$$

where the operators $\mathcal{U}_a$ are the linear combinations that diagonalize the anomalous dimension matrix for the operators with zero transverse spin. c_a and $\tilde{c}$ are coefficients that can be obtained by performing the operator product expansion explicitly. These constants are independent of the state for which we compute the energy correlation. Of course, the terms $\langle \mathcal{U}_a \cdots\rangle$ and $\langle \mathcal{U}_{(il)}\cdots\rangle$ do depend on the state on which we compute the energy correlation function. The coefficients c_a, $\tilde{c}$ start at order λ at weak coupling since it is easy to check explicitly that at tree level there is no contribution to the operator product expansion of two energy flux operators.

In QCD one can do a similar analysis, including the effects of the beta function, see [171] . In that case, the operator made out from scalars in (5.2.22) does not contribute.

Having done one OPE, we could also do a further OPE of the resulting operator with a third energy flux operator. That would give an operator of total spin $j-1=3$, or $j=4$, and so on. More generally, we can consider the case where n energy operators come close together. If we keep the ratios of angles between these n points fixed, then the small angle behavior is given by the anomalous dimension of the operator of spin $j = n+1$. The structure of a jet at weak coupling is largely controlled by these operator product expansions.

Note that, at weak coupling, after we consider the effects of the anomalous dimensions, the energy correlators have small angle singularities that are integrable. In fact, if we do the integral over a small angle θ_0 of the energy two point function we find schematically

$$\int d^2\theta \mathcal{E}(\theta)\mathcal{E}(0) \sim \int_0^{\theta_0} d^2\theta \frac{\lambda}{\theta^{2-\gamma_*\lambda}} \sim (\theta_0)^{\gamma_*\lambda} \tag{5.2.28}$$

where the anomalous twist of the spin three operator is $\tau - 2 = \gamma_*\lambda$ and γ_* is a numerical constant. This expression is schematic because at weak coupling we have to include all the terms in (5.2.27). If θ_0 is fixed and $\lambda \to 0$ then we see that the integral gives a finite answer. This is to be expected since the total integral of one energy insertion over the whole sphere should give the total energy, independently of λ [7]. The fact that we get a finite contribution from this region is consistent with the

[7]Here we are also assuming that the energies are locally positive. For charge correlators we cannot make the same argument because the charge can be positive or negative.

idea that the energy is going out in localized jets. We can also estimate the angular size of jets, by finding a θ_0 in such a way that we get a fixed fraction, f, of the total energy in the jet. This gives an estimate $\theta_0 \sim e^{-c/\lambda}$, where c depends on f. This was originally discussed in [144], see also [171] for a more detailed discussion.

We could also do an OPE of two charge operators, each of which has spin zero, after we integrate the spin one current over y^- (5.2.14). In this case we get operators with total spin $j-1=0$, or $j=1$. Some of these have negative anomalous dimensions. In fact, we expect that charge correlators would be more singular at small angles due to the fact that a gluon can create a pair of oppositely charged particles fairly easily and there is no reason that we couldn't get a divergence when we integrate the charge correlator at small θ.

Finally, we should mention that in QCD the energy-energy correlation two point function was computed for all angles in [146–148], [178, 179], [180], [181].

5.2.3 Energy flux one point functions

In this section we will make some simple and general remarks about the energy flux one point function

$$\langle \mathcal{E}(\vec{n}) \rangle = \frac{\langle 0|\mathcal{O}_q^\dagger \mathcal{E}(\vec{n}) \mathcal{O}_q|0\rangle}{\langle 0|\mathcal{O}_q^\dagger \mathcal{O}_q|0\rangle} \tag{5.2.29}$$

These one point functions are determined up to a few coefficients by Lorentz symmetry, even in non-conformal theories. Here we will consider these in the CFT context in order to make contact with other results in conformal field theories.

The energy flux one point function (5.2.29) amounts to computing a three point function in the CFT. Three point functions in a generic CFT are determined up to a few numbers by conformal symmetry [182], [183].

Let us start with the case that we create the external state with a scalar operator with energy q and zero momentum. Strictly speaking such an operator is not a localized insertion. Thus, more precisely, we will be considering operators of the form

$$\mathcal{O}_q \equiv \int d^4x \mathcal{O}(x) e^{-iqx_0} \exp\{-\frac{x_0^2+x_1^2+x_2^2+x_3^2}{\sigma^2}\} \ , \qquad q\sigma \gg 1 \tag{5.2.30}$$

where the last inequality ensures that the operator is localized, has finite norm and has four momentum approximately $\tilde{q}^\mu = (q,\vec{0})+o(1/\sigma)$. In particular we have $q^0 \sim q$. Once we know this precise form of the operator we see that we can also write it in other coordinate systems by performing the suitable conformal transformation and taking into account the conformal transformation properties of $\mathcal{O}(x)$.

In what follows we will consider field theory states produced by scalar operators, $\mathcal{O} \sim \mathcal{S}$, conserved currents $\mathcal{O} \sim \epsilon_i j_i$, and the stress tensor, $\mathcal{O} \sim \epsilon_{ij} T_{ij}$. In all cases we consider states with essentially zero spatial momentum as in (5.2.30). The case where q^μ is a generic four vector can be obtained by performing a simple boost of the configurations we discuss.

In the case that we insert a scalar operator it is clear by $O(3)$ symmetry that the energy one point function is constant on the two sphere. In addition the integral over the angles should give the total energy. Thus, for a scalar operator we have

$$\langle \mathcal{E}(\vec{n}) \rangle = \frac{q}{4\pi} \tag{5.2.31}$$

Even though we know the answer already, it is possible to do the calculation explicitly by writing down the unique general expression for the three point function of two scalars and the stress tensor [183] . Its normalization is fixed by a Ward identity in terms of the two point function of the two scalars. This Ward identity is another version of the energy conservation argument that we used above. Writing down the three point function and doing the integrals in the limit (5.1.1) we indeed obtain (5.2.31). One has to be careful about the operator ordering. In appendix C we do this explicitly.

We now turn to the case where the external perturbation couples to a conserved current in the CFT. In that case the operator is given by $\mathcal{O}_{\epsilon,q} \sim \epsilon^\mu j_\mu(q)$ where ϵ^μ is a constant polarization vector. Due to the current conservation condition we can identify $\epsilon^\mu \sim \epsilon^\mu + \lambda q^\mu$. So we can choose ϵ to point in the spacelike directions. In this case $O(3)$ symmetry and the energy conservation condition constrain the form of the one point function to

$$\begin{aligned}\langle \mathcal{E}(\vec{n})\rangle =& \frac{\langle 0|(\epsilon^* \cdot j^\dagger)\,\mathcal{E}(\vec{n})\,(j\cdot\epsilon)|0\rangle}{\langle 0|(\epsilon^* \cdot j^\dagger)\,(j\cdot\epsilon)|0\rangle} = \frac{q}{4\pi}\left[1 + a_2\left(\frac{|\vec{\epsilon}\cdot\vec{n}|^2}{|\vec{\epsilon}|^2} - \frac{1}{3}\right)\right] \\ =& \frac{q}{4\pi}\left[1 + a_2(\cos^2\theta - \frac{1}{3})\right]\end{aligned} \tag{5.2.32}$$

where θ is the angle between between the point on the S^2, labeled by n^i, and the direction of the polarization vector ϵ^i.

The fact that we have one free parameter is in agreement with the general analysis of the three point function of two conserved currents and the stress tensor. In fact in [183] it was shown that the three point function is determined by conformal symmetry up to two parameters and one of them is fixed by the Ward identity of the stress tensor.

Note that a_2 in (5.2.32) obeys a constraint that comes from demanding that the expectation value of the energy $\mathcal{E}(\theta)$ is positive, see (5.2.13). This condition leads to the constraint

$$3 \geq a_2 \geq -\frac{3}{2} \tag{5.2.33}$$

This one point function was computed for the electromagnetic current in QCD in [146–148]. To first order in α_s the result is

$$a_2 = -\frac{3}{2} + \frac{9\alpha_s}{2\pi} + \cdots \tag{5.2.34}$$

To the order written in (5.2.34) we can approximate the QCD computation by a conformal field theory with the value of the coupling set by the energy of the process

$\alpha_s = \alpha_s(|q^0|)$.

In the application to e^+e^- collisions that produce a gauge boson which in turns couples to a current the polarization vector of the current depends on the polarization states of the e^+ and e^- as well as the type of gauge boson we are considering (γ or Z, Z', etc). In the case that we consider unpolarized electrons we can express the answer in terms of the angle with respect to the beam axis, θ_b. ($\cos\theta_b = \vec{n}.\hat{z}$ where $\hat{z}$ is the beam axis). The polarization vectors for the current are orthogonal to the beam direction and we should average over them. After doing this average, we find that (5.2.32) becomes

$$\langle \mathcal{E}(\vec{n}) \rangle = \frac{\sum_s \langle 0 | (\epsilon_s^* \cdot j^\dagger) \, \mathcal{E}(\vec{n}) \, (j \cdot \epsilon_s) | 0 \rangle}{\sum_s \langle 0 | (\epsilon_s^* \cdot j^\dagger) \, (j \cdot \epsilon_s) | 0 \rangle} = \frac{q}{4\pi} \left[1 + a_2 (\frac{1}{2} \sin^2\theta_b - \frac{1}{3}) \right] \tag{5.2.35}$$

where we sum over polarization vectors transverse to the beam. For a current that couples to free fermions we find the familiar $(1+\cos^2\theta_b)$ distribution, as we can check from the leading order QCD result (5.2.34).

For a current that couples to free complex bosons of charges q_i^b and Weyl fermions of charges q_i^{wf} we get

$$a_2^{free} = 3 \frac{\sum_i (q_i^b)^2 - (q_i^{wf})^2}{\sum_i (q_i^b)^2 + 2(q_i^{wf})^2} \tag{5.2.36}$$

where we sum over both left and right Weyl fermions. Note that the case where we only have bosons saturates the upper bound in (5.2.33) and free fermions saturate the lower bound in (5.2.33). In fact, going back to (5.2.35) we see that we get the well known distributions proportional to $\sin^2\theta_b$ or $(1+\cos^2\theta_b)$ for free bosons and fermions respectively.

We can consider a similar problem now in an $\mathcal{N}=1$ superconformal theory. If the current is a global symmetry that commutes with supersymmetry (a non-R symmetry) then one can see that $a_2 = 0$. In a free supersymmetric theory we see from (5.2.36) that the bosons and Weyl fermions cancel each other. For an interacting theory this follows from the fact that such a current is in the same multiplet as a scalar operator, and for a scalar operator we do not have any arbitrary parameters [184] . Thus the value of a_2 is fixed by superconformal symmetry. However, since we got $a_2 = 0$ in a free theory, we have $a_2 = 0$ for any global symmetry of a SCFT.

On the other hand we can get a non-zero value of a_2 if we consider the R current[8]. The R current is in a different supermultiplet. In fact, it is in a supermultiplet with the stress tensor. All three point functions among elements of this supermultiplet are determined by two numbers, c and a [184] . These numbers also characterize the anomalies of the R current, which are encoded in parts of the jjj and jTT three point functions [185, 186], [184]. They also contribute to the conformal anomaly on a general background,

$$T^\mu_\mu = \frac{c}{16\pi^2} W_{\mu\nu\delta\sigma} W^{\mu\nu\delta\sigma} - \frac{a}{16\pi^2} E \ , \qquad E = R_{\mu\nu\delta\rho} R^{\mu\nu\delta\rho} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \quad (5.2.37)$$

where W is the Weyl tensor and E the Euler density. The c coefficient is the only constant that appears in the two point functions of the currents and the stress tensor [184]. Thus c appears in the part of the three point function that is fixed by the Ward identity. The coefficient a_2 is given by a linear combination of a and c. The particular linear combination is independent of the theory. It is fixed by supersymmetry. We can compute the precise combination by considering the particular case of free field theories. As we explain in more detail below we find that

$$\langle \mathcal{E}(\theta) \rangle = 1 + 3 \frac{c-a}{c} (\cos^2\theta - \frac{1}{3}) \qquad (5.2.38)$$

This formula was obtained as follows. We used that for a free supersymmetric theory with n_V vector multiplets and n_S chiral multiplets we have $48a = 9n_V + n_S$, and $24c = 3n_V + n_S$ [185, 186], [187], [188]. The vector multiplet has one Weyl fermion of

[8]We thank Scott Thomas for pointing this out.

charge 1 and the scalar multiplet in a free theory has a Weyl fermion of charge $-1/3$ and a boson of charge $2/3$. Then using (5.2.36) we obtain (5.2.38) . Note that even though we used free field theories to fix the numerical coefficients, the final result (5.2.38) is true for a general interacting $\mathcal{N} = 1$ SCFT.

In $\mathcal{N} = 4$ super Yang Mills $a = c$ and the result for the one point function is spherically symmetric. Of course this is not a surprise since U(1) subgroups of the SO(6) symmetry group can also be viewed as global symmetries from the point of view of $\mathcal{N} = 4$ written as an $\mathcal{N} = 1$ theory. Thus, in $\mathcal{N} = 4$ super Yang mills the result is independent of the coupling.

The positivity constraint (5.2.33) , together with (5.2.38) gives $\frac{3c}{2} \geq a \geq 0$.

We can also consider the energy one point function in the case that the state is created by the stress tensor. As above, we take the momentum of the inserted operator in the time like direction. Then the operator that we are considering is characterized by a symmetric polarization tensor ϵ^{ij} which we take to have indices in the purely spacelike directions by using the conservation equations. Since the stress energy tensor is traceless, we also take $\epsilon^{ii} = 0$. By O(3) invariance we see that the most general form of the three point function is

$$\langle \mathcal{E}(\theta) \rangle = \frac{\langle 0 | \epsilon^*_{ij} T_{ij} \mathcal{E}(\theta) \epsilon_{lk} T_{lk} | 0 \rangle}{\langle 0 | \epsilon^*_{ij} T_{ij} \epsilon_{lk} T_{lk} | 0 \rangle} = \frac{q^0}{4\pi} \left[1 + t_2 \left(\frac{\epsilon^*_{ij} \epsilon_{il} n_i n_j}{\epsilon^*_{ij} \epsilon_{ij}} - \frac{1}{3} \right) + t_4 \left(\frac{|\epsilon_{ij} n_i n_j|^2}{\epsilon^*_{ij} \epsilon_{ij}} - \frac{2}{15} \right) \right] \tag{5.2.39}$$

We see that we have two undetermined coefficients. This agrees with the general analysis of stress tensor three point functions in [183] , where they found that conformal symmetry determines the three point function of the stress tensor up to three coefficients, one of which is fixed by a Ward identity. We have chosen the constants in the last two terms in (5.2.39) in such a way that the corresponding terms integrate to zero on the sphere.

By demanding that $\langle \mathcal{E} \rangle$ is positive we get the constraints

$$\begin{aligned}
&(1 - \frac{t_2}{3} - \frac{2t_4}{15}) \geq 0 \\
&2(1 - \frac{t_2}{3} - \frac{2t_4}{15}) + t_2 \geq 0 \\
&\frac{3}{2}(1 - \frac{t_2}{3} - \frac{2t_4}{15}) + t_2 + t_4 \geq 0
\end{aligned} \tag{5.2.40}$$

We obtain these constraints by using $O(3)$ invariance in (5.2.39) to set $\vec{n} = \hat{z}$. We then view the resulting equation as a bilinear form on the space of ϵ's. This space can be divided into three orthogonal parts according to their $O(2)$ invariance properties (the spin of ϵ along the $\hat{z}$ axis). We have an $SO(2)$ scalar, a vector and a symmetric traceless tensor. On each of these subspaces we get each of the constraints (5.2.40). Each of this limits in saturated in a free theory with no vectors, no fermions or no bosons respectively. The fact that the first equation is saturated in a theory without vectors is clear. In that case, if we consider a stress tensor insertion with spin +2 in the $\hat{z}$ direction we cannot have emission of bosons or fermions in the $\hat{z}$ direction due to the orbital angular momentum wavefunctions. It is also possible to write a general bound on the two coefficients that appear in the conformal anomaly (5.2.37) , see appendix C.

In an $\mathcal{N} = 1$ supersymmetric theory we find that

$$t_2 = 6(c - a)/c \ , \qquad t_4 = 0 \tag{5.2.41}$$

By requiring that (5.2.39) is positive for all choices of traceless ϵ_{ij} we find

$$\frac{3}{2}c \geq a \geq \frac{c}{2} \tag{5.2.42}$$

Of course $c > 0$ due to the positivity of the two point functions. The bounds are saturated by free theories with only vector supermultiplets (upper bound) or only chiral supermultiplets (lower bound). It is interesting that the lower bound that we obtain in this way is precisely the same as the bound obtained in [189] (see also [190]) based on causality for a gravity theory that contains only the Einstein term and a R^2 term[9]. In the theory considered in [189] one would also have $t_4 = 0$, though it is not clear whether it corresponds to any dual quantum field theory. Here we have only used general field theory considerations.

For a non-supersymmetric theory it is also possible to derive a bound from (5.2.40) . As explained in appendix C we find

$$\frac{31}{18} \geq \frac{a}{c} \geq \frac{1}{3} \tag{5.2.43}$$

where the lower bound is saturated by a free theory with only scalar bosons and the upper bound by a free theory with only vectors. Note that the bound in supersymmetric theories (5.2.42) is more stringent than in non-supersymmetric theories (5.2.43). Let us also add that the results from appendix C also allow us to calculate this bound for $\mathcal{N} = 2$ supersymmetric theories. In this case we can obtain the bound by taking the operator $\mathcal{O}$ to be one of the $SU(2)$ R-symmetry generators and demanding that the energy one point function is positive. The result is in this case

$$\frac{5}{4} \geq \frac{a}{c} \geq \frac{1}{2} \tag{5.2.44}$$

This is a smaller window than for the $\mathcal{N} = 1$ case, as expected. The upper bound corresponds to a free theory with vector supermultiplets only while the lower bound corresponds to a free theory with hypermultiplets only. This agrees with results in [192].

We can make similar remarks for operators that involve charge correlations. For

[9] In order to see this one has to set $\lambda_{GB} \to 9/100$ (the bound in [190]) into the expressions for a and c [191] (see eqn. (5.1) of [190]).

example, we could consider a theory with an $SU(2)$ global symmetry and then select one $U(1) \subset SU(2)$ to form the charge flow operator $\mathcal{Q}$ that we measure at infinity. We could consider a charged state state created by the current $\epsilon \cdot J^+$, where the plus indicates that it carries charge plus one. As in the energy correlations the charge correlations have a form

$$\langle \mathcal{Q}(\vec{n}) \rangle = \frac{\langle 0|\vec{\epsilon}^* \cdot \vec{J}^- \, \mathcal{Q}(\vec{n}) \, \vec{\epsilon} \cdot \vec{J}^+ |0\rangle}{\langle 0|\vec{\epsilon}^* \cdot \vec{J}^- \, \vec{\epsilon} \cdot \vec{J}^+ |0\rangle} = \frac{1}{4\pi} \left[1 + \tilde{a}_2 \left(\frac{|\vec{\epsilon} \cdot \vec{n}|^2}{|\vec{\epsilon}|^2} - \frac{1}{3} \right) \right] \tag{5.2.45}$$

Again, the coefficient $\tilde{a}_2$ is related to the fact that there are two possible (parity preserving) structures for the three point function of three currents [182], [183]. One of them is fixed by the Ward identities in terms of the two point functions, a fact we used in (5.2.45). We note that in a supersymmetric theory where these currents are global symmetries $\tilde{a}_2 = 0$. One can show this as follows. First note from [184] that there is only one parity preserving structure for current three point functions in supersymmetric theories. This means that the value of $\tilde{a}_2$ is fixed by supersymmetry. One can show that is vanishes by computing it in a particular theory, such as a free theory or $\mathcal{N} = 4$ super Yang Mills at strong coupling.

There are some cases where there are parity odd structures that can contribute. Such parity odd parts of three point functions are related to anomalies. For example, in the case of three currents these are related to the usual anomaly [182] . Consider the case that we have an external state produced by a current and we measure a charge one point function distribution. For example, we can consider a $U(1)$ current that has a cubic anomaly. A concrete example is the R current in superconformal theories. We consider a state obtained by acting with this current on the vacuum and we measure the charge flux far away. We find

$$\langle \mathcal{Q}(\vec{n}) \rangle = \frac{\langle 0|\vec{\epsilon}^* \cdot \vec{j}^\dagger \, \mathcal{Q}(\vec{n}) \, \vec{\epsilon} \cdot \vec{j} |0\rangle}{\langle 0|\vec{\epsilon}^* \cdot \vec{j}^\dagger \, \vec{\epsilon} \cdot \vec{j} |0\rangle} = i \, \alpha \, \epsilon_{jlk} \epsilon_j^* \epsilon_l n_k \sim \alpha \cos \chi \tag{5.2.46}$$

Note that this is non-zero only if ϵ is complex. This happens, for example, when we consider a circularly polarized state. Then χ is the angle between the direction of the spin of the current and the calorimeter. This leads to a charge flow asymmetry. Such asymmetries are extensively studied in e^+e^- collisions that produce a Z boson that then decays. Here we are pointing out that the charge flow asymmetries are related to the anomaly. Of course the full electroweak symmetry is not anomalous. But if one focuses only in decays of the Z into leptons, then the fact that the purely leptonic theory is anomalous leads to the charge flow asymmetry. The fact that tree level processes plus unitarity fix the anomaly was pointed out in [193].

The three point function of two stress tensors and a current has a term that reflects the mixed gravitational anomaly [194] . Consider the case where the stress tensor creates the state and we measure the charge distribution of the current that has a mixed anomaly. The charge distribution has the structure

$$\langle \mathcal{Q}(\theta) \rangle = \frac{\langle 0 | \epsilon^*_{ij} T_{ij} \mathcal{Q}(\theta) \epsilon_{lk} T_{lk} | 0 \rangle}{\langle 0 | \epsilon^*_{ij} T_{ij} \epsilon_{lk} T_{lk} | 0 \rangle} = i\beta \frac{\epsilon_{ljk} \epsilon^*_{rl} \epsilon_{sj} n^r n^s n^k}{|\epsilon_{ij}|^2} \tag{5.2.47}$$

where β is related to the anomaly coefficient. For a supersymmetric theory, and for $\mathcal{Q}$ given by the R-current, we have that $\beta \sim (a - c)$ [185, 186], [194]. Notice that there is, in principle, another tensor structure consistent with $O(3)$ symmetry that could have contributed to (5.2.47), namely $\epsilon_{ljk} \epsilon^*_{rl} \epsilon_{rj} n^k$. This term is, however, absent from the three point function once conformal symmetry and the Ward identities are imposed [194]. Thus in a theory with a gravitational mixed anomaly there is charge asymmetry for the states produced by the graviton.

5.2.4 Relation to deep inelastic scattering

In this section we explore the relation between the energy correlation functions and the deep inelastic scattering cross sections.

The deep inelastic cross section for the scattering of an electron from a proton can be factorized into the electromagnetic process and the strong interactions process. At lowest order in the electromagnetic coupling, but exactly in α_s, the strong interactions part of the cross section can be written in terms of the expectation value of two currents in the state of the target (which is traditionally a proton, but can be generalized to any other particle)

$$\begin{aligned}\tilde{W}^{\mu\nu} =& \int d^4y e^{iqy} \langle p| J^\mu(0) J^\nu(y) |p\rangle = \\ =&\tilde{F}_1(x, q^2/p^2) \left(g_{\mu\nu} - \frac{q_\nu q_\nu}{q^2} \right) + \frac{2x}{q^2} \tilde{F}_2(x, q^2/p^2) \left(p_\mu + \frac{q_\mu}{2x} \right) \left(p_\nu + \frac{q_\nu}{2x} \right) \\ & \text{where} \qquad x \equiv -\frac{q^2}{2p.q} \ , \qquad\quad q^2 > 0\end{aligned} \tag{5.2.48}$$

We are imagining that we have a plane wave state in the y coordinates with timelike momentum $p^2 < 0$. The tensor (5.2.48) is nonvanishing only if we create a timelike state $s = -(q + p) \geq 0$ with the current. For these values of p and q our definition of $\tilde{W}^{\mu\nu}$, (5.2.48), coincides with the ordinary one [195] , which involves a commutator of the currents.

We would like to relate these formulas to the ones appearing in the energy correlators. Let us consider the charge operator Q evaluated in the y coordinates

$$\mathcal{Q}(\vec{y}) = \int dy^- j_-(y^+ = 0, y^-, \vec{y}) \tag{5.2.49}$$

where $\vec{y}$ denotes two transverse dimensions. These two transverse dimensions are related to the angles on the two sphere by (5.2.12). In the y coordinates, we can Fourier transform this operator. We, then, have something similar to the current appearing above, except that the current in (5.2.48) is in momentum space also in the y^+ direction. Note also that $q_- = 0$, due to the y^- integral in the definition of the charge flux operator $\mathcal{Q}$. Since q_- is zero, we find that $q^2 = (\vec{q})^2 > 0$ and independent of q_+. However x depends on q_+ since

$$\frac{1}{x} = \frac{-2p.q}{q^2} = \frac{-(p+q)^2 + p^2 + q^2}{q^2} = \frac{4p_- q_+}{q^2} + \cdots \tag{5.2.50}$$

where the dots indicate terms that are independent of q_+. In order to produce the $\delta(y^+)$ that is present in the charge flux operator we need to integrate over q_+. This integral translates into an integral over x. The range of integration can be determined by the condition that $-(p+q)^2 \geq 0$. Thus we end up integrating between $x = 0$ and x_{max} with $1/x_{max} = 1 + p^2/q^2$. In the limit $p^2/q^2 \to 0$ we get the usual boundary $x = 1$.

We then have the following relation between the two quantities

$$\begin{aligned} \int d^2\vec{y} e^{i\vec{q}.\vec{y}} {}_p\langle \mathcal{Q}(\vec{0})\mathcal{Q}(\vec{y})\rangle_p = & \int_{-\infty}^{\infty} \frac{dq_+}{2\pi} W_{--}(q_+, q_- = 0, \vec{q}; p) = \\ = & \frac{(-p_-)}{4\pi} \int_0^{x_{max}} \frac{dx}{x} F_2(x, q^2/p^2) \end{aligned} \tag{5.2.51}$$

This is a particular moment of the parton distribution functions. More precisely it is the moment $M_1^{(2)}$. As it is well known, the even moments M_{2k} can be expressed in terms of the expectation values of local operators with spins $j = 2k$ [195] , via a dispersion relation argument. In fact, the moments $M_j^{(2)}$ can also be expressed in terms of the expectation values of the non-local light-ray operators with spin j for any j, see [196] for a general discussion.

In (5.2.51) the charge correlation is evaluated on a state with definite momentum in the y coordinates. This implies, in particular, that the charge two point function is also translation invariant in the transverse space, ${}_p\langle \mathcal{Q}(\vec{y})\mathcal{Q}(y')\rangle_p = {}_p\langle \mathcal{Q}(\vec{y}-\vec{y}')\mathcal{Q}(0)\rangle_p$.

In this work we have been mainly considering states which are in momentum eigenstates in the x coordinates, related to the y coordinates via (5.2.4). This does not lead to momentum eigenstates in the y coordinates. However, they do have definite momentum in the p_- direction. To the extent that we can neglect other components of the momentum in the y coordinates we see that the charge correlator has a simple relation to the deep inelastic scattering amplitude and the ordinary parton distribution functions. In the general case we will need to evaluate expectation values of the form $\langle p'|JJ|p\rangle$. These require generalized parton distribution functions [197,198]. Thus, if we have a state with definite momentum in the original x-coordinates, we will have a superposition of momenta in the y coordinates and the charge two point function will be related to integrals over generalized parton distribution functions. We will not write a detailed expression here.

Notice that the integral over x is divergent at small x. We think that this is due to the fact that the integral over $\vec{y}$ is also divergent for the charge correlator since the small angle singularity is not integrable. This divergence, though, is local in $\vec{q}$ and can probably be extracted without changing the overall picture. We have not checked this in detail. This problem is not present if we consider the energy correlation functions and the relation to the deep inelastic amplitudes probed by gravitons. In that case all quantities are manifestly finite.

The fact that in our problem we do not have ordinary plane wave wavefunctions in y has an interesting consequence. It was shown in [141] that, in the gravity regime, the leading power of q, which governs the short distance behavior in $\vec{y}$, is controlled by a double trace operator. We will show below that this contribution is highly suppressed for operators that have definite momentum in x-space. We expect that this double trace contribution will also be suppressed at weak coupling when we consider plane wave states in x-space.

Of course, everything we said here can be repeated for the energy correlation function, except that we should consider a deep inelastic process where we scatter gravitons from the field theory excitations.

5.2.5 Energy correlations and the C parameter

Let us make here a side comment on the relation between the energy correlators and other usually considered event shape variables. Event shape variables are certain functions of the four momenta of the observed particles which are infrared safe. One concrete example is the C parameter, defined as [199]

$$C = \frac{3}{2E^2} \int d^2\Omega_1 d^2\Omega_2 \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2) \sin^2\theta_{12} \tag{5.2.52}$$

where E is the total energy (and we assume that the total momentum vanishes). We see that the expectation value of C is given by an integral over the energy two point correlation function.

On the other hand, it is common to compute the cross section as a function of C (see for example [200]). This is just the probability of measuring various values of C, $\frac{d\sigma}{dC}$. This calculation involves more input than the two point correlation function, since we would need to know all the moments of C, $\langle C^n \rangle$, to reconstruct $\frac{d\sigma}{dC}$.

The point of this short remark is to stress that, even though the C parameter is given by a product of energies, the computation of the cross-section as a function of C involves knowledge of the n point energy-correlation functions. Of course, in practice, $\frac{d\sigma}{dC}$ is computed directly rather than going through the energy correlation functions.

5.3 Energy correlation functions in theories with gravity duals

In this section we consider energy correlation functions in conformal field theories that have gravity duals. We first start with some general remarks on the energy correlators and the basic ingredients necessary to calculate them. Then, we will present explicit calculations for the energy one point functions, which are given in terms of three point functions in the gravitational theories. Finally, we add a general prescription for computing arbitrary n point functions.

5.3.1 General remarks and basic ingredients of the calculation

The general prescription for computing correlation functions of local operators in the CFT using the gravity dual was derived in [4], [3]. Computations of the expectation values of the stress tensor for falling objects include [142], [201]. Since energy flux correlation functions are given in terms of stress tensor correlators, we simply need to perform the integral over time and take the limit in (5.1.1). In order to simplify the computations it is useful to consider some coordinate changes.

Let us start by writing AdS_5 using the coordinates

$$-(W^{-1})^2 - (W^0)^2 + (W^1)^2 + (W^2)^2 + (W^3)^2 + (W^4)^2 = -1 \tag{5.3.53}$$

The boundary of AdS_5 corresponds to the region where $W^M \to \infty$. In that regime we can forget about the -1 in (5.3.53) and we recover the coordinates Z^M that we described around (5.2.5). It will be convenient to introduce three possible sets of coordinates which are natural from different points of view. The first two are

$$\begin{aligned} \text{Original}: \ & \frac{1}{z} = W^{-1} + W^4 \ , \quad W^\mu = \frac{x^\mu}{z} \ , \quad \mu = 0,1,2,3 \\ \text{Easy}: \ & \frac{1}{y_5} \equiv W^0 + W^3 \ , \qquad W^{-1} = -y^0 \ , \quad W^4 = -y^3 \ , \quad W_{1,2} = \frac{y_{1,2}}{y_5} \end{aligned} \tag{5.3.54}$$

Of course the metrics are simply

$$\begin{aligned} \text{Original}: \ ds^2 &= \frac{dx^2 + dz^2}{z^2} \\ \text{Easy}: ds^2 &= \frac{-dy^+ dy^- + dy_1^2 + dy_2^2 + dy_5^2}{y_5^2} \end{aligned} \tag{5.3.55}$$

It is also convenient to introduce a third set of coordinates, which is defined as follows. We first choose three coordinates describing the H_3 subspace $-(W^0)^2 + (W^1)^2 + (W^2)^2 + (W^3)^2 = -r^2$ for a fixed r^2. The two other coordinates are chosen as

$$W^{\pm} = W^{-1} \pm W^4 \tag{5.3.56}$$

Then $r^2 = 1 - W^+ W^-$ and the metric is

$$\text{Hyperbolic}: \quad ds^2 = -dW^+ dW^- - \frac{1}{4} \frac{(W^- dW^+ + W^+ dW^-)^2}{1 - W^+ W^-} + (1 - W^+ W^-) ds_{H_3}^2 \tag{5.3.57}$$

The advantage of this coordinate system is that it makes the $SO(1,3)$ symmetry of the problem manifest. This $SO(1,3)$ symmetry are the isometries of H_3. In addition, the dilatation symmetry in the original coordinates becomes a boost in the $W^{\pm}$ plane, which is also a clear symmetry of the metric in this parametrization.

The surface that is at the boundary of four dimensional Minkowski space can be extended to the interior in a unique way so that it is invariant under the symmetries that preserve the boundary of Minkowski space. In fact, this surface is simply given by $W^+ = 0$.

The insertion of the stress tensor operator corresponds to a non-normalizable perturbation of the metric in the bulk. It will be convenient to derive first the expressions for the momentum in the y coordinates introduced in (5.2.4). Since all the generators in (5.2.8) are given in terms of the integral of $T_{--}(y')$ over a line along y'^-, let us compute this first. We insert $T_{--}(y')$ on the boundary at $y'^+ = 0$ and

$y'^1 = y'^2 = 0$ but at an arbitrary value of y'^-. We denote the boundary points with primes and the bulk points without primes. This induces the following fluctuation in the metric of AdS_5, $g_{MN} \to g_{MN} + h_{MN}$,

$$h_{MN}dx^M dx^M \sim (dy^+)^2 \frac{y_5^2}{[-y^+(y^- - y'^-) + y_1^2 + y_2^2 + y_5^2 + i\epsilon]^4} \tag{5.3.58}$$

We can now perform the integral over y'^-. We use the formula

$$\int_{-\infty}^{\infty} dy'^- \frac{1}{[y^+y'^- + A + i\epsilon]^4} \sim \delta(y^+)\frac{1}{A^3} \tag{5.3.59}$$

Now, by a simple translation, we can set the energy operator at any other value of $y'^{1,2}$. We obtain

$$\begin{aligned} h_{MN}dX^M dX^N \sim & \delta(y^+)(dy^+)^2 \frac{y_5^2}{(y_5^2 + (y_1 - y_1')^2 + (y_2 - y_2')^2)^3} dy_1' dy_2' = \\ \sim & \delta(W^+)(dW^+)^2 \frac{y_5^3}{(y_5^2 + (y_1 - y_1')^2 + (y_2 - y_2')^2)^3} dy_1' dy_2' \\ \sim & \delta(W^+)(dW^+)^2 \frac{(Z^0 + Z^3)}{(W \cdot Z)^3} dZ_1 dZ_2 \end{aligned} \tag{5.3.60}$$

where in the last line we have represented the boundary coordinates using (5.2.6), with $Z^+ = 0$, in order to get the answer in a form that will make it easy to make coordinate changes.

For example, if we wish to express the result in the hyperbolic coordinates (5.3.57) all we need to do is to express W and Z in terms of such coordinates. Let us define $\vec{n} = (n_1, n_2, n_3)$ to be a unit vector on a two sphere. On the surface $W^+ = 0$ we can parametrize $W^0 = \cosh\zeta$, $W_i = \sinh\zeta n_i$. It is then natural to take boundary coordinates $Z^0 = 1$, $Z^i = n'^i$. We take the limit $\zeta \to \infty$ and we then have that

$$W^0 + W^3 \sim e^\zeta(1 + n_3) \sim e^\zeta \left(Z^0 + Z^3\right) \ , \qquad y'_{1,2} = \frac{Z_{1,2}}{Z^0 + Z^3} = \frac{n'_{1,2}}{(1 + n'_3)} \tag{5.3.61}$$

The last equation is simply the change of coordinates (5.2.12). We then find the following expression for the generators

$$\begin{aligned}
E &\longrightarrow h^{E}_{MN} dX^N dX^M \sim \delta(W^+)(dW^+)^2 \frac{1}{(W^0 - W_i n'_i)^3} d\Omega'_2 \\
P^i &\longrightarrow h^{P^i}_{MN} dX^N dX^M \sim \delta(W^+)(dW^+)^2 n'_i \frac{1}{(W^0 - W_i n'_i)^3} d\Omega'_2 \\
\mathcal{E}(\vec{n}') &\longrightarrow h^{\mathcal{E}(\vec{n}')}_{MN} dX^N dX^M \sim \delta(W^+)(dW^+)^2 \frac{1}{(W^0 - W_i n'_i)^3}
\end{aligned} \tag{5.3.62}$$

In summary, we have computed the metric fluctuation that corresponds to the integrated insertion of the stress tensor that measures the energy deposited in the idealized calorimeters that we are placing at infinity at some position $\vec{n}'$ on the two-sphere. In the original AdS coordinates the corresponding insertion would be localized on the horizon of AdS in Poincare coordinates ($z = \infty$). We found it convenient to express the results in a couple of different coordinate systems that are regular at $z = \infty$ in order avoid having to take a limit. In these other coordinates we see that we are performing a measurement on the $W^+ = 0$ surface. This amounts to sampling the wavefunction of the particles in the bulk at $W^+ = 0$. At $W^+ = 0$ we have an H^3 subspace plus the null direction parametrized by W^-. The boundary of H^3 corresponds to the two sphere at infinity where we place the calorimeters.

The form of the equations (5.3.62) does not make explicit the Lorentz covariance of the expressions. In order to see this explicitly we can rewrite them as

$$P^\nu \rightarrow h^{P^\nu}_{MN} dX^N dX^M \sim \delta(W^+)(dW^+)^2 \frac{Z^\nu}{(-W.Z)^3} \frac{dS^0{}_\lambda Z^\lambda}{Z^0} \tag{5.3.63}$$

where we sit at $Z^+ = 0$ and we are integrating over a spacelike surface inside $\sum_{\mu=0}^{3} Z_\mu Z^\mu = 0$. The integration surface differential is defined to be such that $dS^\nu{}_\lambda Z^\lambda$ is parallel to Z^ν. Therefore the transformation of $dS^0_\lambda Z^\lambda$ cancels the transformation of Z^0 [10] .

[10] This is completely analog to the fact in classical electrodynamics that the power radiated by an

In the same way that we have discussed the graviton associated to energy flux measurements we can also consider the U(1) gauge field configurations associated to charge flow measurements on the boundary theory. The operator that corresponds to putting a counter at infinity at some specific location and measuring the charge corresponds to the following bulk gauge field configuration

$$\mathcal{Q}(\vec{n}') \to A_M dx^M \sim dW^+ \delta(W^+) \frac{1}{(W^0 - W^i n'_i)^2} \tag{5.3.64}$$

Having discussed the properties of the probe gravitons or gauge fields that represent our measurement, let us now turn to the field in the bulk that describes the state that we insert with the operator $\mathcal{O}$. We can think of a scalar source to make things simpler, although our results will be quite general. We are interested in obtaining the field configuration, ϕ, in the bulk of AdS_5 created by the insertion of the operator $\mathcal{O}$ of dimension Δ. If we insert the operator $\int d^4x' \phi_0(x') \mathcal{O}(x')$ on the boundary theory, then the bulk field configuration is given by [4], [3]

$$\begin{aligned}
\phi(x,z) &= \int d^4x' \phi_0(x') \frac{z^\Delta}{[(x-x')^2 + z^2]^\Delta} = \\
&= \lim_{z' \to 0} \int d^4x' \phi_0(x') \frac{1}{(z')^\Delta (W.W')^\Delta} \\
\phi_q(W^+ = 0, W^-, W^\mu) &= \int d^4x \frac{e^{iq.x'}}{\left[-\frac{W^-}{2} - W^0 x'^0 + W^i x'^i - i\epsilon\right]^\Delta}
\end{aligned} \tag{5.3.65}$$

where we have first rewritten the result in a way that allow us to easily change coordinates. In the last line we wrote the expression for the bulk field at $W^+ = 0$ in the case that $\phi_0(x') = e^{iq.x'}$. Notice that we only need the wavefunction at $W^+ = 0$ since that is where the graviton perturbation is localized. It is not hard to do the integral in (5.3.65) explicitly. There is, however, a very simple way to see what the answer should be. We are creating a state that is a momentum eigenstate. For the

accelerating charge is a Lorentz scalar.

moment let us set $q^\mu = (q^0, \vec{0})$. The momentum generator corresponds to a bulk isometry generated by a Killing vector that becomes simpler at $W^+ = 0$,

$$P_x^\mu|_{W^+=0} = -2iW^\mu \partial_{W^-} \tag{5.3.66}$$

Of course, this is similar to the corresponding boundary statement (5.2.7). A wave function that diagonalizes all four of these operators has to be a plane wave in W^- and should be localized in the W^μ coordinates. In other words we have

$$\phi_q(W^+ = 0, W^-, W^\mu) \sim \left(q^0\right)^{\Delta-4} e^{iq^0 W^-/2} \delta^3(\vec{W}) \tag{5.3.67}$$

where $\vec{W}$ refers to a parametrization of the hyperboloid given by W^i with $i = 1, 2, 3$. In these coordinates, W^0 is just a function of $\vec{W}$ since $W_\mu W^\mu = -1$ at $W^+ = 0$. $\left(q^0\right)^{\Delta-4}$ is a normalization constant that can be obtained from (5.3.65) by considering the dilatation operator acting on both sides. In other words, $\phi_0(x')$ is dimensionless so that ϕ scales as $\left(q^0\right)^{\Delta-4}$. The overall constant in (5.3.67) cancels out when we compute energy correlations. Note that in the y-AdS coordinates (5.3.54) the wavefunction with definite x-momentum, (5.3.67), is localized at $y_1 = y_2 = 0$, $y_5 = 1$ when $y^+ = 0$.

Therefore, the general result is that an incoming plane wave (with no spatial momentum) gives us a very peculiar wavefunction that is δ function localized in the H_3 subspace at the origin $W^i = 0$. In addition we find that the momentum in the W^- direction is proportional to the original energy. The wavefunction for an external operator with a generic value of the momentum q^μ can be obtained by performing a boost of this solution. The end result is again a wavefunction that is localized at a point in H_3. It is localized at $\frac{\vec{W}}{W^0} = \frac{\vec{q}}{q^0}$. The momentum in the W^- direction is now $-\frac{\sqrt{-q_\mu q^\mu}}{2}$. The wavefunctions corresponding to plane waves have a divergent norm since a plane wave wavefunction has a divergent norm. One can consider the regularized external wavefunction in (5.2.30). In that case we find a finite norm. We discuss this case in more detail in appendix B. As expected, one finds that the delta function is smeared over a region $|\vec{W}| \sim \frac{1}{\sigma q}$. We will continue to discuss wavefunctions for plane waves, but having in mind that we will eventually smear the δ function in (5.3.67), as in (5.2.30).

Once we have the bulk wavefunction we can compute the energy flux one point function. By considering the effects of the metric perturbation (5.3.62), and considering an operator that creates the bulk wavefunction ϕ, we obtain the expression

$$\begin{aligned} \langle \mathcal{E}(\vec{n}') \rangle =& N^{-2} \int dW^- d\Sigma_3 \frac{1}{4\pi(W^0 - \vec{W}.\vec{n}')^3} \left[(2i\partial_{W^-}\phi^*)(-2i\partial_{W^-}\phi) \right]|_{W^+=0} \\ N^2 =& \int dW^- \int d\Sigma_3 \left[\phi^*(-i\partial_{W^-}\phi) + c.c. \right]|_{W^+=0} \end{aligned} \tag{5.3.68}$$

where Σ_3 denotes the integral over the three dimensional hyperbolic space parametrized by W^μ, with $W_\mu W^\mu = -1$. The last factor, N^2, is simply the total production cross section and it is related to the two point function of the operator insertion. In other words, the two point function $\langle 0|\mathcal{O}_q^\dagger \mathcal{O}_q|0\rangle = ||\mathcal{O}_q|0\rangle||^2$ is the norm of the state. This norm is given in the bulk by the expression for N^2. When we insert the wavefunction (5.3.67) we see that a single point in the integral over hyperbolic space contributes. We finally get the expected result $\langle \mathcal{E} \rangle = \frac{q}{4\pi}$, (5.2.31).

5.3.2 Energy flux one point functions in theories with gravity duals

Using the results above we are ready to calculate the energy one point functions for different type of sources. For a scalar source the symmetries imply the result (5.2.31). Because there are no free parameters we know this is the correct result and we did not need to go throught the previous discussion. The situation is more interesting for current sources. In a theory that has a gravity dual the three point function of two currents and a stress tensor can be computed from the bulk interaction between two bulk photons and a bulk graviton that follows from the bulk Maxwell action

$$S = -\frac{1}{4g^2}\int d^5x\sqrt{g}F^2 \tag{5.3.69}$$

where g is the bulk gauge coupling. This term in the action also determines the two point function. Thus, we can see that the three point function will be determined and we will get a particular value for a_2 in (5.2.32). We can find this value by noticing that for $\mathcal{N} = 4$ Super Yang Mills we had $a_2 = 0$. Thus, any theory that has a gravity dual gives us $a_2 = 0$, as long as the two derivative approximation (5.3.69) is valid. In general there will be higher derivative corrections to this action. Up to field redefinitions there is a unique higher order operator that can contribute to the three point function

$$S = -\frac{1}{4g^2}\int d^5x\sqrt{g}F^2 + \frac{\alpha_1}{g^2M_*^2}\int d^5x\sqrt{g}W^{\mu\nu\delta\rho}F_{\mu\nu}F_{\delta\rho} \tag{5.3.70}$$

where $W^{\mu\nu\delta\rho}$ is the Weyl tensor. Here M_* is some mass scale in the gravity theory determining the strength of the correction relative to the strength of the Maxwell term in the action. In order to see that this is the only operator that contributes we consider the possible three point vertices between two photons and a graviton in flat space. It turns out that there are only two possible structures. This onshell vertex is so constrained because there is no kinematic invariant that we can make purely with the external momenta, which all square to zero. In fact the two possible interaction vertices consistent with gauge invariance are

$$\begin{aligned} v_1 =& \epsilon_{\mu\nu}\left[\epsilon_1^\mu k_2^\nu(\epsilon_2.k_1) + (1\leftrightarrow 2) - k_1^\mu k_2^\nu(\epsilon_1.\epsilon_2)\right] \\ v_2 =& \epsilon_{\mu\nu}\, k_1^\mu k_2^\nu\,(\epsilon_1.k_2)\,(\epsilon_2.k_1) \end{aligned} \tag{5.3.71}$$

where $\epsilon_{1,2}^{\mu}$ are the polarization vectors of the gauge bosons and $\epsilon_{\mu\nu}$ the polarization vector of the graviton. They are all transverse $\epsilon_1 \cdot k_1 = 0$ and $\epsilon_{\mu\mu} = 0$. The first arises from the quadratic action (5.3.69) and the second from the higher order correction in (5.3.70).

We expect that the higher derivative corrections give us a deviation from a perfectly spherical energy distribution for the state created by the currents. Notice that the higher order correction will not contribute to the angle independent term in the energy one point function. The reason is that this term is related by the Ward identities to the two point function and the two point function is not corrected by the presence of the higher order operator in (5.3.70) because the Weyl tensor vanishes is AdS. A detailed computation in appendix D shows that the higher order term indeed contributes to the anisotropic contribution to the energy correlation function

$$a_2 = -\frac{48\alpha_1}{R_{AdS}^2 M_*^2} \tag{5.3.72}$$

Notice, in particular, that in non-supersymmetric weakly coupled QCD we expect that the higher derivative corrections are comparable to the radius of AdS since a_2 is of order one for weak coupling (5.2.34).

This anisotropy is intimately related to the anisotropy in the gravitational field that is produced by a fast moving photon. Let us consider a photon with high momentum $|p_-| \gg 1$. We focus on the problem in flat space for the moment. Such a fast moving particle produces a metric of the form

$$ds^2 = -dx^+ dx^- + d\vec{x}^2 + \delta(x^-)(dx^-)^2 h(\vec{x}) \ , \qquad \nabla^2 h = 0 \tag{5.3.73}$$

where ∇^2 is the flat laplacian in the transverse directions. This metric is an exact solution of Einstein's equations (with zero cosmological constant) and arbitrary higher derivative corrections [202] . The particular form of the solution for h depends on the coupling of the photons to the graviton. For the lowest order action (5.3.69) we find that $h_0 \sim \frac{p_-}{|x|}$ which is independent of the spin of the photon. Here we are focusing on the five dimensional case, so that we have three transverse directions $\vec{x}$. On the other hand, the second interaction (5.3.70) gives a function h of the form

$$h_1 \sim \frac{p_-}{M_*^2} \epsilon_i^* \epsilon_j \partial_i \partial_j \frac{1}{|x|} = \frac{3p_-}{M_*^2} \frac{|n_i \epsilon_i|^2 - \frac{1}{3}|\epsilon|^2}{|x|^3} \tag{5.3.74}$$

where $n_i = x_i/|x|$. Note that this contribution to the gravitational field is sensitive to the spin of the photon and it has a quadrupole form. In the case that we have a large number of photons, this quadrupole tensor would be proportional to the polarization density matrix of the photons.

Even though we've discussed the case of a photon, all that we have said so far can be extended to the case that we have a non-abelian gauge theory in the bulk, which corresponds to a non-abelian global symmetry in the boundary theory.

We have a similar story in the case that the inserted external operator is the stress tensor itself. Then there are three possible vertices and three parameters specifying the stress tensor three point function. One of these parameters is fixed by the Ward identities and it multiplies the three point function that we expect from the gravity action. The other two parameters multiply higher order gravity corrections. In fact, the three possible gravity vertices in five dimensions are

$$\begin{aligned} v_1 =& k_2^\mu \epsilon^1{}_{\mu\nu} \epsilon^{2\nu}{}_\delta \epsilon^{3\delta}{}_\rho k_2^\rho + \frac{1}{4} \epsilon^1{}_{\mu\nu} \epsilon^{2\mu\nu} \epsilon^3{}_{\delta\rho} k_1^\delta k_2^\rho + \text{cyclic} \\ v_2 =& (k_3^\mu \epsilon^1{}_{\mu\nu} \epsilon^{2\nu}{}_\delta k_3^\delta)\, (\epsilon^3{}_{\rho\sigma} k_1^\rho k_2^\sigma) + \text{cyclic} \\ v_3 =& (\epsilon^1{}_{\mu\nu} k_2^\mu k_2^\nu)(\epsilon^2{}_{\delta\sigma} k_3^\delta k_3^\sigma)(\epsilon^3{}_{\rho\gamma} k_1^\rho k_1^\gamma) \end{aligned} \tag{5.3.75}$$

Such vertices arise from terms in the action of the form

$$S = \frac{M_{pl}^3}{2} \left[\int d^5x \sqrt{g} R + \frac{\gamma_1}{M_{pl}^2} W_{\mu\nu\delta\sigma} W^{\mu\nu\delta\sigma} + \frac{\gamma_2}{M_{pl}^4} W_{\mu\nu\delta\sigma} W^{\delta\sigma\rho\gamma} W_{\rho\gamma}{}^{\mu\nu} \right] \tag{5.3.76}$$

This is one way to parametrize the higher derivative corrections. In principle we can have another curvature cubed term but it does not contribute to the three point function [203].

In fact, in [203] such corrections were computed for various string theories. They found that γ_1 and γ_2 are non-zero in the bosonic string, only γ_1 is nonzero in the heterotic string and both $\gamma_1 = \gamma_2 = 0$ in the type II superstrings. Incidentally, $\frac{1}{N}$ corrections to this action, yielding an R^4 term, were computed for type IIB superstrings in $AdS_5 \times S^5$ in [204] and their effect on the 3 point function of stress energy tensors in $\mathcal{N} = 4$ SYM was discussed.

One can compute the contributions of the higher order terms in the action to t_2 and t_4 as defined in (5.2.39). After performing some calculation described in appendix D, we find

$$\begin{aligned} t_2 =& \frac{48\gamma_1}{R_{AdS}^2 M_{pl}^2} + o(\frac{\gamma_2}{R_{AdS}^4 M_{pl}^4}) \\ t_4 =& \frac{4320\gamma_2}{R_{AdS}^4 M_{pl}^4} \end{aligned} \tag{5.3.77}$$

to leading order in the γ_i. In addition we have assumed that the contribution of the W^3 operator to t_2 will be smaller than the one from the W^2 operator. This is expected in the large radius limit because W^3 has more derivatives. For an $\mathcal{N} = 1$ supersymmetric theory $t_4 = 0$. Using (5.3.77) and (5.2.41) we get the expression for the R^2 coefficient that was derived in [205], [191].

Notice that the presence of the first correction to the action (5.3.76) can also change the angular independent part of the energy flux one point function. This change should be compensated precisely by a change in the stress tensor two point function in order to obey the Ward identity.

We could also consider charge one point functions. In that case there are two (parity preserving) structures for the three point function [182], [183], [206]. The coefficient of one of them is determined by the Ward identities and arises only when we have a non-abelian gauge symmetry in the bulk. It comes from the usual bulk term of the form $\int Tr[F^2]$. The second structure arises from a bulk term of the form $\int Tr[F_{\mu\nu}F^{\nu\delta}F_{\delta}{}^{\mu}]$, or more generally, from a bulk coupling of the form $\int f_{abc}F^a_{\mu\nu}F^{b\nu\delta}F^c{}_{\delta}{}^{\mu}$ with a totaly antisymmetric f_{abc}. Notice that these terms do not necessarily come from a non-abelian gauge symmetry. They could come from a coupling between three different $U(1)$ gauge field strengths in the bulk.

The parity odd terms come from Chern Simons couplings in five dimensions [4] [206] . For example, for a gauge field we can have $\int A \wedge F \wedge F$ or its non-abelian generalization.

5.3.3 Comments on the n point functions

After this discussion on one point functions, let us move on to n point functions. All we need to do is to consider metric fluctuations which contain several insertions of the energy flux operator. In general we would have to worry about the bulk tree level interactions among the bulk gravitons corresponding to the insertions of the operators. In our case, there is an important simplification. This is due to the fact that the following plane wave solutions[11] are exact solutions of Einstein's bulk equations [208], [209]

$$ds^2 = ds^2_{AdS_5} + (dW^+)^2\delta(W^+)h(w) \tag{5.3.78}$$

where $h(w)$ is a function defined on the transverse space, which in this case is a hyperbolic space H_3 of radius one, given by $-(W^0)^2 + (W^1)^2 + (W^2)^2 + (W^3)^2 = -1$. The function $h(w)$ obeys the Laplace equation on this hyperbolic space

$$\vec{\nabla}^2_w h = 3h \tag{5.3.79}$$

[11] For applications of this type of solutions in confining backgrounds see [207].

Of course, one can check that[12]

$$h_{\vec{n}'} = \frac{2i}{4\pi(W^0 - W^i n'_i)^3} \tag{5.3.80}$$

is a solution and so will be an arbitrary superposition

$$h = \sum_{j=1}^{n} h_{\vec{n}'_j} \tag{5.3.81}$$

which represents the insertion of n calorimeters at angular positions given by $\vec{n}'_j$, $j = 1, \cdots, n$. This is summing all the gravity tree diagrams. We should now consider the propagation of the wavefunction on the background of this plane wave geometry. We want to consider the effects of each $h_{\vec{n}'_j}$ to first order in its strength but the combined effect of all of them. Let us recall how we would analyze this problem in flat space first. We consider a flat space plane wave of the form

$$ds^2 = -dx^+ dx^- + (dx^+)^2 f(x^+) h(\vec{x}) + d\vec{x}^2 \tag{5.3.82}$$

and we will eventually take the limit where $f(x^+) \to \delta(x^+)$. The scalar field obeys the equation

$$-4\partial_- \partial_+ \phi - 4 f(x^+) h(\vec{x}) \partial_-^2 \phi + \vec{\nabla}^2 \phi - m^2 \phi = 0 \tag{5.3.83}$$

We now assume that $f(x^+)$ is nonzero only within some small neighborhood of the origin $-\epsilon < x^+ < \epsilon$. This implies that $f(x^+)$ varies rapidly. Thus we assume that this rate of variation is much faster than the rate of variation of the wavefunction along the rest of the coordinates. In that region we can then approximately solve the wave equation (5.3.83) as

$$\phi(x^+ = \epsilon) = e^{-\int_{-\epsilon}^{\epsilon} f(x^+) h \partial_-} \phi(x^+ = -\epsilon) \to \phi(x^+ = \epsilon) = e^{-h\partial_-} \phi(x^+ = -\epsilon) \tag{5.3.84}$$

[12] Here the normalization factor is fixed such that we obtain the total energy upon integration.

Generalizing this method to our case of interest we find that

$$\phi(W^+ = \epsilon, W^-, W^\mu) = e^{-h\partial_{W^-}} \phi(W^+ = -\epsilon, W^-, W^\mu) \tag{5.3.85}$$

where W^μ denotes a point in H_3. This is nothing else than a translation of magnitude h in the W^- direction. The same type of behavior was observed for scattering of particles off shock waves in [210–212] and was used to study four point functions in the context of the AdS/CFT correspondence in [213].

The computation we want to do involves the overlap of the final state with the initial state in the background deformed by the insertion of the plane wave. In addition, we need to divide by the norm (5.3.68). If we write $h = \sum_j h_{\vec{n}'_j}$ and we expand in each of the $h_{\vec{n}'j}$ to first order we get the n point function

$$\langle \prod_j \mathcal{E}(\vec{n}'_j) \rangle = N^{-2} \int_{W^+=0} dW^- d\Sigma_3 \left[(i\partial_{W^-}\phi^*) \prod_{j=1}^n h_{\vec{n}'_j}(W) [(-\partial_{W^-})^n \phi] + c.c. \right] \tag{5.3.86}$$

where the integral over $d\Sigma_3$ is over the hyperboloid $W^\mu W_\mu = -1$, $\mu = 0, 1, 2, 3$. Let us specialize this expression to the case that we have a plane wave external state, which leads to (5.3.67). In that case we find that all the $h_{\vec{n}'_j}$ are evaluated at $\vec{W} = 0$ so that they become independent of the angle. Thus we get that not only the one point function is uniform but also all the n point functions are uniform as well. This implies that there are no fluctuations in the energy and an observer would see a uniform energy deposition in all the detectors. In other words, we get

$$\langle \mathcal{E}(\vec{n}'_1) \cdots \mathcal{E}(\vec{n}'_n) \rangle = \left(\frac{q}{4\pi} \right)^n \tag{5.3.87}$$

This is what we would expect by thinking that fragmentation is very rapid at strong coupling as suggested in [141], [143]. For a state with a generic, but definite, momentum we find

$$\mathcal{E}_{q^\mu}(\vec{n}') = \frac{1}{4\pi}\frac{(q^2)^2}{(q^0 - \vec{q}.\vec{n}')^3} \tag{5.3.88}$$

which is simply the boosted version of the uniform distribution $\mathcal{E} = \frac{q^0}{4\pi}$ that we get for the case where $\vec{q} = 0$.

This is the result for plane wave states. If one considers a generic state, then there can be fluctuations, but such fluctuations are parametrized by the fact that we have a wavefunction for momentum. In other words, one can write the wavefunction $\phi_0(x)$ appearing in (5.3.65) in momentum space as $\tilde{\phi}_0(p) \equiv \int d^4x e^{-ip.x}\phi_0(x)$. We consider only wavefunctions which are nonvanishing in the forward light-cone $p^2 < 0,\ p^0 > 0$. We could consider other wavefunctions but the corresponding operators vanish when they act on the vacuum and will not contribute. Thus, in the formulas below we imagine that p^μ is restricted to be in the forward light-cone. Then we can write the bulk wavefunction as

$$\phi(W^+ = 0, W^-, W^\mu) = \int_0^\infty \frac{d\lambda}{\lambda}\lambda^\Delta e^{i\lambda W^-/2}\tilde{\phi}_0(\lambda W^\mu) \tag{5.3.89}$$

We see that for a plane wave with purely timelike momentum we should set $\tilde{\phi}_0 = \delta(p^0 - q^0)\delta^3(\vec{p})$ and we recover (5.3.67). Inserting (5.3.89) into (5.3.86) we obtain

$$\begin{aligned} \langle\prod_{i=1}^n \mathcal{E}(\vec{n}'_j)\rangle =& N^{-2}\int d^4p\,\rho(p)\prod_{i=1}^n \frac{p^4}{4\pi(p^0 - \vec{p}\vec{n}'_i)^3} \\ \rho(p) =& N^{-2}(p^2)^{\Delta-2}|\tilde{\phi}_0(p)|^2\ , \qquad N^2 = \int d^4p(p^2)^{\Delta-2}|\tilde{\phi}_0(p)|^2 \end{aligned} \tag{5.3.90}$$

The factor of $(p^2)^{\Delta-2}$ appears when we consider the norm of a state that has momentum p, see appendix A. This factor is determined by the dilatation operator. In appendix B we compute ρ for the localized wavefunction $\phi_0(x)$ given in (5.2.30).

The final picture is that for a generic operator insertion we have a superposition of the results for each momentum, given by (5.3.88) with a probability weight given by $\rho(p)$ which is giving us the probability of exciting the mode with momentum p^μ in the conformal field theory (or in the bulk gravity theory).

For a generic $\tilde{\phi}_0(p)$ (5.3.90) gives non-trivial functions of the angles. The final picture for what we would see in each event is actually very simple. After we measure the energy on four of the calorimeters in each event, we can determine the value of p that is contributing and, therefore, the energies in all other calorimeters is determined. See appendix B for a longer discussion of this point. In other words, from event to event, we have some random variations which are completely captured by the distribution of momenta $\rho(p)$. In the bulk picture, we have a pointlike particle in the bulk with some wavefunction. Measuring all the energies is tantamount to measuring the position of the particle on H_3 and its momentum in the direction W^- when it crosses $W^+ = 0$. We can view this as the horizon of AdS. We can say that we are simply measuring the momentum of the particle as it crosses the AdS_5 horizon. In the approximation that we have a pointlike particle we have a small number of random (quantum) variables characterizing the event. We only have the position or momentum of the particle when it crosses the horizon. When we consider a string we have an infinite number of degrees of freedom and we can have much more variation in the energy deposition patterns.

5.4 Stringy corrections

In this section we study stringy corrections to the gravity results. First we consider a flat space problem that is closely related to the problem we need to solve in AdS. We then use these results to compute the leading order stringy corrections to the gravity results. Finally we study the small angle behavior of the two point function and we find the stringy version of the operator product expansion we discussed above.

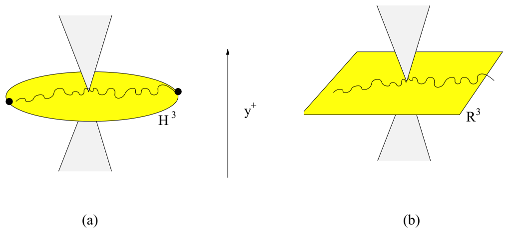


Figure 5.4: (a) The AdS computation of the energy correlators involves gravitons that propagate from the boundary to the interior on an H_3 subspace of the full AdS_5 space. The gravitons originate on the boundary of H_3, at the point where the calorimeter is inserted, and propagate on H^3 to the interior. (b) Since the falling string state is localized on H_3 we can approximate the computation by a flat space one.

5.4.1 Strings probed by plane waves

Let us first make the approximation that the AdS space is weakly curved and let us approximate the problem as that of strings in flat space, see (5.4). In fact, we have seen that the state created by the operator insertion is localized on the transverse surface, the H_3 subspace. In addition, the energy flux operator corresponds to a graviton localized at $W^+ = 0$. Thus, we can just look at the problem in a neighborhood of this point and approximate it as a flat space problem where we have a particle, or a string, with nonzero p_- which it is being probed by gravitons with $p_- = 0$ that are localized along y^+. Note that the probe gravitons are extended in the y^- direction.

More explicitly, we can consider a flat space problem where we have a string with a non-zero value of p_- which crosses a gravitational plane wave of the form

$$ds^2 = -dy^+dy^- + (dy^+)^2\delta(y^+)h + d\vec{y}^2 \tag{5.4.91}$$

where h is a function of the transverse coordinates obeying $\vec{\nabla}^2 h = 0$. Due to the symmetries of the problem it is convenient to choose light cone gauge where $y^+ = -2\alpha' p_- \tau$. Recall that $p_- < 0$ is the momentum conjugate to y^-. Following the usual steps that lead to light cone quantization we find that we get the following light cone gauge Lagrangian for the transverse dimensions

$$S = \frac{1}{4\pi\alpha'}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma d\tau [(\partial_\tau \vec{y})^2 - (\partial_\sigma \vec{y})^2] - \frac{1}{2\pi} p_- \int_0^{2\pi} d\sigma h(\vec{y}(\tau = 0, \sigma)) \quad (5.4.92)$$

Notice that the h dependent term is localized at $\tau = 0$, at a single value of the worldsheet time. Thus, the string propagates freely in flat space away from $y^+ = 0$, or $\tau = 0$. We will also assume that near $y^+ \sim 0$ the string is localized near $\vec{y} = 0$ in the transverse directions. We can then compute correlation functions from the expression

$$\langle \Psi | e^{-ip_- \int_0^{2\pi} \frac{d\sigma}{2\pi} h(\vec{y}(\sigma))|_{\tau=0}} | \Psi \rangle \quad (5.4.93)$$

where $|\Psi\rangle$ the full wavefunction of the string state in the light cone gauge theory at $\tau = 0$. We consider a function h which is a sum of a finite number of plane waves, $h = \sum_j h_j e^{i\vec{k}_j \vec{y}}$, and we expand to linear order in each perturbation h_j. The mass shell condition is $\vec{k}_j^2 = 0$. This implies that $\vec{k}$ is complex. (When we go back to the AdS problem it will be natural to take the component of k along the radial direction to be purely imaginary and the others to be real.) Let us assume that the string state corresponds to the ground state for the bosonic oscillators excitations, at least for the bosonic transverse directions where the momenta $\vec{k}$ are nonzero. We then find that we have to compute correlation functions of the form

$$\begin{aligned} &(-ip_-)^n \langle \psi_{cm} | \prod_j e^{i\vec{k}_j \vec{y}} | \psi_{cm} \rangle \langle 0 | \prod_j \int \frac{d\sigma_j}{2\pi} e^{i\vec{k}_j \vec{y}_{osc}(\sigma)} | 0 \rangle \sim \\ &\sim (-ip_-)^n \langle \psi_{cm} | \prod_j e^{i\vec{k}_j \vec{y}} | \psi_{cm} \rangle \prod_j \int \frac{d\sigma_j}{2\pi} \prod_{j<i} |2 \sin \frac{\sigma_i - \sigma_j}{2}|^{\alpha' \vec{k}_i . \vec{k}_j} \end{aligned} \quad (5.4.94)$$

where we have separated out the contribution from the center of mass and the oscillators. Since the center of mass wavefunction is well localized we expect no contribution from it. Namely, we imagine a wavefunction which is localized near $\vec{y} = 0$ an thus we simply need to evaluate the plane waves in (5.4.94) at zero which just gives one.

Namely, $\langle\psi_{cm}|\prod_j e^{i\vec{k}_j\vec{y}}|\psi_{cm}\rangle \sim 1$. Note that if we neglect the oscillator contributions we recover the gravity result following from (5.3.85). Therefore, the nontrivial contribution comes from the oscillators. Notice that these integrals are convergent if the k's are all small enough. In the case of the two point function we have

$$\begin{aligned}\int_0^{2\pi}\frac{d\sigma}{2\pi}|2\sin\frac{\sigma}{2}|^{\alpha' k_1.k_2} =&\frac{2^{\alpha' k_1.k_2}}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}+\frac{\alpha' k_1.k_2}{2})}{\Gamma(1+\frac{\alpha' k_1.k_2}{2})} = \\ =&1+\frac{\pi^2}{24}(\alpha' k_1.k_2)^2+\cdots\end{aligned} \tag{5.4.95}$$

In the second line, the 1 corresponds to the gravity result and the second term is the first correction. Naively, one might have expected the first correction to be of order α'. However, the first term vanishes and the order α'^2 term is the first non vanishing one.

It is convenient to rewrite this result in position space. We find that the gravity result plus the leading order correction can be written as

$$\begin{aligned}N^{-2}\int_{y^+=0} dy^- d^3y\left[i\phi^*\partial^3_{y^-}\phi+c.c.\right]\left[h_1(y)h_2(y)+\frac{\pi^2}{24}\alpha'^2(\partial_i\partial_j h_1(y)\partial_i\partial_j h_2(y))\right]\ , \\ \text{where}\qquad N^2=\int_{y^+=0} dy^- d^3y\left[i\phi^*\partial_{y^-}\phi+c.c.\right]\end{aligned} \tag{5.4.96}$$

where ϕ is the wavefunction of the center of mass of the closed string state and h_1 and h_2 are two graviton plane wave states. We have also normalized the result.

In fact, from our discussion we can easily see the origin of the α' corrections to the three graviton vertex in various string theories. These were computed in [203] . We first consider the case where there is just one probe graviton in (5.4.94). α' corrections can only arise if the initial state contains bosonic oscillators in the transverse directions. For a graviton in the superstring we have no bosonic oscillators in the initial state, only fermion zero modes. Thus the vertex is the same as the gravity one. In the case of the heterotic string the graviton contains fermion zero modes for the right movers and a bosonic oscillator for the left movers. Such an oscillator can give rise to momentum dependent terms of the from given by the second vertex in (5.3.75) , but not like the third in (5.3.75). Finally, in the case of the bosonic string a graviton with indices in the transverse directions involves bosonic oscillators for both left and right movers and gives rise to a vertex like the third in (5.3.75) (plus the first two, of course).

It is interesting that the string result is finite. One might have worried that since we are using $\delta(y^+)$ wavefunctions we would obtain divergencies. As we will see in more detail below, the Regge behavior of the scattering amplitudes in string theory ensures that the results are finite.

5.4.2 Leading order α' corrections to the two point function

We now generalize this result to curved space. We can simply replace the ordinary derivatives in (5.4.96) by covariant derivatives. However, in the AdS context h obeys an equation of the form $\nabla^2 h = 3h$, or more precisely $\nabla^2 h = \frac{3}{R^2_{AdS}} h$, so that terms that would have been zero in flat space are non-zero in AdS, so we seem to be faced with an ambiguity. However, these ambiguities only affect terms that do not have angular dependence, at least for the first correction. Thus, such terms only correct the constant part of the energy correlations. We can fix such corrections by demanding that we obey the energy conservation conditions. It is convenient to think about the problem in the hyperbolic coordinates (5.3.57). The graviton wavefunction associated to the insertion of a calorimeter at $\vec{n}'$ on the two sphere is given by (5.3.68)

$$h \sim \frac{1}{(W^0 - \vec{W}.\vec{n}')^3} \sim \frac{1}{(1 + \frac{|\vec{W}|^2}{2} - \vec{W}.\vec{n}')^3} \tag{5.4.97}$$

where we have have expanded the result around $\vec{W} \sim 0$. We have already seen that the center of mass wavefunction is localized near the origin of hyperbolic space if we have a state created by an operator with zero spatial momentum on $R^{1,3}$, see (5.3.67). Thus, we can evaluate the derivatives in (5.4.96) and then set $\vec{W} = 0$. This gives

$$\langle \mathcal{E}(\vec{n}'_1)\mathcal{E}(\vec{n}'_2)\rangle = (\frac{q^0}{4\pi})^2 \left[1 + \frac{\pi^2}{24}\frac{\alpha'^2}{R^4_{AdS}} \left(\partial_i\partial_j h \partial_i\partial_j h|_{\vec{W}=0} + \text{const}\right)\right] \qquad (5.4.98)$$

where the constant is an angle independent term that we cannot compute purely in flat space. It can be fixed so that we obey the energy conservation condition. In the end we find

$$\langle \mathcal{E}(\vec{n}'_1)\mathcal{E}(\vec{n}'_2)\rangle = (\frac{q^0}{4\pi})^2 \left[1 + \frac{6\pi^2}{\lambda}(\cos^2\theta_{12} - \frac{1}{3}) + \cdots\right] \qquad (5.4.99)$$

for $\mathcal{N} = 4$ super Yang Mills. where $\cos\theta_{12} = \vec{n}'_1.\vec{n}'_2$. We see that, as expected here, the distribution rises in the forward and backward regions. We have fixed the constant term in the correction by demanding that the integral over one of the angles gives the total energy. The dots in (5.4.99) denote higher order terms in the $1/\sqrt{\lambda}$ expansion.

In this derivation we have assumed that the state we are considering has no oscillators excited along the three transverse AdS directions. In the case of the superstring we can still have a massless mode with indices in the transverse AdS directions since those can be accounted for by fermion zero modes on the string worldsheet in light cone gauge. The result (5.4.99) is very general and holds for any theory with a ten dimensional weakly coupled dual with an AdS_5 factor if we replace $1/\lambda \to \alpha'^2/R^4_{AdS}$, under the assumption that we are creating a ten dimensional massless closed string with the external operator.

5.4.3 Corrections to the n point function

We now consider the n point function $\langle \mathcal{E}(\vec{n}_1)\cdots\mathcal{E}(\vec{n}_n)\rangle$. We have seen that the gravity result is just a constant. Let us compute the stringy corrections. The leading deviation can be computed by expanding the full expression (5.4.94) up to quadratic order in products of $k_i.k_j$. The resulting correction is basically the same as the one contributing to the two point function (5.4.99). In order to see something new we can go to cubic order in the products $k_i.k_j$. In the end this gives us a correction to the n point function which looks like

$$\langle \mathcal{E}(\vec{n}_1)\cdots\mathcal{E}(\vec{n}_n)\rangle = \left(\frac{q}{4\pi}\right)^n \Bigg[1 + \sum_{i<j}\frac{6\pi^2}{\lambda}[(\vec{n}_i.\vec{n}_j)^2 - \frac{1}{3}] + \\ + \frac{\beta}{\lambda^{3/2}}[\sum_{i<j<k}(\vec{n}_i.\vec{n}_j)(\vec{n}_j.\vec{n}_k)(\vec{n}_i.\vec{n}_k) + \cdots] + o(\lambda^{-2})\Bigg] \tag{5.4.100}$$

where β is a numerical coefficient[13] and the dots denote terms that are necessary to ensure that the integral over each of the angles gives zero as well as a term that corrects the coefficient of the $(\vec{n}_i\vec{n}_j)^2$ term by an order $\lambda^{-3/2}$ amount.

Thus, we find that for a strongly coupled field theory the energy distribution is uniform with small fluctuations which have an amplitude of order $1/\sqrt{\lambda}$. In other words, $\delta\mathcal{E}/\mathcal{E} \sim \frac{1}{\sqrt{\lambda}}$. The two point function of these fluctuations is given by the first non-constant term in (5.4.100). One might have thought that these flucutuations would be gaussian. However, we find that the three point function of the fluctuations is of order $\lambda^{-3/2}$. Thus, when we normalize the two point functions to one, the three point functions are of order one. Thus we conclude that the fluctuations are not even approximately gaussian. More explicitly, we can define a fluctuation operator

$$\delta = \frac{\mathcal{E} - \langle\mathcal{E}\rangle}{\langle\mathcal{E}\rangle}\ , \qquad \text{and} \qquad \hat{\delta} = \sqrt{\lambda}\delta \tag{5.4.101}$$

where the operator $\hat{\delta}$ is defined so that its two point function is independent of λ. We

[13] $\beta = -1728\int_0^{2\pi}\frac{d\sigma_1 d\sigma^2}{(2\pi)^2}\log[2\sin\frac{\sigma_1}{2}]\log[2\sin\frac{\sigma_2}{2}]\log[2|\sin\frac{(\sigma_1-\sigma_2)}{2}|] \sim 518 \pm 5$.

then have

$$\begin{aligned}\langle\hat{\delta}(\vec{n}_1)\rangle =& 0\\ \langle\hat{\delta}(\vec{n}_1)\hat{\delta}(\vec{n}_2)\rangle =& 6\pi^2[(\vec{n}_1.\vec{n}_2)^2-\frac{1}{3}][1+o(\lambda^{-1/2})] \\ \langle\hat{\delta}(\vec{n}_1)\hat{\delta}(\vec{n}_2)\hat{\delta}(\vec{n}_3)\rangle =& \beta[(\vec{n}_1.\vec{n}_2)(\vec{n}_1.\vec{n}_3)(\vec{n}_2.\vec{n}_3)+\cdots]\end{aligned} \tag{5.4.102}$$

We see that the three point function is not parameterically suppressed relative to the two point function. Of course, they are both suppressed relative to the gravity result.

5.4.4 Stringy corrections to charge two point functions

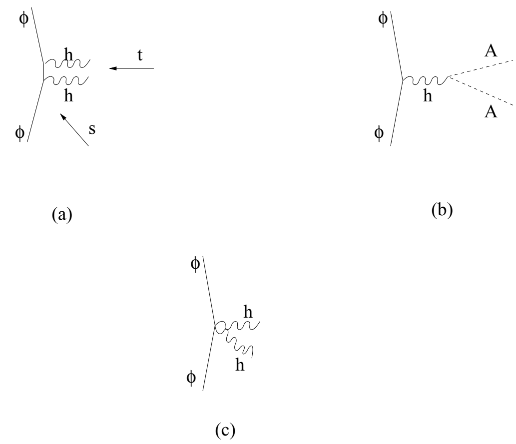


Figure 5.5: (a) Feynman diagrams that lead to energy correlation functions. The gravitons do not interact before they touch the falling state, ϕ. We have also indicated the t and s channels as we define them in the text. (b) Diagram that leads to a divergence in the charge correlation function. The intermediate state is a graviton and the AAh vertex comes from the Maxwell term in the action. This divergence is cured by going to string theory and exploiting the Regge behavior of the amplitudes. In (c) we draw a diagram that can arise due to a higher derivative contact interaction in the gravity theory which could lead to a divergence in the gravity approximation.

We now consider the two point function of a charge that is dual to a closed string mode. For example, we can pick one of the $SO(6)$ currents in $\mathcal{N}=4$ super Yang Mills. More generally we consider a current associated to a symmetry that is carried by fields in the adjoint representation in the dual field theory. Imagine that the current comes from Kaluza Klein reduction. Then the corresponding vertex operator, in light cone gauge, has the form $A_+ \to \partial_\tau \varphi e^{ik.x}$ where φ is one of the internal dimensions. We assume that the state we are measuring, $|\Psi\rangle$, does not have any oscillator excited in the φ direction and that it is not charged. Thus the one point function of the charge, is zero. The two point function is actually infinite in the gravity approximation. This is due to the Feynman diagram in 5.5 (b). This is intimately related to the fact that (5.3.64) is not an exact solution of the gravity equations, but it sources a gravitational plane wave proportional to F_{+i}^2, which leads to the square of a $\delta(y^+)$ function. We did not run into this problem in the gravity theory because there are no diagrams of this form due to the fact that the gravitational shock wave is an exact solution of the theory. Even in the gravity theory we could have run into this problem if we had had a higher derivative contact interaction that brings together two gravitons, as in 5.5 (c).

Let us now compute the two point function in string theory. The corresponding flat space expression is similar to (5.4.95), but with an extra factor coming from the contractions of the $\partial_\tau \varphi$ field coming from the two vertex operators. We get a result proportional to

$$\int_0^{2\pi} \frac{d\sigma}{(2\pi)} |2 \sin \frac{\sigma}{2}|^{\alpha' k_1.k_2 - 2} = \frac{2^{\alpha' k_1.k_2 - 2}}{\sqrt{\pi}} \frac{\Gamma(-\frac{1}{2} + \frac{\alpha' k_1.k_2}{2})}{\Gamma(\frac{\alpha' k_1.k_2}{2})} \sim -\frac{\alpha' k_1.k_2}{4} + \cdots \tag{5.4.103}$$

We have defined the integral by analytic continuation in $k_1.k_2$ We see that we get a perfectly finite answer in string theory. The string theory answer even goes to zero as we take the small momentum limit. Translating this flat space result (5.4.103) to AdS as we did above we get

$$\langle \mathcal{Q}(\vec{n}_1) \mathcal{Q}(\vec{n}_2) \rangle = \frac{\gamma}{\sqrt{\lambda}} \vec{n}_1.\vec{n}_2 = \frac{\gamma}{\sqrt{\lambda}} \cos \theta_{12} \tag{5.4.104}$$

where γ is a positive numerical coefficient. This result has the angular dependence that one would intuitively expect, with the two oppositely charged particles going in opposite directions.

Let us emphasize once more an important point. Due to the fact that we are considering shock waves which are highly localized, with $\delta(W^+)$ wavefunctions, it is important to perform the computation in string theory rather than first taking the low energy limit of string theory and then doing the computation. We will revisit this point later.

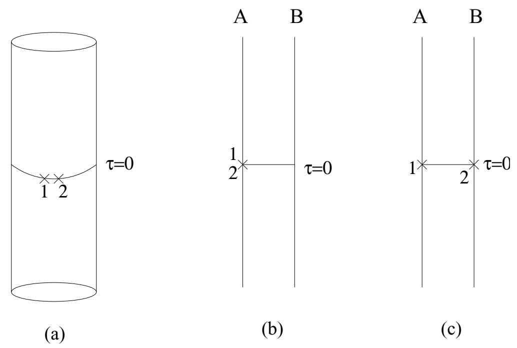


Figure 5.6: (a) Worldsheet vertex operator insertions for charge correlators associated to closed string gauge fields. These charges are carried by the bulk of the worldsheet. In (b) and (c) we consider charges carried by open strings. We consider an open string stretching between two different branes called A and B. In (b) we consider the charge two point function for the $U(1)_A$ living on the brane A. The two vertex operators are inserted at the same point. In (c) we consider the charge two point function for charge living on the brane A, $U(1)_A$, and the charge living on brane B, $U(1)_B$. The vertex operators are inserted on different boundaries and the result is non-singular.

Another interesting situation arises when we consider currents that act only on fields in the fundamental representation, such as flavor symmetries. In this case the currents live on D-branes in the bulk. For simplicity let us assume that we have two D-branes with two different $U(1)$ gauge fields in the bulk. Let us call them $U(1)_A$ and $U(1)_B$. We could imagine a QCD-like theory where $U(1)_A$ and $U(1)_B$ are different flavor number symmetries. At leading order in N we detect these charges in the detector only if we create a mesonic operator that contains the corresponding quarks. Consider a situation where we have a lorentz scalar meson where the quark is charged under $U(1)_A$ and the anti-quark under $U(1)_B$. In such a situation we expect to find only one charge of each type in the detector. In this case the charge one point functions are $\langle \mathcal{Q}_A(\vec{n}) \rangle = \langle \mathcal{Q}_B(\vec{n}) \rangle = \frac{1}{4\pi}$. What is the charge two point function for $U(1)_A$? From the boundary field theory point of view we expect it to be zero for generic angles since the charged quark can be detected only at one particular angle, since we are working to leading order in N where we do not create quark anti-quark pairs. On the other hand, to leading order in N, from the gravity plus maxwell theory in the bulk we get get a completely spherically symmetric distribution of charge. In this case, the divergent term coming from the Feynman diagram in 5.5(b), is subleading in N and we do not consider it (string theory ought to make this $1/N$ correction finite too). In other words, the two point function for the charges is $\langle \mathcal{Q}_A(\vec{n})\mathcal{Q}_A(\vec{n}') \rangle_{gravity} = \frac{1}{(4\pi)^2}$. This contradicts the field theory expectations. The resolution is that the stringy corrections are so large that they completely change the gravity result. Let us first see how this works in the flat space case. Here, we can quantize the open string in light cone gauge and we will get an action very similar to the one we had for the closed string except that the photon vertex operator, which is inserted at $\tau = 0$, is also inserted at $\sigma = 0$ at one of the boundaries of the open string. We find that

$$\langle \psi_{cm} | e^{i\vec{k}_1 \vec{y}} e^{i\vec{k}_2 \vec{y}} | \psi_{cm} \rangle \langle 0 | e^{i\vec{k}_1 \vec{y}_{osc}(0,0)} e^{i\vec{k}_2 \vec{y}_{osc}(0,0)} | 0 \rangle \qquad (5.4.105)$$

If we ignore the oscillators we go back to the gravity result. However, the contribution from the oscillators involves a singularity, since both vertex operators are evaluated at the same point. Formally, this gives a contribution of the form $0^{2\alpha' k_1.k_2}$. If $k_1.k_2 > 0$, then we see that this vanishes. Thus, if we define the answer by analytic continuation we get zero for all values of $k_1.k_2$, including the physical values of $k_1.k_2$ for our problem (which are negative). The reason that stringy corrections have such a large effect is that we start with singular wavefunctions for the photon, which contain a $\delta(y^+)$. If we had started with a smooth wavefunction in the x^+ dimension we would have integrated the vertex operators along the τ direction on the boundary of the open string worldsheet and we would have obtained a non-vanishing function of $k_1.k_2$.

On the other hand, if we compute the two point function for the two different $U(1)$ charges, the charge carried by the quark and the charge carried by the antiquark, then we get the vertex operators at opposite points of the string and we obtain a finite answer

$$\langle \psi_{cm}| e^{i\vec{k}_1 \vec{y}} e^{i\vec{k}_2 \vec{y}} |\psi_{cm}\rangle \langle 0| e^{i\vec{k}_1 \vec{y}_{osc}(0,0)} e^{i\vec{k}_2 \vec{y}_{osc}(0,\sigma=\pi)} |0\rangle \sim 2^{2\alpha' k_1.k_2} \tag{5.4.106}$$

In this case the leading order α' correction to the two point function reads

$$\langle \mathcal{Q}_A(\vec{n}_1) \mathcal{Q}_B(\vec{n}_2)\rangle = \frac{1}{(4\pi)^2}\left[1 - \frac{8\log 2}{\sqrt{\lambda}}\cos\theta_{12}\right] \tag{5.4.107}$$

We see that there is a tendency for the two charges to go in opposite directions, as one naively expects. Of course, at weak coupling the quark and the antiquark fly in opposite directions (if we have the simplest operator which contains only a quark anti-quark pair). If we were to consider a higher point function we would get zero again.

The general lesson is that when we compute charge correlators it is very important to understand the effects of stringy corrections.

Once we consider finite N corrections we do not expect the two point functions for the same $U(1)$ to be exactly zero.

5.4.5 Small angle behavior of the two point function and the operator product expansion

In this section we study the small angle behavior of the two point functions using string theory. The leading order correction to the energy flux two point function (5.4.99) is analytic at small angles, i.e. when $\theta_{12} \to 0$. As we will explain below this is no longer the case once all the α' corrections are included. We will show that at small angles there is a non-analytic term of the form $|\theta_{12}|^p$ with a power, p, that we will compute. This power is intimately related to the singularities in the first line of (5.4.95) as a function of $k_1.k_2$.

Let us first understand how the singularities in the flat space answer (5.4.95) arise. These singularities are at $\alpha' k_1.k_2 = -1 - 2n$. We can rewrite this condition as

$$t \equiv -(k_1 + k_2)^2 = \frac{2 + 4n}{\alpha'} \ , \qquad n = 0, 1, 2, \cdots \tag{5.4.108}$$

Similar looking singularities are a well known feature of string scattering amplitudes and they arise when an invariant, such as t, is equal to the mass of a string state. In that case we can view them as arising from the production of an on-shell closed string state. In our case, however, there are no states in the closed string spectrum with masses given by (5.4.108). Thus we seem to have a puzzle. We will argue that we indeed have certain string states, but of a non-local kind.

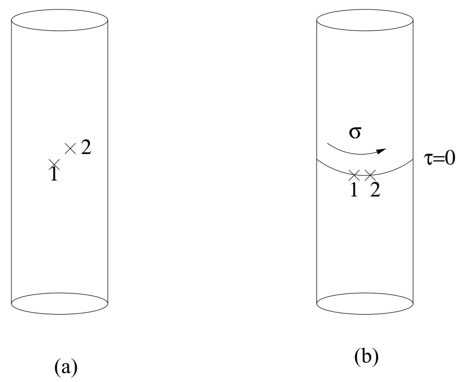


Figure 5.7: (a) The poles of the ordinary closed string amplitude arise from the region where the two vertex operators are close to each other but are integrated over both τ and σ. (b) Wordsheet OPE for the problem we are considering where we have wavefunctions localized in x^+. In light cone gauge this results in operators localized at $\tau = 0$. Thus we get singularities from the region of the integral where $\sigma_{12} \to 0$.

Let us first understand the worldsheet origin of these singularities. For concreteness, let us focus on the first singularity at $\alpha' k_1.k_2 = -1$. We see that at this point the integral in (5.4.95) diverges at $\sigma = 0$ like $\int \frac{d\sigma}{\sigma}$. At $\sigma \sim 0$ the two closed string vertex operators of the external gravitons come close together, see 5.7(b). This looks similar to the ordinary OPE region of a closed string worldsheet which produces the usual closed string state poles. The crucial difference is that in our case the integral runs only over the sigma direction (see 5.7(b)), while in the ordinary case it runs over τ and σ (see 5.7(a). For this reason the position of the poles has been shifted compared to the ordinary closed string poles. Schematically we have

$$\begin{aligned} \text{Usual Case :} \quad & \int dz^2 |z|^{\alpha' k_1 . k_2} \sim \frac{1}{\alpha' k_1 . k_2 + 2} \\ \text{Our Case :} \quad & \int d\sigma |\sigma|^{\alpha' k_1 . k_2} \sim \frac{1}{\alpha' k_1 . k_2 + 1} \end{aligned} \tag{5.4.109}$$

We have now understood how the singularities arise from the worldsheet computation. Before moving on, let us clarify further a confusing aspect of these singularities. All we are doing is to scatter four string states: the two gravitons, the state we are measuring and its complex conjugate. So, why are the singularities different than singularities of the ordinary closed string scattering amplitude? What happens is that we are choosing very peculiar wavefunctions for the two gravitons. These wavefunctions contain $\delta(y^+)$ factors which implicitly carry an infinite amount of momentum. More precisely, in order to go from the usual momentum space result to our expression for $\delta(y^+)$ wavefunctions we should integrate the momentum space result over k_{1+} and k_{2+}. The four point amplitude is characterized by $t = -(k_1 + k_2)^2$ and $s = -(p + k_1)^2$, where p is the momentum of the incoming closed string state with non-zero p_-. Since $k_{i-} = 0$, we have that t is independent of k_{i+} and it continues to be given by the transverse components of k_i, $t = -(\vec{k}_1 + \vec{k}_2)^2$. On the other hand s contains a contribution of the form $s = 4p_- k_{1+} + \cdots$. For the polarizations of the gravitons that we are choosing here the amplitude has the form [14]

$$\mathcal{A}_4 = p_-^4 \left(\frac{1}{s} + \frac{1}{u} \right) \frac{\Gamma(1-\frac{\alpha' s}{4})\Gamma(1-\frac{\alpha' t}{4})\Gamma(1-\frac{\alpha' u}{4})}{\Gamma(1+\frac{\alpha' s}{4})\Gamma(1+\frac{\alpha' t}{4})\Gamma(1+\frac{\alpha' u}{4})} , \qquad u = -s - t \qquad (5.4.110)$$

In the gravity limit we recover the results that we expect from the diagram in 5.5(a) and a crossed version. We can take p_- to be fixed. Then the integral over k_{1+} translates into an integral over s [15]. At large s, the four point amplitude is controlled by Regge behavior. The amplitude goes as

$$\mathcal{A}_4 \sim s^{-2+\frac{\alpha' t}{2}} \qquad (5.4.111)$$

So the integral over s converges at large values of s for small t [16]. As we increase t, the integral over s first diverges when the amplitude goes like $1/s$, $\mathcal{A}_4 \sim 1/s$. This condition is precisely the $n = 0$ case in (5.4.108). We get the higher order ones by a similar reasoning by expanding the amplitude to higher orders in the $1/s$ expansion. One minor subtlety is that only even powers of $1/s$ give rise to $1/s$ terms in the full amplitude, after we adjust t appropriately[17]. Odd powers of $1/s$ would lead to extra singularities beyond those given by (5.4.108), see more on this below. Thus, we see that the poles (5.4.108) are associated to the high energy behavior of the string amplitude. This is to be expected since by probing the string at $y^+ = 0$ we are taking a snapshot of the string state and this requires high energy scattering. The fact that the amplitudes we are computing are finite is related to the fact that the high energy scattering displays Regge behavior. In conclusion, the result we obtained in

[14] We take the momenta to be nonvanishing only in the first five dimensions. The falling string state is taken to be a graviton which has indices in the remaining five dimensions. Of course the two probe gravitons have indices in the $++$ directions.

[15] The integral over k_{2+} simply gets rid of the momentum conservation delta function in the k_+ direction.

[16] There are poles along the real s axis. As usual, we give these poles a small positive or negative imaginary part so that we are analyzing the amplitude in the physical sheet. Thus, the poles along the real s axis do not lead to divergences in the amplitude.

[17] In other words, $A_4 \sim s^{-2+\alpha' t/2} \sum_{n=0}^{\infty} c_n(\alpha' t)/s^n$, but $c_{2k+1}(\alpha' t = 4k + 4) = 0$.

lightcone gauge is perfectly consistent with the usual structure of the Shapiro-Virasoro amplitude.

A related remark that we can make at this point is the following. Let us go back to the case where we consider a neutral falling state probed by two closed string gauge bosons. Going to ten dimensions we can view the gauge bosons as Kaluza Klein gravitons. In that case the flat space amplitude is very similar to (5.4.110) except that we now have

$$\mathcal{A}_g = p_-^2 \frac{\Gamma(1-\frac{\alpha' s}{4})\Gamma(1-\frac{\alpha' t}{4})\Gamma(1-\frac{\alpha' u}{4})}{\Gamma(1+\frac{\alpha' s}{4})\Gamma(1+\frac{\alpha' t}{4})\Gamma(1+\frac{\alpha' u}{4})} \tag{5.4.112}$$

If we take the small momentum limit of this amplitude we get a constant. If we integrate this constant with respect to s, in order to go to δ function wavefunctions, then we get an infinity. This is the infinity that we mentioned above as coming from the field theory diagram in 5.5(b). Of course the full string amplitude is not constant. Thus, once we go to string theory we should integrate the full string amplitude (5.4.110), which goes as $\mathcal{A}_g \sim s^{-1+\frac{\alpha' t}{2}}$ and converges if t is negative. We can define it by analytic continuation for other values of t. Thus, we see that for this particular case, taking the low energy limit of the amplitude first and then doing the s integral gives a very different answer than doing first the s integral and then the low energy limit, which gives (5.4.103). In the case of energy correlations, if we first take the low energy limit of (5.4.111) and then we do the integral we get the same answer as doing first the integral and then the low energy limit. In this case this happens because the contribution is coming mainly from the $s \sim 0$ and $u \sim 0$ region.

It is useful to perform explicitly the worldsheet operator product expansion of the two graviton vertex operators, see figure 5.7(b). We obtain[18]

$$p_- e^{ik_1 . y(\tau=0,\sigma)} p_- e^{ik_2 . y(0,0)} \sim p_-^2 |\sigma|^{\alpha' k_1 . k_2} [e^{i(k_1+k_2)y(0,0)} + \ldots] \tag{5.4.113}$$

[18]In the following expression it is convenient to redefine the range of σ to $[-\pi, \pi]$, such that insertions close to the operator at zero are given by small $|\sigma|$.

The pole arises when the power of σ is precisely $1/\sigma$. This gives rise to the $n = 0$ case in (5.4.108). The operator that appears at this point has the form

$$p_-^2 e^{ik.y} \,, \qquad m^2 = -k^2 = \frac{2}{\alpha'} \tag{5.4.114}$$

This is the operator that appears on the worldsheet in light cone gauge. One is tempted to write an operator in conformal gauge that would reduce to (5.4.114) in light-cone gauge. Due to its peculiar p_- dependence, we are forced to write an expression of the form[19]

$$(\partial_\alpha y^+ \partial_\alpha y^+)^{\frac{3}{2}} \delta(y^+) e^{ik.y} \tag{5.4.115}$$

with k obeying the condition (5.4.114). This operator is formally a Virasoro primary but is not a proper local operator on the string worldsheet. Similar operators were shown to control Regge physics in [214] . Of course, our regime is closely connected with Regge physics so it is not a surprise that similar operators appear. The operator (5.4.115), without the delta function, has spin $j = 3$ in the y^+, y^- plane, as we had in the field theory discussion. This is related to the factor of p_-^2 that appears in (5.4.113). Notice that the complete operator (5.4.115), including the delta function, has total spin 2. It corresponds to the field theory operator $\mathcal{U}_{j-1}$ with $j = 3$.

The operator (5.4.114) is the leading contribution in the worldsheet OPE (5.4.113). As we expand the exponentials at higher orders we pick up new operators which contain derivatives with respect to the transverse directions. Some of these operators can have transverse spin. These strings states have higher masses. They all have spin $j = 3$ in the $y^{\pm}$ directions. Notice, however, that terms that come with odd powers of σ in the $\ldots$ in (5.4.113) vanish upon integration over σ. The reason is a Z_2 symmetry from interchanging $\sigma \to -\sigma$. This is completely analogous to the fact that terms that are not symmetric under the interchange of $z \to \bar{z}$ in the usual case where we

[19] Notice that the extra power of p_- appears to compensate the one appearing from the delta function $\delta(y^+) \sim \frac{\delta(\tau)}{p_-}$.

integrate over the whole complex plane vanish upon integration. This is nothing else than the level matching condition. These terms are the same as the ones discussed above in connection with the singularities for odd powers of $\frac{1}{s}$. Now we understand why these poles are absent from (5.4.95).

In the end, the singularities arise from a fairly ordinary worldsheet operator product expansion in light cone gauge. On the other hand, we cannot associate the corresponding worldsheet operator to an ordinary closed string state. This is related to the fact that the operator product expansion of two energy flux operators in the field theory lead to non-local operators in the y^- direction. In the field theory we were not too disturbed by the appearance of operators that are non-local in the y^- direction, so we should also not be surprised that in string theory we also get string states that are non-local in the y^- direction. These string states are localized at $y^+ = 0$, carry zero p_- and are local in the transverse directions.

The final conclusion of this discussion is that we should interpret the singularities in (5.4.95) as arising from the propagation, in the transverse space, of non-local string states created by the operators like (5.4.114) or (5.4.115).

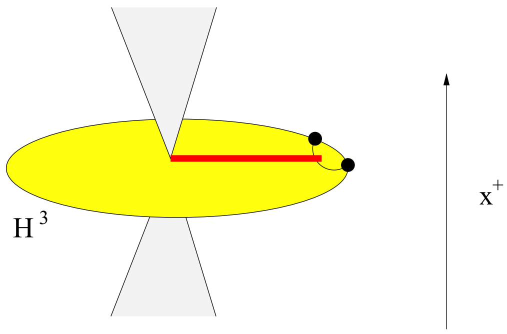


Figure 5.8: Small angle expansion of the energy correlation function. The expansion is dominated by the propagation of a spin three non-local string state, denoted here by a thick red line.

We will now argue that the short distance singularity of the energy flux two point function is governed by the operator associated to the first singularity in (5.4.95). Since we are working in the regime of large but fixed λ we might imagine that we could always expand the two point function as in the second line of (5.4.95). This is not correct if the angle is very small. In that case the relevant relative momenta are of order $t \sim \frac{1}{|\theta_{12}|^2}\frac{1}{R_{AdS}^2}$. Thus, at angles of order $\theta \sim \lambda^{-1/4}$ we cannot use the approximations leading to (5.4.99). However, we can use the interpretation given above to the poles in t to write the flat space result in the first line of (5.4.95) as a sum over contributions of poles. Then each pole corresponds to the contribution of a physical (but non-local) string state that is localized in the y^+ direction but propagating in the transverse directions. We generalize this result to AdS by replacing the transverse space by H^3. Now the non-local string states propagate on the H_3 subspace of AdS_5. These states propagate from the center of H_3, where the string state created by the localized operator insertion is concentrated, to a region near the H_3 boundary, near the insertion of the two energy flux operators, see 5.8. At large distances from the H_3 center we expect that the wavefunction of the non-local string state goes as $1/|\vec{W}|^\Delta$, $|\vec{W}| \gg 1$, with

$$\Delta \sim mR_{AdS} \sim \sqrt{2}\lambda^{1/4} + \cdots \tag{5.4.116}$$

where m is given, to a good approximation, by the mass of the flat space state computed in (5.4.114). Incidentally, we can calculate the conformal weight Δ of other (generally non local) operators with arbitrary spin in the same manner. They correspond to the string states

$$(\partial_\alpha y^+ \partial_\alpha y^+)^{\frac{j}{2}} \delta(y^+) e^{ik.y} \tag{5.4.117}$$

The mass of these states is given in flat space by $m^2 = -k^2 = \frac{2}{\alpha'}(j-2)$. Therefore

$$\Delta(j) \sim \sqrt{2}\sqrt{j-2}\lambda^{1/4} + \cdots \tag{5.4.118}$$

This formula is expected to be a good approximation only for $j \ll \lambda^{1/2}$, since it was derived assuming the flat space approximation. For very large values of j we get a logarithmic behavior in j, see [44]. Of course this is simply the analytic continuation of the leading Regge trajectory.

The dots in (5.4.116) and (5.4.118) denote terms independent of λ as well as higher order corrections. We then need to compute the overlap of a wavefunction which decays like $1/|\vec{W}|^{\Delta}$ for large $|\vec{W}|$ with the wavefunctions of the two gravitons associated to the energy flux insertions. We find a behavior

$$\langle \mathcal{E}(\theta_1)\mathcal{E}(\theta_2)\cdots\rangle \sim \theta_{12}^{\Delta-6}\langle \mathcal{U}_{3-1}(\theta_2)\cdots\rangle \tag{5.4.119}$$

where $\mathcal{U}_{3-1}$ is related to the lightest spin 3 non-local operator, with zero p_-, which at strong coupling has a large dimension (5.4.116). In string theory this expectation value is computed by inserting the operator (5.4.114).

In conclusion, the structure of the OPE is precisely what we expected from general principles in any conformal field theory. At weak coupling the operator product expansion is dominated by operators of twist slightly bigger than two. This leads to correlation functions that are highly localized along certain jet directions. For any value of the coupling the operator, or string state, that dominates has zero p_- and spin $j = 3$. At strong coupling, the operator acquires a large twist given in (5.4.116). The fact that operators with spin $j > 2$ have large dimensions at strong coupling is seen to be intimately related with the fact that the energy distribution is uniform. Of course, this fact is also connected with the validity of the gravity approximation in the bulk.

5.5 Discussion

Let us summarize some of our results.

We studied energy correlation functions in conformal field theories. Energy correlation functions are an infrared finite quantity that is useful for characterizing the states produced by localized operator insertions in a field theory [147,148], [178,179].

They can be computed for all values of the coupling since they involve the stress tensor operator [149] and make no reference to a partonic description. This is more manifest at strong coupling where the partons are difficult to see in the gravity or string description.

After a conformal transformation these energy correlation functions amount to measuring the state along a null surface. More precisely, each "calorimeter" insertion corresponds to an integral of the stress tensor along a lightlike line, $\int dy^- T_{--}$, (5.2.8).

We have argued that the small angle behavior of the energy correlation functions is controlled by an operator product expansion which features non-local light-ray operators of definite spin. When two calorimeters come close to each other we have a spin three operator $\langle \mathcal{E}(\theta_1)\mathcal{E}(\theta_2)\cdots\rangle \sim |\theta_{12}|^{\tau_3-4}\langle \mathcal{U}_{3-1}(\theta_2)\cdots\rangle$. These operators can be discussed for any coupling. We recalled the weak coupling expression for the twist [172] (5.2.24), and we also computed the twist at strong coupling $\tau \sim \sqrt{2}\lambda^{1/4}$ (5.4.116), after having identified the string states that are dual to the operator $\mathcal{U}_{3-1}$. These are not ordinary closed string states. They are peculiar string states localized along $x^+ = 0$ that have non-local vertex operators on the covariant worldsheet but do have a local description on the worldsheet in light-cone gauge. Closely related string states appear in the Regge limit [214]. Despite their unfamiliar features they control the short distance singularities of energy correlation functions.

The light-ray operators that appear in the small angle behavior of the correlator are related to the ones that control the moments of the parton distribution functions. In fact, one can write a precise relation between the energy correlation functions on a special state and a particular moment of the functions that govern the deep inelastic scattering amplitude (5.2.51).

We have seen that energy flux one point functions in states created by currents or stress tensor insertions have an "antenna" pattern[20] which is determined by the three point functions in the conformal field theory (5.2.35) (5.2.39). In the gravity description this pattern is spherical but as we include higher order corrections to the gravity action we start seeing deviations from the spherical pattern (5.3.72) (5.3.77). These deviations are sensitive to the spin of the operator that created the excitation in the conformal field theory. In the particular case of $\mathcal{N} = 4$ super Yang Mills, the energy one point function is spherical for all values of the coupling. In more general $\mathcal{N} = 1$ superconformal theories we find that the antenna pattern, (5.2.38) (5.2.41), depends on the parameters a and c that characterize the three point functions in the current/stress tensor multiplet [185, 186] [184]. These results are exact expressions, valid for any coupling. They depend only on the two anomaly coefficients a and c defined in [185,186]. Demanding that the energy that calorimeters measure is positive we get a constraint on a and c, $|a - c| \leq c/2$, which is saturated for free field theories (of course, $c > 0$). A deeper discussion on this feature of the preoblem can be found in the next chapter of this thesis. There, it is understood how the requirement translates to the dual gravitational theory. Also, a more formal argument is presented in favor of the positivity of the energy operator. The result guarantees that this should be the case for any UV complete Quantum Field Theory.

On the gravitational side, we present in this chapter a general prescription for computing the energy correlation functions on the from this dual perspective. The operator insertion in the field theory produces a string state that falls into the AdS horizon. Energy correlation functions depend on the wavefunction of this string state at the AdS horizon. The falling string is probed by particular shock waves associated to the insertion of each calorimeter. This can be computed in a simple way by choosing a coordinate system in AdS_5 that is non-singular at the horizon. We can

[20] Remember that individual events do not present this pattern. This refers to the one point functions which consist of averages over events.

view the computation of the energy correlation functions as taking a snapshot of the falling string state as it crosses the horizon. In the gravity approximation the result depends only on the momentum distribution of the initial state and it is independent of the spin or any other property of the string state we consider. If the state carries a purely timelike momentum $q^\mu = (q^0, \vec{0})$, then the energy distribution on the detector is perfectly spherical with no fluctuations. As we include stringy corrections we find small fluctuations that are inversely proportional to the square of the radius of AdS in string units (or $1/\sqrt{\lambda}$) (5.4.99) (5.4.100). These fluctuations are small but they are not gaussian (5.4.102). Since the shock waves we are considering are infinitely localized one might worry that this leads to divergences. In fact, they would lead to divergent answers in a field theory context (at least in some cases). The Regge behavior of string amplitudes at large energies ensures that the results we obtain are finite.

5.6 Appendix A: Positivity of $\int dy^- T_{--}$

Let us consider first free field theories. The classical expression for the stress tensor for the Maxwell field, $T_{--} \sim \sum_{i=1,2} F_{-i}F_{-i}$, is explicitly positive since it is a sum of squares. On the other hand, the quantum expectation value of T_{--} can be negative. Let us recall why this happens. Formally, we also have the sum of squares of hermitian operators, so that we would also expect a positive answer. However, when we normal order we subtract and infinite constant. Then the normal ordered expression is not a sum of squares of hermitian operators. In fact, we have schematically $T_{--} \sim (a^\dagger)^2 + a^\dagger a + a^2$ where we have separated the operator into terms with different numbers of creation and annihilation operators. By considering a state of the rough form $|\Psi\rangle = |0\rangle + \epsilon a_1^\dagger a_2^\dagger |0\rangle$, and using that the vacuum expectation value of T_{--} is zero we find that $\langle\Psi|T_{--}|\Psi\rangle \sim Re[c\epsilon] + o(\epsilon^2)$ where c is some number. By taking ϵ to be a small complex number we see that we can make T_{--} negative at a point [215].

Let us now consider the integrated expression $\mathcal{E} = \int dy^- F_{-i} F_{-i}$. This expression has the schematic form

$$\mathcal{E} \sim \int_0^\infty dp^+ p^+ (a_{p^+}(\vec{y}))^\dagger a_{p^+}(\vec{y}) \tag{5.6.120}$$

we thus see that we have the integral of products of operators and their adjoints. This is an explicitly positive operator. We have used the variable $p^+ \sim -2p_-$ which is positive. Notice that terms with two $a^\dagger$ or two a operators have disappeared from (5.6.120) due to the following argument. The integral over y^- enforces that the total p^+ should be zero. However, creation operators can only increase p^+, thus we do not obey the $p^+ = 0$ constraint with only creation operators. Further discussion on the null energy condition for free fields can be found in [153].

Let us consider now an interacting field theory. If we choose the gauge $A_- = 0$, then the stress tensor T_{--} continues to be quadratic in the fields and the above argument would hold. Of course, this argument is not too convincing since we might be ignoring renormalization subtleties or problems with the gauge choice.

A more general argument that applies to any UV complete theory is discussed in Chapter 6.

5.7 Appendix B: Energy distributions in gravity for generic states

5.7.1 Energy distributions for general states

Here we show how to go between a discussion of n point functions and the computation of probabilities for seing various energy distributions on the detector. Sometimes one might be interested in computing the probability functional $\rho[\mathcal{E}(\theta)]$ for measuring a particular pattern of energy deposition on the calorimeters. When one computes jet amplitudes one is computing probabilities of this kind, where one integrates over certain regions, such as the low energy region between two jets, etc.

If we are given ρ we can compute the n point functions, as $\langle \mathcal{E}(\theta_1)\cdots\mathcal{E}(\theta_n)\rangle = \int \mathcal{D}\mathcal{E}\rho[\mathcal{E}]\mathcal{E}(\theta_1)\cdots\mathcal{E}(\theta_n)$.

Formally one can also go in the other direction by computing the generating functional for energy correlation functions, $\langle e^{i\int d^2\theta\lambda(\theta)\mathcal{E}(\theta)}\rangle$. This expression is a functional of $\lambda(\theta)$ and its expansion in powers of λ gives us the n point functions. Then ρ is given by

$$\rho[\mathcal{E}'] = \int \mathcal{D}\lambda e^{-i\int d^2\theta\lambda(\theta)\mathcal{E}'(\theta)}\langle e^{i\int d^2\theta\lambda(\theta)\mathcal{E}(\theta)}\rangle \qquad (5.7.121)$$

Just in order to see how this works, let us start with the n point functions given in (5.3.90). We can easily compute the expression

$$\langle e^{i\int d^2\theta\lambda(\theta)\mathcal{E}(\theta)}\rangle = \int d^4q\rho(q)e^{i\int d^2\theta\lambda(\theta)\mathcal{E}_{q^\mu}(\theta)} \qquad (5.7.122)$$

where $\mathcal{E}_{q^\mu}(\theta)$ is the function in (5.3.88) and ρ is defined in (5.3.90). After doing the functional integral in (5.7.121) we get

$$\rho[\mathcal{E}] = \int d^4q\rho(q)\prod_\theta \delta[\mathcal{E}(\theta) - \mathcal{E}_q(\theta)] \qquad (5.7.123)$$

We see that we have a continuum of δ functions, one for each angle. But we are integrating only over four variables. Thus, once we fix the energy at four points, the energy at all other points is also fixed.

The general, formal, string theory expression for the energy correlators has a similar form, except that we have to integrate over the infinite number of variables specifying the string wavefunction. At finite N we would also have many string states.

5.7.2 Bulk wavefunction for a localized state

Here we start with the wavefuntion $\phi_0(x) \sim e^{-iq^0 t} e^{-\frac{(t^2+\vec{x}^2)}{\sigma^2}}$ that we mentioned in (5.2.30). Its Fourier transform is

$$\begin{aligned} \tilde{\phi}_0(p) &= \int d^4x e^{-ipx} \phi_0(x) = \int d^4x e^{ip^0 t - i\vec{p}\vec{x}} e^{-iq^0 t - \frac{(t^2+\vec{x}^2)}{\sigma^2}} \\ &\sim \sigma^4 e^{-\frac{\sigma^2}{4}[(p^0-q^0)^2+(\vec{p})^2]} \end{aligned} \tag{5.7.124}$$

The bulk wavefunction then has the form

$$\phi(W^+=0, W^-, W^\mu) \sim (q^0)^\Delta \int_0^\infty d\tilde{\lambda} \tilde{\lambda}^{\Delta-1} e^{i\tilde{\lambda} W^- q^0/2} e^{-\frac{(\sigma q^0)^2}{4}[(\tilde{\lambda} W^0 - 1)^2 + |\tilde{\lambda} \vec{W}|^2]} \tag{5.7.125}$$

We are considering the case that $q^0\sigma \gg 1$. We then see that as soon as $|\vec{W}| \gg 1/(\sigma q^0)$ the answer is exponentially suppressed. In the region with large $|\vec{W}| \sim W^0 \gg 1$ we can do the integral (5.7.125) by saddle point approximation and we find

$$\phi(W) \sim (q^0)^\Delta (W^0)^{-\Delta} \frac{1}{(\sigma q^0)} e^{iq^0 W^-/(2W^0)} e^{-\frac{(\sigma q^0)^2}{8}} \tag{5.7.126}$$

We then insert this in (5.3.86) to find an expression of the approximate form

$$\langle \mathcal{E}(\vec{n}_1') \mathcal{E}(\vec{n}_2') \rangle \sim q_0^2 e^{-\frac{(\sigma q^0)^2}{4}} \int d\Sigma_3 \frac{1}{(W^0)^{\Delta+2}} \frac{1}{(W^0 - \vec{W}.\vec{n}_1')^3} \frac{1}{(W^0 - \vec{W}.\vec{n}_2')^3} \tag{5.7.127}$$

where we used that N^2 in (5.3.86) is not exponentially small, and in (5.7.127) we have kept only the leading exponential behavior in σq^0. We have also approximated the integrand in the large W^0 region which we expect to dominate for the singular small angle behavior of the two point function. Finally we find the singular small angle behavior

$$\langle \mathcal{E}(\vec{n}_1') \mathcal{E}(\vec{n}_2') \rangle \sim |\theta|^{2\Delta-4} e^{-(\sigma q^0)^2/4} \tag{5.7.128}$$

This is precisely the power we expect for the double trace contribution as was discussed in [216]. This term is exponentially suppressed when we consider a state with definite momentum. Thus, the term that gave the largest contribution in the deep inelastic scattering analysis in [216] does not contribute to energy correlators when we consider states created with definite momentum. They do contribute if the state does not have definite momentum in x-space. In fact, we saw that a state with definite momentum in y space is directly connected with the deep inelastic scattering amplitudes (5.2.51). Such a state does not have definite momentum in x-space and will not have the exponential suppression that we get in (5.7.128).

We should emphasize that the contribution to (5.7.128) is coming from the region where the particle is crossing the horizon at a position that is close to where the calorimeters are inserted.

5.8 Appendix C: Computation of the energy and charge one point function in terms of the three point functions of the CFT

We denote by $\mathcal{O}$ any operator, which could be a scalar operator $\mathcal{S}$ or a vector $\epsilon.j$ or a tensor $\epsilon_{ij}T_{ij}$.

Let us start by recalling some formulas for two point functions

$$\begin{aligned}\langle 0|S(t,x)S(0,0)|0\rangle =& \frac{1}{[-(t-i\epsilon)^2+|\vec{x}|^2]^{\Delta}}\\ \langle 0|T(S(t,x)S(0,0))|0\rangle =& \frac{1}{[-t^2+|\vec{x}|^2+i\epsilon]^{\Delta}}\end{aligned} \tag{5.8.129}$$

where the first is not time ordered and the second is time ordered. Of course the same prescription works for vector or tensor operators. The operator insertion with a definite timelike momentum $q^\mu = (q^0, \vec{0})$ can be written as

$$\mathcal{O}_q|0\rangle = \int dt e^{-iq^0 t}\mathcal{O}(t)|0\rangle \tag{5.8.130}$$

and it creates a state with energy $E = q^0 > 0$. The fourier transform of the two point function is

$$\int d^4x e^{-iq.x}\frac{1}{[-(t-i\epsilon)^2+|x|^2]^\Delta} = c(\Delta)\theta(q^0)(-q^2)^{\Delta-2}\ , \qquad c(\Delta) = \frac{(2\pi)^3(\Delta-1)}{4^{\Delta-1}\Gamma(\Delta)^2} \tag{5.8.131}$$

This is the norm of the state that (5.8.130) creates. This will also give us the total production cross section if the operator $\mathcal{O}$ couples to the standard model. As remarked in [217] the positive norm condition implies $\Delta \geq 1$.

We are interested in starting from the ordinary expressions for the correlation functions in position space and extracting the limit that corresponds to the energy or charge correlators. In doing so, it is important to order the operators appropriately. For the non-time-ordered three point function the correct prescription is

$$\begin{aligned}&\langle 0|\mathcal{S}(x_2)\mathcal{S}(x_1)\mathcal{S}(x_3)|0\rangle = \\ &=\frac{1}{\{[-(t_{23}-i\epsilon)^2+(\vec{x}_{23})^2][-(t_{13}-i\epsilon)^2+(\vec{x}_{13})^2][-(t_{21}-i\epsilon)^2+(\vec{x}_{21})^2]\}^{\Delta/2}}\end{aligned} \tag{5.8.132}$$

If one considers tensor operators we get similar denominators and we choose the same $i\epsilon$ prescription. This $i\epsilon$ prescription is a simple way to enforce the right ordering of the operators. Another way to say this is that an operator that is to the 'left' of another should have a more negative imaginary part in the time direction. When one does perturbation theory, it might be convenient to use time ordering along a Keldysh contour. However, for our purposes this simple prescription suffices.

Let us first show how to extract the energy correlation for a state created from a scalar operator with fixed momentum, or at least fairly well defined momentum, as in (5.2.30). In this case, we know that the answer is independent of the angles and that the overall coefficient is determined by energy conservation. Nevertheless it is instructive to discuss this case in detail since the computation is the simplest and one can apply a similar method for other cases. Our method is not too elegant and there is probably a more direct and elegant method than the one we applied here.

We extract the energy correlation by directly performing the limit and the integral in (5.1.1). We use translation invariance to fix the position of the first operator at $x_3 = 0$. For simplicity we place the detector along the direction z, so that $x_1 = (t, 0, 0, r)$. We will take the limit $r \to \infty$. If t is generic, then the three point function will decay as $1/r^8$ since it would be determined by the dimension of the stress tensor and the operator product expansion. This would be decaying too rapidly in order to give a finite large r limit. However, there is a larger contribution from the region $t \sim r$, the region on the light-cone of the inserted operators. This is the region that will contribute. Of course, this is precisely what we would expect in a theory of massless particles. It is convenient to define coordinates $x^\pm = t \pm r$. We will find that the region with finite x^- will contribute. In addition, only the T_{--} component of the stress tensor can contribute. The integral over t can be traded for an integral over x^-. So we first take the $r \to \infty$ limit. We then do the integral over x^- and at the end we do the integral over x_2.

Let us see what each of these steps gives us. In order to follow this appendix the reader would need to have a copy of the paper by Osborn and Petkos [183], since we will make frequent reference to it. We start with the correlation function for

$$\langle 0|\mathcal{S}(x_2)T_{--}(x_1)\mathcal{S}(x_3)|0\rangle \sim \frac{1}{x_{23}^{2\Delta-2}x_{12}^2x_{13}^2}\left(\frac{x_{12}^+}{x_{12}^2} - \frac{x_{13}^+}{x_{13}^2}\right)^2 \qquad (5.8.133)$$

from equation (3.1) of [183]. We are not going to keep track of overall numerical coefficients. After multiplying by r^2 and taking the $r \to \infty$ limit we get

$$\lim_{r\to\infty} r^2 \langle 0|\mathcal{S}(x_2)T_{--}(x_1)\mathcal{S}(0)|0\rangle \sim \frac{(x_2^-)^2}{(x_2^2)^{\Delta-1}} \frac{1}{(x^- - i\epsilon)^3(x^- + i\epsilon - x_2^-)^3} \tag{5.8.134}$$

We now perform the integral over x^-. Note that we can close the contour on either the upper or lower x^- plane and pick one of the two poles in (5.8.134). We then find

$$\lim_{r\to\infty} r^2 \langle 0|\mathcal{S}(x_2) \int dx^- T_{--}(x_1)\mathcal{S}(0)|0\rangle \sim \frac{1}{(x_2^2)^{\Delta-1}} \frac{1}{(x_2^- - 2i\epsilon)^3} \tag{5.8.135}$$

We now integrate over the two transverse x_2 coordinates and use the wavefunction in (5.8.130) does not depend on them. We find

$$\begin{aligned}\lim_{r\to\infty} r^2 \int d^4x_2 e^{iq^0 t_2} \langle 0|\mathcal{S}(x_2) \int dx^- T_{--}(x_1)\mathcal{S}(0)|0\rangle \sim \\ \sim \int dt_2 dz_2 e^{iq^0 t_2} \frac{1}{[-(t_2 - i\epsilon)^2 + z_2^2]^{\Delta-2}} \frac{1}{(t_2 - z_2 - 2i\epsilon)^3} \sim \theta(q^0)(q^0)^{2\Delta-3}\end{aligned} \tag{5.8.136}$$

where we have also done the remaining two integrals. When we divide by the two point function (5.8.131) we get $\langle \mathcal{E} \rangle \sim q^0$. The numerical coefficient can also be computed at each step. Of course, this gives the right answer $\langle \mathcal{E} \rangle = \frac{q^0}{4\pi}$ due to the Ward identity which fixes the coefficient of the three point function (5.8.133) in terms of the coefficient of the two point function (5.8.129), see eqns (6.15), (6.20) of [183].

This procedure can be repeated replacing the operator $\mathcal{S}$ by a current $\epsilon.j$. The computations are identical but with more indices and we use a computer. In this case the three point function of a stress tensor and two currents is fixed by conformal invariance, plus the Ward identity, up to one unknown coefficient. Conformal invariance leaves two possible structures and the Ward identity fixes the coefficient of one of them. In this case we find, as expected, that the energy correlation function depends on the angle with respect to the vector ϵ. Here we simply quote the value of the parameter a_2, introduced in (5.2.32), in terms of the parameters $\hat{e}, \hat{c}$ defined in

(3.13) and (3.14) of [183] [21]. We find that

$$a_2 = \frac{3(8\hat{e}-\hat{c})}{2(\hat{e}+\hat{c})} \quad \rightarrow \quad 3\frac{\sum_i (q_i^b)^2 - (q_i^{wf})^2}{\sum_i (q_i^b)^2 + 2(q_i^{wf})^2} \tag{5.8.137}$$

where we indicated the value for a free theory with bosons and Weyl fermions of charges q_i^b and q_i^{wf}. The combination $(\hat{e}+\hat{c})$ is fixed in terms of the coefficient of the two point function of two currents via 6.26 of [183].

We can do this also for the correlation functions of the form $\langle 0|\epsilon_{ij}^* T_{ij}\mathcal{E}\epsilon_{ij}T_{ij}|0\rangle$. We can then compute the coefficients t_2, t_4 introduced in (5.2.39)

$$\begin{aligned} t_2 =& \frac{30(13\hat{a}+4\hat{b}-3\hat{c})}{14\hat{a}-2\hat{b}-5\hat{c}} \quad \rightarrow \quad \frac{15(-4n_v+n_{wf})}{(n_b+12n_v+3n_{wf})} \\ t_4 =& -\frac{15(81\hat{a}+32\hat{b}-20\hat{c})}{2(14\hat{a}-2\hat{b}-5\hat{c})} \quad \rightarrow \quad \frac{15(n_b+2n_v-2n_{wf})}{2(n_b+12n_v+3n_{wf})} \end{aligned} \tag{5.8.138}$$

where $\hat{a},\hat{b},\hat{c}$ are defined in (3.19)-(3.21) of [183]. We have also indicated the result for n_v, n_b and n_{wf} free vectors, real bosons, and Weyl fermions (one complex dirac fermion would give $n_{wf}=2$)[22]. Again, the combination appearing in the denominator is fixed in terms of the stress tensor two point function, see (6.42) of [183].

Two combinations of these three coefficients are related to the values of a and c defined through the conformal anomaly[23]

$$T_\mu^\mu = \frac{c}{16\pi^2}W^2 - \frac{a}{16\pi^2}E \tag{5.8.139}$$

where W is the Weyl tensor and $E = R_{\mu\nu\delta\rho}R^{\mu\nu\delta\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the Euler density. c also sets the two point function of the stress tensor. The coefficient a can be expressed

[21] Here we added hats to the parameters e and c in [183] so that they are not confused with other parameters in the present work. Note also that [183] uses some of these letters with multiple meanings through their paper.

[22] $\hat{a},\hat{b},\hat{c}$ here should not be confused with the $\hat{e},\hat{c}$ of the previous paragraph. In particular the two $\hat{c}$ are not the same [183].

[23] Do not confuse these a and c with the parameters $\hat{a}$ and $\hat{c}$ of the previous paragraph.

in terms of the three parameters in (5.8.138) as

$$\frac{a}{c}=\frac{(9\hat{a}-2\hat{b}-10\hat{c})}{3(14\hat{a}-2\hat{b}-5\hat{c})} \quad \rightarrow \quad \frac{2n_b+124n_v+11n_{wf}}{6n_b+72n_v+18n_{wf}} \tag{5.8.140}$$

This follows from (8.37) in [183] . The values for a free theories were computed in [187], [188]. From the positivity conditions (5.2.40) it is possible to get general bounds on this ratio. We find

$$\frac{31}{18} \geq \frac{a}{c} \geq \frac{1}{3} \tag{5.8.141}$$

where the lower bound is saturated by a free theory with only scalar bosons and the upper bound by a free theory with only vectors. Notice that this bound holds for any conformal theory while the more restrictive bound (5.2.42) holds for supersymmetric theories. We can also add that for a $\mathcal{N}=2$ supersymmetric theory we can find a similar bound by using as limiting cases free theories with only vector supermultiplets ($\frac{a}{c}=\frac{5}{4}$) and free theories with hypermultiplets ($\frac{a}{c}=\frac{1}{2}$). Therefore $\frac{5}{4} \geq \frac{a}{c} \geq \frac{1}{2}$. This agrees with results from [192].

In an $\mathcal{N}=1$ supersymmetric theory there is a relation between the three parameters $\hat{a}, \hat{b}, \hat{c}$ which is obtained by setting $t_4=0$ in (5.8.138). In this case, the two coefficients in (5.8.139) specify completely the three point functions of the stress tensor. In a non-supersymmetric theory, we have one more parameter beyond the two in (5.8.139).

Finally, we can repeat this exercise for the correlation function of three currents to find that the coefficient introduced in (5.2.45)

$$\tilde{a}_2=\frac{3}{2}\frac{5\hat{a}-4\hat{b}}{(\hat{a}+4\hat{b})} \quad \rightarrow \quad 3\frac{\sum_i C(r_i)_b-C(r_i)_{wf}}{\sum_i C(r_i)_b+2C(r_i)_{wf}} \tag{5.8.142}$$

where $\hat{a}, \hat{b}$ are defined in eqn. (3.9) of [183], and are not the same as the ones in the previous paragraphs. Here r_i are the representations of the bosons and Weyl fermions.

And $C(r)$ is defined as $tr_r[T^aT^b] = C(r)\delta^{ab}$. Again, the combination $(\hat{a} + 4\hat{b})$ sets the two point function of the current. And $\tilde{a}_2$ vanishes in a supersymmetric theory since there is only one structure contributing for the supersymmetric case [184] and it vanishes for a free supersymmetric theory.

One can also take similar limits of the parity odd part of the three point function of three currents and one obtains the result in (5.2.46). Finally, starting from the correlation functions for two stress tensors and a current derived in [194] one can derive the charge distribution function in (5.2.47). Both of these results relate the corresponding anomaly to a charge asymmetry.

5.9 Appendix D: Energy one point functions in theories with a gravity dual

In this appendix we present the calculation of energy one point functions for states created by current operators and the stress energy operator.

5.9.1 One point function of the energy with a current source

We wish to compute the contributions to the energy one point function (5.2.32) for a state created by a current operator at strong coupling. The AdS/CFT dictionary says we need to compute the bulk three point function between two bulk photons and the graviton. The bulk action (5.3.70) contains two terms. The first term contributes also to the current two point function while the second term in (5.3.70) does not. Thus, the first term contributes to the part of the energy one point function which is determined by the Ward identity and the second one to the angular dependent term which is not fixed by the Ward identity. In principle, the first term could also contribute to the angularly dependent part, but we have argued, based on the results for $\mathcal{N} = 4$ SYM that it contributes only to the constant part. Thus in order to compute the coefficient a_2 in (5.2.32) we need to compute the ratio of the contribution of the second term in (5.3.70) and the contribution of the first term in (5.3.70).

Since the graviton is localized in W^+ and the photon is localized in the other transverse directions if the state has definite four dimensional momentum, we can approximate the computation as a flat space computation. In this particular case, this approximation will be exact, but that will not be the case when we discuss the three graviton vertex. Thus, we evaluate the vertex expanding the flat space action, but we will insert the AdS wavefunctions for the external states.

In flat space our coordinates are $(x^+, x^-, x^{1,2,3})$. The metric is $ds^2 = -dx^+dx^- + dx^i dx^i$ (latin indices i and j go from 1 to 3).

We want to collect terms that are of first order in the perturbation h. There are two such terms, one per factor of $g_{\mu\nu}$ in the action. The determinant g does not receive corrections and $\sqrt{g} = \frac{1}{2}$. Our perturbation is of the form $h = h_{++}(x^i, n^i)\delta(x^+)(dx^+)^2$. We made explicit the fact that h_{++} depends on the transverse coordinates and on a unit vector n^i in the transverse space that represents the position of our calorimeter.

Therefore, we want to calculate

$$S_1 = -\frac{1}{4g^2}\int \frac{dx^+dx^-d^3x}{2} 2h_{++}F^{+i}F^{+j}g_{ij} = -\frac{1}{4g^2}\int dx^+dx^-d^3x h_{++}F^{+i}F^{+j}g_{ij} \tag{5.9.143}$$

Notice that the contraction of Fs is restricted to the 3 dimensional transverse space as the metric element g_{--} is zero. We can do the x^+ integral easily as h_{++} is localized in this direction. Also, we will use the fact that the wave function of the photon is localized in the transverse space (5.3.67). We represent this fact by writing

$$F^{+i}F^{+j}g_{ij} = \alpha(x^+, x^-)\delta^3(x^i)\epsilon^i\epsilon^j g_{ij} = \alpha(x^+, x^-)\delta^3(x^i) \tag{5.9.144}$$

where ϵ^i represents the (normalized) polarization of the photon. Notice that we choose the polarization in the transverse directions. Therefore $F^{+i} \sim \partial^+ A^i$. Using these facts and performing the integrals we get

$$S_1 = -\frac{1}{4g^2}\int dx^+ dx^- d^3x h_{++} F^{+i}F^{+j}g_{ij} = -\frac{1}{4g^2}h_{++}(0,n^i)\int dx^-\alpha(0,x^-) \quad (5.9.145)$$

In fact, h_{++} evaluated at $x^i = 0$ does not depend on n^i, see (5.3.62) at $W^i = 0$. Therefore, this term does not contribute to the angular dependence of the correlation function.

Let us now look at the other term in (5.3.70). We first need to compute the Weyl tensor. Let us start with the Riemann tensor. This tensor has terms that go as $\frac{1}{2}\partial^2 g$ and terms that go as $g\Gamma\Gamma$. Since we are in a flat space background only the first type of term contributes. This yields

$$R_{+i+j} = \frac{1}{2}\partial_i\partial_j h_{++} \quad (5.9.146)$$

All other terms are given by symmetry properties (i.e. $R_{+i+j} = -R_{i++j} = -R_{+ij+} = R_{i+j+}$) or vanish. The Weyl tensor also contain terms of the form $\frac{1}{3}g_{\lambda\nu}R_{\mu\kappa}$. But,

$$R_{\mu\nu} = g^{\lambda\rho}R_{\lambda\mu\rho\nu} \quad \longrightarrow \quad R_{++} = g^{ij}R_{+i+j} = \frac{1}{2}g^{ij}\partial_i\partial_j h_{++} \quad (5.9.147)$$

We see there is only one non vanishing term that is proportional to the laplacian inside the transverse space. There are also terms proportional to the Ricci scalar inside the Weyl tensor, but we can see that these vanish in our case. The Weyl tensor is, then, given by

$$C_{+i+j} = \frac{1}{2}\left(\partial_i\partial_j - \frac{1}{3}g_{ij}\partial^k\partial_k\right)h_{++} \quad (5.9.148)$$

The other components either vanish or are given by symmetry properties. There are four possible positions for the two plus signs, so we will have four terms in the second term in (5.3.70) (we are also using symmetry properties of F^{+i}). That is

$$\begin{aligned} S_2 =& \frac{\alpha}{g^2 M_*^2} \int \frac{dx^+ dx^- d^3x}{2} 4C_{+i+j} F^{+i} F^{+j} = \\ =& \frac{\alpha}{g^2 M_*^2} \int dx^+ dx^- d^3x F^{+i} F^{+j} \left(\partial_i \partial_j - \frac{1}{3} g_{ij} \partial^k \partial_k \right) h_{++} \end{aligned} \tag{5.9.149}$$

Once again we can perform the integrals to obtain

$$S_2 = \frac{\alpha}{g^2 M_*^2} \left(\partial_i \partial_j - \frac{1}{3} g_{ij} \partial^k \partial_k \right) h_{++}(x^i, n^i) \big|_{x^i=0} \epsilon^i \epsilon^j \int dx^- \alpha(0, x^-) \tag{5.9.150}$$

We are interested in the quotient between the angularly dependent term (5.9.150) and the spherically symmetric term (5.9.145)

$$-\frac{4\alpha}{M_*^2} \frac{\left(\partial_i \partial_j - \frac{1}{3} g_{ij} \partial^k \partial_k \right) h_{++}(x^i, n^i) \big|_{x^i=0} \epsilon^i \epsilon^j}{h_{++}(0, n^i)} \tag{5.9.151}$$

We use the explicit form of the perturbation (5.3.62) and we get the result

$$a_2^{AdS} = -48 \frac{\alpha}{M_*^2} \tag{5.9.152}$$

This gives the gravity result for the anisotropic part of the one point function (5.2.32) of a state produced by a current.

5.9.2 One point function of the energy with a stress tensor source

Now we want to repeat this calculation for the case where we have the stress tensor as a source. In this case we need to consider 3 graviton interactions. There are 3 operators that contribute to this vertex. A natural parametrization is given by the action (5.3.76).

We will do first the calculation in a flat space background. As in the computation we did above we will get derivatives acting on the perturbation h. When we go to the AdS background we could get terms involving the background curvature. Such terms are isotropic and will not contribute to the terms that have maximal angular momentum. But they do give contributions to the terms that have smaller values of the angular momentum. The computations we do here give only the leading contribution for t_2 and t_4 in (5.2.39). We start from the action in (5.3.76) and we expand each term to cubic order. We focus on terms with highest angular momentum in the transverse dimensions (see [203]). We use the fact that we need one of the metric perturbations to be h_{++} while the other two only have purely transverse indices. We find

$$\begin{aligned} R = &-\frac{1}{2}h_{++}h^{ij}_{(1)}(\partial^+)^2h^{ij}_{(2)}, \quad R_{\mu\nu\delta\sigma}R^{\mu\nu\delta\sigma} = -2\partial_i\partial_j h_{++}h^{ik}_{(1)}(\partial^+)^2h^{jk}_{(2)}, \\ R_{\mu\nu\delta\sigma}R^{\delta\sigma\rho\gamma}R_{\rho\gamma}{}^{\mu\nu} &= -6\partial_i\partial_j\partial_k\partial_\ell h_{++}h^{ij}_{(1)}(\partial^+)^2h^{k\ell}_{(2)} \end{aligned} \tag{5.9.153}$$

Notice that expanding the determinant of the metric in the action does not contribute to the three point function. If we now use that the wave function is going to be of the form

$$h^{ij}_{(1)}\partial^{+2}h^{k\ell}_{(2)} = \beta(x^+,x^-)\delta^3(\vec{x})\epsilon^{ij}\epsilon^{k\ell} \tag{5.9.154}$$

we can perform the integrals and calculate the quotients of the contribution to the three point function. After taking the derivatives and evaluating at $\vec{x}=0$ we obtain the ratios

$$t_2 = 48\frac{\gamma_1}{R^2_{AdS}M^2_{pl}} \tag{5.9.155}$$

$$t_4 = 4320\frac{\gamma_2}{R^4_{AdS}M^4_{pl}} \tag{5.9.156}$$

Due to the issues we discussed above, there are terms contributing to t_2 which are of first order in $\gamma_2/(R_{AdS}M_{pl})^4$, coming from the term with six derivatives in the action, which we neglected compared to the contribution of the four derivative terms. These formulas are also valid only to first other in the γ_i.

Chapter 6

Higher Derivative Gravity, Causality and Positivity of Energy in a UV complete QFT

In this chapter we discuss the relation between the constraints imposed by causality in the bulk of AdS and the condition of positivity of the energy measured in ideal calorimeters in a collider experiment in the dual CFT. We first extend the analysis in the literature and recover all bounds imposed by causality of the boundary theory in the bulk dynamics for all polarizations of the graviton and the gauge boson field. These results translate to specific bounds for the ratio of central charges $\frac{a}{c}$ in the dual CFT, already found by analyzing the energy one point function. Then, we generalize this discussion and we study shock wave backgrounds in which we make manifest the relation between causality in the bulk and the three point function in the dual field theory. We remark that particular care has to be given to the exponentiation procedure of the three point function when solving the classical equations of motion in the higher gravity theory, as it is not clear that every theory will present causality problems. Finally, we present a field theoretic argument explaining the positivity of energy condition in any UV complete QFT.

This work in this chapter is contained in [218].

6.1 Preliminaries

In the last few years we have seen a great deal of progress in understanding fundamental properties of quantum field theories and quantum gravity through the AdS/CFT correspondence [2–4]. Because this is a weak-strong coupling duality, it is a common characteristic of the correspondence that some features of the theory under consideration may appear in different guises as we study different regimes. While we have a weak coupling description in terms of a local quantum field theory, on the other hand we study the strong coupling regime as a classical theory of gravity (in the strong 't hooft coupling limit). It is clear that in these two extreme cases the fundamental degrees of freedom of the theory become reorganized and we obtain very different descriptions of the same phenomena. This is one of the reasons the AdS/CFT correspondence is so powerful.

The main phenomenon we would like to discuss in this Chapter was discovered from two different perspectives. On the one side, one of the most interesting new insights discovered through gauge/gravity duality is the presence of a universal behavior for the ratio of the shear viscosity (η) and entropy (s) density for field theories with an Einstein gravity dual. Because at strong coupling (when one would expect this ratio to be lower for a theory, as the mean free path goes to zero) all theories with an Einstein gravity dual converge to the same value of $\frac{\eta}{s}$, namely, $\frac{1}{4\pi}$, it was proposed, early on, that $\frac{\eta}{s}$ could be bound from below (KSS) [31, 32]. This observation led the authors of [33] and [189, 190] to consider what would happen if one considered higher derivative corrections to Einstein gravity[1]. It was understood in these papers that, unless some bound was imposed on the coefficients in the action for the higher derivative terms, the KSS bound would be violated. It was suggested in [189, 190] that requiring the theory on the boundary to be causal imposed such a

[1]In [219], gauge theories with large central charges were studied where these higher derivative terms are present in the dual gravitational description. For certain values of the central charges the KSS bound was found to be violated.

bound[2]. This restriction was insufficient to preserve the KSS bound. The new bound obtained in [189, 190] for a theory which is dual to five dimensional Gauss-Bonnet (GB) gravity was $\frac{\eta}{s} \geq \frac{16}{25}\frac{1}{4\pi}$ [189,190]. It is now understood that the KSS bound can't be correct and it is not clear whether any other bound could exist at all.

The action of five dimensional GB gravity is:

$$S_{GB} = \frac{1}{16\pi G_N} \int dx^5 \sqrt{g} \left[R + 12 + \frac{\lambda}{2} \left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right) \right] \qquad (6.1.1)$$

where the AdS radius of the $\lambda = 0$ solution was normalized to 1. This theory is special in the sense that it is the only one among 4 derivative actions of gravity in five dimensions in which the equations of motion for a perturbation propagating in a given background have only 2 derivatives. This makes the classical propagation of perturbations a well defined initial conditions problem. The causality bound in [189, 190] was obtained by propagating a helicity 2 graviton state[3] in a black hole background of Gauss-Bonnet gravity and noticing that if such a state were to be "dropped" from the boundary it would bounce back to it and land outside the light cone of the dual field theory. Therefore, causality in the boundary implied a bound on the coefficient of Gauss-Bonnet gravity in the bulk, $\lambda \leq \frac{9}{100}$. The upshot of this discussion is that certain restrictions of consistency in the field theory constrain the dual gravity action.

Having discussed the constraints in the gravitational theory, what is the easiest way to study the same physics in the field theory?

The other road that led to the discovery of equivalent physical bounds was the study of collider physics of conformal theories in Chapter 5 and [140]. With the

[2] Previously, it had been noticed in [220] that a less restrictive bound could be obtained by requiring that the entropy of black holes be positive.

[3] We define helicity as the eigenvalue of the state under $SO(2)$ rotations in the transverse coordinates. Notice that the theory also possesses helicity 1 and helicity 0 degrees of freedom as the 5 dimensional massless graviton multiplet has 5 excitations. We will make use of this fact shortly.

era of the LHC around the corner it makes sense to ask what insights can we get from AdS/CFT into collider physics. This is particularly interesting for (beyond the Standard) Models where there there is a hidden conformal sector, as in [34,150,152]. This type of systematic study was undertaken in [140] where the most general form of the energy and charge correlation functions [146–148] were calculated for a CFT. Because any UV complete theory is either conformal or free, this analysis describes the most general behavior in field theory at short distances[4]. The energy one point function determines the expected value of the energy accumulated in a calorimeter at the end of a collider experiment. Therefore, one expects this quantity to be positive. If the state in which we calculate the expectation value is created by the energy-momentum tensor and we have $\mathcal{N} = 1$ supersymmetry, then, the correlation function depends only on the central charges of the theory. Furthermore, the same is true for states created by R currents, which are in the same supermultiplet. It turns out, as remarked in [140], that one of the bounds obtained from the requirement that the energy deposited in calorimeters is positive coincides, for $\mathcal{N} = 1$ theories, with the bound discussed above in Gauss-Bonnet gravity. It is of great interest to note that while energy correlation functions are observables that were introduced because of their relevance to experimental setups, they seem to give us some new insights on the fundamental structure of our theories, imposing physical constraints.

We will discuss how these two approaches are related. It is a fact that we should look at (6.1.1) as an effective action specifically engineered to reproduce the two and three point functions of the energy-momentum tensor in the dual field theory. All other information contained in (6.1.1) should not be trusted at λ of order 1. The knowledge of two and three point functions amounts to knowing the energy one point function, which is the observable we discuss in the CFT. In general, classical

[4]The behavior of the theory at long distances can certainly change the readings at calorimeters in a collider experiment. This happens in a similar fashion to the way hadronization affects high energy QCD results.

calculations using higher derivative gravity actions involve more information than two and three point functions. We will show that peculiarities of GB gravity imply that the causality problem is directly connected with the energy one point function alone. Incidentally, there are other ways to parameterize the 4 derivative gravity action, such that the energy one point function is reproduced. We will discuss them below, but it is important to stress that they are not equivalent to GB gravity at the nonlinear level. Here, as the exponentiation procedure in the classical solutions is more involved, it is not clear whether there are any causality problems of the type discussed for GB gravity.

6.2 Some results for energy one point functions

In this section we recall some results concerning energy one point functions in CFTs and the dual gravitational actions that reproduce them. We follow the discussion in [140] closely but stress the results important for the problem at hand.

The n point energy correlation function is defined as

$$\langle \mathcal{E}(\theta_1)\cdots\mathcal{E}(\theta_n)\rangle_{\mathcal{O}} \equiv \frac{\langle 0|\mathcal{O}^\dagger \mathcal{E}(\theta_1)\cdots\mathcal{E}(\theta_n)\mathcal{O}|0\rangle}{\langle 0|\mathcal{O}^\dagger\mathcal{O}|0\rangle} \tag{6.2.2}$$

where $\mathcal{O}$ is an operator that creates an initial state and the energy operator $\mathcal{E}$ is given by

$$\mathcal{E}(\vec{n}) = \lim_{r\to\infty} r^2 \int_{-\infty}^{\infty} dt\, n^i T^0_{\ i}(t, r\vec{n}^i) \tag{6.2.3}$$

We point out that, for states created by local operators, energy n point functions are connected to $n+2$ correlation functions in the field theory: n energy momentum tensors and 2 extra insertions given by the operator $\mathcal{O}$ (which could be the energy momentum tensor as well).

It is easy to see that the expectation value corresponding to the one point function represents the total energy deposited in a calorimeter at a solid angle θ at the end of an experiment where the initial state is $\mathcal{O}|0\rangle$. As it is clear from expression (6.2.2), the energy one point function can be obtained from integration over the three point function of local operators (if $\mathcal{O}$ is local) and the energy-momentum tensor. Because 3 point functions are fixed in a conformal theory, except for a finite number of parameters [183], we can calculate the most general form of the energy one point function. It makes sense to consider as operators $\mathcal{O}$ a conserved current J^μ or the energy-momentum tensor itself $T^{\mu\nu}$ as these are generally present in arbitrary theories. The most general form for this one point functions is

$$\begin{aligned} \langle\mathcal{E}(\vec{n})\rangle_{J^i\epsilon_i} &= \frac{q}{4\pi}\left[1 + a_2(\cos^2\theta - \frac{1}{3})\right] && (6.2.4) \\ \langle\mathcal{E}(\vec{n})\rangle_{T^{ij}\epsilon_{ij}} &= \frac{q}{4\pi}\left[1 + t_2\left(\frac{\epsilon^*_{ij}\epsilon_{il}n_i n_j}{\epsilon^*_{ij}\epsilon_{ij}} - \frac{1}{3}\right) + t_4\left(\frac{|\epsilon_{ij}n_i n_j|^2}{\epsilon^*_{ij}\epsilon_{ij}} - \frac{2}{15}\right)\right] && (6.2.5) \end{aligned}$$

where θ is the angle between $\vec{n}$ and the (space-like) polarization of the current ϵ^i and q is the total energy of the state; ϵ_{ij} is the polarization tensor of the state created by the energy-momentum tensor. Although this form is fixed in a CFT, the parameters a_2, t_2 and t_4 are not fixed by symmetries. There is, however, a set of constraints that has to be satisfied if one demands that calorimeters can only pick up positive energies. These are:

$$\begin{aligned} 3 - a_2 &\geq 0 & (6.2.6) \\ a_2 + \frac{3}{2} &\geq 0 & (6.2.7) \\ (1 - \frac{t_2}{3} - \frac{2t_4}{15}) &\geq 0 & (6.2.8) \\ 2(1 - \frac{t_2}{3} - \frac{2t_4}{15}) + t_2 &\geq 0 & (6.2.9) \\ \frac{3}{2}(1 - \frac{t_2}{3} - \frac{2t_4}{15}) + t_2 + t_4 &\geq 0 & (6.2.10) \end{aligned}$$

It is very interesting to note that the three conditions on t_2 and t_4 given in equations (6.2.8),(6.2.9) and (6.2.10) come, respectively, from the helicity 2, 1 and 0 components of the polarization tensor ϵ_{ij} with respect to $SO(2)$ around $\vec{n}$, while equations (6.2.6) and (6.2.7) come from the helicity 1 and 0 components of ϵ_i. For reasons that will be clear momentarily we will be interested in theories with at least $\mathcal{N} = 1$ supersymmetry. In that case it is possible to write the coefficients a_2, t_2 and t_4 as a function of the central charges of the the theory, a and c. These charges are defined by computing the trace anomaly as

$$T^\mu_\mu = \frac{c}{16\pi^2} W_{\mu\nu\delta\sigma} W^{\mu\nu\delta\sigma} - \frac{a}{16\pi^2} \left(R_{\mu\nu\delta\rho} R^{\mu\nu\delta\rho} - 4R_{\mu\nu} R^{\mu\nu} + R^2 \right) \qquad (6.2.11)$$

where W is the Weyl tensor and R represents the curvature tensors of the background. Imposing $\mathcal{N} = 1$ supersymmetry implies [140]

$$t_2 = 6\left(1 - \frac{a}{c}\right) \qquad ; \qquad t_4 = 0 \qquad (6.2.12)$$

while

$$a_2 = 0 \qquad \text{If } J^\mu \text{ is a non R current} \tag{6.2.13}$$

$$a_2^{U(1)} = 3\left(1 - \frac{a}{c}\right) \qquad \text{If } J^\mu \text{ is a U(1) R current} \tag{6.2.14}$$

These results have been obtained in [140] in the following manner. We first use that, for free theories,

$$a_2^{free} = 3\frac{\sum_i (q_i^b)^2 - (q_i^{wf})^2}{\sum_i (q_i^b)^2 + 2(q_i^{wf})^2} \tag{6.2.15}$$

where we sum over the charges of all complex bosons and Weyl fermions for a given global symmetry. Then the result can be expressed as a function of a and c and, thus, has to be valid for all conformal theories, whether free or not.

We can repeat the same calculation here for a theory with $\mathcal{N} = 2$ symmetry where there is an $SU(2)$ R current. In that case the result is

$$a_2^{SU(2)} = 6\left(1 - \frac{a}{c}\right) \tag{6.2.16}$$

Using formulae (6.2.12), (6.2.14) and (6.2.16) in inequalities (6.2.6)-(6.2.10), coming from requiring positivity of the energy one point function in different states, we obtain

$$\frac{a}{c} \begin{cases} \geq 0, & \text{for U(1) R current, helicity 1 state;} \\ \leq \frac{3}{2}, & \text{for U(1) R current, helicity 0 state;} \\ \geq \frac{1}{2}, & \text{for SU(2) R current, helicity 1 state;} \\ \leq \frac{5}{4}, & \text{for SU(2) R current, helicity 0 state;} \\ \geq \frac{1}{2}, & \text{for energy-momentum, helicity 2 state;} \\ \leq 2, & \text{for energy-momentum, helicity 1 state;} \\ \leq \frac{3}{2}, & \text{for energy-momentum, helicity 0 state.} \end{cases} \tag{6.2.17}$$

Putting together this information we obtain the bounds in [140]

$$\frac{3}{2} \geq \frac{a}{c} \geq \frac{1}{2} \qquad \text{for } \mathcal{N} = 1 \tag{6.2.18}$$

$$\frac{5}{4} \geq \frac{a}{c} \geq \frac{1}{2} \qquad \text{for } \mathcal{N} = 2 \tag{6.2.19}$$

We would now like to discuss the gravitational higher derivative actions that reproduce these results by using the AdS/CFT dictionary. We parameterize the actions by using field redefinitions and only pay attention to reproducing the exact energy one point functions. In that case the general form of the action is [140]:

$$S = \frac{M_{pl}^3}{2}\left[\int d^5x\sqrt{g}\left(R + 12 + \frac{t_2}{48}W_{\mu\nu\delta\sigma}W^{\mu\nu\delta\sigma}\right)\right] - \frac{1}{4g^2}\left[\int d^5x\sqrt{g}\left(F^2 + \frac{a_2}{12}W^{\mu\nu\delta\rho}F_{\mu\nu}F_{\delta\rho}\right)\right] \tag{6.2.20}$$

where we have normalized the radius of the AdS solution to 1. In writing (6.2.20) we have assumed $t_4 = 0$, otherwise we would have had R^3 contributions. This is the main reason we choose to consider supersymmetric theories, as their gravity duals can be restricted to 4 derivatives actions, as far as the energy one point function is considered. Notice that we have only included one generic field strength field in (6.2.20). When it comes to the fields that couple to the R currents of the dual field theory we need to consider a $U(1)$ gauge field when there is $\mathcal{N} = 1$ supersymmetry at the boundary. We also must include an $SU(2)$ gauge field if there is $\mathcal{N} = 2$ supersymmetry. The corresponding coefficients a_2 are given by (6.2.14) and (6.2.16).

The action (6.2.20) is only to be understood as a *machine* to reproduce the exact energy one point functions (three point functions of local operators). Then, it is alright to consider coefficients a_2 and t_2 of order unity while ignoring higher derivative terms. It is not correct to take (6.2.20) as full nonlinear general description.

A reflection of this fact is that (6.2.20) posses ghost modes that violate unitarity semiclassically. These modes are present for generic values of the couplings and are not connected to the causality problems discussed before in the introduction. In fact, they should be *canceled* in the full string theory description.

It is important to stress that although this action is exact in a_2, it might have higher order correction in t_2. As a matter of fact, we can improve this result to the exact expression. In order to do that, we use the results in [191] where the central charges a and c were calculated for the dual theories to higher derivative gravity actions. From (6.2.12) we can express t_2 as a function of a and c and obtain an exact expression for the action, up to four derivative terms. The result is

$$S = \frac{M_{pl}^3}{2}\left[\int d^5x\sqrt{g}\left(R + 12 + \frac{t_2}{48 - 8t_2} W_{\mu\nu\delta\sigma}W^{\mu\nu\delta\sigma}\right)\right] - \frac{1}{4g^2}\left[\int d^5x\sqrt{g}\left(F^2 + \frac{a_2}{12} W^{\mu\nu\delta\rho}F_{\mu\nu}F_{\delta\rho}\right)\right] \tag{6.2.21}$$

Another important point is that other parameterizations are possible (although not completely equivalent as we will discuss below). The authors of [189, 190] have considered a GB term (6.1.1) instead of the W^2 term in (6.2.21). In that case the map between $\frac{a}{c}$ and λ in (6.1.1) is [189–191]:

$$\frac{a}{c} = 3 - \frac{2}{\sqrt{1-4\lambda}} \tag{6.2.22}$$

We are now in a position to use actions (6.2.21) and (6.1.1) to discuss how bounds (6.2.6)-(6.2.10) come about.

6.3 Black hole backgrounds in higher derivative gravity

It was argued in [189,190] that Gauss Bonnet gravity, for certain values of the coupling constant, is such that propagation of gravitons in the bulk lead to causality violations in the boundary dual theory. The setup is such that we have a black hole in the bulk of AdS. Then, it happens that we can shoot particles from the boundary to the black hole in such a way that they bounce back to the boundary and end up traveling faster that massless particles in the boundary. The argument uses that the velocity in a direction parallel to the boundary is greater than 1 in the bulk. Because the effective black hole potential has a maximum, semiclassical (WKB) particles will stay there for a long time, making their average velocity equal to the velocity at the maximum. The authors of [189, 190] showed that the GB coupling constant λ must be $\lambda \leq \frac{9}{100}$ if we are to avoid this behavior. The black hole metric for Gauss Bonnet gravity is:

$$ds^2 = -\frac{1}{2}\left(1+\sqrt{1-4\lambda}\right) f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 dx^i dx^i \tag{6.3.23}$$

where

$$f(r) = \frac{r^2}{2\lambda}\left(1 - \sqrt{1 - 4\lambda + \frac{4\lambda}{r^4}}\right) \tag{6.3.24}$$

and $i = 1, 2, 3$. The study was performed for gravitons of helicity 2 (i.e. $h = h_{12} = h_{21}$ for a state that propagates in the 3 direction in the boundary and also inside the bulk (in the r direction)). Authors of [189, 190] were interested in a bound that would constrain positive values of λ as the viscosity to entropy ratio falls below $\frac{1}{4\pi}$ for $\lambda > 0$. They did not study, however, the possibility of a lower bound for λ. We would like to study the causality constraints more systematically and make a connection with the energy one point function argument coming from the boundary theory.

It turns out that we can map the bound on λ to a bound on $\frac{a}{c}$ in the dual CFT by using (6.2.22), $\frac{a}{c} = 3 - \frac{2}{\sqrt{1-4\lambda}}$. If $\lambda > \frac{1}{4}$ the theory does not admit an AdS solution. Therefore, we assume $\lambda < \frac{1}{4}$. In this range $\frac{a}{c}$ is monotonous in λ. Then, the bound $\lambda < \frac{9}{100}$ implies $\frac{a}{c} > \frac{1}{2}$.

It is interesting that this bound matches the lower bound of $\frac{a}{c}$ obtained in [140] by demanding that the expectation value of the energy measured in a calorimeter at some angle from some operator insertion is positive in a CFT. Notice that this bound matches the expected result for the helicity 2 component of the energy-momentum tensor in the dual boundary theory, (6.2.17). Therefore, we should expect similar bounds coming from the analysis of other helicities of the graviton. In particular, the other optimal bound should come from the helicity 0 component. If we use (6.2.22) we should expect a causality problem unless $\lambda > -\frac{7}{36}$.

Before showing that this is actually the case let us make another comment. An interesting remark is that the way the bound was obtained originally in [189, 190] makes it hard to see that this feature is connected to the UV properties of the boundary field theory. The reason is that the particle must explore deep into the bulk of AdS to violate causality with the arguments presented in [189, 190].

One could try to calculate the exact geodesics in the effective GB metric for particles that don't go too deep into AdS. Instead we could try to consider a different background given by a shock wave. This corresponds to a localized insertion in the CFT and is the actual dual of the energy one point function, as discussed in [140]. Hopefully, we won't have to explore too deep into the bulk to see a problem, meaning this is related to UV properties of the field theory. Another motivation to do this is that the black hole corresponds to heating up the CFT to a certain temperature. We would like to understand the CFT at zero temperature, instead.

In the remainder of this section we will discuss several computations in the black hole background while in the following section we discuss the shock wave approach.

6.3.1 Bounds on $\frac{a}{c}$ from causality for all graviton polarizations

The basic object we would like to study, following [189, 190], is $c^2(r)$, the square of the speed of light in the 3-direction as a function of the AdS coordinate[5] r for a given perturbation of the metric, $h_{\mu\nu}$. Because a massless particle does not follow geodesics in Gauss Bonnet gravity, the way to calculate this is to perturb the black hole metric by $h_{\mu\nu}(r, x^0, x^3)dx^\mu dx^\nu$ and obtain the quadratic effective action for it. After appropriate gauge fixing and separation of the modes, this will result in an action of the type of a scalar in a non trivial metric. We can read off the components of the metric from the effective action as:

$$S_{eff}[h] = \int d^5x g^{rr}\partial_r h \partial_r h + g^{33}\partial_3 h \partial_3 h + g^{00}\partial_0 h \partial_0 h + \dots \tag{6.3.25}$$

From this action we can read the speed of light in the 3-direction:

$$c^2 = \left(\frac{dx^3}{dx^0}\right)^2 = -\frac{g^{00}}{g^{33}} \tag{6.3.26}$$

Alternatively we could look at the equation of motion for the perturbation.

Because c^2 is the effective potential for our semiclassical particle [189, 190], one possible approach is to look for a maximum of $c^2(r)$ such that $c^2(r) > 1$. Then we can tune the probe particle to spend most of its trajectory here at speed $\sqrt{c^2_{max}}$ and then bounce back to the boundary. In that case we will have a violation of causality as the average speed of the particle will be greater than 1.

The result for the helicity 2 mode h_{12} was obtained in [189, 190]

$$c_2^2(r) = \frac{1+\sqrt{1-4\lambda}}{2r} f(r) \frac{1-\lambda f''(r)}{r-\lambda f'(r)} \tag{6.3.27}$$

[5]This coordinate is such that the horizon of the black hole is at $r = 1$ and the boundary of AdS is at $r = \infty$.

The 2 as a lower index indicates that this calculation was performed for the helicity 2 mode of the graviton.

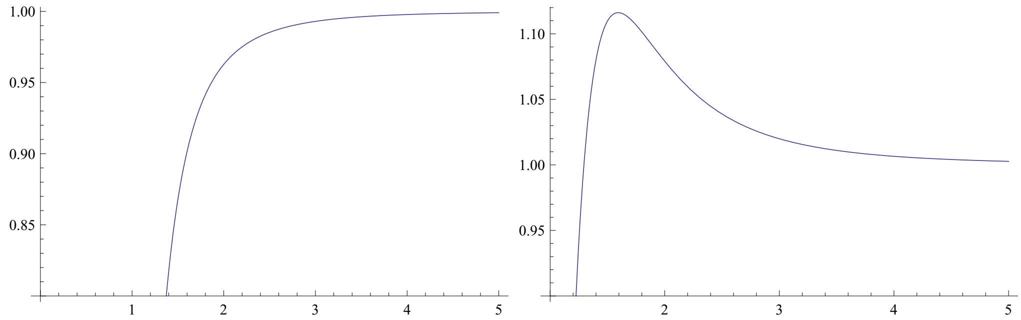


Figure 6.1: Plots of $c_2^2(r)$. (a) Plot for $\lambda = 0.05 < \frac{9}{100}$. (b) Plot for $\lambda = 0.15 > \frac{9}{100}$. Notice that $c_2^2 > 1$ for r close enough to the boundary $r \to \infty$ when $\lambda > \frac{9}{100}$.

By analyzing $c_2^2(r)$ we can see that it is such that $c_2^2 < 1$ for all r for $\lambda < \frac{9}{100}$ and it is monotonous. There is no turning point. For $\lambda > \frac{9}{100}$, c_2^2 becomes greater than 1 in part of the range and develops a unique maximum where $c_2^2 > 1$. Therefore, causality would be broken at the boundary if $\lambda > \frac{9}{100}$.

We can make an interesting comment at this point. If we look in detail at figure 6.1(b) we see that $c^2 > 1$ for all values of r close enough to the boundary. This fact suggests that any problems encountered should be seen by looking at the UV behavior of our field theory.

We would now like to repeat this calculation for a different polarization. The results in [140] indicate that a lower bound on λ might be obtained by studying a state with helicity 0 in the CFT. The symmetries are not enough in this case to select only one component of the metric as our degree of freedom. If we pick a gauge where $h_{\mu 0} = 0$, we still have to consider the dynamics of four components of the metric perturbation. Namely, $h_{11} = h_{22}$, h_{33}, h_{rr} and h_{3r}. Solving the equations of motion for these components is complicated in general, but luckily we are only interested in the value of the speed of light in the 3-direction. Then, we can keep only terms of leading order in ∂_0 and ∂_3. For a plane wave solution $e^{i\omega x^0 - ikx^3}$, the equations of motion become algebraic equations and can be solved. This yields:

$$\begin{aligned}
h_{3r} &= 0 \\
h_{rr} &= \frac{4r(r-\lambda f'(r))}{-2r^2+4\lambda f(r)}h_{11} \\
h_{33} &= \left(-2+\frac{6(r-\lambda f'(r))^2}{3r^2+2\lambda f(r)-8r\lambda f'(r)+4\lambda^2 f'^2(r)+\lambda(r^2-2\lambda f(r))f''(r)}\right)h_{11}
\end{aligned} \tag{6.3.28}$$

The remaining equation of motion imposes

$$\frac{\omega^2}{k^2}=c_0^2=\frac{(1+\sqrt{1-4\lambda})f(r)(3r^2+2\lambda f(r)-8r\lambda f'(r)+4\lambda^2 f'^2(r)+\lambda(r^2-2\lambda f(r))f''(r))}{6r(r^2-2\lambda f(r))(r-\lambda f'(r))} \tag{6.3.29}$$

This is the expression for the speed of light in the 3-direction for the helicity 0 modes. If we again repeat the analysis of the curve given by $c_0^2(r)$ as done for the helicity 2 mode we find a similar behavior[6]. The main difference is that in this case we observe acausal behavior (a maximum in c_0^2 with $c_0^2>1$) for $\lambda<-\frac{7}{36}$. This corresponds to $\frac{a}{c}>\frac{3}{2}$, as expected. We therefore see that if we extend the analysis of [189, 190] to this case we match the bound obtained in [140].

Interestingly enough, we can also study the helicity 1 mode of the graviton. Although, this is not an optimal bound, we also obtain the expected results predicted by (6.2.17). Namely, $\frac{a}{c}<2\rightarrow\lambda>-\frac{3}{4}$. We quote the speed of light in this case.

$$c_1^2=\frac{(1+\sqrt{1-4\lambda})(r-\lambda f'(r))f(r)}{2r(r^2-2\lambda f(r))} \tag{6.3.30}$$

There are two independent perturbations of the metric in this case after gauge fixing: h_{31} and h_{r1}. The latter is set to zero by the equations of motion, while the

[6]The corresponding plots look qualitatively the same as the ones presented in figure 6.1 and we omit them for brevity.

former is the physical degree of freedom.

We have seen that this analysis allows one to calculate that a consistent conformal field theory must have $\frac{1}{2} < \frac{a}{c} < \frac{3}{2}$ if bulk dynamics are to induce causal behavior at the boundary. This is the expected window for a CFT with $\mathcal{N} = 1$. How do we get the stronger bound found for $\mathcal{N} = 2$ theories in [140, 192]? At the level of the low energy four derivative action (6.2.21) $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories are equal except for the fact that $\mathcal{N} = 2$ theories possess an additional $SU(2)$ gauge field in their dual gravitational description. If we are to find the narrower window $\frac{1}{2} < \frac{a}{c} < \frac{5}{4}$ we should look at the propagation of the helicity 0 mode of the $SU(2)$ gauge field, as suggested by (6.2.17). We do this in the next subsection.

6.3.2 Gauge bosons in the black hole background

Now we would like to obtain bounds (6.2.6) and (6.2.7) from considering the propagation of gauge bosons in a black hole background. The simplest way to do this calculation is to consider Einstein gravity with no higher derivative corrections and add Maxwell terms (including the correction proportional to W) as in (6.2.21). We will comment on the full action (6.2.21) when we discuss shock waves further below. We will obtain the same bounds by using action (6.2.21) in this context.

In our coordinates, the (Einstein) metric of the black hole takes the form

$$ds^2 = -\left(\frac{r^4 - 1}{r^2}\right) dt^2 + \left(\frac{r^2}{r^4 - 1}\right) dr^2 + r^2 dx^i dx^i \tag{6.3.31}$$

We can now use the electromagnetic part of the action (6.2.21) in this background to obtain effective actions for the 0 and 1 helicity modes of the gauge field. Up to quadratic order in the electromagnetic perturbation it does not make a difference whether we have an abelian or non abelian gauge group. For the helicity 1 mode we can set $A = A_1(r) e^{ikx^3 - i\omega x^0} dx^1$. The effective equation of motion for this mode at

$|k|, \omega >> |\partial_r|$ is:

$$\left((-1+r^4)(6r^4+a_2)k^2 + r^4(-6r^4+a_2)\omega^2\right) A_1 = 0 \tag{6.3.32}$$

Therefore,

$$c_{1A}^2 = \frac{(r^4-1)(6r^4+a_2)}{r^4(6r^4-a_2)} \tag{6.3.33}$$

The behavior of c_{1a}^2 is analogous to that of c_i^2 discussed before. We find acausal behavior for[7] $a_2 > 3$. This matches exactly the expectations from (6.2.6). We can now use the relations (6.2.14) and (6.2.16) to express this bound as a function of $\frac{a}{c}$. We find that if we consider a $U(1)$ gauge field this yields $\frac{a}{c} > 0$ while and $SU(2)$ field yields $\frac{a}{c} > \frac{1}{2}$. These are the predicted results form the energy one point function analysis listed in (6.2.17). This result does not improve the window coming from the gravitational excitations. In order to do that we have to consider the 0 helicity mode. In that case we can gauge fix $A_0 = 0$. The equations of motion impose $A_3 = 0$ and give us the following value for the speed of light squared:

$$c_{0A}^2 = \frac{(r^4-1)(6r^4-a_2)}{3r^4(2r^4+a_2)} \tag{6.3.34}$$

Once again, we observe a similar behavior, indicating problems with causality for $a_2 < -\frac{3}{2}$. This time we see that while for the case of a $U(1)$ field this implies $\frac{a}{c} < \frac{3}{2}$, and no new bound is found, the presence of an $SU(2)$ gauge field imposes $\frac{a}{c} < \frac{5}{4}$, as predicted in [140, 192]. Therefore gravitational theories which are dual to $\mathcal{N} = 2$ supersymmetric conformal field theories have a narrower causal window.

[7]This behavior has also been discussed in [221].

6.3.3 Are all higher derivative gravities equal?

Up to this point all seems well and in agreement between the field theory results and the higher derivative gravity calculation. One small exercise one could attempt to understand whether all parameterizations of the 4 derivative gravity actions are equivalent is to reproduce the results in section 6.3.1 using action (6.2.21). Here, the scene is complicated by the fact that the equations of motion for the gravitational perturbations include terms with up to four derivatives. We remind the reader that the absence of these terms is what makes Gauss-Bonnet gravity special. Although, this can be done, it involves dealing with the black hole solutions of other gravity theories. Instead of pursuing this road, we will study a different background in the next section, namely, the shock wave. We will explain in the next section several reasons why this is a good idea. In this setup, it will be easier to compare different actions.

At this point we should spoil the punch line and say that we won't see the same results when looking at classical solutions using actions (6.1.1) and (6.2.21).

One might be tempted to argue that there are other metric perturbations in the theory that might be responsible for the problem. Although this modes are problematic on their own, as being responsible for non unitarity in the quantum theory (unless we embed the model in a UV complete theory), they are dual to operators with different weight in the dual CFT. In other words, they are massive fields in the gravitational theory. The only point when they do become massless gravitons corresponds to $\frac{a}{c} = \infty$, when they *cancel* the usual gravitons and the theory becomes topological Chern-Simons. Then, except for $\frac{a}{c} = \infty$ we can always imagine imposing boundary conditions such that the only mode we excite is the one that couples to the energy-momentum tensor. See more on this in section 6.4.5.

We will see in the next section that one explanation for the different behavior in this case is that graviton scattering of a classical background involves higher energy point functions for a general 4 derivative gravity theory. It is only in Gauss-Bonnet gravity where the scattering of gravitons of a shock wave involves only the energy one point functions of the dual CFT. Remember that the gravitational actions of the type discussed in [140] are engineered to reproduce the energy one point function. When higher point functions become relevant we need to consider higher derivative terms and we can't truncate the form of the action to (6.2.21).

6.4 Shock wave backgrounds

It seems like studying a black hole background is overkill for the type of problem we have been discussing. This amounts to putting the theory in a thermal bath. It is natural to think that the simplest thing to consider is the interaction of gravitons with shock waves that propagate on top of AdS space-time. This matches naturally with the observables considered in [140], where it was explained that this is the gravitational configuration dual to the energy one point function in the CFT. Let us first check that shock waves are indeed a solution to the equations of motion, modified by higher derivative terms.

6.4.1 Solutions to Gauss Bonnet gravity and linearity

Although this is a trivial fact, we recall that AdS is a solution to Gauss Bonnet gravity. This can be checked by taking the zero mass limit of the black hole solution discussed in [189, 190]. We can also check it explicitly from the equations of motion. It is important to remember (and this is why we are discussing this trivial fact) that the higher derivative correction changes the curvature of the AdS space. For a given value of the cosmological constant, normalized such that the unperturbed AdS space has radius 1, the radius of the AdS space in Gauss-Bonnet gravity is given by:

$$R^2 = \frac{1}{2}\left(1+\sqrt{1-4\lambda}\right) \tag{6.4.35}$$

This reduces to 1 as $\lambda \to 0$ and shows explicitly that there is no AdS solution for $\lambda > \frac{1}{4}$. We choose to parameterize the AdS metric with Poincare coordinates as[8]:

$$ds^2_{AdS} = \frac{1}{2}\left(1+\sqrt{1-4\lambda}\right)\frac{dz^2 + d\vec{x}^2}{z^2} \tag{6.4.36}$$

We can now add a shock wave propagating on top of this metric. We define coordinates $x^{\pm} = x^0 \pm x^3, x = x^1, y = x^2$ and take the following form for a general shock wave metric

$$ds^2_{Shock} = ds^2_{AdS} + \delta(x^+)w(x,y,z)dx^{+2} \tag{6.4.37}$$

While $\delta(x^-)$ is in principle an arbitrary function we will consider it to be a localized delta function. By plugging (6.4.37) into the equations of motion for Gauss-Bonnet gravity we find that this is always a solution provided

$$4w - z\partial_z w - z^2\left(\partial_z^2 w + \partial_x^2 w + \partial_y^2 w\right) = 0 \tag{6.4.38}$$

This is the same condition we find in usual Einstein gravity for propagation of a shock wave on AdS. We can see that this equation is linear, so we can superimpose as many solutions as we want. This result means that, classically, shock waves do not interact with each other. This result was already briefly discussed in [140], in the context of stringy corrections to the energy correlation functions. The fact that the shock wave solution receives no corrections from higher derivative terms was discussed at length in [209].

Some interesting types of solutions to this equation are:

[8] The z coordinate corresponds to $\frac{1}{r}$.

$$
\begin{aligned}
w^A &= \alpha \frac{1}{2}\left(1+\sqrt{1-4\lambda}\right) z^2 && (6.4.39)\\
w^B &= \alpha \frac{1}{2}\left(1+\sqrt{1-4\lambda}\right) \frac{1}{z^2} && (6.4.40)\\
w^C &= \alpha \frac{1}{2}\left(1+\sqrt{1-4\lambda}\right) \frac{z^2}{(z^2+x^2+y^2)^3} && (6.4.41)
\end{aligned}
$$

where α is just an arbitrary normalization. Notice that these are exact solutions of Gauss Bonnet gravity, therefore this is not an expansion in α.

While solutions A (6.4.39) and C (6.4.41) represent interesting physical situations, B (6.4.40) is just a coordinate redefinition. We will see this explicitly when we study the propagation of perturbations on top of AdS space with shock waves of these types.

6.4.2 From the Black Hole to the shock wave

One way to make contact between the black hole solution and the shock wave is to realize that the shock wave solution can be obtained by boosting a black hole solution while keeping its energy $E = \frac{m}{\sqrt{1-v^2}}$ constant [209, 210, 222]. In our particular case, we are studying a black brane, which is invariant under translations in the boundary coordinates. It is possible to check that by boosting this solution we obtain shock wave w^A (6.4.39), which retains the symmetries preserved by the boost. This solution corresponds to the deformation of the background caused by some source deep inside AdS. In this case the normalization constant α is proportional to the energy density (which we keep constant) of the black brane and it is positive if the solution has a positive mass.

We can now calculate the effective linearized equation of motion for gravitons with helicity 2 (as in [189, 190]) in this background. The resulting equation of motion for the perturbation $h(x^+, x^-, z)dx\, dy$ is

$$\frac{3}{z}\partial_z h - \partial_z^2 h + 4\partial_+\partial_- h + 4\alpha\delta(x^+)\left[1 - 16g(\lambda)\right] z^4 \partial_-^2 h = 0 \tag{6.4.42}$$

with $g(\lambda) = \frac{\lambda}{1-4\lambda+\sqrt{1-4\lambda}}$.

In the limit of very high energies we can consider a wave packet that moves with definite momentum. By integrating over the discontinuity at $x^+ = 0$ we can extend a solution $h_<$ at $x^+ < 0$ to the other side of the shock wave ($x^+ > 0$) as

$$h_> = e^{-iP_-\alpha[1-16g(\lambda)]z^4} h_< \tag{6.4.43}$$

where we used $P_- = -i\partial_-$. From this we can read there is a shift $\Delta x^- = \alpha\left[1 - 16g(\lambda)\right] z^4$ after the collision. Also, we can act on (6.4.43) with $P_z = -i\partial_z$. We obtain the shift in the momentum in the z direction as:

$$P_z^> = P_z^< - 4P_-\alpha\left[1 - 16g(\lambda)\right] z^3 \tag{6.4.44}$$

Notice that for a particle going inside AdS, $P_z > 0$. Therefore, if we want our particle to come back to the boundary we need $[1 - 16g(\lambda)] < 0$. This comes about from the fact that $P_- < 0$ and $\alpha > 0$ if our original black hole had a positive mass. But if this is the case, then $\Delta x^- < 0$. Notice that the shock wave vanishes on the boundary (6.4.39). Therefore, the shock wave does not affect the light cone of particles that move on the boundary of AdS ($z = 0$). The end result is that particles that come back to the boundary end up landing outside the light cone.

If we require that $1 - 16g(\lambda) > 0$, we recover the result from [189, 190] $\lambda < \frac{9}{100}$. If the bound is satisfied, particles that go into the bulk of AdS do not return to the boundary as found in [189, 190] (also, look at the form of c^2 in figure 6.1(a)), and no problem is encountered.

6.4.3 Shock waves that change the boundary metric

Let us now look at cases where we actually insert a perturbation at the boundary. This corresponds to non-normalizable shock waves of the type B (6.4.40) and C (6.4.41). We are interested in this possibility as we want to make a connection to the calculation of correlation functions in the boundary theory.

The first thing we can check is that, because B corresponds to a change of coordinates, the equation of motion of a gravitational perturbation will not depend on λ or the point z inside AdS where the collision occurs. In accordance with our expectations, the equation of motion for the perturbation is, in this case:

$$\frac{3}{z}\partial_z h - \partial_z^2 h + 4\partial_+\partial_- h + 4\alpha\delta(x^+)\partial_-^2 h = 0 \tag{6.4.45}$$

Therefore, the discontinuity of the light cone is the same for all propagating modes and there is no conflict with causality. Also, particles can't bounce back to the boundary.

The shock wave given by w^C is more interesting. It corresponds to the insertion of a stress energy tensor over a light like line at the boundary. In this case

$$\frac{3}{z}\partial_z h - \partial_z^2 h + 4\partial_+\partial_- h + \frac{4\alpha\delta(x^+)z^4}{(z^2+x^2+y^2)^3}\left([1-16g(\lambda)] + 96g(\lambda)\frac{z^2(x^2+y^2)}{(z^2+x^2+y^2)^2}\right)\partial_-^2 h = 0 \tag{6.4.46}$$

The situation is a little bit more complicated in this case as there is a dependence on where we put our particle in the transverse space (x, y). The particle trajectory is shifted by

$$\Delta x^- = \frac{\alpha z^4}{(z^2+x^2+y^2)^3}\left([1-16g(\lambda)] + 96g(\lambda)\frac{z^2(x^2+y^2)}{(z^2+x^2+y^2)^2}\right) \tag{6.4.47}$$

If we study this problem for a particle at $x = y = 0$ we find

$$\Delta x^- = \frac{\alpha}{z^2}\left[1 - 16g(\lambda)\right] \tag{6.4.48}$$

and

$$P_z^> = P_z^< + 2P_- \frac{\alpha}{z^3}\left[1 - 16g(\lambda)\right] \tag{6.4.49}$$

This implies that we need $\alpha\left[1 - 16g(\lambda)\right] > 0$ for our particle to come back to the boundary. In order to see whether we have problems with causality we need to study the light cone of the boundary theory. In this particular case the metric of the boundary theory is affected, so we need to be careful.

Because the shock wave at the boundary does not depend on λ we expect the light cone to be changed in a λ independent manner. It is clear from the equation of motion (6.4.46) that the shock wave behaves as $\sim \delta(x^+)\delta(x, y)$ at the boundary. We need to check what the overall multiplicative factor is. In order to check this we will integrate the shock wave over x and y at a fixed z to obtain $w_{eff}(z)$ and then take the $z \to 0$ limit. Therefore,

$$\begin{aligned} w_{eff}(z) &= \int dxdy \frac{\alpha z^4}{(z^2 + x^2 + y^2)^3}\left(\left[1 - 16g(\lambda)\right] + 96g(\lambda)\frac{z^2(x^2 + y^2)}{(z^2 + x^2 + y^2)^2}\right) && (6.4.50) \\ &= \frac{\alpha}{4}\left[1 - 16g(\lambda)\right] + \frac{\alpha}{24}96g(\lambda) = \frac{\alpha}{4} && (6.4.51) \end{aligned}$$

We can see that this result is independent of z and λ. Therefore the light cone shift at the boundary is:

$$\Delta x^-\Big|_{z=0} \sim \alpha\delta(x, y) \tag{6.4.52}$$

This infinite shift at $x = y = 0$ is a consequence of considering a localized discontinuous perturbation of the boundary and could be regularized by a smooth deformation of the metric. If the light cone shifted towards $x^- < 0$ by an infinite amount, then every event with $x^+ > 0$ would be inside the future cone of our null geodesic. Therefore, we would like to consider the case in which $\Delta x^-\big|_{z=0} > 0$ so we can have a possible violation of causality. The schematic picture of the situation is the following

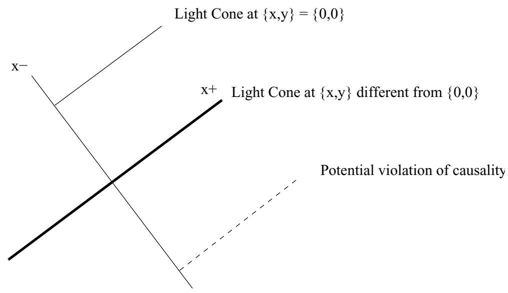


Figure 6.2: Light cone structure of our boundary space time. The thick line represents the unchanged null geodesic away from the shock wave in transverse space $\{x, y\} \neq \{0, 0\}$. The thin line is the displaced geodesic at the position of the shock wave $\{x, y\} = \{0, 0\}$. Notice that if we have a δ function type shock wave this geodesic sits at $x^- = \infty$. If we find that $\Delta x^- < 0$ for the metric perturbation (dashed line), we will encounter a causality problem.

Having the boundary light cone shift in the positive x^- direction implies $\alpha > 0$. Therefore we need $1 - 16g(\lambda) > 0$ for the particle to come back to the boundary and, according to (6.4.48) we don't have a violation of causality.

We can now consider a particle at $x, y \neq 0$. In that case and for $z \ll x, y$ we can write the shift in x^- as

$$\Delta x^- = \frac{\alpha z^4}{(x^2 + y^2)^3} \left[1 - 16g(\lambda)\right] \tag{6.4.53}$$

This time

$$P_z^{>} = P_z^{<} - 4P_{-}\alpha \left[1 - 16g(\lambda)\right] \frac{z^3}{(x^2 + y^2)^3} \tag{6.4.54}$$

Now this case is almost identical to the one discussed for the shock wave coming from the black hole solution. We will observe a violation in causality for $1 - 16g(\lambda) <$

0. Notice that in this case there is also a small deflection in the x, y directions. This effect is however suppressed as $\frac{z}{\sqrt{x^2+y^2}}$ with respect to the deflection in z.

6.4.4 General remarks about results for the shock wave background

The results of the last two subsections make it clear that the causality problems found from the gravity theory are directly connected with the value of the energy one point function. The calculation of the energy one point function is given by the scattering of a shock wave and we have shown that here is where the problem lies. Notice also that the equations of motion for the gravitational perturbation (6.4.42) and (6.4.46) are exactly linear in α. This indicates that only the 3 point function involving two gravitons and a shock wave play a part in this discussion. As explained in [140], this means that only the energy one point function is of relevance here.

Let us explain in more detail why it is only the three point function that it is involved. The equations of motion studied, (6.4.42) and (6.4.46), are of the form generic form:

$$(\Delta - \delta_{sw}\partial_-^2)h = 0 \tag{6.4.55}$$

where Δ is the usual differential operator in AdS for the propagation of a perturbation and δ_{sw} represents the shock wave. Therefore, the classical (tree level) propagator is:

$$P \sim \frac{1}{\Delta - \delta_{sw}\partial_-^2} = \frac{1}{\Delta}\sum_n \left(\frac{\delta_{sw}\partial_-^2}{\Delta}\right)^n \tag{6.4.56}$$

This exponentiation corresponds to the following scattering picture, where, as we can see, only the three point function is involved.

Finally let us add that, because we have found a problem for scattering that occurs arbitrarily near the boundary ($z \sim 0$) in (6.4.53) , the calculation makes manifest that

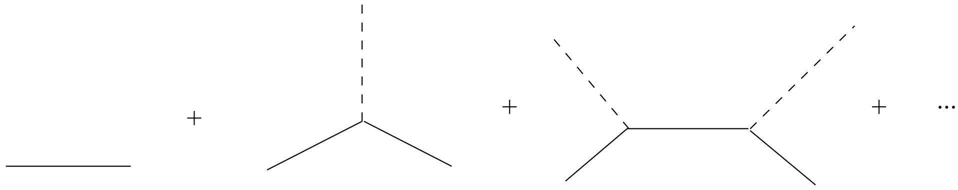


Figure 6.3: Only the three point function contributes to the exponentiation given by the classical solution. Solid lines represent the perturbation while the dashed lines represents the shock wave.

any potential problem reflects the UV properties of the field theory. We could have reached a similar conclusion by following the analysis of [189, 190] and avoiding the argument using the maximum in the effective potential. In this case, calculating the exact classical trajectories would have shown the same problem. All this is clear from the fact, already noted in section 6.3.1, that figure 6.1 shows that $c^2 > 1$ for small values of z. The upshot of this discussion is that the positivity of energy condition explained in [140] must apply, not only to CFTs, but to any UV complete QFT (i.e. field theories that are asymptotically free or have a UV fixed point). Needless to say, we could repeat the same analysis for the other graviton polarizations and obtain similar results.

6.4.5 The shock wave background for W^2 higher derivative gravity

In this part we would like to study the problem of scattering perturbations of a shock wave by using a gravity action of the form (6.2.21). We will only consider a shock wave of the form w^A in (6.4.39). This solution is more symmetrical and further detail won't be necessary.

First let us briefly study the problem for gauge boson perturbations so we can be sure that what was discussed in section 6.3.2 can be extended with no problems to the full action (6.2.21). The argument is fast and simple. The shock wave remains a solution of the full equations of motion including the higher derivative terms. Because, once we have a background, all we need is the Maxwell part of the action, the calculation is the same we would have in Einstein gravity or GB gravity. We spare the reader the details in this case and just note that studying helicity 1 and 0 modes we recover the results (6.2.6) and (6.2.7).

In the case of gravitational perturbations the story is more complicated. The action (6.2.21) yields fourth order equations and, as we shall see, also involves a vertex involving two shock waves. If we are only interested in the high momentum limit we can neglect the Einstein term and just study the W^2 contribution. We see right away that, because the shock wave solution is independent of t_2, the equations of motion will not depend on t_2 at all. This already shows that it will not be possible to match the bounds on t_2 listed in (6.2.8),(6.2.9) and (6.2.10). Let us understand where the problem comes from. The equations of motion in the $\partial_-, \partial_+ \to \infty$ limit for a helicity 2 component of the metric $h = h_{12}$ are:

$$\left(\partial_+^2 \partial_-^2 + z^4 \delta'(x^+) \partial_-^3 + 2z^4 \delta(x^+) \partial_+ \partial_-^4 + z^8 \delta(x^+)^2 \partial_-^2\right) h = 0 \tag{6.4.57}$$

This can be rewritten as:

$$\left(\partial_+ \partial_- + z^4 \delta(x^+) \partial_-^2\right)^2 h = 0 \tag{6.4.58}$$

This expression makes manifest that there are no problems of causality of the type discussed before. The shift in x^- is always positive. Notice that in this limit the metric has two independent spin 2 degrees of freedom. If we had kept the contributions from subleading terms in ∂_-, ∂_+, we would see that one of these modes becomes massive while the other is the usual graviton. Any problems induced by a possible tachyonic mass are visible at low momentum and, in any case, represent a different phenomenon. As we explained, the only point where both modes are massless is the Chern-Simons theory ($\frac{a}{c} \to \infty$). The limit of infinite mass, $\frac{a}{c} \to 1$, is such that the theory is tachyonic for $\frac{a}{c}$ slightly below 1 and has $m^2 > 0$ for $\frac{a}{c}$ slightly above 1. This makes manifest that these problems are unrelated with the positivity of energy in the dual CFT.

Let us look at these statements in more detail. The equation of motion for a perturbation of the metric of helicity 2, $h(z)e^{ikx^3-i\omega x^0}dx^1dx^2$, over AdS given by the action (6.2.21) is

$$\begin{aligned} 0 \;=\; & \left[21z(\omega^2-k^2)+8\beta z^3(\omega^2-k^2)(\omega^2-k^2-\frac{1}{z^2})\right]h(z)+ \\ & \left[-9+32\beta z^2(\omega^2-k^2)\right]h'(z)+\left[3z+16\beta z(1-2z^2(\omega^2-k^2))\right]h''(z)- \\ & 16\beta z^2 h'''(z)+8\beta z^3 h''''(z) \end{aligned} \tag{6.4.59}$$

where $\beta = \frac{t_2}{48-8t_2} = \frac{1}{8}\left(\frac{c}{a}-1\right)$.

If we solve this equation asymptotically near the boundary, $z \to 0$ with an ansatz of the form z^Δ we find the following solutions:

$$\Delta_-^1 = 0 \qquad \Delta_+^1 = 4 \qquad \Delta_-^2 = 2-\sqrt{1-\frac{3}{8\beta}} \qquad \Delta_+^2 = 2+\sqrt{1-\frac{3}{8\beta}} \tag{6.4.60}$$

Therefore, $\Delta_+^{1,2}$ are the conformal dimensions of the spin 2 operators dual to the degrees of freedom contained in the metric. While Δ_+^1 corresponds to the usual stress-energy tensor the other gravitational mode is dual to a different spin 2 operator of weight Δ_+^2. As promised, one can see that the second degree of freedom becomes *another* graviton for $\beta = -\frac{1}{8}$ ($\frac{a}{c} = \infty$) where they cancel and no degree of freedom remains. Also, the massive mode is such that $m^2 \sim \left(\Delta_+^2\right)^2$ for $\Delta_+^2 \to \infty$. A small value of β yields $m^2 \to \infty$ for $\beta < 0$ and $m^2 \to -\infty$ for $\beta > 0$. We therefore confirm that the peculiarities of the second spin 2 mode are not directly related to the causality problems discussed above.

Going back to our discussion of the large momentum limit, the main reason why we expect things to be different from GB gravity is that the equations of motion (6.4.57) include vertices that involve two shock waves (δ^2). This goes beyond the graviton 3 point function where we can trust our general description by the action (6.2.21). The main point is that GB gravity is special in the sense that it only involves the 3 point function when we exponentiate tree level diagrams, while a generic $2n$ derivative theory involves up to $2+n$ graviton vertices.

One might add that the reason why no δ^2 contributions appear in (6.4.42) is a combination of 2 derivative equations of motion with Lorentz symmetry. As we can see from (6.4.37), the shock wave w has two $+$ lower indices. The only way to obtain a Lorentz invariant expression is to be able to construct a term with as many lower $-$ indices. If we want a δ^2 term we need to have ∂_-^4 terms available. As these can't appear in Gauss-Bonnet gravity, only the three particle vertex can contribute. The same reasoning proves that any $2n$ derivative theory of gravity has at most δ^n terms.

The end result of this discussion is that, while actions (6.2.21) and (6.1.1) are engineered to represent the same energy one point function in their dual theories (they yield the same value for $\frac{a}{c}$, that is), when we consider classical solutions the exponentiation procedure is different and one theory involves higher point vertices than the other. Therefore, one must be careful about what consistency results are expected from arbitrary higher derivative theories of gravity.

6.5 Positivity of energy in any CFT from field theory arguments

In this section we would like to add a short argument explaining why, besides expectations coming from the experimental collider setup, the energy one point function should be positive in any CFT. The reader should take the following arguments as a *physicist's proof* as opposed to a formal proof (as in Axiomatic Quantum Field Theory, say).

The first thing that should be said is that results in [140] show that, using conformal transformations, the positivity of the energy operator (6.2.3) is equivalent to the positivity of the following energy operator written in new coordinates:

$$\mathcal{E}(\vec{y}) = \int dy^- T_{--}\left(y^-, y^+, \vec{y}\right) \tag{6.5.61}$$

where the flat space metric is $ds^2 = -dy^+ dy^- + d\vec{y}^2$. Therefore, what we are trying to prove is a version of the averaged null energy condition for CFTs in flat space.

It was pointed out in [140] that proving the positivity of (6.5.61) is an easy task for free theories. Using the creation and annihilation operators one can show that the integral in y^- precisely takes care of the terms in T_{--} responsible for negative energy densities [153]. We are, therefore, interested in writing the argument in a general language common to all CFTs that we can apply to strongly coupled theories. Furthermore, we want to use minimum structure, as the argument should apply to all CFTs. We are led to study the operator product expansions (OPEs) of energy momentum tensors. Although this structure is present in every theory, the OPEs in $d > 2$ are not as thoroughly determined as their $d = 2$ counterparts. An example of this indeterminacy are Schwinger terms. We will, however, make use of all information available to constraint our results.

The general strategy will be the following. We will consider non local operators similar to (6.5.61). Then we consider the theory with euclidean time y^+ and we regard y^- as a space variable. If we can find one operator such that its OPE with itself yields (6.5.61) at leading order in y^+, then the standard argument of positivity of norms[9] proves the positivity of our energy operator (6.5.61).

[9] See [217] for an interesting recent discussion of this old argument.

Schematically, if

$$\mathcal{O}(0)\mathcal{O}(y^+) \sim \frac{\gamma}{y^{+n}} \int dy^- T_{--}(0) + \dots \tag{6.5.62}$$

then $\int dy^- T_{--}(0)$ is a positive operator, provided $\gamma > 0$. This means that any N point function of the form

$$\langle \alpha | \mathcal{E}(\vec{y}_1) \dots \mathcal{E}(\vec{y}_N) | \alpha \rangle > 0 \tag{6.5.63}$$

for any state $|\alpha\rangle$.

Let us find candidate operators $\mathcal{O}$. Equation (6.5.62) plus invariance under the full conformal group imply

$$2\Delta\,(\mathcal{O}) = n + 3 \tag{6.5.64}$$

$$2S\,(\mathcal{O}) = -n + 1 \tag{6.5.65}$$

where $\Delta\,(\mathcal{O})$ is the conformal weight and $S\,(\mathcal{O})$ is the Lorentz spin of $\mathcal{O}$ in the $+-$ plane. This implies

$$\Delta\,(\mathcal{O}) + S\,(\mathcal{O}) = 2 \tag{6.5.66}$$

We want to build an OPE from an operator $\mathcal{O}$ available in any theory. Natural candidates are $\int dy^- T_{\mu\nu}$, which have $\Delta = 3$. Then, we need $S = -1$, which fixes $\mu\nu = +-$. Our task is now to compute

$$\int dy^- T_{-+}(0) \int dy^- T_{-+}(y^+) \sim \dots + \frac{\gamma}{y^{+3}} \int dy^- T_{--}(0) + \dots \tag{6.5.67}$$

Our argument will be complete if we show that: a) the first group of ... representing terms more divergent than $\frac{1}{y^{+3}}$ is actually zero; b) $\gamma > 0$.

More divergent contributions would be of the form $\frac{1}{y^{+k}}\mathcal{O}'_{\Delta',S'}$. Symmetry under the conformal group implies $\Delta' + k = 6$ and $S' - k = -2$. Then,

$$k > 3 \qquad \longrightarrow \qquad \Delta' < 2 + S' \quad \text{and} \quad S' > 1 \tag{6.5.68}$$

If we assume that only integrals of local operators can appear, then this contribution would violate the unitarity bound [223]. It is interesting to notice that some local operators appear in the local OPE $T_{+-}T_{+-}$ but these contributions vanish when we integrate over y^-. This explains why local densities can be negative while (6.5.61) is positive and is reminiscent of the situation in the free theory, discussed in [153].

Of course, non local operators can appear in the OPE in principle (as they appear in a similar OPE in [140]). We are not aware of a similar bound for this class of operators. If they do satisfy a similar bound, we know the leading term will be $\int dy^- T_{--}$. In that case, a) is true.

We can now show that b) is true by integrating (6.5.67) over the transverse coordinates $\vec{y}$. Then, the left hand side of (6.5.67) is still positive and in the right hand side we have $\sim \frac{\gamma}{y^{+3}}P^+$, where P^+ is the total momentum $P^+ \sim \int dy^- d\vec{y}^2 T_{--} > 0$. Therefore γ should be positive. We have checked this explicitly by using the 3 point functions in [183] for the supersymmetric case and using the final result $\frac{3}{2}c > a > \frac{1}{2}c$ [140]. In the explicit calculation, with $\frac{3}{2}c > a > \frac{1}{2}c$, it is the case that $\gamma > 0$, but the general argument does not rule out $\gamma = 0$.

6.6 Discussion

In this chapter we have discussed the connection between the constraints imposed by causality of the boundary theory on bulk dynamics and the positivity of the energy one point function in the field theory. Considering different propagating modes in the bulk leads to different causality constraints and there is a precise match between these bounds and the ones expected from the field theory analysis in [140]. In particular we have reproduced the expected bounds for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric CFTs by using gravitational and gauge boson perturbations in the bulk.

These calculations were performed for the black hole background and the shock wave background in higher derivative gravity. By performing the calculation in the shock wave background, several features of the problem become manifest. In the first place, this is the natural setup to perform the calculation. The AdS/CFT dictionary establishes that the shock wave background is the correct dual configuration to compute the energy one point function at zero temperature. Therefore, the connection between the energy one point function and the causality problem becomes evident. It is the graviton three point function (which upon integration yields the energy one point function) that is responsible for the scattering of the bulk modes outside the boundary theory light cone. Furthermore, it becomes evident that this is connected to properties of the theory at zero temperature.

This description of the problem also makes it clear that causality violation reflects the UV properties of the dual CFT as the interaction can occur in the asymptotic region close to the boundary. This means that any UV complete QFT which flows to fixed point at high energies will encounter the same problems, even when the black hole solutions discussed in [189, 190] change dramatically. Therefore the positivity of energy must be a statement valid for all UV complete QFTs. While this statement is obvious for asymptotically free theories, as discussed in [140, 153], it is certainly non trivial for theories that flow to strongly coupled fixed points. One interesting point is that the argument can be reversed and used as a test of UV completeness for a candidate theory.

It is also important to add that it is rather miraculous that the classical calculation in GB gravity yields the correct bounds. As was discussed here, different parameterizations of the 4 derivative gravity action that yield the same energy one point function do not show evident causality problems for propagation of perturbations on classical backgrounds. The case of a W^2 correction to the Einstein theory was studied in detail here. The small miracle of GB gravity comes about in the following way. When there is a problem with the three point function, one can observe scattering of perturbations only infinitesimally outside the light cone. This is hard to pick up. When one does a classical calculation the result from n point functions is exponentiated. It is a special property of GB gravity that only the three point function contributes to this exponentiation. Therefore, the effect becomes amplified and can be observed macroscopically. A general gravity theory will have other contributions and the exponentiation can obscure the microscopic result. This is specially important to remember when using effective actions of the type (6.1.1) or (6.2.21) as they are built to reproduce the three point function exactly but can't be trusted if one is interested in higher point functions. The GB action is special in the sense that the classical calculation isolates the effect of the three point function.

Finally, we presented an argument in favor of the positivity of energy condition in a CFT. This result implies that the operator $\int dy^- T_{--}$ should be positive in any UV complete QFT (including strongly coupled and non supersymmetric theories). It would be interesting to have a more rigorous version of this proof. In particular, it would be nice to have a more detailed analysis of the contributions of non local operators.

Chapter 7

Conclusions

In this work we have studied different aspects of gauge/gravity duality. We have focused on the specific case of the AdS_5/CFT_4 Correspondence. Within this framework, we have discussed two separate lines of research.

In the first place, we have studied certain fundamental aspects for the particular case of String Theory on $AdS_5 \times S^5$ / $\mathcal{N} = 4$ Super Yang Mills. The bulk of the work presented here in this direction was directly connected to the following question: Can we find the complete spectrum of operator dimensions of planar $\mathcal{N} = 4$ SYM for all values of 't Hooft coupling λ?

In the infinite $SO(6)$ charge J limit the fundamental excitation is the "magnon" which we have identified on both the Gauge and the String Theory in Chapter 2. Once this object is understood, the basic observable of relevance is the scattering matrix. Integrability guarantees that if the two magnon S-matrix is known then the n body problem can be solved and we have a formal solution to the large J spectrum of the theory. In this work we have analyzed the non trivial phase factor of the S-matrix at strong coupling. Also, in Chapter 3, we have studied the analytic structure of the complete S-matrix and have shown that the spectrum given by BPS magnon states is compatible with the proposal in [24, 25]. In particular, the origin of the double poles in the magnon S-matrix is the same as the origin of the double poles in the sine Gordon S-matrix. The position of the poles is determined by the spectrum of BPS particles of the model.

The most important problem that is still not well understood in this area is the solution of the spectrum at finite J. While this problem is tractable in the Bethe Ansatz approach when interactions have finite range, the same techniques are not available for this infinite range problem. The main obstacle resides in the wrapping interactions that go around the spin chain, in Gauge Theory language. Nonetheless there has been progress in this problem. Examples of recent work in this area are [224, 225].

A separate direction of importance is the understanding of other models which have a similar behavior and where the same techniques could be applied. Such an example is given, at strong coupling, by String Theory on $AdS_4 \times CP^3$. This theory also appears to be integrable and is connected to a specific field theory described in [29]. Examples of work towards understanding the integrable structure of the theory are [30, 226].

If one is to understand the complete spectrum of String Theory excitations, it is also of importance to understand other objects that can appear in the theory. In particular, D-branes are natural objects that serve as boundary conditions for worldsheet excitations. This idea was developed in Chapter 4. The dual picture involves long operators attached to baryonic operators given by determinants of order N fields. Several different configurations can be considered and we showed that, for the cases studied, the integrable structure of the closed string sector is maintained. Furthermore, the same techniques were applied and the reflection matrix was found alongside equations determining the corresponding phase factor. This phase was actually found explicitly by solving these equations in [227].

It is of interest to consider other type of boundary conditions as well. The case of D5 and D7 branes was studied, for example, in [228]. Another case where similar technology could be applied is the expectation value of Wilson loops in the gauge theory. Here, perturbations from the simpler solvable cases can be interpreted as the attachment of *words* to the loop. The idea is discussed in general terms in [113].

The second area that was studied in this thesis corresponds to applications of AdS/CFT to collider physics. The natural infrared safe observables to study in a collider experiment involving conformal matter are given by the energy correlation functions. Of great importance is the fact that these observables are well defined for any value of the coupling constant and do not rely on a partonic description. In this language, there is a simple procedure, which we discussed in detail in Chapter 5, to calculate these quantities at strong coupling in the dual String Theory.

We have also investigated the short distance singularities of energy correlation functions and find that they are governed by the presence of non local light ray operators in the field theory. Also here, we have identified the analogous phenomenon in the String Theory and described the string states dual to these non local operators.

Some further directions of study in this topic include: extensions to the $2+1$ dimensional case where we might find condensed matter applications; the understanding of finite N corrections; generalization to finite size initial states more similar to baryons; a phenomenological model of hadronization before the interaction with calorimeters; possible implications for black hole physics, since energy correlations are a way of measuring the final state of Hawking radiation and its non-thermal properties.

A very interesting byproduct of this investigation was the realization that the energy operators defined in Chapter 5 had to be positive in order to have positive energies measured in our calorimeters in a collider experiment. The consequence of this statement in a supersymmetric theory is that the central charges a and c must live in small window given by $\frac{1}{2} < \frac{a}{c} < \frac{3}{2}$. This is a great example of how studying phenomenological applications can feed back into fundamental theoretical issues.

This positivity condition was studied in greater detail in Chapter 6. Here it was understood how this story is connected to certain bounds on the coefficients of higher derivative gravities found in [189, 190]. We have also presented in this Chapter an argument in favor of the positivity of energy condition from fundamental properties of any UV complete Quantum Field Theory.

A possible generalization of this analysis would be to understand contributions of higher order terms in the gravity action. In particular, non supersymmetric theories require the presence of R^3 terms in the actions. It would be of interest to see if there is a classical gravity calculation that can reproduce the bounds proposed in Chapter 5 for this case.

It is clear from the variety of topics discussed in this thesis that the applications of the AdS/CFT Correspondence are many and its consequences are deep. This approach is, at this point, the more powerful tool we posses to understand the dynamics of strongly coupled gauge theories. The use of this framework in related problems in Condensed Matter Physics and Quark Gluon Plasmas, for example, is already very promising. Furthermore, the Correspondence gives us a preferred window into the realm of Quantum Gravity. We are merely starting to discover the main implications of gauge/gravity duality. Who knows what surprises lie ahead of us in the near future?

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