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# The Magic of Being Exceptional

**Alessio Marrani**

Museo Storico della Fisica e Centro Studi e Ricerche Enrico Fermi, Via Panisperna 89A,  
I-00184, Roma, Italy

E-mail: jazzphyzz@gmail.com

**Piero Truini**

Dipartimento di Fisica and INFN, via Dodecaneso 33, I-16146 Genoa - Italy

E-mail: truini@ge.infn.it

**Michael Rios**

Dyonica ICMQG, 5151 State University Drive, Los Angeles, CA 90032, USA

E-mail: mrios@dyonicatech.com

**Abstract.** Starting from the Jordan algebraic interpretation of the “*Magic Star*” embedding within the exceptional sequence of simple Lie algebras, we exploit the so-called spin factor embedding of rank-3 Jordan algebras and its consequences on the Jordan algebraic Lie symmetries, in order to provide another perspective on the origin of the *Exceptional Periodicity (EP)* and its “*Magic Star*” structure. We also highlight some properties of the special class of Vinberg’s rank-3 (dubbed *exceptional*) T-algebras, appearing on the tips of the “*Magic Star*” projection of *EP*(-generalized, finite-dimensional, exceptional) *algebras*.

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## 1. The Magic Star and the Exceptional Sequence

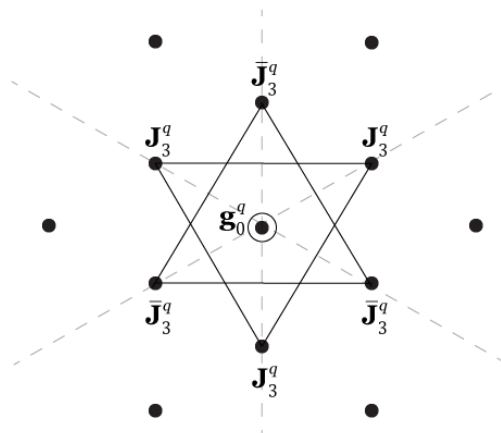
Given a finite-dimensional exceptional Lie algebra, the so-called “*Magic Star*” projection (depicted<sup>1</sup> in Fig. 1) of the corresponding root lattice onto a plane determined by an  $\mathbf{a}_2$  root sub-lattice has been introduced by Mukai<sup>2</sup> [2], and later investigated in depth in [3] (see also [4]), with a different approach exploiting Jordan Pairs [5]; in the case of  $\mathbf{e}_8$ , it has been also recalled in another contribution to Group32 Proceedings [1].

Here, we focus on the properties of the corresponding (not necessarily maximal, generally

<sup>1</sup> Note the slight change of notation with respect to [1], in which also the equivalent notation  $\mathfrak{C}$  (Cayley numbers) was used for octonions  $\mathbb{O}$ .

<sup>2</sup> Mukai used the name “ $\mathfrak{g}_2$  decomposition” to denote the *Magic Star* structure of exceptional Lie algebras.





**Figure 1.** The “*Magic Star*” of finite-dimensional exceptional Lie algebras [2, 3].  $\mathbf{J}_3^q$  denotes a simple Jordan algebra of rank-3, parametrized by  $q = \dim_{\mathbb{R}} \mathbb{A} = 1, 2, 4, 8$  for  $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ , corresponding to  $\mathbf{f}_4, \mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_8$ , respectively. In the case of  $\mathbf{g}_2$  (corresponding to  $q = -2/3$ ), the Jordan algebra is trivially the identity element :  $\mathbf{J}_3^{-2/3} \equiv \mathbb{I} := \text{diag}(1, 1, 1)$ .

non-symmetric) “*Magic Star*” embedding<sup>3</sup>

$$\mathbf{qconf}(\mathbf{J}_3^q) \supset \mathbf{a}_2 \oplus \mathbf{str}_0(\mathbf{J}_3^q), \tag{1.1}$$

and on its real forms related to a Jordan-algebraic interpretations.

Over  $\mathbb{C}$ , for all finite-dimensional exceptional Lie algebras (and some classical Lie algebras) the branching corresponding to (1.1) reads as follows [3, 4] :

$$\mathbf{qconf}(\mathbf{J}_3^q) = \mathbf{a}_2 \oplus \mathbf{str}_0(\mathbf{J}_3^q) \oplus \mathbf{3} \times \mathbf{J}_3^q \oplus \overline{\mathbf{3}} \times \overline{\mathbf{J}_3^q}. \tag{1.2}$$

By suitably varying the parameter  $q$ , the sequence given in Table 1 is obtained. The sequence

$q$	8	4	2	1	0	-1/3	-2/3	-1
$\mathbf{qconf}(\mathbf{J}_3^q)$	$\mathbf{e}_8$	$\mathbf{e}_7$	$\mathbf{e}_6$	$\mathbf{f}_4$	$\mathbf{d}_4$	$\mathbf{b}_3$	$\mathbf{g}_2$	$\mathbf{a}_2$
$\mathbf{str}_0(\mathbf{J}_3^q)$	$\mathbf{e}_6$	$\mathbf{a}_5$	$\mathbf{a}_2 \oplus \mathbf{a}_2$	$\mathbf{a}_2$	$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}$	$\emptyset$	-

**Table 1.** The exceptional sequence

$\{\mathbf{qconf}(\mathbf{J}_3^q)\}_{q=8,4,2,1,0,-2/3,-1}$  is usually named *exceptional sequence* (or *exceptional series*; cfr. e.g.[8]), with the addition of an  $\mathbf{a}_1$  corresponding to  $q = -4/3$  (which we disregard<sup>4</sup>, because it is irrelevant for our treatment).

<sup>3</sup> We briefly recall that, in the context of supergravity, the “*Magic Star*” embedding (1.1) has been discussed in [6] (where it was named *Jordan pairs’ embedding*), and later identified as the  $D = 5$  case of the so-called *super-Ehlers embedding* in [7]. In [6] the “*Magic Star*” embedding for (suitable real forms of) orthogonal Lie algebras was discussed, as well.

<sup>4</sup> On the other hand, we add a  $\mathbf{b}_3$  corresponding to  $q = -1/3$ , which is absent in the treatment e.g. of [8].

$\mathbf{J}_3^q$  denotes the rank-3 Jordan algebra (cfr. e.g. [9], and Refs. therein) associated to the parameter  $q$ ; such a Jordan algebra is simple for  $q = 8, 4, 2, 1$  and  $-2/3$  [10], and for  $q = 8, 4, 2, 1$  the parameter  $q$  has the meaning of real dimension of the division algebra  $\mathbb{A}$  on which the corresponding Jordan algebra is realized as the algebra of Hermitian matrices :  $q = \dim_{\mathbb{R}} \mathbb{A} = 8, 4, 2, 1$  for  $\mathbb{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}$ , resp., and  $\mathbf{J}_3^q \equiv \mathbf{J}_3^{\mathbb{A}} \equiv H_3(\mathbb{A})$  in these cases. Also, it should be remarked that for  $q = 8, 4, 2, 1$   $\mathbf{qconf}(\mathbf{J}_3^q)$  and  $\mathbf{str}_0(\mathbf{J}_3^q)$  respectively span the fourth and second row/column of the Freudenthal-Rosenberg-Tits Magic Square [11, 12]. Moreover<sup>5</sup>,  $\mathbf{J}_3^0 \equiv \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$  is the completely factorized (triality symmetric) rank-3 Jordan algebra, and  $\mathbf{J}_3^{-1/3} \equiv \mathbb{C} \oplus \mathbb{C}$  and  $\mathbf{J}_3^{-2/3} \equiv \mathbb{C}$  are its partial and total degenerations, respectively.  $\mathbf{qconf}(\mathbf{J}_3^q)$  and  $\mathbf{str}_0(\mathbf{J}_3^q)$  stand for the quasi-conformal Lie algebra resp. the reduced structure Lie algebra of the rank-3 Jordan algebra  $\mathbf{J}_3^q$  (cfr. e.g. [13, 14]).

Over  $\mathbb{R}$  (as we shall consider throughout the present paper), there are two non-compact real forms of the above “enlarged” exceptional sequence  $\{\mathbf{qconf}(\mathbf{J}_3^q)\}_{q=8,4,2,1,0,-1/3,-2/3,-1}$  which enjoy an immediate Jordan-algebraic interpretation; they are reported in Tables 2 and Table 3. In both these cases, the real form of the  $\mathfrak{a}_2$  defining the plane onto which the Magic Star projection is defined is maximally non-compact (i.e., split), and the following branching (non-compact, real form of (1.2)) correspondingly holds:

$$\mathbf{qconf}(\mathbf{J}_3^q) = \mathfrak{sl}_{3,\mathbb{R}} \oplus \mathbf{str}_0(\mathbf{J}_3^q) \oplus \mathfrak{3} \times \mathbf{J}_3^q \oplus \mathfrak{3}' \times \mathbf{J}_3^{q'}. \quad (1.3)$$

$q$	8	4	2	1	0	$-1/3$	$-2/3$	$-1$
$\mathbf{qconf}(\mathbf{J}_3^q)$	$\mathfrak{e}_{8(8)}$	$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{6(6)}$	$\mathfrak{f}_{4(4)}$	$\mathfrak{so}_{4,4}$	$\mathfrak{so}_{4,3}$	$\mathfrak{g}_{2(2)}$	$\mathfrak{sl}_{3,\mathbb{R}}$
$\mathbf{str}_0(\mathbf{J}_3^q)$	$\mathfrak{e}_{6(6)}$	$\mathfrak{sl}_{6,\mathbb{R}}$	$\mathfrak{sl}_{3,\mathbb{R}} \oplus \mathfrak{sl}_{3,\mathbb{R}}$	$\mathfrak{sl}_{3,\mathbb{R}}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}$	$\emptyset$	$-$

**Table 2.** The maximally non-compact (split) real form of the exceptional sequence. In this case, for  $q = 8, 4, 2, 1$ ,  $\mathbf{J}_3^q \equiv \mathbf{J}_3^{\mathbb{A}_s} \equiv H_3(\mathbb{A}_s)$ , where  $\mathbb{A}_s$  is the split form of  $\mathbb{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}$ , respectively.

$q$	8	4	2	1	0	$-1/3$	$-2/3$	$-1$
$\mathbf{qconf}(\mathbf{J}_3^q)$	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{f}_{4(4)}$	$\mathfrak{so}_{4,4}$	$\mathfrak{so}_{4,3}$	$\mathfrak{g}_{2(2)}$	$\mathfrak{sl}_{3,\mathbb{R}}$
$\mathbf{str}_0(\mathbf{J}_3^q)$	$\mathfrak{e}_{6(-26)}$	$\mathfrak{su}_6^*$	$(\mathfrak{sl}_{3,\mathbb{C}})_{\mathbb{R}}$	$\mathfrak{sl}_{3,\mathbb{R}}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}$	$\emptyset$	$-$

**Table 3.** Another (non-split) non-compact real form of the exceptional sequence

## 2. The Spin-Factor Embedding and the Exceptional Periodicity

Let us now consider the following (maximal) algebraic embeddings ( $q = 8, 4, 2, 1$ ):

$$\mathbf{J}_3^{\mathbb{A}} \supset \mathbb{R} \oplus \mathbf{J}_2^{\mathbb{A}}; \quad (2.1)$$

$$\mathbf{J}_3^{\mathbb{A}_s} \supset \mathbb{R} \oplus \mathbf{J}_2^{\mathbb{A}_s}, \quad (2.2)$$

<sup>5</sup> Note that the  $q = -1$  case is a limiting case, included for completeness' sake. In fact, in such a case there is no corresponding rank-3 Jordan algebra, and so the corresponding reduced structure symmetry is ill defined.

realized as<sup>6</sup> ( $r_i \in \mathbb{R}$ ,  $A_i \in \mathbb{A}$  or  $\mathbb{A}_s$ ,  $i = 1, 2, 3$ )

$$\mathbf{J}_3^{\mathbb{A}} \ni J = \begin{pmatrix} r_1 & A_1 & \bar{A}_2 \\ \bar{A}_1 & r_2 & A_3 \\ A_2 & \bar{A}_3 & r_3 \end{pmatrix} \mapsto J' = \begin{pmatrix} r_1 & A_1 & 0 \\ \bar{A}_1 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix} \in \mathbb{R} \oplus \mathbf{J}_2^{\mathbb{A}}. \quad (2.3)$$

By noticing that  $r_1$  and  $r_2$  can be associated to lightcone degrees of freedom,

$$r_1 := x_+ + x_-, \quad r_2 := x_+ - x_-, \quad (2.4)$$

it can be proved that (cfr. e.g. [15])

$$\mathbf{J}_2^{\mathbb{A}} \sim \mathbf{\Gamma}_{1,q+1}; \quad (2.5)$$

$$\mathbf{J}_2^{\mathbb{A}_s} \sim \mathbf{\Gamma}_{q/2+1,q/2+1}, \quad (2.6)$$

where  $\mathbf{\Gamma}_{1,q+1}$  and  $\mathbf{\Gamma}_{q/2+1,q/2+1}$  respectively are  $(q+2)$ -dimensional spin factors with Lorentzian and split (Kleinian) signature, respectively; thus, we will dub (2.1)-(2.2) the *spin-factor embeddings*.

Let us now restrict to consider the embedding (2.1) for  $\mathbb{A} = \mathbb{O}$ , i.e. for  $q = 8$ , and let us analyze its consequences at the level of the symmetry algebras of its l.h.s. and r.h.s. :

(i) at the level of the Lie algebra of derivations  $\mathfrak{der}$ , it holds that<sup>7</sup>

$$\mathfrak{der}(\mathbf{J}_3^{\mathbb{O}}) \supset^{m,s} \mathfrak{der}(\mathbb{R} \oplus \mathbf{J}_2^{\mathbb{O}}) \Leftrightarrow \begin{cases} \mathfrak{f}_{4(-52)} \supset^{m,s} \mathfrak{so}_9; \\ \mathbf{52} = \mathbf{36} \oplus \mathbf{16}, \end{cases} \quad (2.7)$$

where  $\mathbf{16}$  is the real spinor irrepr. of  $\mathfrak{so}_9$ .

(ii) at the level of the reduced structure Lie algebra  $\mathfrak{str}_0$ , it holds that

$$\mathfrak{str}_0(\mathbf{J}_3^{\mathbb{O}}) \supset^{m,s} \mathfrak{str}_0(\mathbb{R} \oplus \mathbf{J}_2^{\mathbb{O}}) \Leftrightarrow \begin{cases} \mathfrak{e}_{6(-26)} \supset^{m,s} \mathfrak{so}_{9,1} \oplus \mathbb{R}; \\ \mathbf{78} = \mathbf{16}'_{-1} \oplus (\mathbf{45} \oplus \mathbf{1})_0 \oplus \mathbf{16}_1, \\ \text{or} \\ \mathbf{78} = \mathbf{16}_{-1} \oplus (\mathbf{45} \oplus \mathbf{1})_0 \oplus \mathbf{16}'_1, \end{cases} \quad (2.8)$$

where  $\mathbf{16}$  and  $\mathbf{16}'$  are the Majorana-Weyl (MW) spinors of  $\mathfrak{so}_{9,1}$ ; the indeterminacy denoted by “or” depends on the *spinor polarization* [16].

(iii) at the level of the conformal Lie algebra  $\mathfrak{conf}$ , it holds that

$$\mathfrak{conf}(\mathbf{J}_3^{\mathbb{O}}) \supset^{m,s} \mathfrak{conf}(\mathbb{R} \oplus \mathbf{J}_2^{\mathbb{O}}) \Leftrightarrow \begin{cases} \mathfrak{e}_{7(-25)} \supset^{m,s} \mathfrak{so}_{10,2} \oplus \mathfrak{sl}_{2,\mathbb{R}}; \\ \mathbf{133} = (\mathbf{78}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus ((\mathbf{32}^{(\prime)}, \mathbf{2})), \end{cases} \quad (2.9)$$

where  $\mathbf{32}$  is the MW spinor of  $\mathfrak{so}_{10,2}$ , and the possible priming (denoting spinorial conjugation) depends on the choice of the spinor polarization [16]. By further branching the  $\mathfrak{sl}_{2,\mathbb{R}}$ , one gets the following 5-grading of contact type :

$$\mathfrak{e}_{7(-25)} \supset \mathfrak{so}_{10,2} \oplus \mathbb{R}; \quad (2.10)$$

$$\mathbf{133} = \mathbf{1}_{-2} \oplus \mathbf{32}_{-1}^{(\prime)} \oplus (\mathfrak{so}_{10,2} \oplus \mathbb{R})_0 \oplus \mathbf{32}_1^{(\prime)} \oplus \mathbf{1}_2.$$

<sup>6</sup> The bar denotes the conjugation in  $\mathbb{A}$  or in  $\mathbb{A}_s$ .

<sup>7</sup> The upper scripts “ $m$ ” and “ $s$ ” respectively denote the maximality and symmetricity of the embedding under consideration.

(iv) at the level of the quasi-conformal Lie algebra  $\mathfrak{qconf}$ , it holds that

$$\mathfrak{qconf}\left(\mathbf{J}_3^{\oplus}\right) \supset^{m,s} \mathfrak{qconf}\left(\mathbb{R} \oplus \mathbf{J}_2^{\oplus}\right) \Leftrightarrow \begin{cases} \mathfrak{e}_{8(-24)} \supset^{m,s} \mathfrak{so}_{12,4}; \\ \mathbf{248} = \mathbf{120} \oplus \mathbf{128}^{(\prime)}, \end{cases} \quad (2.11)$$

where  $\mathbf{128}$  is the MW spinor of  $\mathfrak{so}_{12,4}$ , and again the possible priming (denoting spinorial conjugation) depends on the choice of the spinor polarization [16]. By further branching the  $\mathfrak{so}_{12,4}$ , one gets the following 5-grading of “extended Poincaré” type [17]:

$$\begin{aligned} \mathfrak{e}_{8(-24)} &\supset \mathfrak{so}_{11,3} \oplus \mathbb{R}; \\ \mathbf{248} &= \begin{cases} \mathbf{14}_{-2} \oplus \mathbf{64}'_{-1} \oplus (\mathfrak{so}_{11,3} \oplus \mathbb{R})_0 \oplus \mathbf{64}_1 \oplus \mathbf{14}_2; \\ \text{or} \\ \mathbf{14}_{-2} \oplus \mathbf{64}_{-1} \oplus (\mathfrak{so}_{11,3} \oplus \mathbb{R})_0 \oplus \mathbf{64}'_1 \oplus \mathbf{14}_2, \end{cases} \end{aligned} \quad (2.12)$$

where  $\mathbf{64}$  is the MW spinor of  $\mathfrak{so}_{11,3}$  and, again, there is some indeterminacy depending on the spinor polarization [16].

By exploiting Bott periodicity for the spinor irreprs., one can formally define the “*Exceptional Periodicity generalizations*” [18, 1] of the real forms of exceptional Lie algebras  $\mathfrak{detr}\left(\mathbf{J}_3^{\oplus}\right) = \mathfrak{f}_{4(-52)}$ ,  $\mathfrak{str}_0\left(\mathbf{J}_3^{\oplus}\right) = \mathfrak{e}_{6(-26)}$ ,  $\mathfrak{conf}\left(\mathbf{J}_3^{\oplus}\right) = \mathfrak{e}_{7(-25)}$  and  $\mathfrak{qconf}\left(\mathbf{J}_3^{\oplus}\right) = \mathfrak{e}_{8(-24)}$  (or, more briefly, the following real forms of *Exceptional Periodicity algebras*), as follows<sup>8</sup> ( $n \in \mathbb{N} \cup \{0\}$  throughout<sup>9</sup>):

(i) Exceptional periodization of level  $\mathfrak{detr}$ :

$$\mathfrak{f}_{4(-52)}^{(n)} := \mathfrak{so}_{9+8n} \oplus \psi_{\mathfrak{so}_{9+8n}}, \quad (2.13)$$

where  $\psi_{\mathfrak{so}_{9+8n}} \equiv \mathbf{2}^{4+4n}$  is the real spinor of  $\mathfrak{so}_{9+8n}$ .

(ii) Exceptional periodization of level  $\mathfrak{str}_0$ :

$$\mathfrak{e}_{6(-26)}^{(n)} := \psi'_{\mathfrak{so}_{9+8n,1,-1}} \oplus (\mathfrak{so}_{9+8n,1} \oplus \mathbb{R})_0 \oplus \psi_{\mathfrak{so}_{9+8n,1,1}}, \quad (2.14)$$

where  $\psi_{\mathfrak{so}_{9+8n,1}} \equiv \mathbf{2}^{4+4n}$  is the MW spinor of  $\mathfrak{so}_{9+8n,1}$ .

(iii) Exceptional periodization of level  $\mathfrak{conf}$ :

$$\begin{aligned} \mathfrak{e}_{7(-25)}^{(n)} &:= (\mathfrak{so}_{10+8n,2} \oplus \mathfrak{sl}_{2,\mathbb{R}}(2, \mathbb{R})) \oplus (\psi_{\mathfrak{so}_{10+8n,2}}, \mathbf{2}) = \\ &\mathbf{1}_{-2} \oplus \psi_{\mathfrak{so}_{10+8n,2,-1}} \oplus (\mathfrak{so}_{10+8n,2} \oplus \mathfrak{sl}_{2,\mathbb{R}}(2, \mathbb{R}))_0 \oplus \psi_{\mathfrak{so}_{10+8n,2,1}} \oplus \mathbf{1}_2, \end{aligned} \quad (2.15)$$

where  $\psi_{\mathfrak{so}_{10+8n,2}} \equiv \mathbf{2}^{5+4n}$  is the MW spinor of  $\mathfrak{so}_{10+8n,2}$ .

<sup>8</sup> In the EP generalization, for simplicity’s sake, we assume to have chosen a spinor polarization, so as to remove the *a priori* indeterminacy in the vector space grading structure of the EP algebras.

<sup>9</sup> Note that the index  $n$  used here is actually  $n - 1$ , where  $n$  is the index used in [1] and [18]. In other words, the exceptional Lie algebras (trivial level of EP) are obtained for  $n = 0$  in the present paper, whereas they are obtained for  $n = 1$  in [1] and [18].

(iv) Exceptional periodization of level  $q\text{conf}$  :

$$\begin{aligned} \mathbf{e}_{8(-24)}^{(n)} &:= \mathfrak{so}_{12+8n,4} \oplus \psi_{\mathfrak{so}_{12+8n,4}} \\ &= (\mathbf{14} + \mathbf{8n})_{-2} \oplus \psi'_{\mathfrak{so}_{11+8n,3,-1}} \oplus (\mathfrak{so}_{11+8n,3} \oplus \mathbb{R})_0 \oplus \psi_{\mathfrak{so}_{11+8n,3,1}} \oplus (\mathbf{14} + \mathbf{8n})_2, \end{aligned} \quad (2.16)$$

where  $\psi_{\mathfrak{so}_{12+8n,4}} \equiv \mathbf{2}^{7+4n}$  and  $\psi_{\mathfrak{so}_{11+8n,3}} \equiv \mathbf{2}^{6+4n}$  respectively denote the MW spinors of  $\mathfrak{so}_{12+8n,4}$  and of  $\mathfrak{so}_{11+8n,3}$ .

So far, (2.13)-(2.14) is just a bunch of definitions, exploiting Bott periodicity. On  $\mathbb{C}$ , in [18] (see also [1]) the rigorous definitions of *EP algebras*  $\mathbf{f}_4^{(n)}$ ,  $\mathbf{e}_6^{(n)}$ ,  $\mathbf{e}_7^{(n)}$  and  $\mathbf{e}_8^{(n)}$  was dealt with by introducing the generalized roots and by defining the commutations relations of the corresponding generators with (a suitably generalized) Kac's asymmetry function [19, 20]. Here, we would like to recall that EP algebras are not simply non-reductive, spinorial extensions of Lie algebras, but rather they are characterized by a non-translational (i.e., non-Abelian) nature of their spinorial sector; this implies that they are Lie algebras only for  $n = 0$ , i.e. at the trivial level of EP, whereas for  $n \geq 1$  they are not Lie algebras, because the Jacobi identity is violated in the spinorial sector itself [18].

The above treatment on  $\mathbb{R}$ , based on the EP generalization of the symmetry Lie algebras of  $\mathbf{J}_3^\circ$ , allowed to determine some of the real forms of the *EP algebras*  $\mathbf{f}_4^{(n)}$ ,  $\mathbf{e}_6^{(n)}$ ,  $\mathbf{e}_7^{(n)}$  and  $\mathbf{e}_8^{(n)}$ . The other (compact and non-compact) real forms can be analogously<sup>10</sup> defined e.g. by considering  $\mathbf{J}_3^{\circ s}$ ; we plan to deal with a rigorous definition of the real forms of EP algebras, through the generalized roots and the definition of suitable involutions in the corresponding EP lattice [18], in a forthcoming investigation.

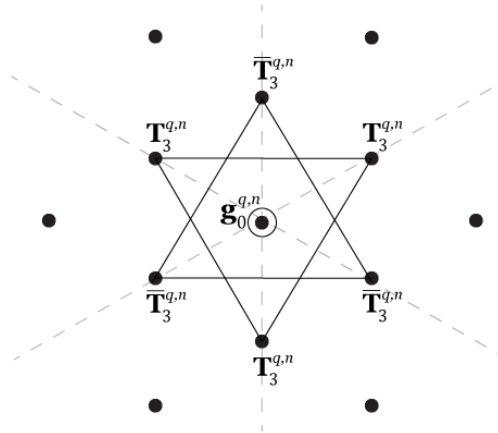
The crucial result, which all the above construction and the corresponding construction in the EP lattice non-trivial, is the following [18]: there exists an<sup>11</sup>  $\mathbf{a}_2$  projection of the EP algebras (namely, of the corresponding EP lattice), such that a *Magic Star structure persists*, with suitable generalizations of rank-3 Jordan algebras  $\mathbf{J}_3^n$  (given by rank-3 T-algebras of *special type* [22]) occurring on the tips of the persisting Magic Star! The resulting, EP-generalized Magic Star is depicted in Fig. 2.

### 3. Vinberg's T-Algebras in EP : a Glimpse

In the treatment of previous Section, we fixed  $q = 8$  and dealt with the EP-generalization of the Lie algebras which are symmetries of the corresponding rank-3 simple Jordan algebra  $\mathbf{J}_3^\circ$ .

<sup>10</sup> Apart from the EP-generalization  $\mathbf{f}_{4(-52)}^{(n)}$  (2.13) of the compact real form  $\mathbf{f}_{4(-52)}$ , the EP-generalized non-compact real forms  $\mathbf{e}_{6(-26)}^{(n)}$  (2.14),  $\mathbf{e}_{7(-25)}^{(n)}$  (2.15) and  $\mathbf{e}_{8(-24)}^{(n)}$  (2.16), albeit being well defined, are characterized by a threefold degree of arbitrariness. Indeed, at the  $n$ -th level of EP, we understood to enlarge by  $8n$  only the spacelike dimensions in the  $(s, t)$ -signature of the reductive, pseudo-orthogonal part of the aforementioned EP algebras, thus obtaining  $\mathfrak{so}_{9+8n,1}$ ,  $\mathfrak{so}_{10+8n,2}$  and  $\mathfrak{so}_{12+8n,4}$  (and then  $\mathfrak{so}_{11+8n,3}$ ), resp. in  $\mathbf{e}_{6(-26)}^{(n)}$ ,  $\mathbf{e}_{7(-25)}^{(n)}$  and  $\mathbf{e}_{8(-24)}^{(n)}$ . Nevertheless, the conjugation and reality properties of spinors depend only on  $D = s + t$  and on  $\rho := s - t$  (cfr. e.g. [21]), thus at the  $n$ -th level of EP, the implementation of Bott (i.e.,  $\text{mod}(8n)$ ) periodicity could also be made by increasing by  $8n$  only the timelike dimensions, or also by increasing by  $4n$  both spacelike and timelike dimensions. In the former case, one would obtain  $\mathfrak{so}_{9,1+8n}$ ,  $\mathfrak{so}_{10,2+8n}$  and  $\mathfrak{so}_{12,4+8n}$  (and then  $\mathfrak{so}_{11,3+8n}$ ) resp. in  $\mathbf{e}_{6(-26)}^{(n)}$ ,  $\mathbf{e}_{7(-25)}^{(n)}$  and  $\mathbf{e}_{8(-24)}^{(n)}$ , whereas in the latter case one would obtain  $\mathfrak{so}_{9+4n,1+4n}$ ,  $\mathfrak{so}_{10+4n,2+4n}$  and  $\mathfrak{so}_{12+4n,4+4n}$  (and then  $\mathfrak{so}_{11+4n,3+4n}$ ) resp. in  $\mathbf{e}_{6(-26)}^{(n)}$ ,  $\mathbf{e}_{7(-25)}^{(n)}$  and  $\mathbf{e}_{8(-24)}^{(n)}$ . Such a threefold degeneracy of the implementation of Bott periodicity (yielding spinors with the same dimensions, reality properties and conjugation properties) can in principle be applied at any level of EP, also in a different way from the way it was implemented at the previous level; this allows to span a large variety of  $(s, t)$ -signatures in the  $\mathfrak{so}_{s,t}$  reductive part of the non-compact real forms of EP algebras.

<sup>11</sup> Such a projection is not unique; as for the Magic Star of Lie algebras, depicted in Figure 1, also for the Magic Star of EP algebras, depicted in Figure 2, there are four possible equivalent projections.



**Figure 2.** The “*Magic Star*” of EP-generalized, finite-dimensional exceptional Lie algebras [18].  $\mathbf{T}_3^{q,n}$  denotes a T-algebra of rank-3 and of *special type* [22], parametrized by  $q = \dim_{\mathbb{R}} \mathbb{A} = 1, 2, 4, 8$  for  $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ , and  $n \in N \cup \{0\}$ , corresponding to  $\mathbf{f}_4^{(n)}, \mathbf{e}_6^{(n)}, \mathbf{e}_7^{(n)}, \mathbf{e}_8^{(n)}$ , respectively. The smallest exceptional Lie algebra  $\mathbf{g}_2$  (corresponding to  $q = -2/3$ ) cannot be EP-generalized, because it does not enjoy a spin factor embedding, and because  $\mathbf{J}_3^{-2/3} \equiv \mathbb{I}$ .

Actually, due to the symmetry of the Freudenthal-Rosenberg-Tits Magic Square [11, 12] (over  $\mathbb{C}$ ), the  $\mathbf{det}, \mathbf{str}_0, \mathbf{conf}$  and  $\mathbf{qconf}$  Lie symmetries (sitting in fourth row of the Magic Square) also corresponds to the fourth column of the Magic Square itself, and thus to  $q = 1, 2, 4, 8$ , respectively. Such a correspondence, pointed out in the caption of Fig. 2, will be exploited in the treatment below.

Concerning the EP-generalized real forms treated in the previous Section, namely  $\mathbf{e}_{8(-24)}^{(n)}, \mathbf{e}_{7(-25)}^{(n)}, \mathbf{e}_{6(-26)}^{(n)}$  and  $\mathbf{f}_{4(-52)}^{(n)}$  (corresponding to  $q = 8, 4, 2, 1$ , respectively), the  $3 \times 3$  matrix algebras  $\mathbf{T}_3^{q,n}$  occurring on the tips of the EP-generalized Magic Star depicted in Fig. 2 are of the following form :

$$\mathbf{T}_3^{q,n} := \begin{pmatrix} r_1 & \mathbf{V}_{\mathbf{so}_{q+8n}} & \psi_{\mathbf{so}_{q+8n}} \\ \overline{\mathbf{V}}_{\mathbf{so}_{q+8n}} & r_2 & \psi'_{\mathbf{so}_{q+8n}} \\ \overline{\psi}_{\mathbf{so}_{q+8n}} & \overline{\psi}'_{\mathbf{so}_{q+8n}} & r_3 \end{pmatrix}, \tag{3.1}$$

where  $\mathbf{V}_{\mathbf{so}_{q+8n}} \equiv (\mathbf{q} + \mathbf{8n}, \mathbf{1})$  and  $\psi_{\mathbf{so}_{q+8n}} \equiv (2^{q/2+4n-1}, \mathbf{Fund}(\mathcal{S}_q))$  are representations of  $\mathbf{so}_{q+8n} \oplus \mathcal{S}_q$ , with

$$\mathcal{S}_q := \mathbf{tri}_{\mathbb{A}} \ominus \mathbf{so}_{\mathbb{A}} = \emptyset, \mathbf{so}_2, \mathbf{su}_2, \emptyset \text{ for } q = 1, 2, 4, 8 \text{ (i.e., for } \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}, \text{ resp.)} \tag{3.2}$$

being the coset algebra of the triality symmetry  $\mathbf{tri}_{\mathbb{A}}$  of  $\mathbb{A}$  [23]:

$$\begin{aligned} \mathbf{tri}_{\mathbb{A}} : &= \{(A, B, C) \mid A(xy) = B(x)y + xC(y), A, B, C \in \mathbf{so}_{\mathbb{A}}, x, y \in \mathbb{A}\} \\ &= \emptyset, \mathbf{so}_2^{\oplus 2}, \mathbf{so}_3^{\oplus 3}, \mathbf{so}_8 \text{ for } \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \end{aligned}$$

modded by the norm preserving symmetry  $\mathbf{so}_{\mathbb{A}}$  of  $\mathbb{A}$  :

$$\mathbf{so}_{\mathbb{A}} := \mathbf{so}_q = \emptyset, \mathbf{so}_2, \mathbf{so}_4, \mathbf{so}_8 \text{ for } \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}. \tag{3.3}$$

$\mathbf{Fund}(\mathcal{S}_q)$  denotes the smallest non-trivial representation of the simple Lie algebra  $\mathcal{S}_q$  (if any) :

$$\mathbf{Fund}(\mathcal{S}_q) = -, \mathbf{2}, \mathbf{2}, - \text{ for } q = 1, 2, 4, 8, \quad (3.4)$$

with real dimension

$$fund_q := \dim_{\mathbb{R}} \mathbf{Fund}(\mathcal{S}_q) = 1, 2, 2, 1 \text{ for } q = 1, 2, 4, 8. \quad (3.5)$$

Thus, the total real dimension of  $\mathbf{T}_3^{q,n}$  is

$$\dim_{\mathbb{R}}(\mathbf{T}_3^{q,n}) = q + 3 + 8n + fund_q 2^{[q/2]+4n+\delta_{q,1}}, \quad (3.6)$$

where the square brackets denote the integer part.

$\mathbf{T}_3^{q,n}$  (3.1) is a rank-3 T-algebra, introduced by Vinberg as a generalization of cubic Jordan algebras in the study of homogeneous convex cones [22]. In particular,  $\mathbf{T}_3^{q,n}$  is a rank-3 T-algebra of *special* type; it is dubbed *exceptional* T-algebra in Sec. 4.3 of [1]. Such algebras, slightly generalized, also occur in the study of homogeneous non-symmetric real special manifolds<sup>12</sup>, which are non-compact Riemannian spaces occurring as (vector multiplets') scalar manifolds of  $\mathcal{N} = 2$  Maxwell-Einstein supergravity theories in  $D = s + t = 4 + 1$  space-time dimensions [25].

It is immediate to check that *triality* is broken in  $\mathbf{T}_3^{q,n}$  whenever  $n$  is positive; in fact, triality holds whenever

$$\begin{aligned} \dim_{\mathbb{R}} \mathbf{V} &= \dim_{\mathbb{R}} \psi \\ &\Updownarrow \\ q + 8n &= fund_q \cdot 2^{[q/2]+4n-1+\delta_{q,1}} \\ &\Updownarrow \\ n = 0 &: \mathbf{T}_3^{q,0} \equiv \mathbf{J}_3^q. \end{aligned} \quad (3.7)$$

Interestingly, a kind of *weaker* (or *generalized*) form of triality exists for<sup>13</sup>  $\mathfrak{so}_{24}$  : this corresponds to  $q = 8$  and  $n = 2$ , i.e. to the second non-trivial level of EP over  $\mathbb{O}$ . In this case,  $\mathcal{S}_8 = \emptyset$ , and from definition (3.1)  $\mathbf{T}_3^{8,2}$  reads as follows :

$$\mathbf{T}_3^{8,2} = \begin{pmatrix} r_1 & \mathbf{V}_{\mathfrak{so}_{24}} & \psi_{\mathfrak{so}_{24}} \\ \overline{\mathbf{V}}_{\mathfrak{so}_{24}} & r_2 & \psi'_{\mathfrak{so}_{24}} \\ \overline{\psi}_{\mathfrak{so}_{24}} & \overline{\psi}'_{\mathfrak{so}_{24}} & r_3 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{24} & \mathbf{2048} \\ \mathbf{24} & \mathbf{1} & \overline{\mathbf{2048}} \\ \mathbf{2048} & \overline{\mathbf{2048}} & \mathbf{1} \end{pmatrix} \text{ of } \mathfrak{so}_{24}. \quad (3.8)$$

Remarkably, there exists an irreducible representation of  $\mathfrak{so}_{24}$  whose dimension is exactly the difference between the dimensions of the semispinor (**2048**) and of the vector (**24**) of  $\mathfrak{so}_{24}$  : this is the 3-form representation  $\wedge^3 \mathbf{24} = \mathbf{2024}$ . Thus, one could argue the existence of an “augmented” T-algebra

$$\widetilde{\mathbf{T}}_3^{8,2} := \begin{pmatrix} r_1 & \mathbf{V}_{\mathfrak{so}_{24}} \oplus \wedge^3_{\mathfrak{so}_{24}} & \psi_{\mathfrak{so}_{24}} \\ \overline{\mathbf{V}}_{\mathfrak{so}_{24}} \oplus \wedge^3_{\mathfrak{so}_{24}} & r_2 & \psi'_{\mathfrak{so}_{24}} \\ \overline{\psi}_{\mathfrak{so}_{24}} & \overline{\psi}'_{\mathfrak{so}_{24}} & r_3 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{24} \oplus \mathbf{2024} & \mathbf{2048} \\ \mathbf{24} \oplus \mathbf{2024} & \mathbf{1} & \overline{\mathbf{2048}} \\ \mathbf{2048} & \overline{\mathbf{2048}} & \mathbf{1} \end{pmatrix} \text{ of } \mathfrak{so}_{24}, \quad (3.9)$$

exhibiting a manifest triality, albeit with a reducible nature of the bosonic representations. It would be interesting to investigate the properties of “augmented” T-algebras of the type (3.9). However, it is here worth stressing that triality is rigorously holding only for the trivial level of

<sup>12</sup> And, of course, in their images under R-map and c-map (cfr. e.g. [24], and Refs. therein).

<sup>13</sup> This was communicated to us by Eric Weinstein during an informal discussion in 2016.

EP, i.e. for  $n = 0$ . For instance, when  $n = 0$  and  $q = 8$ , one obtains the well known case of  $\mathfrak{so}(8)$  triality among the off-diagonal blocks of the Albert algebra  $\mathbf{J}_3^8 \equiv \mathbf{J}_3^\mathbb{O}$  ( $o_i \in \mathbb{O}$ ,  $r_i \in \mathbb{R}$ ):

$$\mathbf{T}_3^{8,0} \equiv \mathbf{J}_3^\mathbb{O} \ni \begin{pmatrix} r_1 & o_1 & \bar{o}_2 \\ \bar{o}_1 & r_2 & o_3 \\ o_2 & \bar{o}_3 & r_3 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{8}_v & \mathbf{8}_s \\ \mathbf{8}_v & \mathbf{1} & \mathbf{8}_c \\ \mathbf{8}_s & \mathbf{8}_c & \mathbf{1} \end{pmatrix}, \quad (3.10)$$

where  $\mathbf{8}_v$ ,  $\mathbf{8}_s$  and  $\mathbf{8}_c$  are the vector, semispinor and conjugate semispinor of  $\mathbf{tri}_\mathbb{O} = \mathfrak{so}_8$ , respectively.

For what concerns the invariant structures of the special class of rank-3 T-algebras defined in (3.1), let us define ( $\mu = 0, 1, \dots, q + 1 + 8n$ )

$$V^\mu := (r_1, r_2, \mathbf{V}_{\mathfrak{so}_{q+8n}}),$$

which, by recalling (2.4), is recognized to be a vector in a  $(q + 2 + 8n)$ -dimensional space, with Lorentzian signature  $(s, t) = (q + 1 + 8n, 1)$ ; also, let us denote the corresponding spinor of  $\mathfrak{so}_{q+1+8n,1}$  (which is chiral for  $q = 2, 4, 8$ ), of real dimension  $fund_q \cdot 2^{\lfloor q/2 \rfloor + 4n + \delta_{q,1}}$ , by  $\Psi^{\alpha A}$  (where  $\alpha = 1, \dots, 2^{\lfloor q/2 \rfloor + 4n + \delta_{q,1}}$  and  $A = 1, fund_q$ ). Then, an invariant structure constructed with the corresponding T-algebra  $\mathbf{T}_3^{q,n}$  is formally given by the ‘‘determinant’’ of such a  $3 \times 3$  Hermitian matrix, defining the cubic norm  $\mathbf{N}$  of  $\mathbf{T}_3^{q,n}$  itself :

$$\mathbf{N}(\mathbf{T}_3^{q,n}) := \frac{1}{2} \eta_{\mu\nu} \left[ r_3 V^\mu V^\nu + \gamma_{\alpha\beta}^\mu \Psi^{\alpha A} \Psi_A^\beta V^\nu \right], \quad (3.11)$$

where  $\eta_{\mu\nu}$  is the symmetric bilinear invariant of the  $\mathfrak{q} + \mathbf{2} + \mathbf{8n}$  irrepr.  $V$  of  $\mathfrak{so}_{q+1+8n,1}$  and  $\gamma_{\alpha\beta}^\mu$  are the gamma matrices of  $\mathfrak{so}_{q+1+8n,1}$ . Consequently, as for the Jordan algebras, one can classify the elements of  $\mathbf{T}_3^{q,n}$  depending on their *rank*, defined as follows [18]:

$$\begin{aligned} \text{rank-3} & : \mathbf{N} \neq 0; \\ \text{rank-2} & : \mathbf{N} = 0; \\ \text{rank-1} & : \partial \mathbf{N} = 0. \end{aligned} \quad (3.12)$$

In the  $\mathcal{N} = 2$  Maxwell-Einstein supergravity theories in  $D = s + t = 4 + 1$  space-time dimensions based on  $\mathbf{T}_3^{q,n}$  [25], the Bekenstein-Hawking entropy  $S_{BH}$  of the extremal black holes simply reads

$$S_{BH} = \pi \sqrt{|\mathbf{N}|}. \quad (3.13)$$

## References

- [1] Truini P, Marrani A and Rios M 2018 *Magic Star and Exceptional Periodicity: an approach to Quantum Gravity*, contribution to the Proceedings of the *32nd International Colloquium on Group Theoretical Methods in Physics*, Prague, July 9-13, 2018.
- [2] Mukai S 1996 *Simple Lie algebra and Legendre variety*, Nagoya Sūri Forum **3**, 1-12
- [3] Truini P 2012 *Exceptional Lie Algebras, SU(3) and Jordan Pairs*, Pacific J. Math. **260**, 227
- [4] Marrani A and Truini P 2014 *Exceptional Lie Algebras, SU(3) and Jordan Pairs Part 2: Zorn-type Representations*, J. Phys. **A47** (2014) 265202
- [5] Loos O 1975 *Jordan Pairs*. Lecture Notes in Mathematics **460**, (Berlin, Heidelberg, New York: Springer)
- [6] Ferrara S, Marrani A and Zumino B 2013 *Jordan Pairs, E<sub>6</sub> and U-Duality in Five Dimensions*, J. Phys. **A46** (2013) 065402
- [7] Ferrara S, Marrani A and Trigiante M, *Super-Ehlers in Any Dimension*, JHEP **1211** (2012) 068
- [8] Landsberg J M and Manivel L 2002 *Triality, Exceptional Lie Algebras and Deligne Dimension Formulas*, Adv. Math. **15**, 59
- [9] McCrimmon K 1978 *Jordan Algebras and their Applications* Bull. Amer. Math. Soc. **84**, N. 4, 612-627
- [10] Jordan P von Neumann J and Wigner E 1934 *On an algebraic generalization of the quantum mechanical formalism*, Ann. Math. **35**, 29

- [11] Tits J 1962 *Une classe d'algèbres de Lie en relation avec les algèbres de Jordan*, Nederl. Akad. Wetensch. Proc. Ser. **A 65** = Indagationes Mathematicae **24**, 530
- [12] Freudenthal H 1953 *Sur le groupe exceptionnel E7* Nederl. Akad. Wetensch. Proc. Ser. A. **56**=Indagationes Math. **15** 81-89
- [13] Günaydin M Koepsell K and Nicolai H 2001 *Conformal and Quasiconformal Realizations of Exceptional Lie Groups*, Commun. Math. Phys. **221**, 57
- [14] Günaydin M and Pavlyk O 2005 *Minimal Unitary Realizations of Exceptional U-Duality Groups and their Subgroups as Quasiconformal Groups*, JHEP **0501**, 019
- [15] Günaydin M 2010 *Lectures on Spectrum Generating Symmetries and U-duality in Supergravity, Extremal Black Holes, Quantum Attractors and Harmonic Superspace*, Springer Proc. Phys. **134**, 31
- [16] Minchenko A N 2006 *The Semisimple Subalgebras of Exceptional Lie Algebras*, Trans. Moscow Math. Soc., 225–259 (AMS, Providence RI)
- [17] Cantarini N, Ricciardo A and Santi A 2018 *Classification of simple linearly compact Kantor triple systems over the complex numbers*, J. Algebra **514**, 468
- [18] Truini P, Rios M and Marrani A 2018 *The magic star of exceptional periodicity*, contribution to the Proceedings of the *4th Mile High Conference on Nonassociative Mathematics*, Denver, CO USA, July 29 - August 5, 2017, to appear on AMS Contemporary Mathematics, e-Print: arXiv:1711.07881 [hep-th]
- [19] de Graaf W A 2000 *Lie Algebras: Theory and Algorithms*, North-Holland Mathematical Library **56** (Elsevier, Amsterdam)
- [20] Kac V G 1990 *Infinite Dimensional Lie Algebras*, third edition (Cambridge University Press, Cambridge)
- [21] D'Auria R Ferrara S Lledó M A Varadarajan V S 2001 *Spinor Algebras*, J. Geom.Phys. **40**, 101-128
- [22] Vinberg E B 1963 *The theory of Convex Homogeneous Cones*, Trans. Moscow Math. Soc., 340–403 (AMS, Providence RI)
- [23] Barton C H and Sudbery T 2003 *Magic squares and matrix models of Lie algebras*, Adv. Math. **180**, 596-647
- [24] de Wit B Vanderseypen F and Van Proeyen A 1993 *Symmetry structure of special geometries*, Nucl. Phys. **B400**, 463-524
- [25] Cecotti S 1989 *Homogeneous Kähler Manifolds and T Algebras in  $\mathcal{N}=2$  Supergravity and Superstrings*, Commun. Math. Phys. **124**, 23-55